Linear programming
Background material

Fabian Bastin
fabian.bastin@umontreal.ca
Université de Montréal – CIRRELT – IVADO – Fin-ML

Linear program

Linear program, standard form:

$$\begin{aligned} \min_x \; & c^T x \\ \text{s.t. } & Ax = b, \\ & x \geq 0, \end{aligned}$$

with $x, c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$.

Assumptions: rank(A) = m, $m \leq n$.

Standard form

Any linear program can be transformed to the standard form. Consider for instance the program

$$\min_x c^T x \text{ s.t. } Ax \geq b.$$

Note that there is no bound constraints on x.

We first add a vector z of surplus variables:

$$\begin{aligned} \min_x \ & c^T x \\ \text{s.t. } & Ax - z = b, \\ & z \geq 0. \end{aligned}$$

We next decompose x as the difference of two non-negative variables: $x = x^+ - x^-$, where $x^+ = \max\{x, 0\} \geq 0$, and $x^- = \max\{-x, 0\} \geq 0$.

Standard form (cont'd)

The problem becomes

$$
\begin{aligned}
\min_{x} \quad & \begin{pmatrix} c & -c & 0 \end{pmatrix} \begin{pmatrix} x^+ \\ x^- \\ z \end{pmatrix}, \\
\text{s.t.} \quad & \begin{pmatrix} A & -A & -I \end{pmatrix} \begin{pmatrix} x^+ \\ x^- \\ z \end{pmatrix} = b, \begin{pmatrix} x^+ \\ x^- \\ z \end{pmatrix} \geq 0.
\end{aligned}
$$

The inequalities constraints of the form $x \leq u$ or $Ax \leq b$ can be handled by adding slack variables:

$$
\begin{aligned}
x \leq u &\Leftrightarrow x + w = u, \; w \geq 0, \\
Ax \leq b &\Leftrightarrow Ax + y = b, \; y \geq 0.
\end{aligned}
$$

Basic solutions

Without loss of generality, suppose that the first m columns are independent, and form

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{B} & \boldsymbol{D} \end{pmatrix}$$

Basic solution: $\boldsymbol{x} = (\boldsymbol{x}_b \; 0)$, with $\boldsymbol{B}\boldsymbol{x}_b = \boldsymbol{b}$.
Degenerated basic solution: if some components of $\boldsymbol{x}_b$ are equal to zero.
Feasible basic solution: basic solution such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq 0$.

Fundamental theorem of linear programming

Consider an LP under standard form, with $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, and rank($\boldsymbol{A}$) = m.

- If there exists a feasible solution, then there exists a feasible basic solution.
- If there exists a feasible optimal solution, then there exists a feasible optimal basic solution.

Duality

- Primal program

$$\min_{x \geq 0} c^T x$$
$$\text{s.t. } Ax \geq b$$

- Dual program

$$\max_{\lambda \geq 0} \lambda^T b$$
$$\text{s.t. } A^T \lambda \leq c$$

- $A \in \mathbb{R}^{m \times n}$, $c, x, \in \mathbb{R}^n$, $\lambda, b \in \mathbb{R}^m$.

Duality

- x: primal variables
- λ: dual variables
- Dual of the dual:

$$\begin{aligned} \min_x \; & c^T x \\ \text{s.t. } & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Duality: standard form

The program

$$\begin{aligned} \min_x \; & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq 0 \end{aligned}$$

is equivalent to

$$\begin{aligned} \min_x \; & c^T x \\ \text{s.t. } & Ax \geq b \\ & -Ax \geq -b \\ & x \geq 0 \end{aligned}$$

Duality: standard form

The dual problem can then be written as

$$\begin{aligned}\max_{u,v}\ & u^T b - v^T b \\ \text{s.t. }\ & u^T A - v^T A \le c^T \\ & u \ge 0 \\ & v \ge 0\end{aligned}$$

or, with $\lambda = u - v$,

$$\begin{aligned}\max_{\lambda}\ & \lambda^T b \\ \text{s.t. }\ & \lambda^T A \le c^T\end{aligned}$$

Asymmetric from $\lambda \in \mathbb{R}^m$.

Primal-dual conversion

Minimization	**Maximization**
Constraints	Variables
$\geq$	≥ 0
$\leq$	≤ 0
$=$	unconstrained
Variables	Constraints
≥ 0	$\leq$
≤ 0	$\geq$
unconstrained	$=$

Weak duality

(Asymmetric or symmetric form)

If x and λ are feasible for the primal and the dual, respectively, then

$$c^T x \geq \lambda^T b$$

Proof.

For any primal feasible x and dual feasible λ, we have

$$\lambda^T b \leq \lambda^T A x \leq c^T x.$$

$\square$

Corollary

If x_0 and λ_0 are feasible for the primal and the dual, respectively, and if

$$c^T x_0 = \lambda_0^T b,$$

then x_0 and λ_0 are optimal for their respective problem.

But we have said nothing about the feasibility of one problem with respect to the other one!

Strong duality

If one of the problems, primal or dual, has a finite optimal solution, the other problem also has a finite optimal solution, and the corresponding values of the objective functions are equal. If one of the problems has an unbounded objective, the other problem has no feasible solution.

If a program is feasible, this does not imply that its dual is unbounded. It can also be unfeasible.

Primal / Dual	Bounded	Unbounded	Unfeasible
Bounded	possible	impossible	impossible
Unbounded	impossible	impossible	possible
Unfeasible	impossible	possible	possible

Optimality et duality

The optimality can be deduced from KKT conditions.

In LP, we do not need a constraint qualification to apply KKT conditions.

Lagrangian:

$$\mathcal{L} = c^T x - \pi^T (Ax - b) - s^T x.$$

KKT conditions: Lagrangian vectors π and s s.t.

$$\begin{aligned} A^T \pi + s &= c, \\ Ax &= b, \\ x &\geq 0, \\ s &\geq 0, \\ x_i s_i &= 0, \; i = 1, 2, \ldots, n. \end{aligned}$$

Dual problem

Let (x^*, π^*, s^*) be a vector satisfying the KKT contions. We have that

$$c^T x^* = (A^T \pi^* + s^*)^T x^* = (Ax^*)^T \pi^* = b^T \pi^*.$$

It is furthermore easy to show that the (necessary) KKT conditions are sufficient.

Dual problem:

$$\max_{\pi} b^T \pi, \text{ s.t. } A^T \pi \leq c,$$

or, equivalently,

$$\max_{\pi} b^T \pi, \text{ s.t. } A^T \pi + s = c, \; s \geq 0.$$