Stochastic optimization
Properties of recourse models

Fabian Bastin

Two-stage linear programming problem, fixed recourse

Consider again the problem

$$\min c^T x + E_{\boldsymbol{\xi}}[q(\xi)^T y(\xi)]$$

subject to the constraints

$$\begin{aligned} Ax &= b, \\ T(\xi)x + Wy(\xi) &= h(\xi) \qquad \forall \xi \in \Xi, \\ x &\in X, \\ y(\xi) &\in Y, \end{aligned}$$

where $\boldsymbol{\xi}$ is a random vector defined on the random space $(\Omega, \mathcal{F}, P)$, and Ξ is the support of $\boldsymbol{\xi}$.

Let

$$Q(x, \xi(\omega)) = \min_{y \in Y} \left\{ q(\xi)^T y : Wy = h(\xi) - T(\xi)x \right\}.$$

Reformulation(s)

$$\min_{x \in X \,|\, Ax=b} \left\{ c^T x + E_{\boldsymbol{\xi}} \left[\min_{y \in Y} \{ q(\boldsymbol{\xi})^T y \mid Wy = h(\boldsymbol{\xi}) - T(\boldsymbol{\xi})x \} \right] \right\}$$

Second-stage function, or recourse function, $v : \Xi \times \mathbb{R}^m \to \mathbb{R}$:

$$v(\xi, z) \stackrel{def}{=} \{ q(\xi)^T y \mid Wy = z \}.$$

Given a "policy" x and a realization of the random vector $\boldsymbol{\xi}$, z measures the deviation of the first stage, i.e. $z = h(\xi) - T(\xi)x$, $v(\xi, z)$ is the minimum cost to "correct" the decision in order to satisfy the constraints again.

Recourse function

The expected recourse function, or the function of minimum expected recourse, $\mathcal{Q} : \mathbb{R}^n \to \mathbb{R}$, for any policy $x \in \mathbb{R}^n$:

$$\mathcal{Q} \stackrel{def}{=} E_{\boldsymbol{\xi}}[Q(x, \boldsymbol{\xi})],$$

describes the recourse cost expectation, with

$$Q(x, \xi) = v(\xi, h(\xi) - T(\xi)x).$$

With these definitions, the problem can be rewritten as:

$$\min_{x \in X} c^T x + \mathcal{Q}(x) \text{ such that } Ax = b.$$

It is a nonlinear program over $\mathbb{R}^n$. Properties?

Summary

Summarize our formulations.

$$\min_{x \in \mathbb{R}^n_+ \mid Ax=b} \left\{ c^T x + E_\xi \left[\min_{y \in \mathbb{R}^p_+} \{ q(\xi)^T y \mid Wy = h(\xi) - T(\xi)x \} \right] \right\}$$

$$\min_{x \in \mathbb{R}^n_+ \mid Ax=b} \left\{ c^T x + E_\xi \left[v(\xi, h(\xi) - T(\xi)x) \right] \right\}$$

$$\min_{x \in \mathbb{R}^n_+ \mid Ax=b} \left\{ c^T x + E_\xi \left[Q(x, \xi) \right] \right\}$$

$$\min_{x \in \mathbb{R}^n_+} \left\{ c^T x + \mathcal{Q}(x) \mid Ax = b \right\}$$

Notations

- First-stage feasible set:

$$K_1 = \{x \in \mathbb{R}^n_+ \mid Ax = b\}.$$

- Second-stage feasible set:

$$K_2 = \{x \mid \mathcal{Q}(x) < \infty\}.$$

Therefore we can rewrite the problem as

$$\min_x \{c^T x + \mathcal{Q}(x) \mid x \in K_1 \cap K_2\}.$$

Relatively complete recourse

A problem is said to have a relatively complete recourse if $K_1 \subseteq K_2$. Advantage: $\forall x$ feasible in the first stage, we have $\mathcal{Q}(x) < \infty$, so we do not have to consider the case $Q(x, \xi) = \infty$.

We can also define the set of second-stage feasible points, given a realization ξ:

$$K_2(\xi) = \{x \mid Q(x, \xi) < \infty\}.$$

Define

$$K_2^P = \cap_{\xi \in \Xi} K_2(\xi).$$

Clearly $K_2 = K_2^P$ if ξ has a finite support. Is it still the case when ξ follows a continuous distribution?

Relatively complete recourse (cont'd)

We have the following results.

Theorem

If ξ has finite second order moments, $P[\xi \mid Q(x,\xi) < \infty] = 1$ implies $\mathcal{Q}(x) < \infty$.

In other words, we must have that $Q(x,\xi)$ is upper bounded almost surely. Proof: technical!

Reminder: almost surely, or with probability one. An event A is said to occur almost surely if $P[A] = 1$.

More interestingly, we have

Theorem

For a stochastic program with fixed recourse, where ξ has finite second order moments, the sets K_2 and K_2^p are the same.

Complete recourse

The relatively complete recourse is very useful in practice and on a theoretical point of view, but it can be difficult to identify. A particular case of relatively complete recourse can however often be identified from the structure of W.

We say that a problem has a complete recourse if $\forall\, z \in \mathbb{R}^m$, $\forall \xi$, $v(\xi, z) < +\infty$. In other terms, $\forall\, z \in \mathbb{R}^m$, $\exists\, y \in \mathbb{R}^m_+$ such that $Wy = z$, i.e. if the matrix W satisfies

$$\{z \mid z = Wy,\ y \geq 0\} = \mathbb{R}^m.$$

This also implies that $\forall\, x$, $T(\xi)$, $h(\xi)$, $Q(x, \xi) < \infty$, as $z = h - Tx$.

Simple recourse

A particular case of complete recourse is the simple recourse, for which we have

$$W = \begin{pmatrix} I & -I \end{pmatrix},$$

with I the identiy matrix, of order m.

In this case, the second stage program can be read as

$$\begin{aligned} Q(x,\xi) = \min_y\ & q^+(\xi)^T y^+ + q^-(\xi)^T y^- \\ \text{s.t. } & y^+ - y^- = h(\xi) - T(\xi)x, \\ & y^+,\ y^- \geq 0. \end{aligned}$$

That is, for $q^+(\xi) + q^-(\xi) \geq 0$, the recourse variables y^+ and y^- can be chosen to measure the absolute violations in the stochastic constraints.

Simple recourse (cont'd)

Theorem

Assume that the two-stage (linear) stochastic program is feasible and has a simple recourse, and that ξ has finite second-order moments. Then $\mathcal{Q}(x)$ is finite if and only if $\boldsymbol{q}_i^+ + \boldsymbol{q}_i^- \geq 0$ with probability one.

Simple recourse (cont'd)

Proof.

($\Rightarrow$) Assume by contradiction that $\mathcal{Q}$ is finite, but for some component i, $q_i^+(\xi(\omega)) + q_i^-(\xi(\omega)) < 0$ for $\omega \in \Omega_1$ with $P[\Omega_1] > 0$. Then, for any feasible x, for all $\omega \in \Omega_1$ with $h_i(\xi(\omega)) - T_i(\xi(\omega))x > 0$, define

$$y_i^+(\xi(\omega)) = h_i(\xi(\omega)) - T_i(\xi(\omega))x + u, \; y_i^-(\xi(\omega)) = u.$$

Therefore,

$$y_i^+ - y_i^-(\xi(\omega)) = h_i(\xi(\omega)) - T_i(\xi(\omega))x, \; y_i^+ \geq 0, \; y_i^- \geq 0.$$

Moreover, since $\mathcal{Q}$ is finite, $Q(x, \xi(\omega))$ is feasible almost surely, so, almost surely, we can choose y_j^+ and y_j^- feasible, $j \neq i$.

$\square$

Simple recourse (cont'd)

Proof.

$(\Rightarrow)$

When $u \to \infty$, $Q(x, \xi(\omega)) \to -\infty$ since $q_i^+(\xi(\omega))y_i^+ + q_i^-(\xi(\omega))y_i^- \to -\infty$.

A similar argument can be applied if $h_i(\xi(\omega)) - T_i(\xi(\omega))x \leq 0$, by taking

$$y_i^+(\xi(\omega)) = u, \; y_i^-(\xi(\omega)) = -h_i(\xi(\omega)) + T_i(\xi(\omega))x + u.$$

By componing these two cases, we conclude that $\mathcal{Q}$ is not finite. $\square$

Simple recourse (cont'd)

Proof.

($\Leftarrow$) Assume $\boldsymbol{q}_i^+ + \boldsymbol{q}_i^- \geq 0$ with probability one, $\forall i$. Any feasible solution satisfies

$$y^+(\omega) - y^-(\omega) = h(\xi(\omega)) - T(\xi(\omega))x, \; y^+(\omega) \geq 0, \; y^-(\omega) \geq 0.$$

Therefore for almost every ω, $Q(x, \xi(\omega))$ is bounded below by 0, and from the fundamental theorem of linear programming, we can choose as optimal solution

$$\begin{aligned} y^+(\omega) &= (h(\xi(\omega)) - T(\xi(\omega))x)^+ \,, \\ y^-(\omega) &= (-h(\xi(\omega)) + T(\xi(\omega))x)^+ \,, \end{aligned}$$

where $a^+ = \max\{0, a\}$. $\square$

Simple recourse (cont'd)

Proof.

($\Leftarrow$) Thus,

$$Q(x, \xi(\omega)) = \sum_{i=1}^{m} (q_i^+(\xi(\omega))(h_i(\xi(\omega)) - T_i(\xi(\omega))x)^+ + \\ q_i^-(\xi(\omega))(-h_i(\xi(\omega)) + T_i(\xi(\omega))x)^+)$$

Consequently $Q(x, \xi(\omega))$ is finite for almost every ω and bounded below by 0.

Therefore, $\mathcal{Q}(x)$ is bounded below by 0, and according to the previous results, $\mathcal{Q}(x) < \infty$. This implies that $\mathcal{Q}(x)$ is finite. $\square$

Exercise

Consider the second stage program

$$Q(x, \xi) = \min_y \{y \mid \xi y = 1 - x, y \geq 0\}.$$

We assume that $\boldsymbol{\xi}$ follows a triangular distribution on [0, 1], with $P[\boldsymbol{\xi} \leq u] = u^2$.

(a) Is the recourse fixed? Why?

The recourse is not fixed, as $W \equiv \boldsymbol{\xi}$, and therefore, W is random. Moreover, as ξ can take the value 0, the transformation

$$y = 1/\xi - x/\xi,$$

is not properly defined on $\Xi = [0, 1]$; this also means that

$$W = \begin{cases} 0 & \text{si } \xi = 0; \\ 1 & \text{si } \xi \neq 0. \end{cases}$$

Exercise (cont'd)

(b) Express $K_2(\xi)$ for all ξ in $[0,1]$.

We have to consider two cases: $\xi = 0$ or $\xi \in (0,1]$.

1. $\xi \in (0,1]$ In this case, as $y, \xi \geq 0$, $1 - x$ has to be non-negative in order to have a well-defined problem:

$$K_2(\xi) = \{x \mid x \leq 1\}.$$

The value and optimal solutions are

$$Q^*(x,\xi) = (1-x)/\xi, \quad y^* = (1-x)/\xi.$$

2. $\xi = 0$ There exists no y such that $0.y = 1 - x$, except if $x = 1$, so

$$K_2(0) = \{1\}.$$

Exercise (cont'd)

(c) Express K_2, K_2^P and $\mathcal{Q}$.

From the previous point, we have

$$K_2^P = \{x \,|\, x \leq 1\} \cap \{1\} = \{1\}.$$

We also have, as $P[\xi = 0] = 0$,

$$\mathcal{Q}(x) = \int_0^1 \frac{1-x}{\xi} 2\xi d\xi = 2(1-x), \forall x \leq 1.$$

Consequently $K_2^P \subset K_2 = \{x \leq 1\}$.

The difference comes from the fact that a point is not in K_2^P as soon as it is not feasible for a given value of ξ, but K_2 does not consider unfeasible situations that occur with a null probability.

Recourse function

Let y_1^* and y_2^* be two optimal solutions of $v(\xi, z)$, associated to $z = z_1$ and $z = z_2$, respectively. Then, the convex combination $y_\alpha^* = \alpha y_1^* + (1-\alpha)y_2^*$, $\alpha \in [0,1]$, is feasible with respect to $z_\alpha = \alpha z_1 + (1-\alpha)z_2$, as $\alpha y_1^* + (1-\alpha)y_2^* \geq 0$, and

$$W(\alpha y_1^* + (1-\alpha)y_2^*) = \alpha W y_1^* + (1-\alpha) W y_2^* = \alpha z_1 + (1-\alpha)z_2 = z_\alpha.$$

Moreover,

$$\begin{aligned} v(\xi, z_\alpha) &= q(\xi)^T y_\alpha^* \leq q(\xi)^T(\alpha y_1^* + (1-\alpha)y_2^*) \\ &= \alpha q(\xi)^T y_1^* + (1-\alpha) q(\xi)^T y_2^* \\ &= \alpha v(\xi, z_1) + (1-\alpha) v(\xi, z_2). \end{aligned}$$

In other words, v is a convex function w.r.t. $z \in \mathbb{R}^m$.

Convexity of $Q(x, \xi)$?

$$Q(x, \xi) = v(\xi, h(\xi) - T(\xi)x).$$

$$\begin{aligned}
\lambda Q(x_1, \xi) &+ (1-\lambda) Q(x_2, \xi) \\
&= \lambda v(\xi, h(\xi) - T(\xi)x_1) + (1-\lambda) v(\xi, h(\xi) - T(\xi)x_2) \\
&\geq v(\xi, \lambda(h(\xi) - T(\xi)x_1) + (1-\lambda)(h(\xi) - T(\xi)x_2)) \\
&= v(\xi, h(\xi) - T(\xi)(\lambda x_1 + (1-\lambda)x_2)) \\
&= Q(\lambda x_1 + (1-\lambda)x_2, \xi).
\end{aligned}$$

Therefore $Q(x, \xi)$ if convex w.r.t. x, given ξ. More generally

Theorem

If A if a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, and $f(x)$ is a convex function on $\mathbb{R}^m$, the composite function $(fA)(x) \stackrel{def}{=} f(Ax)$ is a convex function on $\mathbb{R}^n$.

Convexity of second-stage function

We have the following result (Birge and Louveaux, Chapter 3, Theorem 5).

Theorem

For a stochastic program with fixed recourse, $Q(x, \xi)$ is

(a) *a piecewise convex linear function in (h, T),*

(b) *a piecewise concave linear function in q,*

(c) *a piecewise convex linear function in x, for all x in $K = K_1 \cap K_2$.*

Convexity of second-stage function (cont'd)

Proof.

In order to show convexity in (a) and (c), it is sufficient to prove that $v(\xi, z) = \min\{q(\xi)^T y \mid Wy = z\}$ is convex, which has already been done. We can proceed similarly to show concavity w.r.t. q.

The piecewise linearity follows from the fact that the number of different optimal bases for a linear program is finite. $\square$

Convexity of the recourse?

$$\mathcal{Q}(x) = E_{\boldsymbol{\xi}}[Q(x, \xi)].$$

Suppose for now that ξ has a finite support, i.e. $\Xi = \{\xi_1, \xi_2, \ldots, \xi_m\}$. Then

$$\mathcal{Q}(x) = \sum_{i=1}^{m} P[\boldsymbol{\xi} = \xi_i] Q(x, \xi_i).$$

Convexity of the recourse

Theorem

If $f(x)$ is convex, and $\alpha \geq 0$, $g(x) = \alpha f(x)$ is convex.

Theorem

If $f_k(x)$, $k = 1, 2, \dots, K$, are convex functions, then $g(x) = \sum_{k=1}^{K} f_k(x)$ is convex.

$\mathcal{Q}(x)$ is therefore a convex function w.r.t. x.

What is happening in the continuous case?

We have the following result: if $g(x, y)$ is convex w.r.t. x, then $\int g(x, y) dy$ is convex w.r.t. x. Since

$$\mathcal{Q}(x) = \int_{\Xi} Q(x, t) dF(t),$$

$\mathcal{Q}(x)$ is convex.

An example...

Consider the second-stage function $Q(x, \xi)$ defined as:

$$\min y^+ + y^- \text{ s.t. } y^+ - y^- = \xi - x,\ y^+ \geq 0,\ y^- \geq 0.$$

In other terms:

$$y = \begin{pmatrix} y^+ \\ y^- \end{pmatrix} \quad q = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, W = \begin{pmatrix} 1 & -1 \end{pmatrix} \quad h(\xi) = \xi \quad T(\xi) = 1.$$

Dual:

$$\begin{aligned} &\max\ (\xi - x)\pi \\ &\text{s.t. } \pi \leq 1,\ -\pi \leq 1 \end{aligned}$$

or

$$\begin{aligned} &\max\ (\xi - x)\pi \\ &\text{s.t. } \pi + s_1 = 1 \\ &\qquad -\pi + s_2 = 1 \\ &\qquad s_1 \geq 0,\ s_2 \geq 0 \end{aligned}$$

An example: optimality conditions

The recourse is simple, and the primal-dual/KKT conditions give

$$\begin{aligned}
\begin{pmatrix}1\\1\end{pmatrix} &= \begin{pmatrix}1\\-1\end{pmatrix}\pi + \begin{pmatrix}s_1\\s_2\end{pmatrix}\\
y^+ - y^- &= \xi - x\\
y^+ &\geq 0,\ y^- \geq 0\\
s_1 &\geq 0,\ s_2 \geq 0\\
s_1 y^+ &= 0,\ s_2 y^- = 0.
\end{aligned}$$

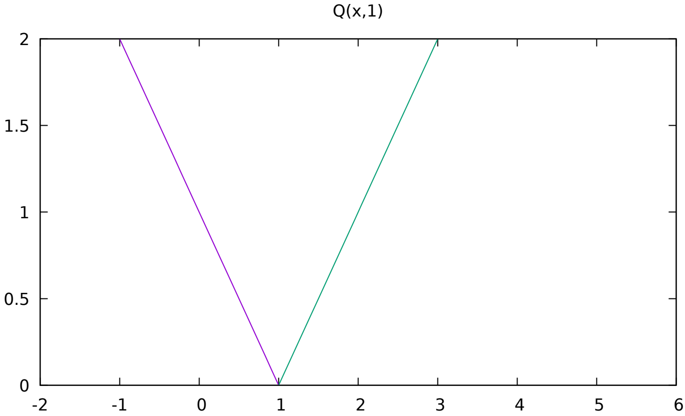
An example (cont'd)

- The first condition implies that we cannot have $s_1 = s_2 = 0$.
- From the complementarity conditions, we have that $y^+ = 0$ or $y^- = 0$.
- We have to consider two cases:
  - $x \leq \xi$: in this situation, we have
$$y^+ = \xi - x, \quad y^- = 0.$$
  - $x \geq \xi$: then,
$$y^- = x - \xi, \quad y^+ = 0.$$
- Consequently,

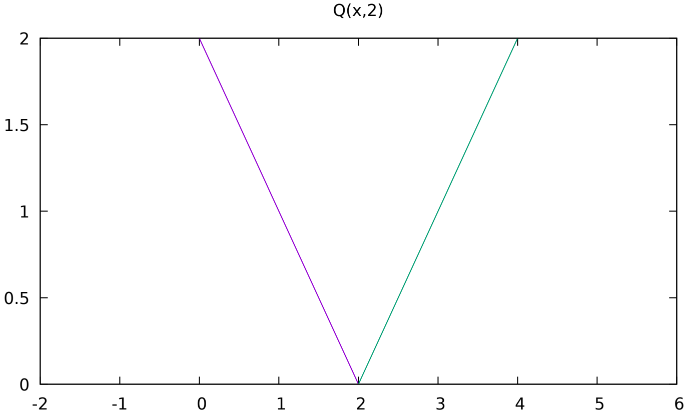
$$Q(x, \xi) = \begin{cases} \xi - x & \text{si } x \leq \xi, \\ x - \xi & \text{si } x \geq \xi. \end{cases}$$

Graphically?

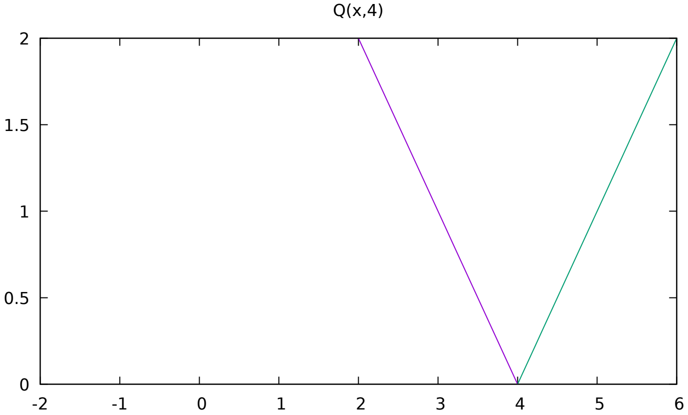
Assume that ξ can take the realizations 1, 2, 4.



Graphically (cont'd)



Graphically (cont'd)



$\mathcal{Q}(x)$

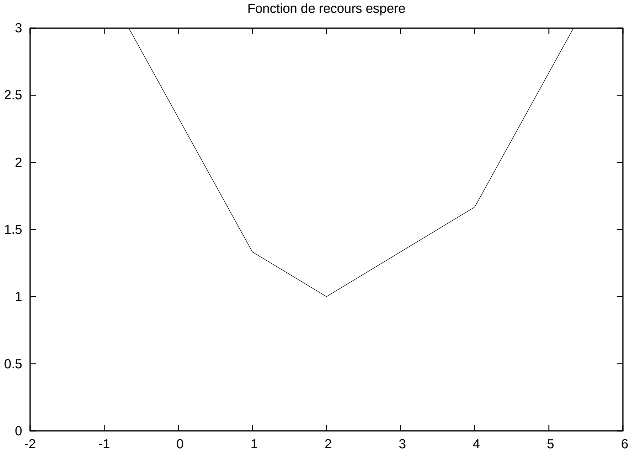
What about $\mathcal{Q}(x)$?

Assume that the three realizations have the same probability.

We have to consider 4 cases:

1. $x \leq 1$: $\mathcal{Q}(x) = 7/3 - x$;
2. $1 \leq x \leq 2$: $\mathcal{Q}(x) = 5/3 - x/3$;
3. $2 \leq x \leq 4$: $\mathcal{Q}(x) = x/3 + 1/3$;
4. $4 \leq x$: $\mathcal{Q}(x) = x - 7/3$;

Graphically



Properties of $\mathcal{Q}(x)$

We note that $\mathcal{Q}(x)$ is convex and piecewise linear. As $\mathcal{Q}(x)$ is a finite weighted sum of piecewise linear functions when the support of ξ is finite, we have the following result.

Theorem

For a stochastic program with fixed recourses where ξ has finite second-order moments,

(a) *$\mathcal{Q}(x)$ is a convex Lipschitz function and is finite over K_2;*

(b) *when ξ has a finite support, $\mathcal{Q}(x)$ is piecewise linear.*

Reminder: a function f is Lipschitz if there exists some $M < \infty$ such that for all x, y,

$$|f(x) - f(y)| \leq M\|x - y\|.$$

Differentiability of the recourse

Is $\mathcal{Q}(x)$ also differentiable?

The recourse function is partially differentiable with respect to x_j at $(\hat{x}, \hat{\xi})$ if the directional derivative exists for the direction e_j. In other terms, there exists a function $\frac{\partial Q(x,\xi)}{\partial x_j}$ such that

$$\frac{Q(\hat{x} + he_j, \hat{\xi}) - Q(\hat{x}, \hat{\xi})}{h} = \frac{\partial Q(x,\xi)}{\partial x_j} + \frac{\rho_j(\hat{x}, \hat{\xi}; h)}{h},$$

with

$$\frac{\rho_j(\hat{x}, \hat{\xi}; h)}{h} \to 0, \text{ as } h \to 0.$$

We will assume from now that $\nabla_x Q(x,\xi) = \left(\frac{\partial Q(x,\xi)}{\partial x_1}, \dots, \frac{\partial Q(x,\xi)}{\partial x_n}\right)$ exists.

Differentiability of the recourse (cont'd)

What about the differentiability of $\mathcal{Q}(x)$?

$$
\begin{aligned}
\frac{\mathcal{Q}(\hat{x}+he_j)-\mathcal{Q}(\hat{x})}{h} &= \int_{\Xi} \frac{Q(\hat{x}+he_j,\xi)-Q(\hat{x},\xi)}{h}dP \\
&= \int_{\Xi\setminus N_\delta} \frac{\partial Q(\hat{x},\xi)}{\partial x_j}dP + \int_{\Xi\setminus N_\delta} \frac{\rho_j(\hat{x},\xi;h)}{h}dP,
\end{aligned}
$$

where N_δ is measurable and $P[N_\delta]=0$. Therefore, we have

Theorem

If $Q(x,\xi)$ if partially differentiable almost everywhere, if its partial derivative $\frac{\partial Q(\hat{x},\xi)}{\partial x_j}$ is integrable and if the residual satisfies $(1/h)\int_{\Xi\setminus N_\delta}\rho_j(\hat{x},\xi;h)dP \overset{h\to 0}{\to} 0$, then $\frac{\partial \mathcal{Q}(\hat{x})}{\partial x_j}$ exists and

$$
\frac{\partial \mathcal{Q}(\hat{x})}{\partial x_j} = \int_{\Xi} \frac{\partial Q(\hat{x},\xi)}{\partial x_j}dP.
$$

Differentiability of the recourse: discrete case

But how to prove $(1/p)\int_{\Xi\setminus N_\delta}\rho_j(\hat{x},\xi;h)dP \overset{h\to 0}{\to} 0$?

If we stay in the linear framework with fixed recourse, and vectors $\boldsymbol{\xi}$ with finite second-order moments, we have seen that for $\boldsymbol{\xi}$ with finite support, $\mathcal{Q}(x)$ is piecewise linear. Therefore $\mathcal{Q}(x)$ is not differentiable.

Differentiability of the recourse: continuous case

If $\boldsymbol{\xi}$ is continuous, $\mathcal{Q}(x)$ is obtained as an integral over the $Q(x, \xi)$'s, that are not differentiable as they are piecewise linear given ξ. However, it is x that has to be fixed, not ξ. It is possible to show that (the proof is quite technical)

Theorem

For a stochastic program with fixed recourse where $\boldsymbol{\xi}$ has finite second-order moments, if $\boldsymbol{\xi}$ is continuous, $\mathcal{Q}(x)$ is differentiable over K_2.

Intuitively, the function $\mathcal{Q}(x)$ is "smoothed" by the superposition of an infinite number of functions $Q(x, \xi)$.

Two-stage non-linear problems

Now consider the general program

$$\min_{x \in X} E_{\boldsymbol{\xi}}[f_0(x, \boldsymbol{\xi})] = \min_{x \in X} E_{\boldsymbol{\xi}}[g_0(x, \boldsymbol{\xi}) + Q(x, \boldsymbol{\xi})].$$

Theorem

If $g_0(\cdot, \xi)$ and $Q(\cdot, \xi)$ are convex with respect to x, $\forall\, \xi \in \Xi$, and if X is a convex set, the aforementioned program is convex.

Proof.

For x, $y \in X$, $\lambda \in (0, 1)$ and $z = \lambda x + (1 - \lambda)y$, we have

$$\begin{aligned} & g_0(z, \xi) + Q(z, \xi) \\ & \qquad \leq \lambda(g_0(x, \xi) + Q(x, \xi)) + (1 - \lambda)(g_0(y, \xi) + Q(y, \xi)). \end{aligned}$$

The result follows by taking the expectation. $\square$

In a more standard form

Inspired from Birge et Louveaux, Section 3.4.

We consider the problem

$$\begin{aligned} \inf z = {} & f^1(x) + \mathcal{Q}(x), \\ \text{s.t. } & g_i^1(x) \leq 0,\ i = 1, \ldots, \overline{m}_1, \\ & g_i^1(x) = 0,\ i = \overline{m}_1 + 1, \ldots, m_1, \end{aligned}$$

where $\mathcal{Q}(x) = E_\omega[Q(x,\omega)]$ and

$$\begin{aligned} Q(x,\omega) = {} & \inf f^2(y(\omega),\omega), \\ & \text{s.t. } t_i^2(x,\omega) + g_i^2(y(\omega),\omega) \leq 0,\ i = 1, \ldots, \overline{m}_2, \\ & \phantom{\text{s.t. }} t_i^2(x,\omega) + g_i^2(y(\omega),\omega) = 0,\ i = \overline{m}_2 + 1, \ldots, m_2, \end{aligned}$$

We say that the recourse is additive (why?).

In a more standard form (cont'd)

The functions $f^2(\cdot,\omega)$, $t_i^2(\cdot,\omega)$, and $g_i^2(\cdot,\omega)$ are continuous for any given ω, and measurable w.r.t. ω for any given first argument. This allows to prove that $Q(x,\omega)$ is measurable, and therefore that $\mathcal{Q}(x)$ is well defined.

Reintroduce K_1, $K_2(\omega)$ and K_2.

$$
\begin{aligned}
K_1 = \{x \mid\ & g_i^1(x) \leq 0,\ i = 1,\ldots,\overline{m}_1, \\
& g_i^1(x) = 0,\ i = \overline{m}_1 + 1,\ldots,m_1\}, \\
K_2(\omega) = \{x \mid\ & \exists\, y(\omega) \text{ t.q. } t_i^2(x,\omega) + g_i^2(y(\omega),\omega) \leq 0,\ i = 1,\ldots,\overline{m}_2, \\
& t_i^2(x,\omega) + g_i^2(y(\omega),\omega) = 0,\ i = \overline{m}_2 + 1,\ldots,m_2\}, \\
K_2 = \{x \mid\ & \mathcal{Q}(x) < \infty\}.
\end{aligned}
$$

Remarks

The formulation is not yet totally general. We will consider more general forms when we will discuss sampling techniques.

Here, there is no more fixed recourse, but the first-stage decision x acts separately in the constraints. Goal: extend the previous results.

Questions: convexity, differentiability, optimality. We will also consider the concept of lower semi-continuity.