# Stochastic optimization
# Two-stage stochastic programming with recourse

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# Formalization

Uncertainty: representation by means of random elements. The realizations are denoted by $\omega$, and they are drawn form the sample space $\Omega$.

A event $A$ is a subset of $\Omega$; the collection of random of random events is denoted by $\mathcal{A}$. The event $A \in \mathcal{A}$ occurs if the output of the experiment is an element from $A$.

# A random linear program

Consider the (toy) problem

$$\begin{aligned} \min_{x} \; & x_1 + x_2 \\ \text{s.t.} \; & \omega_1 x_1 + x_2 \geq 7 \\ & \omega_2 x_1 + x_2 \geq 4 \\ & x_1, x_2 \geq 0, \end{aligned}$$

where $\omega_1 \sim U[1, 4]$, $\omega_2 \sim U[1/3, 1]$.

# What to do?

- How to solve this problem?
- What is the *meaning* of solving this problem?
- Is it possible to decide on $x$ *after* having observed the realization of the random vector $\omega$?
  We then talk of an wait-and-see approach. The problem is then easier to solve (we have here a simple linear program).
- But this approach is rarely appropriate!!! We usually have to decide on $x$ before we know the realizations of $\omega$!
- Usually, the “wait-and-see” approach is not appropriate to model the reality behavior: we have to decide on $x$ before we know the realizations from $\omega$.
- Three suggestions:
  1. try to estimate, predict, the uncertainty;
  2. chance-constraints;
  3. penalties on deviations.

# A random linear program

Consider the following linear program, parametrized by the random vector $\omega$:

$$\begin{aligned} \min_x \ & c^T x \\ \text{s.t. } & Ax = b \\ & T(\omega)x = h(\omega) \\ & x \in X, \end{aligned}$$

with $X = \{x \in \mathcal{R}^n | l \leq x \leq u\}$. Example:

$$\begin{aligned} \min_x \ & x_1 + x_2 \\ \text{s.t. } & \omega_1 x_1 + x_2 \geq 7 \\ & \omega_2 x_1 + x_2 \geq 4 \\ & x_1, x_2 \geq 0. \end{aligned}$$

# Remove the randomness?

A popular approach consists to look for reasonable values for $\omega_1$ and $\omega_2$. How?

Propositions:

- unbiased: choose the mean values for each random variable;
- pessimistic: choose the worst-case values for $\omega$;
- optimistic: choose the best-case values for $\omega$.

Each approach will deliver a different optimal solution!

# Penalization of violations

Again, we have to deal with decision problems where the decision $x$ has to be taken before we know the realization of $\omega$

- We nevertheless have to know the distribution of $\omega$ over $\Omega$. We assume for now that $\Omega$ is finite.
- In models with recourse, the random constraints are "soft". They can be violated, but the violation cost will influence the choice of $x$.
- In fact, a second stage linear program is introduced; it describes how the violated random constraints are handled.

# The new problem...

In the simplest case, we can simply penalize the constraints deviations by vectors of penalty coefficients $q_+$ and $q_-$.

$$\begin{aligned}
\min\ & c^T x + q_+^T s(\omega) + q_-^T t(\omega) \\
\text{s.t. } & Ax = b, \\
& T(\omega)x + s(\omega) - t(\omega) = h(\omega), \\
& x \in X.
\end{aligned}$$

But it is still not possible to solve the problem!

# The new optimization problem

A reasonable, and solvable, problem is then

$$
\begin{aligned}
\min\ & c^T x + E_\omega[q_+^T s(\omega) + q_-^T t(\omega)] \\
\text{s.t. } & Ax = b, \\
& T(\omega)x + s(\omega) - t(\omega) = h(\omega),\ \forall \omega \in \Omega \\
& x \in X.
\end{aligned}
$$

- In general, we can react in a correct (and maybe optimal) way: we have a recourse to “correct” the first decision once the uncertainty is removed.
- A recourse structure in linear progamming is provided by 3 elements:
  - a set $Y \subset \mathcal{R}^p$ that describes the feasible set of recourse actions, for instance $Y = \{y \in \mathcal{R}^p | y \geq 0\}$;
  - $q$: a vector of recourse costs;
  - $W$: a matrix $m \times p$, called the recourse matrix.

# Recourse formulation

The previous considerations lead us to formulate the following program:

$$\begin{aligned} \min\ & c^T x + E_\omega[q^T y] \\ \text{s.t. } & Ax = b, \\ & T(\omega)x + Wy(\omega) = h(\omega),\ \forall \omega \in \Omega \\ & x \in X, \\ & y \in Y,\ \forall \omega. \end{aligned}$$

We could have $W$ varying with the realization $\omega$. If $W$ is unique, as in the previous formulation, we speak of fixed recourse: the recourse does not change with the scenario.
But how to decide on $y$?

# Some definitions

$$\min_{x \in X | Ax=b} \left\{ c^T x + E_\omega \left[ \min_{y \in Y} q^T y | Wy = h(\omega) - T(\omega)x \right] \right\}.$$

- Second stage function, or recourse function (penalty) $v : \mathcal{R}^m \to \mathcal{R}$:

  $$v(z) \stackrel{def}{=} \min_{y \in Y} \{q^T y | Wy = z\};$$

  this function describes the costs related to any vector $z$ representing the "deviations from the random constraints $T(\omega)x = f(\omega)$".
- Expected value function, or recourse of minimum expectation $\mathcal{Q} : \mathcal{R}^n \to \mathcal{R}$:

  $$\mathcal{Q}(x) = E_\omega[v(h(\omega) - T(\omega)x)].$$

  It describes the expected recourse cost, for any policy $x \in \mathcal{R}^n$.

# The two-stage linear stochastic problem (SP)

Using the previous definitions, we can rewrite the stochastic programming problem with recourse in terms of $x$ only:

$$\min_{x \in X} \{c^T x + \mathcal{Q}(x) \mid Ax = b\}.$$

It is a (nonlinear) mathematical programming problem in $\mathcal{R}^n$. The properties of $\mathcal{Q}(x)$ influence the solution techniques.

Is $\mathcal{Q}(x)$

- linear?
- convex?
- continuous?
- differentiable?

# Expression in terms of $y$'s

$$
\begin{aligned}
\min_{x,\ y(\omega)} \ & E_\omega[c^T x + q^T y(\omega)] & \\
\text{s.t. } & Ax = b & \text{first-stage constraints} \\
& T(\omega)x + Wy(\omega) = h(\omega),\ \forall \omega \in \Omega & \text{second-stage contraints} \\
& x \in X,\ y(\omega) \in Y. &
\end{aligned}
$$

Consider the (discrete) case where $\Omega = \{\omega_1,\ \omega_2, \ldots, \omega_S\} \subset \mathcal{R}^r$.

$$
P(\omega = \omega_s) = p_s,\ s = 1, 2, \ldots ..., S
$$
$$
T_s = T(\omega),\ h_s = h(\omega)
$$

# Deterministic equivalent

Develop along the $S$ scenarios.

$$
\begin{aligned}
\min_{x, y_1, \ldots, y_S} \quad & c^T x + p_1 q^T y_1 + p_2 q^T y_2 + \ldots p_S q^T y_S \\
\text{s.t.} \quad & \\
& \begin{array}{lllll}
Ax & & & & = b \\
T_1 x & + W y_1 & & & = h_1 \\
T_2 x & & + W y_2 & & = h_2 \\
\vdots & & & \ddots & \\
T_S x & & & + W y_s & = h_s
\end{array} \\
& x \in X, \; y_1 \in Y, \; y_2 \in Y, \ldots, y_s \in Y.
\end{aligned}
$$

# Deterministic equivalent (cont'd)

- $y_s = y(\omega_s)$ is the recourse action to take if the scenario $\omega_s$ occurs.
- Advantage: it is a linear program.
- Drawback: it is a linear program of (very) large dimension:
  - $n + pS$ variables;
  - $m_1 + mS$ constraints.
- Advantage: the linear program matrix has a special structure (stairway shape).
  Can we exploit it?

# Large scale,... and?

Assume that we have $r$ random variables ($\Omega \subset \mathcal{R}^r$).

- Consider the following problem (source: Linderoth). A Telecom company want to expand its network in order to meet an unknown (random) demand.
- There are 86 unknown demands. Each demand is independant and take a value in a set of 7 values. Consequently

$$S = |\Omega| = 7^{86} \approx 4.77 \times 10^{72}.$$

  ...number of subatomic particles in the universe!
- It can be even worse...
  If $\Omega$ is not finite, but holds an infinite number of elements? It is especially true with continuous random variables. Our "deterministic equivalent" would have an infinite number of variables and constraints!
- We can solve an approximate problem, obtained by sampling over the random vector.

# An example (cont'd)

Consider again our toy problem

$$\begin{aligned} \min_x \ & x_1 + x_2 \\ \text{s.t. } & \omega_1 x_1 + x_2 \geq 7 \\ & \omega_2 x_1 + x_2 \geq 4 \\ & x_1, x_2 \geq 0, \end{aligned}$$

where $\omega_1 \sim U[1, 4]$, $\omega_2[1/3, 1]$.

How to build the deterministic equivalent?

# Example: recourse formulation

Assume for now that $\Omega$ is finite, with $S$ scenarios.

$$
\begin{aligned}
\min_x \ & x_1 + x_2 + \sum_{s \in S} p_s \lambda (y_{1s} + y_{2s}) \\
\text{s.t. } & \omega_{1s} x_1 + x_2 + y_{1s} \geq 7 \\
& \omega_{2s} x_1 + x_2 + y_{1s} \geq 4 \\
& x_1, x_2 \geq 0, \\
& y_{1s}, y_{2s} \geq 0.
\end{aligned}
$$

A difficulty is therefore to decide how to construct the deterministic equivalent. How to choose $\lambda$?

How to construct the scenarios? We can proceed with Monte Carlo sampling, with $p_s = 1/N$, $\forall s$. We will explore this approach in more details later.

# Example: recourse formulation (cont'd)

More generally, we can build the program

$$\begin{aligned} \min_x \ & x_1 + x_2 + E_\omega[Q(x)] \\ \text{s.t. } & x_1, x_2 \geq 0, \end{aligned}$$

and

$$\begin{aligned} Q(x) = \min_y \ & q_1 y_1 + q_2 y_2 \\ \text{s.t. } & \omega_1 x_1 + x_2 + y_1 \geq 7, \\ & \omega_2 x_1 + x_2 + y_2 \geq 4. \end{aligned}$$

# Two-stage linear programming problem, fixed recourse

More generally, consider the problem

$$\min c^T x + E_{\boldsymbol{\xi}}[q(\xi)^T y(\xi)]$$

subject to the constraints

$$\begin{aligned} Ax &= b, \\ T(\xi)x + Wy(\xi) &= h(\xi) \qquad \forall \xi \in \Xi, \\ x &\in X, \\ y(\xi) &\in Y, \end{aligned}$$

where $\boldsymbol{\xi}$ is a random vector defined on the random space $(\Omega, \mathcal{F}, P)$, and $\Xi$ is the support of $\boldsymbol{\xi}$.

Let

$$Q(x, \xi(\omega)) = \min_{y \in Y} \left\{ q(\xi)^T y : Wy = h(\xi) - T(\xi)x \right\}.$$

# Reformulation(s)

$$\min_{x \in X \mid Ax=b} \left\{ c^T x + E_{\boldsymbol{\xi}} \left[ \min_{y \in Y} \{ q(\boldsymbol{\xi})^T y \mid Wy = h(\boldsymbol{\xi}) - T(\boldsymbol{\xi}) x \} \right] \right\}$$

Second-stage function, or recourse function, $v : \Xi \times \mathcal{R}^m \to \mathcal{R}$:

$$v(\xi, z) \stackrel{def}{=} \{ q(\xi)^T y \mid Wy = z \}.$$

Given a "policy" $x$ and a realization of the random vector $\boldsymbol{\xi}$, $z$ measures the deviation of the first stage, i.e. $z = h(\xi) - T(\xi)x$, $v(\xi, z)$ is the minimum cost to "correct" the decision in order to satisfy the constraints again.

# Recourse function

The expected recourse function, or the function of minimum expected recourse, $\mathcal{Q} : \mathcal{R}^n \to \mathcal{R}$, for any policy $x \in \mathcal{R}^n$:

$$\mathcal{Q} \stackrel{def}{=} E_{\boldsymbol{\xi}}[Q(x, \boldsymbol{\xi})],$$

describes the recourse cost expectation, with

$$Q(x, \xi) = v(\xi, h(\xi) - T(\xi)x).$$

With these definitions, the problem can be rewritten as:

$$\min_{x \in X} c^T x + \mathcal{Q}(x) \text{ such that } Ax = b.$$

It is a nonlinear program over $\mathcal{R}^n$. Properties?

# Summary

Summarize our formulations.

$$\min_{x \in \mathcal{R}^n_+ \mid Ax=b} \left\{ c^T x + E_{\boldsymbol{\xi}} \left[ \min_{y \in \mathcal{R}^p_+} \{ q(\xi)^T y \mid Wy = h(\boldsymbol{\xi}) - T(\boldsymbol{\xi})x \} \right] \right\}$$

$$\min_{x \in \mathcal{R}^n_+ \mid Ax=b} \left\{ c^T x + E_{\boldsymbol{\xi}} \left[ v(\boldsymbol{\xi}, h(\boldsymbol{\xi}) - T(\boldsymbol{\xi})x) \right] \right\}$$

$$\min_{x \in \mathcal{R}^n_+ \mid Ax=b} \left\{ c^T x + E_{\boldsymbol{\xi}} \left[ Q(x, \boldsymbol{\xi}) \right] \right\}$$

$$\min_{x \in \mathcal{R}^n_+} \left\{ c^T x + \mathcal{Q}(x) \mid Ax = b \right\}$$

# Notations

- First-stage feasible set:

$$K_1 = \{x \in \mathcal{R}^n_+ \mid Ax = b\}.$$

- Second-stage feasible set:

$$K_2 = \{x \mid \mathcal{Q}(x) < \infty\}.$$

Therefore we can rewrite the problem as

$$\min_x \{c^T x + \mathcal{Q}(x) \mid x \in K_1 \cap K_2\}.$$

# Relatively complete recourse

A problem is said to have a relatively complete recourse if $K_1 \subseteq K_2$. Advantage: $\forall x$ feasible in the first stage, we have $\mathcal{Q}(x) < \infty$, so we do not have to consider the case $Q(x, \xi) = \infty$.

We can also define the set of second-stage feasible points, given a realization $\xi$:

$$K_2(\xi) = \{x \mid Q(x, \xi) < \infty\}.$$

Define

$$K_2^P = \cap_{\xi \in \Xi} K_2(\xi).$$

Clearly $K_2 = K_2^P$ if $\boldsymbol{\xi}$ has a finite support. Is it still the case when $\boldsymbol{\xi}$ follows a continuous distribution?

# Relatively complete recourse (cont'd)

We have the following results.

**Theorem**

*If $\xi$ has finite second order moments, $P[\xi \mid Q(x,\xi) < \infty] = 1$ implies $\mathcal{Q}(x) < \infty$.*

In other words, we must have that $Q(x,\xi)$ is upper bounded almost surely. Proof: technical!

Reminder: almost surely, or with probability one. An event $A$ is said to occur almost surely if $P[A] = 1$.

More interestingly, we have

**Theorem**

*For a stochastic program with fixed recourse, where $\xi$ has finite second order moments, the sets $K_2$ and $K_2^p$ are the same.*

# Complete recourse

The relatively complete recourse is very useful in practice and on a theoretical point of view, but it can be difficult to identify. A particular case of relatively complete recourse can however often be identified from the structure of $W$.

We say that a problem has a complete recourse if $\forall\, z \in \mathcal{R}^m$, $\forall \xi$, $v(\xi, z) < +\infty$. In other terms, $\forall\, z \in \mathcal{R}^m$, $\exists\, y \in \mathcal{R}^m_+$ such that $Wy = z$, i.e. if the matrix $W$ satisfies

$$\{z \mid z = Wy,\ y \geq 0\} = \mathcal{R}^m.$$

This also implies that $\forall\, x$, $T(\xi)$, $h(\xi)$, $Q(x, \xi) < \infty$, as $z = h - Tx$.

# Simple recourse

A particular case of complete recourse is the simple recourse, for which we have

$$W = \begin{pmatrix} I & -I \end{pmatrix},$$

with $I$ the identiy matrix, of order $m$.

In this case, the second stage program can be read as

$$\begin{aligned} Q(x, \xi) = \min_y \; & q^+(\xi)^T y^+ + q^-(\xi)^T y^- \\ \text{s.t. } & y^+ - y^- = h(\xi) - T(\xi)x, \\ & y^+, \; y^- \geq 0. \end{aligned}$$

That is, for $q^+(\xi) + q^-(\xi) \geq 0$, the recourse variables $y^+$ and $y^-$ can be chosen to measure the absolute violations in the stochastic constraints.

# Simple recourse (cont'd)

**Theorem**

*Assume that the two-stage (linear) stochastic program is feasible and has a simple recourse, and that $\xi$ has finite second-order moments. Then $\mathcal{Q}(x)$ is finite if and only if $\boldsymbol{q}_i^+ + \boldsymbol{q}_i^- \geq 0$ with probability one.*

# Simple recourse (cont'd)

**Proof.**
($\Rightarrow$) Assume by contradiction that $\mathcal{Q}$ is finite, but for some component $i$, $q_i^+(\xi(\omega)) + q_i^-(\xi(\omega)) < 0$ for $\omega \in \Omega_1$ with $P[\Omega_1] > 0$. Then, for any feasible $x$, for all $\omega \in \Omega_1$ with $h_i(\xi(\omega)) - T_i(\xi(\omega))x > 0$, define

$$y_i^+(\xi(\omega)) = h_i(\xi(\omega)) - T_i(\xi(\omega))x + u, \;\; y_i^-(\xi(\omega)) = u.$$

Therefore,

$$y_i^+(\xi(\omega)) - y_i^-(\xi(\omega)) = h_i(\xi(\omega)) - T_i(\xi(\omega))x, \;\; y_i^+ \geq 0, \;\; y_i^- \geq 0.$$

Moreover, since $\mathcal{Q}$ is finite, $Q(x, \xi(\omega))$ is feasible almost surely, so, almost surely, we can choose $y_j^+$ and $y_j^-$ feasible, $j \neq i$.

□

# Simple recourse (cont'd)

**Proof.**

($\Rightarrow$)

When $u \to \infty$, $Q(x, \xi(\omega)) \to -\infty$ since $q_i^+(\xi(\omega))y_i^+ + q_i^-(\xi(\omega))y_i^- \to -\infty$.

A similar argument can be applied if $h_i(\xi(\omega)) - T_i(\xi(\omega))x \leq 0$, by taking

$$y_i^+(\xi(\omega)) = u, \; y_i^-(\xi(\omega)) = -h_i(\xi(\omega)) + T_i(\xi(\omega))x + u.$$

By componing these two cases, we conclude that $\mathcal{Q}$ is not finite. $\square$

# Simple recourse (cont'd)

## Proof.

($\Leftarrow$) Assume $\boldsymbol{q}_i^+ + \boldsymbol{q}_i^- \geq 0$ with probability one, $\forall i$. Any feasible solution satisfies

$$y^+(\xi(\omega)) - y^-(\xi(\omega)) = h(\xi(\omega)) - T(\xi(\omega))x, \; y^+(\xi(\omega)) \geq 0, \; y^-(\xi(\omega)) \geq 0$$

Therefore for almost every $\omega$, $Q(x, \xi(\omega))$ is bounded below by 0, and from the fundamental theorem of linear programming, we can choose as optimal solution

$$\begin{aligned} y^+(\xi(\omega)) &= (h(\xi(\omega)) - T(\xi(\omega))x)^+ \,, \\ y^-(\xi(\omega)) &= (-h(\xi(\omega)) + T(\xi(\omega))x)^+ \,, \end{aligned}$$

where $a^+ = \max\{0, a\}$. $\square$

# Simple recourse (cont'd)

Proof.
($\Leftarrow$) Thus,

$$Q(x,\xi(\omega)) = \sum_{i=1}^{m} (q_i^+(\xi(\omega))(h_i(\xi(\omega)) - T_i(\xi(\omega))x)^+ + q_i^-(\xi(\omega))(-h_i(\xi(\omega)) + T_i(\xi(\omega))x)^+)$$

Consequently $Q(x,\xi(\omega))$ is finite for almost every $\omega$ and bounded below by 0.

Therefore, $\mathcal{Q}(x)$ is bounded below by 0, and according to the previous results, $\mathcal{Q}(x) < \infty$. This implies that $\mathcal{Q}(x)$ is finite. $\square$

# Exercise

Consider the second stage program

$$Q(x, \xi) = \min_y \{y \mid \xi y = 1 - x, y \geq 0\}.$$

We assume that $\boldsymbol{\xi}$ follows a triangular distribution on $[0, 1]$, with $P[\boldsymbol{\xi} \leq u] = u^2$.

(a) Is the recourse fixed? Why?

The recourse is not fixed, as $W \equiv \boldsymbol{\xi}$, and therefore, $W$ is random. Moreover, as $\xi$ can take the value 0, the transformation

$$y = 1/\xi - x/\xi,$$

is not properly defined on $\Xi = [0, 1]$; this also means that

$$W = \begin{cases} 0 & \text{si } \xi = 0; \\ 1 & \text{si } \xi \neq 0. \end{cases}$$

## Exercise (cont'd)

(b) Express $K_2(\xi)$ for all $\xi$ in $[0, 1]$.

We have to consider two cases: $\xi = 0$ or $\xi \in (0, 1]$.

1. $\xi \in (0, 1]$ In this case, as $y, \xi \geq 0$, $1 - x$ has to be non-negative in order to have a well-defined problem:

$$K_2(\xi) = \{x \mid x \leq 1\}.$$

The value and optimal solutions are

$$Q^*(x, \xi) = (1 - x)/\xi, \quad y^* = (1 - x)/\xi.$$

2. $\xi = 0$ There exists no $y$ such that $0.y = 1 - x$, except if $x = 1$, so

$$K_2(0) = \{1\}.$$

# Exercise (cont'd)

(c) Express $K_2$, $K_2^P$ and $\mathcal{Q}$.

From the previous point, we have

$$K_2^P = \{x \,|\, x \leq 1\} \cap \{1\} = \{1\}.$$

We also have, as $P[\xi = 0] = 0$,

$$\mathcal{Q}(x) = \int_0^1 \frac{1-x}{\xi} 2\xi d\xi = 2(1-x), \forall x \leq 1.$$

Consequently $K_2^P \subset K_2 = \{x \leq 1\}$.

The difference comes from the fact that a point is not in $K_2^P$ as soon as it is not feasible for a given value of $\xi$, but $K_2$ does not consider unfeasible situations that occur with a null probability.

# Recourse function

Let $y_1^*$ and $y_2^*$ be two optimal solutions of $v(\xi, z)$, associated to $z = z_1$ and $z = z_2$, respectively. Then, the convex combination $y_\alpha \stackrel{def}{=} \alpha y_1^* + (1-\alpha)y_2^*$, $\alpha \in [0,1]$, is feasible with respect to $z_\alpha = \alpha z_1 + (1-\alpha)z_2$, as $\alpha y_1^* + (1-\alpha)y_2^* \geq 0$, and

$$W(\alpha y_1^* + (1-\alpha)y_2^*) = \alpha W y_1^* + (1-\alpha) W y_2^* = \alpha z_1 + (1-\alpha) z_2 = z_\alpha.$$

Moreover,

$$\begin{aligned} v(\xi, z_\alpha) &= q(\xi)^T y_\alpha^* \leq q(\xi)^T(\alpha y_1^* + (1-\alpha)y_2^*) \\ &= \alpha q(\xi)^T y_1^* + (1-\alpha) q(\xi)^T y_2^* \\ &= \alpha v(\xi, z_1) + (1-\alpha) v(\xi, z_2). \end{aligned}$$

In other words, $v$ is a convex function w.r.t. $z \in \mathcal{R}^m$.

# Convexity of $Q(x, \xi)$?

$$Q(x, \xi) = v(\xi, h(\xi) - T(\xi)x).$$

$$\begin{aligned}
\lambda Q(x_1, \xi) &+ (1-\lambda) Q(x_2, \xi) \\
&= \lambda v(\xi, h(\xi) - T(\xi)x_1) + (1-\lambda) v(\xi, h(\xi) - T(\xi)x_2) \\
&\geq v(\xi, \lambda(h(\xi) - T(\xi)x_1) + (1-\lambda)(h(\xi) - T(\xi)x_2)) \\
&= v(\xi, h(\xi) - T(\xi)(\lambda x_1 + (1-\lambda)x_2)) \\
&= Q(\lambda x_1 + (1-\lambda)x_2, \xi).
\end{aligned}$$

Therefore $Q(x, \xi)$ if convex w.r.t. $x$, given $\xi$. More generally

**Theorem**

*If $A$ if a linear transformation from $\mathcal{R}^n$ to $\mathcal{R}^n$, and $f(x)$ is a convex function on $\mathcal{R}^m$, the composite function $(fA)(x) \stackrel{def}{=} f(Ax)$ is a convex function on $\mathcal{R}^n$.*

# Convexity of second-stage function

We have the following result (Birge and Louveaux, Chapter 3, Theorem 5).

## Theorem

*For a stochastic program with fixed recourse, $Q(x, \xi)$ is*

(a) *a piecewise convex linear function in $(h, T)$,*

(b) *a piecewise concave linear function in $q$,*

(c) *a piecewise convex linear function in $x$, for all $x$ in $K = K_1 \cap K_2$.*

## Convexity of second-stage function (cont'd)

### Proof.

In order to show convexity in (a) and (c), it is sufficient to prove that $v(\xi, z) = \min\{q(\xi)^T y \mid Wy = z\}$ is convex, which has already been done. We can proceed similarly to show concavity w.r.t. $q$.

The piecewise linearity follows from the fact that the number of different optimal bases for a linear program is finite. $\square$

Convexity of the recourse?

$$\mathcal{Q}(x) = E_{\boldsymbol{\xi}}[Q(x, \xi)].$$

Suppose for now that $\xi$ has a finite support, i.e. $\Xi = \{\xi_1, \xi_2, \ldots, \xi_m\}$. Then

$$\mathcal{Q}(x) = \sum_{i=1}^{m} P[\boldsymbol{\xi} = \xi_i] Q(x, \xi_i).$$

# Convexity of the recourse

**Theorem**

*If $f(x)$ is convex, and $\alpha \geq 0$, $g(x) = \alpha f(x)$ is convex.*

**Theorem**

*If $f_k(x)$, $k = 1, 2, \dots, K$, are convex functions, then $g(x) = \sum_{k=1}^{K} f_k(x)$ is convex.*

$\mathcal{Q}(x)$ is therefore a convex function w.r.t. $x$.

What is happening in the continuous case?

We have the following result: if $g(x, y)$ is convex w.r.t. $x$, then $\int g(x, y)dy$ is convex w.r.t. $x$. Since

$$\mathcal{Q}(x) = \int_{\Xi} Q(x, t)dF(t),$$

$\mathcal{Q}(x)$ is convex.

# An example...

Consider the second-stage function $Q(x, \xi)$ defined as:

$$\min y^+ + y^- \text{ s.t. } y^+ - y^- = \xi - x,\ y^+ \geq 0,\ y^- \geq 0.$$

In other terms:

$$y = \begin{pmatrix} y^+ \\ y^- \end{pmatrix} \quad q = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, W = \begin{pmatrix} 1 & -1 \end{pmatrix} \quad h(\xi) = \xi \quad T(\xi) = 1.$$

Relying on the fundamental theorem of linear programming, we are looking from an optimal basis solution, implying that $y^+ = 0$ or $y^- = 0$.

# An example...

We immediately see that

$$y^+ = \begin{cases} \xi - x & \text{if } \xi - x \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$y^- = \begin{cases} -\xi + x & \text{if } \xi - x < 0, \\ 0 & \text{otherwise,} \end{cases}$$

# Alternative approach

Dual:

$$\begin{aligned} &\max\ (\xi - x)\pi \\ &\text{s.t. } \pi \leq 1,\ -\pi \leq 1 \end{aligned}$$

or

$$\begin{aligned} &\max\ (\xi - x)\pi \\ &\text{s.t. } \pi + s_1 = 1 \\ &\quad -\pi + s_2 = 1 \\ &\quad s_1 \geq 0,\ s_2 \geq 0 \end{aligned}$$

Consequently,

$$Q(x, \xi) = \begin{cases} \xi - x & \text{si } x \leq \xi, \\ x - \xi & \text{si } x \geq \xi. \end{cases}$$

## An example: optimality conditions

The recourse is simple, and the primal-dual/KKT conditions give

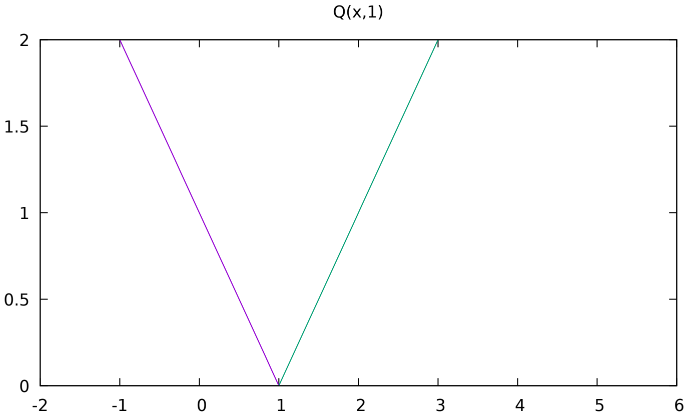
$$
\begin{aligned}
\begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \pi + \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \\
y^+ - y^- &= \xi - x \\
y^+ &\geq 0, \;\; y^- \geq 0 \\
s_1 &\geq 0, \;\; s_2 \geq 0 \\
s_1 y^+ &= 0, \;\; s_2 y^- = 0.
\end{aligned}
$$

# An example (cont'd)

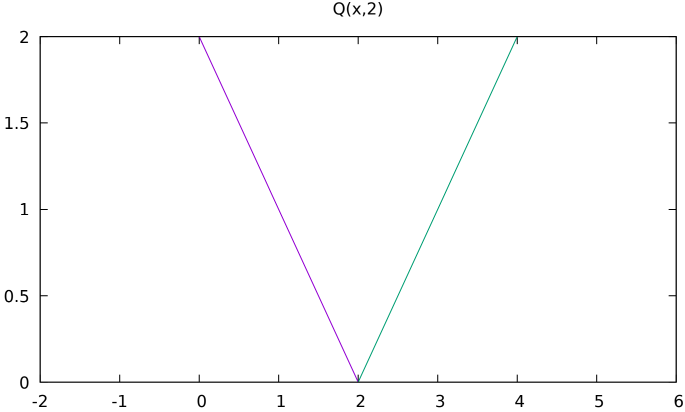
- The first condition implies that we cannot have $s_1 = s_2 = 0$.
- From the complementarity conditions, we have that $y^+ = 0$ or $y^- = 0$.
- We have to consider two cases:
  - $x \leq \xi$: in this situation, we have
$$y^+ = \xi - x, \quad y^- = 0.$$
  - $x \geq \xi$: then,
$$y^- = x - \xi, \quad y^+ = 0.$$

# Graphically?

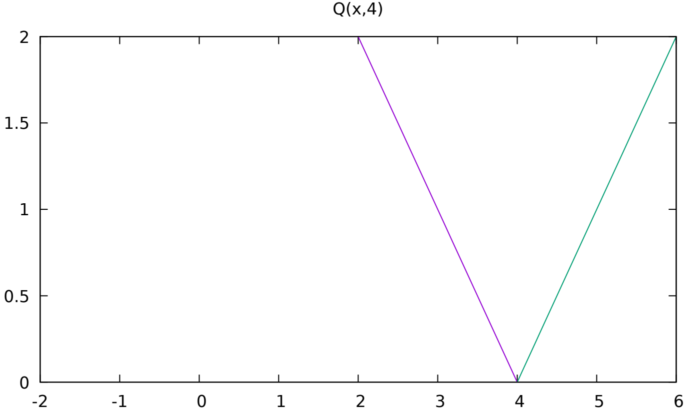
Assume that $\xi$ can take the realizations 1, 2, 4.



# Graphically (cont'd)



# Graphically (cont'd)



# $\mathcal{Q}(x)$

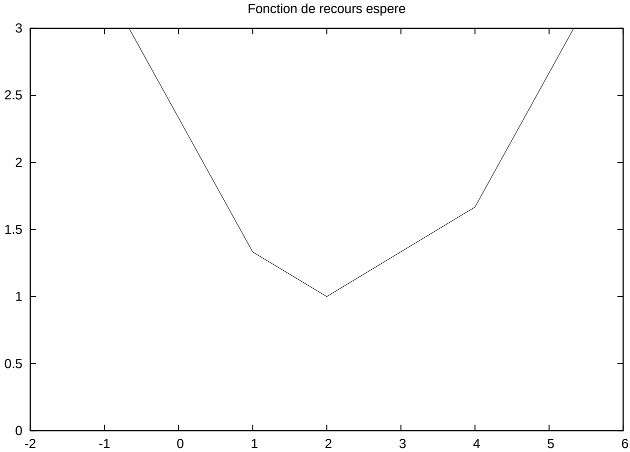
What about $\mathcal{Q}(x)$?

Assume that the three realizations have the same probability.

We have to consider 4 cases:

1. $x \leq 1$: $\mathcal{Q}(x) = 7/3 - x$;
2. $1 \leq x \leq 2$: $\mathcal{Q}(x) = 5/3 - x/3$;
3. $2 \leq x \leq 4$: $\mathcal{Q}(x) = x/3 + 1/3$;
4. $4 \leq x$: $\mathcal{Q}(x) = x - 7/3$;

# Graphically



# Properties of $\mathcal{Q}(x)$

We note that $\mathcal{Q}(x)$ is convex and piecewise linear. As $\mathcal{Q}(x)$ is a finite weighted sum of piecewise linear functions when the support of $\xi$ is finite, we have the following result.

## Theorem

*For a stochastic program with fixed recourses where $\xi$ has finite second-order moments,*

(a) *$\mathcal{Q}(x)$ is a convex Lipschitz function and is finite over $K_2$;*

(b) *when $\xi$ has a finite support, $\mathcal{Q}(x)$ is piecewise linear.*

Reminder: a function $f$ is Lipschitz if there exists some $M < \infty$ such that for all $x$, $y$,

$$|f(x) - f(y)| \leq M\|x - y\|.$$

# Differentiability of the recourse

Is $\mathcal{Q}(x)$ also differentiable?

The recourse function is partially differentiable with respect to $x_j$ at $(\hat{x}, \hat{\xi})$ if the directional derivative exists for the direction $e_j$. In other terms, there exists a function $\frac{\partial Q(x,\xi)}{\partial x_j}$ such that

$$\frac{Q(\hat{x} + he_j, \hat{\xi}) - Q(\hat{x}, \hat{\xi})}{h} = \frac{\partial Q(x, \xi)}{\partial x_j} + \frac{\rho_j(\hat{x}, \hat{\xi}; h)}{h},$$

with

$$\frac{\rho_j(\hat{x}, \hat{\xi}; h)}{h} \to 0, \text{ as } h \to 0.$$

We will assume from now that $\nabla_x Q(x, \xi) = \left( \frac{\partial Q(x,\xi)}{\partial x_1}, \ldots, \frac{\partial Q(x,\xi)}{\partial x_n} \right)$ exists.

# Differentiability of the recourse (cont'd)

What about the differentiability of $\mathcal{Q}(x)$?

$$\begin{aligned}\frac{\mathcal{Q}(\hat{x}+he_j)-\mathcal{Q}(\hat{x})}{h} &= \int_{\Xi}\frac{Q(\hat{x}+he_j,\xi)-Q(\hat{x},\xi)}{h}dP \\ &= \int_{\Xi\setminus N_\delta}\frac{\partial Q(\hat{x},\xi)}{\partial x_j}dP+\int_{\Xi\setminus N_\delta}\frac{\rho_j(\hat{x},\xi;h)}{h}dP,\end{aligned}$$

where $N_\delta$ is measurable and $P[N_\delta]=0$. Therefore, we have

## Theorem

*If $Q(x,\xi)$ if partially differentiable almost everywhere, if its partial derivative $\frac{\partial Q(\hat{x},\xi)}{\partial x_j}$ is integrable and if the residual satisfies $(1/h)\int_{\Xi\setminus N_\delta}\rho_j(\hat{x},\xi;h)dP \stackrel{h\to 0}{\to} 0$, then $\frac{\partial \mathcal{Q}(\hat{x})}{\partial x_j}$ exists and*

$$\frac{\partial\mathcal{Q}(\hat{x})}{\partial x_j}=\int_{\Xi}\frac{\partial Q(\hat{x},\xi)}{\partial x_j}dP.$$

# Differentiability of the recourse: discrete case

But how to prove $(1/p) \int_{\Xi \setminus N_\delta} \rho_j(\hat{x}, \xi; h) dP \stackrel{h \to 0}{\to} 0$?

If we stay in the linear framework with fixed recourse, and vectors $\boldsymbol{\xi}$ with finite second-order moments, we have seen that for $\boldsymbol{\xi}$ with finite support, $\mathcal{Q}(x)$ is piecewise linear. Therefore $\mathcal{Q}(x)$ is not differentiable.

# Differentiability of the recourse: continuous case

If $\boldsymbol{\xi}$ is continuous, $\mathcal{Q}(x)$ is obtained as an integral over the $Q(x, \xi)$'s, that are not differentiable as they are piecewise linear given $\xi$. However, it is $x$ that has to be fixed, not $\xi$. It is possible to show that (the proof is quite technical)

**Theorem**

*For a stochastic program with fixed recourse where $\boldsymbol{\xi}$ has finite second-order moments, if $\boldsymbol{\xi}$ is continuous, $\mathcal{Q}(x)$ is differentiable over $K_2$.*

Intuitively, the function $\mathcal{Q}(x)$ is "smoothed" by the superposition of an infinite number of functions $Q(x, \xi)$.

# Two-stage non-linear problems

Now consider the general program

$$\min_{x \in X} E_{\boldsymbol{\xi}}[f_0(x, \boldsymbol{\xi})] = \min_{x \in X} E_{\boldsymbol{\xi}}[g_0(x, \boldsymbol{\xi}) + Q(x, \boldsymbol{\xi})].$$

## Theorem

*If $g_0(\cdot, \xi)$ and $Q(\cdot, \xi)$ are convex with respect to $x$, $\forall\, \xi \in \Xi$, and if $X$ is a convex set, the aforementioned program is convex.*

## Proof.

For $x$, $y \in X$, $\lambda \in (0, 1)$ and $z = \lambda x + (1 - \lambda)y$, we have

$$\begin{aligned} & g_0(z, \xi) + Q(z, \xi) \\ & \qquad \leq \lambda(g_0(x, \xi) + Q(x, \xi)) + (1 - \lambda)(g_0(y, \xi) + Q(y, \xi)). \end{aligned}$$

The result follows by taking the expectation. □

# In a more standard form

Inspired from Birge et Louveaux, Section 3.4.

We consider the problem

$$
\begin{aligned}
\inf z &= f^1(x) + \mathcal{Q}(x), \\
\text{s.t. } & g_i^1(x) \le 0, \ i = 1, \ldots, \overline{m}_1, \\
& g_i^1(x) = 0, \ i = \overline{m}_1 + 1, \ldots, m_1,
\end{aligned}
$$

where $\mathcal{Q}(x) = E_\omega[Q(x,\omega)]$ and

$$
\begin{aligned}
Q(x,\omega) &= \inf f^2(y(\omega),\omega), \\
\text{s.t. } & t_i^2(x,\omega) + g_i^2(y(\omega),\omega) \le 0, \ i = 1, \ldots, \overline{m}_2, \\
& t_i^2(x,\omega) + g_i^2(y(\omega),\omega) = 0, \ i = \overline{m}_2 + 1, \ldots, m_2,
\end{aligned}
$$

We say that the recourse is additive (why?).

# In a more standard form (cont'd)

The functions $f^2(\cdot,\omega)$, $t_i^2(\cdot,\omega)$, and $g_i^2(\cdot,\omega)$ are continuous for any given $\omega$, and measurable w.r.t. $\omega$ for any given first argument. This allows to prove that $Q(x,\omega)$ is measurable, and therefore that $\mathcal{Q}(x)$ is well defined.

Reintroduce $K_1$, $K_2(\omega)$ and $K_2$.

$$
\begin{aligned}
K_1 &= \{x \mid g_i^1(x) \leq 0,\ i = 1, \ldots, \overline{m}_1, \\
&\qquad\quad g_i^1(x) = 0,\ i = \overline{m}_1 + 1, \ldots, m_1\}, \\
K_2(\omega) &= \{x \mid \exists\, y(\omega) \text{ t.q. } t_i^2(x,\omega) + g_i^2(y(\omega),\omega) \leq 0,\ i = 1, \ldots, \overline{m}_2, \\
&\qquad\quad t_i^2(x,\omega) + g_i^2(y(\omega),\omega) = 0,\ i = \overline{m}_2 + 1, \ldots, m_2\}, \\
K_2 &= \{x \mid \mathcal{Q}(x) < \infty\}.
\end{aligned}
$$

# Remarks

The formulation is not yet totally general. We will consider more general forms when we will discuss sampling techniques.

Here, there is no more fixed recourse, but the first-stage decision $x$ acts separately in the constraints. Goal: extend the previous results.

Questions: convexity, differentiability, optimality. We should also consider the concept of lower semi-continuity.