Stochastic optimization
Chance constrained programming

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Motivation

We consider the toy problem (taken from J. Linderoth)

$$\begin{aligned}
\min_{x} \; & x_1 + x_2 \\
\text{s.t. } & \xi_1 x_1 + x_2 \geq 7 \\
& \xi_2 x_1 + x_2 \geq 4 \\
& x_1, x_2 \geq 0,
\end{aligned}$$

where $\xi_1 \sim U(1,4)$, $\xi_1 \sim U(1/3,1)$.

Instead of requiring that a constraint holds for all the scenarios, we can require a sufficiently large probability to satisfy a constraint.

Chance constraints

1. Separate chance constraints

$$P[\xi_1 x_1 + x_2 \geq 7] \geq \alpha_1$$
$$P[\xi_2 x_1 + x_2 \geq 4] \geq \alpha_2$$

2. Joint (integrated) chance constraint

$$P[\xi_1 x_1 + x_2 \geq 7 \cap \xi_2 x_1 + x_2 \geq 4] \geq \alpha$$

Example: joint chance constraints

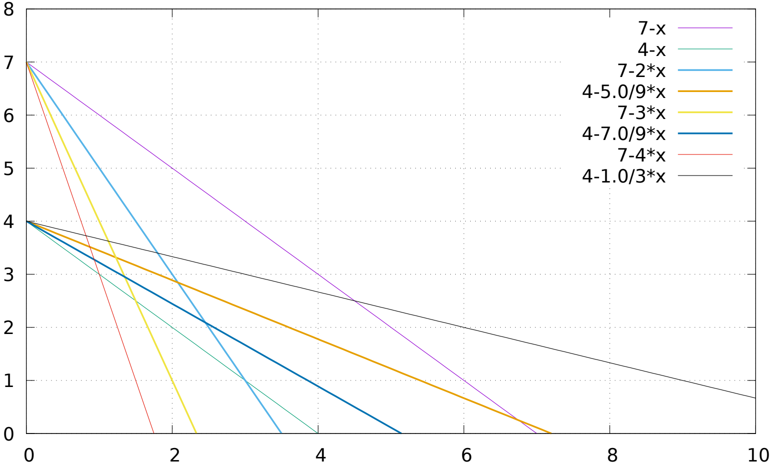
$$P[(\xi_1, \xi_2) = (1, 1)] = 0.1 \tag{1}$$
$$P[(\xi_1, \xi_2) = (2, 5/9)] = 0.4 \tag{2}$$
$$P[(\xi_1, \xi_2) = (3, 7/9)] = 0.4 \tag{3}$$
$$P[(\xi_1, \xi_2) = (4, 1/3)] = 0.1 \tag{4}$$

Assume that $\alpha \in (0.8, 0.9]$, and we have the joint constraint

$$P[\xi_1 x_1 + x_2 \geq 7 \cap \xi_2 x_1 + x_2 \geq 4] \geq \alpha$$

We then have to satisfy constraints (2) and (3) and either (1) or (4).

Example: graph



Properties

Feasible set

$$K_1(\alpha) = \{x \mid P[T(\xi)x \geq h(\xi)] \geq \alpha\}$$

$K_1(\alpha)$ is not necessarily convex.

Theorem

Suppose $T(\xi) = T$ is fixed, and $h(\xi)$ has a quasi-concave probability measure P. Then $K_1(\alpha)$ is convex for $0 \leq \alpha \leq 1$.

A function $P : D \to \mathcal{R}$ defined on a domain D is quasi-concave if $\forall$ convex sets $U, V \subseteq D$, and $0 \leq \lambda \leq 1$,

$$P[(1-\lambda)U + \lambda V] \geq \min\{P[U], P[V]\}.$$

Quasi-concave probability distributions

- Uniform

$$f(x) = \begin{cases} 1/\mu(S), & x \in S \\ 0 & \text{otherwise,} \end{cases}$$

  where $\mu(S)$ is the measure of S.
- Exponential density

$$f(x) = \lambda e^{-\lambda x}$$

- Multivariate normal density:

$$f(x) = \frac{1}{\sqrt{(2\pi)^n/2\,\det(\Sigma)}} e^{-\frac{1}{2}(x-\mu)'\Sigma(x-\mu)}$$

If you have such a density, you can

- use Lagrangian techniques
- use a reduced-gradient technique (see Kall & Wallace, Section 4.1)

Single constraint: easy case

The situation in the single constraint case is somewhat more simple.

Suppose again that $T_i(\xi) = T_i$ is constant. Then

$$P[T_i x \geq h_i(\xi)] = F(T_i x) \geq \alpha$$

so the deterministic equivalent is

$$T_i x \geq F^{-1}(\alpha)$$

... linear constraint! The resulting problem is still linear.

Recall that the inverse of the cdf is defined as

$$F^{-1}(\alpha) = \min\{x : F(x) \geq \alpha\}.$$

Other "solvable" cases

Let $h(\xi) = h$ be fixed, $T(\xi) = (\xi_1, \xi_2, \ldots, \xi_n)$, with $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ a multivariate normal distribution with mean $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ and variance-covariance matrix Σ. Then

$$K_1(\alpha) = \{x \mid \mu' x \geq h + \Phi^{-1}(\alpha)\sqrt{x'\Sigma x}\},$$

where Φ is the standard normal cdf.

$K_1(\alpha)$ is a convex set for $\alpha \geq 0.5$.

It is possible to express it as a second order cone constraint:

$$\|\Sigma^{1/2} x\|_2 \leq \frac{1}{\Phi^{-1}(\alpha)}(\mu' x - h)$$

Second-order cone programming

A second-order cone program (SOCP) is a convex optimization problem of the form

$$\begin{aligned} \min_{x} \;& f^T x \\ \text{s.t.} \;& \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \;\; i = 1, \ldots, m \\ & Fx = g \end{aligned}$$

where $x \in \mathcal{R}^n$, $f, c_i \in \mathcal{R}^n$, $A_i \in \mathcal{R}^{n_i \times n}$, $b_i \in \mathcal{R}^{n_i}$, $d_i \in \mathcal{R}$, $F \in \mathcal{R}^{p \times n}$, and $g \in \mathcal{R}^p$.

SOCPs can be solved by interior point methods.

Example: robust portfolio optimization

(Taken from S. Boyd and J. Linderoth)
Suppose we want to invest in n assets, providing return rates $\beta_1, \beta_2, \ldots, \beta_n$.

The β_i's are random variables. Assume that they are following a multivariate normal distribution with means β_i and covariance matrix Σ.

Suppose that we want to ensure a return of at least T. We cannot guarantee it all the time, but we want it to occur most of the time.

Example: robust portfolio optimization (cont'd)

Let $x_i \geq 0$ the part of portfolio to invest in stock i. We have the constraints

$$P\left[\sum_{i=1}^{n} \beta_i x_i \geq T\right] \geq \alpha$$
$$\sum_{i=1}^{n} x_i \leq x$$
$$x_i \geq 0,\ i = 1, \ldots, n$$

where x is the total amount to invest.

The chance constraint can be rewritten as

$$\beta' x - \Phi^{-1}(\alpha)\sqrt{x'\Sigma x} \geq T.$$

Example: robust portfolio optimization (cont'd)

We can also interpret x_i as proportion of the portfolio (position of asset i), by normalizing $\|x\|_1$ to 1. T is now the minimum return rate of the portfolio and x is the portfolio allocation.

We can add some constraints on the x_i to ensure diversification. We summarize them by requiring $x \in \mathcal{C}$.

A complete program can now be expressed as

$$\begin{aligned} \max_x \; & E[\beta' x] \\ \text{s.t.} \; & P\left[\beta' x \geq T\right] \geq \alpha \\ & \sum_{i=1}^{n} x_i = 1 \\ & x \in \mathcal{C} \end{aligned}$$

Example: loss constraint

Setting T to 0 means that we want to ensure that we will no suffer from loss with some probability. Typicially, α is set to 0.9, 0.95, 0.99,...

The chanced-constraint can also be expressed as

$$P\left[\beta' x \leq 0\right] \leq 1 - \alpha = \gamma.$$

We can also allow the sale of some parts of the portfolio by allowing some x_i to be negative.

Numerical illustration

(Taken from S. Boyd – `http://ee364a.stanford.edu/lectures/chance_constr.pdf`)

$n = 10$ assets, $\alpha = 0.95$, $\beta = 0.05$, $\mathcal{C} = \{x | x \succeq -0.1\}$

Compare

- optimal portfolio
- optimal portfolio without loss risk constraint
- uniform portfolio $(1/n)\mathbf{1}$

portfolio	$E[\beta' x]$	$P[\beta' x \leq 0])$
optimal	7.51	5.0%
w/o loss constraint	10.66	20.3%
uniform	3.41	18.9%

Other situations

Usually very hard.

Use a bounding approximation or sample average approximation (SAA).