Stochastic optimization
Chance constrained programming

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A long story

Introduced in 1959 by Charnes and Cooper
https://dl.acm.org/doi/10.1287/mnsc.6.1.73

And also a bit improbable. Cooper dropped high-school to support his family, and became a professional boxer. He became an accountant for Eric Louis Kohler, met while hitchhiking. Kohler financed his bachelor at University of Chicago. At 26, he enrolled at Columbia University and finished his coursework and dissertation. Due to its clam that decision making was not a centralized process, he never received his PhD. The collaboration with Charnes was however successful, with more than 200 publications, and led a successful academic carrer. Source: https://www.informs.org/Explore/History-of-O.R.-Excellence/Biographical-Profiles/Cooper-William-W

Cooper and Charnes

INFORMS John Von Neumann prize (with Richard J. Duffin)

Motivation

Source: J. Linderoth `https://homepages.cae.wisc.edu/~linderot/classes/ie495/lecture22.pdf`

We consider the toy problem

$$\begin{aligned} \min_{x} \ & x_1 + x_2 \\ \text{s.t. } & \xi_1 x_1 + x_2 \geq 7 \\ & \xi_2 x_1 + x_2 \geq 4 \\ & x_1, x_2 \geq 0, \end{aligned}$$

where $\xi_1 \sim U(1, 4)$, $\xi_1 \sim U(1/3, 1)$.

Instead of requiring that a constraint holds for all the scenarios, we can require a sufficiently large probability to satisfy a constraint.

Chance constraints

1. Separate chance constraints

$$P[\xi_1 x_1 + x_2 \geq 7] \geq \alpha_1$$
$$P[\xi_2 x_1 + x_2 \geq 4] \geq \alpha_2$$

2. Joint (integrated) chance constraint

$$P[\xi_1 x_1 + x_2 \geq 7 \cap \xi_2 x_1 + x_2 \geq 4] \geq \alpha$$

Example: joint chance constraints

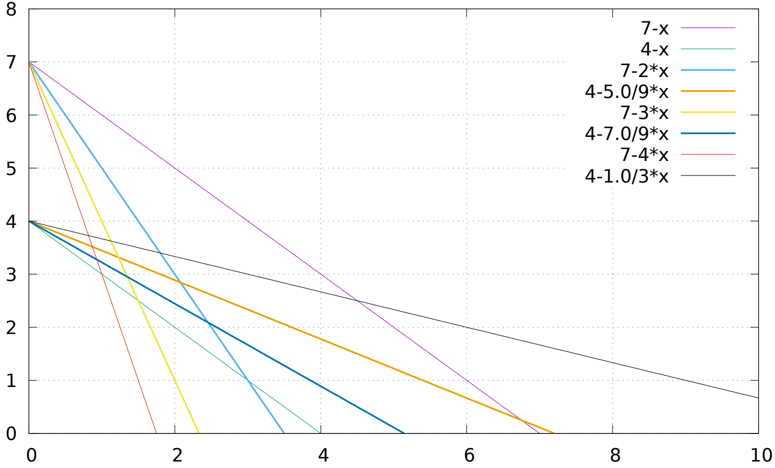
$$P[(\xi_1, \xi_2) = (1, 1)] = 0.1 \quad (1)$$
$$P[(\xi_1, \xi_2) = (2, 5/9)] = 0.4 \quad (2)$$
$$P[(\xi_1, \xi_2) = (3, 7/9)] = 0.4 \quad (3)$$
$$P[(\xi_1, \xi_2) = (4, 1/3)] = 0.1 \quad (4)$$

Assume that $\alpha \in (0.8, 0.9]$, and we have the joint constraint

$$P[\xi_1 x_1 + x_2 \geq 7 \cap \xi_2 x_1 + x_2 \geq 4] \geq \alpha$$

We then have to satisfy constraints (2) and (3) and either (1) or (4).

Example: graph



Properties

Feasible set

$$K_1(\alpha) = \{x \mid P[T(\xi)x \geq h(\xi)] \geq \alpha\}$$

$K_1(\alpha)$ is not necessarily convex.

Theorem

Suppose $T(\xi) = T$ is fixed, and $h(\xi)$ has a quasi-concave probability measure P. Then $K_1(\alpha)$ is convex for $0 \leq \alpha \leq 1$.

A function $P : D \to \mathcal{R}$ defined on a domain D is quasi-concave if $\forall$ convex sets $U, V \subseteq D$, and $0 \leq \lambda \leq 1$,

$$P[(1-\lambda)U + \lambda V] \geq \min\{P[U], P[V]\}.$$

Quasi-concave probability distributions

- Uniform

$$f(x) = \begin{cases} 1/\mu(S), & x \in S \\ 0 & \text{otherwise,} \end{cases}$$

  where $\mu(S)$ is the measure of S.
- Exponential density

$$f(x) = \lambda e^{-\lambda x}$$

- Multivariate normal density:

$$f(x) = \frac{1}{\sqrt{(2\pi)^n/2 \det(\Sigma)}} e^{-\frac{1}{2}(x-\mu)'\Sigma(x-\mu)}$$

If you have such a density, you can

- use Lagrangian techniques
- use a reduced-gradient technique (see Kall & Wallace, Section 4.1)

Single constraint: easy case

The situation in the single constraint case is somewhat more simple.

Suppose again that $T_i(\xi) = T_i$ is constant. Then

$$P[T_i x \geq h_i(\xi)] = F(T_i x) \geq \alpha$$

so the deterministic equivalent is

$$T_i x \geq F^{-1}(\alpha)$$

. . . linear constraint! The resulting problem is still linear.

Recall that the inverse of the cdf is defined as

$$F^{-1}(\alpha) = \min\{x : F(x) \geq \alpha\}.$$

Other "solvable" cases

Let $h(\xi) = h$ be fixed, $T(\xi) = (\xi_1, \xi_2, \ldots, \xi_n)$, with $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ a multivariate normal distribution with mean $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ and variance-covariance matrix Σ. Then

$$K_1(\alpha) = \{x \mid \mu' x \geq h + \Phi^{-1}(\alpha)\sqrt{x'\Sigma x}\},$$

where Φ is the standard normal cdf.

$K_1(\alpha)$ is a convex set for $\alpha \geq 0.5$.

It is possible to express it as a second order cone constraint:

$$\|\Sigma^{1/2} x\|_2 \leq \frac{1}{\Phi^{-1}(\alpha)}(\mu' x - h)$$

Second-order cone programming

A second-order cone program (SOCP) is a convex optimization problem of the form

$$
\begin{aligned}
\min_x \; & f^T x \\
\text{s.t. } & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \; i = 1, \ldots, m \\
& Fx = g
\end{aligned}
$$

where $x \in \mathcal{R}^n$, $f, c_i \in \mathcal{R}^n$, $A_i \in \mathcal{R}^{n_i \times n}$, $b_i \in \mathcal{R}^{n_i}$, $d_i \in \mathcal{R}$, $F \in \mathcal{R}^{p \times n}$, and $g \in \mathcal{R}^p$.

SOCPs can be solved by interior point methods.

Example: robust portfolio optimization

(Taken from S. Boyd and J. Linderoth)
Suppose we want to invest in n assets, providing return rates $\beta_1, \beta_2, \ldots, \beta_n$.

The β_i's are random variables. Assume that they are following a multivariate normal distribution with means β_i and covariance matrix Σ.

Suppose that we want to ensure a return of at least T. We cannot guarantee it all the time, but we want it to occur most of the time.

Example: robust portfolio optimization (cont'd)

Let $x_i \geq 0$ the part of portfolio to invest in stock i. We have the constraints

$$P\left[\sum_{i=1}^{n} \beta_i x_i \geq T\right] \geq \alpha$$
$$\sum_{i=1}^{n} x_i \leq x$$
$$x_i \geq 0, \; i = 1, \ldots, n$$

where x is the total amount to invest.

The chance constraint can be rewritten as

$$\beta' x - \Phi^{-1}(\alpha)\sqrt{x' \Sigma x} \geq T.$$

Example: robust portfolio optimization (cont'd)

We can also interpret x_i as proportion of the portfolio (position of asset i), by normalizing $\|x\|_1$ to 1. T is now the minimum return rate of the portfolio and x is the portfolio allocation.

We can add some constraints on the x_i to ensure diversification. We summarize them by requiring $x \in \mathcal{C}$.

A complete program can now be expressed as

$$\begin{aligned} \max_{x} \ & E[\beta' x] \\ \text{s.t. } & P\left[\beta' x \geq T\right] \geq \alpha \\ & \sum_{i=1}^{n} x_i = 1 \\ & x \in \mathcal{C} \end{aligned}$$

Example: loss constraint

Setting T to 0 means that we want to ensure that we will no suffer from loss with some probability. Typicially, α is set to 0.9, 0.95, 0.99,...

The chanced-constraint can also be expressed as

$$P\left[\beta' x \leq 0\right] \leq 1 - \alpha = \gamma.$$

We can also allow the sale of some parts of the portfolio by allowing some x_i to be negative.

Numerical illustration

(Taken from S. Boyd – `http://ee364a.stanford.edu/lectures/chance_constr.pdf`)
$n = 10$ assets, $\alpha = 0.95$, $\gamma = 0.05$, $\mathcal{C} = \{x | x \succeq -0.1\}$

Compare

- optimal portfolio
- optimal portfolio without loss risk constraint
- uniform portfolio $(1/n)\mathbf{1}$

portfolio	$E[\beta' x]$	$P[\beta' x \leq 0])$
optimal	7.51	5.0%
w/o loss constraint	10.66	20.3%
uniform	3.41	18.9%

Other situations

Usually very hard.

Use a bounding approximation or sample average approximation (SAA).

We will discuss about it in more details when introducing Monte Carlo techniques.

Value at Risk

Source: `https://web.stanford.edu/class/ee364a/lectures/chance_constr.pdf`

Value-at-risk of random variable Z, at level η:

$$\mathrm{VaR}(Z;\eta) = \inf\{\gamma \mid P[Z \leq \gamma] \geq \eta\}$$

Therefore, the value-at-risk is simply the inverse of the cdf evaluated at η!

$$\mathrm{VaR}(Z;\eta) = F_Z^{-1}(\eta).$$

Conditional Value at Risk

$$\mathrm{CVaR}(Z;\eta) = \inf_{\beta}\left(\beta + \frac{1}{1-\eta}\mathbb{E}\left[(Z-\beta)_+\right]\right).$$

Assume that the distribution of Z is continuous.

The solution β^* in the CVaR definition can be obtained by searching for the root ot the derivative w.r.t. β:

$$0 = \frac{d}{d\beta}\left(\beta + \frac{1}{1-\eta}\mathbb{E}\left[(Z-\beta)_+\right]\right) = 1 - \frac{1}{1-\eta}P[Z \geq \beta],$$

leading to

$$P[Z \geq \beta] = 1 - \eta.$$

As Z is continuous, this last equation can be rewritten as

$$P[Z \leq \beta] = \eta = \mathrm{VaR}(Z;\eta).$$

Expected shortfall

Conditional tail expectation (or expected shortfall)

$$\begin{aligned}\mathbb{E}[z \mid z \geq \beta^*] &= \mathbb{E}[\beta^* + (z - \beta^*) \mid z \geq \beta^*] \\ &= \beta^* + \frac{\mathbb{E}\left[(z - \beta^*)_+\right]}{P[z \geq \beta^*]} \\ &= \mathrm{CVaR}(z; \eta)\end{aligned}$$