Stochastic programming
Multi-stage problems

Fabian Bastin
fabian.bastin@cirrelt.ca
Université de Montréal – CIRRELT

Motivation : example

- Assume that we invest in the market. We would to reach a nominal goal G after Y years.
- The portfolio is rebalanced each v years, so there are $T = Y/v$ times where we have to decide what are the investments to do.
  There are T stages in our problem.
- We face a space of N investment decisions (i.e. we have N possible actions) ; we have a set $\mathcal{T} = \{1, 2, \ldots, T\}$ of investment periods.
- If we go over the target G, we will have an interest rate q on the surplus.
- If we do not meet the target, we have to borrow the difference at an interest r.
- The initial wealth is b.

Variables

- x_{it}, $i \in N$, $t \in \mathcal{T}$: amount of money to invest in action i during the period t ;
- y : excess amount at the end of the horizon ;
- w : lack of money at the end of the horizon ;
- In the current framework, we aim to take the corrective actions in order to reach the target of G dollars.

Deterministic formulation

$$
\begin{aligned}
\max \;\; & qy - rw \\
\text{s.t.} \; & \sum_{i \in N} x_{i1} = b, \\
& \sum_{i \in N} \omega_{it} x_{i,t-1} = \sum_{i \in N} x_{it}, \quad \forall t \in \mathcal{T} \backslash 1, \\
& \sum_{i \in N} \omega_{iT} x_{iT} - y + w = G, \\
& x_{it} \geq 0, \quad \forall i \in N, \; t \in \mathcal{T}, \\
& y, w \geq 0.
\end{aligned}
$$

Random returns

- The investment returns can be viewed as random variables, as we do not know their realizations in advance.
- Imagine that for each action, at each time t, the are R possible realizations.
- The scenarios are made of all the possible sequences of realizations.

Example : assume $R = 4$ and $T = 3$. Scenarios :

$t = 1$	$t = 2$	$t = 3$
1	1	1
1	1	2
1	1	3
1	1	4
1	2	1
$\vdots$	$\vdots$	$\vdots$
4	4	4

Stochastic program formulation

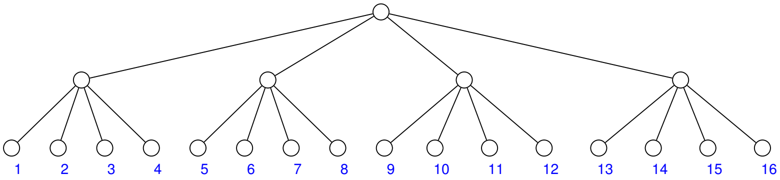
- Let x_{its}, $t \in \mathcal{T}$, $s \in S$ be the amount to invest in action i during the period t in scenario s ;
- y_s : excess amount at the end of the horizon for scenario s ;
- w_s : lack of money at the end of the horizon for scenario s.
- Let ω_{it} be the random return ; it can be seen as function of the scenario s.

Stochastic version

$$
\begin{aligned}
\max \sum_{s=1}^{S} p_s(q y_s - r w_s) \\
\text{s.t.} \sum_{i \in N} x_{i1} = b, \\
\sum_{i \in N} \omega_{its} x_{i,t-1,s} = \sum_{i \in N} x_{its}, \quad \forall t \in \mathcal{T} \setminus 1, \forall s \in S \\
\sum_{i \in N} \omega_{iTs} x_{iT} - y_s + w_s = G, \quad \forall s \in S, \\
x_{its} \geq 0, \quad \forall i \in N,\ t \in \mathcal{T}, \forall s \in S, \\
y_s, w_s \geq 0 \quad \forall s \in S.
\end{aligned}
$$

Scenario tree

Conceptually, the sequence of events can be arranged in a tree. There is one leaf per scenario.



The scenarios can have different probabilities, the variables can be correlated,. . .

Non-anticipativity

- The conversion between the deterministic model and the stochastic version is easy...
- but the built model is wrong!!!
- We have to enforce the non-anticipativity :

Let S_s^t be the set of scenarios that are identical to scenario s at time t. We have to guarantee that

$$x_{its} = x_{its'}, \ \forall i \in N, \ \forall t \in T, \ \forall s \in S, \ \forall s' \in S_s^t.$$

Non-anticipativity (cont'd)

Equivalently,

$$\left(\sum_{s' \in S_s^t} p'_s \right) x_{its} = \sum_{s' \in S_s^t} p'_s x_{its'} \quad \forall\, i, t, s,$$

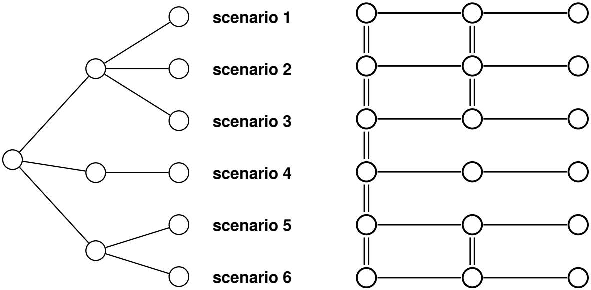
or

$$\begin{aligned} x_{its} &= \frac{\sum_{s' \in S_s^t} p'_s x_{its'}}{\sum_{s' \in S_s^t} p'_s} \quad \forall\, i, t, s \\ &= E[x_{its'} | s' \in S_s^t] \quad \forall\, i, t, s. \end{aligned}$$

The amount invested in scenario s, multiplied by the probability to reach a scenario equivalent to s, has to be equal to the expected amount that would be invested in any scenario equivalent to s.
There are many formulations of these conditions !

Back to the scenario tree

- We have to enforce nonanticipativity as we have created copies of the variables for each scenario, at each period. Graphically :



Nodal approach

- We can also create a vector of variables for each node in the tree.
- This vector corresponds to what should be our decision given the known realizations of the random variables upon this stage (i.e. at this node).
- Index the nodes by $l = 1, 2, \ldots, \mathcal{L}$.
- We have to know the ancestor of each node. Let $A(l)$ the ancestor of node $l \in \mathcal{L}$ in the scenarios tree.

Reformulation

$$
\begin{aligned}
\max \; & \sum_{s\in S} p_s(qy_s - rw_s) \\
\text{s.t.} \; & \sum_{i\in N} x_{i1} = b, \\
& \sum_{i\in N} \omega_{il} x_{i,A(l)} = \sum_{i\in N} x_{il}, \quad \forall l \in \mathcal{L}\backslash 1, \\
& \sum_{i\in N} w_{iA(s)} x_{iA(s)} - y_s + w_s = G, \quad \forall s \in S, \\
& x_{il} \geq 0, \quad \forall i \in N, \; \forall l \in \mathcal{L}, \\
& y_s, w_s \geq 0 \quad \forall s \in S.
\end{aligned}
$$

Multi-period production planification

- A firm produces various different goods.
- Ressources (for instances manual and machine work hours) are required. We assume that these requirements are known for each product.
- The demand must be met at the end of each period, but this demand is not perfectly known.
- A too large or too small inventory induces costs (storage, additional purchases to satisfy the demand,...).
- It is possible to add machine and work hours, within predefined limits.
- There is a hiring and firing cost related to changes in the workforce.

Decision problem

- We have to decide now on
  - the quantity of each product to manufacture during each period ;
  - the additional capacity to use during each period ;
  - to hire or to fire staff during each period ;
- A random demand occurs. Conceptually, this happens at each period.
- After having observed the random demands, we can decide on how to stock the products in the inventory, or to buy from an external source.

Definitions

1. Sets :
   - T : number of periods, also seen as a set ;
   - N : products ;
   - M : ressources.
2. Variables :
   - x_{jt} : production of product $j \in N$ during the period $t \in T$;
   - u_{it} : additional amount of ressource $i \in M$ to buy during the period $t \in T$;
   - $z^{+}_{t-1,t}$, $z^{-}_{t-1,t}$: planned augmentation or reduction of the labor force ;
   - $y^{+}_{j,t}$, $y^{-}_{j,t}$: surplus, deficit of product $j \in N$ at the end of the period $t \in T$.

Parameters

- All the above variables have associated costs (α, β, γ, δ).
- ω_{jt} : demand of product $j \in N$ during the period $t \in T$.
- U_{it} : upper bound on u_{it}.
- a_{ij} : amount of ressource $i \in M$ required to produce one unit of the good $j \in N$.
- b_{it} : amount of ressource $i \in M$ available at time $t \in T$.

Model

$$\min \sum_{j\in N}\sum_{t\in T} \alpha_{jt} x_{jt} + \sum_{i\in M}\sum_{t\in T} \beta_{it} u_{it} + \sum_{t\in T\backslash 1} \left(\gamma^{+}_{t-1,t} z^{+}_{t-1,t} + \gamma^{-}_{t-1,t} z^{-}_{t-1,t}\right)$$

$$+ E_{\omega}\left[\min\left(\sum_{j\in N}\sum_{t\in T}(\delta^{+}_{jt} y^{+}_{jt} + \delta^{-}_{jt} y^{-}_{jt})\right)\right],$$

such that

$$\sum_{j\in N} a_{ij} x_{jt} \le b_{it} + u_{it} \quad \forall i \in M,\ \forall t \in T,$$

$$u_{it} \le U_{it} \qquad \forall i \in M,\ \forall t \in T,$$

$$z^{+}_{t-1,t} - z^{-}_{t-1,t} = \sum_{j\in N} a_{jt}(x_{jt} - x_{j,t-1}) \quad \forall t \in T\backslash 1,$$

$$x_{jt} + y^{+}_{j,t-1} - y^{+}_{jt} + y^{-}_{jt} = \omega_{jt} \qquad \forall j \in N,\ \forall t \in T \text{ (demand satisfaction)},$$

Extended form of the recourse model

$$\min \sum_{j \in N} \sum_{t \in T} \alpha_{jt} \boldsymbol{x}_{jt} + \sum_{i \in M} \sum_{t \in T} \beta_{it} \boldsymbol{u}_{it} + \sum_{t \in T \setminus 1} \left(\gamma^{+}_{t-1,t} \boldsymbol{z}^{+}_{t-1,t} + \gamma^{-}_{t-1,t} \boldsymbol{z}^{-}_{t-1,t} \right)$$
$$+ \sum_{s \in S} \sum_{j \in N} \sum_{t \in T} p_s (\delta^{+}_{jt} y^{+}_{jts} + \delta^{-}_{jt} y^{-}_{jts}),$$

such that

$$\sum_{j \in N} a_{ij} \boldsymbol{x}_{jt} \le b_{it} + \boldsymbol{u}_{it} \quad \forall i \in M, \ \forall t \in T,$$
$$\boldsymbol{u}_{it} \le U_{it} \qquad \forall i \in M, \ \forall t \in T,$$
$$\boldsymbol{z}^{+}_{t-1,t} - \boldsymbol{z}^{-}_{t-1,t} = \sum_{j \in N} a_{jt} (\boldsymbol{x}_{jt} - \boldsymbol{x}_{j,t-1}) \quad \forall t \in T \setminus 1,$$
$$\boldsymbol{x}_{jt} + y^{+}_{j,t-1,s} - y^{+}_{jts} + y^{-}_{jts} = \omega_{jts} \qquad \forall j \in N, \ \forall t \in T,$$

nonanticipativity constraints.