Stochastic optimization
Chance constrained programming - SAA

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Motivation

We consider the problem

$$\min_{x \in \mathcal{X}} f(x)$$
$$\text{s.t. } P[G(x,\xi) \leq 0] \geq 1 - \alpha,$$

where $G : \mathcal{R}^n \times \Xi \rightarrow \mathcal{R}^m$.
This allows to consider separate and joint chanced constrained.
We can reformulate this problem as

$$\min_{x \in \mathcal{X}} f(x)$$
$$\text{s.t. } P[G(x,\xi) > 0] \leq \alpha,$$

If $P[G(x,\xi) \leq 0]$ has no closed form, we can always approximate it by sample average approximation.

SAA

With the indicator function of $(0, \infty)$

$$\mathcal{I}_{(0,\infty)} = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases}$$

we can rewrite the chance constraint as

$$p(x) = E\left[\mathcal{I}_{(0,\infty)}\left(G(x,\xi)\right)\right] \leq \alpha.$$

Let $\xi_1, \ldots, \xi_N$ be an independent identically distributed (iid) sample of N realizations of ξ. We approximate the chance constraint by

$$\hat{p}_N = \frac{1}{N}\sum_{j=1}^{N} \mathcal{I}_{(0,\infty)}\left(G(x,\xi_j)\right) \leq \gamma$$

SAA (cont'd)

$\hat{p}_N(x)$ is equal to the proportion of times that $G(x, \xi_j) > 0$. *gamma* is not necessarily equal to α, especially when we want to enlarge the feasible of the approximate problem.

The approximate problem is

$$\min_{x \in \mathcal{X}} f(x)$$
$$\text{s.t. } \hat{p}_N(x) \leq \gamma.$$

Theory

For theoretical justifications, see

> *B.K. Pagnoncelli, S. Ahmed, and A. Shapiro, "Sample Average Approximation Method for Chance Constrained Programming: Theory and Applications", Journal of Optimization Theory and Applications 142 (2), pp. 399–416, 2009.*

Especially, under some regularity conditions, we have

$$sup_{x \in C} |\hat{p}_N(x) - p(x)| \to 0,$$

w.p.1, as $N \to \infty$.

Convergence

Assume that there is an optimal solution $\overline{x}$ of the true problem such that for any $\epsilon > 0$ there is $x \in \mathcal{X}$ with $\|x - \overline{x}\| \leq \epsilon$ and $p(x) < \alpha$.

Theorem

Suppose $\gamma = \alpha$, $\mathcal{X}$ compact, and $f(x)$ continuous. Then $\hat{v}_N \to v^*$, where $\hat{v}_N$ and v^* are the optimal values of the approximate and true problems, respectively, and $D(\hat{S}_N, S) \to 0$ almost surely as $N \to \infty$.

Application: call centers

Work in progress with Thuy Anh Ta and Pierre L'Ecuyer. The next slides are due Thuy Anh Ta.

A call center: an office used for receiving or transmitting customers requests by telephone.

- A call represents a customer and it is classified by type (K types).
- Customers are served by agents.
- Agents sharing a common set of tasks form a group of agents (I groups).
- A day is divided into periods (P periods).
- The work schedule of an agent is determined by her shift (Q shift configurations).

Performance measure

- Performance measures (PM) allow to assess the quality of service (QoS) and efficiency of a call center, e.g., Service level (SL) and average waiting time (AWT), abandonment ratio, occupancy ratio, etc.
- Service level (SL): the percentage of calls that are answered in a defined waiting threshold.
- Average waiting time (AWT): the average (or mean) time a customer waited to have a service.
- QoS can be defined in the long run (expectations), or over a given time period (random variable).

Staffing, Scheduling and Routing

- The objective is to minimize the operating cost of a call center under a set of constraints on certain PMs (SL or AWT).
- Staffing: deciding the number of agents $y_{i,p}$ of each group for each period in a day.
- Scheduling: determining how many agents $x_{i,q}$ to assign to a set of predefined shifts.
- Routing: determining which agents are allowed to handle the call, and how agents are chosen when several agents are free.

Modeling a call center

- The calls arrive according to arbitrary stochastic processes that could be non-stationary (e.g. homogeneous Poisson process), and perhaps doubly stochastic (e.g. Poisson gamma process, the arrival rate is a random variable of gamma distribution).
- Service times: exponential, log-normal distributions.
- Delay time, patience time, etc.
- When number of skills is low: Arrivals are served in a first come first serve (FCFS) order and/or a longest idle server first rules.
- Otherwise, skills based routing may be used to route calls to appropriate agents.

$$g(y) = \frac{\mathbb{E}[\#\text{of served call that waited at most } \tau]}{\mathbb{E}[\# \text{ Total } \# \text{ of calls}]}$$

Scheduling problem

$$\min c^T x = \sum_{i=1}^{I} \sum_{q=1}^{Q} c_{i,q} x_{i,q}$$

subject to:

$$\begin{aligned} Ax &\geq y \\ g_{k,p}(y) &\geq l_{k,p} \ \forall k, p, \\ g_p(y) &\geq l_p \ \forall p, \\ g_k(y) &\geq l_k \ \forall k, \\ g(y) &\geq l, \\ x, y &\geq 0 \text{ and integer.} \end{aligned} \tag{P1}$$

Staffing problem

$$\min_{y} {c'}^T y = \sum_{i=1}^{I} \sum_{p=1}^{P} c'_{i,p} y_{i,p}$$

subject to:

$$\begin{aligned} g_{k,p}(y) &\geq l_{k,p} \ \forall k, p, \\ g_p(y) &\geq l_p \ \forall p, \\ g_k(y) &\geq l_k \ \forall k, \\ g(y) &\geq l, \\ y &\geq 0 \text{ and integer.} \end{aligned} \tag{P2}$$

SL $S(\tau, y)$ and AWT $W(y)$ are considered as random variables in a given period.
The chance-constrained scheduling problem for multiskill call centers is formulated as:

$$\min c^T x = \sum_{i=1}^{I} \sum_{q=1}^{Q} c_{i,q} x_{i,q}$$

subject to:

$$\begin{aligned} Ax &\geq y \\ \mathbb{P}[S_{k,p}(\tau_{k,p}, y) \geq s_{k,p}] &\geq r_{k,p} \qquad \forall k, p \\ \mathbb{P}[W_{k,p}(y) \leq w_{k,p}] &\geq v_{k,p} \qquad \forall k, p \\ x, y &\geq 0 \text{ and integer.} \end{aligned} \tag{P3}$$

Sample average approximate problem

$$\min c^T x = \sum_{i=1}^{I} \sum_{q=1}^{Q} c_{i,q} x_{i,q}$$

subject to:

$$\begin{aligned}
Ax &\geq y \\
\frac{1}{N} \sum_{d=1}^{N} \mathbb{I}[\hat{S}^d_{N,k,p}(\tau_{k,p}, y) \geq s_{k,p}] &\geq r_{k,p} \qquad \forall k, p \\
\frac{1}{N} \sum_{d=1}^{N} \mathbb{I}[\hat{W}^d_{N,k,p}(y) \leq w_{k,p}] &\geq v_{k,p} \qquad \forall k, p \\
x, y &\geq 0 \text{ and integer .}
\end{aligned} \tag{SAA}$$

Under some specific conditions, the optimal value and the set of optimal solutions of the SAA problem converge to those of the true problem with probability one as N approaches infinity.