

# A short tutorial of random numbers generation

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## U[0,1]-distributed random numbers

- A good uniform random generator on the interval  $[0, 1]$  is a major component of any good random generator library.
- Draws from other distributions are usually obtained by adequately transform an uniformly distributed sample.

Define a **transition function**  $f : \mathcal{S} \rightarrow \mathcal{S}$ , where  $\mathcal{S}$  is the **state space**. The cardinality of  $\mathcal{S}$  is assumed to be finite.

The initial state is denoted by  $s_0$ , and we will write

$$s_n = f(s_{n-1}).$$

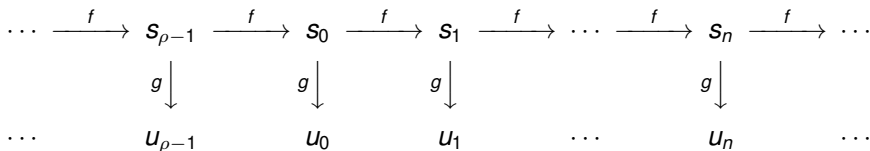
We will furthermore assume that  $f$  is periodic for all  $n$  greater or equal to some known  $\tau$  (often equal to 0), with period  $\rho$ :

$$s_{n+\rho} = s_n, \quad \forall n \geq \tau.$$

## U[0,1]-distributed random numbers (cont'd)

- Output space:  $\mathcal{U}$ .
- We assume here that  $\mathcal{U} = (0, 1)$ .
- Output function  $g : \mathcal{S} \rightarrow \mathcal{U}$ .

It transforms the state  $s_n$  into an output value  $u_n$ .



**How to choose  $f$  and  $g$ ?**

**Goals:** large  $\rho$ , good uniformity, “random” behavior.

# Linear congruential generators (LCGs)

- Introduced by Lehmer (1951).
- State  $s = x \in \mathbb{N}_{\geq 0}$ .
- Recursive formula:

$$x_i = f(x_{i-1}) = (ax_{i-1} + c) \mod m,$$

where  $\mod$  is the modulo operator.

Given two numbers,  $a$  (the dividend) and  $n$  (the divisor),

$$a \mod n = a - n \left\lfloor \frac{a}{n} \right\rfloor.$$

## Linear congruential generators: full period?

Full period:  $m$  is  $c \neq 0$ ,  $m - 1$  otherwise (if  $c = 0$ , 0 is a fixed point for the recurrence). Consider the case  $c \neq 0$ .

### Theorem (Period)

*The LCG has full period iff the following three conditions hold:*

- 1. the only positive integer that (exactly) divides both  $m$  and  $c$  is 1;*
- 2. if  $q$  is a prime number that divides  $m$ , then  $q$  divides  $a - 1$ ;*
- 3. if 4 divides  $m$ , then 4 divides  $a - 1$ .*

**"Standard minimal"** (Park and Miller, 1988):

$$x_{n+1} = 16807x_n \bmod 2147483647.$$

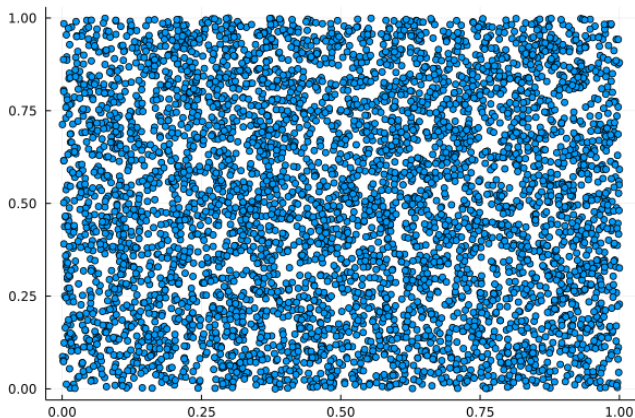
Observe that  $2147483647 = 2^{31} - 1$ ; on 32-bit architectures, the largest representable (signed) integer is  $2^{31}$ .

# The Standard Minimal Generator

```
function getlcg(seed::Integer, a::Integer, c::Integer,
               m::Integer)
    state = seed
    am_mil = 1.0/m
    return function lcgrand()
        state = mod(a * state + c, m)
        return state*am_mil # produce a number in (0,1)
    end
end

stdmin = getlcg(1234, 16807, 0, 2^31-1)
```

# Standard minimal generator: illustration



10000 generated points on the unit square.

# Multiple Recursive Generator (MRG)

But we want better! Generalize the LCG:

$$x_n = (a_1 x_{n-1} + \cdots + a_k x_{n-k}) \bmod m, \quad u_n = x_n/m.$$

In practice, we will take  $u_n = (x_n + 1)/(m + 1)$ , or  $u_n = x_n/(m + 1)$  if  $x_n > 0$  and  $u_n = m/(m + 1)$  otherwise, but the structure remains the same, and is easier when studying theoretical properties. This kind of generators is very popular.

State at step  $n$ :

$$s_n = (x_{n-k+1}, \dots, x_n)^T.$$

State space:  $\mathcal{Z}_m^k$ , of cardinality  $m^k$ .

The maximal period if  $\rho = m^k - 1$ .



## Period of MRG's

It can be shown that for  $k > 1$ , it is sufficient to have at least two non-zero coefficient, including  $a_k$ , in order to get the maximal period.

The cheapest recurrence has therefore the form

$$x_n = (a_r x_{n-r} + a_k x_{n-k}) \mod m.$$

But how to choose  $a_r$  and  $a_k$ ?

It is possible to study theoretical properties of MRG's, and excluding directly some generators that have known strong deficiencies.

# Choosing a good MRG's

## Example: Lagged-Fibonacci

$$x_n = (\pm x_{n-r} \pm x_{n-k}) \mod m.$$

It can be shown the vectors  $(u_n, u_{n+k-r}, u_{n+k})$  are all contained in two plans! We therefore know without additional tests that the numbers cannot be considered as random.

In practice, we can impose various conditions on the coefficients, and compute theoretically appealing generators by maximizing some quality measure. This maximization is numerically expensive.

## Combined MRG's

Consider two (or more) MRG's working in parallel:

$$\begin{aligned}x_{1,n} &= (a_{1,1}x_{1,n-1} + \cdots + a_{1,k}x_{1,n-k}) \bmod m_1, \\x_{2,n} &= (a_{2,1}x_{2,n-1} + \cdots + a_{2,k}x_{2,n-k}) \bmod m_2.\end{aligned}$$

We define the two **combinations**

$$\begin{aligned}z_n &:= (x_{1,n} - x_{2,n}) \bmod m_1; & u_n &:= z_n/m_1; \\w_n &:= (x_{1,n}/m_1 - x_{2,n}/m_2) \bmod 1.\end{aligned}$$

The sequence  $\{w_n, n \geq 0\}$  is the output of another MRG, of module  $m = m_1 m_2$ , and  $\{u_n, n \geq 0\}$  is nearly the same sequence if  $m_1$  and  $m_2$  are close.

We can achieve the period  $(m_1^k - 1)(m_2^k - 1)/2$ .

## MRG32k3a

The following combined MRG was proposed by L'Ecuyer, and is amongst the most popular and efficient known generators. It combines 2 MRG's.

$$\begin{aligned}k &= 3, \\m_1 &= 2^{32} - 209, a_{11} = 0, a_{12} = 1403580, a_{13} = -810728, \\m_2 &= 2^{32} - 22853, a_{21} = 527612, a_{22} = 0, a_{23} = -1370589.\end{aligned}$$

Combination:  $z_n = (x_{1,n} - x_{2,n}) \bmod m_1$ .

$$\begin{aligned}\text{Corresponding MRG: } k &= 3, \\m &= m_1 m_2 = 18446645023178547541, \\a_1 &= 18169668471252892557, \\a_2 &= 3186860506199273833, \\a_3 &= 8738613264398222622.\end{aligned}$$

$$\text{Périod } \rho = (m_1^3 - 1)(m_2^3 - 1)/2 \approx 2^{191}.$$

# MRG32k3a: implementation

```
function rand(rng::MRG32k3a)

    p1::Int64 = (a12 * rng.Cg[2] + a13 * rng.Cg[1]) % m1
    p1 += p1 < 0 ? m1 : 0

    rng.Cg[1] = rng.Cg[2]
    rng.Cg[2] = rng.Cg[3]
    rng.Cg[3] = p1

    p2::Int64 = (a21 * rng.Cg[6] + a23 * rng.Cg[4]) % m2
    p2 += p2 < 0 ? m2 : 0

    rng.Cg[4] = rng.Cg[5]
    rng.Cg[5] = rng.Cg[6]
    rng.Cg[6] = p2

    u::Float64 = p1 > p2 ? (p1 - p2) * norm :
        (p1 + m1 - p2) * norm
end
```

## Random numbers generators on $\mathcal{F}_2$

Alternatives to MRG's: random numbers generators based on linear recurrence in  $\mathcal{F}_2$ .

Galois field  $\mathcal{F}_2$ : set  $\{0, 1\}$  on which we define addition and multiplication operation modulo 2.

We construct two sequence of bits vectors  $\mathbf{x}_n$  and  $\mathbf{y}_n$  with the linear recurrences

$$\mathbf{x}_n = X\mathbf{x}_{n-1} \quad (\text{state vector, } k \text{ bits}),$$

and

$$\mathbf{y}_n = B\mathbf{x}_n \quad (\text{output vector, } w \text{ bits}),$$

$$u_n = \sum_{j=1}^w y_{n,j-1} 2^{-j} = .y_{n,0} y_{n,1} y_{n,2} \cdots \quad (\text{sortie}).$$

## Random numbers generators on $\mathcal{F}_2$ (2)

The implementation of  $\mathcal{F}_2$ -generators is often quite complex, but it is possible to operate bitwise, so that they are numerically very fast.

The LFSR (linear feedback shift register), while known to have important deficiencies, gives an illustration of such generators.

We use the relations (with  $a_k \neq 0$ )

$$u_n = \sum_{l=1}^w x_{n\nu+j-1} 2^{-l} = .x_{n\nu} x_{n\nu+1} x_{n\nu+2} \dots x_{n\nu+l-1}$$
$$X = \begin{pmatrix} & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ a_k & a_{k-1} & \dots & a_1 \end{pmatrix}^{\nu} \text{ et } B = I.$$

## Tausworthe Generator (2)

**Maximum period.**  $\rho = 2^k - 1$  iff

$Q(z) = z^k - a_1 z^{k-1} - \dots - a_{k-1} z - a_k$  is primitive and  $\text{pgcd}(\nu, 2^k - 1) = 1$ .

In most applications, only two coefficients are nonzero in order to simplify implementation, so we obtain

$$Q(z) = z^k - a_r z^{k-r} - a_k.$$

Since we are working in  $\mathcal{F}_2$ , the recurrence on  $x_n$  becomes

$$x_n = (x_{n-r} + x_{n-k}) \mod 2.$$

The execution of the addition modulo 2 is equivalent to the instruction exclusive-or (xor) on the bits:

$$x_n = \begin{cases} 0 & \text{si } x_{n-r} = x_{n-k}, \\ 1 & \text{si } x_{n-r} \neq x_{n-k}. \end{cases}$$



# Implementations

More generally, we construct a fast implementation by using shifts, xor's, masks, . . . We can also combine them.

Most popular:

- Mersenne Twister MT19937 (Matsumoto and Nishimura); period of  $2^{19937} - 1$
- xoshiro: <https://prng.di.unimi.it/>

## Jump ahead

- A very useful possibly proposed by some implementation is the possibility to make a jump of  $m$  positions in the random number sequences, with  $m$  very large.
- This allows to easily define independant random variables.
- Useful in simulation when relying on common random random numbers.

The MRG32k3a implementation proposes functions to generate independent streams and substreams.

# RDST library

`https://github.com/JLChartrand/RDST.jl`

This package proposes an implementation of MRG32k3a with streams and substreams.

## Non-uniform random variables generation

A good reference: Luc Devroye, *Non-Uniform Random Variate Generation*,

<http://luc.devroye.org/rnbookindex.html>.

Assume that we have a good uniform random variates generator, but we want to generate random variables following various probability laws: Normal, Weibull, Poisson, binomial,...

The desired properties are:

- correct method (or good approximation);
- as simple as possible, but as fast as possible;
- low memory consumption;
- robust;
- compatible with variance reduction technique (as quasi-Monte Carlo).

## Inversion

This is the preferred method, if it can be applied. The reason is that it is compatible with variance reduction.

Consider a random variable  $X$  with cumulative distribution function  $F$ . Let  $U \sim U(0, 1)$  and

$$X = F^{-1}(U) = \min\{x : F(x) \geq U\}.$$

Then

$$P[X \leq x] = P[F^{-1}(U) \leq x] = P[U \leq F(x)] = F(x),$$

i.e.,  $X$  has the desired distribution. Indeed,

- in the continuous case,  $F(X) \sim U[0, 1]$ ;
- in the discrete case, it is easy to prove that  $P[X = x_i] = p(x_i)$ , for all  $i$ , and we assume  $x_1 < x_2 < \dots < x_n$ ;
- The principle still works for mixte distributions.

## Inversion (2)

- **Advantage:** monotone, only one  $U$  for all  $X$ .
- **Weakness:** for some laws,  $F$  is very difficult to invert. But we can often approximate  $F^{-1}$ .

**Example:** normal law.

If  $Z \sim N(0, 1)$ , then  $X = \sigma Z + \mu : N(\mu, \sigma^2)$ .

It is therefore sufficient to be able to generate a  $N(0, 1)$ , of density  $f(x) = (2\pi)^{-1/2} e^{-x^2/2}$ .

We do not have any formula for  $F(x)$  or  $F^{-1}(x)$ . Efficient codes however exist to approximate  $F^{-1}(x)$ .

Chi-square, gamma, beta, etc.: it is much more complicated since the form of  $F^{-1}$  depends of the distribution parameters.

## Inversion for discrete distributions

Recall that

$$p(x_i) = P[X = x_i]; \quad F(x) = \sum_{x_i \leq x} p(x_i).$$

We have to generate  $U$ , search  $I = \min\{i | F(x_i) \geq U\}$  and return  $x_I$ .

Various algorithms perform this search. Their efficiency depends of the distribution.

**Initialization:** store the  $x_i$  and  $F(x_i)$  in arrays, for  $i = 1, \dots, n$ .

1. **Linear search** (time in  $O(n)$ ):  $U \leftarrow U(0, 1)$ ;  $i \leftarrow 1$ ;  
while  $F(x_i) < U$  do  $i \leftarrow i + 1$ ; return  $x_i$ .

2. **Binary search** (time in  $O(\log(n))$ ):

$U \leftarrow U(0, 1)$ ;  $L \leftarrow 0$ ;  $R \leftarrow n$ ;

while  $L < R - 1$

$m \leftarrow \lfloor (L + R)/2 \rfloor$ ;

if  $F(x_m) < U$  then  $L \leftarrow m$  otherwise  $R \leftarrow m$ ;

return  $x_R$ .

## Other approaches: composition

Assume that  $F$  is a convex combination of several cumulative distribution functions:

$$F(x) = \sum_{j=0}^{\infty} p_j F_j(x),$$

and that it is easier to invert  $F_j$ ,  $j = 0, \dots, \infty$  than  $F$ .

Generate  $J = j$  with the probability  $p_j$ , then generate  $X$  following  $F_j$ .

The method therefore requires two uniforms for each random variable, and exploit the decomposition

$$P[X \leq x] = \sum_{j=1}^{\infty} P[X \leq x | J = j] P[J = j] = \sum_{j=1}^{\infty} F_j(x) p_j.$$



# Convolution

**Convolution.** Assume that

$$X = Y_1 + Y_2 + \dots + Y_n,$$

where the  $Y_i$  are independent, of given laws. We generate the  $Y_i$ ,  $i = 1, \dots, n$ , and we sum.

Examples: Erlang (sum of exponentials with same mean), binomial.

**Acceptance/rejection:** the most important technique after inversion.

We consider the case where  $X$  is continuous (the discrete case is analogous). Let  $f(x)$  be the density of  $X$ , and let  $t$  be a "hat" function that majors  $f$ , i.e.  $f(x) \leq t(x) \forall x$ .

## Acceptance/rejection

We can normalize  $t$  in a density  $r$ :

$$r(x) = t(x)/a, \text{ where } a = \int_{-\infty}^{\infty} t(s)ds.$$

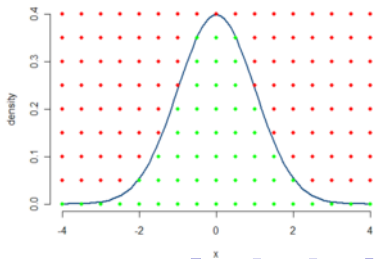
We choose  $t$  so that

1. it is easy to generate random variables of density  $r$ ;
2.  $a$  is small (close 1), or in other terms,  $t(x)$  is close to  $f(x)$ .

The choice of  $t$  may be automatized.

**Algorithm:** Repeat

1. generate  $Y$  of density  $r(x)$ ;
2. generate  $U : U(0, 1)$  independantly of  $Y$ ;
3. until  $U \leq f(Y)/t(Y)$ ;
4. return  $Y$ .



## Particular cases

Sometimes, we can benefit of mathematical transformations. The main weakness is that they are seldom compatible with variance reduction techniques.

**Example:** Box-Muller method for the normal law.

**Idea:** it is easier to generate a point  $(X, Y)$  from the bivariate normal law, of density on  $\mathbb{R}^2$

$$f(x, y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}.$$

We change the cartesian coordinates  $(X, Y)$  by the polar coordinates  $(R, \Theta)$ :

$$R^2 = X^2 + Y^2; Y = R \sin \Theta.$$

It gives an elegant approach, but incompatible with variance reduction techniques and can be slower than inversion.

# Box-Muller Algorithm

1. Independently draw  $U_1, U_2$  from a  $U(0, 1)$  distribution.
2. Set

$$R = \sqrt{-2 \log(U_1)}$$

$$\theta = 2\pi U_2$$

3. Set

$$X = R \cos(\theta)$$

$$Y = R \sin(\theta)$$