

Stochastic optimization

Two-stage stochastic programming with recourse

Fabian Bastin

`fabian.bastin@umontreal.ca`

Université de Montréal – CIRRELT – IVADO – Fin-ML

Formalization

Uncertainty: representation by means of **random elements**.

The realizations are denoted by ω , and they are drawn from the sample space Ω .

A **event** A is a subset of Ω ; the collection of random events is denoted by \mathcal{A} . The event $A \in \mathcal{A}$ occurs if the output of the experiment is an element from A .

A random linear program

Consider the linear program (LP), parametrized by the random (r.v.) $\xi : \Omega \rightarrow \mathbb{R}^2$:

$$\begin{aligned} \min_x \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & T(\xi)x = h(\xi) \\ & x \in X, \end{aligned}$$

with $X = \{x \in \mathbb{R}^n | l \leq x \leq u\}$. Example:

$$\begin{aligned} \min_x \quad & x_1 + x_2 \\ \text{s.t.} \quad & \xi_1 x_1 + x_2 \geq 7 \\ & \xi_2 x_1 + x_2 \geq 4 \\ & x_1, x_2 \geq 0, \end{aligned}$$

where $\xi_1 \sim U[1, 4]$, $\xi_2 \sim U[1/3, 1]$.

What to do?

- How to solve this problem?
- What is the *meaning* of solving this problem?
- Can we decide on x *after* having observed the realization of the r.v. ξ ?

We then talk of an **wait-and-see** approach. The problem is then easier to solve (we have here a simple linear program).

- But this approach is rarely appropriate!!! We usually have to decide on x **before** we know the realizations of ξ !
- Usually, the “wait-and-see” approach is not appropriate to model the reality behavior: we have to decide on x before we know the realizations from ξ .
- Three suggestions:
 1. try to estimate, predict, the uncertainty;
 2. chance-constraints;
 3. penalties on deviations.

Remove the randomness?

A popular approach consists to look for reasonable values for ξ_1 and ξ_2 . How?

Propositions:

- unbiased: choose the mean values for each random variable;
- pessimistic: choose the worst-case values for ξ ;
- optimistic: choose the best-case values for ξ .

Each approach will deliver a different optimal solution!

Penalization of violations

Again, we have to deal with decision problems where the decision x has to be taken before we know the realization of ξ . In the simplest case, we can simply penalize the constraints deviations by vectors of penalty coefficients q_+ and q_- .

$$\begin{aligned} \min \quad & c^T x + q_+^T s(\xi) + q_-^T t(\xi) \\ \text{s.t.} \quad & Ax = b, \\ & T(\xi)x + s(\xi) - t(\xi) = h(\xi), \\ & x \in X. \end{aligned}$$

But it is still not possible to solve the problem!

The new optimization problem

A reasonable, and solvable, problem is then

$$\begin{aligned} \min \quad & c^T x + E_{\xi}[q_+^T s(\xi) + q_-^T t(\xi)] \\ \text{s.t.} \quad & Ax = b, \\ & T(\xi)x + s(\xi) - t(\xi) = h(\xi), \text{ for a.e. } \xi \\ & x \in X, \end{aligned}$$

where *a.e.* stands for “almost every”.

- In general, we can react in a correct (and maybe optimal) way: we have a recourse to “correct” the first decision once the uncertainty is removed.
- A LP recourse structure is provided by 3 elements:
 - a set $Y \subset \mathbb{R}^p$ that describes the feasible set of recourse actions, for instance $Y = \{y \in \mathbb{R}^p \mid y \geq 0\}$;
 - q : a vector of recourse costs;
 - $W \in \mathbb{R}^{m \times p}$: **recourse matrix**.

Recourse formulation

The previous considerations lead us to formulate the more general following program:

$$\begin{aligned} \min \quad & c^T x + E_{\xi}[q^T(\xi)y(\xi)] \\ \text{s.t.} \quad & Ax = b, \\ & T(\xi)x + Wy(\xi) = h(\xi), \text{ for a.e.} \\ & x \in X, \\ & y(\xi) \in Y, \text{ for a.e. } \xi. \end{aligned}$$

We could have W varying with the realization ξ . If W is unique, as in the previous formulation, we speak of **fixed recourse**: the recourse does not change with the scenario.

But how to decide on y ?

The two-stage linear stochastic problem (SP)

Using the previous definitions, we can rewrite the stochastic programming problem with recourse in terms of x only:

$$\min_{x \in X} \{c^T x + Q(x) \mid Ax = b\}.$$

It is a (nonlinear) mathematical programming problem in \mathbb{R}^n .
The properties of $Q(x)$ influence the solution techniques.

Is $Q(x)$

- linear?
- convex?
- continuous?
- differentiable?

Expression in terms of y 's

$$\min_{x, y(\xi)} E_{\xi}[c^T x + q^T y(\xi)]$$

$$\text{s.t. } Ax = b$$

first-stage constraints

$$T(\xi)x + Wy(\xi) = h(\xi), \text{ for a.e. } \xi$$

second-stage constraints

$$x \in X, y(\xi) \in Y \text{ for a.e. } \xi.$$

Consider the (discrete) case where $\Omega = \{\omega_1, \omega_2, \dots, \omega_S\} \subset \mathbb{R}^r$.

$$P(\omega = \omega_s) = p_s, \quad s = 1, 2, \dots, S$$

$$T_s = T(\xi(\omega_s)), \quad h_s = h(\xi(\omega_s))$$

Deterministic equivalent

Develop along the S scenarios.

$$\min_{x, y_1, \dots, y_S} c^T x + p_1 q^T y_1 + p_2 q^T y_2 + \dots p_S q^T y_S$$

s.t.

$$\begin{array}{ll} Ax & = b \\ T_1 x & + Wy_1 = h_1 \\ T_2 x & \quad + Wy_2 = h_2 \\ \vdots & \quad \quad \ddots \\ T_S x & \quad \quad + Wy_S = h_S \\ x \in X, y_1 \in Y, y_2 \in Y, \dots, y_S \in Y. \end{array}$$

Deterministic equivalent (cont'd)

- $y_s = y(\xi(\omega_s))$ is the recourse action to take if the scenario s occurs.
- Advantage: it is a linear program.
- Drawback: it is a linear program of (very) large dimension:
 - $n + pS$ variables;
 - $m_1 + mS$ constraints.
- Advantage: the linear program matrix has a special structure (stairway shape).
Can we exploit it?

Large scale,... and?

Assume that we have r random variables ($\Omega \subset \mathbb{R}^r$).

- Consider the following problem (source: Linderöth). A Telecom company want to expand its network in order to meet an unknown (random) demand.
- There are 86 unknown demands. Each demand is independant and take a value in a set of 7 values. Consequently

$$S = |\Omega| = 7^{86} \approx 4.77 \times 10^{72}.$$

... number of subatomic particles in the universe!

- It can be even worse...
If Ω is not finite, but holds an infinite number of elements?
It is especially true with continuous random variables. Our “deterministic equivalent” would have an infinite number of variables and constraints!
- We can solve an approximate problem, obtained by sampling over the random vector.

An example (cont'd)

Consider again our toy problem

$$\begin{aligned} \min_x \quad & x_1 + x_2 \\ \text{s.t.} \quad & \xi_1 x_1 + x_2 \geq 7 \\ & \xi_2 x_1 + x_2 \geq 4 \\ & x_1, x_2 \geq 0, \end{aligned}$$

where $\xi_1 \sim U[1, 4]$, $\xi_2[1/3, 1]$.

How to build the deterministic equivalent?

Example: recourse formulation

Assume for now that Ω is finite, with S scenarios.

$$\begin{aligned} \min_x \quad & x_1 + x_2 + \sum_{s \in S} p_s \lambda (y_{1s} + y_{2s}) \\ \text{s.t.} \quad & \xi_{1s} x_1 + x_2 + y_{1s} \geq 7 \\ & \xi_{2s} x_1 + x_2 + y_{1s} \geq 4 \\ & x_1, x_2 \geq 0, \\ & y_{1s}, y_{2s} \geq 0. \end{aligned}$$

A difficulty is therefore to decide how to construct the deterministic equivalent. How to choose λ ?

How to construct the scenarios? We can proceed with Monte Carlo sampling, with $p_s = 1/N$, $\forall s$. We will explore this approach in more details later.

Example: recourse formulation (cont'd)

More generally, we can build the program

$$\begin{aligned} \min_x \quad & x_1 + x_2 + E_{\xi}[Q(x, \xi)] \\ \text{s.t.} \quad & x_1, x_2 \geq 0, \end{aligned}$$

and

$$\begin{aligned} Q(x, \xi_i) = \min_y \quad & q_1 y_1 + q_2 y_2 \\ \text{s.t.} \quad & \xi_1 x_1 + x_2 + y_1 \geq 7, \\ & \xi_2 x_1 + x_2 + y_2 \geq 4. \end{aligned}$$

Two-stage linear programming problem, fixed recourse

More generally, consider the problem

$$\min c^T x + E_{\xi}[q(\xi)^T y(\xi)]$$

subject to the constraints

$$\begin{aligned} Ax &= b, \\ T(\xi)x + Wy(\xi) &= h(\xi) \quad \text{for a.e. } \xi \in \Xi, \\ x &\in X, \\ y(\xi) &\in Y, \end{aligned}$$

where ξ is a random vector defined on the random space (Ω, \mathcal{F}, P) , and Ξ is the support of ξ .

Let

$$Q(x, \xi) = \min_{y \in Y} \left\{ q(\xi)^T y : Wy = h(\xi) - T(\xi)x \right\}.$$

Reformulation(s)

$$\min_{x \in X \mid Ax=b} \left\{ c^T x + E_{\xi} \left[\min_{y \in Y} \{ q(\xi)^T y \mid Wy = h(\xi) - T(\xi)x \} \right] \right\}$$

Second-stage function, or **recourse function**, $v : \Xi \times \mathbb{R}^m \rightarrow \mathbb{R}$:

$$v(\xi, z) \stackrel{\text{def}}{=} \{ q(\xi)^T y \mid Wy = z \}.$$

Given a “policy” x and a realization of the random vector ξ , z measures the deviation of the first stage, i.e. $z = h(\xi) - T(\xi)x$, $v(\xi, z)$ is the minimum cost to “correct” the decision in order to satisfy the constraints again.

Recourse function

The **expected recourse function**, or the function of minimum expected recourse, $Q : \mathbb{R}^n \rightarrow \mathbb{R}$, for any policy $x \in \mathbb{R}^n$:

$$Q \stackrel{\text{def}}{=} E_{\xi}[Q(x, \xi)],$$

describes the recourse cost expectation, with

$$Q(x, \xi) = v(\xi, h(\xi) - T(\xi)x).$$

With these definitions, the problem can be rewritten as:

$$\min_{x \in X} c^T x + Q(x) \text{ such that } Ax = b.$$

It is a nonlinear program over \mathbb{R}^n . Properties?

Summary

Summarize our formulations.

$$\min_{x \in \mathbb{R}_+^n \mid Ax=b} \left\{ c^T x + E_\xi \left[\min_{y \in \mathbb{R}_+^p} \{ q(\xi)^T y \mid Wy = h(\xi) - T(\xi)x \} \right] \right\}$$

$$\min_{x \in \mathbb{R}_+^n \mid Ax=b} \left\{ c^T x + E_\xi [v(\xi, h(\xi) - T(\xi)x)] \right\}$$

$$\min_{x \in \mathbb{R}_+^n \mid Ax=b} \left\{ c^T x + E_\xi [Q(x, \xi)] \right\}$$

$$\min_{x \in \mathbb{R}_+^n} \left\{ c^T x + Q(x) \mid Ax = b \right\}$$

Notations

- First-stage feasible set:

$$K_1 = \{x \in \mathbb{R}_+^n \mid Ax = b\}.$$

- Second-stage strong feasible set:

$$K_2^s = \{x \mid Q(x) < \infty\}.$$

Therefore we can rewrite the problem as

$$\min_x \{c^T x + Q(x) \mid x \in K_1 \cap K_2^s\}.$$

Weak feasible set

See Walkup, D. W. and Wets, R. J.-B., *Stochastic programs with recourse*, SIAM Journal on Applied Mathematics, 15(5):1299–1314, 1967.

Positive hull (or conical hull)

$$\text{pos } W = \{z \mid z = Wy, y \in \mathbb{R}_+^p\}$$

Weak second-stage feasible set

$$\begin{aligned} K_2 &= \{x \in \mathbb{R}^n \mid Q(x, \xi) < +\infty \text{ a.s.}\} \\ &= \{x \in \mathbb{R}^n \mid (h(\xi) - T(\xi)x) \in \text{pos } W \text{ a.s.}\} \end{aligned}$$

Relatively complete recourse

A problem is said to have a **relatively complete recourse** if $K_1 \subseteq K_2$.

Advantage: the second-stage problem is feasible $\forall x$ feasible in the first stage, almost surely.

Issue: $K_2^s \subseteq K_2$. We would like $K_2^s = K_2$.

Complete recourse

- The relatively complete recourse is very useful in practice and on a theoretical point of view, but it can be difficult to identify.
- A particular case of relatively complete recourse can however often be identified from the structure of W .
- **Complete recourse:** $\text{pos } W = \mathbb{R}^m$.
- The complete recourse property implies that $\forall x, T(\xi), h(\xi), Q(x, \xi) < \infty$, as $z = h(\xi) - T(\xi)x$.
- Complete recourse \Rightarrow relatively complete recourse.

$$K_2 \neq K_2^s$$

Consider the second-stage problem

$$\begin{aligned} \min_y \quad & 2y_1 + y_2 \\ \text{s.t.} \quad & y_1 + y_2 \geq 1 - x_1, \\ & y_1 \geq \xi - x_1 - x_2, \\ & y_1, y_2 \geq 0, \end{aligned}$$

with

$$P[\xi = 2^n] = \frac{1}{2^{n+1}}, \quad n = 0, 1, 2, \dots$$

Note that $P[\xi = 2^n] \in (0, 1)$, $n = 0, 1, 2, \dots$, $\Xi = \{2^n, n \in \mathbb{N}_+\}$, and

$$\sum_{n=0}^{+\infty} P[\xi = 2^n] = \sum_{n=0}^{+\infty} 2^{-n-1} = \frac{1}{2} \frac{1}{1 - \frac{1}{2}} = 1.$$

$K_2 \neq K_2^s$ (cont'd)

Given $x \in \mathbb{R}^2$, we have

$$\begin{aligned} Q(x) &= \mathbb{E}_\xi \left[\min_{y \geq 0} 2y_1 + y_2 \mid y_1 + y_2 \geq 1 - x_1, y_1 \geq \xi - x_1 - x_2 \right] \\ &\geq \mathbb{E}_\xi \left[\min_{y_1 \geq 0} 2y_1 \mid y_1 \geq \xi - x_1 - x_2 \right] \\ &= 2\mathbb{E}_\xi [\max\{0, \xi - x_1 - x_2\}] \\ &= 2 \sum_{\xi \in \Xi} P[\xi = \xi] \max\{0, \xi - x_1 - x_2\} \\ &= 2 \sum_{n=0}^{+\infty} 2^{-n-1} \max\{0, 2^n - x_1 - x_2\} \\ &= \sum_{n=0}^{+\infty} 2^{-n} \max\{0, 2^n - x_1 - x_2\} = +\infty \end{aligned}$$

$K_2 \neq K_2^s$ (cont'd)

Thus, $K_2^s = \emptyset$.

The problem, set under standard form, can be rewritten as

$$\begin{aligned} \min_{y,u} \quad & 2y_1 + y_2 \\ \text{s.t.} \quad & y_1 + y_2 - u_1 = 1 - x_1, \\ & y_1 - u_2 = \xi - x_1 - x_2, \\ & y_1, y_2, u_1, u_2 \geq 0, \end{aligned}$$

and

$$W = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

$\text{pos } W = ?$

$$K_2 \neq K_2^s \text{ (cont'd)}$$

Complete recourse if $\text{pos } W = \mathbb{R}^2$, i.e.

$$\forall z \in \mathbb{R}^2, \exists y \geq 0, u \geq 0 \text{ s.t. } W \begin{pmatrix} y \\ u \end{pmatrix} = z.$$

We have that

$$W \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} y_1 + y_2 - u_1 \\ y_1 - u_2 \end{pmatrix} = z$$
$$\Leftrightarrow \begin{cases} y_1 + y_2 - u_1 &= z_1, \\ y_1 - u_2 &= z_2. \end{cases}$$

Thus, we can take

$$\begin{cases} y_1 = z_2, u_2 = 0, & \text{if } z_2 \geq 0, \\ y_1 = 0, u_2 = -z_2, & \text{otherwise.} \end{cases}$$

$K_2 \neq K_2^s$ (cont'd)

Then, writing $y_2 - u_1 = z_1 - y_1$, we take

$$\begin{cases} y_2 = z_1 - y_1, & u_1 = 0, & \text{if } z_1 - y_1 \geq 0, \\ y_2 = 0, & u_1 = y_1 - z_1, & \text{otherwise.} \end{cases}$$

Thus,

- $\text{pos}W = \mathbb{R}^2$, and the recourse is complete,
- $K_2 = \mathbb{R}^2$, and the recourse is relatively complete,
- $K_2 \neq K_2^s$.

Note that

$$\mathbb{E}[\xi] = \sum_{n=0}^{+\infty} P[\xi = 2^n] 2^n = \sum_{n=0}^{+\infty} \frac{2^n}{2^{n+1}} = \sum_{n=0}^{+\infty} \frac{1}{2} = +\infty.$$

$K_2 \neq K_2^s$ (cont'd)

$\mathbb{E}[\xi]$ not finite does not necessarily imply that $\mathcal{Q}(x) = +\infty$.
Consider

$$\begin{aligned} \min_y \quad & y_1 - y_2 \\ \text{s.t.} \quad & y_1 \geq \xi \\ & y_2 \leq \xi. \end{aligned}$$

We have that $\forall \xi, Q(x, \xi) = 0$, and therefore $\mathcal{Q}(x) = 0$.

$$K_2 = K_2^s$$

Theorem

If ξ has finite second order moments, then

$$P[\xi \mid Q(x, \xi) < \infty] = 1 \implies Q(x) < \infty,$$

and consequently

$$K_2 = K_2^s.$$

Reminder: almost surely, or with probability one. An event A is said to occur almost surely if $P[A] = 1$.

Elementary feasible set

- Given a realization ξ , the **elementary second-stage feasible set** is defined as:

$$K_2(\xi) = \{x \mid Q(x, \xi) < \infty\}.$$

- Define **possibility interpretation** of the second-stage feasibility as

$$K_2^P = \bigcap_{\xi \in \Xi} K_2(\xi).$$

- Clearly $K_2 = K_2^P$ if ξ has a finite support. Is it still the case when ξ follows a continuous distribution?

Theorem

For a stochastic program with fixed recourse, where ξ has finite second order moments,

$$K_2 = K_2^S = K_2^P.$$

Simple recourse

A particular case of complete recourse is the **simple recourse**, for which we have

$$W = \begin{pmatrix} I & -I \end{pmatrix},$$

with I the identity matrix, of order m .

It is easy to see that $\text{pos } W = \mathbb{R}^m$ as, writing

$$y = \begin{pmatrix} y^+ \\ y^- \end{pmatrix},$$

where $y \in \mathbb{R}^{2m}$, $y^+, y^- \in \mathbb{R}^m$, for $z \in \mathbb{R}^m$, we can solve the system

$$Wy = z, \quad y \geq 0,$$

by choosing $y^+ = \max\{z, 0\}$, $y^- = \max\{-z, 0\}$.

Simple recourse (cont'd)

The second stage program can be read as

$$\begin{aligned} Q(x, \xi) = \min_y & q^+(\xi)^T y^+ + q^-(\xi)^T y^- \\ \text{s.t. } & y^+ - y^- = h(\xi) - T(\xi)x, \\ & y^+, y^- \geq 0. \end{aligned}$$

That is, for $q^+(\xi) + q^-(\xi) \geq 0$, the recourse variables y^+ and y^- can be chosen to measure the absolute violations in the stochastic constraints.

Simple recourse (cont'd)

- Since the recourse is complete, it is feasible for any first-stage decision x .
- The following results establishes conditions to have a finite recourse (i.e. $Q > -\infty$).

Theorem

Assume that the two-stage (linear) stochastic program is feasible and has a simple recourse, and that ξ has finite second-order moments. Then $Q(x)$ is finite if and only if $\mathbf{q}_i^+ + \mathbf{q}_i^- \geq 0$ with probability one.

Simple recourse (cont'd)

Proof.

(\Rightarrow) Assume by contradiction that \mathcal{Q} is finite, but for some component i , $q_i^+(\xi(\omega)) + q_i^-(\xi(\omega)) < 0$ for $\omega \in \Omega_1$ with $P[\Omega_1] > 0$. Then, for any feasible x , for all $\omega \in \Omega_1$ with $h_i(\xi(\omega)) - T_i(\xi(\omega))x > 0$, define

$$y_i^+(\xi(\omega)) = h_i(\xi(\omega)) - T_i(\xi(\omega))x + u, \quad y_i^-(\xi(\omega)) = u.$$

Therefore,

$$y_i^+(\xi(\omega)) - y_i^-(\xi(\omega)) = h_i(\xi(\omega)) - T_i(\xi(\omega))x, \quad y_i^+ \geq 0, \quad y_i^- \geq 0.$$

Moreover, since \mathcal{Q} is finite, $Q(x, \xi(\omega))$ is feasible almost surely, so, almost surely, we can choose y_j^+ and y_j^- feasible, $j \neq i$.



Simple recourse (cont'd)

Proof.

(\Rightarrow)

When $u \rightarrow \infty$, $Q(x, \xi(\omega)) \rightarrow -\infty$ since
 $q_i^+(\xi(\omega))y_i^+ + q_i^-(\xi(\omega))y_i^- \rightarrow -\infty$.

A similar argument can be applied if $h_i(\xi(\omega)) - T_i(\xi(\omega))x \leq 0$,
by taking

$$y_i^+(\xi(\omega)) = u, \quad y_i^-(\xi(\omega)) = -h_i(\xi(\omega)) + T_i(\xi(\omega))x + u.$$

By composing these two cases, we conclude that Q is not finite. □

Simple recourse (cont'd)

Proof.

(\Leftarrow) Assume $\mathbf{q}_i^+ + \mathbf{q}_i^- \geq 0$ with probability one, $\forall i$. Any feasible solution satisfies

$$\mathbf{y}^+(\xi(\omega)) - \mathbf{y}^-(\xi(\omega)) = \mathbf{h}(\xi(\omega)) - \mathbf{T}(\xi(\omega))\mathbf{x}, \mathbf{y}^+(\xi(\omega)) \geq 0, \mathbf{y}^-(\xi(\omega)) \geq 0$$

Therefore for almost every ω , $\mathbf{Q}(\mathbf{x}, \xi(\omega))$ is bounded below by 0, and from the fundamental theorem of linear programming, we can choose as optimal solution

$$\begin{aligned}\mathbf{y}^+(\xi(\omega)) &= (\mathbf{h}(\xi(\omega)) - \mathbf{T}(\xi(\omega))\mathbf{x})^+, \\ \mathbf{y}^-(\xi(\omega)) &= (-\mathbf{h}(\xi(\omega)) + \mathbf{T}(\xi(\omega))\mathbf{x})^+, \end{aligned}$$

where $\mathbf{a}^+ = \max\{0, \mathbf{a}\}$.



Simple recourse (cont'd)

Proof.

(\Leftarrow) Thus,

$$Q(x, \xi(\omega)) = \sum_{i=1}^m (q_i^+(\xi(\omega))(h_i(\xi(\omega)) - T_i(\xi(\omega))x)^+ + q_i^-(\xi(\omega))(-h_i(\xi(\omega)) + T_i(\xi(\omega))x)^+)$$

Consequently $Q(x, \xi(\omega))$ is finite for almost every ω and bounded below by 0.

Therefore, $Q(x)$ is bounded below by 0, and according to the previous results, $Q(x) < \infty$. This implies that $Q(x)$ is finite. \square

Exercise

Consider the second stage program

$$Q(x, \xi) = \min_y \{y \mid \xi y = 1 - x, y \geq 0\}.$$

We assume that ξ follows a triangular distribution on $[0, 1]$, with $P[\xi \leq u] = u^2$.

(a) Is the recourse fixed? Why?

The recourse is not fixed, as $W \equiv \xi$, and therefore, W is random. Moreover, as ξ can take the value 0, the transformation

$$y = 1/\xi - x/\xi,$$

is not properly defined on $\Xi = [0, 1]$; this also means that

$$W = \begin{cases} 0 & \text{si } \xi = 0; \\ 1 & \text{si } \xi \neq 0. \end{cases}$$

Exercise (cont'd)

(b) Express $K_2(\xi)$ for all ξ in $[0, 1]$.

We have to consider two cases: $\xi = 0$ or $\xi \in (0, 1]$.

1. $\xi \in (0, 1]$ In this case, as $y, \xi \geq 0$, $1 - x$ has to be non-negative in order to have a well-defined problem:

$$K_2(\xi) = \{x \mid x \leq 1\}.$$

The value and optimal solutions are

$$Q^*(x, \xi) = (1 - x)/\xi, \quad y^* = (1 - x)/\xi.$$

2. $\xi = 0$ There exists no y such that $0 \cdot y = 1 - x$, except if $x = 1$, so

$$K_2(0) = \{1\}.$$

Exercise (cont'd)

(c) Express K_2 , K_2^P and \mathcal{Q} .

From the previous point, we have

$$K_2^P = \{x \mid x \leq 1\} \cap \{1\} = \{1\}.$$

We also have, as $P[\xi = 0] = 0$,

$$\mathcal{Q}(x) = \int_0^1 \frac{1-x}{\xi} 2\xi d\xi = 2(1-x), \forall x \leq 1.$$

Consequently $K_2^P \subset K_2 = \{x \leq 1\}$.

The difference comes from the fact that a point is not in K_2^P as soon as it is not feasible for a given value of ξ , but K_2 does not consider unfeasible situations that occur with a null probability.

Recourse function

Let y_1^* and y_2^* be two optimal solutions of $v(\xi, z)$, associated to $z = z_1$ and $z = z_2$, respectively. Then, the convex combination $y_\alpha \stackrel{\text{def}}{=} \alpha y_1^* + (1 - \alpha)y_2^*$, $\alpha \in [0, 1]$, is feasible with respect to $z_\alpha = \alpha z_1 + (1 - \alpha)z_2$, as $\alpha y_1^* + (1 - \alpha)y_2^* \geq 0$, and

$$W(\alpha y_1^* + (1 - \alpha)y_2^*) = \alpha W y_1^* + (1 - \alpha)W y_2^* = \alpha z_1 + (1 - \alpha)z_2 = z_\alpha.$$

Moreover,

$$\begin{aligned} v(\xi, z_\alpha) &= q(\xi)^T y_\alpha^* \leq q(\xi)^T (\alpha y_1^* + (1 - \alpha)y_2^*) \\ &= \alpha q(\xi)^T y_1^* + (1 - \alpha)q(\xi)^T y_2^* \\ &= \alpha v(\xi, z_1) + (1 - \alpha)v(\xi, z_2). \end{aligned}$$

In other words, v is a convex function w.r.t. $z \in \mathbb{R}^m$.

Convexity of $Q(x, \xi)$?

$$Q(x, \xi) = v(\xi, h(\xi) - T(\xi)x).$$

$$\begin{aligned} \lambda Q(x_1, \xi) + (1 - \lambda)Q(x_2, \xi) &= \lambda v(\xi, h(\xi) - T(\xi)x_1) + (1 - \lambda)v(\xi, h(\xi) - T(\xi)x_2) \\ &\geq v(\xi, \lambda(h(\xi) - T(\xi)x_1) + (1 - \lambda)(h(\xi) - T(\xi)x_2)) \\ &= v(\xi, h(\xi) - T(\xi)(\lambda x_1 + (1 - \lambda)x_2)) \\ &= Q(\lambda x_1 + (1 - \lambda)x_2, \xi). \end{aligned}$$

Therefore $Q(x, \xi)$ is **convex** w.r.t. x , given ξ . More generally

Theorem

If A is a linear transformation from \mathbb{R}^n to \mathbb{R}^n , and $f(x)$ is a convex function on \mathbb{R}^m , the composite function $(fA)(x) \stackrel{\text{def}}{=} f(Ax)$ is a convex function on \mathbb{R}^n .

Convexity of second-stage function

We have the following result (Birge and Louveaux, Chapter 3, Theorem 5).

Theorem

For a stochastic program with fixed recourse, $Q(x, \xi)$ is

- (a) a piecewise convex linear function in (h, T) ,*
- (b) a piecewise concave linear function in q ,*
- (c) a piecewise convex linear function in x , for all x in $K = K_1 \cap K_2$.*

Convexity of second-stage function (cont'd)

Proof.

In order to show convexity in (a) and (c), it is sufficient to prove that $v(\xi, z) = \min\{q(\xi)^T y \mid Wy = z\}$ is convex, which has already been done. We can proceed similarly to show concavity w.r.t. q .

The piecewise linearity follows from the fact that the number of different optimal bases for a linear program is finite. □

Convexity of the recourse?

$$Q(x) = E_{\xi}[Q(x, \xi)].$$

Suppose for now that ξ has a finite support, i.e.

$\Xi = \{\xi_1, \xi_2, \dots, \xi_m\}$. Then

$$Q(x) = \sum_{i=1}^m P[\xi = \xi_i] Q(x, \xi_i).$$

Convexity of the recourse

Theorem

If $f(x)$ is convex, and $\alpha \geq 0$, $g(x) = \alpha f(x)$ is convex.

Theorem

If $f_k(x)$, $k = 1, 2, \dots, K$, are convex functions, then $g(x) = \sum_{k=1}^K f_k(x)$ is convex.

$Q(x)$ is therefore a convex function w.r.t. x .

What is happening in the continuous case?

We have the following result: if $g(x, y)$ is convex w.r.t. x , then $\int g(x, y) dy$ is convex w.r.t. x . Since

$$Q(x) = \int_{\Xi} Q(x, t) dF(t),$$

$Q(x)$ is convex.

An example...

Consider the second-stage function $Q(x, \xi)$ defined as:

$$\min y^+ + y^- \text{ s.t. } y^+ - y^- = \xi - x, y^+ \geq 0, y^- \geq 0.$$

In other terms:

$$y = \begin{pmatrix} y^+ \\ y^- \end{pmatrix} \quad q = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, W = \begin{pmatrix} 1 & -1 \end{pmatrix} \quad h(\xi) = \xi \quad T(\xi) = 1.$$

Relying on the fundamental theorem of linear programming, we are looking from an optimal basis solution, implying that $y^+ = 0$ or $y^- = 0$.

An example...

We immediately see that

$$y^+ = \begin{cases} \xi - x & \text{if } \xi - x \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$y^- = \begin{cases} -\xi + x & \text{if } \xi - x < 0, \\ 0 & \text{otherwise,} \end{cases}$$

Alternative approach

Dual:

$$\begin{aligned} \max & (\xi - x)\pi \\ \text{s.t. } & \pi \leq 1, \quad -\pi \leq 1 \end{aligned}$$

or

$$\begin{aligned} \max & (\xi - x)\pi \\ \text{s.t. } & \pi + s_1 = 1 \\ & -\pi + s_2 = 1 \\ & s_1 \geq 0, \quad s_2 \geq 0 \end{aligned}$$

Consequently,

$$Q(x, \xi) = \begin{cases} \xi - x & \text{si } x \leq \xi, \\ x - \xi & \text{si } x \geq \xi. \end{cases}$$

An example: optimality conditions

The recourse is simple, and the primal-dual/KKT conditions give

$$\begin{aligned}\begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \pi + \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \\ y^+ - y^- &= \xi - x \\ y^+ &\geq 0, \quad y^- \geq 0 \\ s_1 &\geq 0, \quad s_2 \geq 0 \\ s_1 y^+ &= 0, \quad s_2 y^- = 0.\end{aligned}$$

An example (cont'd)

- The first condition implies that we cannot have $s_1 = s_2 = 0$.
- From the complementarity conditions, we have that $y^+ = 0$ or $y^- = 0$.
- We have to consider two cases:
 - $x \leq \xi$: in this situation, we have

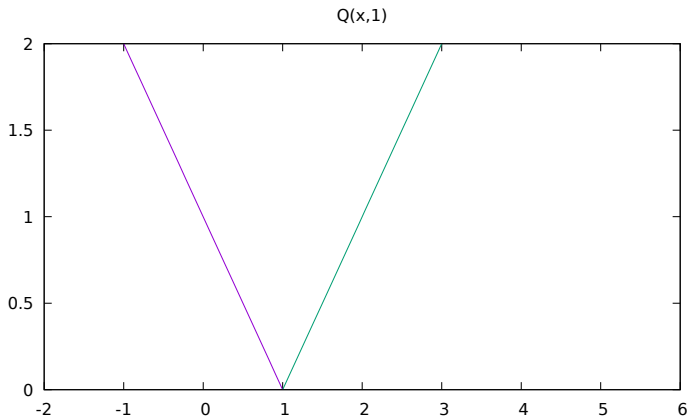
$$y^+ = \xi - x, \quad y^- = 0.$$

- $x \geq \xi$: then,

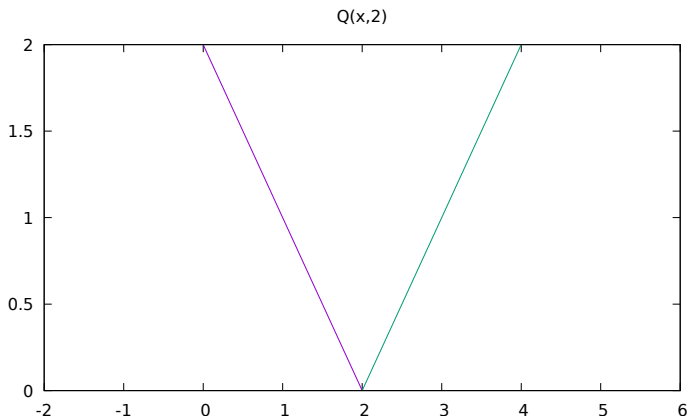
$$y^- = x - \xi, \quad y^+ = 0.$$

Graphically?

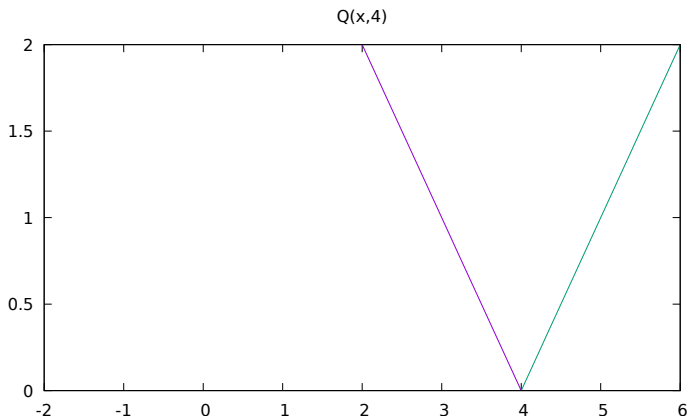
Assume that ξ can take the realizations 1, 2, 4.



Graphically (cont'd)



Graphically (cont'd)



$$Q(x)$$

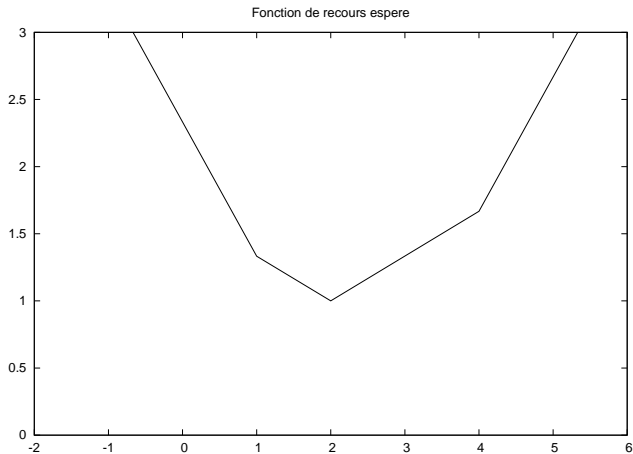
What about $Q(x)$?

Assume that the three realizations have the same probability.

We have to consider 4 cases:

1. $x \leq 1$: $Q(x) = 7/3 - x$;
2. $1 \leq x \leq 2$: $Q(x) = 5/3 - x/3$;
3. $2 \leq x \leq 4$: $Q(x) = x/3 + 1/3$;
4. $4 \leq x$: $Q(x) = x - 7/3$;

Graphically



Properties of $Q(x)$

We note that $Q(x)$ is convex and piecewise linear. As $Q(x)$ is a finite weighted sum of piecewise linear functions when the support of ξ is finite, we have the following result.

Theorem

For a stochastic program with fixed recourses where ξ has finite second-order moments,

- (a) $Q(x)$ is a convex Lipschitz function and is finite over K_2 ;*
- (b) when ξ has a finite support, $Q(x)$ is piecewise linear.*

Reminder: a function f is Lipschitz if there exists some $M < \infty$ such that for all x, y ,

$$|f(x) - f(y)| \leq M\|x - y\|.$$

Differentiability of the recourse

Is $Q(x)$ also differentiable?

The recourse function is partially differentiable with respect to x_j at $(\hat{x}, \hat{\xi})$ if the directional derivative exists for the direction e_j . In other terms, there exists a function $\frac{\partial Q(x, \xi)}{\partial x_j}$ such that

$$\frac{Q(\hat{x} + h e_j, \hat{\xi}) - Q(\hat{x}, \hat{\xi})}{h} = \frac{\partial Q(x, \xi)}{\partial x_j} + \frac{\rho_j(\hat{x}, \hat{\xi}; h)}{h},$$

with

$$\frac{\rho_j(\hat{x}, \hat{\xi}; h)}{h} \rightarrow 0, \text{ as } h \rightarrow 0.$$

We will assume from now that

$\nabla_x Q(x, \xi) = \left(\frac{\partial Q(x, \xi)}{\partial x_1}, \dots, \frac{\partial Q(x, \xi)}{\partial x_n} \right)$ exists.

Differentiability of the recourse (cont'd)

What about the differentiability of $Q(x)$?

$$\begin{aligned}\frac{Q(\hat{x} + he_j) - Q(\hat{x})}{h} &= \int_{\Xi} \frac{Q(\hat{x} + he_j, \xi) - Q(\hat{x}, \xi)}{h} dP \\ &= \int_{\Xi \setminus N_\delta} \frac{\partial Q(\hat{x}, \xi)}{\partial x_j} dP + \int_{\Xi \setminus N_\delta} \frac{\rho_j(\hat{x}, \xi; h)}{h} dP,\end{aligned}$$

where N_δ is measurable and $P[N_\delta] = 0$. Therefore, we have

Theorem

If $Q(x, \xi)$ is partially differentiable almost everywhere, if its partial derivative $\frac{\partial Q(\hat{x}, \xi)}{\partial x_j}$ is integrable and if the residual satisfies

$(1/h) \int_{\Xi \setminus N_\delta} \rho_j(\hat{x}, \xi; h) dP \xrightarrow{h \rightarrow 0} 0$, then $\frac{\partial Q(\hat{x})}{\partial x_j}$ exists and

$$\frac{\partial Q(\hat{x})}{\partial x_j} = \int_{\Xi} \frac{\partial Q(\hat{x}, \xi)}{\partial x_j} dP.$$

Differentiability of the recourse: discrete case

But how to prove $(1/p) \int_{\Xi \setminus N_\delta} \rho_j(\hat{x}, \xi; h) dP \xrightarrow{h \rightarrow 0} 0$?

If we stay in the linear framework with fixed recourse, and vectors ξ with finite second-order moments, we have seen that for ξ with finite support, $Q(x)$ is piecewise linear. Therefore $Q(x)$ is not differentiable.

Differentiability of the recourse: continuous case

If ξ is continuous, $Q(x)$ is obtained as an integral over the $Q(x, \xi)$'s, that are not differentiable as they are piecewise linear given ξ . However, it is x that has to be fixed, not ξ . It is possible to show that (the proof is quite technical)

Theorem

For a stochastic program with fixed recourse where ξ has finite second-order moments, if ξ is continuous, $Q(x)$ is differentiable over K_2 .

Intuitively, the function $Q(x)$ is “smoothed” by the superposition of an infinite number of functions $Q(x, \xi)$.

Two-stage non-linear problems

Now consider the general program

$$\min_{x \in X} E_{\xi}[f_0(x, \xi)] = \min_{x \in X} E_{\xi}[g_0(x, \xi) + Q(x, \xi)].$$

Theorem

If $g_0(\cdot, \xi)$ and $Q(\cdot, \xi)$ are convex with respect to x , $\forall \xi \in \Xi$, and if X is a convex set, the aforementioned program is convex.

Proof.

For $x, y \in X$, $\lambda \in (0, 1)$ and $z = \lambda x + (1 - \lambda)y$, we have

$$\begin{aligned} g_0(z, \xi) + Q(z, \xi) \\ \leq \lambda(g_0(x, \xi) + Q(x, \xi)) + (1 - \lambda)(g_0(y, \xi) + Q(y, \xi)). \end{aligned}$$

The result follows by taking the expectation. □

In a more standard form

Inspired from Birge et Louveaux, Section 3.4.

We consider the problem

$$\begin{aligned} \inf z &= f^1(x) + Q(x), \\ \text{s.t. } g_i^1(x) &\leq 0, \quad i = 1, \dots, \bar{m}_1, \\ g_i^1(x) &= 0, \quad i = \bar{m}_1 + 1, \dots, m_1, \end{aligned}$$

where $Q(x) = E_\omega[Q(x, \omega)]$ and

$$\begin{aligned} Q(x, \omega) &= \inf f^2(y(\omega), \omega), \\ \text{s.t. } t_i^2(x, \omega) + g_i^2(y(\omega), \omega) &\leq 0, \quad i = 1, \dots, \bar{m}_2, \\ t_i^2(x, \omega) + g_i^2(y(\omega), \omega) &= 0, \quad i = \bar{m}_2 + 1, \dots, m_2, \end{aligned}$$

We say that the recourse is additive (why?).

In a more standard form (cont'd)

The functions $f^2(\cdot, \omega)$, $t_i^2(\cdot, \omega)$, and $g_i^2(\cdot, \omega)$ are continuous for any given ω , and measurable w.r.t. ω for any given first argument. This allows to prove that $Q(x, \omega)$ is measurable, and therefore that $\mathcal{Q}(x)$ is well defined.

Reintroduce K_1 , $K_2(\omega)$ and K_2 .

$$K_1 = \{x \mid g_i^1(x) \leq 0, i = 1, \dots, \bar{m}_1, \\ t_i^1(x) = 0, i = \bar{m}_1 + 1, \dots, m_1\},$$

$$K_2(\omega) = \{x \mid \exists y(\omega) \text{ t.q. } t_i^2(x, \omega) + g_i^2(y(\omega), \omega) \leq 0, i = 1, \dots, \bar{m}_2, \\ t_i^2(x, \omega) + g_i^2(y(\omega), \omega) = 0, i = \bar{m}_2 + 1, \dots, m_2\},$$

$$K_2 = \{x \mid \mathcal{Q}(x) < \infty\}.$$

Remarks

The formulation is not yet totally general. We will consider more general forms when we will discuss sampling techniques.

Here, there is no more fixed recourse, but the first-stage decision x acts separately in the constraints. Goal: extend the previous results.

Questions: convexity, differentiability, optimality. We should also consider the concept of lower semi-continuity.