Chapter 1. Open Sets, Closed Sets, and Borel Sets

Section 1.4. Borel Sets

Note. Recall that a set of real numbers is *open* if and only if it is a countable disjoint union of open intervals. Also recall that:

1. a countable union of open sets is open, and
2. a countable intersection of closed sets is closed.

These two properties are the main motivation for studying the following.

Definition. A collection $\mathcal{A}$ of subsets of a set X is an *algebra* (or *Boolean algebra*) of sets if:

1. $A, B \in \mathcal{A}$ implies $A \cup B \in \mathcal{A}$.
2. $A \in \mathcal{A}$ implies $\tilde{A} = X \sim A \in \mathcal{A}$ ($\tilde{A}$ is the complement of A).
3. $A, B \in \mathcal{A}$ implies $A \cap B \in \mathcal{A}$ (this follows from (1) and (2) by DeMorgan's Laws).

We also require that $\varnothing, X \in \mathcal{A}$. (This last condition, absent in previous editions of Royden, insures that an algebra is nonempty.)

Example. $\mathcal{A} = \{\varnothing, \mathbb{N}, \text{evens}, \text{odds}\}$ is an algebra on $\mathbb{N}$.

Note. By induction, (1) and (3) hold for any finite collection of elements of $\mathcal{A}$.

Theorem. Given any collection $\mathcal{C}$ of subsets of X, there exists a smallest algebra $\mathcal{A}$ which contains $\mathcal{C}$. That is, if $\mathcal{B}$ is any algebra containing $\mathcal{C}$, then $\mathcal{B}$ contains $\mathcal{A}$.

Definition. The smallest algebra containing $\mathcal{C}$, a collection of subsets of a set X, is called the *algebra generated by* $\mathcal{C}$.

Definition. An algebra $\mathcal{A}$ of sets is a *σ-algebra* (or a *Borel field*) if every union of a countable collection of sets in $\mathcal{A}$ is again in $\mathcal{A}$.

Example. Let $X = \mathbb{R}$ and $\mathcal{A} = \{A \subset \mathbb{R} \mid A \text{ is finite or } \tilde{A} \text{ is finite}\}$. Then $\mathcal{A}$ is an algebra but not a σ-algebra (since $\mathbb{N} = \cup\{n\}$ but $\mathbb{N} \notin \mathcal{A}$).

Proposition 1.13. Let $\mathcal{C}$ be a collection of subsets of a set X. Then the intersection $\mathcal{A}$ of all σ-algebras of subsets of X that contain $\mathcal{C}$ is a σ-algebra that contains $\mathcal{C}$. Moreover, it is the smallest σ-algebra of subsets of X that contain $\mathcal{C}$ in the sense that if $\mathcal{B}$ is a σ-algebra containing $\mathcal{C}$, then $\mathcal{A} \subset \mathcal{B}$.

Definition. The σ-algebra of Proposition 1.13 is the σ-algebra *generated* by $\mathcal{C}$.

Recall. A countable union of closed sets of real numbers need not be closed:

$$\bigcup_{n=1}^{\infty}\left[0+\frac{1}{n},2-\frac{1}{n}\right]=(0,2).$$

In fact, a countable union of closed sets may be neither open nor closed: $\bigcup_{i=1}^{\infty}\{r_i\}=\mathbb{Q}$ where the rationals are enumerated as $\mathbb{Q}=\{r_i \mid i\in\mathbb{N}\}$. We are interested in describing (or at least naming) the sets we get from countable unions, intersections, and complements of open sets. More specifically, we are interested in the "Borel sets."

Definition. The collection $\mathcal{B}$ of *Borel sets* is the smallest σ-algebra that contains all open sets of real numbers.

Note. How many Borel sets are there: $|\mathcal{B}|=$? According to Corollary 4.5.3 of Inder Rana's *An Introduction to Measure and Integration* (2nd Edition, AMS Graduate Studies in Mathematics, Volume 45, 2002), $|\mathcal{B}|=c=|\mathbb{R}|$ ($=\aleph_1$ if you buy the Continuum Hypothesis). This is bad (why?).

Note. What do Borel sets "look like"? We can describe some of them.

Definition. A set which is a countable union of closed sets is an F_σ *set*. A set which is a countable intersection of open sets is a G_δ *set*.

Note. According to Wikipedia (hmm...), "F" is for *ferme* (French for "closed") and σ for *somme* (French for "sum" or "union"). "G" is for *gebiet* (German for "neighborhood") and δ for *durchschnitt* (German for "intersection").

Note. A countable set is F_σ since it is a countable union of the singletons which compose it. Of course closed sets are F_σ. Since a countable collection of countable sets is countable, a countable union of F_σ sets is again F_σ. Every open interval is F_σ:

$$(a,b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$$

(a and b could be $\pm\infty$), and hence every open set is F_σ (this is Problem 1.37).

Notice. The complement of an F_σ set is a G_δ set (and conversely).

Theorem. Young's Theorem. (Problem 1.56)
Let f be a real valued function defined on all of $\mathbb{R}$. The set of points at which f is continuous is a G_δ set.

Note. The converse of Young's Theorem also holds:
Theorem. (From *Real Functions* by Hahn, and *Counterexamples in Analysis* by Gelbrum and Olmstead.) If $A \subset \mathbb{R}$ is a G_δ set, then there exists $f : \mathbb{R} \to \mathbb{R}$ such that f is continuous at each point of A and discontinuous at each point of $\mathbb{R} \setminus A$.

Note. With δ for intersection and σ for union, we can construct (for example) a countable intersection of F_σ sets, denoted as an $F_{\sigma\delta}$ set. Similarly, we can discuss $F_{\sigma\delta\sigma}$ sets or $G_{\delta\sigma}$ and $G_{\delta\sigma\delta}$ sets. These classes of sets are subsets of the collection of Borel sets, but not every Borel set belongs to one of these classes.

Theorem. (Problem 1.57.)
Let $\{f_n\}$ be a sequence of continuous functions defined on $\mathbb{R}$. Then the set of points x at which the sequence $\{f_n(x)\}$ converges to a real number is the intersection of a countable collection of F_σ sets (i.e., is an $F_{\sigma\delta}$ set).

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