Global GL_2 Hecke-Baxter operator

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July 16, 2025

Abstract. We construct a global Hecke-Baxter operator for integrable systems of arithmetic type associated with the group GL_2. It is an element of a global Hecke algebra associated with the double coset space $GL_2(\mathbb{Z})\backslash GL_2(\mathbb{R})/O_2$. Eigenvalues of the global Hecke-Baxter operator acting on the GL_2-Eisenstein series are given by the corresponding global L-factors. This construction generalizes our previous construction of the Hecke-Baxter operators over local completions $\mathbb{R}$ and $\mathbb{Q}_p$ of the number field $\mathbb{Q}$. Presumably, zeroes of the corresponding global L-factors should be subjected to an arithmetic version of the Bethe ansatz equations.

1 Introduction

An interpretation of a wide class of integrable systems in terms of representation theory provides important insights both into theory of integrable systems and into representations theory allowing transferring various techniques from one area of research into the other. Among various examples we would like to mention the formalism of the Baxter operator [**?**] which was properly placed in representation theory perspective using Hecke algebras formalism in [**?**] (see also [**?**]). We coin the term the Hecke-Baxter operator for a one-parameter family of element of the Hecke algebra reproducing Baxter operator of various quantum integrable systems. Starting with the case of spherical principles series representations of $GL_{\ell+1}(\mathbb{R})$ the construction of the Hecke-Baxter operator is recently extended to general principle series representations of $GL_{\ell+1}(\mathbb{R})$ (for details see [**?**] and reference therein). A remarkable fact is that the Hecke-Baxter operators are directly related to the Archimedean L-factors attached to the corresponding representations of $GL_{\ell+1}(\mathbb{R})$. Precisely the Archimedean L-factors appear as eigenvalues of the Hecke-Baxter operators acting on the spherical and Whittaker functions given by specific matrix elements in the spherical principle series representations. As a direct consequence the local L-factors enter the integral representations of the Whittaker functions expressed via a version of the Gelfand-Tsetlin construction of irreducible representations of $GL_{\ell+1}(\mathbb{R})$ [**?**]. Note that the $GL_{\ell+1}(\mathbb{R})$-Whittaker functions are eigenfunctions of the quantum $GL_{\ell+1}(\mathbb{R})$-Toda chains, one of the most well-studied finite-dimensional integrable systems. Actually the Hecke-Baxter operator (depending on auxiliary parameter) provides an alternative formulation of the quantum Toda chain. Notice in this regard that an advantage of the Hecke algebra formulation of integrable systems is in a unified treatment of both continues and discrete symmetries of the systems.

Not surprisingly, proper counterpart of the Hecke-Baxter operator exists in the case of representations theory over non-Archimedean fields. The case of spherical principle series representations of $GL_{\ell+1}(\mathbb{Q}_p)$ was considered in [**?**] together with the corresponding integrable systems governing spherical and Whittaker functions over $\mathbb{Q}_p$. Connection with local L-factors still holds in this case, and local non-Archimedean L-factors show up as eigenvalues of the non-Archimedean Hecke-Baxter operators acting on $GL_{\ell+1}(\mathbb{Q}_p)$-Whittaker functions.

It is natural to expect that the Hecke-Baxter operator formalism may be further generalized to the case of (global) number fields. This is indeed so, and in this short note we consider one representative example of the global Hecke-Baxter operator: we introduce a GL_2 Hecke-Baxter operator over compactification $\overline{\mathrm{Spec}(\mathbb{Z})}$ of $\mathrm{Spec}(\mathbb{Z})$ acting on the non-ramified GL_2-automorphic functions (i.e. functions on the double coset $GL_2(\mathbb{Z})\backslash GL_2(\mathbb{R})/O_2$). The following result is proven in Theorem **??** in Section 4. The automorphic functions represented by matrix elements of the spherical principle series $GL_2(\mathbb{R})$-representations are eigenfunctions of the proposed global Hecke-Baxter operator with the eigenvalues given by the corresponding global L-functions generalizing the completed form of the Riemann zeta function $\zeta(s)$:

$$\hat{\zeta}(s) \,=\, \zeta(s)\,\pi^{-\frac{s}{2}}\,\Gamma\Big(\frac{s}{2}\Big)\,. \tag{1.1}$$

This result complements the constructions of [**?**].

Let us stress that in terms of quantum integrable systems we basically consider a hyperbolic billiard on upper half-plane modulo action of the modular group $PSL_2(\mathbb{Z})$. These quantum systems are integrable and are deeply connected with the quantum $SL_2(\mathbb{R})$-Toda chains: harmonics of the quantum billiard eigenfunctions are given by solutions of the quantum Toda chain for integer coupling constants. On the other hand this quantum billiard is a generalization of the Euclidean billiard arsing in the tropical limit of the $SL_2(\mathbb{R})$-Toda chain [**?**]. This provides an interesting number theoretic perspective on an interpretation of the tropical limit proposed in [**?**].

One curious point worth mentioning is as follows. In the case of the integrable systems with discrete spectrum, the Baxter operator is instrumental in finding the spectrum given by common eigenvalues of quantum Hamiltonians. Precisely the eigenvalues of quantum Hamiltonian are expressed in a simple way through zeroes of the eigenvalues of the Baxter operators (considered as functions of an auxiliary parameter). In turn, zeroes of the eigenvalues of the Baxter operator satisfy a set of equations called the Baxter equations. Our interpretation of the global L-functions as eigenvalues of the Hecke-Baxter operators points to the problem of finding an analog of the Baxter equations in the arithmetic setup. This might extend a traditional optics for looking at analogs of Riemann hypotheses for global L-functions as well as various conjectures on the special values of global L-functions. This suggestion seems close to the Faddeev-Pavlov approach [**?**] to studying analytic properties of the Riemann zeta-functions via scattering theory. This line of research seems still worth to pursue.

Let us also note that the Hecke-Baxter operators (and more general elements of Hecke algebras) are examples of averaging operators that are ubiquitous in various areas of Mathematics and Physics. One interesting example of the averaging operator appears in the Kadanov approach to the renormalization (semi)group in lattice quantum field theories (see e.g. [?]). Fixed points of the renormalization group flow corresponding to the eigenvalues of the Kadanov operators describe continuum limit of the lattice theory. The analogy between the constructions of [?] and this paper is very fruitful and will be considered in details elsewhere. As an obvious next step we are going to generalize the results of this note to the global ramified case for the groups $GL_{\ell+1}$ of higher ranks.

Acknowledgements: The research of S.V.O. is partially supported by the Beijing Natural Science Foundation grant IS24004.

2 Automorphic forms and global Hecke-Baxter operator: GL_1

In this Section we consider the almost trivial case of the Lie group GL_1. Our goal is to introduce basic elements of the construction to proceed in the following Sections with a more involved case of GL_2.

Let us define the non-ramified GL_1 automorphic functions as functions on $GL_1(\mathbb{R})$ invariant under the left action of $GL_1(\mathbb{Z})$ and right action of the orthogonal subgroup $O_1 \subset GL_1(\mathbb{R})$ i.e. these functions may be considered as functions on the double coset space

$$\mathcal{M}_1 = GL_1(\mathbb{Z})\backslash GL_1(\mathbb{R})/O_1 \,. \tag{2.1}$$

Note that $\mathcal{M}_1$ is a $GL_1(\mathbb{Z})$-orbifold as $GL_1(\mathbb{Z})$ acts trivially on the coset space $GL_1(\mathbb{R})/O_1$. We however are interested in the space of functions on $\mathcal{M}_1$ and thus might ignore this subtlety by considering functions on $GL_1(\mathbb{R})/O_1$ (invariant under trivial action of $GL_1(\mathbb{Z})$) as functions on $\mathcal{M}_1$. Thus taking into account the identifications

$$GL_1(\mathbb{R}) \simeq \mathbb{R}^*, \qquad GL_1(\mathbb{Z}) \simeq \mu_2, \qquad O_1 \simeq \mu_2\,, \qquad \mu_2 = \{\pm 1\}\,. \tag{2.2}$$

the GL_1-automorphic functions may be identified with functions on $\mathbb{R}_+ = \mathbb{R}^*/\mu_2$.

The double cosets space (??) allows an interpretation as a moduli space of circles S^1 supplied with S^1-invariant metrics. Indeed $\mathcal{M}_1$ may be presented in the following factorized form

$$\mathcal{M}_1 = GL_1(\mathbb{Z})\backslash GL_1(\mathbb{R}) \times_{GL_1(\mathbb{R})} GL_1(\mathbb{R})/O_1 \,. \tag{2.3}$$

The first factor $GL_1(\mathbb{Z})\backslash GL_1(\mathbb{R})$ should be identified with the space of lattices L in $\mathbb{R}$

$$L = \{nv | n \in \mathbb{Z}\}, \qquad v \in \mathbb{R} - \{0\}\,, \tag{2.4}$$

characterized by non-zero real numbers v with group $GL_1(\mathbb{Z}) = \mu_2$ of automorphisms of L acting by effectively changing the sign of v. By action of $GL_1(\mathbb{R}) \simeq \mathbb{R}^*$ any lattice may transformed into the standard one $L = \mathbb{Z} \subset \mathbb{R}$. The second factor $GL_1(\mathbb{R})/O_1$ in (**??**) is identified with the space of constant metrics on $\mathbb{R}$ with O_1 being stabilizer of a reference metric. From this description we infer that the space (**??**) is the moduli space of $GL_1(\mathbb{R})$-equivalence classes of pairs of lattices and constant metrics on $\mathbb{R}$ or equivalently as the moduli space of circles $S^1 = \mathbb{R}/\mathbb{Z}$ supplied with constant metrics. Algebraically $\mathcal{M}_1$ may be understood as a moduli space of rank one $\mathbb{Z}$-modules L supplied with a metric on its real extension $L \otimes_{\mathbb{Z}} \mathbb{R}$. The interpretation of $\mathcal{M}_1$ as moduli space metricized circles provides us with a canonical coordinate on $\mathcal{M}_1$, the volume of the corresponding circle

$$|x| = \mathrm{Vol}_h(\mathbb{R}/L)\,. \tag{2.5}$$

Here x is the canonical coordinate on $GL_1(\mathbb{R}) = \mathbb{R}^*$ identified with the moduli space of the oriented circles S^1_{or} supplied with a constant metric h. Given a pair (S^1_{or}, h) we might consider corresponding volume one-form ω so that the coordinate x would be a period of this form

$$x = \int_{\mathbb{R}/L} \omega\,. \tag{2.6}$$

We are interested in a particular bases in the space of GL_1-automorphic functions given by GL_1 analog of the Eisenstein functions. This bases may be defined precisely in various ways but having in mind subsequent generalizations to the case of GL_2 we construct these functions using representation theory approach. Precisely we define GL_1 Eisenstein functions as matrix elements of $GL_1(\mathbb{Z})$ and O_1 invariant vectors in unitary spherical principle series representations $GL_1(\mathbb{R})$. Let $(\pi_\gamma, \mathcal{V}_\gamma)$, $\gamma \in \mathbb{R}$ be a one-dimensional unitary spherical unitary representation of $GL_1(\mathbb{R})$, $\langle\,,\,\rangle$ be the corresponding Hermitian pairing and $v \in \mathcal{V}_\gamma$ be such that $\langle v, v\rangle = 1$. By definition spherical representations of $GL_1(\mathbb{R}) \simeq \mathbb{R}^*$ are factored through homomorphysm $\mathbb{R}^* \to \mathbb{R}_+$ and are given by

$$\pi_\gamma :\; x \longrightarrow |x|^{\imath\gamma}\,, \qquad x \in \mathbb{R}^*\,. \tag{2.7}$$

Consider the following matrix elements in representation $(\pi_\gamma, \mathcal{V}_\gamma)$

$$\psi_\gamma(x) = \langle v, \pi_\gamma(x)\, v\rangle = |x|^{\imath\gamma}\,, \qquad x \in GL_1(\mathbb{R}) \simeq \mathbb{R}^*\,. \tag{2.8}$$

It is a μ_2-invariant function on $GL_1(\mathbb{R})$ and thus $\psi_\gamma(x)$ is a lift of a function on $\mathcal{M}_1 = \mathbb{R}_+$. Corresponding function on $\mathcal{M}_1$ will be called the $GL_1(\mathbb{R})$-Eisenstein series associated with the representation $(\pi_\gamma, \mathcal{V}_\gamma)$. In the following we will consider interchangeably automorphic eigenfunctions as functions on $\mathcal{M}_1$ depending on $|x|$ or as μ_2-invariant functions on $GL_1(\mathbb{R})$ depending on x.

The GL_1-Eisenstein functions may be defined also as eigenfunctions of some operators. In the following we will be interested in characterization of the GL_1-Eisenstein functions as common eigenfunctions of elements of the Hecke algebra associated with the space of double cosets (this formulation especially useful as takes into account the invariance under both discrete and continues groups). Define the Hecke algebra associated with the double coset space (**??**) as a tensor product of two convolution algebras $\mathcal{H}(GL_1(\mathbb{Q}), GL_1(\mathbb{Z}))$ and $\mathcal{H}(GL_1(\mathbb{R}), O_1)$. Recall that Hecke algebra $\mathcal{H}(G, K)$ associated with the pair $K \subset G$ is an associative algebra of the proper subset of K-biinvariant functions on G under convolution. It is natural to consider the maximal subset of functions on G such that the convolution operation is defined. In the case when (G, K) is a Gelfand pair i.e. K is a fixed point of an involution of G the corresponding algebra is commutative. The power of the Hecke algebra formalism is in the fact that $\mathcal{H}(G, K)$ in general is not a group algebra but replaces it in various representation theory constructions.

The Hecke algebra $\mathcal{H}(GL_1(\mathbb{R}), O_1)$, the algebra of O_1-biinvraint functions on $GL_1(\mathbb{R})$, acts naturally on the functions on $GL_1(\mathbb{R})/O_1$ and in particular on the functions on double coset $\mathcal{M}_1$ via convolution. Note that it does not take into account $\mathbb{Z}$ (and thus $\mathbb{Q}$) structure responsible for the lattice moduli space interpretation of $\mathcal{M}_1$. To take into account this arithmetic structure we consider another Hecke algebra $\mathcal{H}(GL_1(\mathbb{Q}), GL_1(\mathbb{Z}))$ which we identify with the convolution algebra of $GL_1(\mathbb{Z})$-biinvariant generalized function on $GL_1(\mathbb{R})$ with the support at $GL_1(\mathbb{Q}) \subset GL_1(\mathbb{R})$. It is easy to verify that the algebras $\mathcal{H}(GL_1(\mathbb{Q}), GL_1(\mathbb{Z}))$ and $\mathcal{H}(GL_1(\mathbb{R}), O_1)$ are (mutually) commutative associative algebras acting on functions on $\mathcal{M}_1$ from the right and the left correspondingly. Note that in the considered case of GL_1 the Hecke algebras are actually groups algebras of the quotient groups $GL_1(\mathbb{Q})/GL_1(\mathbb{Z})$ and $GL_1(\mathbb{R})/O_1$.

It is instructive to describe the $\mathcal{H}(GL_1(\mathbb{Q}), GL_1(\mathbb{Z}))$-action considering $\mathcal{M}_1$ as a moduli space of metricized circles. Given a lattice $L \subset \mathbb{R}$, we can consider a sublattice $L_p \subset L$ of index $[L : L_p] = p$, and also we consider $L_{1/q} \subset \mathbb{R}$, such that $L \subset L_{1/q}$ with $[L_{1/q} : L] = q$. Now we define, for any $p/q \in \mathbb{Q}_+^*$, an operation $T_{p/q}$ on lattices by first taking $L \subset L_{1/q}$ such that $[L_{1/q} : L] = q$, and then considering a sublattice $L_{p/q} \subset L_{1/q}$ of index $[L_{1/q} : L_{p/q}] = p$. Thus we have the following operator acting on functions on the space of lattices:

$$(T_{p/q} \cdot f)(L) = \sum_{L \subset L_{1/q} \supset L_{p/q}} f(L_{p/q}), \qquad [L_{1/q} : L] = q, \quad [L_{1/q} : L_{p/q}] = p\,. \tag{2.9}$$

In terms of the functions of the coordinate $|x| \in \mathbb{R}_+$ this reduces to a simple mupltiplication operation

$$(T_{p/q} \cdot f)(|x|) = f(p/q \cdot |x|)\,. \tag{2.10}$$

These operators obviously belongs to the algebra $\mathcal{H}(GL_1(\mathbb{Q}), GL_1(\mathbb{Z})$ and satisfy the following relations:

$$T_{p_1/q_1} \circ T_{p_2/q_2} = T_{(p_1p_2)/(q_1q_2)}\,. \tag{2.11}$$

Let us remark that the collection of operators $T_{p/q}$, $p/q \in \mathbb{Q}_+^*$ provides a GL_1-analog of the modular tower structure arising in the case of GL_2.

To construct a meaningful generating function we consider a multiplicative semigroup $\mathbb{Z}_+ \subset \mathbb{Q}_+^*$. Corresponding elements of the Hecke algebra $\mathcal{H}(GL_1(\mathbb{Q}), GL_1(\mathbb{Z})$ act as follows

$$(T_n \cdot f)(L) = f(nL)\,, \qquad n \in \mathbb{Z}_+\,, \tag{2.12}$$

or equivalently, in terms of functions on $GL_1(\mathbb{R})$

$$(T_n \cdot f)(x) = f(nx)\,, \qquad n \in \mathbb{Z}_+\,, \qquad x \in \mathbb{R}^*\,. \tag{2.13}$$

These operators may be conveniently combined into the generating series

$$Q_s^{GL_1(\mathbb{Z})} = \sum_{n=1}^{\infty} \frac{1}{n^s} T_n \,. \tag{2.14}$$

Its action on the functions of x is given by

$$(Q_s^{GL_1(\mathbb{Z})} \cdot f)(x) = \sum_{n=1}^{\infty} n^{-s} \, f(nx) \,. \tag{2.15}$$

Now we consider a kind of generating functions for the elements of the Hecke algebra $\mathcal{H}(GL_1(\mathbb{R}), O_1)$ providing a proper analog for the generating function (**??**). Such generating functions were introduced (for more general case of $GL_{\ell+1}(\mathbb{R})$) in [**?**] under the name the Hecke-Baxter operator. Precisely the $GL_1(\mathbb{R})$ Hecke-Baxter operator is the integral operator,

$$(Q_s^{GL_1(\mathbb{R})} \cdot f)(x) = \int_{\mathbb{R}^*} d\mu_{\mathbb{R}^*}(y) \; |y|^s \, f(y^{-1}x) \,, \quad d\mu_{\mathbb{R}^*}(y) = e^{-\pi y^2} \, \frac{dy}{y} \,, \tag{2.16}$$

acting by convolution with O_1-biinvariant function on $GL_1(\mathbb{R})$

$$Q_s^{GL_1(\mathbb{R})}(y) = |y|^s \, e^{-\pi y^2} \,. \tag{2.17}$$

In the following, for brevity, we identify suitable functions on Lie groups, operators obtained by the actions of these functions via convolution and the corresponding integral kernels.

Proposition 2.1 *The matrix elements* (**??**)

$$\psi_\gamma(x) = \langle v, \pi_\gamma(x) \, v \rangle = |x|^{\imath\gamma} \,, \tag{2.18}$$

are common eigen-functions of the operators $Q_s^{GL_1(\mathbb{Z})}$ *and* $Q_s^{GL_1(\mathbb{R})}$*:*

$$\big(Q_s^{GL_1(\mathbb{Z})} \cdot \psi_\gamma\big)(x) = \zeta(s - \imath\gamma) \, \psi_\gamma(x) \,, \tag{2.19}$$

$$\big(Q_s^{GL_1(\mathbb{R})} \cdot \psi_\gamma\big(x\big) = L^{\mathbb{R}}(s - \imath\gamma) \, \psi_\gamma(x) \,. \tag{2.20}$$

The eigenvalues are given by

$$\zeta(s) = \sum_{n \in \mathbb{Z}_+} \frac{1}{n^s} \,, \qquad L^{\mathbb{R}}(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \,, \tag{2.21}$$

where we impose $\mathrm{Re}(s) > 1$ *for convergence.*

Proof. Using

$$(T_n \cdot \psi_\gamma)(x) = \psi_\gamma(nx) = n^{\imath\gamma} \, \psi_\gamma(x) \,, \tag{2.22}$$

we indeed have

$$\big(Q_s^{GL_1(\mathbb{Z})}\cdot\psi_\gamma\big)(x)\;=\;\left(\sum_{n\in\mathbb{Z}_+}\frac{1}{n^{s-\imath\gamma}}\right)\;\psi_\gamma(x)\;=\;\zeta(s-\imath\gamma)\,\psi_\gamma(x)\,. \tag{2.23}$$

The analogous statement for the Archimedean Hecke-Baxter operator $Q_s^{GL_1(\mathbb{R})}$ acting on matrix element (**??**) via

$$\big(Q_s^{GL_1(\mathbb{R})}\cdot\psi_\gamma\big)(x)\;=\int\limits_{\mathbb{R}^*}\frac{dy}{y}\;|y|^s\,e^{-\pi y^2}\,\psi_\gamma(y^{-1}x)\;=\;\pi^{-\frac{s-\imath\gamma}{2}}\,\Gamma\left(\frac{s-\imath\gamma}{2}\right)\;\psi_\gamma(x)\,, \tag{2.24}$$

reduces basically to the integral representation of the Gamma-function. □

Now we introduce main object of our considerations in this Section, global Hecke-Baxter operator $\widehat{Q}_s$.

Definition 2.1 *The GL_1 global Hecke-Baxter operator is the operator acting in the space of functions on $GL_1(\mathbb{R})\simeq\mathbb{R}^*$ via convolution with the function*

$$\widehat{Q}_s^{GL_1}(x)\;=\;\frac{1}{2}\,|x|^s\Big(\Theta(0|\imath x^2)-1\Big)\,,\qquad x\in\mathbb{R}^*\,, \tag{2.25}$$

where the theta-constant is given by

$$\Theta(0|\tau)=\sum_{n\in\mathbb{Z}}e^{\imath\pi\tau n^2}\,. \tag{2.26}$$

Proposition 2.2 *Consider the matrix element* (**??**)

$$\psi_\gamma(x)=\langle v,\pi_\gamma(x)\,v\rangle=|x|^{\imath\gamma}\,, \tag{2.27}$$

in the unitary spherical principle series representation $(\pi_\gamma,\mathcal{V}_\gamma)$ of $GL_1(\mathbb{R})$. Define completed zeta-function as follows

$$\hat{\zeta}(s)\;=\;\zeta(s)\,\pi^{-\frac{s}{2}}\,\Gamma\Big(\frac{s}{2}\Big)\,. \tag{2.28}$$

Then the global Hecke-Baxter operator (**??**) *acts on* (**??**) *via multiplication by a shifted completed zeta-function*

$$\big(\widehat{Q}_s^{GL_1}\cdot\psi_\gamma\big)(x)\;=\hat{\zeta}(s-\imath\gamma)\,\psi_\gamma(x)\,,\qquad \mathrm{Re}(s)>1\,. \tag{2.29}$$

Proof. We have

$$\begin{aligned}
\big(\widehat{Q}_s^{GL_1}*\psi_\gamma\big)(x)\;&=\int\limits_{\mathbb{R}^*}\frac{dy}{y}\;|y|^s\;\frac{1}{2}\Big(\Theta(0|\imath y^2)-1\Big)\,\psi_\gamma(y^{-1}x)\\
&=\sum_{n\in\mathbb{Z}_+}\int\limits_{\mathbb{R}^*}\frac{dy}{y}\,|y|^s\,e^{-\pi|ny|^2}\,|y|^{-\imath\gamma}\,\psi_\gamma(x)\\
&=\;\pi^{-\frac{s-\imath\gamma}{2}}\,\Gamma\Big(\frac{s-\imath\gamma}{2}\Big)\;\left(\sum_{n\in\mathbb{Z}_+}\frac{1}{n^{s-\imath\gamma}}\right)\;\psi_\gamma(x)\\
&=\;\pi^{-\frac{s-\imath\gamma}{2}}\,\Gamma\Big(\frac{s-\imath\gamma}{2}\Big)\,\zeta(s-\imath\gamma)\;\psi_\gamma(x)\,,
\end{aligned} \tag{2.30}$$

thus arriving at the required identity. □

Let us notice that in the simple case of the trivial representation $(\pi_{\gamma=0}, \mathcal{V}_{\gamma=0})$ the identity (**??**) reduces to the standard integral expression for the completed zeta-function:

$$\hat{\zeta}(s) \,=\, \sum_{n\in\mathbb{Z}_+} \int\limits_{\mathbb{R}_+} \frac{dt}{t}\, t^{\frac{s}{2}}\, e^{-\pi t n^2}\,. \tag{2.31}$$

The fundamental property of the completed Riemann zeta-function (**??**) is the functional relation

$$\hat{\zeta}(1-s) = \hat{\zeta}(s)\,. \tag{2.32}$$

The matrix elements (**??**) also respect an analogous reflection symmetry

$$\psi_{-\gamma}(x^\tau) = \psi_\gamma(x)\,, \tag{2.33}$$

where $x^\tau := x^{-1}$ is the involution on the group $GL_1(\mathbb{R})$. Taking into account that matrix elements (**??**) are eigenfunctions of the Hecke-Baxter operator $\hat{Q}_s^{GL_1}$ with the eigenvalues expressed through completed zeta-function one expects that the kernel of the Hecke-Baxter integral operator should also satisfy a form of functional equation. Indeed we have the following relation

$$\widehat{Q}_{1-s}^{GL_1}(x^\tau) + \frac{1}{2}|x^\tau|^{1-s} = \widehat{Q}_s^{GL_1}(x) + \frac{1}{2}|x|^s\,, \tag{2.34}$$

where the terms $|x|^s$ compensate the correction terms entering the expression (**??**) of the kernel via theta-constant. The functional relation (**??**) is a direct consequence of the modular properties of the theta-constant verified using the Poisson summation formula. Thus we have a deep connection between properties of the global Hecke-Baxter operator and analytic properties of the completed Riemann zeta-function.

It is possible to interpolate between the global and Archimedean Hecke-Baxter operators via considering a GL_1-analog of the congruence (semi)groups. Let us introduce the following semigroup $\mathbb{Z}_+^{(N)} \subset \mathbb{Q}_+^*$:

$$\mathbb{Z}_+^{(N)} \,=\, \{\eta \in \mathbb{Q}_+^* | \eta = 1 + Nm,\, m \in \mathbb{Z}_+\}\,. \tag{2.35}$$

Then the generating function of the elements of the Hecke algebra $\mathcal{H}(GL_1(\mathbb{Q}), GL_1(\mathbb{Z}))$ reads

$$Q_{s,N}^{GL_1(\mathbb{Z})} = \sum_{n=1}^{\infty} \frac{1}{(1+nN)^s}\, T_{1+nN}\,. \tag{2.36}$$

Therefore the modified kernel of the global Hecke-Baxter operator is given by

$$\widehat{Q}_{s,N}^{GL_1}(x) \,=\, \frac{1}{2}\,|x|^s\; \Theta^{(N)}(0|\imath x^2)\,, \qquad x \in \mathbb{R}^*\,, \qquad N > 1\,. \tag{2.37}$$

Here

$$\Theta^{(N)}(0|\tau) = \sum_{n\in\mathbb{Z}} e^{\imath\pi\tau(1+Nn)^2} = e^{\imath\pi\tau} \sum_{n\in\mathbb{Z}} e^{\imath\pi\tau N^2 n^2 + 2\pi\imath N n\tau} = e^{\imath\pi\tau}\, \Theta_{N^2}\Big(\frac{\tau}{N}\Big|\tau\Big)\,, \tag{2.38}$$

where the level k theta function is defined by:

$$\Theta_k(z|\tau) = \sum_{n\in\mathbb{Z}} e^{\imath\pi k\tau n^2+2\pi\imath knz} . \tag{2.39}$$

Now it is easy to check that taking the limit $N \to +\infty$ the kernel (**??**) of the modified global Hecke-Baxter operator turns into the kernel (**??**) of the Archimedean Hecke-Baxter operator. This provides a kind of regularization of the Archimedean Hecke-Baxter operator.

3 GL_2-automorphic forms

In this Section we recall a construction of the Eisenstein functions for GL_2 (for a review see e.g. [**?**]). Let us start with the double coset space

$$\mathcal{M}_2 = GL_2(\mathbb{Z})\backslash GL_2(\mathbb{R})/O_2 . \tag{3.1}$$

The space (**??**), similar to (**??**) in the previous Section is an orbifold. We will define the space of functions on (**??**) as functions on $GL_2(\mathbb{R})/O_2$ invariant under left action of $GL_2(\mathbb{Z})$. The double coset space $\mathcal{M}_2$ allows interpretation as a moduli space of two-tori T^2 supplied with T^2-invariant metrics. Indeed it may be identified with the space of pairs of lattices L and a constant metrics h in $\mathbb{R}^2$ modulo simulations action of $GL_2(\mathbb{R})$. Taking into account that any lattice in $\mathbb{R}^2$ may be transformed by linear transformations into the standard one $\mathbb{Z}^2 \subset \mathbb{R}^2$ we arrive at the identification of $\mathcal{M}_2$ with the moduli space of metricizes tori. Constant metric on T^2 defines a conformal metric, and therefore a complex structure, supplying T^2 with a structure of elliptic curve $E(\mathbb{C})$. As a result the space $\mathcal{M}_2$ is naturally fibred over the moduli space $\mathcal{M}_2^c$ of elliptic curves. The fiber of the projection $\mathcal{M}_2 \to \mathcal{M}_2^c$ may be identified with $\mathbb{R}_+$ supplied with the natural coordinate, the volume of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ in the considered metric. In turn, the space $\mathcal{M}_2^c$ of complex structures has the double coset description as the upper complex half-plane

$$\mathcal{H}_+ = PSL_2(\mathbb{R})/SO(2) , \tag{3.2}$$

modulo action of the discrete group $PSL_2(\mathbb{Z})$. In the standard linear coordinates on $\mathcal{H}_+ = \{\tau \in \mathbb{C}|\mathrm{Im}(\tau) > 0\}$ the isomorphism (**??**) may be described as SO_2-projection of the following lift

$$\tau = (\tau_1 + \imath\tau_2) \in \mathcal{H}_+ \longrightarrow \frac{1}{\sqrt{\tau_2}} \begin{pmatrix} \tau_2 & \tau_1 \\ 0 & 1 \end{pmatrix} \in PSL_2(\mathbb{R}) , \tag{3.3}$$

while the left action of $PSL_2(\mathbb{Z})$ on $\mathcal{H}_+$ is realized by the fractional linear transformations.

Now we introduce a special kind of GL_2-automorphic functions, the Eisenstein functions. The GL_2-Eisenstein functions are associated with spherical principle series representations entering the decomposition of the $GL_2(\mathbb{R})$-representation $L^2(GL_2(\mathbb{Z})\backslash GL_2(\mathbb{R}))$ acting from the right:

$$\big(\pi(g) \cdot f\big)(\tilde{g}) = f(\tilde{g} \cdot g), \qquad f \in L^2(GL_2(\mathbb{Z})\backslash GL_2(\mathbb{R})) . \tag{3.4}$$

The irreducible components corresponding to spherical principle series representations are in one to one correspondence with the elements of $L^2(GL_2(\mathbb{Z})\backslash GL_2(\mathbb{R}))$ invariant under the subgroup $O_2 \subset GL_2(\mathbb{R})$. This correspondence follows by the uniqueness of spherical vectors in spherical principle series representations (see e.g. [**?**]).

Let $(\pi_\gamma, \mathcal{V}_\gamma)$ be a unitary spherical principle series representation of $GL_2(\mathbb{R})$ realized via induction from the Borel subgroup $B \subset GL_2(\mathbb{R})$ (identified with the subgroup of lower triangular matrices) via the spherical character of B

$$\pi_\gamma = \mathrm{Ind}_B^{GL_2(\mathbb{R})} \chi_\gamma^B, \quad \chi_\gamma^B(b) = \prod_{j=1}^{2} |b_{jj}|^{\imath\gamma_j - \rho_j}, \quad \rho = \left(\frac{1}{2}, -\frac{1}{2}\right). \tag{3.5}$$

Therefore the representation space $\mathcal{V}_\gamma$

$$\mathcal{V}_\gamma = \left\{ f \in \mathrm{Fun}(GL_2(\mathbb{R})) \; : \quad f(bg) = \chi_\gamma^B(b)\, f(g), \;\; b \in B \right\}, \tag{3.6}$$

supports the right $GL_2(\mathbb{R})$-action. Using the Bruhat decomposition, the representation $(\pi_\gamma, \mathcal{V}_\gamma)$ may be realized in the space of functions on $B\backslash GL_2(\mathbb{R}) = \mathbb{P}^1(\mathbb{R})$, which in turn can be identified with the compactification the (opposite) unipotent subgroup $N_+ \subset GL_2(\mathbb{R})$:

$$N_+ \;=\; \left\{ n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \Big| \quad x \in \mathbb{R} \right\}. \tag{3.7}$$

Explicitly, the $GL_2(\mathbb{R})$ action in $\mathcal{V}_\gamma \subset L^2(B\backslash GL_2(\mathbb{R}))$, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, is given by

$$[\pi_\gamma(g) \cdot f](x) \;=\; f(n_x g) \;=\; |\det g|^{\imath\gamma_2 + \frac{1}{2}}\, |a + xc|^{\imath(\gamma_1 - \gamma_2) - 1}\, f\Big(\frac{b + xd}{a + xc}\Big), \tag{3.8}$$

providing the following $GL_2(\mathbb{R})$-action on $\mathbb{R}P^1$:

$$g \cdot x \;=\; \frac{b + xd}{a + xc}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.9}$$

The Hilbert space structure on the space of functions on N_+ is defined via the pairing

$$\langle \phi_1 \, , \phi_2 \rangle \;=\; \int\limits_{N_+} dn_x\, \overline{\phi_1(n_x)}\; \phi_2(n_x)\,. \tag{3.10}$$

We supply the Hilbert space $\mathcal{V}_\gamma$ with a structure of the rigged Hilbert spaces $\mathcal{V}_\gamma^{(t)} \subset \mathcal{V}_\gamma \subset \mathcal{V}_\gamma^{(g)}$ (the Gelfand triple) where $\mathcal{V}_\gamma^{(t)}$ is the subspace of smooth test functions and $\mathcal{V}_\gamma^{(g)}$ is the space of generalized functions. The pairing (**??**) extends to the duality between $\mathcal{V}_\gamma^{(t)}$ and $\mathcal{V}_\gamma^{(g)}$. We would like to represent the GL_2-Eisenstein functions in terms of matrix elements of the spherical principle series representations. As we will see the corresponding matrix elements are not well-defined for the unitary principle series and requires analytic continuation of the representation parameters $\gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2$. In turn this implies a replacement of the structure of Hilbert space $\mathcal{V}_\gamma$ by a pair of dual spaces. Thus we consider $\mathcal{V}_\gamma^{(t)}$ and $\mathcal{V}_\gamma^{(g)}$ to be the dual $GL_2(\mathbb{R})$-modules with representation parameters having a small imaginary part. To construct matrix element representation of the Eisenstein functions we start with the explicit construction $GL_2(\mathbb{Z})$- and O_2-invariant vectors. Let $\phi_{O_2} \in \mathcal{V}_\gamma^{(t)}$ be the unique (up to normalization) spherical vector, i.e. vector invariant under the action of $O_2 \subset GL_2(\mathbb{R})$ and let $\phi_{GL_2(\mathbb{Z})}$ be the unique (up to normalization) $GL_2(\mathbb{Z})$-invariant vector in $\mathcal{V}_\gamma^{(g)}$.

Lemma 3.1 *In the representation* $(\pi_\gamma, \mathcal{V}_\gamma)$ *given by* (**??**) *the* O_2*-invariant vector* $\phi_{O_2} \in \mathcal{V}_\gamma^{(t)}$ *and the* $GL_2(\mathbb{Z})$*-invariant vector* $\phi_{GL_2(\mathbb{Z})} \in \mathcal{V}_\gamma^{(g)}$ *may be chosen in the following form*

$$\phi_{O_2}(x) \;=\; (1+x^2)^{\frac{\imath(\gamma_1-\gamma_2)-1}{2}}\,, \tag{3.11}$$

$$\phi_{GL_2(\mathbb{Z})}(x) = \sum_{(m,n)\in\mathcal{P}} |n|^{-\imath(\gamma_1-\gamma_2)}\,\delta(m+nx)\,, \tag{3.12}$$

where

$$\mathcal{P} = \{(m,n)\in\mathbb{Z}^2 - \{0\}|\ \gcd(m,n)=1,\ (m,n)\sim(-m,-n)\}\,. \tag{3.13}$$

Proof. Elements of $O_2 \subset GL_2(\mathbb{R})$ may be written in the following form

$$k = \begin{pmatrix} \cos\theta & (-1)^\epsilon \sin\theta \\ -\sin\theta & (-1)^\epsilon\,\cos\theta \end{pmatrix} \in O_2\,, \quad 0\leq\theta<2\pi\,, \qquad \epsilon\in\{0,1\}\,. \tag{3.14}$$

By (**??**), a direct calculation gives

$$1+(k\cdot x)^2 \;=\; 1+\Big(\frac{(-1)^\epsilon\sin\theta + x(-1)^\epsilon\cos\theta}{\cos\theta - x\sin\theta}\Big)^2\,. \tag{3.15}$$

Taking into account $|\det k| = 1$ we infer from (**??**) the O_2-invariance of the vector (**??**).

Next, we find a $GL_2(\mathbb{Z})$-invariant vector $\phi_{GL_2(\mathbb{Z})}$ by solving the following equation,

$$[\pi_\gamma(g)\cdot\phi_{GL_2(\mathbb{Z})}](x) \;=\; \phi_{GL_2(\mathbb{Z})}(n_x g) \;=\; \phi_{GL_2(\mathbb{Z})}(n_x)\,, \qquad g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in GL_2(\mathbb{Z})\,.$$

The group $GL_2(\mathbb{Z})$ is generated by

$$T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right), \qquad S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right), \qquad R = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1\end{smallmatrix}\right), \tag{3.16}$$

so by (**??**) each generator is acting in $\mathcal{V}_\gamma \subset L^2(B\backslash GL_2(\mathbb{R}))$ via

$$\begin{gathered}[\pi_\gamma(T)\,f](x) \;=\; f(x+1), \qquad [\pi_\gamma(R)\,f](x) \;=\; f(-x), \\ [\pi_\gamma(S)\,f](x) \;=\; |x|^{\imath(\gamma_1-\gamma_2)-1}\,f(x^{-1})\,.\end{gathered} \tag{3.17}$$

Considering the expression (**??**), its R-invariance reduces to change $n\to -n$ of the summation variable, and the invariance under T is compensated the change of the variable $m\to m-n$ (which does not spoil the condition $(m,n)=1$). To check the invariance under S we take into account the following identity

$$|x|^{\imath(\gamma_1-\gamma_2)-1}\,\delta(m+x^{-1}n) \;=\; |x|^{\imath(\gamma_1-\gamma_2)}\,\delta(xm+n) \;=\; \frac{|n|^{\imath(\gamma_1-\gamma_2)}}{|m|^{\imath(\gamma_1-\gamma_2)}}\,\delta(xm+n)\,. \tag{3.18}$$

This completes a verification of the required properties of (**??**) and (**??**). □

Now the Eisenstein automorphic function associated with $(\pi_\gamma, \mathcal{V}_\gamma)$ is defined as a matrix element

$$\Phi_\gamma(g) = \langle \phi_{GL_2(\mathbb{Z})}, \pi_\gamma(g)\phi_{O(2)} \rangle \,. \tag{3.19}$$

Note that $\Phi_\gamma(g)$ obviously defines a function on the double coset $\mathcal{M}_2$. Explicit realization of the principle series representation allows to obtain explicit expression for the GL_2-automorphic form (**??**).

Proposition 3.1 *For the GL_2-Eisenstein function given by the matrix element (**??**), the following series representation holds,*

$$\Phi_\gamma(g) = \langle \phi_{GL_2(\mathbb{Z})}, \pi_\gamma(g)\, \phi_{O_2} \rangle$$
$$= |\det g|^{\imath\gamma_2+\frac{1}{2}} \sum_{(n,m)\in\mathcal{P}} \Big|(na+mc)^2 + (nb+md))^2\Big|^{\frac{\imath(\gamma_1-\gamma_2)-1}{2}}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.20}$$

provided $\mathrm{Im}(\gamma_1-\gamma_2) > \frac{1}{2}$ *for convergence.*

Proof. Using (**??**), (**??**) and (**??**) we have

$$\begin{aligned} \Phi_\gamma(g) \;=\; \langle \phi_{GL_2(\mathbb{Z})},\, \pi_\gamma(g)\, \phi_{O_2} \rangle \;&=\int\limits_{N_+} dn_x\, \overline{\varphi_{GL_2(\mathbb{Z})}(n_x)}\; \phi_{O_2}(n_x g) \\ = |\det(g)|^{\imath\gamma_2+\frac{1}{2}} \sum_{(n,m)\in\mathcal{P}} |n|^{\imath(\gamma_1-\gamma_2)} &\int\limits_{\mathbb{R}} dx\; \delta(m+xn) \\ \times \Big|(a+xc)^2 + (b+xd)^2\Big|^{\frac{\imath(\gamma_1-\gamma_2)-1}{2}} & \\ = |\det(g)|^{\imath\gamma_2+\frac{1}{2}} \sum_{(n,m)\in\mathcal{P}} |n|^{\imath(\gamma_1-\gamma_2)-1} \left|\Big(a-\frac{mc}{n}\Big)^2 + \Big(b-\frac{md}{n}\Big)^2\right|^{\frac{\imath(\gamma_1-\gamma_2)-1}{2}}, & \end{aligned} \tag{3.21}$$

which gives (**??**) after changing the summation variable $m \mapsto -m$. □

The Eisenstein functions are right O_2-invariant and thus are defined as functions of $GL_2(\mathbb{R})/O_2$ via a choice of a section of the projection $GL_2(\mathbb{R}) \to GL_2(\mathbb{R})/O_2$. We chose the following section

$$g(x,y,t) = t^{\frac{1}{2}} y^{-\frac{1}{2}} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{R})\,, \qquad x \in \mathbb{R}, \quad t, y \in \mathbb{R}_+\,. \tag{3.22}$$

Evaluation of the Eisenstein function of elements (**??**) gives

$$\Phi_\gamma(\tau, t) = t^{\imath(\gamma_1+\gamma_2)} \sum_{(n,m)\in\mathcal{P}} \frac{(\mathrm{Im}(\tau))^{\frac{\imath(\gamma_2-\gamma_1)+1}{2}}}{|m+n\tau|^{\imath(\gamma_2-\gamma_1)+1}}\,, \qquad \tau = x + \imath y\,. \tag{3.23}$$

The Eisenstein series may be written in a form that makes its $GL_2(\mathbb{Z})$-invariance obvious.

Lemma 3.2 *Let us define a subgroup $B(\mathbb{Z}) \subset GL_2(\mathbb{Z})$ as follows*

$$B(\mathbb{Z}) = \left\{ g = \begin{pmatrix} (-1)^{\epsilon_1} & r \\ 0 & (-1)^{\epsilon_2} \end{pmatrix} \Big| \, \epsilon_{1,2} \in \{0,1\}, r \in \mathbb{Z} \right\}. \tag{3.24}$$

Then we have the following coset decomposition

$$GL_2(\mathbb{Z}) = \bigsqcup_{(m,n)\in\mathcal{P}} B(\mathbb{Z}) \cdot \gamma_{(m,n)}\,, \tag{3.25}$$

where

$$\gamma_{(m,n)} = \begin{pmatrix} k & l \\ m & n \end{pmatrix}, \qquad \gcd(m,n) = 1 \qquad m,n,k,l \in \mathbb{Z}\,, \tag{3.26}$$

where we take k, l to be unique solutions of the equation

$$|kn - ml| = 1\,, \qquad 0 \leq k < m\,. \tag{3.27}$$

*In the expression (**??**) we imply a fixed lift along the projection $(\mathbb{Z}^2 - \{0\}) \to (\mathbb{Z}^2 - \{0\})/\mathbb{Z}_2$ of $(m,n) \in \mathcal{P}$.*

Proof. Taking into account the explicit form of the left action of elements of $B(\mathbb{Z})$

$$\begin{pmatrix} (-1)^{\epsilon_1} & r \\ 0 & (-1)^{\epsilon_2} \end{pmatrix} \cdot \begin{pmatrix} k & l \\ m & n \end{pmatrix} = \begin{pmatrix} (-1)^{\epsilon_1}k + rm & (-1)^{\epsilon_1}l + rn \\ (-1)^{\epsilon_2}m & (-1)^{\epsilon_2}n \end{pmatrix}, \tag{3.28}$$

and determinant condition

$$|kn - ml| = 1\,, \tag{3.29}$$

we infer that the set of $B(\mathbb{Z})$-orbits is projected to $\mathcal{P}$. Let us now check that fibers of this projection allow transitive action of $B(\mathbb{Z})$. For $m \neq 0$ one might chose $m > 0$ and $n \in \mathbb{Z} - \{0\}$. Moreover we can chose $0 < k < m$. Then for fixed m and n the elements l and k are uniquely defined as solution of the equation

$$|kn - ml| = 1\,, \qquad 0 \leq k < m\,. \tag{3.30}$$

For $m = 0$ we take $n \in \mathbb{Z}_+$ and then from $|nk| = 1$ derive that $n = 1$. Thus we can take $k = 1$ and $l = 0$. This exhausts the set of representatives in the right $B(\mathbb{Z})$-cosets. $\square$

Proposition 3.2 *The Eisenstein function (**??**) allows the following presentation*

$$\Phi_\gamma(\tau, t) = t^{\imath(\gamma_1+\gamma_2)} \sum_{\gamma\in B(\mathbb{Z})\backslash GL_2(\mathbb{Z})} \mathrm{Im}(\gamma \cdot \tau)^{\frac{\imath(\gamma_2-\gamma_1)+1}{2}}\,, \tag{3.31}$$

where the action of $GL_2(\mathbb{Z})$ is given by

$$\gamma \cdot \tau \, = \, \frac{a + b\tau}{c + d\tau}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})\,. \tag{3.32}$$

Proof. In the summation we may chose the representatives (**??**). Then the expression (**??**) follows from the following simple identity

$$\mathrm{Im}(\gamma_{(m,n)} \cdot \tau) \, = \, \frac{\mathrm{Im}(\tau)}{|m + n\tau|^2}\,. \tag{3.33}$$

This completes our proof. $\square$

4 Global GL_2 Hecke-Baxter operator

To construct global GL_2 Hecke-Baxter operator we start with the description of the elements of the Hecke algebras $\mathcal{H}(GL_2(\mathbb{Q}), GL_2(\mathbb{Z}))$ and $\mathcal{H}(GL_2(\mathbb{R}), O_2)$ acting on the space of the GL_2-automorphic forms, functions on $\mathcal{M}_2$ (by convolutions form the right and form the left). Taking into account the interpretation of $\mathcal{M}_2$ as space equivalence classes of lattices L in $\mathbb{R}^2$ we introduce the following averaging operators T_n as analogs of (**??**)

$$(T_n \cdot f)(L) = \sum_{[L:L']=n} f(L') \,, \tag{4.1}$$

where the sum goes over sub-lattices $L' \subset L$ of index n. The double coset description of $\mathcal{M}_2$ allows to rewrite the operators (**??**) as follows

$$(T_n \cdot f)(g) = \sum_{\gamma \in SL_2(\mathbb{Z}) \backslash \mathrm{Mat}_2^{(n)}(\mathbb{Z})} f(\gamma \cdot g) \,, \tag{4.2}$$

where

$$\mathrm{Mat}_2^{(n)}(\mathbb{Z}) \; = \; \{\gamma \in \mathrm{Mat}_2(\mathbb{Z}) | \det \gamma = n\} \,. \tag{4.3}$$

This action may be written more explicitly using a specific choice of representatives of the coset space $SL_2(\mathbb{Z}) \backslash \mathrm{Mat}_2^{(n)}(\mathbb{Z})$.

Lemma 4.1 *For $n \in \mathbb{Z}_+$, the space $\mathrm{Mat}_2^{(n)}(\mathbb{Z})$ defined in* (**??**) *allows the following coset decomposition:*

$$\mathrm{Mat}_2^{(n)}(\mathbb{Z}) \; = \bigsqcup_{i=1}^{\sigma(n)} SL_2(\mathbb{Z}) \, \alpha_i \,, \tag{4.4}$$

where

$$\alpha_i = \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix}, \qquad a_i, b_i, d_i > 0, \quad a_i d_i = n, \quad 0 \leq b_i < d_i \,, \tag{4.5}$$

and the number $\sigma(n)$ of the coset representatives $SL_2(\mathbb{Z}) \backslash \mathrm{Mat}_2^{(n)}(\mathbb{Z})$ equals the positive divisors sum:

$$\sigma(n) \; = \sum_{d|n} d \,. \tag{4.6}$$

Proof. See e.g. [**?**], Chapter II, §1. □

As a consequence of the previous Lemma we obtain the following presentation for the operators T_n

$$(T_n \cdot f)(g) \; = \sum_{\substack{a,d>0 \\ ad=n}} \sum_{b=0}^{d-1} f\Big(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) g \Big) \,. \tag{4.7}$$

Let us combine the operators T_n, $n \in \mathbb{Z}_+$ into the generating series

$$Q_s^{GL_2(\mathbb{Z})} = \sum_{n=1}^{\infty} \frac{1}{n^{s+\frac{1}{2}}} T_n \,, \tag{4.8}$$

acting on a function on the space of lattices $L \subset \mathbb{R}^2$ in the following way

$$(Q_s^{GL_2(\mathbb{Z})} \cdot f)(L) \;=\; \sum_{L' \subset L}^{\infty} \frac{1}{[L:L']^{s+\frac{1}{2}}}\, f(L') \,, \tag{4.9}$$

(note that the shift $s \to s + \frac{1}{2}$ is a special case of the general expression $s + \frac{\ell}{2}$ for $GL_{\ell+1}$). Equivalently,

$$\big(Q_s^{SL_2(\mathbb{Z})} \cdot f\big)(g) \;=\; \sum_{\alpha \in SL_2(\mathbb{Z}) \backslash \mathrm{Mat}_2^*(\mathbb{Z})} \frac{1}{|\det \alpha|^{s+\frac{1}{2}}}\, f(\alpha \cdot g) \,, \tag{4.10}$$

where α runs through the set of $SL_2(\mathbb{Z})$-coset representatives (**??**) of the space

$$\mathrm{Mat}_2^*(\mathbb{Z}) \;=\mathrm{Mat}_2(\mathbb{Z}) \cap GL_2^+(\mathbb{Q}) = \bigsqcup_{n \in \mathbb{Z}_+} \mathrm{Mat}_2^{(n)}(\mathbb{Z}) \,. \tag{4.11}$$

One has an analog of the Riemann zeta-function for GL_2:

$$\zeta^{GL_2}(s) = \sum_{\alpha \in SL_2(\mathbb{Z}) \backslash \mathrm{Mat}_2^*(\mathbb{Z})} \frac{1}{|\det \alpha|^{s+\frac{1}{2}}} \,. \tag{4.12}$$

Lemma 4.2 *The following identity holds*

$$\zeta^{GL_2}(s) = \sum_{\alpha \in SL_2(\mathbb{Z}) \backslash \mathrm{Mat}_2^*(\mathbb{Z})} \frac{1}{|\det \alpha|^{s+\frac{1}{2}}} = \zeta\left(s + \frac{1}{2}\right) \cdot \zeta\left(s - \frac{1}{2}\right) \,, \tag{4.13}$$

where $\zeta(s)$ is the Riemann zeta-function given by by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \,, \qquad \mathrm{Re}(s) > 1 \,. \tag{4.14}$$

Proof. Using the set of representatives for the coset we have

$$\sum_{\alpha \in SL_2(\mathbb{Z}) \backslash \mathrm{Mat}_2^*(\mathbb{Z})} \frac{1}{|\det \alpha|^{s+\frac{1}{2}}} = \sum_{\substack{a,d>0 \\ 0 \le b < d}} \frac{1}{a^{s+\frac{1}{2}}} \cdot \frac{1}{d^{s+\frac{1}{2}}} \,. \tag{4.15}$$

Summing over b we obtain

$$\sum_{\alpha \in SL_2(\mathbb{Z}) \backslash \mathrm{Mat}_2^*(\mathbb{Z})} \frac{1}{|\det \alpha|^{s+\frac{1}{2}}} = \sum_{a,d>0} \frac{1}{a^{s+\frac{1}{2}}} \cdot \frac{1}{d^{s-\frac{1}{2}}} = \zeta\left(s + \frac{1}{2}\right) \cdot \zeta\left(s - \frac{1}{2}\right) \,. \tag{4.16}$$

□

Below we encounter a generalization of (**??**) associated with spherical principle series representations. The following result is a direct consequence of the well-known expressions for action of the Hecke algebra generators T_n (**??**) on the Eisenstein functions.

Proposition 4.1 *The action of the operator $Q_s^{GL_2(\mathbb{Z})}$ on the GL_2-Eisenstein functions (**??**) associated with a spherical principle series representation $(\pi_\gamma, \mathcal{V}_\gamma)$ is given by*

$$\big(Q_s^{GL_2(\mathbb{Z})} \cdot \Phi_\gamma\big)(g) \,=\, \zeta^{GL_2}(s|\gamma)\ \Phi_\gamma(g)\,, \tag{4.17}$$

where

$$\zeta^{GL_2}(s|\gamma) = \zeta(s-\imath\gamma_1)\,\zeta(s-\imath\gamma_2)\,. \tag{4.18}$$

Proof. Application of the Hecke-Baxter operator (**??**) gives

$$\big(Q_s^{GL_2(\mathbb{Z})} \cdot \Phi_\gamma\big)(g) = \sum_{\alpha\in SL_2(\mathbb{Z})\backslash \mathrm{Mat}_2^*(\mathbb{Z})} \frac{1}{|\det\alpha|^{s+\frac{1}{2}}}\ \Phi_\gamma(\alpha\cdot g)\,. \tag{4.19}$$

Using the presentation (**??**), (**??**) we have

$$\begin{gathered}\big(Q_s^{GL_2(\mathbb{Z})} \cdot \Phi_\gamma\big)(\tau,t)\\ =\, t^{\imath(\gamma_1+\gamma_2)} \sum_{\alpha\in SL_2(\mathbb{Z})\backslash \mathrm{Mat}_2^*(\mathbb{Z})} \frac{|\det\alpha|^{\frac{\imath(\gamma_1+\gamma_2)}{2}}}{|\det\alpha|^{s+\frac{1}{2}}} \sum_{\gamma\in B(\mathbb{Z})\backslash GL_2(\mathbb{Z})} (\mathrm{Im}(\gamma\cdot\alpha\cdot\tau))^{\frac{\imath(\gamma_2-\gamma_1)+1}{2}}\,.\end{gathered}$$

We formally extend the summation domain in the first sum

$$\begin{gathered}\big(Q_s^{GL_2(\mathbb{Z})} \cdot \Phi_\gamma\big)(\tau,t)\\ = \frac{t^{\imath(\gamma_1+\gamma_2)}}{|SL_2(\mathbb{Z})|\cdot|B(\mathbb{Z})|} \sum_{\alpha\in \mathrm{Mat}_2^*(\mathbb{Z})} \frac{|\det\alpha|^{\frac{\imath(\gamma_1+\gamma_2)}{2}}}{|\det\alpha|^{s+\frac{1}{2}}} \sum_{\gamma\in GL_2(\mathbb{Z})} (\mathrm{Im}(\gamma\cdot\alpha\cdot\tau))^{\frac{\imath(\gamma_2-\gamma_1)+1}{2}}\\ = \frac{t^{\imath(\gamma_1+\gamma_2)}}{|SL_2(\mathbb{Z})|\cdot|B(\mathbb{Z})|} \sum_{\alpha\in \mathrm{Mat}_2^*(\mathbb{Z})} \frac{|\det\alpha|^{\frac{\imath(\gamma_1+\gamma_2)}{2}}}{|\det\alpha|^{s+\frac{1}{2}}} \sum_{\gamma\in GL_2(\mathbb{Z})} (\mathrm{Im}(\alpha\cdot\gamma\cdot\tau))^{\frac{\imath(\gamma_2-\gamma_1)+1}{2}}\,.\end{gathered}$$

Let us split the summation domain using the right hand side analog of the decomposition (**??**)

$$\mathrm{Mat}_2^*(\mathbb{Z}) \,=\, \bigsqcup_{n>0}\bigsqcup_i\ \tilde{\alpha}_i\, SL_2(\mathbb{Z})\,, \tag{4.20}$$

to obtain

$$\begin{gathered}\big(Q_s^{GL_2(\mathbb{Z})} \cdot \Phi_\gamma\big)(\tau,t)\\ = t^{\imath(\gamma_1+\gamma_2)} \sum_{\alpha\in \mathrm{Mat}_2^*(\mathbb{Z})/SL_2(\mathbb{Z})} \frac{|\det\alpha|^{\frac{\imath(\gamma_1+\gamma_2)}{2}}}{|\det\alpha|^{s+\frac{1}{2}}} \sum_{\gamma\in B(\mathbb{Z})\backslash GL_2(\mathbb{Z})} (\mathrm{Im}(\alpha\cdot\gamma\cdot\tau))^{\frac{\imath(\gamma_2-\gamma_1)+1}{2}}\,.\end{gathered}$$

Now using the identity

$$\mathrm{Im}(\alpha\cdot\gamma\cdot z) = \frac{a}{d}\cdot\mathrm{Im}(\gamma\cdot z)\,, \tag{4.21}$$

and the fact that $\det\alpha = ad$ for the considered representatives of the coset space we obtain

$$\begin{aligned}&\big(Q_s^{GL_2(\mathbb{Z})}\cdot\Phi_\gamma\big)(\tau,t)\\ = t^{\imath(\gamma_1+\gamma_2)}&\left(\sum_{a,d>0}\sum_{b=0}^{d-1}\frac{a^{\frac{\imath(\gamma_1+\gamma_2)}{2}}}{a^{s-\frac{\imath(\gamma_2-\gamma_1)}{2}}}\frac{d^{\frac{\imath(\gamma_1+\gamma_2)}{2}}}{d^{s+\frac{\imath(\gamma_2-\gamma_1)}{2}+1}}\right)\sum_{\gamma\in B(\mathbb{Z})\backslash GL_2(\mathbb{Z})}(\mathrm{Im}(\gamma\cdot\tau))^{\frac{\imath(\gamma_2-\gamma_1)+1}{2}}\\ &=\left(\sum_{a,d>0}\sum_{b=0}^{d-1}\frac{1}{a^{s-\imath\gamma_2}}\frac{1}{d^{s-\imath\gamma_1+1}}\right)\cdot\Phi_\gamma(\tau,t)\,.\end{aligned}$$

Thus for the eigenvalue we have

$$\sum_{a,d>0}\sum_{b=0}^{d-1}\frac{1}{a^{s-\imath\gamma_1}}\frac{1}{d^{s-\imath\gamma_2+1}}=\sum_{a,d>0}\frac{1}{a^{s-\imath\gamma_1}}\frac{1}{d^{s-\imath\gamma_2}}=\zeta(s-\imath\gamma_1)\,\zeta(s-\imath\gamma_2)\,. \tag{4.22}$$

This completes the proof. □

The Archimedean counterpart of the one-parameter family $Q_s^{GL_2(\mathbb{Z})}$ of elements in $\mathcal{H}(GL_2(\mathbb{Q}),GL_2(\mathbb{Z})$ given by (**??**) is the $GL_2(\mathbb{R})$ Hecke-Baxter operator [**?**]:

$$\begin{gathered}\big(Q_s^{GL_2(\mathbb{R})}\cdot f(g)\;=\;\int\limits_{GL_2(\mathbb{R})}d\mu^G(\tilde g)\;|\det\tilde g|^{s+\frac12}f(\tilde g^{-1}g)\,,\\ d\mu^G(\tilde g)=e^{-\pi\mathrm{Tr}\,\tilde g^\top\tilde g}\,d\mu(\tilde g)\,,\end{gathered} \tag{4.23}$$

acting by convolution with O_2-biinvariant function on $GL_2(\mathbb{R})$

$$Q_s^{GL_2(\mathbb{R})}(g)=|\det g|^{s+\frac12}\,e^{-\pi\mathrm{Tr}\,g^\top g}\,. \tag{4.24}$$

Its action on the matrix element (**??**) (actually on any matrix element with the right O_2-invariant vector) was calculated in [**?**] and is given by

$$\big(Q_s^{GL_2(\mathbb{R})}\cdot\Phi_\gamma\big)(g)\;=\;L^{GL_2(\mathbb{R})}(s|\gamma)\,\Phi_\gamma(g) \tag{4.25}$$

where

$$L^{GL_2(\mathbb{R})}(s|\gamma)=\prod_{j=1}^{2}\pi^{-(s-\imath\gamma_j)/2}\,\Gamma((s-\imath\gamma_j)/2)\,. \tag{4.26}$$

Let us note that while the operator $Q_s^{GL_2(\mathbb{Z})}$ acts by convolution on the functions on the double coset space $GL_2(\mathbb{Z})\backslash GL_2(\mathbb{R})/O_2$ from the left the operator $Q_s^{O_2}$ acts by convolution on the functions on the double coset space $GL_2(\mathbb{Z})\backslash GL_2(\mathbb{R})/O_2$ from the right. Still it is reasonable to consider its combination acting by simultaneous via left/right convolution. Let us define the following integral operator acting on GL_2-automorphic functions

$$\big(\widehat{Q}_s^{GL_2}\bullet\Phi\big)(g)\;=\;\int\limits_{GL_2(\mathbb{R})}d\mu(h)\;\widehat{Q}_s^{GL_2}(g,h)\;\Phi(h^{-1})\,, \tag{4.27}$$

where the kernel of the integral operator is given by

$$\widehat{Q}_s^{GL_2}(g,h) = |\det g|^{s+\frac{1}{2}} \sum_{\alpha\in SL_2(\mathbb{Z})\backslash \mathrm{Mat}_2^*(\mathbb{Z})} |\det h|^{s+\frac{1}{2}}\, e^{-\pi \mathrm{Tr}\,(h^\top h\,\alpha g g^\top\,\alpha^\top)}\,, \tag{4.28}$$

where sum goes over representatives (**??**). This operator is a global analog (in the sense of arithmetic geometry of $\overline{\mathrm{Spec}(\mathbb{Z})}$) of the local spherical Hecke-Baxter operators considered above. Our previous considerations may be summarized in the following form.

Theorem 4.1 *The global Hecke-Baxter operator* (**??**) *acts on the Eisenstein functions represented by matrix elements* (**??**) *of the spherical principle series representation* $(\pi_\gamma,\mathcal{V}_\gamma)$ *of* $GL_2(\mathbb{R})$ *by multiplication on the corresponding global completed zeta-functions*

$$\hat{\zeta}^{GL_2}(s|\gamma) = \pi^{-\frac{s-\imath\gamma_j}{2}}\,\Gamma\Big(\frac{s-\imath\gamma_j}{2}\Big)\prod_{j=1}^{2}\zeta(s-\imath\gamma_j)\,. \tag{4.29}$$

Proof. The action of $\widehat{Q}_s^{GL_2}$ on (**??**) is given by the following integral

$$\big(\widehat{Q}_s^{GL_2}\bullet\Phi_\gamma\big)(g)$$
$$= |\det g|^{s+\frac{1}{2}} \sum_{\alpha\in SL_2(\mathbb{Z})\backslash \mathrm{Mat}_2^*(\mathbb{Z})} \int\limits_{GL_2(\mathbb{R})} d\mu(h)\; |\det h|^{s+\frac{1}{2}}\, e^{-\pi \mathrm{Tr}\,(h^\top h\,\alpha g g^\top\,\alpha^\top)}\;\Phi_\gamma(h^{-1})\,.$$

Using the change of integration variable $h\to h\alpha^{-1}g^{-1}$ we obtain

$$\big(\widehat{Q}_s^{GL_2}\bullet\Phi_\gamma\big)(g)$$
$$= \sum_{\alpha\in SL_2(\mathbb{Z})\backslash \mathrm{Mat}_2^*(\mathbb{Z})} \int\limits_{GL_2(\mathbb{R})} d\mu(h)\; \frac{|\det h|^{s+\frac{1}{2}}}{\det\alpha|^{s+\frac{1}{2}}}\; e^{-\pi \mathrm{Tr}\,(h^\top h)}\;\Phi_\gamma(\alpha g h^{-1})\,. \tag{4.30}$$

This is a combination of $Q_s^{GL_2(\mathbb{Z})}$ and $Q_s^{GL_2(\mathbb{R})}$. Therefore the required result follows from (**??**) and (**??**). □

There is global analog of representation (**??**) that may be described as follows. Let us define the GL_2-analog of the theta constant as follows

$$\Theta_\alpha(0|\imath g g^\top) = 1 + \sum_{\alpha\in SL_2(\mathbb{Z})\backslash \mathrm{Mat}_2^*(\mathbb{Z})} e^{-\pi \mathrm{Tr}\, g g^\top \alpha^\top\alpha}\,. \tag{4.31}$$

Then the following expression for the global zeta-function (**??**) for $\gamma=0$ holds

$$\hat{\zeta}^{GL_2}(s|0) = \int\limits_{GL_2(\mathbb{R})} d\mu(g)\; |\det g|^{s}\,\Big(\Theta_\alpha(0|\imath g g^\top)-1\Big)\,. \tag{4.32}$$

Indeed substituting (**??**) into (**??**) and making the change of integration variable $g \to \alpha^{-1} g$ leads to factorization summation and integration. Thus (**??**) reduces to the product of zeta-functions and Gamma-factor (**??**) with $\gamma = 0$. Note that the essential part of the integral kernel (**??**) is expressed in terms of the following generalization of the classical theta constant

$$\hat{\Theta}(0|A, B) = \sum_{\alpha \in SL_2(\mathbb{Z}) \backslash \mathrm{Mat}_2^*(\mathbb{Z})} e^{-\pi \mathrm{Tr}\ A\alpha B\alpha^\top} . \tag{4.33}$$

where A and B are symmetric positive (2×2)-matrices. This kind of theta series is instrumental for the verification of the analog of the functional equation for the global GL_2 Hecke-Baxter operator extending the relations (**??**) for GL_1. The functional equations for the global Hecke-Baxter operator are compatible with the functional equations for the completed GL_2 zeta-function (**??**)

$$\hat{\zeta}^{GL_2}(1-s|-\gamma) = \hat{\zeta}^{GL_2}(s|\gamma) . \tag{4.34}$$

Similar functional equations hold (when supplied with the Cartan involution $g \to g^\tau = (g^\top)^{-1}$ of $GL_2(\mathbb{R})$) for GL_2-Eisenstein function.

As a final remark let us note that in the case of GL_2 (as well as in the case of GL_1, see Section 2) one might construct interpolation of the global and Archimedean Hecke-Baxter operators by considering congruence semigroups

$$\Gamma(N) = \mathrm{Id} + N\mathrm{Mat}_2(\mathbb{Z}) \subset GL_2(\mathbb{Z}) . \tag{4.35}$$

This might be useful to compare this approach with the results by D. Kazhdan [**?**].

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