MULTIDIMENSIONAL STATISTICS FOR FINITE ORBITS OF GENERALISED CONTINUED FRACTIONS

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Abstract. We statistically compare the relationships between frequencies of digits in continued fraction expansions of typical rational points in the unit interval and higher dimensional generalisations. This takes the form of a Large Deviation and Central Limit Theorem, including multidimensional results for random vectors. These results apply to classical multidimensional continued fraction transformations including Brun's algorithm and the Jacobi–Perron algorithm, and more generally for maps satisfying mild contraction hypothesis on the inverse branches. We prove in particular that the finite trajectories capture the generic ergodic behaviour of infinite trajectories.

1. Introduction

The study of continued fraction expansions of real numbers has a long and rich history that famously includes seminal contributions by Gauss. An important modern aspect of this theory relates to the frequencies of the digits in the expansion, with basic estimates for typical points coming from ergodic theorems (See e.g., [?]). In this article, we describe more subtle statistical results such as Central Limit Theorems (CLT) for finite trajectories. This allows us to show that the expansions of rational numbers capture the generic ergodic behaviour of the expansions of real numbers, as first established in Baladi–Vallée [?].

More generally, we formulate natural and verifiable conditions for a map defined on a compact subset of $\mathbb{R}^m$ to exhibit nice statistical behaviour of the random vectors associated to symbolic expansions induced by the map. Before we present the main results in Section ??, we motivate them by highlighting new consequences with a concrete example.

For $(q,r,s) \in \mathbb{N}^3$ with $q > r > s$, the Brun map in dimension $m = 2$ (See [?, ?, ?]) yields an algorithm for finding a Greatest Common Divisor (GCD) by dividing the largest entry by the second largest entry:

$$(q,r,s) \longmapsto (q_1,r_1,s_1)$$

with $q_1,r_1,s_1 \in \{q - \lfloor q/r \rfloor r, r, s\}$ reordered in a descending order. This map is the integer version of Brun's multidimensional continued fraction algorithm, playing the same role as Euclid algorithm for continued fractions, and has been studied for several applications (See [?, ?]). This algorithm terminates uniquely in finite time when it reaches $(q_n,0,0)$ (where q_n is the GCD of the initial triple), and gives a sequence of partial quotients $a_i = \lfloor q_{i-1}/r_{i-1} \rfloor \geq 1$ for $1 \leq i \leq n$ where $(q_0,r_0,s_0) := (q,r,s)$. For $j \in \mathbb{N}$ let

$$N_j(q,r,s) := \#\{1 \leq i \leq n : a_i = j\}$$

be the number of occurrences of the prescribed partial quotients in the execution. This can be extended to $m > 2$ in a canonical way. For Brun's algorithm, our result can be informally stated as follows.

Theorem 1.1. *For $(j_1,\cdots,j_d) \in \mathbb{N}^d$, consider a random vector $\overline{N} = (N_{j_1},\cdots,N_{j_d}) \in \mathbb{N}^d$. Then, up to suitable normalisations, the values*

$$\{\overline{N}(t_1,\cdots,t_{m+1}) : 1 \leq t_{m+1} < \cdots < t_1 \leq Q, (t_1,\cdots,t_{m+1}) = 1\}$$

become equidistributed according to a large deviation theorem and a Gaussian limit law as $Q \to \infty$.

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The motivation for our general results is to provide an easily applicable theory that can be used to study the multidimensional statistics of finite orbits of generalised continued fraction algorithms in higher dimensions. Under mild hypothesis, we significantly simplify and generalise the transfer operator methods initially formulated in [?] for the Gauss map, where they required a delicate analysis on uniform polynomial decay of operator norms. Verifying that these estimates hold is challenging as it needs precise knowledge of the underlying dynamics, which need to be checked case by case for different maps. We discuss this point further in Section **??**.

There are many algorithms, especially in higher dimensions for which a priori assumptions and estimates are difficult to show and have not been obtained. These include the very well-known Brun and Jacobi–Perron algorithms. In fact, there were no previous results known on the distribution laws for their finite orbits. This has lead us to formulated a flexible framework with great generality that can be applied to a variety of continued fraction maps, including the Brun and Jacobi–Perron algorithms. Since our hypotheses are easily verified we expect further applications to follow in the future (See Section **??**).

We now move on to discuss our general results and approach.

1.1. **The setting.** Let us introduce the necessary notation to present the dynamical formulation of our main results.

Let $I \subset \mathbb{R}^m$ be a compact connected subset and T be a self-map on I. Let $\{I_j\}_{j\in J}$ be a countable partition (modulo a null set) of pairwise disjoint open subsets such that

$$I = \bigcup_{j\in J} \overline{I_j} \quad \text{and} \quad T|_{I_j} : I_j \to T(I_j) \text{ is bijective.}$$

We further assume that the restriction $T|_{I_j}$ can be extended to a $C^{1+\mathrm{Lip}}$ map on its closure $\overline{I_j}$. However, we do not necessarily assume that T is a full branch map, i.e., we do not require that $T(\overline{I_j}) = I$ for all $j \in J$.

Let $\mathcal{H} = \{h_j = T|_{I_j}^{-1} : j \in J\}$ denote the set of inverse branches of the map T. For $n \geq 1$ and $(j_1, \cdots, j_n) \in J^n$, we inductively define

$$I_{j_1,\cdots,j_n} = \{x \in I_{j_1} : T(x) \in I_{j_2,\cdots,j_n}\}.$$

It follows that $h = h_{j_1} \circ \cdots \circ h_{j_n}$ induces a bijection from $T^n(I_{j_1,\cdots,j_n})$ onto $I_{j_1,\cdots,j_n}$. We call h an *admissible inverse branch of depth* n and denote by $\mathcal{H}^n$ the set of all such inverse branches. We also set $\mathcal{H}^* = \cup_{n\geq 1}\mathcal{H}^n$.

Throughout, we make the following uniform contraction, distortion and Markovian assumptions on inverse branches. We emphasise that these correspond to natural, mild conditions that are sufficient to deduce the existence of an absolutely continuous invariant measure (See e.g., [?]).

Assumption 1.2 (**Markov**)**.** Let (I, T) be as above. We assume that the map T is *topologically transitive* (i.e., there is a point with a dense orbit) and that it satisfies the following.

(1) The inverse branches are *uniformly contracting*, i.e., there is a uniform constant $0 < \rho < 1$ such that for all $h \in \mathcal{H}^n$ and for $n \geq 1$
$$|h(x) - h(y)| \leq \rho^n |x - y|, \ \ x, y \in \mathrm{Dom}(h)$$
where $\mathrm{Dom}(h)$ denotes the domain of h.
(2) The inverse branches have *bounded distortion* of their Jacobians, i.e., there is a uniform constant $M > 0$ such that for all $h \in \mathcal{H}^n$ and for $n \geq 1$
$$\left| \frac{J_h(x)}{J_h(y)} - 1 \right| \leq M|x - y|, \quad x, y \in \mathrm{Dom}(h).$$
Here $|J_h(x)| = |\det Dh(x)|$ denotes the Jacobian determinant of h.

(3) There is a finite partition (modulo a null set) $\mathcal{P} = \{P_a\}_{a\in A}$ of pairwise disjoint open subsets for I such that $I = \bigcup_{a\in A} \overline{P_a}$ and $\{P_a\}_{a\in A}$ is compatible with $\{I_j\}_{j\in J}$ as follows. Each $T(I_j)$ is a union of the closure of partition elements. Further, if $I_j \cap P_a \neq \varnothing$, then $I_j \subset P_a$. Lastly, for $j \in J$ and $a \in A$, there either exists a unique $b \in A$ such that

$$h_j(P_a) \subset P_b \cap I_j, \text{ or } h_j(P_a) \cap I = \varnothing.$$

We now describe the subspace of I on which our estimates apply in terms of finite backward orbits under the map T of a given point. Fix a point $x_0 \in I$ and write

$$X = \bigcup_{n\geq 1} X_n \quad \text{where} \quad X_n = \{x \in I : x = h(x_0) \text{ for some } h \in \mathcal{H}^n\}. \tag{1}$$

Given an element $x = h(x_0) \in X$ with $h \in \mathcal{H}^*$, we assign to it the *weight*

$$w(x) = -\log |J_h(x_0)|. \tag{2}$$

We impose a further assumption, called *non-arithmeticity.*

Assumption 1.3 (Non-arithmeticity). We assume that the function w does not take values lying in a lattice. That is, there does not exist $c \in \mathbb{R}$ such that

$$\{w(x) = -\log |J_h(x)| : x \in I, h \in \mathcal{H}^* \text{ and } h(x) = x\} \subset c\mathbb{Z}.$$

A familiar motivating example is provided by the Gauss map.

Example 1.4. Let $I = [0,1]$ and let $T : [0,1] \to [0,1]$ denote the Gauss map defined by

$$T(x) = \begin{cases} \frac{1}{x} - \lfloor \frac{1}{x} \rfloor = \{\frac{1}{x}\} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0. \end{cases}$$

For $j \in \mathbb{N}_{\geq 1}$, set $I_j := (1/(j+1), 1/j)$. With the notation from Assumption **??**, the set J equals $\mathbb{N}$, and in this case $\mathcal{P}$ consists of a single element $(0,1)$. We have $I = \bigcup_{j\in J} \overline{I_j} = \bigcup_{P\in\mathcal{P}} \overline{P}$ with $\mathrm{Leb}(\partial I_j) = \mathrm{Leb}(\partial P) = 0$. The inverse branch of T associated with $j \in J$ is $h_j : (0,1) \to I_j$ given by $x \mapsto \frac{1}{j+x}$.

In the particular case that $x_0 = 0$, we can identify the set X from (**??**) as $X = \mathbb{Q}$. Moreover, for any $n \geq 1$, the image of x_0 under the elements in $\mathcal{H}^n$ correspond to rationals p/q in I with $(p,q) = 1$ and for which the Euclidean algorithm stops exactly after n steps. One gets a unique expansion with the convention that the last partial quotient in the corresponding continued fraction expansion is larger than or equal to 2. More precisely, for $x \in (0,1) \cap \mathbb{Q}$, we have $j_1, \cdots, j_{n-1} \geq 1$ and $j_n \geq 2$ such that for all $i \in \{1, \cdots, n-1\}$

$$T^{i-1}(x) \in I_{j_i}, \ T^{n-1}(x) \in \overline{I_{j_n}}, \quad \text{and } T^n(x) = 0.$$

This yields

$$x := p/q = [0; j_1, \ldots, j_n] = h_{j_1} \circ \cdots \circ h_{j_n}(0).$$

Note that we can identify $h \in \mathcal{H}^n$ with the linear fractional transformation

$$x \mapsto \frac{rx+p}{sx+q} \text{ and the matrix } \begin{bmatrix} r & p \\ s & q \end{bmatrix} = \begin{bmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & j_1 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & j_n \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}).$$

Thus we have $q = |(h_{j_1} \circ \cdots \circ h_{j_n})'(0)|^{\frac{1}{2}}$ and the associated weight becomes $w(p/q) = 2\log q$. We will check in Section **??** that Assumptions **??** and **??** are satisfied.

We aim to study distributional results for $h(x_0)$, for $h \in \mathcal{H}^n$. As far as we know, there are only a few examples where related results have been established, which include the work [**?**] where they successfully adopted transfer operator analysis for the limit laws of the costs associated to continued fractions on the set of rationals with bounded denominator. Their method may generalise to other settings, and does have error estimates, but unfortunately, their proof is quite technical and requires strong hypothesis. In addition to Assumptions **??** and **??**, their arguments need topological mixing and a Uniform Non-Integrability (UNI) property for Dolgopyat estimate that can be very difficult to verify in other cases.

1.2. **The results.** The purpose of this article is to present a general "Dolgopyat estimate free" proof that the even more general vector-valued costs associated to dynamical system (I,T) satisfying Assumption **??** and **??** admit a large deviation principle and a Gaussian limit distribution as random variables on a discrete subspace X of I.

This not only allows us to have the basic CLTs for finite backward orbits simply by checking a priori mild assumptions, but also to observe interesting multidimensional statistics. In particular, we compare frequencies of digits in the symbolic expansions in terms of the associated covariance matrix.

We assign to each $x \in I$ a cost via the locally constant functions $c_j : I \to \mathbb{R}_{\geq 0}$ defined for $j \in J$ by

$$c_j(x) = \begin{cases} 1 & \text{if } x \in I_j \\ 0 & \text{otherwise.} \end{cases}$$

In this definition, the weight 1 can be replaced by any $c > 0$, however we keep $c = 1$ to explicitly remark later on the genericity for the behaviour of finite orbits.

We define a *counting function* N_j on X in a natural way: If $x = h(x_0)$ with $h \in \mathcal{H}^n$, then we write

$$N_j(x) = \#\{0 \leq i \leq n-1 : T^i(x) \in I_j\} = \sum_{0 \leq i \leq n-1} c_j(T^i(x)).$$

From now on, we assume (I,T) and $w(\cdot)$ satisfy Assumptions **??** and **??**. Our first result describes a frequency result for elements of X with respect to the non-arithmetic weight w. The following law of large numbers shows in particular that the set X is not too thin.

Theorem 1.5. *There exists $C > 0$ such that*

$$\#\{x \in X : w(x) < Q\} \sim Ce^Q$$

as $Q \to \infty$. Moreover, for each $j \in J$, there exists $\Lambda_j > 0$ such that

$$\lim_{Q \to \infty} \frac{1}{\#\{x \in X : w(x) < Q\}} \sum_{w(x) < Q} \frac{N_j(x)}{Q} = \Lambda_j.$$

In particular, we have

$$\sum_{w(x) < Q} N_j(x) \sim C\,\Lambda_j\,Q\,e^Q \ \ \textit{as } Q \to \infty.$$

Here and throughout the rest of the paper, the notation '$\sim$' represents the asymptotic limit: that is if $f, g : \mathbb{R} \to \mathbb{R}$ are real valued functions then $f(t) \sim g(t)$ as $t \to \infty$ means that $f(t)/g(t) \to 1$ as $t \to \infty$.

Example 1.6. We revisit Example **??** with the Gauss map. Writing

$$x = [0; j_1, \ldots, j_n] = \frac{1}{j_1 + \frac{1}{j_2 + \cdots + \frac{1}{j_n}}}$$

as a finite continued fraction expansion, this yields $N_j(x) = \#\{1 \leq i \leq n : j_i = j\}$, i.e., $N_j(x)$ counts the number of occurrences of the prescribed partial quotient j in the expansion of x. We will see that Λ_j is given by $\int_{I_j} g(x)dx$ (up to an absolute positive constant $2/h(T)$, where $h(T)$ is the entropy of the Gauss map), and $g(x) = \frac{1}{\log 2(1+x)}$ is the invariant density.

In fact, we can also deduce Theorem **??** from the following large deviation principle and central limit theorem.

Theorem 1.7. *For each $j \geq 1$ and for any $\epsilon > 0$*

$$\limsup_{Q \to \infty} \frac{1}{Q} \log\left(\frac{1}{\#\{x \in X : w(x) < Q\}} \# \left\{ x \in X : w(x) < Q, \left| \frac{N_j(x)}{Q} - \Lambda_j \right| > \epsilon \right\} \right) < 0.$$

Furthermore, for each $j \geq 1$ there exists $\sigma_j^2 > 0$ such that for any $a \in \mathbb{R}$

$$\frac{1}{\#\{x \in X : w(x) < Q\}} \# \left\{ x \in X : w(x) < Q \quad \text{and} \ \frac{N_j(x)}{\sqrt{Q}} \leq a \right\} \to \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{a} e^{-u^2/2\sigma_j^2}\, du.$$

Let us now turn to the multidimensional version. Let $d \geq 1$ and fix distinct numbers $j_1, \dots, j_d$ in J that are admissible in the sense of Assumption **??**.(3). We now want to compare the counting functions $N_{j_1}, \dots, N_{j_d}$. To simplify the notation, we write

$$\overline{N}(x) = (N_{j_1}(x), \dots, N_{j_d}(x)) \in \mathbb{N}^d \quad \text{for } x \in X. \tag{3}$$

We also consider the *centred counting functions*

$$\varphi_i(x) = N_{j_i}(x) - w(x)\Lambda_{j_i}$$

and write

$$\overline{\varphi}(x) = (\varphi_1(x), \dots, \varphi_d(x)) \in \mathbb{R}^d. \tag{4}$$

We then have the following Central Limit Theorem, which formalises Theorem **??** for general maps satisfying Assumption **??** and **??**.

Theorem 1.8. *There exists a positive definite, symmetric matrix $\Sigma \in \mathrm{GL}_d(\mathbb{R})$ such that, for any non-empty open $U \subset \mathbb{R}^d$, we have*

$$\frac{1}{\#\{x \in X : w(x) < Q\}} \# \left\{ x \in X : w(x) < Q \quad \text{and} \ \frac{\overline{\varphi}(x)}{\sqrt{Q}} \in U \right\} \to \frac{1}{(2\pi \det(\Sigma))^{d/2}} \int_U e^{-\frac{1}{2}\langle u, \Sigma u \rangle}\, du$$

as $Q \to \infty$. Moreover, the covariance matrix Σ is described by a Hessian of $\overline{\varphi}$.

Example 1.9. For the Gauss map, we have $T(I_j) = (0,1)$ for all $j \in J$. This means that there is no correlation among the admissible digits, hence the covariance matrix Σ is given by a diagonal matrix. In fact, the covariance matrix encodes explicitly the relations provided by the Markovian condition from Assumption **??**.(3).

As an immediate consequence of Theorem **??**, we also deduce the following multidimensional large deviation principle.

Theorem 1.10. *For each $\epsilon > 0$*

$$\limsup_{Q \to \infty} \frac{1}{Q} \log \left(\frac{1}{\#\{x \in X : w(x) < Q\}} \# \left\{ x \in X : w(x) < Q, \ \left\| \frac{\overline{\varphi}(x)}{Q} \right\| > \epsilon \right\} \right) < 0$$

where $\| \cdot \|$ is any fixed norm on $\mathbb{R}^d$.

We remark that we are counting the points in X up to finite multiplicity in Theorem **??**-**??** due to boundary issues for the function space we take in Definition **??** if $|\mathcal{P}| > 1$. To remove this restriction, we may fix a point $x_0 \in I \backslash \{T^n(x) : x \in \partial P_a, P_a \in \mathcal{P}, n \geq 1\}$.

For an arbitrary choice of the initial point x_0, we believe that this issue can be resolved by adapting [?]. They suggested a way of introducing a finite CW-cell structure on the space of piecewise C^1-functions on I with respect to $\mathcal{P}$ to deduce the exact counting result for the cost functions associated to complex continued fractions (See Remark **??**).

1.3. **An overview of the proofs.** We now briefly describe our approach and compare it with earlier approaches. The original argument employed by Baladi–Vallée holds for the Gauss map when $m = d = 1$. They applied a Tauberian theorem (Perron's formula of order 2) to complex Dirichlet series identified using the resolvent of transfer operators, and accordingly obtained a "Quasi-power behaviour" of the moment generating function of centred counting functions $\overline{\varphi}$ from (**??**) associated to classical continued fractions.

The uniform quasi-power expression is useful because of the Hwang's Quasi-power Theorem [?].

Theorem 1.11 (Hwang's Quasi-power Theorem). *Suppose that a sequence of random variables Y_n has the property that the moment generating functions $\lambda_n(s) = \mathbb{E}(e^{sY_n})$ are analytic in a disc $|s| < \rho$ for some $\rho > 0$. Further assume that there is a uniform expression*

$$\lambda_n(s) = e^{\beta_n U(s)+V(s)}\left(1 + O\left(\frac{1}{k_n}\right)\right)$$

where β_n, k_n are sequences tending to ∞, U, V are analytic in $|s| \le \rho$ and $U''(0) \neq 0$.

Then, the distribution of $Z_n := (Y_n - \beta_n U'(0))/\beta_n U''(0)$ is asymptotically Gaussian, the speed of convergence to the Gaussian limit being $O(k_n^{-1} + \beta_n^{-1/2})$.

In order to deduce the quasi-power estimate, one has to apply sophisticated Tauberian theorems that provide not just an upper bound, but also an asymptotic of precisely the right order. The use of Perron's formula in [?] requires finer analysis on a Dirichlet series, in particular, uniformity and analytic continuation to $\Re(s) \ge 1 - \epsilon$ with no other pole on the vertical line $\Re(s) = 1$, except a unique pole at $s = 1$ of order 1. To have so, strong spectral properties are needed, namely that the transfer operator satisfies the Dolgopyat estimate and has a spectral gap with a unique dominant eigenvalue that is simple.

Here, for all (I,T) under Assumption **??** and **??**, it is possible to deduce a spectral gap on the piecewise Lipschitz space $\oplus_{a\in A} C^{1+\mathrm{Lip}}(P_a)$ possibly without the uniqueness of the largest eigenvalue and without a Dolgopyat-type estimate. Motivated by Morris [?], this is sufficient to have Theorem **??** by generalising the method of moments. However, we remark that our result does not give any information regarding the speed of convergence, in contrast to the Baladi–Vallée theorem [?].

This article is organised as follows. In Section **??**, we first present applications of the main theorems to some examples of multidimensional continued fraction algorithm by checking the a priori hypotheses. In Section **??**, we introduce complex functions and transfer operators, and study their spectral properties. Then the law of large numbers is proved in Section **??** with the proof of Theorem **??**. Theorem **??** on large deviations is proved in Section **??**. This leads to CLTs based on the moments estimates in Sections **??** and **??**.

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2. Application: Examples

In this section, we present CLTs and multidimensional statistics for the Gauss map and generalised continued fractions simply by checking the hypotheses in Assumptions **??** and **??**. Our examples include classical continued fraction (dimension $m = 1$), Brun and Jacobi-Perron multidimensional continued fractions ($m \ge 2$).

2.1. **Continued fractions.**

In Example **??**, we have seen that each $x \in (0,1) \cap \mathbb{Q}$ admits a unique continued fraction expansion

$$x = [0; j_1, \ldots, j_n]$$

with $j_1, \cdots, j_{n-1} \ge 1$, $j_n \ge 2$, which is identified by a depth n inverse branch $h = h_{j_1} \circ \cdots \circ h_{j_n} \in \mathcal{H}^n$ of the Gauss map T on $I = [0,1]$.

We now check that (I,T) satisfies all of the required hypotheses from Assumption **??** and **??**.

Proposition 2.1. *The Gauss map satisfies the Markov and Non-arithmeticity assumptions.*

Proof. For all positive integer j and $I_j = (\frac{1}{j+1}, \frac{1}{j})$, we have $T(\overline{I_j}) = I$ which immediately shows that T is topologically mixing, so transitive.

Using the explicit form $h_j(x) = \frac{1}{j+x}$, it is not difficult to see the following by induction. For all $n \geq 1$ and $h \in \mathcal{H}^n$, we have for all $x \in I$

$$|h'(x)| \ll \left(\frac{1}{2}\right)^n \quad \text{and} \quad \frac{|h''(x)|}{|h'(x)|} \ll M$$

for some $M > 0$, which imply the Markov condition from Assumption **??** by simple applications of the mean value theorem. One can also use the explicit form of $h \in \mathcal{H}^n$, that is,

$$h(x) = \frac{p_{n-1}x + p_n}{q_{n-1}x + q_n}, \text{ where } p_n, q_n \text{ satisfy } \begin{bmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & j_1 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & j_n \end{bmatrix},$$

which gives $h'(x) = \frac{-1}{(q_{n-1}x+q_n)^2}$, that is, the derivative of the (unimodular) homography h is provided by minus the inverse of the square of its denominator. (This is a crucial property providing a simple expression of the derivatives of the inverse branches in terms of denominators of the corresponding homographies, that will also hold in the higher dimensional case.)

For the non-arithmeticity condition from Assumption **??**, it is sufficient to consider the quadratic numbers with purely periodic continued fraction expansions $\frac{\sqrt{5}-1}{2} = [0; 1, \ldots, 1]$ and $\sqrt{2} - 1 = [0; 2, \ldots, 2]$, which are respectively fixed points of the inverse branches $h_1 = (1+x)^{-1}$ and $h_2 = (2+x)^{-1}$, which gives

$$\log \left| h_1'\left(\frac{\sqrt{5}-1}{2}\right) \right| = -2\log\left(\frac{\sqrt{5}+1}{2}\right) \text{ and } |h_2'(\sqrt{2}-1)| = -2\log(1+\sqrt{2}).$$

We can then easily see that $\log(1+\sqrt{2})/\log((1+\sqrt{5})/2) \notin \mathbb{Q}$ concluding the proof. □

The quantity $N_j(x)$ counts the number of occurrences of the prescribed partial quotient j in the continued fraction expansion for all $x \in X$. Theorem **??** then gives

$$\lim_{Q\to\infty} \frac{1}{\#\{p/q : 2\log q < Q\}} \sum_{2\log q < Q} \frac{N_j(p/q)}{2\log q} = \Lambda_j,$$

where

$$\Lambda_j = \frac{\int_{I_j} g(x)dx}{h(T)} = \frac{2}{h(T)\log 2} \log \frac{(j+1)^2}{j(j+2)}$$

and $g(x) = (\log 2(1+x))^{-1}$ denotes the Gauss invariant density for (I, T). Indeed one has

$$-\widetilde{\lambda}_s(1,0) = \int_I \log|T'(x)| g(x)dx = h(T) = \frac{\pi^2}{6\log 2}$$

by Rohlin's formula for the entropy $h(T)$ of the Gauss map T, and

$$\widetilde{\lambda}_t(1,0) = \int_{I_j} g(x)dx = \frac{1}{\log 2} \log \frac{(j+1)^2}{j(j+2)}.$$

We recover the classical result on the number of steps in Euclid algorithm. Moreover we can compare this with the classical result of Lévy from 1937 who showed that for Lebesgue typical points x the frequency of the digit j in the continued fraction expansion is $\frac{1}{\log 2} \log \frac{(j+1)^2}{j(j+2)}$.

Theorem **??** gives that for the distinct $j_1, \ldots, j_d$, a random vector $\overline{\varphi} = (\varphi_1, \ldots, \varphi_d)$, where each $\varphi_i(x) = N_{j_i}(x) - w(x)\Lambda_{j_i}$, has limit Gaussian distribution with the covariance matrix given by the identity matrix. This is because there is no correlation among admissible digits, i.e., we have $T(\overline{I_j}) = I$ for all $j \in J$.

2.2. **Multidimensional continued fractions.** We consider unimodular continued fraction algorithms such as defined in [**?**]; See also [**?**]. These algorithms associate with some given vector an infinite sequence of matrices with determinant ±1, and one can consider the quality of convergence of this product of matrices which correspond to the contraction property in Assumption **??**.(1). In this setting, the inverse branches are homographies (as in the regular continued fraction case) and the Jacobian determinant has a simple formulation in terms of the denominator of the homography (See e.g., [**?**, Proposition 5.2]), which allows one to deduce easily the bounded distortion in Assumption **??**.(2).

2.2.1. *Brun's algorithm.* We consider here the continued fraction version of the Brun GCD algorithm discussed in the introduction. This is one of the most classical multidimensional continued fraction algorithms (See [**?**, **?**, **?**, **?**] and [**?**] for mean results on rational trajectories). The m-dimensional *Brun algorithm* $T_{\mathrm{B}} : [0,1]^m \to [0,1]^m$ is defined for $(x_1,\dots,x_m) \in [0,1]^m$ by

$$T_{\mathrm{B}}(x_1,\dots,x_m) = \begin{cases} \left(\frac{x_{i+1}}{x_i},\dots,\frac{x_m}{x_i},\ \left\{\frac{1}{x_i}\right\},\frac{x_1}{x_i},\dots,\frac{x_{i-1}}{x_i}\right) & \text{if } x_i \neq 0 \\ (0,\dots,0) & \text{otherwise,} \end{cases}$$

where $x_i = \max_k\{x_k\}$. Its density function is explicitly given in [**?**, **?**] by

$$\sum_{\sigma\in\mathfrak{S}_m}\prod_{i=1}^{m}\frac{1}{1+x_{\sigma(1)}+\ldots+x_{\sigma(i)}},$$

where $\mathfrak{S}_m$ is the set of permutations on m elements. This transformation satisfies all of the required hypotheses.

Proposition 2.2. *Brun's algorithm satisfies the Markov and Non-arithmeticty assumptions.*

Proof. The partition (I_j) is indexed by the set $J = \mathbb{N}$ of positive integers. Inverse branches in $\mathcal{H}$ are homographies of the form

$$\left(\frac{x_{i+1}}{j+x_i},\dots,\frac{x_m}{j+x_i},\frac{1}{j+x_i},\frac{x_1}{j+x_i},\dots,\frac{x_{i-1}}{j+x_i}\right) \tag{5}$$

where $x_i = \max_k\{x_k\}$ and $j = \lfloor 1/x_i \rfloor$. This is a full branch algorithm so that $\mathcal{P}$ consists of a single element $(0,1)^m$. Transitivity follows immediately. The contraction is proved by exhibiting using the Hilbert distance a suitable metric that is contracted by the homographies (See [**?**, Annexe]). Distortion properties are proved thanks to the expression of the Jacobian as $|J_h(x)| = \frac{1}{(j+x_i)^{m+1}}$ for h as in (**??**).

For the non-arithmeticity, consider the unique root τ_m with $0 < \tau_m < 1$ of $x^{m+1} + x - 1 = 0$, and the unique root ϱ_m with $0 < \varrho_m < 1$ of $x^{m+1} + 2x - 1 = 0$, for $m \geq 2$. The algebraic number $(\tau_m^m, \tau_m^{m-1}, \dots, \tau_m)$ has purely periodic expansion. Indeed one has

$$T_B(\tau_m^m,\tau_m^{m-1},\dots,\tau_m) = (\{1/\tau_m\},\tau_m m-1,\dots,\tau_m) = (\tau_m^m,\tau_m^{m-1},\dots,\tau_m).$$

Similarly, one checks that $(\varrho_m^m, \varrho_m^{m-1}, \dots, \varrho_m)$ has purely periodic expansion. One then observes that $\log(1+\tau_m)$ and $\log(2+\varrho_m)$ are rationally independent. We conclude by recalling that the Jacobians are given by powers of the denominators of the homographies in $\mathcal{H}$. □

2.2.2. *Jacobi–Perron algorithm.* The Jacobi–Perron algorithm is also one of the most famous multidimensional continued fraction algorithms; See [**?**, **?**, **?**, **?**] or [**?**, Chapter 4]. There is no known expression of the absolutely continuous invariant measure of the Jacobi–Perron algorithm, however its (piecewise) analyticity has been established in [**?**]. However note that with our approach we do not need to have an explicit form of its absolutely continuous invariant measure.

Let $m \geq 2$. The m-dimensional *Jacobi-Perron algorithm* $T_{\mathrm{JP}} : [0,1]^m \to [0,1]^m$ is defined as

$$T_{\mathrm{JP}}(x_1,\dots,x_m) = \begin{cases} \left(\left\{\frac{x_2}{x_1}\right\},\left\{\frac{x_3}{x_1}\right\},\dots,\left\{\frac{1}{x_1}\right\}\right) & \text{if } x_1 \neq 0 \\ (0,\dots,0) & \text{otherwise.} \end{cases}$$

The inverse branches are of the form

$$\left(\frac{1}{a_m + x_m}, \frac{x_1 + a_1}{a_m + x_m}, \dots, \frac{x_{m-1} + a_{m-1}}{a_m + x_m}\right)$$

where $a_i = \lfloor x_{i+1}/x_1 \rfloor$ for $1 \le i \le m-1$, and $a_m = \lfloor \frac{1}{x_1} \rfloor$.

We check that this transformation satisfies all of the required hypotheses.

Proposition 2.3. *The Jacobi-Perron algorithm satisfies the Markov and Non-arithmeticty assumptions.*

Proof. This algorithm is not a full branch map but is topologically mixing since we have $T_{\mathrm{JP}}^m(\overline{I_j}) = I$, where the partition is indexed by the proper subset J of $\mathbb{Z}_{\ge 0}^m$ that is described below. Hence the transitivity follows immediately.

The contracting property from Assumption **??**.(1) is given in [**?**]; See also [**?**]. To check (2), one again uses the fact that the Jacobian determinant has a convenient expression since we have homographies. Regarding (3), the map T_{JP} satisfies explicit Markov assumptions; See e.g., [**?**, Proposition 8] and [**?**, Prop. 2.12]. The Markov partition is given by the following set indexed by permutations σ on m elements:

$$P_\sigma = \{(x_1, \dots, x_m) \in [0,1]^m : 0 \le x_{\sigma(1)} \le \dots \le x_{\sigma(m)} \le 1\}.$$

We give an explicit description of the above facts in the case of $m = 2$. The map T_{JP} satisfies

$$T_{\mathrm{JP}}(\xi, \eta) = \begin{cases} (\{\frac{\eta}{\xi}\}, \{\frac{1}{\xi}\}) & \text{if } \xi \neq 0 \\ (0, \eta) & \text{if } \xi = 0. \end{cases}$$

Denote by $I_{a,b}$, for $(a,b) \in \mathbb{Z}^2$ with $0 \le a \le b$ and $b \ge 1$, the sets

$$I_{a,b} = \begin{cases} \{(\xi,\eta) \in (0,1)^2 : 1/(b+1) < \xi < 1/b,\ a\xi < \eta < (a+1)\xi\} & \text{if } a \neq b \\ \{(\xi,\eta) \in (0,1)^2 : 1/(b+1) < \xi < 1/b,\ a\xi < \eta < 1\} & \text{if } a = b, \end{cases}$$

which form a disjoint partition for $[0,1]^2$. The map T_{JP} is not a full branch map and $\mathcal{P}$ consists of two elements $P_1 = \{(\xi,\eta) \in [0,1]^2 : \xi < \eta\}$ and $P_2 = \{(\xi,\eta) \in [0,1]^2 : \xi > \eta\}$. More precisely, we have

$$T_{\mathrm{JP}}(I_{a,b}) = \begin{cases} (0,1)^2 & \text{if } a \neq b \\ P_1 & \text{if } a = b. \end{cases}$$

Let $h_{a,b} : T_{\mathrm{JP}}(I_{a,b}) \to I_{a,b}$ be the inverse branch associated with $I_{a,b}$. For $(\xi,\eta) \in (0,1)^2 \cap \mathbb{Q}^2$, we have an admissible sequence $(a_1,b_1), \dots, (a_n,b_n)$ with $b_n \ge 2$ such that for all $i \in \{1,\dots,n-1\}$

$$T_{\mathrm{JP}}^{i-1}(\xi,\eta) \in I_{a_i,b_i},\ T_{\mathrm{JP}}^{n-1}(\xi,\eta) \in \overline{I_{a_n,b_n}},\ \text{and } T_{\mathrm{JP}}^n(\xi,\eta) = (0,0).$$

This yields

$$(\xi,\eta) = h_{a_1,b_1} \circ \dots \circ h_{a_n,b_n}(0,0).$$

In the particular case that $x_0 = (0,0)$, we thus can identify $X = \mathbb{Q}^2$ and the image of x_0 under the elements in $\mathcal{H}^n$ corresponds to a pair of rationals $(\frac{p}{q}, \frac{r}{q})$ with $(p,q,r) = 1$. Moreover, the associated weight becomes $w(p/q, r/q) = 3 \log q$.

Next, the non-arithmeticity is proved by comparing two denominators for convergents of periodic orbits as in the previous cases. Indeed, by [**?**], in every real number field $\mathbb{K}$ of degree $m+1$, there exists $(x_1,\dots,x_m) \in (0,1)^m$ having a purely periodic Jacobi–Perron expansion such that $(1, x_1, \dots, x_m)$ is a basis of $\mathbb{K}$. Consider two distinct real number field $\mathbb{K}$ and $\mathbb{K}'$, and $(x_1,\dots,x_m)$, $(x_1',\dots,x_m')$ having purely periodic Jacobi–Perron expansions, with $(1,x_1,\dots,x_m)$ being a basis of $\mathbb{K}$ and $(1,x_1',\dots,x_m')$ being a basis of $\mathbb{K}'$. The Jacobians are given by the common denominator of the homographies involved in the inverse branches. The denominators of the homographies in $\mathcal{H}^*$ are of the form $q_{n-m}x_1 + \dots + q_{n-1}x_m + q_n$ with $(q_n)_n$ taking positive integer values, and similarly for $(1, x'_1, \dots, x'_m)$, the denominators of the homographies are of the form $q'_{n'-m}x'_1 + \dots + q'_{n'-1}x'_m + q'_{n'}$ with $(q'_n)_n$ taking integer values. One sees that

$$\frac{\log(q'_{n'-m}x'_1 + \dots + q'_{n'-1}x'_m + q'_{n'})}{\log(q_{n-m}x_1 + \dots q_{n-1}x_m + q_n)} \notin \mathbb{Q}$$

since $(1, x_1, \dots, x_m)$ and $(1, x_1, \dots, x_m)$ belong to the distinct field extensions. □

2.3. **Remarks.** Though we have restricted our focus to the generalised continued fraction maps in higher dimensions, the same ideas might apply to a wide family of instances such as suitable accelerations of the Rauzy–Veech map in the setting of interval exchanges, or induced Bowen–Series maps (e.g., Romik's map [?], Rosen's map [?]). In these examples, Assumption **??** is known or easily verifiable, however, Assumption **??** has to be checked case by case as our space X and periodic orbits are not completely characterisable (cf. [?] for "Rosen's cusp challenge"), hence the above tricks might not work for some cases.

We stress the fact there are some algorithms for which the current method would not apply. For example, Selmer's algorithm does not admit natural accelerations and the invariant density is unbounded by [?, Chapter 7], while our transfer operators need the density eigenfunction to lie in the set of Lipschitz bounded functions. We refer to Cantrell–Pollicott [?] and [?] for other geometric contexts where the Dolgopyat estimate is not accessible.

As a final remark, our motivation also comes from the applications to number theory. We recall the recent progress by Bettin–Drappeau [?] and Lee–Sun [?] on Mazur–Rubin's conjectural statistics on modular symbols and L-functions of $\mathrm{GL}(2)$, where they presented dynamical proofs by generalising [?]. In particular, the main technical difficulties heavily depend on verifying the Dolgopyat estimates and topological mixing property of a variant of the Gauss map. Our result overcomes this point in a precise way, hence would yield simpler proofs. We expect further extensions in this direction.

3. Spectral properties of transfer operators

We first derive from our assumptions basic spectral properties of the transfer operators associated with the dynamical system (I, T).

Let $\mathrm{Lip}(P_a)$ be the space of functions $f : P_a \to \mathbb{C}$ which can be extended to Lipschitz functions on the compact closure $\overline{P_a}$. Any $f \in \mathrm{Lip}(P_a)$ has a Lipschitz constant

$$\mathrm{Lip}(f|_{P_a}) = \sup_{x \neq y \in P_a} \frac{|f(x) - f(y)|}{|x - y|}.$$

Definition 3.1. We define the space of functions $B := \oplus_{a \in A} C^{1+\mathrm{Lip}}(P_a)$ with the norm

$$\|f\| := \|f\|_\infty + \max_{a \in A} \mathrm{Lip}(f|_{P_a})$$

where $\|f\|_\infty = \sup_{x \in I} |f(x)|$.

We note that B is a Banach space because $\mathrm{Lip}(P_a)$ is a Banach space for any $a \in A$ using the fact that any function in $\mathrm{Lip}(P_a)$ has a unique extension to the closure $\overline{P_a}$ with the same Lipschitz constant. Further, the norm $\|\cdot\|$ is pre-compact for the topology of the norm $\|\cdot\|_\infty$ by the Arzela–Ascoli Theorem.

To prove Proposition **??** we introduce families of transfer operators. Let $s \in \mathbb{C}$ and $t \in \mathbb{C}^d$. To proceed, we define an extended notion of (**??**) as follows. For $h \in \mathcal{H}^*$ and $y \in \mathrm{Im}(h)$, set

$$w(y) := -\log |J_h(x)|, \quad y = h(x), x \in \mathrm{Dom}(h)$$

by abusing the notation, as for all $y \in X$ we always have $x = x_0$.

Definition 3.2. Let $\widetilde{\mathcal{L}}_{s,t} : B \to B$ be the transfer operator defined by

$$\widetilde{\mathcal{L}}_{s,t} f(x) = \sum_{\substack{y : y = h(x) \\ h \in \mathcal{H}, x \in \mathrm{Dom}(h)}} e^{s w(y) + \langle t, \overline{N}(y) \rangle} f(y).$$

We denote by $\widetilde{\mathcal{L}}^{\sharp}_{s,t}$ the same operator defined by the inverse branches corresponding to $\mathcal{H}^{\sharp} \subset \mathcal{H}$ for the unique representation for $x \in X$ in case that there is a duplication.

Remark 3.3. Recall Example **??** that for the Gauss map, it is given by

$$\mathcal{H}^{\sharp} = \{h_j : j \geq 2\}$$

which corresponds to the unique terminating condition for finite continued fraction expansions of the rational. In other cases, it is possible to have $\mathcal{H}^{\sharp} = \mathcal{H}$.

It will be convenient to normalise the transfer operator. We use the centred counting function $\overline{\varphi}$ defined in (**??**), that is, $\overline{\varphi}(x) = \overline{N}(x) - w(x)\overline{\Lambda}$, where $\overline{\Lambda} = (\Lambda_{j_1}, \ldots, \Lambda_{j_d}) \in \mathbb{R}^d$.

Definition 3.4. We define the normalised transfer operator $\mathcal{L}_{s,t} : B \to B$ as the operator defined by

$$\mathcal{L}_{s,t} f(x) = \sum_{\substack{y: y = h(x) \\ h \in \mathcal{H}, x \in \mathrm{Dom}(h)}} e^{sw(y) + \langle t, \overline{\varphi}(y) \rangle} f(y).$$

Similarly we denote by $\mathcal{L}^{\sharp}_{s,t}$ the final operator corresponding to $\mathcal{H}^{\sharp}$.

By the choice of our weight $w(\cdot)$ given by the Jacobian determinant, one can easily see that our transfer operators converge for $\mathfrak{Re}(s)$ and $\mathfrak{Im}(t)$ belonging to a real neighborhood of $(1,0)$ by Assumption **??**.(1) together with the use of the mean value theorem.

Remark 3.5. In [**?**], they identified $\mathcal{P}$ as a CW-complex as follows. For $0 \leq i \leq m = \dim(I)$, let $\mathcal{P}[i]$ be the set of open cells of real dimension i. Then one has a finite structure $\mathcal{P} = \cup_{i=0}^{m} \mathcal{P}[i]$ that gives a decomposition of the function space $B = \oplus_{i=0}^{m} B(\mathcal{P}[i])$ equipped with reasonable norms and of the operator $\mathcal{L} := \mathcal{L}_{s,t}$ as a lower-triangular matrix

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_m^m & 0 & \cdots & 0 \\ \mathcal{L}_m^{m-1} & \mathcal{L}_{m-1}^{m-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{L}_m^0 & \mathcal{L}_{m-1}^0 & \cdots & \mathcal{L}_0^0 \end{bmatrix}$$

where $\mathcal{L}_k^i : B(\mathcal{P}[k]) \to B(\mathcal{P}[i])$ with $0 \leq i, k \leq m$ is the component operator. This allows the boundaries of P's, $P \in \mathcal{P}$ when $|\mathcal{P}| > 1$, to be taken into account, which means the elements in B are honest functions rather than equivalence classes of them.

For complex continued fraction maps ($m = 2$) and transfer operators acting on C^1-functions, they obtained spectral properties that enable to deduce the counting statistics without multiplicity. We speculate that a similar argument would apply to our setting for general expanding maps on $\mathbb{R}^m$ and analysis on Lipschitz functions. In our setting, we can track the multiplicities thanks to Assumption **??**.(3); $I_j \subset P_a$, so we prefer not to try to work in greater generality.

To establish the required properties on the spectrum of $\widetilde{\mathcal{L}}_{s,t}$, we first state the following result on the density of backward orbits.

Lemma 3.6. *For any $x_0 \in I$ we have that*

$$\overline{\bigcup_{n \geq 1} \mathcal{H}^n(x_0)} = I,$$

that is, the collection of pre-images of x_0 under T is dense in I.

Proof. Fix $\epsilon > 0$ and $x \in I$. Now take n sufficiently large so that $x_0 \in I_{j_1,\ldots,j_n}, x \in I_{k_1,\ldots,k_n}$ and $|I_{k_1,\ldots,k_n}|, |I_{j_1,\ldots,j_n}| \leq \epsilon/2$. Then by the transitivity assumption from Assumption **??**, there exists $N \geq 1$ such that $\mathrm{int}(I_{k_1,\ldots,k_n}) \cap T^N \mathrm{int}(I_{j_1,\ldots,j_n}) \neq \varnothing$ and so there exist $h \in \mathcal{H}^N, y \in I$ so that $y \in \mathrm{int}(I_{j_1,\ldots,j_n}) \cap h(\mathrm{int}(I_{k_1,\ldots,k_n}))$. Hence

$$|h(x) - x_0| \leq |h(x) - y| + |y - x_0| \leq \epsilon$$

and we are done. $\square$

Our first result on the positive real transfer operator is then as follows. Let K be a real neighbourhood of 1, or the region $\sigma > 1 - \epsilon$ for some $\epsilon > 0$, where $\sum_{h\in\mathcal{H}} \sup |J_h|^\sigma$ converges.

Proposition 3.7. *For each $\sigma \in K$ the operator $\widetilde{\mathcal{L}}_{\sigma,0}$ has finite spectral radius $R_\sigma > 0$. Furthermore, $\widetilde{\mathcal{L}}_{\sigma,0}$ has a simple, maximal eigenvalue $\widetilde{\lambda}_\sigma$ with a corresponding strictly positive eigenfunction h_σ. All eigenvalues of modulus R_σ are simple and all other eigenvalues lie in a disk $\{z \in \mathbb{C} : |z| < \theta_\sigma R_\sigma\}$ for some $0 < \theta_\sigma < 1$.*

In particular, $\widetilde{\mathcal{L}}_{1,0}$ has a strictly positive eigenfunction $h \in B$ associated to the eigenvalue 1.

Proof. The proof follows the proof [**?**, Theorem 1.5]. We explain how the assumptions on our dynamical systems are used in the proof.

For the existence of $\widetilde{\lambda}_\sigma$, we show the operator satisfies the Lasota–Yorke inequality. For all $n \geq 1$, $a \in A$ and $x, y \in P_a$, one has

$$\begin{aligned}
|\widetilde{\mathcal{L}}^n_{\sigma,0} f(x) - \widetilde{\mathcal{L}}^n_{\sigma,0} f(y)| &\leq \sum_{\substack{h\in\mathcal{H}^n \\ x,y\in \mathrm{Dom}(h)}} \left(|J_h(x) - J_h(y)|^\sigma |f \circ h(x)| + |J_h(y)|^\sigma |f \circ J_h(x) - f \circ J_h(y)|\right) \\
&\leq \sum_{\substack{h\in\mathcal{H}^n \\ x,y\in \mathrm{Dom}(h)}} |J_h(y)|^\sigma \left|\frac{J_h(x)}{J_h(y)} - 1\right| |f \circ h(x)| + \sum_{\substack{h\in\mathcal{H}^n \\ x,y\in \mathrm{Dom}(h)}} |J_h(y)|^\sigma \|f\| |h(x) - h(y)| \\
&\ll_K M\|f\|_\infty + \rho^n \|f\|
\end{aligned}$$

by Assumption **??**, where the implicit constants depend only on (I, T).

By Hennion's criterion [**?**, Theorem XIV.3], the operator $\widetilde{\mathcal{L}}_{\sigma,0}$ is then quasi-compact with the maximal eigenvalue $\widetilde{\lambda}_\sigma > 0$. All other eigenvalues lie in a disk $\{z \in \mathbb{C} : |z| < \theta_\sigma R_\sigma\}$ for some $0 < \theta_\sigma < 1$. We also have $R_\sigma \leq \widetilde{\lambda}_\sigma = \lim_{n\to\infty} \|\widetilde{\mathcal{L}}^n_{\sigma,0} 1\|_\infty^{1/n}$ and conversely

$$R_\sigma \geq \lim_{n\to\infty} \|\widetilde{\mathcal{L}}^n_{\sigma,0} 1\|^{1/n} \geq \lim_{n\to\infty} \|\widetilde{\mathcal{L}}^n_{\sigma,0} 1\|_\infty^{1/n} = \widetilde{\lambda}_\sigma > 0.$$

Suppose now that $h_\sigma(x) = 0$ for some $x \in X$. Then for any $n \geq 1$

$$0 = \widetilde{\mathcal{L}}^n_{\sigma,0} h_\sigma(x) = \sum_{\substack{y=h(x) \\ h\in\mathcal{H}^n, x\in\mathrm{Dom}(h)}} e^{-\sigma w(y)} h_\sigma(y).$$

Since $e^{-\sigma w(y)} > 0$ for all y and h_σ is continuous, the eigenfunction h_σ should be identically zero, which contradicts Lemma **??**. Similarly the eigenvalues of modulus R_σ are simple again due to Lemma **??**, as one can deduce that if $h_{\sigma,1}$ and $h_{\sigma,2}$ are two eigenfunctions for R_σ then they are $\mathbb{R}$-linearly dependent.

For the last item, we observe that $\widetilde{\mathcal{L}}_{1,0}$ is simply weighted by $1/|J_h|$ hence the change of variable directly shows that 1 is the dominant eigenvalue with a strictly positive eigenfunction $h \in B$. □

Normalisation assumption: Moving forward we will assume, without loss of generality, that $R_1 = 1$.

Lemma 3.8 (Spectral properties)**.** *The operator $\widetilde{\mathcal{L}}_{s,t}$ satisfies the following:*

(i) For (s,t) in a neighbourhood of $(1,0)$, the operators $\widetilde{\mathcal{L}}_{s,t}$ have a simple maximal eigenvalue $\widetilde{\lambda}(s,t)$ that varies analytically and the rest of the spectrum is contained in a disk of strictly smaller radius that $|\widetilde{\lambda}(s,t)|$ uniformly in s,t.

(ii) For any fixed $s \neq 1$ with $\Re(s) \geq 1$, there exists $\epsilon(s) > 0$ such that if $|t| < \epsilon(s)$, then $\widetilde{\mathcal{L}}_{s,t}$ has spectral radius at most 1 and does not have 1 as an eigenvalue.

Proof. Part (i) is now a consequence of Proposition **??** and analytic perturbation theory (See e.g., [**?**]).

For part (ii), we claim that for $\xi \neq 0$ we have that 1 is not in the spectrum of $\widetilde{\mathcal{L}}_{1+i\xi,0}$. Assume for a contradiction that 1 is in the spectrum. Then by the quasi-compactness from (i), it would correspond to an eigenvalue $\widetilde{\mathcal{L}}_{1+i\xi,0} g = g$. Since $\widetilde{\mathcal{L}}_{1,0} 1 = 1$ (this is by the Normalisation assumption and by replacing g by $w + \log g \circ T - \log g$) and $|\widetilde{\mathcal{L}}_{1+i\xi,0}g(x)| = |g(x)|$, a simple convexity argument (See e.g., [?, Chapter 4]) shows that $|g| = 1$ and whence w is cohomologous to a function taking values in $c\mathbb{Z}$ for some $c \in \mathbb{R}$. However this gives a contradiction by evaluating on periodic points in view of the non-arithmeticity Assumption **??**. □

Remark 3.9. Theorem **??**.(iii) yields the existence of an ergodic absolutely continuous invariant measure for the map T. It thus makes sense to compare the behaviour of finite orbits to the behaviour of generic truncated orbits.

Remark 3.10. It is clear that the normalised transfer operator $\mathcal{L}_{s,t}$ admits the same spectral properties as $\widetilde{\mathcal{L}}_{s,t}$ as described in Lemma **??**. We further have $\lambda(s,t) = \widetilde{\lambda}(s - \langle\overline{\Lambda}, t\rangle, t)$ for (s,t) in a neighbourhood of $(1,0)$.

It is then easy to show the following.

Lemma 3.11. *We have that* $\lambda_s(1,0) < 0$, *and writing* $t = (t_1, \ldots, t_d)$, *we have that* $\lambda_{t_i}(1,0) = 0$ *for each* $i = 1, \ldots, d$. *Furthermore the matrix* $(\sigma_{i,k})$ *where*

$$\sigma_{i,k} = -\frac{\lambda_{t_i,t_k}(1,0)}{\lambda_s(1,0)}$$

for $i, k \in \{1, \ldots, d\}$, *is positive definite.*

Proof. Since the operator $\mathcal{L}_{1,0} : B \to B$ has isolated simple maximal eigenvalue 1 the associated eigenvalue $\lambda_{s,t}$ and associated eigenfunction $h_{s,t}$ for $\mathcal{L}_{s,t} : B \to B$ have an analytic dependence (s,t) for $|s-1|, \|t\|$ sufficiently small.

We can differentiate the eigenfunction $\mathcal{L}_{s,0}g_{s,0} = \lambda_{s,0}g_{s,0}$ in s to get

$$\mathcal{L}_{s,0}(-wg_{s,0}) + \mathcal{L}_{s,0}(\partial_s g_{s,0}) = \partial_s\lambda_{s,0}g_{s,0} + \lambda_{s,0}\partial_s g_{s,0}.$$

We can set $s = 0$ and apply the dual eigenvalue equation $\mathcal{L}_{1,0}^*\mu = \lambda_{1,0}\mu$ and cancelling terms to deduce that

$$\lambda_s(1,0) := \partial_{s=1}\lambda_{s,0} = \mu\left(\sum_{j\in J}\int_{I_j} -\log|J_{h_j}(x)|g_{1,0}(x)\ dx\right) < 0.$$

A similar argument differentiating in the variables t_i shows that $\lambda_{t_i}(1,0) := \partial_{t_i=0}\lambda_{1,t} = \mu(\varphi_i) = 0$ for each i.

The expressions for the second derivatives start from the eigenvalue identities $\mathcal{L}_{1,t}^n g_{1,t} = \lambda_{1,t}^n g_{1,t}$ (for $n \geq 1$). Differentiating once in t_i (at $s = 0$) gives

$$\mathcal{L}_{1,t}^n((S_n\varphi_i)h_{1,t}) + \mathcal{L}_{1,t}^n(\partial_t h_{1,t}) = n\lambda_{1,t,0}^{n-1}\partial_t\lambda_{1,t,0}h_{1,t,0} + \lambda_{1,t,0}^n\partial_t h_{1,t,0}.$$

where $S_n\varphi_i = \varphi_i + \varphi_i \circ T + \cdots + \varphi_i \circ T^{n-1}$ and differentiating again in t_i gives

$$\begin{aligned}&\mathcal{L}_{1,t}^n((S_n\varphi_i)^2h_{1,t}) + 2\mathcal{L}_{1,t}^n((S_n\varphi_i)\partial_t g_{1,t}) + \mathcal{L}_{1,t}^n(\partial_t^2 g_{1,t})\\ &= n(n-1)\lambda_{1,t}^{n-2}(\partial_t\lambda_{1,t})^2 g_{1,t} + n\lambda_{1,t}^{n-1}\partial_t^2\lambda_{1,t}g_{1,t} + 2n\lambda_{1,t}^{n-1}\partial_t\lambda_{1,t}\partial_t g_{1,t} + \lambda_{1,t}^n\partial_t^2 g_{1,t}.\end{aligned}$$

Setting $t = 0$, applying the dual eigenvalue equation and cancelling terms, dividing by n and letting $n \to \infty$ we have the expression

$$\lambda_{t_i,t_i}(1,0) = \lim_{n\to\infty}\frac{1}{n}\mu\left((S_n\varphi_i)^2\right)$$

at $t = 0$. A similar argument gives

$$\lambda_{t_i,t_j}(1,0) = \lim_{n\to+\infty}\frac{1}{n}\mu\left((S_n\varphi_i)(S_n\varphi_j)\right).$$

at $t = 0$. To show that the matrix $A = (\lambda_{t_i,t_j}(1,0))_{i,j}$ is positive define we can consider any vector $v = (v_1, \ldots, v_d) \in \mathbb{R}^d\backslash\{0\}$ and then we can write

$$vAv^T = \lim_{n\to\infty}\frac{1}{n}\mu\left(\sum_{i=1}^d v_i S_n\varphi_i\right)^2 \geq 0.$$

□

We would like to know when the matrix $(\sigma_{i,k})$ is strictly positive definite.

Proposition 3.12. *Suppose that $d = 1$, i.e., we are working with a single cost function. Then $\lambda_{t,t}(1,0) > 0$.*

Proof. Following a well-known argument for transfer operators, we see that $\lambda_{t,t}(1,0) = 0$ if and only if when $x \in I$ satisfies that $h(x) = x$ for $h = h_1 \circ \cdots \circ h_n \in \mathcal{H}^n$ then

$$N_j(x) - \Lambda_j w(x) = \sum_{0 \le i \le n-1} c_j(T^i(x)) + \Lambda_j \log |J_h(x)|$$

is equal to Cn. However, it is easy to see that we must have $C = 0$ by our choice of Λ_j. This would contradict Assumption **??** as $\sum_{0 \le i \le n-1} c_j(T^i(x))$ takes values in $\mathbb{Z}$. □

It is easy to generalise this to the following multidimensional version.

Proposition 3.13. *The matrix $(\sigma_{i,k})$ from Lemma **??** is strictly positive definite if and only if there does not exist $v \in \mathbb{R}^d$ such that*

$$\langle v, \overline{\varphi}(x) - w(x)\overline{\Lambda} \rangle = 0$$

for all $x \in I$ such that $h(x) = x$ for some $h \in \mathcal{H}$.

4. Proofs of the main results

4.1. **Law of large numbers.** The proof of Theorem **??** relies on the following classical result due to Delange [?] which will allow us to convert results on complex functions to asymptotic formulae.

Proposition 4.1 (Delange Tauberian Theorem)**.** *For a monotone increasing function $\phi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, we set*

$$f(s) = \int_0^\infty e^{-sQ}\, d\phi(Q).$$

Suppose that there is $\delta > 0$ such that

(1) $f(s)$ is analytic on $\{\mathfrak{Re}(s) \ge \delta\} \backslash \{\delta\}$; and,
(2) there are positive integers $n, k \ge 1$, δ a real number, an open neighbourhood U of δ, non-integer numbers $0 < \mu_1, \ldots, \mu_k < n$ and analytic maps $g, h, l_1, \ldots, l_k : U \to \mathbb{C}$ such that

$$f(s) = \frac{g(s)}{(s-\delta)^n} + \sum_{j=1}^{k} \frac{l_j(s)}{(s-\delta)^{\mu_j}} + h(s) \quad \text{for } s \in U \text{ and } g(\delta) > 0.$$

Then

$$\phi(Q) \sim \frac{g(\delta)}{(n-1)!} Q^{n-1} e^{\delta Q}$$

as $Q \to \infty$.

We will use this result and follow the main arguments present in [?].

Proof of Theorem **??**. Let X be defined as in (**??**). Fix $j \in J$. We define the two-variable series

$$\zeta(s,t) = \sum_{x \in X} e^{-sw(x) + tN_j(x)}$$

for $s, t \in \mathbb{C}$. Note that

$$\zeta(s,t) = \widetilde{\mathcal{L}}^{\sharp}_{s,t} \circ \sum_{n \ge 0} \widetilde{\mathcal{L}}^n_{s,t} \mathbf{1}(x_0)$$

where $\mathbf{1}$ stands for the function taking constant value 1 in B.

It follows from Lemma **??** that for any $s_0 \in \mathbb{C}$ with $\mathfrak{Re}(s_0) \ge 1$, there exist $\epsilon(s_0), \delta(s_0) > 0$ such that $\zeta(s,t)$ is bi-analytic for the set of (s,t) satisfying $|s - s_0| \le \delta(s_0)$ and $|t| \le \epsilon(s_0)$. For (s,t) close to $(1,0)$ we can write

$$\widetilde{\mathcal{L}}^n_{s,t} \mathbf{1}(x_0) = \widetilde{\lambda}(s,t)^n \widetilde{Q}(s,t) \mathbf{1}(x_0) + O(\theta^n)$$

where $0 < \theta < 1$ is independent of s, t. It follows that

$$\zeta(s,t) = \frac{\widetilde{\lambda}(s,t)\widetilde{Q}(s,t)\mathbf{1}(x_0)}{1 - \widetilde{\lambda}(s,t)} + R(s,t) \tag{6}$$

where $R(s,t)$ is bi-analytic in a neighbourhood of $(1,0)$. Set $F(s,t) = \widetilde{\lambda}(s,t)\widetilde{Q}(s,t)\mathbf{1}(x_0)$ and note that $F(1,0) > 0$ (as $\widetilde{\mathcal{L}}_{1,0}$ has strictly positive eigenfunction associated to the eigenvalue 1).

When $t = 0$, we deduce that

$$\zeta(s,0) = \frac{-\widetilde{\lambda}_s(1,0)^{-1}F(1,0)}{s-1} + \tilde{R}(s)$$

for $\mathfrak{Re}(s) \geq 1$, where $\tilde{R}$ is analytic on this domain. It follows from Proposition **??** (here $\delta = 1$) together with Lemma **??** that

$$\#\{x \in X : w(x) < Q\} \sim -\widetilde{\lambda}_s(1,0)^{-1}F(1,0)\, e^{Q}$$

as $Q \to \infty$.

We now differentiate (**??**) with respect to t at $t = 0$. This yields

$$\sum_{n\geq 1} N_j(x)e^{-sw(x)} = \zeta_t(s,0) = \frac{-\widetilde{\lambda}_t(s,0)\widetilde{\lambda}(s,0)\widetilde{Q}(s,0)\mathbf{1}(x_0)}{(1-\lambda(s,0))^2} + L(s)$$

where L is analytic on $\mathfrak{R}(s) \geq 1$ apart from a possible simple pole at $s = 1$. It follows easily that $\zeta_s(s,0)$ has a pole of residue $\widetilde{\lambda}_s(1,0)^{-2}\widetilde{\lambda}_t(1,0)\widetilde{\lambda}(1,0)\widetilde{Q}(1,0)\mathbf{1}(x_0)$ at $s = 1$. Applying Proposition **??** gives that

$$\sum_{w(x)<Q} N_j(x) \sim \widetilde{\lambda}_s(1,0)^{-2}\widetilde{\lambda}_t(1,0)F(1,0)Qe^{Q}$$

as $G \to \infty$. Combining this with our asymptotic above shows that

$$\lim_{Q\to\infty} \frac{1}{\#\{x \in X : w(x) < T\}} \sum_{w(x)<Q} \frac{N_j(x)}{Q} = -\frac{\widetilde{\lambda}_t(1,0)}{\widetilde{\lambda}_s(1,0)} = \Lambda_j$$

as required. $\square$

4.2. **Large deviations.** We will deduce our large deviation theorem (Theorem **??**) from the following local Gärtner-Ellis type Theorem.

Lemma 4.2. *[Lemma 3.11 in [**?**]] Let Z_n be a sequence of real random variables and μ_n be a sequence of probability measures on a space X. Suppose that there exists $\eta > 0$ with*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}_n(e^{tZ_n}) = c(t)$$

for all $t \in [-\eta, \eta]$ where $\mathbb{E}_n$ represents the expectation with respect to μ_n. If c is continuously differentiable and strictly convex on $[-\eta, \eta]$ and $c'(0) = 0$, then, for $0 < \epsilon < \eta$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(Z_n > n\epsilon) \leq -\sup_{0<t<\eta}\{t\epsilon - c(t)\} < 0.$$

We also need the following asymptotic result. This can be seen as a weak version of the Hwang Quasi-power Theorem mentioned in the introduction.

Proposition 4.3. *Let $j \in J$. There exists an open real neighbourhood U with $0 \in U$, a real analytic function $\sigma : U \to \mathbb{R}_{>0}$ such that the following holds. For any $t_0 \in U$, there is a positive constant C_{t_0} such that*

$$\sum_{w(x)<Q} e^{t_0\varphi_j(x)} \sim C_{t_0}e^{\sigma(t_0)Q}$$

as $Q \to \infty$. Furthermore σ is strictly convex on U.

Proof. We begin by applying the Implicit Function Theorem to find a real, open neighbourhood U of $t = 0$ and a real analytic function $\sigma : U \to \mathbb{R}$ such that $\lambda(\sigma(t), t) = 1$ for all $t \in U$.

The Implicit Function Theorem also implies that

$$\sigma'(0) = -\frac{\lambda_t(1,0)}{\lambda_s(1,0)} = 0$$

and

$$\sigma''(0) = -\frac{\lambda_{t,t}(1,0)}{\lambda_s(1,0)} > 0$$

by Lemma **??**. In particular, σ is strictly convex. Then for any fixed $t_0 \in U$ we can find $s_0 = \sigma(t_0)$ and a neighbourhood of s_0 such that

$$\zeta(s,t_0) = \frac{\lambda(s,t_0)Q(s,t_0)\mathbf{1}(x_0)}{1-\lambda(s,t_0)} + R(s,t_0) \tag{7}$$

in this neighbourhood (here $R(s,t_0)$ is analytic in s). In particular, $\zeta(s,t_0)$ has a simple pole $s = s_0$. Furthermore, the expression

$$\zeta(s,t_0) = \mathcal{L}^{\sharp}_{s,t} \circ \sum_{n\geq 0} \mathcal{L}^n_{s,t_0}\mathbf{1}(x_0) = \sum_{x\in X} e^{-sw(x)+t_0\varphi_j(x)}$$

shows us that $\zeta(s,t_0)$ does not have any other poles in the line $\mathfrak{R}(s) = s_0$. To see this, note that the operator $\mathcal{L}_{s,t_0}$ has simple maximal real eigenvalue 1 when $s = s_0$. Then as in the proof of Lemma **??** we see that $\mathcal{L}_{s_0+i\xi,t_0}$ can not have 1 as an eigenvalue as this would imply that w takes values in a lattice which contradicts Assumption **??**.

We also see that

$$\zeta(s,t_0) = \frac{\lambda_s(\sigma(t_0),t_0)^{-1}\lambda(s,t_0)Q(s,t_0)\mathbf{1}(x_0)}{\sigma(t)-s} + \widetilde{R}(s)$$

where $\widetilde{R}$ is analytic in a neighbourhood of s_0. Note that $\lambda_s(\sigma(t_0),t_0) > 0$ by the same argument as in previous lemma. To summarise we have shown that $\zeta(s,t_0)$ is analytic in a neighbourhood of $\mathfrak{Re}(s) \geq s_0$ apart from a simple pole with positive residue at $s = s_0$.

Then we see by Proposition **??** that there exists a positive constant C_{t_0} depending on t_0 such that

$$\sum_{w(x)<Q} e^{t_0\varphi_j(x)} \sim C_{t_0}e^{\sigma(t_0)Q}$$

as $Q \to \infty$. □

We are now ready to prove our large deviation theorem.

Proof of Theorem **??**. Combining the above results show that there is a neighbourhood U of 0 and an analytic function $\sigma : U \to \mathbb{R}$ such that if $t \in U$ then

$$\lim_{n\to\infty} \frac{1}{n}\log\left(\frac{1}{\#\{x \in X : w(x) < n\}} \sum_{w(x)<n} e^{t\varphi_j(x)}\right) = \sigma(t) - \sigma(0).$$

Here we are taking a discrete limit as n runs through the integers (despite the fact the limit exists as a continuous limit in Q). We are doing this so that we can apply Lemma **??** (note that $\sigma''(0) > 0$) to deduce that

$$\limsup_{n\to\infty} \frac{1}{n}\log\left(\frac{1}{\#\{x\in X : w(x) < n\}}\#\left\{x \in X : w(x) < n, \left|\frac{N_j(x)}{n} - \Lambda_j\right| > \epsilon\right\}\right) < 0.$$

Here the random variables are $Z_n(x) = \varphi(x)$ if $w(x) < n$ and $Z_n(x) = 0$ otherwise and the measures μ_n are the uniform counting measures on $\{x \in X : w(x) < n\}$. To conclude the proof of the theorem we just need to note that

$$\#\left\{x \in X : w(x) < Q \left|\frac{N(x)}{Q} - \Lambda_j\right| > \epsilon\right\} \leq \#\left\{x \in X : w(x) < \lfloor Q\rfloor + 1 \left|\frac{N(x)}{\lfloor Q\rfloor + 1} - \Lambda_j\right| > \frac{\epsilon}{2}\right\}$$

for all Q sufficiently large. It is then easy to deduce the required exponential decay in the continuous limit $Q \to \infty$. □

4.3. Central limit theorems.

To prove our central limit theorem we follow method presented in Section 3.7 of [?]. We include all details for the convenience of the reader.

We begin by defining the following function.

Definition 4.4. For each $s \in \mathbb{C}, t \in \mathbb{C}^d$ we formally define

$$\eta(s,t) = \sum_{x \in X} e^{-sw(x)+\langle t, \overline{\varphi}(x)\rangle}.$$

Furthermore, for each $\widehat{p} = (p_1, \dots, p_d) \in \mathbb{N}^d$, we can formally define

$$\eta_{\widehat{p}}(s) = \sum_{x \in X} \varphi_{\widehat{p}}(x) e^{-sw(x)}$$

where $\varphi_{\widehat{p}}(x) = \varphi_1^{p_1}(x) \cdots \varphi_d^{p_d}(x)$.

Both of these series converge to analytic functions providing the real part of s is sufficiently large and t is bounded.

Notation. Given $\widehat{p} = (p_1, \dots, p_d) \in \mathbb{N}^d$, we write $|\widehat{p}| = p_1 + \cdots + p_d$.

The first pole of $\eta(s,0)$ will appear on the real axis at $\lambda > 0$. The following gives the required properties of the complex function $\eta_{\widehat{p}}(s)$.

Proposition 4.5. *Given $\widehat{p} \in \mathbb{N}^d$ the function $\eta_{\widehat{p}}(s)$ is analytic in the plane $\Re(s) > 1$. Furthermore, $\eta_{\widehat{p}}(s)$ is analytic on $\Re(s) \geq 1$ apart from at $s = 1$. The nature of the singularity at 1 depends on the parity of $p = |\widehat{p}|$ as follows:*
Case 1: p is odd. *Then $\eta_{\widehat{p}}(s)$ has possibly finitely many integer order poles at $s = 1$ and is analytic otherwise. These poles have order at most $(p+1)/2$.*
Case 2: p is even. *In this case, there exists a positive definite, symmetric matrix $\Sigma = (\sigma_{i,j})_{i,j=1}^n$ such that the following holds. For s in a neighbourhood of 1, we can write*

$$\eta_{\widehat{p}}(s) = \frac{R_{\widehat{p}}(s)}{(s-1)^{1+\frac{p}{2}}}$$

where each $R_{\widehat{p}}(s)$ is analytic and

$$R_{\widehat{p}}(1) = C\left(\frac{p}{2}\right)! \sum_{i_1,\dots,i_p} \sigma_{l(i_1),l(i_2)}\sigma_{l(i_3),l(i_4)} \cdots \sigma_{l(i_{p-1}),l(i_p)}$$

where:

(1) the sum over $i_1, \dots, i_p$ is over the partition the numbers $1, \dots, p$ into disjoint pairs $(i_1, i_2), \dots, (i_{p-1}, i_p)$; and
(2) $l : \{1, \dots, p\} \to \{1, 2, \dots, p\}$ sends the set $\{1, \dots, p_1\}$ to 1, the set $\{p_1+1, \dots, p_1+p_2\}$ to 2 and continues in this way until finally sending $\{p_1 + \cdots + p_{d-1} + 1, \dots, p\}$ to d.

Proof of Proposition **??**. We first show that $\eta_{\widehat{p}}$ is analytic at $\mathfrak{Re}(s) \geq 1$ other than $s = 1$. To do so we notice that

$$\eta(s,t) = \mathcal{L}_{s,t}^{\sharp} \circ \sum_{n \geq 0} \mathcal{L}_{s,t}^n \mathbf{1}(x_0).$$

Using Lemma **??**, we see that $\eta(s,t)$ has the right domain of analyticity and we can differentiate to get what we need. Studying the pole is the harder bit. For (s,t) close to $(1,0)$ we can write

$$\mathcal{L}_{s,t}^n \mathbf{1}(x_0) = \lambda(s,t)^n Q(s,t) \mathbf{1}(x_0) + O(\theta^n)$$

where $0 < \theta < 1$ is independent of s, t. It follows that

$$\eta(s,t) = \frac{\lambda(s,t) Q(s,t) \mathbf{1}(x_0)}{1 - \lambda(s,t)} + R(s,t)$$

where $R(s,t)$ is multianalytic in a neighbourhood of $(1,0)$. Set $F(s,t) = \lambda(s,t)Q(s,t)\mathbf{1}(x_0)$ and note that $F(1,0) > 0$. We now want to study the partial derivatives

$$\frac{\partial^n}{\partial t_1^{n_1}\partial t_2^{n_2}\cdots\partial t_d^{n_d}}\bigg|_{(s,0)} \frac{F(s,t)}{1-\lambda(s,t)}$$

for $\widehat{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{N}$ with $|\widehat{n}| = n_1 + n_2 + \cdots + n_d = n$. Since we are only interested in the largest order poles, it suffices to study

$$\frac{\partial^n}{\partial t_1^{n_1}\partial t_2^{n_2}\cdots\partial t_d^{n_d}}\bigg|_{(s,0)} \frac{1}{1-\lambda(s,t)} \text{ where } n_1 + n_2 + \cdots + n_d = n.$$

Using Faà di Bruno's formula we have that these derivatives are given by

$$\sum_{\pi\in\Pi_n} \frac{|\pi|!}{(1-\lambda(s,0))^{|\pi|+1}} \prod_{B\in\pi} \lambda_B(s,0)$$

where

(1) Π_n is the set of partitions of $\{1,\ldots,n\}$;
(2) $|\pi|$ is the number of blocks (i.e., cyles) in π;
(3) in the product B runs over the blocks in π; and,
(4) $\lambda_B(s,0)$ is the partial derivative of λ over the block B: if $B = \{b_1,\ldots,b_k\} \subset \{1,\ldots,n\}$ then
$$\lambda_B(s,0,0) = \frac{\partial^n\lambda}{\partial t_1^{l_1}\partial t_2^{l_2}\cdots\partial t_d^{l_d}}(s,0)$$
where $l_1 = \#(B\cap\{1,\ldots,n_1\}), l_2 = \#(B\cap\{n_1+1,\ldots,n_1+n_2\}),\ldots,l_d = \#(B\cap\{n_1+\cdots+n_{d-1}+1,\ldots,n\}$.

Now notice that by Lemma **??** for any block B of length 1, $\lambda_B(1,0) = 0$. Hence, the Faà di Bruno formula shows that when we are searching for the poles of highest order we can ignore terms coming from partitions that contain blocks of a single number.

It therefore follows that when n is odd, the highest order pole coming from the derivative above has order at most $|\pi| + 1 = (n-3)/2 + 1 + 1 = (n+1)/2$. Indeed, one considers a cycle of order 3 and $(n-3)/2$ transpositions.

When n is even the highest order poles come from partitions of $\{1,\ldots,n\}$ into pairs. We deduce that, in this case, the highest order poles are of the form

$$\frac{(n/2)!}{(1-\lambda(s,0))^{n/2+1}} \sum_{\pi\in\Pi_n(2)} \prod_{B\in\pi} \lambda_B(s,0)$$

where $\Pi_n(2)$ represents all partitions of $\{1,\ldots,n\}$ into blocks of size 2. Using Lemma **??** we can then write

$$\frac{(n/2)!}{(1-\lambda(s,0))^{n/2}} \sum_{\pi\in\Pi_n(2)} \prod_{B\in\pi} \lambda_B(s,0) = \frac{(-\lambda_s(1,0))^{n/2}(n/2)!}{(1-\lambda(s,0))^{n/2+1}} \sum_{i_1,\ldots,i_n} \sigma_{l(i_1),l(i_2)}\cdots\sigma_{l(i_{n-1}),l(t_n)} + g(s)$$

where $g(s)$ has poles of integer orders strictly less that $n/2+1$ and $l : \{1,\ldots,p\} \to \{1,2,\ldots,n\}$ sends the set $\{1,\ldots,n_1\}$ to 1, the set $\{n_1+1,\ldots,n_1+n_2\}$ to 2 and continues in this way until finally sending $\{n_1+\cdots+n_{d-1}+1,\ldots,n\}$ to d (as in the statement of the proposition).

To conclude the proof we sum over $0 \le n_1 \le p_1, 0 \le n_2 \le p_2$. To see that when $|\widehat{p}|$ is odd, $\eta_{\widehat{p}}$ has the required pole structure and when $|\widehat{p}| = p$ is even

$$\eta_{\widehat{p}}(s) = \frac{F(s,0)(-\lambda_s(1,0))^{p/2}(p/2)!}{(1-\lambda(s,0))^{p/2+1}} \sum_{i_1,\ldots,i_p} \sigma_{l(i_1),l(i_2)}\cdots\sigma_{l(i_{p-1}),l(t_p)} + G(s)$$

where G is analytic other than integer order poles of order at most $p/2$. To conclude we note that

$$\frac{F(1,0)(-\lambda_s(1,0))^{p/2}(p/2)!}{(1-\lambda(s,0))^{p/2+1}} = \frac{F(1,0)(p/2)!}{(-\lambda_s(1,0))(s-1)^{p/2+1}} + H(s)$$

where $H(s)$ is analytic on $\Re(s) \geq 1$ except for finite integer order poles at $s = 1$ of order at most $p/2$. This concludes the proof with

$$C = -\frac{F(1,0)}{\lambda_s(1,0)} > 0 \text{ and the } \sigma_{i,k} \text{ as defined above.}$$

□

4.4. **Deducing the Central Limit Theorem.** In this section we will employ the estimates on the complex function described in the previous section to deduce the central limit theorem. The approach is to use the method of moments, following an approach inspired by Morris [?].

Definition 4.6. For each d-tuple $\widehat{p} = (p_1, \ldots, p_d) \in \mathbb{N}^d$, we define

$$\pi_{\widehat{p}}(Q) = \sum_{w(x)<Q} \varphi_{\widehat{p}}(x) \text{ where } \varphi_{\widehat{p}}(x) = \varphi_1^{p_1}(x) \cdots \varphi_d^{p_d}(x).$$

We can now use Proposition **??** in the proof of the following moment estimate.

Proposition 4.7. *When $|\widehat{p}| = p$ is even, we have that*

$$\lim_{Q\to\infty} \frac{1}{\#\{x \in X : w(x) < Q\}} \sum_{w(x)<Q} \left(\frac{\varphi_{\widehat{p}}(x)}{\sqrt{Q}}\right)^p = \sum_{i_1,\ldots,i_p} \sigma_{\pi(i_1),\pi(i_2)}\sigma_{\pi(i_3),\pi(i_4)} \cdots \sigma_{\pi(i_{p-1}),\pi(i_p)}.$$

Proof. When all of the p_i are even we can apply Proposition **??** to immediately deduce the result. When some of the p_i are odd we have to work harder. There are a further 2 sub-cases.

Sub-case 1*: $p/2$ is even.* When this is the case we define $G_1(s), G_2(s)$ and $G_3(s)$ by

$$\sum_{x\in X} (\varphi_{2\widehat{p}}(x) + w(x)^p)\, e^{-sw(x)}, \sum_{x\in X} \left(\varphi_{\widehat{p}}(x) + w(x)^{p/2}\right)^2 e^{-sw(x)} \text{ and } \sum_{x\in X} w(x)^{p/2}\varphi_{\widehat{p}}(x)e^{-sw(x)}$$

respectively. We note that $G_2 = G_1 + 2G_3$. Using Proposition **??** in combination with Proposition **??** we see that

$$\frac{1}{\#\{x \in X : w(x) < Q\}} \sum_{w(x)<Q} \varphi_{2\widehat{p}}(x) + w(x)^p \sim \frac{Q^p\,(R_{2\widehat{p}}(1) + R_{0,0}(1)(p!/(p/2)!))}{Cp!}$$

as $Q \to \infty$. We also have that (since $p/2$ is even)

$$\eta_{\widehat{p}}^{(p/2)}(s) = \sum_{x\in X} w(x)^{p/2}\varphi_{\widehat{p}}(x)e^{-sw(x)} = G_3(x).$$

Then using that $G_2 = G_1 + 2G_3$ we see that

$$G_2(s) = \frac{R_{2\widehat{p}}(1) + R_{0,0}(1)(p!/(p/2)!) + 2R_{\widehat{p}}(1)(p!/(p/2)!)}{(s-\lambda)^{1+p}} + f(s)$$

where $f(s)$ is analytic other than integer poles of order at most p. Therefore

$$\frac{1}{\#\{x \in X : w(x) < Q\}} \sum_{w(x)<Q} \left(\varphi_{\widehat{p}}(x) + w(x)^{p/2}\right)^2$$

grows asymptotically like

$$\frac{R_{2\widehat{p}}(1) + R_{0,0}(1)(p!/(p/2)!) + 2R_{p_1,p_2}(1)(p!/(p/2)!)}{Cp!} Q^p.$$

This implies that

$$\frac{1}{\#\{x \in X : w(x) < Q\}} \sum_{w(x)<Q} w(x)^{p/2}\varphi_{wp}(x) \sim \frac{R_{\widehat{p}}(1)Q^p}{C(p/2)!}.$$

We want the same expression but with $w(x)^{p/2}$ replaced with $Q^{p/2}$. We now remove this weighting term. Note that

$$\sum_{w(x)<Q} w(x)^{p/2}\varphi_{\widehat{p}}(x) = \int_0^Q t^{p/2}\ d\pi_{\widehat{p}}(t) = Q^{p/2}\pi_{\widehat{p}}(Q) - \frac{p}{2}\int_0^Q t^{p/2-1}\pi_{\widehat{p}}(t)\ dt.$$

Without loss of generality we can assume that $p_1,\dots,p_{2k}$ are odd and $p_{2k}+1,\dots,p_d$ are even for some $k \le \lfloor p/2 \rfloor$.

We now would like to use the elementary inequality for real numbers $x_1,\dots,x_{2k}$,

$$\sum_{B\subset\{1,\dots,2k\},|B|=k} x_B^2 \ge |x_1\cdots x_{2k}|$$

where for $B = \{b_1,\dots,b_k\} \subset \{1,\dots,2k\}$ we set $x_B = x_{b_1}\cdots x_{b_k}$.

Using this we see that

$$|\pi_{\widehat{p}}(t)| \le \sum_{B\subset\{1,\dots,2k\},|B|=k} \pi_{B(\widehat{p})}(t)$$

where $B(\widehat{p}) \in \mathbb{N}^d$ is the vector with entries p_i for $i \notin \{1,\dots,2k\}$, p_i+1 if $i \in B \cap \{1,\dots,2k\}$ and $p_i - 1$ if $i \in \{1,\dots,2k\}\backslash B$. Since all of the entries in $B(\widehat{p})$ are even and $|B(\widehat{p})| = |\widehat{p}| = p$ we can apply Proposition **??** to deduce that

$$\sum_{B\subset\{1,\dots,2k\},|B|=k} \pi_{B(\widehat{p})}(t) = O(t^{p/2}e^t)$$

as $t\to\infty$. It therefore follows that

$$\int_0^Q t^{p/2-1}\pi_{p_1,p_2}(t)\ dt = o\left(Q^{p/2}e^Q\right)$$

as $Q\to\infty$. This implies the required asymptotic in this sub-case.

Sub-case 2: $p/2$ is odd. In this case we define $H_1(s), H_2(s)$ and $H_3(s)$ by

$$\sum_{x\in X}\left(\varphi_{2\widehat{p}}(x) + w(x)^p\right)e^{-sw(x)},\ \sum_{x\in X}\left(\varphi_{\widehat{p}}(x) + w(x)^{p/2}\right)^2 e^{-sw(x)} \text{ and } \sum_{x\in X} -w(x)^{p/2}\varphi_{\widehat{p}}(x)e^{-sw(x)}$$

respectively. Similarly to before $H_2 = H_1 - 2H_3$. Following the same argument as before, we deduce that

$$\frac{1}{\#\{x\in X : w(x)<Q\}}\sum_{w(x)<Q} w(x)^{p/2}\varphi_{\widehat{p}}(x) \sim \frac{R_{p_1,p_2}(1)Q^p}{C(p/2)!}.$$

Again, we can use the Stiltjes integral as before to change this expression into the required asymptotic. $\square$

We now handle the odd sum case.

Proposition 4.8. *When $p = |\widehat{p}|$ is odd, we have that*

$$\lim_{Q\to\infty}\frac{1}{\#\{x\in X : w(x)<Q\}}\sum_{w(x)<Q}\left(\frac{\varphi_{\widehat{p}}(x)}{\sqrt{Q}}\right)^p = 0.$$

Proof. As before we define $K_1(s), K_2(s)$ and $K_3(s)$ by

$$\sum_{x\in X}\left(\varphi_{2\widehat{p}}(x) + w(x)^p\right)e^{-sw(x)},\ \sum_{x\in X}\left(\varphi_{\widehat{p}}(x) + w(x)^{p/2}\right)^2 e^{-sw(x)}, \text{ and } \sum_{x\in X} w(x)^{p/2}\varphi_{\widehat{p}}(x)(x)e^{-sw(x)}$$

respectively and note that $K_2 = K_1 + 2K_3$ and

$$K_3(s) = \eta_{2\widehat{p}}(s) - \eta_0^{(p)}(s) = \frac{g(s)}{(s-1)^{1+p}} + f(s)$$

where $g(1) > 0$ and f, g are analytic. By the Tauberian theorem we deduce that

$$\frac{1}{\#\{x\in X : w(x)<Q\}}\sum_{w(x)<Q}\varphi_{2\widehat{p}}(x) + w(x)^p \sim \frac{g(1)Q^p}{p!}.$$

We now calculate

$$\eta_{\widehat{p}}^{\left(\frac{p+1}{2}\right)}(s) = (-1)^{\left(\frac{p+1}{2}\right)} \sum_{w(x)<Q} w(x)^{\left(\frac{p+1}{2}\right)} \varphi_{\widehat{p}}(x) e^{-sw(x)}.$$

Now using the identity

$$\int_0^\infty t^{-1/2} e^{-tx}\ dt = \sqrt{\pi} x^{-1/2}$$

for $x > 0$ we see that

$$K_3(s) = \frac{(-1)^{\frac{p+1}{2}}}{\sqrt{\pi}} \int_0^\infty \frac{\eta_{\widehat{p}}^{\left(\frac{p+1}{2}\right)}(s+t)}{\sqrt{t}}\ dt$$

and hence K_3 is analytic except for a pole of order at most $p+1$ at $s = 1$. Now, we can write

$$\eta_{\widehat{p}}^{\left(\frac{p+1}{2}\right)}(s) = \sum_{i=1}^{p+1} \frac{a_i}{(s-1)^i} + h(s)$$

where $a_i \in \mathbb{C}$, h is analytic in $\Re(s) \geq 1$. Then using the identity

$$\int_0^\infty \frac{1}{(s+t-1)^i \sqrt{t}}\ dt = \frac{\pi(2i-2)!}{2^{2i-1}(i-1)!^2} \frac{1}{(s-1)^{k-\frac{1}{2}}}$$

we deduce that

$$K_3(x) = \sum_{i=0}^{p+1} \frac{c_i}{(s-1)^{i-\frac{1}{2}}} + l(s)$$

where $c_i \in \mathbb{C}$ and l is analytic in the half plane. Then using that $k_2 = K_1 + 2K_2$ we deduce that

$$K_2(x) = \sum_{i=0}^{p+1} \frac{a_i}{(s-1)^i} + \frac{b_i}{(s-1)^{i-\frac{1}{2}}} + r(s)$$

and $a_{p+1}(1) = g(1)$. The Tauberian theorem now implies that

$$\frac{1}{\#\{x \in X : w(x) < Q\}} \sum_{w(x)<Q} \left(\varphi_{\widehat{p}}(x) + w(x)^{p/2}\right)^2 \sim \frac{g(1)Q^p}{p}$$

and so

$$\frac{1}{\#\{x \in X : w(x) < Q\}} \sum_{w(x)<Q} w(x)^{p/2} \varphi_{\widehat{p}}(x) = o(Q^p).$$

To conclude the proof we need to remove the weighting term $w(x)^{p/2}$. To do so we set

$$\phi(t) = t^{-p/2} \quad \text{and} \quad \widetilde{\pi}(t) = \sum_{w(x)<t} w(x)^{p/2} w(x)^{p/2} \varphi_{\widehat{p}}(x)$$

for $t > 0$ and note that by the above $\widetilde{\pi}(t)$ is $O(t^p e^t)$. It follows that

$$\int_0^Q \widetilde{\pi}(t)\phi'(t)\ dt = O(Q \cdot T^p e^t Q^{-1-p/2}) = O(Q^{p/2} e^Q)$$

as $Q \to \infty$. However we also have that

$$\begin{aligned}
\int_1^Q \widetilde{\pi}(t)\phi'(t)\ dt &= \int_1^Q \sum_{w(x)<Q} \varphi_{\widehat{p}}(x) w(x)^{p/2} \phi'(t)\ dt + O(1) \\
&= \sum_{w(x)<Q} \varphi_{\widehat{p}}(x) w(x)^{p/2} \int_{w(x)}^Q \phi'(t) dt \\
&= Q^{-p/2} \widetilde{\pi}(Q) - \pi_{\widehat{p}}(Q) + O(1)
\end{aligned}$$

as $Q \to \infty$. Rearranging this and using our estimates above gives the required result. □

We are now ready to conclude the proofs of the results in the introduction.

proof of Theorem **??** *and* **??**. We conclude using the method of moments. Indeed, the limits obtained in Proposition **??** and Proposition **??** are the limits of the sequences of moments for the distributions considered in our main theorems. We have shown that these sequences converge to the moments of the Gaussian limit law with mean 0 and co-variances given by the $\sigma_{i,k}$. Along with Proposition **??** and Proposition **??** which characterise non-degeneracy, this concludes the proof using the method of moments. □

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