Spanning subgraphs and spectral radius in graphs

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Abstract

A spanning tree T of a connected graph G is a subgraph of G that is a tree covers all vertices of G. The leaf distance of T is defined as the minimum of distances between any two leaves of T. A fractional matching of a graph G is a function h assigning every edge a real number in $[0,1]$ so that $\sum\limits_{e\in E_G(v)} h(e) \leq 1$ for any $v \in V(G)$, where $E_G(v)$ denotes the set of edges incident with v in G. A fractional matching of G is called a fractional perfect matching if $\sum\limits_{e\in E_G(v)} h(e) = 1$ for any $v \in V(G)$. A graph G with at least $2k+2$ vertices is said to be fractional k-extendable if every k-matching M in G is included in a fractional perfect matching h of G such that $h(e) = 1$ for any $e \in M$. This paper considers a lower bound on the spectral radius of G to guarantee that G has a spanning tree with leaf distance at least d. At the same time, we obtain a lower bound on the spectral radius of G to ensure that G is fractional k-extendable.

Keywords: graph; spectral radius; spanning tree; fractional perfect matching; fractional k-extendable graph.

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1 Introduction

In this paper, all graphs considered are assumed to simple and undirected. Let $G = (V(G), E(G))$ denote a graph with vertex set $V(G)$ and edge set $E(G)$. The order and size of G is denoted by n and $e(G)$, respectively. That is, $n = |V(G)|$ and $e(G) = |E(G)|$. For $v \in V(G)$, the set of vertices adjacent to v in G is called the neighborhood of v and denoted by $N_G(v)$. We denote by $d_G(v) = |N_G(v)|$ the degree of v in G, and by $\delta(G)$ (or δ for short) the minimum degree of G. Let $\alpha(G)$ and $i(G)$ denote the independence number and the number of isolated vertices in G, respectively. For any $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S, and write $G - S = G[V(G) \setminus S]$. The complete graph of order n is denoted by K_n. Let c be a real number. Recall that $\lceil c \rceil$ is the smallest integer satisfying $\lceil c \rceil \geq c$.

Let G_1 and G_2 be two vertex-disjoint graphs. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$, which is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The join $G_1 \vee G_2$ is obtained from $G_1 \cup G_2$ by adding all the edges joining a vertex of G_1 to a vertex of G_2.

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Given a graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, let $A(G)$ denote the adjacency matrix of G. The (i, j)-entry of $A(G)$ is 1 if $v_iv_j \in E(G)$, and 0 otherwise. The eigenvalues of $A(G)$ are called the eigenvalues of G. It is obvious that $A(G)$ is a real symmetric nonnegative matrix. Consequently, its eigenvalues are real, which can be arranged in non-increasing order as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. Notice that the spectral radius of G, denoted by $\rho(G)$, is equal to $\lambda_1(G)$.

A spanning tree T of a connected graph G is a subgraph of G that is a tree covers all vertices of G. For $v \in V(T)$, the vertex v is called a leaf of T if $d_T(v) = 1$. The leaf degree of a vertex $v \in V(T)$ is defined as the number of leaves adjacent to v in T. The leaf degree of T is the maximum leaf degree among all the vertices of T. The leaf distance of T is defined as the minimum of distances between any two leaves of T. In fact, a tree with leaf degree 1 has leaf distance at least 3.

Kaneko [**?**] presented some sufficient conditions for a connected graph to have a spanning tree with leaf distance at least $d = 3$ and conjectured that similar conditions suffice for larger d. Later, Kaneko, Kano and Suzuki [**?**] claimed that Kaneko's conjecture is true for $d = 4$. For $d \geq 4$, Erbes, Molla, Mousley and Santana [**?**] showed that a stronger form of Kaneko's conjecture holds for all n-vertex connected graphs with $\alpha(G) \leq 5$, and proved Kaneko's conjecture for $d \geq \frac{n}{3}$. Zhou, Sun and Liu [**?**] provided two spectral conditions for a connected graph to contain a spanning tree with leaf distance at least $d = 3$. Chen, Lv, Li and Xu [**?**] investigated the existence of spanning trees with leaf distance at least $d = 4$ in connected graphs and obtained three new results. More results on spanning trees can be found in [**?**, **?**, **?**, **?**, **?**, **?**].

A set $M \subseteq E(G)$ is a matching if no two edges share a vertex. A matching of size k is called a k-matching. A matching M is called a perfect matching (or 1-factor) if it covers all the vertices of G. Let $k \geq 0$ be an integer. Then a graph G with at least $2k + 2$ vertices is said to be k-extendable if every k-matching in G can be extended to a perfect matching in G. A fractional matching of a graph G is a function h assigning every edge a real number in $[0, 1]$ so that $\sum\limits_{e \in E_G(v)} h(e) \leq 1$ for any $v \in V(G)$, where $E_G(v)$ denotes the set of edges incident with v in G. A fractional matching of G is called a fractional perfect matching if $\sum\limits_{e \in E_G(v)} h(e) = 1$ for any $v \in V(G)$. Then a graph G with at least $2k + 2$ vertices is said to be fractional k-extendable if every k-matching M in G is included in a fractional perfect matching h of G such that $h(e) = 1$ for any $e \in M$.

The perfect matching and matching extendability attracted much attention. Tutte [**?**] provided a characterization for a graph to contain a perfect matching. Enomoto [**?**] established a connection between toughness and a perfect matching in a graph. Niessen [**?**] presented a neighborhood union condition for a graph to have a perfect matching. O [**?**] obtained a spectral radius condition to guarantee that a connected graph has a perfect matching. Zhang and Lin [**?**] got a distance spectral condition to guarantee the existence of a perfect matching in a connected graph. Plummer [**?**] first introduced the concept of k-extendable graph and obtained some results on k-extendable graphs. Ananchuen and Caccetta [**?**], Lou and Yu [**?**], Cioaba, Koolen and Li [**?**], Robertshaw and Woodall [**?**] investigated the existence of k-extendable graphs. The fractional perfect matching and fractional matching extendability also attracted much attention. Lovász and Plummer [**?**] showed a characterization for the existence of fractional perfect matchings in graphs. Liu and Zhang [**?**] claimed a toughness condition for a graph to contain a fractional perfect matching in a graph. Ma and Liu [**?**] provided a characterization of fractional k-extendable graphs. Zhu and Liu [**?**] established a relationship between binding numbers and fractional k-extendable graphs. Much effort has been devoted to finding sufficient conditions for the existence of spanning subgraphs (see [?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?]).

Motivated by [?, ?, ?] directly, we are to establish a spectral radius condition for the existence of a spanning tree with leaf distance at least d in a connected graph, and propose a lower bound on the spectral radius of a connected graph G to guarantee that G is a fractional k-extendable graph. Our main results are shown in the following.

Theorem 1.1. Let G be a connected graph of order n with $\alpha(G) \leq 5$, and let d be an integer with $16 \leq d^2 \leq n$. If

$$\rho(G) \geq \rho(K_{\lceil \frac{d}{2} \rceil - 1} \vee (K_{n-\lceil \frac{d}{2} \rceil} \cup K_1)),$$

then G has a spanning tree with leaf distance at least d, unless $G = K_{\lceil \frac{d}{2} \rceil - 1} \vee (K_{n-\lceil \frac{d}{2} \rceil} \cup K_1)$.

Theorem 1.2. Let $k \geq 1$ be an integer, and let G be a connected graph of order n with minimum degree δ and $n \geq \max\{2k+9, 5\delta+1\}$. If

$$\rho(G) \geq \max\{\rho(K_{2k} \vee (K_{n-2k-1} \cup K_1)), \rho(K_\delta \vee (K_{n-2\delta+2k-1} \cup (\delta - 2k + 1)K_1))\},$$

then G is fractional k-extendable, unless $G \in \{K_{2k} \vee (K_{n-2k-1} \cup K_1), K_\delta \vee (K_{n-2\delta+2k-1} \cup (\delta - 2k + 1)K_1)\}$.

The proofs of Theorems 1.1 and 1.2 will be provided in Sections 3 and 4, respectively.

2 Preliminary lemmas

In this section, we put forward some necessary preliminary lemmas, which are very important to the proofs of our main results.

For $t \leq \alpha(G)$, let $\delta_t(G)$ be the minimum order of the neighborhood of an independent set of order t in a graph G. Namely, $\delta_t(G) = \min\{|N_G(I)| : I \text{ is an independent set of order } t\}$. Erbes, Molla, Mousley and Santana [?] proved the following two results.

Lemma 2.1 (Erbes, Molla, Mousley and Santana [?]). Let $d \geq 3$ be an integer, and let G be a connected graph. Then $i(G - S) < \frac{2}{d-2}|S|$ for all nonempty $S \subseteq V(G)$ if and only if $\delta_t(G) > \frac{t(d-2)}{2}$ for all t satisfying $1 \leq t \leq \alpha(G)$.

Lemma 2.2 (Erbes, Molla, Mousley and Santana [?]). Let d be an integer with $d \geq 4$, and let G be a connected graph of order n with $n > d$ and $\alpha(G) \leq 5$. If

$$\delta_{2t}(G) > t(d-2)$$

for all t satisfying $1 \leq t \leq \frac{\alpha(G)}{2}$, then G has a spanning tree with leaf distance at least d.

Ma and Liu [?] showed a characterization for a graph to be fractional k-extendable.

Lemma 2.3 ([?]). Let $k \geq 1$ be an integer, and let G be a graph with a k-matching. Then G is fractional k-extendable if and only if

$$i(G - S) \leq |S| - 2k$$

holds for any $S \subseteq V(G)$ such that $G[S]$ contains a k-matching.

Lemma 2.4 ([?]). Let G be a connected graph, and let H be a proper subgraph of G. Then $\rho(G) > \rho(H)$.

Lemma 2.5 (Hong [?]). Let G be a graph with n vertices. Then

$$\rho(G) \le \sqrt{2e(G) - n + 1},$$

where the equality holds if and only if G is a star or a complete graph.

Let M be a real symmetric matrix whose rows and columns are indexed by $V = \{1, 2, \cdots, n\}$. Suppose that M can be written as

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1s} \\ \vdots & \ddots & \vdots \\ M_{s1} & \cdots & M_{ss} \end{pmatrix}$$

in terms of partition $\pi : V = V_1 \cup V_2 \cup \cdots \cup V_s$, wherein M_{ij} is the submatrix (block) of M obtained by rows in V_i and columns in V_j. The average row sum of M_{ij} is denoted by q_{ij}. Then matrix $M_\pi = (q_{ij})$ is said to be the quotient matrix of M. If the row sum of every block M_{ij} is a constant, then the partition is equitable.

Lemma 2.6 ([?]). Let M be a real matrix with an equitable partition π, and let M_π be the corresponding quotient matrix. Then every eigenvalue of M_π is an eigenvalue of M. Furthermore, if M is nonnegative, then the largest eigenvalues of M and M_π are equal.

The subsequent lemma is the well-known Cauchy Interlacing Theorem.

Lemma 2.7 (Haemers [?]). Let M be a Hermitian matrix of order s, and let N be a principal submatrix of M with order t. If $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_s$ are the eigenvalues of M and $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_t$ are the eigenvalues of N, then $\lambda_i \ge \mu_i \ge \lambda_{s-t+i}$ for $1 \le i \le t$.

3 The proof of Theorem 1.1

In order to verify Theorem 1.1, we first prove the following lemma.

Lemma 3.1. Let d is an integer with $d \ge 3$, and let G be a connected graph of order n with $n \ge d^2$. If

$$\rho(G) \ge \rho(K_{\lceil \frac{d}{2} \rceil - 1} \vee (K_{n - \lceil \frac{d}{2} \rceil} \cup K_1)),$$

then $\delta_t(G) > \frac{t(d-2)}{2}$ for all t satisfying $1 \le t \le \alpha(G)$, unless $G = K_{\lceil \frac{d}{2} \rceil - 1} \vee (K_{n - \lceil \frac{d}{2} \rceil} \cup K_1)$.

Proof. Suppose that $\delta_t(G) \le \frac{t(d-2)}{2}$ for some t satisfying $1 \le t \le \alpha(G)$. According to Lemma 2.1, we conclude

$$\frac{(d-2) \cdot i(G-S)}{2} \ge |S|$$

for some nonempty $S \subseteq V(G)$. By the integrity of $|S|$, we see

$$\left\lceil \frac{(d-2) \cdot i(G-S)}{2} \right\rceil \ge |S|$$

for some nonempty $S \subseteq V(G)$. Let $|S| = s$ and $i(G-S) = q$. Then G is a spanning subgraph of $G_1 = K_{\lceil \frac{q(d-2)}{2} \rceil} \vee (K_{n_1} \cup qK_1)$, where $n_1 = n - \lceil \frac{q(d-2)}{2} \rceil - q$. In view of Lemma 2.4, we obtain

$$\rho(G) \leq \rho(K_{\lceil \frac{q(d-2)}{2} \rceil} \vee (K_{n-\lceil \frac{q(d-2)}{2} \rceil - q} \cup qK_1)), \tag{3.1}$$

where the equality holds if and only if $G = K_{\lceil \frac{q(d-2)}{2} \rceil} \vee (K_{n-\lceil \frac{q(d-2)}{2} \rceil - q} \cup qK_1)$.

If $q = 1$, then $G_1 = K_{\lceil \frac{d}{2} \rceil - 1} \vee (K_{n-\lceil \frac{d}{2} \rceil} \cup K_1)$. Using (**??**), we get

$$\rho(G) \leq \rho(K_{\lceil \frac{d}{2} \rceil - 1} \vee (K_{n-\lceil \frac{d}{2} \rceil} \cup K_1)),$$

with equality if and only if $G = K_{\lceil \frac{d}{2} \rceil - 1} \vee (K_{n-\lceil \frac{d}{2} \rceil} \cup K_1)$, a contradiction. In what follows, we consider $q \geq 2$.

Recall that $G_1 = K_{\lceil \frac{q(d-2)}{2} \rceil} \vee (K_{n-\lceil \frac{q(d-2)}{2} \rceil - q} \cup qK_1)$. By a direct computation, we have

$$\begin{aligned} e(G_1) =& \binom{n-q}{2} + q \left\lceil \frac{q(d-2)}{2} \right\rceil \\ =& \frac{(n-q)(n-q-1)}{2} + q \left\lceil \frac{q(d-2)}{2} \right\rceil. \end{aligned} \tag{3.2}$$

It follows from (**??**) and Lemma 2.5 that

$$\begin{aligned} \rho(G_1) \leq& \sqrt{2e(G_1) - n + 1} \\ =& \sqrt{(n-q)(n-q-1) + 2q \left\lceil \frac{q(d-2)}{2} \right\rceil - n + 1} \\ \leq& \sqrt{(n-q)(n-q-1) + 2q \cdot \frac{q(d-2)+1}{2} - n + 1} \\ =& \sqrt{(d-1)q^2 - (2n-2)q + n^2 - 2n + 1}. \end{aligned} \tag{3.3}$$

Let $\psi_1(q) = (d-1)q^2 - (2n-2)q + n^2 - 2n + 1$. Notice that $n \geq \lceil \frac{q(d-2)}{2} \rceil + q \geq \frac{q(d-2)}{2} + q = \frac{qd}{2}$. Then we obtain $2 \leq q \leq \frac{2n}{d}$. By a direct calculation, we have

$$\psi_1\left(\frac{2n}{d}\right) - \psi_1(2) = -\frac{4}{d^2}(n-d)(n-d^2) \leq 0$$

by $n \geq d^2$. Thus, we see that $\psi_1(q)$ attains its maximum value at $q = 2$ for $2 \leq q \leq \frac{2n}{d}$. Together with (**??**), $n \geq d^2$ and $d \geq 3$, we get

$$\begin{aligned} \rho(G_1) \leq& \sqrt{\psi_1(2)} \\ =& \sqrt{4(d-1) - 2(2n-2) + n^2 - 2n + 1} \\ =& \sqrt{(n-2)^2 - 2n + 4d - 3} \\ \leq& \sqrt{(n-2)^2 - 2d^2 + 4d - 3} \\ =& \sqrt{(n-2)^2 - 2(d-1)^2 - 1} \\ <& n - 2. \end{aligned} \tag{3.4}$$

Since K_{n-1} is a proper subgraph of $K_{\lceil \frac{d}{2} \rceil - 1} \vee (K_{n-\lceil \frac{d}{2} \rceil} \cup K_1)$, it follows from Lemma 2.4 that

$$\rho(K_{\lceil \frac{d}{2} \rceil - 1} \vee (K_{n-\lceil \frac{d}{2} \rceil} \cup K_1)) > \rho(K_{n-1}) = n - 2. \tag{3.5}$$

Using (**??**), (**??**) and (**??**), we conclude

$$\rho(G) \le \rho(G_1) < n-2 < \rho(K_{\lceil \frac{d}{2} \rceil -1} \vee (K_{n-\lceil \frac{d}{2} \rceil} \cup K_1)),$$

which contradicts $\rho(G) \ge \rho(K_{\lceil \frac{d}{2} \rceil -1} \vee (K_{n-\lceil \frac{d}{2} \rceil} \cup K_1))$. This completes the proof of Lemma 3.1. $\square$

Next, we prove Theorem 1.1.

Proof of Theorem 1.1. According to Lemma 3.1, we see

$$\delta_k(G) > \frac{k(d-2)}{2}$$

for all even k satisfying $2 \le k \le \alpha(G)$, unless $G = K_{\lceil \frac{d}{2} \rceil -1} \vee (K_{n-\lceil \frac{d}{2} \rceil} \cup K_1)$. Let $k = 2t$. Then we have

$$\delta_{2t}(G) > t(d-2)$$

for all t satisfying $1 \le t \le \frac{\alpha(G)}{2}$, unless $G = K_{\lceil \frac{d}{2} \rceil -1} \vee (K_{n-\lceil \frac{d}{2} \rceil} \cup K_1)$. Combining this with $n \ge d^2 \ge 16$, $\alpha(G) \le 5$ and Lemma 2.2, we see that G has a spanning tree with leaf distance at least d, unless $G = K_{\lceil \frac{d}{2} \rceil -1} \vee (K_{n-\lceil \frac{d}{2} \rceil} \cup K_1)$. Theorem 1.1 is proved. $\square$

4 The proof of Theorem 1.2

In this section, we prove Theorem 1.2.

Proof of Theorem 1.2. Suppose, to the contrary, that G is not fractional k-extendable. According to Lemma 2.3, there exists some nonempty subset S of $V(G)$ such that $|S| \ge 2k$ and $i(G-S) \ge |S|-2k+1$. Then G is a spanning subgraph of $G_1 = K_s \vee (K_{n_1} \cup (s-2k+1)K_1)$, where $|S| = s \ge 2k$ and $n_1 = n-2s+2k-1 \ge 0$. Using Lemma 2.4, we conclude

$$\rho(G) \le \rho(G_1), \tag{4.1}$$

with equality if and only if $G = G_1$. Notice that G has the minimum degree δ. Thus, we have $\delta(G_1) = s \ge \delta(G) = \delta$. Then we proceed by the following two cases.

Case 1. $\delta \le 2k$.

Obviously, $s \ge 2k \ge \delta$. Let $G_2 = K_{2k} \vee (K_{n-2k-1} \cup K_1)$. We are to prove that $\rho(G_1) \le \rho(G_2)$ with equality if and only if $G_1 = G_2$.

It is obvious that $G_1 = G_2$ if $s = 2k$, and so $\rho(G_1) = \rho(G_2)$. Next, we are to consider $s \ge 2k+1$.

In terms of the partition $V(G_1) = V(K_s) \cup V(K_{n-2s+2k-1}) \cup V((s-2k+1)K_1)$, the quotient matrix of $A(G_1)$ is equal to

$$B_1 = \begin{pmatrix} s-1 & n-2s+2k-1 & s-2k+1 \\ s & n-2s+2k-2 & 0 \\ s & 0 & 0 \end{pmatrix}.$$

Then the characteristic polynomial of the matrix B_1 is

$$\varphi_{B_1}(x) = x^3 + (s-2k+3-n)x^2 + (2ks-s^2-2k+2-n)x + s(s-2k+1)(n-2s+2k-2).$$

Since the partition $V(G_1) = V(K_s) \cup V(K_{n-2s+2k-1}) \cup V((s-2k+1)K_1)$ is equitable, it follows from Lemma 2.6 that $\rho(G_1)$ is the largest root of $\varphi_{B_1}(x) = 0$. Namely, $\varphi_{B_1}(\rho(G_1)) = 0$. Let $\gamma_1 = \rho(G_1) \geq \gamma_2 \geq \gamma_3$ be the three roots of $\varphi_{B_1}(x) = 0$ and $Q_1 = \mathrm{diag}(s, n-2s+2k-1, s-2k+1)$. One checks that

$$Q_1^{\frac{1}{2}} B_1 Q_1^{-\frac{1}{2}} = \begin{pmatrix} s-1 & s^{\frac{1}{2}}(n-2s+2k-1)^{\frac{1}{2}} & s^{\frac{1}{2}}(s-2k+1)^{\frac{1}{2}} \\ s^{\frac{1}{2}}(n-2s+2k-1)^{\frac{1}{2}} & n-2s+2k-2 & 0 \\ s^{\frac{1}{2}}(s-2k+1)^{\frac{1}{2}} & 0 & 0 \end{pmatrix}$$

is symmetric, and also contains

$$\begin{pmatrix} n-2s+2k-2 & 0 \\ 0 & 0 \end{pmatrix}$$

as its submatrix. Since $Q_1^{\frac{1}{2}} B_1 Q_1^{-\frac{1}{2}}$ and B_1 have the same eigenvalues, by Lemma 2.7, we conclude

$$\gamma_2 \leq n-2s+2k-2 < n-2. \tag{4.2}$$

Recall that $G_2 = K_{2k} \vee (K_{n-2k-1} \cup K_1)$. Then the quotient matrix of $A(G_2)$ by the partition $V(G_2) = V(K_{2k}) \cup V(K_{n-2k-1}) \cup V(K_1)$ is equal to

$$B_2 = \begin{pmatrix} 2k-1 & n-2k-1 & 1 \\ 2k & n-2k-2 & 0 \\ 2k & 0 & 0 \end{pmatrix},$$

whose characteristic polynomial is

$$\varphi_{B_2}(x) = x^3 + (3-n)x^2 + (2-2k-n)x + 2k(n-2k-2).$$

By virtue of Lemma 2.6, the largest root, say ρ_2, of $\varphi_{B_2}(x) = 0$ is equal to $\rho(G_2)$.

Note that K_{n-1} is a proper subgraph of $G_2 = K_{2k} \vee (K_{n-2k-1} \cup K_1)$, it follows from (**??**) and Lemma 2.4 that

$$\rho_2 = \rho(G_2) > \rho(K_{n-1}) = n-2 \geq \gamma_2. \tag{4.3}$$

Next, we prove $\varphi_{B_1}(\rho_2) = \varphi_{B_1}(\rho_2) - \varphi_{B_2}(\rho_2) > 0$. By a direct calculation, we get

$$\varphi_{B_1}(\rho_2) = \varphi_{B_1}(\rho_2) - \varphi_{B_2}(\rho_2) = (s-2k)f(\rho_2), \tag{4.4}$$

where $f(\rho_2) = \rho_2^2 - s\rho_2 + (s+1)n - 2s^2 + 2ks - 4s - 2k - 2$.

Claim 1. $f(\rho_2) > 0$ for $\rho_2 > n-2$.

Proof. Firstly, we consider $n = 2s - 2k + 1$. Together with $n \geq 2k+9$, we deduce $s \geq 2k+4$. Combining this with (**??**), we get

$$\frac{s}{2} < 2s - 2k - 1 = n - 2 < \rho_2,$$

and so

$$\begin{aligned} f(\rho_2) >& f(n-2) \\ =& n^2 - 3n - 2s^2 + 2ks - 2s - 2k + 2 \end{aligned}$$

$$
\begin{aligned}
&=(2s-2k+1)^2-3(2s-2k+1)-2s^2+2ks-2s-2k+2\\
&=2s^2-(6k+4)s+4k^2\\
&\geq 2(2k+4)^2-(6k+4)(2k+4)+4k^2\\
&=16>0.
\end{aligned}
$$

Now we consider $n \geq 2s-2k+2$. If $s \geq 2k+2$, then it follows from (**??**) that

$$\frac{s}{2} < 2s-2k \leq n-2 < \rho_2,$$

and so

$$
\begin{aligned}
f(\rho_2) >&f(n-2)\\
=&n^2-3n-2s^2+2ks-2s-2k+2\\
\geq&(2s-2k+2)^2-3(2s-2k+2)-2s^2+2ks-2s-2k+2\\
=&2s^2-6ks+4k^2-4k\\
\geq&2(2k+2)^2-6k(2k+2)+4k^2-4k\\
=&8>0.
\end{aligned}
$$

If $s = 2k+1$, then we deduce

$$\frac{s}{2} < 2s-2k \leq n-2 < \rho_2$$

by (**??**) and $n \geq 2s-2k+2$. Combining this with $n \geq 2k+9$, we conclude

$$
\begin{aligned}
f(\rho_2) >&f(n-2)\\
=&n^2-3n-2s^2+2ks-2s-2k+2\\
\geq&(2k+9)^2-3(2k+9)-2(2k+1)^2+2k(2k+1)-2(2k+1)-2k+2\\
=&8k+52>0.
\end{aligned}
$$

Claim 1 is proved. $\square$

According to (**??**), (**??**), $s \geq 2k+1$ and Claim 1, we obtain

$$\varphi_{B_1}(\rho_2) = (s-2k)f(\rho_2) > 0.$$

As $\gamma_2 \leq n-2 < \rho(G_2) = \rho_2$ (see (**??**)), we deduce

$$\rho(G_1) = \gamma_1 < \rho_2 = \rho(G_2).$$

From the above discussion, we have

$$\rho(G_1) \leq \rho(G_2), \tag{4.5}$$

with equality if and only if $G_1 = G_2$. Recall that $G_2 = K_{2k} \vee (K_{n-2k-1} \cup K_1)$. It follows from (**??**) and (**??**) that

$$\rho(G) \leq \rho(K_{2k} \vee (K_{n-2k-1} \cup K_1)),$$

with equality if and only if $G = K_{2k} \vee (K_{n-2k-1} \cup K_1)$, a contradiction.

Case 2. $\delta \geq 2k+1$.

Clearly, $s \geq \delta \geq 2k+1$. Recall that $G_1 = K_s \vee (K_{n-2s+2k-1} \cup (s-2k+1)K_1)$, the adjacency matrix $A(G_1)$ of G_1 has the quotient matrix B_1, and B_1 has the characteristic polynomial $\varphi_{B_1}(x)$. Let $G_3 = K_\delta \vee (K_{n-2\delta+2k-1} \cup (\delta-2k+1)K_1)$, where $n \geq 2\delta - 2k+1$. We are to verify $\rho(G_1) \leq \rho(G_3)$ with equality if and only if $G_1 = G_3$.

It is clear that $G_1 = G_3$ if $s = \delta$, and so $\rho(G_1) = \rho(G_3)$. In what follows, we are to consider $s \geq \delta + 1$.

For the graph G_3, its adjacency matrix $A(G_3)$ has the quotient matrix B_3 which is formed by replacing s with δ in B_1, and B_3 has the characteristic polynomial $\varphi_{B_3}(x)$ which is derived by replacing s with δ in $\varphi_{B_1}(x)$. Thus, we obtain

$$\varphi_{B_3}(x) = x^3 + (\delta - 2k + 3 - n)x^2 + (2k\delta - \delta^2 - 2k + 2 - n)x + \delta(\delta - 2k + 1)(n - 2\delta + 2k - 2).$$

In terms of Lemma 2.6, the largest root, say ρ_3, of $\varphi_{B_3}(x) = 0$ equals the spectral radius of G_3. That is, $\rho(G_3) = \rho_3$.

Notice that $\varphi_{B_3}(\rho_3) = 0$. By plugging the value ρ_3 into x of $\varphi_{B_1}(x) - \varphi_{B_3}(x)$, we have

$$\varphi_{B_1}(\rho_3) = \varphi_{B_1}(\rho_3) - \varphi_{B_3}(\rho_3) = (s - \delta)g(\rho_3), \tag{4.6}$$

where $g(\rho_3) = \rho_3^2 - (s+\delta-2k)\rho_3 - 2s^2 + ns + 6ks - 2\delta s - 4s - 2kn + \delta n + n - 2\delta^2 + 6k\delta - 4\delta - 4k^2 + 6k - 2$. Since $K_{n-\delta+k-1}$ is a proper subgraph of G_3, it follows from Lemma 2.4 that

$$\rho_3 = \rho(G_3) > \rho(K_{n-\delta+k-1}) = n - \delta + k - 2. \tag{4.7}$$

From (**??**), $s \geq \delta + 1$ and $n \geq 2s - 2k + 1$, we deduce

$$\frac{s + \delta - 2k}{2} < n - \delta + k - 2 < \rho_3, \tag{4.8}$$

which leads to

$$\begin{aligned} g(\rho_3) >& g(n - \delta + k - 2) \\ =& n^2 - (2\delta - 2k + 3)n - 2s^2 + (5k - \delta - 2)s + k\delta + 2\delta - k^2 - 2k + 2. \end{aligned} \tag{4.9}$$

We first consider $s \geq \frac{5}{2}\delta + 1$. In view of (**??**) and $n \geq 2s - 2k + 1$, we conclude

$$\begin{aligned} g(\rho_3) >& n^2 - (2\delta - 2k + 3)n - 2s^2 + (5k - \delta - 2)s + k\delta + 2\delta - k^2 - 2k + 2 \\ \geq& (2s - 2k + 1)^2 - (2\delta - 2k + 3)(2s - 2k + 1) \\ & - 2s^2 + (5k - \delta - 2)s + k\delta + 2\delta - k^2 - 2k + 2 \\ =& 2s^2 - (5\delta - k + 4)s + 5k\delta - k^2 + 2k. \end{aligned} \tag{4.10}$$

Let $h(s) = 2s^2 - (5\delta - k + 4)s + 5k\delta - k^2 + 2k$. Note that $\delta \geq 2k + 1$ and

$$\frac{5\delta - k + 4}{4} < \frac{5}{2}\delta + 1 \leq s,$$

which implies that

$$\begin{aligned} h(s) \geq& h\left(\frac{5}{2}\delta + 1\right) \\ =& \left(\frac{15}{2}k - 5\right)\delta - k^2 + 3k - 2 \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{5}{2}k\delta - k^2 + 3k - 2 \\
&\geq \frac{5}{2}k(2k+1) - k^2 + 3k - 2 \\
&= 4k^2 + \frac{11}{2}k - 2 \\
&> 0.
\end{aligned}
$$

Combining this with (**??**), (**??**) and $s \geq \frac{5}{2}\delta + 1$, we infer

$$\varphi_{B_1}(\rho_3) = (s-\delta)g(\rho_3) > (s-\delta)h(s) > 0. \tag{4.11}$$

According to (**??**), $s \geq \frac{5}{2}\delta + 1$ and $n \geq 2s - 2k + 1$, we have

$$
\begin{aligned}
\varphi'_{B_1}(\rho_3) =&(s-\delta)g'(\rho_3) \\
=&(s-\delta)(2\rho_3 - s - \delta + 2k) \\
>&(s-\delta)(2(n-\delta+k-2) - s - \delta + 2k) \\
=&(s-\delta)(2n - 3\delta - s + 4k - 4) \\
\geq&(s-\delta)(2(2s-2k+1) - 3\delta - s + 4k - 4) \\
=&(s-\delta)(3s - 3\delta - 2) \\
>&0.
\end{aligned} \tag{4.12}
$$

The inequalities (**??**) and (**??**) imply

$$\rho(G_1) = \gamma_1 < \rho_3 = \rho(G_3).$$

From the above discussion, we get

$$\rho(G_1) \leq \rho(G_3), \tag{4.13}$$

with equality if and only if $G_1 = G_3$. Recall that $G_3 = K_\delta \vee (K_{n-2\delta+2k-1} \cup (\delta - 2k + 1)K_1)$. By virtue of (**??**) and (**??**), we deduce

$$\rho(G) \leq \rho(K_\delta \vee (K_{n-2\delta+2k-1} \cup (\delta - 2k + 1)K_1)),$$

with equality if and only if $G = K_\delta \vee (K_{n-2\delta+2k-1} \cup (\delta - 2k + 1)K_1)$, a contradiction.

In what follows, we consider $\delta + 1 \leq s < \frac{5}{2}\delta + 1$. Notice that $\frac{5k-\delta-2}{4} < \delta + 1 \leq s < \frac{5}{2}\delta + 1$. According to (**??**), $\delta \geq 2k + 1$ and $n \geq 5\delta + 1$, we obtain

$$
\begin{aligned}
g(\rho_3) >&n^2 - (2\delta - 2k + 3)n - 2s^2 + (5k - \delta - 2)s + k\delta + 2\delta - k^2 - 2k + 2 \\
>&n^2 - (2\delta - 2k + 3)n - 2\left(\frac{5}{2}\delta + 1\right)^2 + (5k - \delta - 2)\left(\frac{5}{2}\delta + 1\right) \\
&+ k\delta + 2\delta - k^2 - 2k + 2 \\
=&n^2 - (2\delta - 2k + 3)n - 15\delta^2 + \frac{27}{2}k\delta - 14\delta - k^2 + 3k - 2 \\
\geq&(5\delta + 1)^2 - (2\delta - 2k + 3)(5\delta + 1) - 15\delta^2 + \frac{27}{2}k\delta - 14\delta - k^2 + 3k - 2
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{47}{2}k-21\right)\delta-k^2+5k-4\\
&\geq \left(\frac{47}{2}k-21\right)(2k+1)-k^2+5k-4\\
&=46k^2-\frac{27}{2}k-25\\
&>0. \qquad (4.14)
\end{aligned}
$$

According to (??), (??) and $\delta+1\leq s<\frac{5}{2}\delta+1$, we have

$$\varphi_{B_1}(\rho_3)=(s-\delta)g(\rho_3)>0. \qquad (4.15)$$

Using (??), $\delta+1\leq s<\frac{5}{2}\delta+1$ and $n\geq 5\delta+1$, we obtain

$$
\begin{aligned}
\varphi'_{B_1}(\rho_3) =&(s-\delta)g'(\rho_3)\\
=&(s-\delta)(2\rho_3-s-\delta+2k)\\
>&(s-\delta)(2(n-\delta+k-2)-s-\delta+2k)\\
=&(s-\delta)(2n-3\delta-s+4k-4)\\
\geq&(s-\delta)(2(5\delta+1)-3\delta-s+4k-4)\\
=&(s-\delta)(7\delta-s+4k-2)\\
>&0. \qquad (4.16)
\end{aligned}
$$

The inequalities (??) and (??) yield

$$\rho(G_1)=\gamma_1<\rho_3=\rho(G_3).$$

From the above discussion, we conclude

$$\rho(G_1)\leq\rho(G_3), \qquad (4.17)$$

with equality if and only if $G_1=G_3$. Recall that $G_3=K_\delta\vee(K_{n-2\delta+2k-1}\cup(\delta-2k+1)K_1)$. It follows from (??) and (??) that

$$\rho(G)\leq\rho(K_\delta\vee(K_{n-2\delta+2k-1}\cup(\delta-2k+1)K_1)),$$

with equality if and only if $G=K_\delta\vee(K_{n-2\delta+2k-1}\cup(\delta-2k+1)K_1)$, a contradiction. This completes the proof of Theorem 1.2. □

Data availability statement

My manuscript has no associated data.

Declaration of competing interest

The authors declare that they have no conflicts of interest to this work.

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