ON A GENERALIZATION OF MOTOHASHI'S FORMULA: NON-ARCHIMEDEAN WEIGHT FUNCTIONS

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Abstract. This is a continuation of the adelic version of Kwan's formula. At non-archimedean places we give a bound of the weight function on the mixed moment side, when the weight function on the $\mathrm{PGL}_3 \times \mathrm{PGL}_2$ side is nearly the characteristic function of a short family. Our method works for any tempered representation Π of PGL_3, and reveals the structural reason for the appearance of Katz's hypergeometric sums in a previous joint work with P.Xi.

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1. Introduction

1.1. **Main Results.** This paper is the first follow-up of our previous [?]. Let $\mathbf{F}$ be a number field with adele ring $\mathbb{A}$. Let ψ be the additive character of $\mathbf{F}\backslash\mathbb{A}$ à la Tate. Let $S_{\mathbf{F}}$ be the set of places of $\mathbf{F}$, and S_∞ be the subset of archimedean places. Fix a cuspidal automorphic representation Π of $\mathrm{PGL}_3(\mathbb{A})$. We established an adelic version of Kwan's spectral reciprocity identity (see [? , Theorem 1.1 & (5.19)-(5.21)]), which we recall in the special case of trivial central characters as follows.

Theorem 1.1. *Let $S_\infty \subset S \subset S_{\mathbf{F}}$ be any finite subset. At every place $v \in S$ there is a pair of weight functions h_v and $\widetilde{h}_v$ with the auxiliary* normalized *ones at finite places $\mathfrak{p} < \infty$*

$$H_{\mathfrak{p}}(\pi_{\mathfrak{p}}) := h_{\mathfrak{p}}(\pi_{\mathfrak{p}}) \frac{L(1, \pi_{\mathfrak{p}} \times \pi_{\mathfrak{p}})}{L(1/2, \Pi_{\mathfrak{p}} \times \pi_{\mathfrak{p}})}, \quad \widetilde{H}_{\mathfrak{p}}(\chi_{\mathfrak{p}}) := \frac{\widetilde{h}_{\mathfrak{p}}(\chi_{\mathfrak{p}})}{L(1/2, \Pi_{\mathfrak{p}} \times \chi_{\mathfrak{p}}^{-1}) L(1/2, \chi_{\mathfrak{p}})}, \tag{1.1}$$

so that the following equation holds (where π runs through cuspidal automorphic forms of $\mathrm{PGL}_2(\mathbb{A})$)

$$\sum_{\pi} \frac{L(1/2, \Pi \times \pi)}{2\Lambda_{\mathbf{F}}(2) L(1, \pi, \mathrm{Ad})} \cdot \prod_{v \mid \infty} h_v(\pi_v) \cdot \prod_{\mathfrak{p} \in S} H_{\mathfrak{p}}(\pi_{\mathfrak{p}}) +$$
$$\sum_{\chi \in \mathbb{R}_+ \widehat{\mathbf{F}^\times \backslash \mathbb{A}^\times}} \int_{-\infty}^{\infty} \frac{|L(1/2 + i\tau, \Pi \times \chi)|^2}{2\Lambda_{\mathbf{F}}(2) |L(1 + 2i\tau, \chi^2)|^2} \cdot \prod_{v \mid \infty} h_v(\pi(\chi_v, i\tau)) \cdot \prod_{\mathfrak{p} \in S} H_{\mathfrak{p}}(\pi(\chi_{\mathfrak{p}}, i\tau)) \frac{\mathrm{d}\tau}{2\pi}$$
$$= \frac{1}{\zeta_{\mathbf{F}}^*} \sum_{\chi \in \mathbb{R}_+ \widehat{\mathbf{F}^\times \backslash \mathbb{A}^\times}} \int_{-\infty}^{\infty} L(1/2 - i\tau, \Pi \times \chi^{-1}) L(1/2 + i\tau, \chi) \cdot \prod_{v \mid \infty} \widetilde{h}_v(\chi_v |\cdot|_v^{i\tau}) \cdot \prod_{\mathfrak{p} \in S} \widetilde{H}_{\mathfrak{p}}(\chi_{\mathfrak{p}} |\cdot|_{\mathfrak{p}}^{i\tau}) \frac{\mathrm{d}\tau}{2\pi} +$$
$$\frac{1}{\zeta_{\mathbf{F}}^*} \sum_{\pm} \mathrm{Res}_{s_1 = \pm\frac{1}{2}} L(1/2 - s_1, \Pi) \zeta_{\mathbf{F}}(1/2 + s_1) \cdot \prod_{v \mid \infty} \widetilde{h}_v(|\cdot|_v^{s_1}) \cdot \prod_{\mathfrak{p} \in S} \widetilde{H}_{\mathfrak{p}}(|\cdot|_{\mathfrak{p}}^{s_1}),$$

where we have used the abbreviation $\pi(\chi_v, s) := \pi(\chi_v |\cdot|_v^s, \chi_v^{-1} |\cdot|_v^{-s})$.

We only mention that the weight functions h_v and $\widetilde{h}_v$ are integrals of a smooth Whittaker function $W_v \in \mathcal{W}(\Pi_v^\infty, \psi_v)$. More details from [?] will be recalled when we need them.

In this paper we focus on the weight functions at a *non-archimedean* place $\mathfrak{p} < \infty$. We omit the subscript $\mathfrak{p}$ for simplicity. Let $\mathbf{F}$ be a non-archimedean local field of characteristic 0 with valuation ring

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$\mathcal{O}_{\mathbf{F}}$, and write $q = \mathrm{Nr}(\mathfrak{p})$ from now on. In the case $\chi = |\cdot|_{\mathbf{F}}^s$, the function $\widetilde{H}(|\cdot|_{\mathbf{F}}^s)$ is a polynomial in $\mathbb{C}[q^s, q^{-s}]$, hence is entire. We introduce its Taylor expansion at any point s_0 as

$$\widetilde{H}(|\cdot|_{\mathbf{F}}^s) = \sum\nolimits_{k \geqslant 0} \widetilde{H}(k; s_0)\, (s - s_0)^k \,. \tag{1.2}$$

We summarize our main results (namely Proposition **??**, **??**, **??** & **??**) as follows.

Theorem 1.2. *Suppose the residual characteristic of* $\mathbf{F}$ *is not* 2. *Let* $\Pi \in \widehat{\mathrm{PGL}_3(\mathbf{F})}$ *be generic and tempered. Let* $\pi_0 \in \widehat{\mathrm{PGL}_2(\mathbf{F})}$ *with* $\mathfrak{c}(\pi_0) > 1$. *We can choose* $W \in \mathcal{W}(\Pi^{\infty}, \psi)$ *so that:*

(1) *The weight function satisfies* $h(\pi) \geqslant 0$ *for any* $\pi \in \widehat{\mathrm{PGL}_2(\mathbf{F})}$ *and* $h(\pi_0) > 0$;

(2) *For any unitary* $\chi \in \widehat{\mathbf{F}^{\times}}$ *and* $\epsilon > 0$ *the dual weight function satisfies*

$$h(\pi_0)^{-1} \widetilde{h}(\chi) \ll_{\epsilon} \mathbf{C}(\Pi)^{2+\epsilon} \cdot \left\{ \mathbb{1}_{\leqslant \max(\lfloor \frac{\mathfrak{c}(\pi_0)}{2} \rfloor, 6\mathfrak{c}(\Pi))}(\mathfrak{c}(\chi)) + q^{\frac{1}{2}} \mathbb{1}_{(-1)^{\frac{q-1}{2}} = \varepsilon_0} \mathbb{1}_{2 \nmid \mathfrak{c}(\chi) = \frac{\mathfrak{c}(\pi_0)}{2} \geqslant 3} \mathbb{1}_{\boldsymbol{\mathcal{E}}(\pi_0)}(\chi^2) \right\},$$

where $\varepsilon_0 = -1$ *if* π_0 *is dihedral supercuspidal, and* $\varepsilon_0 = 1$ *otherwise; and the exceptional set of characters* $\boldsymbol{\mathcal{E}}(\pi_0)$ *of* $\mathcal{O}_{\mathbf{F}}^{\times}$ *has size* $O(q^{-1}\mathbf{C}(\pi_0)^{\frac{1}{2}})$ *and is given below in Lemma* **??** *(2);*

(3) *For any* $k \in \mathbb{Z}_{\geqslant 0}$ *and* $\epsilon > 0$ *the normalized unramified dual weight function satisfies the bounds*

$$h(\pi_0)^{-1} \widetilde{H}(k; \pm 1/2) \ll_{\epsilon} \mathbf{C}(\Pi)^{4+\epsilon} q^{\lceil \frac{\mathfrak{c}(\pi_0)}{2} \rceil + \epsilon}.$$

Remark 1.3. *Our primary goal is to* reveal the structural reason *for the bounds of the dual weight functions. Our main discovery is the* quadratic elementary functions *given in* (**??**)*, which are "building blocks" of the Bessel functions of the relevant representations. See §***??** *for more details. Further extension of our method to the non-dihedral supercuspidal and Steinberg* π_0 *requires only plugging in the integral representation of the Bessel functions of such* π_0*, analogous to [***?** *, Theorem 1.6].*

Remark 1.4. *Specializing to* $\mathbf{F} = \mathbb{Q}_p$ *and* $\Pi = \mathbb{1} \boxplus \mathbb{1} \boxplus \mathbb{1}$*, Theorem* **??** *corresponds to the main local non-archimedean computation of the recent work of Hu–Petrow–Young [***?** *] in the case* $p \neq 2$.

1.2. **Notation and Convention.** For a locally compact group G, let $\widehat{G}$ be the topological dual of unitary irreducible representations. For $\pi \in \widehat{G}$, we write V_{π} for the underlying Hilbert space, and write $V_{\pi}^{\infty} \subset V_{\pi}$ for the subspace of smooth vectors if G carries extra structure to make sense of the notion.

Throughout the paper $\mathbf{F}$ is a local field of characteristic 0, with residual characteristic $\neq 2$. Let $|\cdot|_{\mathbf{F}}$ (resp. $v_{\mathbf{F}}$) be the valuation (resp. normalized additive valuation) of $\mathbf{F}$. Fix ψ an additive character of conductor exponent 0 and normalize the measures accordingly. The valuation ring of $\mathbf{F}$ is $\mathcal{O}_{\mathbf{F}}$, while the valuation ideal is $\mathcal{P}_{\mathbf{F}}$. We choose a uniformizer $\varpi_{\mathbf{F}} \in \mathcal{P}_{\mathbf{F}} - \mathcal{P}_{\mathbf{F}}^2$. Different choices of $\varpi_{\mathbf{F}}$ give different *ramified* quadratic extensions of $\mathbf{F}$. Write $\mathbf{G}_d := \mathrm{GL}_d$ for simplicity, introduce some compact open subgroups of $\mathbf{G}_2(\mathbf{F})$ as

$$\mathbf{K} := \mathrm{GL}_2(\mathcal{O}_{\mathbf{F}}); \quad \mathbf{K}_0[\mathcal{P}_{\mathbf{F}}^n] := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{K} \;\middle|\; c \in \mathcal{P}_{\mathbf{F}}^n \right\}, \ \forall n \in \mathbb{Z}_{\geqslant 0};$$

and some algebraic subgroups of $\mathbf{G}_2(\mathbf{F})$ as

$$\mathbf{Z} = \mathbf{Z}_2(\mathbf{F}) := \left\{ z\mathbb{1}_2 \;\middle|\; z \in \mathbf{F}^{\times} \right\}, \quad \mathbf{N}_2(\mathbf{F}) := \left\{ n(x) := \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \;\middle|\; x \in \mathbf{F} \right\},$$

$$\mathbf{A}_2(\mathbf{F}) := \left\{ \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \;\middle|\; t_1, t_2 \in \mathbf{F}^{\times} \right\}, \quad \mathbf{B}_2(\mathbf{F}) := \mathbf{A}_2(\mathbf{F})\mathbf{N}_2(\mathbf{F}).$$

For integers $n, m \geqslant 1$ we write $\mathcal{S}(n \times m, \mathbf{F})$ for the space of Schwartz–Bruhat functions on $\mathrm{M}(n \times m, \mathbf{F})$ the $n \times m$ matrices with entries in $\mathbf{F}$. The (inverse) ψ-Fourier transform is denoted and defined by

$$\widehat{\Psi}(X) = \mathfrak{F}_{\psi}(\Psi)(-X) = \int_{\mathrm{M}(n \times m, \mathbf{F})} \Psi(Y) \psi\left(\mathrm{Tr}(XY^T)\right) \mathrm{d}Y.$$

If no confusion occurs, we may omit ψ from the notation. If this is the case, then the inverse Fourier transform is denoted by $\overline{\mathfrak{F}} = \mathfrak{F}_{\overline{\psi}} = \mathfrak{F}_{\psi^{-1}}$. We introduce some elementary operators on the space of functions on $\mathbf{F}^{\times}$:

- For functions ϕ on $\mathbf{F}^\times$, its *extension* by 0 to $\mathbf{F}$ is denoted by $\mathrm{e}(\phi)$, and its *inverse* is $\mathrm{Inv}(\phi)(t) := \phi(t^{-1})$; for functions ϕ on $\mathbf{F}$, its *restriction* to $\mathbf{F}^\times$ is denoted by $\mathrm{r}(\phi)$, and the operator i is
$$\mathrm{i} = \mathrm{e} \circ \mathrm{Inv} \circ \mathrm{r}.$$
- For $s \in \mathbb{C}$, $\mu \in \widehat{\mathbf{F}^\times}$ and functions ϕ on $\mathbf{F}$, we introduce the operator $\mathfrak{m}_s(\mu)$ by
$$\mathfrak{m}_s(\mu)(\phi)(t) = \phi(t)\mu(t)|t|_{\mathbf{F}}^s.$$
- For $\delta \in \mathbf{F}^\times$ we introduce the operator $\mathfrak{t}(\delta)$ by
$$\mathfrak{t}(\delta)(\phi)(y) = \phi(y\delta).$$
- Let $I_n = I_{n,\mathbf{F}} : \mathrm{C}(\mathbf{F}^\times) \to \mathrm{C}(\mathrm{GL}_n(\mathbf{F}))$ be given by $I_n(h)(g) := h(\det g)$.

These notation apply to finite (field) extensions of $\mathbf{F}$.

We introduce the standard involution of *inverse-transpose* on $\mathrm{GL}_n(R)$ as $g^\iota := {}^t g^{-1}$.

If $(U, \mathrm{d}u)$ is a measured space with finite total mass, we introduce the *normalized integral* as

$$\fint_U f(u)\mathrm{d}u := \frac{1}{\mathrm{Vol}(U, \mathrm{d}u)} \int_U f(u)\mathrm{d}u. \tag{1.3}$$

For $n \in \mathbb{Z}_{\geqslant 1}$ we will frequently perform the following *process of regularization* to an integral

$$\int_{\mathbf{F}} f(y)\mathrm{d}y = \int_{\mathbf{F}} \left(\fint_{\mathcal{O}_{\mathbf{F}}} f(y(1+\varpi_{\mathbf{F}}^n x))\mathrm{d}x \right) \mathrm{d}y, \tag{1.4}$$

which we shall refer to as the *level n regularization (with respect) to* $\mathrm{d}y$.

1.3. **Outline of Proof.** From the local main result [**?** , Theorem 1.4] the weight $h(\pi)$ and the dual weight $\widetilde{h}(\chi)$ functions are related to each other by a (hidden) relative orbital integral $H(y)$ via

$$h(\pi) = \int_{\mathbf{F}^\times} H(y) \cdot \mathrm{j}_{\widetilde{\pi},\psi^{-1}} \begin{pmatrix} & -y \\ 1 & \end{pmatrix} \frac{\mathrm{d}^\times y}{|y|_{\mathbf{F}}}, \quad \widetilde{h}(\chi) = \int_{\mathbf{F}^\times} \psi(-y)\chi^{-1}(y)|y|_{\mathbf{F}}^{-\frac{1}{2}} \cdot \widetilde{\mathcal{V}}_\Pi(H)(y)\mathrm{d}^\times y \tag{1.5}$$

where $\mathrm{j}_{\widetilde{\pi},\psi^{-1}}$ is the Bessel function of the contragredient representation $\widetilde{\pi}$ in the sense of [**?** , §3.5], and $\widetilde{\mathcal{V}}_\Pi$ is the extended Voronoi transform characterized by the equations

$$\widetilde{\mathcal{VH}}_\Pi := \widetilde{\mathcal{V}}_\Pi \circ \mathfrak{m}_1, \quad \mathfrak{m}_{-2}(\beta) \circ I_3 \circ \widetilde{\mathcal{VH}}_\Pi = \overline{\mathfrak{F}} \circ \mathfrak{m}_{-1}(\beta^\iota) \circ I_3$$

for all smooth matrix coefficients β of Π.

We need to choose $H(y)$ so that $h(\pi) \geqslant 0$ selects a *short family* containing π_0. In other words the weight function $h(\pi)$ should be an approximation of the characteristic functions of $\{\pi_0\}$. From the first formula of (**??**) and the orthogonality of Bessel functions, one should expect that any such $H(y)$ is an approximation of $\mathrm{j}_{\pi_0,\psi} \begin{pmatrix} & -y \\ 1 & \end{pmatrix}$. If π_0 is supercuspidal, the asymptotic analysis of both functions (in §**??** and Lemma **??**) show that the two functions can be *equal*. In general, we construct $H(y)$ from some test function $\phi_0^\iota * \phi_0$ on $\mathrm{PGL}_2(\mathbf{F})$ of positive type in order to ensure $h(\pi) \geqslant 0$; we also require ϕ_0 to be *left covariant* with respect to some character of an open compact subgroup, which determines the (minimal **K**-)*type* of π_0 in the sense of [**?**]. We use ϕ_0 to deduce some integral representation (**??**) of $H(y)$ which approximates the one of $\mathrm{j}_{\pi_0,\psi} \begin{pmatrix} & -y \\ 1 & \end{pmatrix}$ obtained in [**?** , Theorem 1.6]. The integral representation of $H(y)$ in all cases is summarized in Corollary **??**, and is the departure point of our *refined* analysis of the dual weight function $\widetilde{h}(\chi)$ described below.

Our key observation on the dual weight function is the following decomposition into three parts. The first part is the contribution from $H_\infty(y)$, essentially the restriction of $H(y)$ to $v_{\mathbf{F}}(y) \leqslant -\mathfrak{c}(\pi_0)$ (see (**??**)). In this region $H_\infty(y)$ has a *stable* behavior regardless π_0, i.e., the *germ* of any Bessel relative orbital integral at infinity. To treat the corresponding $\widetilde{h}_\infty(\chi)$ we crucially rely on the extension of the

Voronoi–Hankel transform developed in our previous paper [**?** , Theorem 1.3]. The remaining part $\widetilde{h}_c(\chi)$ corresponding to $H_c := H - H_\infty$ can be written as $\widetilde{h}_c(\chi) = \widetilde{h}_c^+(\chi) + \widetilde{h}_c^-(\chi)$, where

$$\widetilde{h}_c^+(\chi) = \int_{\mathcal{O}_{\mathbf{F}}} \chi^{-1}(y)|y|_{\mathbf{F}}^{-\frac{1}{2}} \cdot \widetilde{\mathcal{V}}_\Pi(H)(y)\mathrm{d}^\times y, \quad \widetilde{h}_c^-(\chi) = \int_{\mathbf{F}-\mathcal{O}_{\mathbf{F}}} \psi(-y)\chi^{-1}(y)|y|_{\mathbf{F}}^{-\frac{1}{2}} \cdot \widetilde{\mathcal{V}}_\Pi(H)(y)\mathrm{d}^\times y.$$

The bound of $\widetilde{h}_c^+$ is offered by the local functional equations via Lemma **??**. The method via the local functional equations can also offer a crude bound in Lemma **??** of $\widetilde{h}_c^-$ when $\mathfrak{c}(\pi_0) \ll \mathfrak{c}(\Pi)$.

To *refine* the bound of $\widetilde{h}_c^-$ in the case $\mathfrak{c}(\pi_0) \gg \mathfrak{c}(\Pi)$ we observe that $H_c(y)$ is a linear combination of translations of the following *quadratic elementary functions* (see Definition **??** & **??**, (**??**)-(**??**) & (**??**)-(**??**) for more details)

$$F_n, G_n(y^2) = \mathbb{1}_{v(y)=-n} \cdot \sum\nolimits_{\pm} \eta_{\mathbf{L}/\mathbf{F}}(\pm y)\psi(\pm y), \tag{1.6}$$

where $\mathbf{L}/\mathbf{F}$ is the quadratic algebra extension associated with the type of π_0, and $\eta_{\mathbf{L}/\mathbf{F}}$ is the corresponding quadratic character of $\mathbf{F}^\times$. We essentially change the order of integrations by first computing the dual weight of (the translates of) the quadratic elementary functions. In particular, the *translation pattern* of the above quadratic elementary functions is responsible for the appearance of Katz's hypergeometric sums in our previous joint work with Xi [**?**].

Remark 1.5. *Xi [**?**] has yet another transformation of the algebraic exponential sums in Petrow–Young's work [**?**]. This is mysterious and seems to lie beyond the framework of* GL_2 *or* GL_3*.*

Remark 1.6. *In the case of principal series π_0, our ϕ_0 coincides with Nelson's test function in [**?**]. But we do not view it within the theory of micro-localized vectors, nor does our method of bounding the dual weight function rely on anything in that theory. Our choice of test function follows the idea of a previous work [**?**] in the real case by analogy in terms of "minimal* $\mathbf{K}$*-type" [**?**].*

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2. Local Weight Transforms Revisited

We have expressed the local weight transforms in terms of the extended Voronoi transforms in [**?** , Theorem 1.4]. In that version, a test function $H(y)$ is some integral of a Kirillov function of a generic *unitary* irreducible Π (namely the restriction of a $W \in \mathcal{W}(\Pi^\infty, \psi)$ to the left-upper embedding of $\mathbf{G}_2(\mathbf{F})$). As explained in [**?** , (6.16)], the space of test functions $H(y)$ contains the Bessel orbital integrals of $\mathrm{C}_c^\infty(\mathbf{G}_2(\mathbf{F}))$. We shall restrict to the latter subspace of test functions and get more refined information on the local weight transforms in the case of trivial central characters.

2.1. **Relative Orbital Integrals.**

For any $m \in \mathbb{Z}_{\geqslant 0}$ we introduce a function $E_m \in \mathrm{C}_c^\infty(\mathbf{F}^\times)$ supported in the subset of *square* elements of $\mathbf{F}^\times$ given by

$$E_m(y^2) := \mathbb{1}_{v_{\mathbf{F}}(y)=-m} \cdot |y|_{\mathbf{F}} \sum_{\pm} \int_{\pm 1 + \mathcal{P}_{\mathbf{F}}^{\lfloor \frac{m}{2} \rfloor}} \psi\left(y(u+u^{-1})\right) \mathrm{d}u. \tag{2.1}$$

Recall that for any $f \in \mathrm{C}_c^\infty(\mathrm{GL}_2(\mathbf{F}))$, the Bessel orbital integral [**?** , (5.17)] is given by

$$h(y) = \int_{\mathbf{F}^\times} \int_{\mathbf{F}^2} f\left(\begin{pmatrix} 1 & x_1 \\ & 1 \end{pmatrix} \begin{pmatrix} & -y \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & x_2 \\ & 1 \end{pmatrix} \begin{pmatrix} z & \\ & z \end{pmatrix} \right) \psi(-x_1 - x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}^\times z. \tag{2.2}$$

Proposition 2.1. *As f traverses $\mathrm{C}_c^\infty(\mathbf{G}_2(\mathbf{F}))$, the Bessel orbital integrals h traverses*

$$\mathrm{C}_c^\infty(\mathbf{F}^\times) \bigoplus \mathbb{C}E_{\geqslant n}$$

where $n \in \mathbb{Z}_{\geqslant 0}$ can be chosen arbitrarily and we have written

$$E_{\geqslant n}(y^2) := \sum\nolimits_{m=n}^{\infty} E_m(y^2) = \mathbb{1}_{v_{\mathbf{F}}(y) \leqslant -n} \cdot |y|_{\mathbf{F}} \sum\nolimits_{\pm} \int_{\pm 1 + \mathcal{P}_{\mathbf{F}}^{\lfloor -\frac{v_{\mathbf{F}}(y)}{2} \rfloor}} \psi\left(y(u+u^{-1})\right) \mathrm{d}u. \tag{2.3}$$

Proof. If f traverses $\mathrm{C}_c^{\infty}(\mathbf{B}_2(\mathbf{F})w\mathbf{N}_2(\mathbf{F}))$, then clearly h traverses $\mathrm{C}_c^{\infty}(\mathbf{F}^{\times})$. Consider a function $f \in \mathrm{C}_c^{\infty}(\mathbf{B}_2(\mathbf{F})\mathbf{N}_2(\mathbf{F})^t)$ and let $\phi \in \mathrm{C}_c^{\infty}(\mathbf{F}^{\times} \times \mathbf{F})$ be defined by

$$\phi(y,x) := \int_{\mathbf{F}^{\times}} \int_{\mathbf{F}^2} f\left(\begin{pmatrix} 1 & x_1 \\ & 1 \end{pmatrix} \begin{pmatrix} y & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \begin{pmatrix} z & \\ & z \end{pmatrix} \right) \psi(-x_1) \mathrm{d}x_1 \mathrm{d}^{\times} z.$$

From the equation of matrices

$$\begin{pmatrix} & -y \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & x_2 \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & -y/x_2 \\ & 1 \end{pmatrix} \begin{pmatrix} y/x_2 & \\ & x_2 \end{pmatrix} \begin{pmatrix} 1 & \\ 1/x_2 & 1 \end{pmatrix}$$

we easily deduce that the relative orbital integral (**??**) is given by

$$h(y) = \int_{\mathbf{F}} \phi\left(\frac{y}{x_2^2}, \frac{1}{x_2} \right) \psi\left(-\frac{y}{x_2} - x_2 \right) \mathrm{d}x_2 = \int_{\mathbf{F}} \phi(yu^2, u) \psi(-yu - u^{-1}) |u|_{\mathbf{F}}^{-2} \mathrm{d}u.$$

Let $\tau \in \{1, \varepsilon\}$, we obtain by an obvious change of variables

$$h(\tau y^2) = |y|_{\mathbf{F}} \int_{\mathbf{F}} \phi(\tau u^2, y^{-1}u) \psi(-y(\tau u + u^{-1})) |u|_{\mathbf{F}}^{-2} \mathrm{d}u.$$

Choose $i_0 \in \mathbb{Z}_{\geqslant 1}$ and $k_0, m_0 \in \mathbb{Z}$ such that:

- $\phi(y(1+\delta_1), x(1+\delta_2)) = \phi(y,x), \quad \forall\ \delta_1, \delta_2 \in \mathfrak{p}^{i_0}$;
- for any $y \in \mathfrak{p}^{2+2k_0}$ and any $x \in \mathbf{F}$, we have $\phi(y,x) = 0$;
- for any $x \in \mathfrak{p}^{m_0}$ and any $y \in \mathbf{F}^{\times}$, we have $\phi(y,x) = \phi(y,0)$.

Let $v_{\mathbf{F}}(y) = -m$ for some $m \geqslant \max(m_0, 2i_0 + k_0, 2i_0)$, and take any $n \in \mathbb{Z}_{\geqslant 1}$ satisfying

$$2n \geqslant m + k_0, \quad n \geqslant i_0, \quad m - n \geqslant i_0.$$

Performing the level n regularization to $\mathrm{d}y$ we get

$$\begin{aligned} h(\tau y^2) &= |y|_{\mathbf{F}} \int_{\mathbf{F} - \mathcal{P}_{\mathbf{F}}^{1+k_0}} \phi(\tau u^2, y^{-1}u) \left[\oint_{\mathcal{O}_{\mathbf{F}}} \psi\left(-y\left(\tau u (1 + \varpi_{\mathbf{F}}^n x) + u^{-1}(1 + \varpi_{\mathbf{F}}^n x)^{-1} \right) \right) \mathrm{d}x \right] |u|_{\mathbf{F}}^{-2} \mathrm{d}u \\ &= |y|_{\mathbf{F}} \int_{\mathbf{F} - \mathcal{P}_{\mathbf{F}}^{1+k_0}} \phi(\tau u^2, y^{-1}u) \psi(-y(\tau u + u^{-1})) \left[\oint_{\mathcal{O}_{\mathbf{F}}} \psi\left(-y\varpi_{\mathbf{F}}^n (\tau u - u^{-1}) x \right) \mathrm{d}x \right] |u|_{\mathbf{F}}^{-2} \mathrm{d}u. \end{aligned}$$

The non-vanishing of the inner integral implies

$$v_{\mathbf{F}}(\tau u - u^{-1}) \geqslant m - n \geqslant i_0,$$

which can be satisfied only if τ is a square modulo $\mathcal{P}_{\mathbf{F}}$, i.e., $\tau = 1$. Moreover, we have $v_{\mathbf{F}}(u - u^{-1}) \geqslant i_0 \Leftrightarrow u \in \pm 1 + \mathcal{P}_{\mathbf{F}}^{i_0}$. We therefore get $h(\varepsilon y^2) = 0$ and

$$\begin{aligned} h(y^2) &= \mathbb{1}_{k_0 \geqslant 0} \phi(1,0) \cdot |y|_{\mathbf{F}} \sum_{\pm} \int_{\pm 1 + \mathcal{P}_{\mathbf{F}}^{i_0}} \psi(-y(u + u^{-1})) \mathrm{d}u \\ &= \mathbb{1}_{k_0 \geqslant 0} \phi(1,0) \cdot |y|_{\mathbf{F}} \sum_{\pm} \int_{\pm 1 + \mathcal{P}_{\mathbf{F}}^{i_0}} \psi(-y(u + u^{-1})) \left[\oint_{\mathcal{O}_{\mathbf{F}}} \psi\left(-y\varpi_{\mathbf{F}}^n (u - u^{-1}) x \right) \mathrm{d}x \right] \mathrm{d}u, \end{aligned}$$

where we have performed the level $n = \lceil m/2 \rceil \geqslant i_0$ regularization to $\mathrm{d}u$. Again the inner integral is non-vanishing only if

$$v_{\mathbf{F}}(u - u^{-1}) \geqslant m - n = \lfloor m/2 \rfloor \quad \Leftrightarrow \quad u \in \pm 1 + \mathcal{P}_{\mathbf{F}}^{\lfloor m/2 \rfloor}.$$

We have obtained

$$h(y^2) = \mathbb{1}_{k_0 \geqslant 0} \phi(1,0) \cdot |y|_{\mathbf{F}} \sum_{\pm} \int_{\pm 1 + \mathcal{P}_{\mathbf{F}}^{\lfloor -\frac{v_{\mathbf{F}}(y)}{2} \rfloor}} \psi(-y(u + u^{-1})) \mathrm{d}u,$$

hence h lies in the desired space of functions. Applying a smooth partition of unity to the open covering

$$\mathbf{G}_2(\mathbf{F}) = \mathbf{B}_2(\mathbf{F}) w \mathbf{N}_2(\mathbf{F}) \bigcup \mathbf{B}_2(\mathbf{F}) \mathbf{N}_2(\mathbf{F})^t$$

we conclude the proof. □

Definition 2.2. *We call $E_{\geqslant n}$ the* elementary Bessel orbital integrals*, abbreviated as* EBOIs*.*

2.2. **Voronoi–Hankel Transforms of EBOIs.** We notice that the Mellin transform of E_m is simple.

Lemma 2.3. *Let $m \geqslant 2$ and χ be a quasi-character. We have*

$$\int_{\mathbf{F}^\times} E_m(y)\chi(y)\mathrm{d}^\times y = \mathbb{1}_{\mathfrak{c}(\chi)=m} \cdot \zeta_{\mathbf{F}}(1)\gamma(1/2,\chi^{-1},\psi)^2.$$

Proof. Applying the change of variables $y \mapsto y^2$ and the level $\lceil m/2 \rceil$ regularization to $\mathrm{d}u$ we get

$$\begin{aligned}\int_{\mathbf{F}^\times} E_m(y)\chi(y)\mathrm{d}^\times y &= \frac{1}{2}\int_{\varpi_{\mathbf{F}}^{-m}\mathcal{O}_{\mathbf{F}}^\times} E_m(y^2)\chi^2(y)\mathrm{d}^\times y \\ &= q^m \int_{1+\mathcal{P}_{\mathbf{F}}^{\lfloor \frac{m}{2} \rfloor}} \left(\int_{\varpi_{\mathbf{F}}^{-m}\mathcal{O}_{\mathbf{F}}^\times} \psi\left(y(u+u^{-1})\right)\chi^2(y)\mathrm{d}^\times y \right)\mathrm{d}u \\ &= q^m \int_{\mathcal{O}_{\mathbf{F}}^\times} \left(\int_{\varpi_{\mathbf{F}}^{-m}\mathcal{O}_{\mathbf{F}}^\times} \psi\left(y(u+u^{-1}\right)\chi^2(y)\mathrm{d}^\times y \right)\mathrm{d}u \\ &= \zeta_{\mathbf{F}}(1)q^m \int_{\mathcal{O}_{\mathbf{F}}^\times}\int_{\mathcal{O}_{\mathbf{F}}^\times} \psi\left(\frac{y^2u+u^{-1}}{\varpi_{\mathbf{F}}^m}\right)\chi^2\left(\frac{y}{\varpi_{\mathbf{F}}^m}\right)\mathrm{d}y\mathrm{d}u.\end{aligned}$$

While the level $\lceil m/2 \rceil$ regularization to $\mathrm{d}u$ also implies

$$\int_{\mathcal{O}_{\mathbf{F}}^\times}\int_{\mathcal{O}_{\mathbf{F}}^\times} \psi\left(\frac{\varepsilon y^2u+u^{-1}}{\varpi_{\mathbf{F}}^m}\right)\chi\left(\frac{\varepsilon y^2}{\varpi_{\mathbf{F}}^{2m}}\right)\mathrm{d}y\mathrm{d}u = 0,$$

we can sum the above two equations to get

$$\begin{aligned}\int_{\mathbf{F}^\times} E_m(y)\chi(y)\mathrm{d}^\times y &= \zeta_{\mathbf{F}}(1)q^m \int_{\mathcal{O}_{\mathbf{F}}^\times}\int_{\mathcal{O}_{\mathbf{F}}^\times} \psi\left(\frac{yu+u^{-1}}{\varpi_{\mathbf{F}}^m}\right)\chi\left(\frac{y}{\varpi_{\mathbf{F}}^{2m}}\right)\mathrm{d}y\mathrm{d}u \\ &= \zeta_{\mathbf{F}}(1)q^m \left(\int_{\mathcal{O}_{\mathbf{F}}^\times} \psi\left(\frac{t}{\varpi_{\mathbf{F}}^m}\right)\chi\left(\frac{t}{\varpi_{\mathbf{F}}^m}\right)\mathrm{d}t \right)^2.\end{aligned}$$

The last integral was studied in [**?** , Proposition 4.6], which is non-vanishing if and only if $\mathfrak{c}(\chi) = m$. Its relation with the local gamma factor

$$\int_{\mathcal{O}_{\mathbf{F}}^\times} \psi\left(\frac{t}{\varpi_{\mathbf{F}}^m}\right)\chi\left(\frac{t}{\varpi_{\mathbf{F}}^m}\right)\mathrm{d}t = \gamma(1,\chi^{-1},\psi)$$

is the content of [**?** , Exercise 23.5]. □

Proposition 2.4. *There is $a = a(\Pi) \in \mathbb{Z}_{\geqslant 2}$, called the* stability barrier *of Π, such that for any $m \geqslant a$*

$$\mathcal{VH}_{\Pi,\psi} \circ \mathfrak{m}_{-1}(E_m)(t) = \psi(t) \cdot \mathbb{1}_{\varpi^{-m}\mathcal{O}_{\mathbf{F}}^\times}(t), \quad \widetilde{\mathcal{VH}}_{\Pi,\psi} \circ \mathfrak{m}_{-1}(E_{\geqslant m})(t) = \psi(t) \cdot \mathbb{1}_{\mathcal{P}_{\mathbf{F}}^m}(t^{-1}).$$

Moreover, we have $a \leqslant \max(2\mathfrak{c}(\Pi), 1)$. More precisely, we have:

1. *If Π is equal to or is included in $\mu_1 \boxplus \mu_2 \boxplus \mu_3$ we can take $a = \max(2\mathfrak{c}(\mu_1), 2\mathfrak{c}(\mu_2), 2\mathfrak{c}(\mu_3), 1)$;*
2. *If $\Pi = \pi \boxplus \mu$ for a supercuspidal π of $\mathbf{G}_2(\mathbf{F})$ we can take $a = \max(\mathfrak{c}(\pi), 2\mathfrak{c}(\mu))$;*
3. *If Π is supercuspidal we have $a \leqslant 2\mathfrak{c}(\Pi)$.*

Proof. By Lemma **??** and the local functional equation, the integral

$$\int_{\mathbf{F}^\times} \mathcal{VH}_{\Pi,\psi} \circ \mathfrak{m}_{-1}(E_m)(t)\chi^{-1}(t)|t|_{\mathbf{F}}^{-s}\mathrm{d}^\times t = \gamma(s,\Pi\times\chi,\psi)\int_{\mathbf{F}^\times} E_m(y)\chi(y)|y|_{\mathbf{F}}^{s-1}\mathrm{d}^\times y$$

is vanishing for any χ with $\mathfrak{c}(\chi) \neq m$. By the *stability* of the local gamma factors (see [**?** , Proposition (2.2)], [**?** , Theorem 23.8] and [**?** , Exercise 23.5]), there is $a \in \mathbb{Z}_{\geqslant 2}$ depending only on Π so that the factors

$$\gamma(s,\Pi\times\chi,\psi) = \gamma(s,\chi,\psi)^3$$

depend only on $\omega_\Pi = \mathbb{1}$ if $\mathfrak{c}(\chi) \geqslant a$. Hence for $\mathfrak{c}(\chi) = m \geqslant a$ we have by Lemma **??**

$$\gamma(s, \Pi \times \chi, \psi) \int_{\mathbf{F}^\times} E_m(y)\chi(y)|y|_{\mathbf{F}}^{s-1} \mathrm{d}^\times y = \gamma(s,\chi,\psi)^3 \cdot \zeta_{\mathbf{F}}(1)\gamma(3/2-s,\chi^{-1},\psi)^2 = \zeta_{\mathbf{F}}(1)\gamma(s+1,\chi,\psi).$$

One verifies easily the desired formula for $\mathcal{VH}_{\Pi,\psi}(E_m)$ by comparing their Mellin transforms. The desired formula for $E_{\geqslant m}$ follows easily by taking limits in the sense of tempered distributions on $\mathrm{M}_3(\mathbf{F})$ via $I_3 : \mathrm{C}(\mathbf{F}^\times) \to \mathrm{C}(\mathrm{GL}_3(\mathbf{F}))$ by [**?** , Theorem 1.3]. In the "moreover" part, (1) and (2) follow from the effective versions of the stability theorems for $\mathbf{G}_1$ and $\mathbf{G}_2$ [**?** , Theorem 23.8 & 25.7] together with the multiplicativity of the local gamma factors. For (3) we can take $a = \max(2\mathfrak{c}(\Pi), 6)$ by examining the proof in [**?** , §2]. Now that a supercuspidal Π of $\mathbf{G}_3$ has $\mathfrak{c}(\Pi) \geqslant 3$, we conclude the last assertion. □

Remark 2.5. *It would be interesting to know a sharp bound for a, say in general for stability theorems for $\mathbf{G}_n \times \mathbf{G}_t$. It could follow from the work of Bushnell–Henniart–Kutzko [**?**].*

3. Local Weight Functions

3.1. **Choice of Test Functions.** The target representation $\pi_0 \in \widehat{\mathrm{PGL}_2(\mathbf{F})}$ are:

1. (Split) $\pi(\chi_0, \chi_0^{-1})$ with a *ramified* and ϑ_3-tempered quasi-character χ_0 of $\mathbf{F}^\times$;
2. (Special) the *quadratic* twists St_η of the Steinberg representation St;
3. (Dihedral) π_β with a *unitary* regular character β of $\mathbf{E}^\times$, $\mathbf{E}/\mathbf{F}$ being a quadratic field extension.

Note that St_η is a sub-representation of $\pi(\eta|\cdot|_{\mathbf{F}}^{1/2}, \eta|\cdot|_{\mathbf{F}}^{-1/2})$.

Let $\mathbf{L}$ be a separable quadratic algebra extension of $\mathbf{F}$. The non-trivial element of the group $\mathrm{Aut}_{\mathbf{F}}(\mathbf{L})$ is denoted by $\mathbf{L} \to \mathbf{L}, x \mapsto \bar{x}$. Define

$$\mathrm{Tr} = \mathrm{Tr}_{\mathbf{L}/\mathbf{F}} : \mathbf{L} \to \mathbf{F},\ x \mapsto x + \bar{x}; \quad \mathrm{Nr} = \mathrm{Nr}_{\mathbf{L}/\mathbf{F}} : \mathbf{L}^\times \to \mathbf{F}^\times,\ x \mapsto x\bar{x}; \quad |x|_{\mathbf{L}} := |\mathrm{Nr}(x)|.$$

We associate to each target π_0 a *parameter* $(\mathbf{L}, \beta)$ by:

1. $\pi(\chi_0, \chi_0^{-1})$: Let $\mathbf{L} \simeq \mathbf{F} \oplus \mathbf{F}$ and $\beta : \mathbf{L}^\times \simeq \mathbf{F}^\times \times \mathbf{F}^\times \to \mathrm{S}^1, (t_1, t_2) \mapsto \chi_0(t_1 t_2^{-1})$;
2. St_η: Let $\mathbf{L} \simeq \mathbf{F} \oplus \mathbf{F}$ and $\beta : \mathbf{L}^\times \simeq \mathbf{F}^\times \times \mathbf{F}^\times \to \mathbb{C}^\times, (t_1, t_2) \mapsto \eta(t_1 t_2^{-1})|t_1 t_2^{-1}|_{\mathbf{F}}^{1/2}$;
3. π_β: Let $\mathbf{L} = \mathbf{E}$ and β be the obvious one.

We equip $\mathbf{L}$ with the self-dual Haar measure $\mathrm{d}z$ with respect to $\psi_{\mathbf{L}} := \psi \circ \mathrm{Tr}$. Write

$$\mathcal{O}_{\mathbf{L}} := \{ x \in \mathbf{L} \mid \mathrm{Tr}(x), \mathrm{Nr}(x) \in \mathcal{O}_{\mathbf{F}} \}.$$

Definition 3.1. *Define $\mathcal{P}_{\mathbf{L}} := \varpi_{\mathbf{L}}\mathcal{O}_{\mathbf{L}}$ and $\varpi_{\mathbf{L}} := \varpi_{\mathbf{F}}$ if $\mathbf{L}$ is split, otherwise $\varpi_{\mathbf{L}}$ is a uniformizer of $\mathbf{L}$. For any (quasi-)character β of $\mathbf{L}^\times$ we define its* $\mathbf{F}$-norm-1 conductor *as*

$$\mathfrak{c}_1(\beta) := \min \left\{ n \mid \beta\left(\mathbf{L}^1 \cap (1 + \mathcal{P}_{\mathbf{L}}^n)\right) = \{1\} \right\}.$$

If $\mathfrak{c}_1(\beta) > 0$ we say that β is a regular *character.*

Remark 3.2. *Directly from the definition we have for any (quasi-)character β of $\mathbf{L}^\times$*

- $\mathfrak{c}_1(\beta) \leqslant \mathfrak{c}(\beta)$;
- $\mathfrak{c}_1(\beta) = \mathfrak{c}_1(\beta \cdot (\chi \circ \mathrm{Nr}))$ *for any (quasi-)character χ of $\mathbf{F}^\times$.*

Lemma 3.3. *If β is a regular character of $\mathbf{L}^\times$ so that the restriction $\beta \mid_{\mathbf{F}^\times} = \eta_{\mathbf{L}/\mathbf{F}}$ coincides with the quadratic character associated with the quadratic extension $\mathbf{L}/\mathbf{F}$, then we have $\mathfrak{c}_1(\beta) = \mathfrak{c}(\beta)$.*

Proof. Write $n_0 := \mathfrak{c}(\beta) (\geqslant \mathfrak{c}_1(\beta) \geqslant 1)$. We shall prove that $\mathfrak{c}_1(\beta) < n_0$ is impossible.
(1) If $\mathbf{L} \simeq \mathbf{F} \oplus \mathbf{F}$ is split then $\eta_{\mathbf{L}/\mathbf{F}} = \mathbb{1}$. The assertion follows from $\mathfrak{c}(\chi_0) = \mathfrak{c}(\chi_0^2)$, because taking square is a group automorphism of $1 + \mathcal{P}_{\mathbf{F}}$ and the regularity of β is equivalent with $\mathfrak{c}(\chi_0^2) > 0$.
(2) Assume $\mathbf{L}/\mathbf{F}$ is non-split. Then β coincides with $\eta_{\mathbf{L}/\mathbf{F}}$ on $\mathbf{F}^\times$. If $n_0 = 1$ then the assertion follows from the condition $\beta \mid_{\mathbf{E}^1} \neq \mathbb{1}$ (equivalent to β being *regular*). Assume $n_0 \geqslant 2$ from now on. We choose the uniformizers $\varpi_{\mathbf{F}}$ and $\varpi_{\mathbf{L}}$ of $\mathcal{O}_{\mathbf{F}}$ and $\mathcal{O}_{\mathbf{L}}$ respectively so that

$$\begin{cases} \varpi_{\mathbf{L}} = \varpi_{\mathbf{F}} & \text{if } e = e(\mathbf{L}/\mathbf{F}) = 1 \\ \overline{\varpi_{\mathbf{L}}} = -\varpi_{\mathbf{L}}, \varpi_{\mathbf{F}} = \mathrm{Nr}(\varpi_{\mathbf{L}}) = -\varpi_{\mathbf{L}}^2 & \text{if } e = e(\mathbf{E}/\mathbf{F}) = 2 \end{cases}. \tag{3.1}$$

Claim: We have $2 \mid n_0$ in the ramified case.

Proof of Claim: In fact, if β is trivial on $1+\mathcal{P}_{\mathbf{L}}^{2n+1}$ with $n \geqslant 1$, then for any element $\alpha \in 1+\mathcal{P}_{\mathbf{L}}^{2n}$ we can find $u_0 \in \mathcal{O}_{\mathbf{F}}$ and $u_1 \in \mathcal{O}_{\mathbf{L}}$ such that $\alpha = 1+\varpi_{\mathbf{L}}^{2n}u_0+\varpi_{\mathbf{L}}^{2n+1}u_1$, since the residue class fields of $\mathbf{L}$ and $\mathbf{F}$ are isomorphic. Then $\beta(\alpha)=\eta_{\mathbf{L}/\mathbf{F}}(1+(-\varpi_{\mathbf{F}})^n u_0)=1$. Hence β is also trivial on $1+\mathcal{P}_{\mathbf{L}}^{2n}$. $\square$

With these reductions, we find $\alpha \in 1+\mathcal{P}_{\mathbf{L}}^{n_0-1}$ such that $\beta(\alpha)\neq 1$. But $\mathrm{Nr}(\alpha) \in \mathbf{F}\cap(1+\mathcal{P}_{\mathbf{L}}^{n_0-1}) = 1+\mathcal{P}_{\mathbf{F}}^{\lceil (n_0-1)/e\rceil}$ is a square. Hence $\mathrm{Nr}(\alpha)=k^2$ for some $k\in 1+\mathcal{P}_{\mathbf{F}}^{\lceil (n_0-1)/e\rceil} \subset 1+\mathcal{P}_{\mathbf{L}}^{n_0-1}$, and $\beta(k^{-1}\alpha) = \beta(\alpha)\neq 1$ with $k^{-1}\alpha \in \mathbf{L}^1\cap(1+\mathcal{P}_{\mathbf{L}}^{n_0-1})$, proving the assertion. $\square$

Corollary 3.4. *Any* dihedral supercuspidal representations π_β *with trivial central character (the case (3) in the beginning of this subsection) is twisted minimal.*

Proof. For any (quasi-)character χ of $\mathbf{F}^\times$ we have $\mathfrak{c}(\beta)=\mathfrak{c}_1(\beta)=\mathfrak{c}_1(\beta\cdot(\chi\circ\mathrm{Nr}))\leqslant \mathfrak{c}(\beta\cdot(\chi\circ\mathrm{Nr}))$. For any regular (quasi-)character β of $\mathbf{E}^\times$ we have by [**?** , Theorem 4.7] with $e=e(\mathbf{E}/\mathbf{F})$

$$\mathfrak{c}(\pi_\beta)=\mathfrak{c}(\beta)f(\mathbf{E}/\mathbf{F})+\mathfrak{c}(\psi_{\mathbf{E}})=\tfrac{2n_0}{e}+e-1, \tag{3.2}$$

where $f(\mathbf{E}/\mathbf{F})$ is the residual field index and $\psi_{\mathbf{E}}=\psi_{\mathbf{F}}\circ\mathrm{Tr}_{\mathbf{E}/\mathbf{F}}$. The desired assertion follows readily. $\square$

Proposition 3.5. *Let* $\mathbf{L}^1$ *be the kernel of the norm map* Nr.

(1) *For any* $x\in\mathbf{L}^\times$*, the following map*

$$\iota_x:\mathbf{F}^\times\times\mathbf{L}^1\to\mathbf{L}^\times,\quad (r,\alpha)\mapsto xr\alpha$$

is a 2*-to-*1 *covering map onto an open subset of* $\mathbf{L}^\times$.

(2) *(Polar decomposition) The Haar measure* $\mathrm{d}\alpha$ *on* $\mathbf{L}^1$ *determined by*

$$|z|_{\mathbf{L}}^{-1}\mathrm{d}z=|r|_{\mathbf{F}}^{-1}\mathrm{d}r\mathrm{d}\alpha,\quad z=\iota_x(r,\alpha)=xr\alpha, \tag{3.3}$$

where $\mathrm{d}r$ *is the self-dual Haar measure of* $\mathbf{F}$ *with respect to* ψ*, is independent of* x*, and satisfies*

$$\mathrm{Vol}(\mathbf{L}^1\cap\mathcal{O}_{\mathbf{L}},\mathrm{d}\alpha)=2^{e-1}\frac{\mathrm{Vol}(\mathcal{O}_{\mathbf{L}}^\times,\mathrm{d}z)}{\mathrm{Vol}(\mathcal{O}_{\mathbf{F}}^\times,\mathrm{d}r)},$$

where the number $e=e(\mathbf{L}/\mathbf{F})$ *is the* generalized ramification index *of* $\mathbf{L}/\mathbf{F}$ *given by*

$$e=\begin{cases}1 & \text{if } \mathbf{L}/\mathbf{F} \text{ is not ramified}\\ 2 & \text{if } \mathbf{L}/\mathbf{F} \text{ is ramified}\end{cases}. \tag{3.4}$$

(3) *For any* $n\in\mathbb{Z}_{\geqslant 1}$ *we have* $\mathrm{Vol}(\mathbf{L}^1\cap(1+\mathcal{P}_{\mathbf{L}}^n),\mathrm{d}\alpha)=q^{-\lfloor\frac{n}{e}\rfloor-\frac{e-1}{2}}$.

Proof. The proof of (1) is easy and omitted. The relation $\Im(\iota_x)=x\Im(\iota_1)$ implies the independence of x in (**??**), since the measure $|z|_{\mathbf{L}}^{-1}\mathrm{d}z$ is invariant by multiplication by elements in $\mathbf{L}^\times$. To show that (**??**) implies the stated formula for $\mathrm{Vol}(\mathbf{L}^1\cap\mathcal{O}_{\mathbf{L}},\mathrm{d}\alpha)$, it suffices to show

$$\mathrm{Vol}(\iota_1(\mathcal{O}_{\mathbf{F}}^\times\times(\mathcal{O}_{\mathbf{L}}\cap\mathbf{L}^1)))=2^{e-2}\mathrm{Vol}(\mathcal{O}_{\mathbf{L}}^\times).$$

Now that $\iota_1(\mathcal{O}_{\mathbf{F}}^\times\times(\mathcal{O}_{\mathbf{L}}\cap\mathbf{L}^1))=\mathcal{O}_{\mathbf{L}}^\times\cap\mathrm{Nr}^{-1}((\mathcal{O}_{\mathbf{F}}^\times)^2)$, we get

$$[\mathcal{O}_{\mathbf{L}}^\times:\iota_1(\mathcal{O}_{\mathbf{F}}^\times\times(\mathcal{O}_{\mathbf{L}}\cap\mathbf{L}^1))]=[\mathrm{Nr}(\mathcal{O}_{\mathbf{L}}^\times):(\mathcal{O}_{\mathbf{F}}^\times)^2]=\frac{[\mathcal{O}_{\mathbf{F}}^\times:(\mathcal{O}_{\mathbf{F}}^\times)^2]}{[\mathcal{O}_{\mathbf{F}}^\times:\mathrm{Nr}(\mathcal{O}_{\mathbf{L}}^\times)]}=\frac{2}{2^{e-1}}$$

and conclude the proof of (2). To prove (3) we note that the map

$$\sigma:\mathcal{O}_{\mathbf{L}}^\times\to\mathbf{L}^1\cap\mathcal{O}_{\mathbf{L}},\quad x\mapsto x/\bar{x}$$

is surjective onto $\mathbf{L}^1\cap(1+\mathcal{P}_{\mathbf{L}}^n)$, whose pre-image is $\mathcal{O}_{\mathbf{F}}^\times(1+\mathcal{P}_{\mathbf{L}}^n)$. It is also easy to see that the image of σ is a subgroup of $\mathbf{L}^1\cap\mathcal{O}_{\mathbf{L}}$ with index 2^{e-1}, by a refinement of Hilbert's 90. Therefore we get

$$\frac{\Im(\sigma)}{\mathbf{L}^1\cap(1+\mathcal{P}_{\mathbf{L}}^n)}\simeq\frac{\mathcal{O}_{\mathbf{L}}^\times}{\mathcal{O}_{\mathbf{F}}^\times(1+\mathcal{P}_{\mathbf{L}}^n)}\simeq\frac{(\mathcal{O}_{\mathbf{L}}/\mathcal{P}_{\mathbf{L}}^n)^\times}{(\mathcal{O}_{\mathbf{F}}/\mathcal{P}_{\mathbf{F}}^{\lceil n/e\rceil})^\times}$$

and deduce from it the measure relation

$$\frac{\mathrm{Vol}(\mathbf{L}^1\cap\mathcal{O}_{\mathbf{L}},\mathrm{d}\alpha)}{\mathrm{Vol}(\mathbf{L}^1\cap(1+\mathcal{P}_{\mathbf{L}}^n),\mathrm{d}\alpha)}=2^{e-1}\frac{q^{2n/e}}{q^{\lceil n/e\rceil}}\cdot\frac{\mathrm{Vol}(\mathcal{O}_{\mathbf{L}}^\times,\mathrm{d}z)}{\mathrm{Vol}(\mathcal{O}_{\mathbf{F}}^\times,\mathrm{d}r)}\cdot\frac{\mathrm{Vol}(\mathcal{O}_{\mathbf{F}},\mathrm{d}r)}{\mathrm{Vol}(\mathcal{O}_{\mathbf{L}},\mathrm{d}z)}.$$

We conclude the proof of (3) by comparing with (2) and $\mathrm{Vol}(\mathcal{O}_{\mathbf{L}}, \mathrm{d}z) = q^{-\frac{e-1}{2}}$. □

Let $\eta_{\mathbf{L}/\mathbf{F}}$ be the quadratic character of $\mathbf{F}^\times$ associated with $\mathbf{L}/\mathbf{F}$, which is trivial on $\mathrm{Nr}(\mathbf{L}^\times)$. We have $\beta\mid_{\mathbf{F}^\times} = \eta_{\mathbf{L}/\mathbf{F}}$ since π_0 has trivial central character. Write $\lambda(\mathbf{L}/\mathbf{F}, \psi)$ for the Weil index, which is equal to 1 if $\mathbf{L}/\mathbf{F}$ is not ramified (see [? , Corollary 3.2]). Let τ run through a system of representatives of $\mathcal{O}_{\mathbf{F}}^\times/(\mathcal{O}_{\mathbf{F}}^\times)^2$ (split), resp. $\mathrm{Nr}(\mathbf{L}^\times)/(\mathbf{F}^\times)^2$ (non-split), and choose any $x_\tau \in \mathbf{L}^\times$ s.t. $\mathrm{Nr}(x_\tau) = \tau$; if $\mathbf{L}/\mathbf{F}$ is split we also require $\mathrm{Tr}(x_\tau) = 1 + \tau$. Define a function H on $\mathbf{F}^\times$ with support contained in $\mathrm{Nr}(\mathbf{L}^\times)$ by

$$H(\tau y^2) = \lambda(\mathbf{L}/\mathbf{F}, \psi)|\tau|_{\mathbf{F}}^{\frac{1}{2}} \cdot |y|_{\mathbf{F}} \cdot \eta_{\mathbf{L}/\mathbf{F}}(y) \cdot \int_{\mathbf{L}^1 \cap \mathcal{O}_{\mathbf{L}}} \beta(x_\tau \alpha)\psi(\mathrm{Tr}(x_\tau \alpha)y)\mathrm{d}\alpha. \tag{3.5}$$

It is clear that H is independent of any choice made in the definition. For definiteness, we fix a $\varepsilon \in \mathcal{O}_{\mathbf{F}}^\times - (\mathcal{O}_{\mathbf{F}}^\times)^2$ and choose a system $\{\tau\}$ of representatives of $\mathcal{O}_{\mathbf{F}}^\times/(\mathcal{O}_{\mathbf{F}}^\times)^2$, resp. $\mathrm{Nr}(\mathbf{L}^\times)/(\mathbf{F}^\times)^2$ as

(1) $\{1, \varepsilon\}$ if $\mathbf{L}/\mathbf{F}$ is split or unramified (in the latter case $\mathbf{L} = \mathbf{F}[\sqrt{\varepsilon}]$),
(2) $\{1, -\varpi_{\mathbf{F}}\}$ if $\mathbf{L}/\mathbf{F}$ is ramified (with $\mathbf{L} = \mathbf{F}[\sqrt{\varpi_{\mathbf{F}}}]$ and we fix a uniformizer $\varpi_{\mathbf{L}} = \sqrt{\varpi_{\mathbf{F}}}$ of $\mathbf{L}$).

The (partial) Mellin transform of H will be fundamental for our analysis. We record it as follows.

Lemma 3.6. *Let $\chi \in \widehat{\mathbf{F}^\times}$ and $n \in \mathbb{Z}_{\geqslant 1}$. We have*

$$\varepsilon_n(\chi, H) := \int_{\varpi_{\mathbf{F}}^{-n}\mathcal{O}_{\mathbf{F}}^\times} H(y)\chi(y)\mathrm{d}^\times y = \zeta_{\mathbf{F}}(1)\cdot$$

$$\begin{cases} \mathbb{1}_{2|n, \mathfrak{c}(\chi_0\chi) = \mathfrak{c}(\chi_0^{-1}\chi) = n/2} \cdot \varepsilon(1/2, \chi_0\chi^{-1}, \psi)\varepsilon(1/2, \chi_0^{-1}\chi^{-1}, \psi) & \text{if } \mathbf{L}/\mathbf{F} \text{ split and } \mathfrak{c}(\chi_0\chi), \mathfrak{c}(\chi_0^{-1}\chi) \geqslant 1 \\ -\mathbb{1}_{n=2, \mathfrak{c}(\chi_0)=1, \mathfrak{c}(\chi_0^2)\neq 0} \cdot q^{-1}\chi_0\chi(\varpi_{\mathbf{F}})\varepsilon(1/2, \chi_0^{-1}\chi^{-1}, \psi) & \text{if } \mathbf{L}/\mathbf{F} \text{ split and } \mathfrak{c}(\chi_0\chi) = 0 \neq \mathfrak{c}(\chi_0^{-1}\chi) \\ -\mathbb{1}_{n=2, \mathfrak{c}(\chi_0)=1, \mathfrak{c}(\chi_0^2)\neq 0} \cdot q^{-1}\chi_0^{-1}\chi(\varpi_{\mathbf{F}})\varepsilon(1/2, \chi_0\chi^{-1}, \psi) & \text{if } \mathbf{L}/\mathbf{F} \text{ split and } \mathfrak{c}(\chi_0^{-1}\chi) = 0 \neq \mathfrak{c}(\chi_0\chi) \\ -\mathbb{1}_{n=2, \mathfrak{c}(\chi_0)=1, \mathfrak{c}(\chi_0^2)=0} \cdot q^{-2}\chi^2(\varpi_{\mathbf{F}}) & \text{if } \mathbf{L}/\mathbf{F} \text{ split and } \mathfrak{c}(\chi_0\chi) = 0 = \mathfrak{c}(\chi_0^{-1}\chi) \\ \mathbb{1}_{2|en, \mathfrak{c}(\beta\cdot(\chi\circ\mathrm{Nr})) = en/2+1-e} \cdot \varepsilon(1/2, \pi_{\beta^{-1}} \otimes \chi^{-1}, \psi) & \text{if } \mathbf{L}/\mathbf{F} \text{ non-split} \end{cases}.$$

In particular, we have $|\varepsilon_n(\chi, H)| \leqslant \zeta_{\mathbf{F}}(1)$.

Proof. We divide the domain of integration into cosets of square elements.
(A) In the split case, we have

$$\begin{aligned} \int_{\varpi_{\mathbf{F}}^{-n}\mathcal{O}_{\mathbf{F}}^\times} H(y)\chi(y)\mathrm{d}^\times y &= \mathbb{1}_{2|n} \cdot \frac{1}{2}\sum_{\tau \in \{1,\varepsilon\}} \int_{\varpi_{\mathbf{F}}^{-\frac{n}{2}}\mathcal{O}_{\mathbf{F}}^\times} \chi(\tau y^2)|y| \left(\int_{\mathcal{O}_{\mathbf{F}}^\times} \chi_0(\tau\alpha^2)\psi\left((\tau\alpha + \alpha^{-1})y\right)\mathrm{d}\alpha \right)\mathrm{d}^\times y \\ &= \mathbb{1}_{2|n} \cdot \zeta_{\mathbf{F}}(1)^{-1} \int_{\varpi_{\mathbf{F}}^{-\frac{n}{2}}\mathcal{O}_{\mathbf{F}}^\times \times \varpi_{\mathbf{F}}^{-\frac{n}{2}}\mathcal{O}_{\mathbf{F}}^\times} \chi(t_1t_2)|t_1t_2|^{\frac{1}{2}} \cdot \chi_0(t_1t_2^{-1})\psi(t_1+t_2)\mathrm{d}^\times t_1 \mathrm{d}^\times t_2 \\ &= \mathbb{1}_{2|n} \cdot \zeta_{\mathbf{F}}(1)q^{\frac{n}{2}} \left(\int_{\varpi_{\mathbf{F}}^{-\frac{n}{2}}\mathcal{O}_{\mathbf{F}}^\times} \psi(t) \cdot \chi\chi_0(t)\frac{\mathrm{d}t}{|t|} \right) \left(\int_{\varpi_{\mathbf{F}}^{-\frac{n}{2}}\mathcal{O}_{\mathbf{F}}^\times} \psi(t) \cdot \chi\chi_0^{-1}(t)\frac{\mathrm{d}t}{|t|} \right), \end{aligned}$$

where we have taken into account that for each τ the map

$$\varpi_{\mathbf{F}}^{-\frac{n}{2}}\mathcal{O}_{\mathbf{F}}^\times \times \mathcal{O}_{\mathbf{F}}^\times \to \varpi_{\mathbf{F}}^{-\frac{n}{2}}\mathcal{O}_{\mathbf{F}}^\times \times \varpi_{\mathbf{F}}^{-\frac{n}{2}}\mathcal{O}_{\mathbf{F}}^\times, \quad (y, \alpha) \mapsto (\tau y\alpha, y\alpha^{-1})$$

is 2-to-1. The stated formulae then follow from the relation between the Gauss integrals and the local epsilon-factors given in [? , Exercise 23.5], and a direct computation in the unramified case.
(B) Similarly in the non-split case we have

$$\begin{aligned} \int_{\varpi_{\mathbf{F}}^{-n}\mathcal{O}_{\mathbf{F}}^\times} H(y)\chi(y)\mathrm{d}^\times y &= \mathbb{1}_{2|en} \cdot \frac{\zeta_{\mathbf{F}}(1)}{\mathrm{Vol}(\mathbf{L}^1)} \int_{\varpi_{\mathbf{L}}^{-\frac{en}{2}}\mathcal{O}_{\mathbf{L}}^\times} H(\mathrm{Nr}(z))\chi(\mathrm{Nr}(z))\frac{\mathrm{d}z}{|z|_{\mathbf{L}}} \\ &= \mathbb{1}_{2|en} \cdot \zeta_{\mathbf{F}}(1)q^{\frac{n}{2}}\lambda(\mathbf{L}/\mathbf{F}, \psi) \int_{\varpi_{\mathbf{L}}^{-\frac{en}{2}}\mathcal{O}_{\mathbf{L}}^\times} \beta(z)\chi(\mathrm{Nr}(z)) \cdot \psi_{\mathbf{L}}(z)\frac{\mathrm{d}z}{|z|_{\mathbf{L}}} \\ &= \mathbb{1}_{2|en} \cdot \zeta_{\mathbf{F}}(1)q^{\frac{n}{2}}\lambda(\mathbf{L}/\mathbf{F}, \psi) \cdot \mathbb{1}_{\mathfrak{c}(\beta\cdot(\chi\circ\mathrm{Nr}))+e-1=\frac{en}{2}}\varepsilon(1, \beta^{-1}\cdot(\chi^{-1}\circ\mathrm{Nr}), \psi_{\mathbf{L}}), \end{aligned}$$

where we have applied [**?** , Exercise 23.5] in the last line. Now that [**?** , Theorem 4.7] implies

$$\lambda(\mathbf{L}/\mathbf{F},\psi)\varepsilon(s,\beta\cdot(\chi\circ\mathrm{Nr}),\psi_{\mathbf{L}})=\varepsilon(s,\pi_\beta\otimes\chi,\psi)=q^{n\cdot(\frac{1}{2}-s)}\varepsilon(\tfrac{1}{2},\pi_\beta\otimes\chi,\psi),$$

we conclude the desired formula. Note that in this case $\beta\cdot(\chi\circ\mathrm{Nr})$ is never unramified. □

We give a first asymptotic analysis of H as follows.

Lemma 3.7. *Let $\alpha\in\mathbf{L}^1$ and $n_1\in\mathbb{Z}_{\geqslant 1}$. Then $\alpha-\overline{\alpha}\in\mathcal{P}_{\mathbf{L}}^{n_1}$ if and only if $\alpha\in\pm1+\mathcal{P}_{\mathbf{L}}^{n_1}$.*

Proof. If $\alpha\in\pm1+\mathcal{P}_{\mathbf{L}}^{n_1}$, then it clear that $\alpha-\overline{\alpha}\in\mathcal{P}_{\mathbf{L}}^{n_1}$. The converse is justified as follows.
(a) If $\mathbf{L}/\mathbf{F}$ is split, we write $\alpha=(t,t^{-1})$ with $t\in\mathbf{F}^\times$. Then $t-t^{-1}\in\mathcal{P}_{\mathbf{F}}^{n_1}$. We must have $t\in\mathcal{O}_{\mathbf{F}}^\times$, and $t\equiv t^{-1}\pmod{\mathcal{P}_{\mathbf{F}}}$. Hence $t\in\pm1+\mathcal{P}_{\mathbf{F}}$. Writing $t=\pm1+\varpi_{\mathbf{F}}^k u$ for some $k\in\mathbb{Z}_{\geqslant1}$ and $u\in\mathcal{O}_{\mathbf{F}}^\times$ we infer that $t-t^{-1}\in2\varpi_{\mathbf{F}}^k u+\mathcal{P}_{\mathbf{F}}^{k+1}\subset\mathcal{P}_{\mathbf{F}}^k-\mathcal{P}_{\mathbf{F}}^{k+1}$. Hence $k\geqslant n_1$, and $t\in\pm1+\mathcal{P}_{\mathbf{F}}^{n-1}$.
(b) If $\mathbf{L}/\mathbf{F}$ is a field, then $\overline{\alpha}=\alpha^{-1}$. We repeat the above argument with (subscript) $\mathbf{F}$ replaced by $\mathbf{L}$. □

Lemma 3.8. *There is $c_\beta\in\mathcal{O}_{\mathbf{F}}^\times$, called the* additive parameter *and is uniquely determined up to multiplication by elements in $1+\mathcal{P}_{\mathbf{F}}^{\lceil\frac{n_0}{2e}\rceil}$, so that for all $u\in\mathcal{P}_{\mathbf{L}}^{\lfloor\frac{n_0}{2}\rfloor}$*

$$\beta(1+u)=\begin{cases}\psi\left(\varpi_{\mathbf{F}}^{-n_0}c_\beta(u-\bar u)\right) & \text{if } \mathbf{L}/\mathbf{F} \text{ split and } 2\mid n_0,\\ \psi\left(\varpi_{\mathbf{F}}^{-n_0}c_\beta\left((u-\bar u)-2^{-1}(u^2-\bar u^2)\right)\right) & \text{if } \mathbf{L}/\mathbf{F} \text{ split and } 2\nmid n_0;\\ \psi\left(\varpi_{\mathbf{F}}^{-n_0}\sqrt{\varepsilon}c_\beta(u-\bar u)\right) & \text{if } \mathbf{L}/\mathbf{F} \text{ unramified and } 2\mid n_0,\\ \psi\left(\varpi_{\mathbf{F}}^{-n_0}\sqrt{\varepsilon}c_\beta\left((u-\bar u)-2^{-1}(u^2-\bar u^2)\right)\right) & \text{if } \mathbf{L}/\mathbf{F} \text{ unramified and } 2\nmid n_0;\\ \psi\left(\varpi_{\mathbf{L}}^{-n_0-1}c_\beta(u-\bar u)\right) & \text{if } \mathbf{L}/\mathbf{F} \text{ ramified.}\end{cases}$$

Proof. We only treat the cases $2\nmid n_0:=2m+1$ if $\mathbf{L}/\mathbf{F}$ is not ramified, the even ones being simpler. The following map

$$\log_{\mathbf{F}}:(1+\mathcal{P}_{\mathbf{F}}^m)/(1+\mathcal{P}_{\mathbf{F}}^{2m+1})\to\mathcal{P}_{\mathbf{F}}^m/\mathcal{P}_{\mathbf{F}}^{2m+1},\quad 1+x\mapsto x-2^{-1}x^2$$

is a group isomorphism. If $\mathbf{L}/\mathbf{F}$ is split there is $c_\beta'\in\mathcal{O}_{\mathbf{F}}^\times/(1+\mathcal{P}_{\mathbf{F}}^{m+1})$ such that

$$\beta(1+\varpi_{\mathbf{F}}^m u)=\psi\left(c_\beta'\left(\varpi_{\mathbf{F}}^{-(m+1)}u-2^{-1}\varpi_{\mathbf{F}}^{-1}u^2\right)\right),\quad\forall u\in\mathcal{O}_{\mathbf{L}}\left(\simeq\mathcal{O}_{\mathbf{F}}\times\mathcal{O}_{\mathbf{F}}\right).\tag{3.6}$$

One checks easily that $c_\beta:=c_\beta'$ is the required one. If $\mathbf{L}/\mathbf{F}$ is unramified we get an analogue of (**??**)

$$\beta(1+\varpi_{\mathbf{F}}^m u)=\psi\left(\mathrm{Tr}\left(c_\beta'\left(\varpi_{\mathbf{F}}^{-(m+1)}u-2^{-1}\varpi_{\mathbf{F}}^{-1}u^2\right)\right)\right),\quad\forall u\in\mathcal{O}_{\mathbf{L}}.\tag{3.7}$$

Now that β is trivial on $\mathcal{O}_{\mathbf{F}}^\times$ and $\log_{\mathbf{F}}$ is surjective, implying $\psi\circ\mathrm{Tr}(c_\beta'\mathcal{P}_{\mathbf{F}}^{-m-1})=1$, i.e., $c_\beta'+\overline{c_\beta'}\in\mathcal{P}_{\mathbf{F}}^{m+1}$. Therefore we may write $c_\beta'\in c_\beta\sqrt{\varepsilon}+\mathcal{P}_{\mathbf{F}}^{m+1}$ for some $c_\beta\in\mathcal{O}_{\mathbf{F}}^\times/(1+\mathcal{P}_{\mathbf{F}}^{m+1})$. If $\mathbf{L}/\mathbf{F}$ is ramified, then $2\mid n_0$ by the claim proved in Lemma **??** and $\psi_{\mathbf{L}}:=\psi\circ\mathrm{Tr}_{\mathbf{L}/\mathbf{F}}$ has conductor exponent -1. We similarly get

$$\beta(1+u)=\psi\left(\varpi_{\mathbf{F}}^{-\frac{n_0}{2}}\varpi_{\mathbf{L}}^{-1}(c_\beta'u-\overline{c_\beta'}\bar u)\right),\quad\forall u\in\mathcal{P}_{\mathbf{L}}^{\frac{n_0}{2}}\tag{3.8}$$

for some $c_\beta'\in\mathcal{O}_{\mathbf{L}}^\times/(1+\mathcal{P}_{\mathbf{L}}^{\frac{n_0}{2}})$. Note that β is trivial on $1+\mathcal{P}_{\mathbf{F}}$, implying $\psi\left(\varpi_{\mathbf{F}}^{-\frac{n_0}{2}}\varpi_{\mathbf{L}}^{-1}(c_\beta'-\overline{c_\beta'})\mathcal{P}_{\mathbf{F}}^{\lceil\frac{n_0}{4}\rceil}\right)=1$, i.e., $c_\beta'-\overline{c_\beta'}\in\mathcal{P}_{\mathbf{L}}^{2\lfloor\frac{n_0}{4}\rfloor+1}$. Therefore we may write $c_\beta'\in c_\beta+\mathcal{P}_{\mathbf{L}}^{2\lfloor\frac{n_0}{4}\rfloor+1}$, hence may take $c_\beta'=c_\beta$ in (**??**), for some $c_\beta\in\mathcal{O}_{\mathbf{F}}^\times/(1+\mathcal{P}_{\mathbf{F}}^{\lceil\frac{n_0}{4}\rceil})$. □

Remark 3.9. *In the split case the character β is constructed from a character χ_0 of $\mathbf{F}^\times$. We also call $c_\beta=c_{\chi_0}$ the additive parameter of χ_0.*

Lemma 3.10. *Let e be the generalized ramification index given in (**??**), and write $n_0:=\mathfrak{c}(\beta)$.*

1. *In the domain $v_{\mathbf{F}}(y)\leqslant-4n_0/e-2(e-1)$ we have $H(y)=E_{\geqslant 2n_0/e+e-1}(y)$.*
2. Assume β is regular. *For any τ we have $H(\tau y^2)=0$ if $v_{\mathbf{F}}(y)\geqslant(2-n_0)e^{-1}-1$.*
3. *If $\tau\neq1$, then $H(\tau y^2)=0$ unless $v_{\mathbf{F}}(y)=1-e-e^{-1}n_0$.*

(4) *Suppose $e=1, n_0 \geqslant 2$. If $\tau=1$ and $v_{\mathbf{F}}(y)=-m$ with $n_0 \leqslant m \leqslant 2n_0-1$, then we may put the following condition in the domain of integration in (??):*

$$\mathrm{Tr}(\alpha) \in \begin{cases} \pm 2(1+\mathcal{P}_{\mathbf{F}}^{2(n_0-1)}) & \text{if } m=2n_0-1 \\ \pm 2(1+\varpi_{\mathbf{F}}^{2(m-n_0)}\mathcal{O}_{\mathbf{F}}^{\times}) & \text{if } n_0<m<2n_0-1 \\ \mathcal{O}_{\mathbf{F}} - \bigcup_{\pm} \pm 2(1+\mathcal{P}_{\mathbf{F}}) & \text{if } m=n_0 \end{cases}.$$

(5) *Suppose $e=2$. We may put the following condition in the domain of integration in (??):*

$$\begin{cases} \mathrm{Tr}(\alpha) \in \pm 2(1+\mathcal{P}_{\mathbf{F}}^{n_0-1}) & \text{if } \tau=1, \text{ and } -v_{\mathbf{F}}(y):=m=n_0 \\ \mathrm{Tr}(\alpha) \in \pm 2(1+\varpi_{\mathbf{F}}^{2m-n_0+1}\mathcal{O}_{\mathbf{F}}^{\times}) & \text{if } \tau=1, \text{ and } \frac{n_0}{2}+1 \leqslant -v_{\mathbf{F}}(y):=m \leqslant n_0-1 \end{cases}.$$

Proof. (1) We only consider non-split $\mathbf{L}$, leaving the simpler split case as an exercise. Let $n=-v_{\mathbf{F}}(y) \geqslant 4n_0/e+2(e-1)$. For $\chi \in \widehat{\mathcal{O}_{\mathbf{F}}^{\times}}$, by Lemma ?? the integral $\int_{\varpi_{\mathbf{F}}^{-n}\mathcal{O}_{\mathbf{F}}^{\times}} H(y)\chi(y)\mathrm{d}^{\times}y \neq 0$ is non-zero only if

$$\tfrac{en}{2} = \mathfrak{c}(\beta(\chi\circ\mathrm{Nr})) + \mathfrak{c}(\psi_{\mathbf{L}}) \quad \Leftrightarrow \quad \max(\mathfrak{c}(\beta),\mathfrak{c}(\chi\circ\mathrm{Nr})) \geqslant \mathfrak{c}(\beta(\chi\circ\mathrm{Nr})) = \begin{cases} n-1 & \text{if } e=2 \\ n/2 & \text{if } e=1 \end{cases}.$$

By [? , Proposition V.2.3 & Corollay V.3.3] we have

$$\mathfrak{c}(\chi\circ\mathrm{Nr}) = \begin{cases} \max(2\mathfrak{c}(\chi)-1,0) & \text{if } e=2 \\ \mathfrak{c}(\chi) & \text{if } e=1 \end{cases}. \tag{3.9}$$

In view of the assumption on n (implying $\mathfrak{c}(\beta(\chi\circ\mathrm{Nr}))>n_0$), the non-vanishing condition is equivalent to $2 \mid n$ & $\mathfrak{c}(\chi)=n/2$. Now that by [? , Theorem 4.7] $\ell(\chi)=\mathfrak{c}(\chi)-1=n/2-1>2\ell(\pi_\beta)=\mathfrak{c}(\pi_\beta)-2=2\mathfrak{c}(\beta)/e+e-3=2n_0/e+e-3$, we can apply the stability theorem [? , Theorem 25.7] and obtain

$$\int_{\varpi_{\mathbf{F}}^{-n}\mathcal{O}_{\mathbf{F}}^{\times}} H(y)\chi(y)\mathrm{d}^{\times}y = \mathbb{1}_{2|n,\mathfrak{c}(\chi)=\frac{n}{2}} \cdot \zeta_{\mathbf{F}}(1)\varepsilon(1/2,\chi^{-1},\psi)^2. \tag{3.10}$$

We conclude by comparing with Lemma ?? and applying the Mellin inversion on $\mathcal{O}_{\mathbf{F}}^{\times}$.
(2) We average over the change of variables $\alpha \mapsto \alpha\delta$ for $\delta \in \mathbf{L}^1 \cap (1+\mathcal{P}_{\mathbf{L}}^{n_0-1}) =: U$. Note that

$$yx_\tau\alpha(\delta-1) \in \mathcal{P}_{\mathbf{L}}^{1-e}, \quad \mathrm{Tr}(\mathcal{P}_{\mathbf{L}}^{1-e}) \subset \mathcal{P}_{\mathbf{L}}^{1-e} \cap \mathbf{F} = \mathcal{O}_{\mathbf{F}} \quad \Rightarrow \quad \psi(\mathrm{Tr}(x_\tau\alpha\delta)y) = \psi(\mathrm{Tr}(x_\tau\alpha)y)$$

for y satisfying the stated condition. By Lemma ??, the character β is non-trivial on U, hence

$$H(\tau y^2) = \lambda(\mathbf{L}/\mathbf{F},\psi)|\tau|_{\mathbf{F}}^{\frac{1}{2}} \cdot |y|_{\mathbf{F}} \cdot \eta_{\mathbf{L}/\mathbf{F}}(y) \cdot \int_{\mathbf{L}^1\cap\mathcal{O}_{\mathbf{L}}} \beta(x_\tau\alpha)\psi(\mathrm{Tr}(x_\tau\alpha)y) \cdot \left(\oint_U \beta(\delta)\mathrm{d}\delta \right) \mathrm{d}\alpha = 0.$$

(3) The idea is to average over similar change of variables $\alpha \mapsto \alpha\delta$ for $\delta \in \mathbf{L}^1 \cap (1+\mathcal{P}_{\mathbf{L}}^n) =: U_n$ with

$$n \geqslant \max(n_0,(1-e-ev_{\mathbf{F}}(y))/2) \tag{3.11}$$

and perform a further change of variables $\delta=\delta(u)$

$$\delta = \frac{1+\varpi_{\mathbf{L}}^n u}{1+\overline{\varpi_{\mathbf{L}}^n u}}, \ u \in \mathcal{O}_{\mathbf{L}} \quad \Rightarrow \quad \delta-1 \in \varpi_{\mathbf{L}}^n u - \overline{\varpi_{\mathbf{L}}^n u} + y^{-1}\mathcal{P}_{\mathbf{L}}^{1-e}.$$

Then we have $\beta(\delta)=1$, $\psi(\mathrm{Tr}(x_\tau\alpha(\delta-1))y) = \psi_{\mathbf{L}}(yx_\tau\alpha(\varpi_{\mathbf{L}}^n u - \overline{\varpi_{\mathbf{L}}^n u}))$ and get

$$H(\tau y^2) = \lambda(\mathbf{L}/\mathbf{F},\psi)|\tau|_{\mathbf{F}}^{\frac{1}{2}} \cdot |y|_{\mathbf{F}} \cdot \eta_{\mathbf{L}/\mathbf{F}}(y) \cdot \int_{\mathbf{L}^1\cap\mathcal{O}_{\mathbf{L}}} \beta(x_\tau\alpha)\psi(\mathrm{Tr}(x_\tau\alpha)y) \cdot \left[\oint_{\mathcal{O}_{\mathbf{L}}} \psi\left(y(x_\tau\alpha - \overline{x_\tau\alpha})(\varpi_{\mathbf{L}}^n u - \overline{\varpi_{\mathbf{L}}^n u}) \right) \mathrm{d}u \right] \mathrm{d}\alpha. \tag{3.12}$$

For any $m \in \mathbb{Z}$ we introduce $\mathcal{P}_{\mathbf{L}}^{m,-} := \{x \in \mathcal{P}_{\mathbf{L}}^m \mid \bar{x}=-x\}$. We see that

$$\mathcal{O}_{\mathbf{L}} \to \mathcal{P}_{\mathbf{L}}^{n,-}, \quad u \mapsto \varpi_{\mathbf{L}}^n u - \overline{\varpi_{\mathbf{L}}^n u}$$

is a surjective group homomorphism. Therefore the inner integral in (**??**) is non-vanishing only if

$$y(x_\tau\alpha - \overline{x_\tau\alpha}) \in \mathcal{P}_{\mathbf{L}}^{1-e-n,-} \quad \Leftrightarrow \quad v_{\mathbf{L}}(x_\tau\alpha - \overline{x_\tau\alpha}) \geqslant 1-e-n-ev_{\mathbf{F}}(y). \tag{3.13}$$

We choose $n = 1-2e-ev_{\mathbf{F}}(y)$, which is consistent with the condition (**??**) if

$$v_{\mathbf{F}}(y) \leqslant \lfloor \min(e^{-1}-2-e^{-1}n_0, e^{-1}-3) \rfloor = -e-e^{-1}n_0, \tag{3.14}$$

so that (**??**) becomes $v_{\mathbf{L}}(x_\tau\alpha - \overline{x_\tau\alpha}) \geqslant e$ or $v_{\mathbf{F}}((x_\tau\alpha - \overline{x_\tau\alpha})^2) \geqslant 2$. In view of the equality

$$\tau = x_\tau\alpha \cdot \overline{x_\tau\alpha} = \tfrac{1}{4}\left\{ (x_\tau\alpha + \overline{x_\tau\alpha})^2 - (x_\tau\alpha - \overline{x_\tau\alpha})^2 \right\} \tag{3.15}$$

and $v_{\mathbf{F}}(\tau) \in \{0,1\}$, we deduce that $v_{\mathbf{F}}(\tau) = v_{\mathbf{F}}((x_\tau\alpha + \overline{x_\tau\alpha})^2) \in 2\mathbb{Z}$. Hence $v_{\mathbf{F}}(\tau) = 0$ and τ is a square in $\mathcal{O}_{\mathbf{F}}^\times$. This is possible only if $\tau = 1$. In other words, if $\tau \neq 1$ and $v_{\mathbf{F}}(y)$ satisfies (**??**) then $H(\tau y^2) = 0$. We conclude because the only integer not satisfying (**??**) nor the inequality in (1) is $v_{\mathbf{F}}(y) = 1-e-e^{-1}n_0$.
(4) Suppose $\mathbf{L}/\mathbf{F}$ is unramified. Let $n := \lceil \frac{m}{2} \rceil$. Recall c_β defined in Lemma **??** (1). We average over the change of variables $\alpha \mapsto \alpha(1+u)(1+\bar{u})^{-1}$ for $u \in \mathcal{P}_{\mathbf{L}}^n$ and get

$$\begin{aligned} \int_{\mathbf{L}^1 \cap \mathcal{O}_{\mathbf{L}}} \beta(\alpha)\psi(y\mathrm{Tr}(\alpha))\mathrm{d}\alpha &= \int_{\mathbf{L}^1 \cap \mathcal{O}_{\mathbf{L}}} \beta(\alpha)\psi(y\mathrm{Tr}(\alpha)) \left\{ \oint_{\mathcal{P}_{\mathbf{L}}^n} \psi\left(\tfrac{2c_\beta\sqrt{\varepsilon}(u-\bar{u})}{\varpi_{\mathbf{F}}^{n_0}} + y\mathrm{Tr}\left(\alpha \cdot \tfrac{u-\bar{u}}{1+\bar{u}}\right)\right) \mathrm{d}u \right\} \mathrm{d}\alpha \\ &= \int_{\mathbf{L}^1 \cap \mathcal{O}_{\mathbf{L}}} \beta(\alpha)\psi(y\mathrm{Tr}(\alpha)) \left\{ \oint_{\mathcal{O}_{\mathbf{L}}} \psi\left((2\sqrt{\varepsilon}c_\beta\varpi_{\mathbf{F}}^{n-n_0} + y\varpi_{\mathbf{F}}^n(\alpha-\bar{\alpha}))(u-\bar{u}) \right) \mathrm{d}u \right\} \mathrm{d}\alpha. \end{aligned}$$

The non-vanishing of the inner integral implies

$$\left(2\sqrt{\varepsilon}c_\beta\varpi_{\mathbf{F}}^{n-n_0} + y\varpi_{\mathbf{F}}^n(\alpha-\bar{\alpha})\right)\sqrt{\varepsilon} \in \mathcal{O}_{\mathbf{F}} \ \Rightarrow$$

$$(\alpha-\bar{\alpha})\sqrt{\varepsilon} \in \begin{cases} \varpi_{\mathbf{F}}^{m-n_0}\mathcal{O}_{\mathbf{F}}^\times & \text{if } m < 2n_0-1 \\ \mathcal{P}_{\mathbf{F}}^{n_0-1} & \text{if } m = 2n_0-1 \end{cases} \Rightarrow (\alpha+\bar{\alpha})^2 \in \begin{cases} 4+\mathcal{P}_{\mathbf{F}}^{2(n_0-1)} & \text{if } m = 2n_0-1 \\ 4+\varpi_{\mathbf{F}}^{2(m-n_0)}\mathcal{O}_{\mathbf{F}}^\times & \text{if } n_0 < m < 2n_0-1 \\ \mathcal{O}_{\mathbf{F}} - 4(1+\mathcal{P}_{\mathbf{F}}) & \text{if } m = n_0 \end{cases}$$

from which we conclude. Replacing $\sqrt{\varepsilon}$ by 1 we obtain the proof in the split case.
(5) In the case $\tau = 1$ and $\frac{n_0}{2}+1 \leqslant m \leqslant n_0-1$, we average over the change of variables $\alpha \mapsto \alpha(1+u)(1+\bar{u})^{-1}$ for $u \in \mathcal{P}_{\mathbf{L}}^m$, noting that $\alpha\frac{u-\bar{u}}{1+\bar{u}} \in \alpha(u-\bar{u}) + y^{-1}\mathcal{O}_{\mathbf{L}}$, and get

$$\begin{aligned} \int_{\mathbf{L}^1} \beta(\alpha)\psi(y\mathrm{Tr}(\alpha))\mathrm{d}\alpha &= \int_{\mathbf{L}^1} \beta(\alpha)\psi(y\mathrm{Tr}(\alpha)) \left\{ \oint_{\mathcal{P}_{\mathbf{L}}^m} \psi\left(\tfrac{2c_\beta(u-\bar{u})}{\varpi_{\mathbf{L}}^{n_0+1}} + y\mathrm{Tr}\left(\alpha \cdot \tfrac{u-\bar{u}}{1+\bar{u}}\right)\right) \mathrm{d}u \right\} \mathrm{d}\alpha \\ &= \int_{\mathbf{L}^1} \beta(\alpha)\psi(y\mathrm{Tr}(\alpha)) \left\{ \oint_{\mathcal{P}_{\mathbf{L}}^m} \psi\left(\left(\tfrac{2c_\beta}{\varpi_{\mathbf{L}}^{n_0+1}} + (\alpha-\bar{\alpha})y \right)(u-\bar{u}) \right) \mathrm{d}u \right\} \mathrm{d}\alpha \\ &= \int_{\mathbf{L}^1} \beta(\alpha)\psi(y\mathrm{Tr}(\alpha)) \left\{ \oint_{\mathcal{P}_{\mathbf{F}}^{\lfloor \frac{m}{2} \rfloor}} \psi\left(\left(\tfrac{2c_\beta}{\varpi_{\mathbf{F}}^{n_0/2}} + \varpi_{\mathbf{L}}(\alpha-\bar{\alpha})y \right)u \right) \mathrm{d}u \right\} \mathrm{d}\alpha. \end{aligned}$$

The non-vanishing of the inner integral implies

$$2c_\beta\varpi_{\mathbf{F}}^{-\frac{n_0}{2}} + \varpi_{\mathbf{L}}(\alpha-\bar{\alpha})y \in \mathcal{P}_{\mathbf{F}}^{-\lfloor \frac{m}{2} \rfloor} \ \Rightarrow \ \begin{cases} (\alpha-\bar{\alpha})\varpi_{\mathbf{L}}^{-1} \in \varpi_{\mathbf{F}}^{m-\frac{n_0}{2}-1}\mathcal{O}_{\mathbf{F}}^\times & \text{if } m < n_0 \\ (\alpha-\bar{\alpha})\varpi_{\mathbf{L}}^{-1} \in \varpi_{\mathbf{F}}^{\frac{n_0}{2}-1}\mathcal{O}_{\mathbf{F}} & \text{if } m = n_0 \end{cases}.$$

We conclude by noting that for any $m \in \mathbb{Z}_{\geqslant 0}$ we have

$$(\alpha-\bar{\alpha})\varpi_{\mathbf{L}}^{-1} \in \mathcal{P}_{\mathbf{F}}^m \ \Leftrightarrow \ \mathrm{Tr}(\alpha) \in \pm 2(1+\mathcal{P}_{\mathbf{F}}^{2m+1}),$$

and that $\mathrm{Tr}(\alpha)^2 - 4 = (\alpha-\bar{\alpha})^2$ never has even valuation. □

Remark 3.11. *If β is not regular, which can happen only if $\mathbf{L}/\mathbf{F}$ is split with $\mathfrak{c}(\chi_0^2) = 0$, then (2) fails. In fact, it is easy to see that $H(\tau y^2)$ is proportional to $\chi_0(\tau)|y|$ as $|y| \to 0$. In this case we modify the definition of H by* truncating it *so that (2) still holds.*

Corollary 3.12. *The test function H defined by* (**??**) *and Remark* **??** *is a Bessel orbital integral.*

Proof. This is a direct consequence of Lemma **??** (1) & (2) and Proposition **??**. □

3.2. **Weight Functions.** To every target representation π_0 we have associated a parameter $(\mathbf{L}, \beta)$ and a test function H in (**??**).

Lemma 3.13. *(1) In the split case $\pi_0 = \pi(\chi_0, \chi_0^{-1})$ consider the function $\phi_0 \in \mathrm{C}_c^{\infty}(\mathrm{PGL}_2(\mathbf{F}))$ given by*

$$\phi_0 \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \chi_0 \left(\frac{x_4}{x_1} \right) \mathbb{1}_{\mathbf{ZK}_0[\mathcal{P}_{\mathbf{F}}^{n_0}]} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

The partial integral I_0 defined by

$$I_0(g) := \int_{\mathbf{F}} \phi_0 \left(g \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \right) \psi(-x) \mathrm{d}x$$

has support contained in $\mathbf{ZK}_0[\mathcal{P}_{\mathbf{F}}^{n_0}]\mathbf{N}(\mathbf{F})$, and satisfies

$$I_0(\kappa n(x)) = \phi_0(\kappa)\psi(x), \quad \forall \kappa \in \mathbf{ZK_0}, x \in \mathbf{F}.$$

(2) We can identify H as a Bessel orbital integral

$$H(y) = \int_{\mathbf{F}^2} \phi_0 \left(\begin{pmatrix} 1 & x_1 \\ & 1 \end{pmatrix} \begin{pmatrix} & -y \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & x_2 \\ & 1 \end{pmatrix} \right) \psi(-x_1 - x_2) \mathrm{d}x_1 \mathrm{d}x_2.$$

(3) The weight function $h(\pi)$ is non-negative. We have $h(\pi) \neq 0$ only if $\mathfrak{c}(\pi \otimes \chi_0^{-1}) \leqslant n_0$, in which case we have a lower bound $h(\pi) \gg q^{-n_0}$.

Proof. (1) Note that ϕ_0 is a character upon restriction to $\mathbf{ZK}_0[\mathcal{P}_{\mathbf{F}}^{n_0}]$. Hence $I_0(g)$ satisfies

$$I_0(\kappa g n(x)) = \phi_0(\kappa)\psi(x) I_0(g), \quad \forall \kappa \in \mathbf{ZK_0}, x \in \mathbf{F}.$$

From the Cartan decomposition $\bigsqcup_{n \in \mathbb{Z}_{\geqslant 0}} \mathbf{ZK} \begin{pmatrix} \varpi_{\mathbf{F}}^n & \\ & 1 \end{pmatrix} \mathbf{N}(\mathbf{F})$ we deduce $\mathrm{supp}(I_0) \subset \mathbf{ZKN}(\mathbf{F})$. We have

$$\mathbf{ZKN}(\mathbf{F}) = \mathbf{ZK}_0[\mathcal{P}_{\mathbf{F}}^{n_0}] \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \mathbf{N}(\mathbf{F}) \sqcup \bigcup_{u \in \mathcal{P}_{\mathbf{F}}/\mathcal{P}_{\mathbf{F}}^{n_0}} \mathbf{ZK}_0[\mathcal{P}_{\mathbf{F}}^{n_0}] \begin{pmatrix} 1 & \\ u & 1 \end{pmatrix} \mathbf{N}(\mathbf{F})$$

by the Bruhat decomposition over the residue field of $\mathbf{F}$. We easily verify for $u \notin \mathcal{P}_{\mathbf{F}}^{n_0}$

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \mathbf{N}(\mathbf{F}) \cap \mathbf{ZK}_0[\mathcal{P}_{\mathbf{F}}^{n_0}] = \emptyset, \quad \begin{pmatrix} 1 & \\ u & 1 \end{pmatrix} \mathbf{N}(\mathbf{F}) \cap \mathbf{ZK}_0[\mathcal{P}_{\mathbf{F}}^{n_0}] = \emptyset.$$

Hence we deduce $\mathrm{supp}(I_0) \subset \mathbf{ZK}_0[\mathcal{P}_{\mathbf{F}}^{n_0}]\mathbf{N}(\mathbf{F})$ and conclude by $I_0(\mathbb{1}) = \mathrm{Vol}(\mathcal{O}_{\mathbf{F}}) = 1$.
(2) Let $\phi^{\iota}(g) := \overline{\phi(g^{-1})}$ for all $\phi \in \mathrm{C}_c^{\infty}(\mathrm{PGL}_2(\mathbf{F}))$. We have $\phi_0^{\iota} = \phi_0$ and $\phi_0^{\iota} * \phi_0 = \mathrm{Vol}(\mathbf{ZK}_0[\mathcal{P}_{\mathbf{F}}^{n_0}])\phi_0$ since ϕ_0 is a unitary character upon restriction to $\mathbf{ZK}_0[\mathcal{P}_{\mathbf{F}}^{n_0}]$. Here the convolution is taken over the group $G := \mathrm{PGL}_2(\mathbf{F})$. We can rewrite the relevant Bessel orbital integral as

$$\begin{aligned} &\mathrm{Vol}(\mathbf{ZK}_0[\mathcal{P}_{\mathbf{F}}^{n_0}])^{-1} \int_{\mathbf{F}^2} \int_G \overline{\phi_0(g)} \phi_0 \left(g \begin{pmatrix} 1 & x_1 \\ & 1 \end{pmatrix} \begin{pmatrix} & -y \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & x_2 \\ & 1 \end{pmatrix} \right) \psi(-x_1 - x_2) \mathrm{d}g \mathrm{d}x_1 \mathrm{d}x_2 \\ &= \mathrm{Vol}(\mathbf{ZK}_0[\mathcal{P}_{\mathbf{F}}^{n_0}])^{-1} \int_G \\ &\left\{ \int_{\mathbf{F}} \overline{\phi_0 \left(g \begin{pmatrix} 1 & -x_1 \\ & 1 \end{pmatrix} \right)} \psi(-x_1) \mathrm{d}x_1 \right\} \cdot \left\{ \int_{\mathbf{F}} \phi_0 \left(g \begin{pmatrix} & -y \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & x_2 \\ & 1 \end{pmatrix} \right) \psi(-x_2) \mathrm{d}x_2 \right\} \mathrm{d}g \\ &= \mathrm{Vol}(\mathbf{ZK}_0[\mathcal{P}_{\mathbf{F}}^{n_0}])^{-1} \int_G \overline{I_0(g)} I_0 \left(g \begin{pmatrix} & -y \\ 1 & \end{pmatrix} \right) \mathrm{d}g = \int_{\mathbf{F}} \psi(-u) I_0 \left(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} & -y \\ 1 & \end{pmatrix} \right) \mathrm{d}u. \end{aligned}$$

Considering the support of I_0, the above integral is non-vanishing only if for some $u' \in \mathbf{F}$

$$\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} & -y \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & u' \\ & 1 \end{pmatrix} = \begin{pmatrix} u & uu' - y \\ 1 & u' \end{pmatrix} \in \mathbf{ZK}_0[\mathcal{P}_{\mathbf{F}}^{n_0}],$$

implying $y \in \varpi_{\mathbf{F}}^{-2n}\mathcal{O}_{\mathbf{F}}^{\times}$ for some $n \geqslant n_0$, and $u = \varpi^{-n}u_1$, $u' = \varpi^{-n}u_2$ for some $u_1, u_2 \in \mathcal{O}_{\mathbf{F}}^{\times}$ satisfying $u_1u_2 - \varpi^{2n}y \in \mathcal{P}_{\mathbf{F}}^n$. Writing $y = \varpi_{\mathbf{F}}^{-2n}y_0$ for some $y_0 \in \mathcal{O}_{\mathbf{F}}^{\times}$ we may take $u_2 = u_1^{-1}y_0$. We recognize the Bessel orbital integral as $H(y)$ by the following equation, which is easy to verify

$$q^n \int_{\mathcal{O}_{\mathbf{F}}^{\times}} \chi_0(u_1^{-2}y_0)\psi\left(-\tfrac{u_1+u_1^{-1}y_0}{\varpi_{\mathbf{F}}^n}\right) \mathrm{d}u_1 = H(y).$$

(3) Up to notation this is due to Nelson [? , Theorem 3.1]. Our former work [? , Lemma 4.1] contains more details in a similar situation. □

Lemma 3.14. *Let π be a supercuspidal representation of $\mathrm{GL}_2(\mathbf{F})$. Let $C(g)$ be a matrix coefficient for smooth vectors in π. Take a non-trivial additive character ψ of $\mathbf{F}$. Then the (relative) orbital integral*

$$I(g) := \int_{\mathbf{F}^2} C(n(u_1)gn(u_2))\psi(-u_1 - u_2)du_1du_2$$

is equal to the ψ-Bessel function of π up to a constant.

Proof. Assume $C(g) = \langle \pi(g)v_1, v_2\rangle$ for smooth vectors v_1 and v_2. One checks that the functional

$$V_\pi^\infty \to \mathbb{C}, \quad v \mapsto \int_{\mathbf{F}} \langle \pi(n(u_1)).v, v_2\rangle \psi(-u_1)du_1$$

is a ψ-Whittaker functional. Hence the following function

$$W_{v_1}(g) := \int_{\mathbf{F}} C(n(u_1)g)\psi(-u_1)du_1$$

is the/a Whittaker function of v_1. We then apply [? , Lemma 4.1] to get

$$I(g) = \mathrm{j}_{\pi,\psi}(g) \cdot W_{v_1}(\mathbb{1}), \quad \forall g \in \mathbf{B}(\mathbf{F})w_2\mathbf{B}(\mathbf{F}).$$

The integral converges absolutely because C is smooth of compact support modulo the center. □

Lemma 3.15. *In the dihedral case $\pi_0 = \pi_\beta$ the weight function $h(\pi) \neq 0$ is non-vanishing only if $\pi \simeq \pi_\beta$, in which case it is positive and satisfies a lower bound $h(\pi_\beta) \gg q^{-e^{-1}n_0+1-e}$.*

Proof. Up to some constant approximately equal to 1 we recognize $H(y)$ as $\mathrm{j}_{\pi_\beta,\psi}\begin{pmatrix} & -y \\ 1 & \end{pmatrix}$ by [? , Theorem 1.1]. By Lemma ?? there exists some $\phi_0 \in \mathrm{C}_c^\infty(\mathrm{GL}_2(\mathbf{F}))$, whose integral along the center $\phi_1 \in \mathrm{C}_c^\infty(\mathrm{PGL}_2(\mathbf{F}))$ is a matrix coefficient of π_β, so that H is the Bessel orbital integral of ϕ_0 as given in Lemma ?? (1). Now that $\pi(\phi_1) \neq 0$ only if $\pi \simeq \pi_\beta$ by Schur's lemma, it remains to show that $h(\pi_\beta)$ is positive with the stated lower bound. By (??) the value $h(\pi_\beta)$ is the square of the L^2-norm of $H(y)$ up to a constant essentially equal to 1. Taking Lemma ?? (2) into account we have

$$\begin{aligned}
\int_{\mathbf{F}^\times} |H(y)|^2 \tfrac{\mathrm{d}^\times y}{|y|_{\mathbf{F}}} &= \frac{1}{2}\sum_\tau \int_{\mathbf{F}^\times} \left|H(\tau y^2)\right|^2 \tfrac{\mathrm{d}^\times y}{|\tau y^2|_{\mathbf{F}}} \\
&= \frac{1}{2}\sum_\tau \int_{\mathbf{F}^\times} \left(\int_{\mathbf{L}^1\times\mathbf{L}^1} \beta(x_\tau\alpha_1)\overline{\beta(x_\tau\alpha_2)}\psi_{\mathbf{L}}(x_\tau(\alpha_1-\alpha_2)y)\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \right) \mathrm{d}^\times y \\
&= \frac{1}{2}\sum_{n\geqslant(n_0-2)e^{-1}+2} \int_{\mathbf{L}^1} \beta(\alpha) \left(\sum_\tau \int_{\mathbf{L}^1}\int_{\varpi_{\mathbf{F}}^{-n}\mathcal{O}_{\mathbf{F}}^\times} \psi_{\mathbf{L}}(x_\tau\alpha_2 y(\alpha-1))\mathrm{d}^\times y\mathrm{d}\alpha_2 \right) \mathrm{d}\alpha,
\end{aligned}$$

where we applied the change of variables $\alpha_1 \mapsto \alpha\alpha_2$ in the last line. By measure relation (??) we get

$$\frac{1}{2}\sum_\tau \int_{\mathbf{L}^1}\int_{\varpi_{\mathbf{F}}^{-n}\mathcal{O}_{\mathbf{F}}^\times} \psi_{\mathbf{L}}(x_\tau\alpha_2 y(\alpha-1))\mathrm{d}^\times y\mathrm{d}\alpha_2 = \zeta_{\mathbf{F}}(1)\int_{-en\leqslant v_{\mathbf{L}}(z)<e-en} \psi_{\mathbf{L}}(z(\alpha-1))\frac{\mathrm{d}z}{|z|_{\mathbf{L}}}.$$

Therefore we continue the calculation as

$$\int_{\mathbf{F}^\times} |H(y)|^2 \tfrac{\mathrm{d}^\times y}{|y|_{\mathbf{F}}} = \zeta_{\mathbf{F}}(1)\sum_{n\geqslant n_0-1+e} \int_{\mathbf{L}^1} \beta(\alpha) \left(\int_{\varpi_{\mathbf{L}}^{-n}\mathcal{O}_{\mathbf{L}}^\times} \psi_{\mathbf{L}}(z(\alpha-1))\tfrac{\mathrm{d}z}{|z|_{\mathbf{L}}} \right) \mathrm{d}\alpha$$

$$= \frac{\zeta_{\mathbf{F}}(1)}{\zeta_{\mathbf{L}}(1)} q_{\mathbf{L}}^{-\frac{e-1}{2}} \sum_{n \geqslant n_0-1+e} \int_{\mathbf{L}^1} \beta(\alpha) \left(\mathbb{1}_{\mathcal{P}_{\mathbf{L}}^{n+1-e}}(\alpha-1) - q_{\mathbf{L}}^{-1} \mathbb{1}_{\mathcal{P}_{\mathbf{L}}^{n-e}}(\alpha-1) \right) \mathrm{d}\alpha$$

$$= \frac{\zeta_{\mathbf{F}}(1)}{\zeta_{\mathbf{L}}(1)^2} q_{\mathbf{L}}^{-\frac{e-1}{2}} \sum_{n \geqslant n_0} \mathrm{Vol}(\mathbf{L}^1 \cap (1+\mathcal{P}_{\mathbf{L}}^n)),$$

and conclude the lower bound by Proposition **??** (3) and the fact $e \mid n_0$ (see Claim in Lemma **??**). □

Remark 3.16. *By (**??**), the bounds in Lemma **??** (1) & **??** can be rewritten as* $h(\pi_0) \gg q^{-\lceil \frac{\mathfrak{c}(\pi_0)}{2} \rceil}$.

4. Dual Weight Functions: Common Reductions

4.1. **Infinite Part.** Recall the constant $a(\Pi)$ for the stability range defined in Proposition **??**. Take the "infinite part" of the test function as $H_\infty = E_{\geqslant n_1}$ for some

$$n_1 \geqslant \max(2n_0/e + e - 1, a(\Pi)) (\geqslant 2). \tag{4.1}$$

Recall the formula for the dual weight function

$$\widetilde{h}(\chi) = \int_{\mathbf{F}^\times} \widetilde{\mathcal{VH}}_{\Pi,\psi} \circ \mathfrak{m}_{-1}(H)(t) \cdot \psi(-t) \chi^{-1}(t) |t|^{-\frac{1}{2}} \mathrm{d}^\times t. \tag{4.2}$$

Lemma 4.1. *The partial dual weight function*

$$\widetilde{h}_\infty(\chi) := \int_{\mathbf{F}^\times} \widetilde{\mathcal{VH}}_{\Pi,\psi} \circ \mathfrak{m}_{-1}(H_\infty)(t) \cdot \psi(-t) \chi^{-1}(t) |t|^{-\frac{1}{2}} \mathrm{d}^\times t$$

is non-vanishing only if χ is unramified, in which case we have

$$\widetilde{h}_\infty(\chi) = \frac{\chi(\varpi_{\mathbf{F}})^{n_1} q^{-\frac{n_1}{2}}}{1 - \chi(\varpi_{\mathbf{F}}) q^{-\frac{1}{2}}}.$$

Proof. By Proposition **??** we have

$$\widetilde{\mathcal{VH}}_{\Pi,\psi} \circ \mathfrak{m}_{-1}(E_{\geqslant n_1})(t) = \psi(t) \cdot \mathbb{1}_{\mathcal{P}_{\mathbf{F}}^{n_1}}(t^{-1}).$$

The desired formula follows readily from the above one. □

Taking into account Lemma **??** (2) we introduce the "finite part" of the test function as

$$H_c := H - H_\infty = \sum\nolimits_{\substack{n=\frac{2}{e}(n_0-2)+3 \\ 2|en}}^{2n_1-1} H_n, \qquad H_n := H \cdot \mathbb{1}_{\varpi_{\mathbf{F}}^{-n} \mathcal{O}_{\mathbf{F}}^\times}. \tag{4.3}$$

Since each function $H_n \in \mathrm{C}_c^\infty(\mathbf{F}^\times)$, we may replace the extended Voronoi–Hankel transform with its original version in the dual weight formula and turn to the study of

$$\widetilde{h}_n(\chi) := \int_{\mathbf{F}^\times} \mathcal{VH}_{\Pi,\psi} \circ \mathfrak{m}_{-1}(H_n)(t) \cdot \psi(-t) \chi^{-1}(t) |t|^{-\frac{1}{2}} \mathrm{d}^\times t. \tag{4.4}$$

We divide the domain of integration into two parts: $\mathcal{O}_{\mathbf{F}} - \{0\}$ and $\mathbf{F} - \mathcal{O}_{\mathbf{F}}$, giving the decomposition

$$\widetilde{h}_n(\chi) = \widetilde{h}_n^+(\chi) + \widetilde{h}_n^-(\chi). \tag{4.5}$$

Accordingly the dual weight functions (see (**??**)) have the following decomposition

$$\widetilde{h}(\chi) = \widetilde{h}_\infty(\chi) + \widetilde{h}_c(\chi), \quad \widetilde{h}_c(\chi) = \widetilde{h}_c^+(\chi) + \widetilde{h}_c^-(\chi), \quad \widetilde{h}_c^\pm(\chi) = \sum\nolimits_{\substack{n=\frac{2}{e}(n_0-2)+3 \\ 2|en}}^{2n_1-1} \widetilde{h}_n^\pm(\chi), \tag{4.6}$$

$$\widetilde{H}(\chi) = \widetilde{H}_\infty(\chi) + \widetilde{H}_c(\chi), \quad \widetilde{H}_c(\chi) = \widetilde{H}_c^+(\chi) + \widetilde{H}_c^-(\chi), \quad \widetilde{H}_c^\pm(\chi) = \sum\nolimits_{\substack{n=\frac{2}{e}(n_0-2)+3 \\ 2|en}}^{2n_1-1} \widetilde{H}_n^\pm(\chi). \tag{4.7}$$

We call those (partial) dual weight functions with "+" (resp. "−") the *postive* (resp. *negative*) part.

4.2. **Positive Part.** We first study $\widetilde{h}_n^+(\chi)$. Since ψ is trivial on $\mathcal{O}_{\mathbf{F}}$, we have $\widetilde{h}_n^+(\chi) = \widetilde{h}_n(1/2, \chi)$ for

$$\widetilde{h}_n(s,\chi) := \int_{\mathcal{O}_{\mathbf{F}}-\{0\}} \mathcal{V}\mathcal{H}_{\Pi,\psi} \circ \mathfrak{m}_{-1}(H_n)(t) \cdot \chi^{-1}(t)|t|^{-s}\mathrm{d}^\times t, \tag{4.8}$$

which is simply the partial sum of non-negative powers of $X = q^s$ in the Laurent series expansion of

$$f_n(q^s;\chi,H) = \int_{\mathbf{F}^\times} \mathcal{V}\mathcal{H}_{\Pi,\psi} \circ \mathfrak{m}_{-1}(H_n)(t) \cdot \chi^{-1}(t)|t|^{-s}\mathrm{d}^\times t. \tag{4.9}$$

By the local functional equation we have (see Lemma **??**)

$$f_n(q^s;\chi,H) = \varepsilon_n(\chi,H)q^{n(s-1)} \cdot \varepsilon\left(s, \Pi\otimes\chi,\psi\right) \frac{L(1-s,\widetilde{\Pi}\otimes\chi^{-1})}{L(s,\Pi\otimes\chi)}. \tag{4.10}$$

To relate $f_n(q^s;\chi,H)$ with $\widetilde{h}_n(s,\chi)$ we need the following crucial lemma.

Lemma 4.2. *Let $f(X) = \sum_{n>-\infty} a_n X^n$ be a Laurent series converging in $0<|X|<\rho$ for some $\rho>1$. Let $f_+(X) = \sum_{n>-\infty} a_n X^n$. Let $D = X\frac{\mathrm{d}}{\mathrm{d}X}$. Assume $f(X)$ (hence $f_+(X)$) has a meromorphic continuation to $X\in\mathbb{C}$.*

(0) *For any X and any $\epsilon < \min(1,\rho|X|^{-1})$ we have the relation*

$$f_+(X) = f(X) - \int_{|z|=\epsilon} \frac{f(Xz)}{1-z}\frac{\mathrm{d}z}{2\pi i}.$$

(1) *For $0<|X|<\rho$ and any $1<r<\rho|X|^{-1}$ we have the relation*

$$f_+(X) = \int_{|z|=r} \frac{f(Xz)}{z-1}\frac{\mathrm{d}z}{2\pi i} = f(X) + \int_{|z|=1} \frac{f(Xz)-f(X)}{z-1}\frac{\mathrm{d}z}{2\pi i}.$$

Consequently, for $0<|X|<\rho$ and $n\in\mathbb{Z}_{\geqslant 0}$ we have the bound

$$|(D^n f_+)(X)| \ll |(D^n f)(X)| + \sup\nolimits_{|z|=1}\left|(D^{n+1}f)(Xz)\right|.$$

(2) *Suppose $f(X) = Q(X)P(X)^{-1}$ for some $Q\in\mathbb{C}[X,X^{-1}], P\in\mathbb{C}[X]$. Let*

$$P(X) = \prod\nolimits_{j=1}^r (1-b_jX)^{m_j}$$

with $m_j\in\mathbb{Z}_{\geqslant 1}$ and distinct $b_j\neq 0$. Introduce

$$P_j(X) := \prod\nolimits_{i\neq j}(1-b_iX)^{m_i}, \quad C_{j,k} := \frac{(-1)^k}{k!\cdot b_j^k}\left(\frac{Q}{P_j}\right)^{(k)}(b_j^{-1}).$$

Suppose the highest power X^m in Q satisfies $m<\deg P$. Then we have

$$P(X)f_+(X) = \sum\nolimits_{j=1}^r \sum\nolimits_{k=0}^{m_j-1} C_{j,k}\cdot(1-b_jX)^kP_j(X).$$

Proof. (0) The stated formula follows from the residue theorem via

$$\int_{|z|=\epsilon}\frac{f(Xz)}{1-z}\frac{\mathrm{d}z}{2\pi i} = \int_{|z|=\epsilon}\left(\sum_{n>-\infty}a_nX^nz^n\right)\left(\sum_{k=0}^\infty z^k\right)\frac{\mathrm{d}z}{2\pi i} = \sum_{n<0}a_nX^n,$$

since only the terms for $n+k=-1$ give non-zero contribution.
(1) Clearly the stated formula follows from the one in (0) by the residue theorem. To see the bound, we introduce $g(z) = f(Xz)$ and rewrite the integral as

$$\int_{|z|=1}\frac{f(Xz)-f(X)}{z-1}\frac{\mathrm{d}z}{2\pi i} = \int_{-\pi}^{\pi}\frac{g(e^{i\theta})-g(1)}{e^{i\theta}-1}\frac{\mathrm{d}\theta}{2\pi} = \int_{-\pi}^{\pi}\frac{i}{e^{i\theta}-1}\left(\int_0^\theta g'(e^{it})e^{it}\mathrm{d}t\right)\frac{\mathrm{d}\theta}{2\pi}.$$

Note that $zg'(z) = Df(Xz)$. Therefore we obtain the formula

$$f_+(X) = f(X) + \int_{-\pi}^{\pi}\frac{i}{e^{i\theta}-1}\left(\int_0^\theta (Df)(Xe^{it})\mathrm{d}t\right)\frac{\mathrm{d}\theta}{2\pi}.$$

Applying D^n on both sides we get a formula and conclude the stated bound as

$$\begin{aligned}(D^n f_+)(X) &= (D^n f)(X) + \int_{-\pi}^{\pi} \frac{i}{e^{i\theta}-1} \left(\int_0^{\theta} (D^{n+1} f)(Xe^{it}) \mathrm{d}t \right) \frac{\mathrm{d}\theta}{2\pi} \\ &\ll |(D^n f)(X)| + \sup_{|z|=1} \left| (D^{n+1} f)(Xz) \right| \cdot \int_{-\pi}^{\pi} \left| \frac{\theta}{e^{i\theta}-1} \right| \frac{\mathrm{d}\theta}{2\pi}.\end{aligned}$$

(2) We may assume $0 < |X| < \rho$ and depart from the first equation in (1), then move the contour to $|z| = r$ for $r \gg 1$, picking up the residues. The contour integral tends to 0 as $r \to +\infty$ due to the assumption $m < \deg P$. Multiplying the resulted formula by $P(X)$, we get the stated formula. □

Definition 4.3. *For any generic irreducible representation Π of $\mathrm{GL}_r(\mathbf{F})$, let $d(\Pi)$ be the degree of the L-function $L(s,\Pi)$, namely the degree of the polynomial $L(s,\Pi)^{-1}$ in $X := q^{-s}$. Write*

$$\rho(\Pi) := \mathfrak{c}(\Pi) + d(\Pi).$$

Introduce the set of exponents *of Π by*

$$\mathrm{E}(\Pi) := \left\{ \xi \in \widehat{\mathcal{O}_{\mathbf{F}}^{\times}} \ \middle| \ d(\Pi \otimes \xi) > 0 \right\}.$$

Lemma 4.4. *Let Π be a generic irreducible representation of $\mathrm{GL}_3(\mathbf{F})$. We have $\rho(\Pi) \geqslant 3$, $|\mathrm{E}(\Pi)| \leqslant 3$. For any $\xi \in \mathrm{E}(\Pi)$ we have the bounds*

$$\mathfrak{c}(\xi) \leqslant \mathfrak{c}(\Pi), \quad \mathfrak{c}(\Pi \otimes \xi) \leqslant 2\mathfrak{c}(\Pi), \quad \rho(\Pi \otimes \xi) \leqslant 2\mathfrak{c}(\Pi) + 3.$$

Proof. The upper bound for $\rho(\Pi \otimes \xi)$ obviously follows from the one for $\mathfrak{c}(\Pi \otimes \xi)$ since $d(\Pi \otimes \xi) \leqslant 3$ in any case. For the other bounds we distinguish cases for Π.
(1) If $\mathfrak{c}(\Pi) = 0$, then Π is spherical, $\mathrm{E}(\Pi) = \{\mathbb{1}\}$ and $d(\Pi) = 3$. The stated bounds clearly hold.
(2) If $\mathfrak{c}(\Pi) > 0$ and if $\Pi = \chi_1 \boxplus \chi_2 \boxplus \chi_3$ is induced from the Borel subgroup, then we have

$$\rho(\Pi) = \sum\nolimits_{i=1}^{3} \rho(\chi_i) = \sum\nolimits_{i=1}^{3} \max(\mathfrak{c}(\chi_i), 1) \geqslant 3.$$

We also have $\mathrm{E}(\Pi) \subset \{\xi_1^{-1}, \xi_2^{-1}, \xi_3^{-1}\}$ with $\chi_j \mid_{\mathcal{O}_{\mathbf{F}}^{\times}} = \xi_j$, from which we deduce the bound for $|\mathrm{E}(\Pi)|$. Obviously we have $\mathfrak{c}(\xi_j) \leqslant \mathfrak{c}(\Pi)$. Taking $\xi = \xi_1^{-1}$ for example, we have the second stated bound as

$$\mathfrak{c}(\Pi \otimes \xi) = \mathfrak{c}(\chi_2 \xi_1^{-1}) + \mathfrak{c}(\chi_3 \xi_1^{-1}) \leqslant 2 \max\{\mathfrak{c}(\chi_j) : 1 \leqslant j \leqslant 3\} \leqslant 2\mathfrak{c}(\Pi).$$

(3) If $\mathfrak{c}(\Pi) > 0$ and if $\Pi = \pi \boxplus \chi$ for some supercuspidal π of $\mathrm{GL}_2(\mathbf{F})$, then the central character of π is χ^{-1}. We have $\rho(\Pi) = \mathfrak{c}(\pi) + \max(\mathfrak{c}(\chi), 1) \geqslant 2 + 1 = 3$. We also have $\mathrm{E}(\Pi) = \{\chi^{-1} \mid_{\mathcal{O}_{\mathbf{F}}^{\times}}\}$. For $\xi = \chi^{-1} \mid_{\mathcal{O}_{\mathbf{F}}^{\times}}$ we have $\mathfrak{c}(\xi) = \mathfrak{c}(\chi) \leqslant \mathfrak{c}(\Pi)$ and get the second stated bound via $\pi \otimes \chi^{-1} \simeq \widetilde{\pi}$ as

$$\mathfrak{c}(\Pi \otimes \xi) = \mathfrak{c}(\pi \otimes \chi^{-1}) = \mathfrak{c}(\widetilde{\pi}) = \mathfrak{c}(\pi) \leqslant \mathfrak{c}(\Pi).$$

(4) If $\mathfrak{c}(\Pi) > 0$ and if $\Pi = \mathrm{St}_{\eta} \boxplus \chi$, then $\chi\eta^2 = \mathbb{1}$. By [? , §8 Proposition & §10 Proposition] we have

$$\rho(\Pi) = \begin{cases} 2\mathfrak{c}(\eta) & \text{if } \mathfrak{c}(\eta) \geqslant 1 \\ 1 & \text{if } \mathfrak{c}(\eta) = 0 \end{cases} + d(\eta) + \max(\mathfrak{c}(\chi), 1) \geqslant 3.$$

We also have $\mathrm{E}(\Pi) = \{\chi^{-1} \mid_{\mathcal{O}_{\mathbf{F}}^{\times}}, \eta^{-1} \mid_{\mathcal{O}_{\mathbf{F}}^{\times}}\}$. For $\xi = \chi^{-1} \mid_{\mathcal{O}_{\mathbf{F}}^{\times}}$ we argue the same as in the proof of (3). For $\xi = \eta^{-1} \mid_{\mathcal{O}_{\mathbf{F}}^{\times}}$ we have

$$\mathfrak{c}(\xi) = \mathfrak{c}(\eta) < 1 + \mathfrak{c}(\eta) = \mathfrak{c}(\Pi \otimes \xi) \leqslant \max(2\mathfrak{c}(\eta), 1) + \mathfrak{c}(\chi) = \mathfrak{c}(\Pi).$$

(5) If $\mathfrak{c}(\Pi) > 0$ and if $\Pi = \mathrm{St}_{\eta}$ is a twist of the Steinberg representation of $\mathrm{PGL}_3(\mathbf{F})$, then by [? , §8 Proposition & §10 Proposition] we have

$$\rho(\Pi) = \begin{cases} 3\mathfrak{c}(\eta) & \text{if } \mathfrak{c}(\eta) \geqslant 1 \\ 2 & \text{if } \mathfrak{c}(\eta) = 0 \end{cases} + d(\eta) \geqslant 3.$$

We also have $\mathrm{E}(\Pi)=\{\eta^{-1}\mid_{\mathcal{O}_{\mathbf{F}}^{\times}}\}$. For $\xi=\eta^{-1}\mid_{\mathcal{O}_{\mathbf{F}}^{\times}}$ we have

$$\mathfrak{c}(\xi)=\mathfrak{c}(\eta)\leqslant \max(3\mathfrak{c}(\eta),2)=\mathfrak{c}(\Pi),\quad \mathfrak{c}(\Pi\otimes\xi)=2\leqslant\mathfrak{c}(\Pi).$$

(6) If Π is supercuspidal, then $\mathrm{E}(\Pi)=\emptyset$. We have $\rho(\Pi)=\mathfrak{c}(\Pi)\geqslant 3$. □

The following subset of $\widehat{\mathcal{O}_{\mathbf{F}}^{\times}}$ will turn out to be important for the bound of $\widetilde{h}_n^+(\chi)$:

$$\mathcal{A}_n=\mathcal{A}_n(\beta,\Pi):=\left\{\xi\in\widehat{\mathcal{O}_{\mathbf{F}}^{\times}}\;\middle|\;\rho(\Pi\otimes\xi)\leqslant n,\ \varepsilon_n(\xi,H)\neq 0\right\}. \tag{4.11}$$

Lemma 4.5. *For any $\xi\in\mathcal{A}_n$ we have $\mathfrak{c}(\xi)\leqslant\max\left(\frac{n_0}{e},2\mathfrak{c}(\Pi)\right)$.*

Proof. Suppose $\xi\in\mathcal{A}_n$ and $\mathfrak{c}(\xi)>\max\left(\frac{n_0}{e},2\mathfrak{c}(\Pi)\right)$. In particular, we have $\mathfrak{c}(\xi)\geqslant a(\Pi)$, the stability barrier of Π, by Proposition **??**. Therefore we get $\mathfrak{c}(\Pi\otimes\xi)=3\mathfrak{c}(\xi)$, $d(\Pi\otimes\xi)=0$ and deduce

$$3\mathfrak{c}(\xi)=\rho(\Pi\otimes\xi)\leqslant n. \tag{4.12}$$

On the other hand, we claim that the condition $\mathfrak{c}(\xi)>\frac{n_0}{e}$ and $\varepsilon_n(\xi,H)\neq 0$ imply

$$\mathfrak{c}(\xi)=\tfrac{n}{2}, \tag{4.13}$$

and conclude the proof by comparing (**??**) and (**??**) taking into account $n\geqslant\frac{2}{e}(n_0-2)+3>0$. In fact, the proof of (**??**) is case-by-case check with Lemma **??**: (1) If $\mathbf{L}/\mathbf{F}$ is split, then $e=1$ and $\mathfrak{c}(\xi)>n_0=\mathfrak{c}(\chi_0^{\pm1})$. Only the first case in Lemma **??** is possible, yielding (**??**). (2) If $\mathbf{L}/\mathbf{F}$ is unramified, then $e=1$ and by (**??**) we have $\mathfrak{c}(\xi\circ\mathrm{Nr})=\mathfrak{c}(\xi)>n_0=\mathfrak{c}(\beta)$. The last case in Lemma **??** yields (**??**). (3) If $\mathbf{L}/\mathbf{F}$ is ramified, then $e=2$ and by (**??**) we have $\mathfrak{c}(\xi\circ\mathrm{Nr})\geqslant 2\mathfrak{c}(\xi)-1\geqslant 2(n_0/2+1)-1=n_0+1>\mathfrak{c}(\beta)$. The last case in Lemma **??** yields (**??**). □

Lemma 4.6. *Write $\xi=\chi\mid_{\mathcal{O}_{\mathbf{F}}^{\times}}$. For any $\epsilon>0$ sufficiently small we have the bound*

$$\left|\widetilde{h}_c^+(\chi)\right|\leqslant\sum\nolimits_{\frac{2}{e}(n_0-2)+3\leqslant n<\rho(\Pi\otimes\xi)}\left|\widetilde{h}_n^+(\chi)\right|+\sum\nolimits_{n\geqslant\rho(\Pi\otimes\chi)}\left|\widetilde{h}_n^+(\chi)\right|$$
$$\ll_\epsilon \mathbf{C}(\Pi)^\epsilon q^{-\frac{n_0}{e}+\frac{1-e}{2}}\cdot\mathbb{1}_{\leqslant\max\left(\frac{n_0}{e},2\mathfrak{c}(\Pi)\right)}(\mathfrak{c}(\xi)).$$

Proof. Write the relevant L-functions of $\Pi\otimes\chi$ as

$$L(s,\Pi\otimes\chi)=\prod\nolimits_{j=1}^{d}(1-a_jq^{-s})^{-1}$$

where $d=d(\Pi\otimes\chi)\in\{0,1,2,3\}$ is the degree, and $a_j=a_j(\Pi\otimes\chi)$ are the *generalized Satake parameters* of $\Pi\otimes\chi$ satisfying $|a_j|\in\{1,q^{-\frac{1}{2}},q^{-1}\}$ by temperedness. We can rewrite (**??**) as

$$f_n(X;\chi,H)=\varepsilon_n(\chi,H)\varepsilon\left(\tfrac{1}{2},\Pi\otimes\chi,\psi\right)q^{\frac{\mathfrak{c}(\Pi\otimes\chi)}{2}-n}X^{n-\rho(\Pi\otimes\chi)}\cdot\prod\nolimits_{j=1}^{d}\tfrac{X-a_j}{1-\overline{a_j}q^{-1}X}. \tag{4.14}$$

If $n\geqslant\rho:=\rho(\Pi\otimes\chi)$, then $f_n(X;\chi,H)\neq 0$ implies $\varepsilon_n(\chi,H)\neq 0$, hence $\chi\mid_{\mathcal{O}_{\mathbf{F}}^{\times}}\in\mathcal{A}_n$ by definition. The Laurent expansion of $f_n(X;\chi,H)$ contains only non-negative powers of X. Therefore the right hand side is equal to $\widetilde{h}_n(s,\chi)$ by definition (**??**). Consequently we get the formula and deduce the bound

$$\widetilde{h}_n(s,\chi)=\varepsilon_n(\chi,H)\varepsilon\left(\tfrac{1}{2},\Pi\otimes\chi,\psi\right)q^{\frac{\mathfrak{c}(\Pi\otimes\chi)}{2}-n}q^{s(n-\mathfrak{c}(\Pi\otimes\chi))}\cdot\prod\nolimits_{j=1}^{d}\tfrac{1-a_jq^{-s}}{1-\overline{a_j}q^{s-1}},$$

$$\left|\widetilde{h}_n^+(\chi)\right|=\left|\widetilde{h}_n(1/2,\chi)\right|\ll_{\vartheta_3}\mathbb{1}_{\mathcal{A}_n}(\chi\mid_{\mathcal{O}_{\mathbf{F}}^{\times}})\cdot q^{-\frac{n}{2}}. \tag{4.15}$$

If $n<\rho$, then the Laurent expansion of $f_n(X;\chi,H)$ contains non-negative powers of X only if $d=d(\Pi\otimes\chi)>0$, i.e. $\chi\mid_{\mathcal{O}_{\mathbf{F}}^{\times}}\in\mathrm{E}(\Pi)$ where $\mathrm{E}(\Pi)$ is the set of exponents of Π introduced in Definition **??**. By Lemma **??** and the summation range of n we have

$$\tfrac{2}{e}(n_0-2)+3\leqslant n\leqslant 2\mathfrak{c}(\Pi)+2. \tag{4.16}$$

Note that the Laurent expansion of $f_n(X;\chi,H)$ is absolutely convergent for $|X|<q$. Note also that $Xf_n'(X;\chi,H)$ and $f_n(X;\chi,H)$ have the same type of bound for $|X|=q^{\frac{1}{2}}$. By Lemma **??** (1) and (**??**) we deduce

$$\begin{aligned}(4.17)\qquad \left|\widetilde{h}_n^+(\chi)\right| = \left|\widetilde{h}_n(1/2,\chi)\right| &\leqslant \left|f_n(q^{\frac{1}{2}};\chi,H)\right| + \sup\nolimits_{|X|=q^{\frac{1}{2}}} |Xf_n'(X;\chi,H)| \\ &\ll \mathbb{1}_{\frac{2}{e}(n_0-2)+3\leqslant n\leqslant \rho(\Pi\otimes\chi)-1}\mathbb{1}_{\mathrm{E}(\Pi)}(\chi\mid_{\mathcal{O}_{\mathbf{F}}^\times})\cdot q^{-\frac{n}{2}}\cdot(1+\rho(\Pi\otimes\chi)-n) \\ &\ll \mathbb{1}_{\frac{2}{e}(n_0-2)+3\leqslant n\leqslant 2\mathfrak{c}(\Pi)+2}\mathbb{1}_{\mathrm{E}(\Pi)}(\chi\mid_{\mathcal{O}_{\mathbf{F}}^\times})\cdot q^{-\frac{n}{2}}\cdot(2\mathfrak{c}(\Pi)+3).\end{aligned}$$

by Lemma **??** & **??**. We conclude by summing over n (for $2\mid en$) the bounds (**??**) and (**??**), taking into account the bound for $\mathbb{1}_{\mathcal{A}_n}$ given by Lemma **??**. □

4.3. **Negative Part.** We turn to the study of a "trivial" bound of $\widetilde{h}_n^-(\chi)$. We introduce

$$(4.18)\qquad \mathcal{B}_n=\mathcal{B}_n(\beta,\Pi) := \left\{\xi\in\widehat{\mathcal{O}_{\mathbf{F}}^\times} \;\middle|\; \rho(\Pi\otimes\xi)>n,\ \varepsilon_n(\xi,H)\neq 0\right\}.$$

Lemma 4.7. *Recall $e=e(\mathbf{L}/\mathbf{F})$. We have $\mathcal{B}_n\subset\left\{\xi\in\widehat{\mathcal{O}_{\mathbf{F}}^\times}\;\middle|\;\mathfrak{c}(\xi)\leqslant n/2\right\}$ and the bound $|\mathcal{B}_n|\ll q^{\frac{n}{2}}$.*

Proof. For $\xi\in\mathcal{B}_n$, the condition $\varepsilon_n(\xi,H)\neq 0$ implies $\mathfrak{c}(\xi)\leqslant\frac{n}{2}$ by checking Lemma **??** case-by-case, taking into account (**??**). For example in the case $\mathbf{L}/\mathbf{F}$ is ramified, we have $n\geqslant n_0+1$ and

$$2\mathfrak{c}(\xi)-1\leqslant\mathfrak{c}(\xi\circ\mathrm{Nr})\leqslant\max\left(\mathfrak{c}(\beta),\mathfrak{c}(\beta\cdot(\xi\circ\mathrm{Nr}))\right)=\max(n_0,n-1)=n-1.$$

Therefore we get $|\mathcal{B}_n|\leqslant\left|\mathcal{O}_{\mathbf{F}}^\times/(1+\mathcal{P}_{\mathbf{F}}^{n/2})\right|\ll q^{\frac{n}{2}}$. □

Lemma 4.8. *We have the bound*

$$\widetilde{h}_n^-(\chi)\ll q^{-\frac{1}{2}}\cdot\mathbb{1}_{\mathfrak{c}(\chi)\leqslant\mathfrak{c}(\Pi)+\frac{n}{2}}+\mathbb{1}_{n\leqslant 2\mathfrak{c}(\Pi)+2}\cdot q^{-\frac{n}{2}}\sum\nolimits_{m=1}^{2\mathfrak{c}(\Pi)+3-n}q^{-\frac{m}{2}}\mathbb{1}_{\mathfrak{c}(\chi)\leqslant\max(\mathfrak{c}(\Pi),m)}.$$

Consequently if $\mathfrak{c}(\Pi)>0$ and $n_0\leqslant A\mathfrak{c}(\Pi)$ for some constant $A\geqslant 1$ then we have

$$\widetilde{h}_c^-(\chi)\ll\sum\nolimits_{\substack{n=\frac{2}{e}(n_0-2)+3\\ 2\mid en}}^{2n_1-1}\left|\widetilde{h}_n^-(\chi)\right|\ll\mathfrak{c}(\Pi)\mathbb{1}_{\mathfrak{c}(\chi)\leqslant 3A\mathfrak{c}(\Pi)}.$$

Proof. Write $\chi_0=\chi\mid_{\mathcal{O}_{\mathbf{F}}^\times}$. By definition and the Plancherel formula on $\mathcal{O}_{\mathbf{F}}^\times$ we can write

$$\widetilde{h}_n^-(\chi)=\mathrm{Vol}(\mathcal{O}_{\mathbf{F}}^\times,\mathrm{d}^\times t)^{-1}q^{-n}\sum\nolimits_{m=1}^{\infty}\chi(\varpi_{\mathbf{F}})^m q^{-\frac{m}{2}}\cdot C_m(n,\chi_0^{-1}),$$

$$C_m(n,\chi_0^{-1}):=\sum\nolimits_{\xi\in\widehat{\mathcal{O}_{\mathbf{F}}^\times}}a_m(\xi;n)\cdot b_m(\xi\chi_0^{-1}),$$

where we have put

$$a_m(\xi;n):=\int_{\varpi_{\mathbf{F}}^{-m}\mathcal{O}_{\mathbf{F}}^\times}\mathcal{VH}_{\Pi,\psi}(H_n)(t)\xi^{-1}(t)\mathrm{d}^\times t,\quad b_m(\xi):=\int_{\mathcal{O}_{\mathbf{F}}^\times}\psi(-\varpi_{\mathbf{F}}^{-m}t)\xi(t)\mathrm{d}^\times t.$$

Since $m\geqslant 1$ the integral $b_m(\xi)$ is essentially a Gauss sum, which we can easily bound as

$$(4.19)\qquad b_m(\xi)\ll q^{-\frac{m}{2}}\cdot\mathbb{1}_{\mathfrak{c}(\xi)=m}+\mathbb{1}_{m=1}\cdot q^{-1}\cdot\mathbb{1}_{\mathfrak{c}(\xi)=0}\ll q^{-\frac{m}{2}}\cdot\mathbb{1}_{\mathfrak{c}(\xi)\leqslant m}.$$

The defining formula for $a_m(\xi;n)$ makes sense for all $m\in\mathbb{Z}$, and we have (writing $X:=q^s$)

$$\begin{aligned}\sum\nolimits_{m\in\mathbb{Z}}a_m(\xi;n)X^{-m}&=\int_{\mathbf{F}^\times}\mathcal{VH}_{\Pi,\psi}(H_n)(t)\xi^{-1}(t)|t|^{-s}\mathrm{d}^\times t=\gamma\left(s,\Pi\otimes\xi,\psi\right)\cdot\varepsilon_n(\xi,H)X^n \\ &=\varepsilon_n(\xi,H)\varepsilon\left(\tfrac{1}{2},\Pi\otimes\xi,\psi\right)q^{\frac{\mathfrak{c}(\Pi\otimes\xi)}{2}}X^{n-\rho}\cdot\prod\nolimits_{j=1}^{d}\tfrac{X-a_j}{1-\overline{a_j}q^{-1}X},\end{aligned}$$

which is equivalent to (**??**). If $\xi\notin E(\Pi)$ then $d:=d(\Pi\otimes\xi)=0$, and we get for $m\geqslant 1$

$$(4.20)\qquad a_m(\xi;n)\ll\mathbb{1}_{\mathcal{B}_n}(\xi)\cdot\mathbb{1}_{m+n}(\mathfrak{c}(\Pi\otimes\xi))\cdot q^{\frac{m+n}{2}}.$$

If $\xi \in E(\Pi)$ then we apply the residue theorem (as $|a_j| \in \{1, q^{-\frac{1}{2}}, q^{-1}\}$) to get

$$a_m(\xi; n) = \varepsilon_n(\xi, H)\varepsilon\left(\tfrac{1}{2}, \Pi \otimes \xi, \psi\right) q^{\frac{\mathfrak{c}(\Pi\otimes\xi)}{2}} \int_{|X|=q^{\frac{1}{2}}} X^{n-\mathfrak{c}(\Pi\otimes\xi)+m-1} \prod_{j=1}^{d} \tfrac{1-a_j X^{-1}}{1-\overline{a_j}q^{-1}X} \frac{\mathrm{d}X}{2\pi i}$$

$$\ll \mathbb{1}_{\mathcal{B}_n}(\xi) \cdot \mathbb{1}_{\geqslant m+n}(\rho(\Pi\otimes\xi)) \cdot q^{\frac{n+m}{2}}. \tag{4.21}$$

Combining the bounds (**??**)-(**??**), Lemma **??** & **??** and $\mathfrak{c}(\Pi \otimes \xi) \leqslant \mathfrak{c}(\Pi) + 3\mathfrak{c}(\xi)$ (see [**?**]) we get

$$\begin{aligned} C_m(n, \chi_0^{-1}) &\ll q^{\frac{n}{2}} \sum_{\substack{\xi\in\mathcal{B}_n - E(\Pi) \\ \mathfrak{c}(\xi\chi_0^{-1})\leqslant m}} \mathbb{1}_{\mathfrak{c}(\Pi\otimes\xi)=m+n} + q^{\frac{n}{2}} \sum_{\substack{\xi\in\mathcal{B}_n\cap E(\Pi) \\ \mathfrak{c}(\xi\chi_0^{-1})\leqslant m}} \mathbb{1}_{\geqslant m+n}(\rho(\Pi\otimes\xi)) \\ &\ll q^{\frac{n}{2}} \sum_{\substack{\mathfrak{c}(\xi)\leqslant n/2 \\ \mathfrak{c}(\xi\chi_0^{-1})\leqslant m}} \mathbb{1}_{m\leqslant\mathfrak{c}(\Pi)+\frac{n}{2}} + q^{\frac{n}{2}} \mathbb{1}_{\leqslant 2\mathfrak{c}(\Pi)+3}(m+n) \mathbb{1}_{\leqslant \max(\mathfrak{c}(\Pi),m)}(\mathfrak{c}(\chi_0)) \\ &\leqslant q^{\frac{n}{2}+\min(\frac{n}{2},m)} \mathbb{1}_{m\leqslant\mathfrak{c}(\Pi)+\frac{n}{2}} \mathbb{1}_{\leqslant\mathfrak{c}(\Pi)+\frac{n}{2}}(\mathfrak{c}(\chi_0)) + q^{\frac{n}{2}} \mathbb{1}_{\leqslant 2\mathfrak{c}(\Pi)+3}(m+n) \mathbb{1}_{\leqslant \max(\mathfrak{c}(\Pi),m)}(\mathfrak{c}(\chi_0)), \end{aligned}$$

and conclude the first bound. To derive the second bound, one simply notice n_1 can be taken as $2A\mathfrak{c}(\Pi)$, since under the condition $\mathfrak{c}(\Pi) > 0$ we have $a(\Pi) \leqslant \mathfrak{c}(\Pi)$ by the "moreover" part of Proposition **??**. ☐

Remark 4.9. *The bound established in Lemma **??** is far from being optimal. For example in the case $e = 1$ for $\xi \in \mathcal{B}_n$ the typical size of $\mathfrak{c}(\Pi \otimes \xi)$ should be $3n/2$, hence the term $q^{-\frac{\mathfrak{c}(\Pi\otimes\xi)}{2}}$ could be bounded as $q^{-\frac{3n}{4}}$. But even with this improvement the individual bound of $\widetilde{h}_n^-(\chi)$ is too weak to apply in the case $n_0 \gg \mathfrak{c}(\Pi)$. It would be interesting to recover the cancellation in the sum of $a_m(\xi; n)b_m(\xi)$ over ξ by refining the above method.*

4.4. **For Unramified Characters.** We turn to the unramified case $\chi = |\cdot|_{\mathbf{F}}^s$. According to the decomposition (**??**) of $\widetilde{H}$, the notation $\widetilde{H}_\infty(k; s_0)$, $\widetilde{H}_c(k; s_0)$ and $\widetilde{H}_n^{\pm}(k; s_0)$ have obvious meaning and

$$\widetilde{H}(k; s_0) = \widetilde{H}_\infty(k; s_0) + \sum_{\pm} \widetilde{H}_c^{\pm}(k; s_0), \quad \widetilde{H}_c^{\pm}(k; s_0) = \sum_{\substack{n=\frac{2}{e}(n_0-2)+3 \\ 2|en}}^{2n_1-1} \widetilde{H}_n^{\pm}(k; s_0).$$

Lemma 4.10. *For any $k \in \mathbb{Z}_{\geqslant 0}$ we have the following bounds.*

(1) $\widetilde{H}_\infty(k; \frac{1}{2}) \ll_{k,\epsilon} q^{-n_1(1-\epsilon)}$, $\widetilde{H}_\infty(k; -\frac{1}{2}) \ll_{k,\epsilon} q^{n_1\epsilon}$.

(2) $\widetilde{H}_n^+(k; \pm\frac{1}{2}) = 0$ *unless* $n = \frac{2n_0}{e} + e - 1$. *We have*

$$\widetilde{H}_c^+(k; \tfrac{1}{2}) \ll_{k,\epsilon} \mathbf{C}(\Pi)^{\frac{1}{2}} \cdot q^{\left(\frac{2n_0}{e}+e-1\right)\epsilon}, \quad \widetilde{H}_c^+(k; -\tfrac{1}{2}) \ll_{k,\epsilon} \mathbf{C}(\Pi)^{\frac{1}{2}} \cdot q^{-\left(\frac{2n_0}{e}+e-1\right)(1-\epsilon)}.$$

Proof. (1) By Lemma **??** we have $\widetilde{H}_\infty(s) = q^{-\left(\frac{1}{2}+s\right)n_1} L\left(\frac{1}{2} - s, \widetilde{\Pi}\right)^{-1}$. The stated bounds follow readily.
(2) By Lemma **??** and (**??**) we have

$$f_n(q^{\frac{1}{2}}; |\cdot|_{\mathbf{F}}^s, H) = f_n(q^{s+\frac{1}{2}}; \mathbb{1}, H) =$$
$$q^{n\left(s-\frac{1}{2}\right)}\varepsilon\left(s+\tfrac{1}{2}, \Pi, \psi\right) \frac{L(\frac{1}{2}-s, \widetilde{\Pi})}{L(\frac{1}{2}+s, \Pi)} \cdot \begin{cases} \mathbb{1}_{n=2n_0} \cdot \zeta_{\mathbf{F}}(1)\chi_0(-1) & \text{if } \mathbf{L}/\mathbf{F} \text{ split} \\ \mathbb{1}_{n=\frac{2n_0}{e}+e-1} \cdot \zeta_{\mathbf{F}}(1)\varepsilon(1/2, \pi_\beta, \psi) & \text{if } \mathbf{L}/\mathbf{F} \text{ non-split} \end{cases}.$$

We introduce a_j ($|a_j| \in \{1, q^{-\frac{1}{2}}, q^{-1}\}$) as the parameters of $L(s, \Pi)$ so that $L(\frac{1}{2} - s, \widetilde{\Pi}) = P(X)^{-1}$ with

$$P(X) := \prod_{j=1}^{d(\Pi)} \left(1 - \overline{a_j}q^{-\frac{1}{2}}X\right), \quad f(X) := f_n(q^{\frac{1}{2}}X; \mathbb{1}, H) = X^{n-\rho(\Pi)} \prod_{j=1}^{d(\Pi)} \tfrac{X-a_jq^{-\frac{1}{2}}}{1-\overline{a_j}q^{-\frac{1}{2}}X} \cdot q^{-\frac{n}{2}}\delta$$

for some $|\delta| = 1$. We then rewrite, with $X = q^s$

$$\widetilde{H}_n^+(|\cdot|_{\mathbf{F}}^s) = \frac{f_+(q^s)}{L(\frac{1}{2}-s,\widetilde{\Pi})\zeta_{\mathbf{F}}(\frac{1}{2}+s)} = (1 - q^{-\frac{1}{2}}X^{-1})P(X)f_+(X).$$

The bounds at $s = -1/2$ then follows readily from Lemma **??** (1) via the bounds

$$\sup_{|z|=1} \left|D^k f(q^{-\frac{1}{2}}z)\right| \ll_{k,\epsilon} \mathbf{C}(\Pi)^{\frac{1}{2}}q^{-n(1-\epsilon)}, \quad D^k\left((1 - q^{-\frac{1}{2}}X^{-1})P(X)\right)|_{X=q^{-\frac{1}{2}}} \ll_k 1.$$

We now consider the bounds at $s = 1/2$. If $n = \frac{2n_0}{e} + e - 1 \geqslant \rho(\Pi)$, then $f_+(X) = f(X)$, implying

$$\widetilde{H}_n^+(|\cdot|_{\mathbf{F}}^s) = q^{-\frac{n}{2}}\delta \cdot X^{n-\mathfrak{c}(\Pi)}(1-q^{-\frac{1}{2}}X^{-1}) \prod_{j=1}^{d(\Pi)} \left(1 - a_j q^{-\frac{1}{2}} X^{-1}\right),$$

$$D^k \widetilde{H}_n^+(|\cdot|_{\mathbf{F}}^s) \mid_{s=1/2} \ll_{k,\epsilon} \mathbf{C}(\Pi)^{-\frac{1}{2}} q^{n\epsilon}.$$

If $n = \frac{2n_0}{e} + e - 1 < \rho(\Pi) =: \rho$, then $Q(X) := P(X)f(X)$ has highest term X^m with $m < d(\Pi) = \deg P$. We also note that $f_+(X) = 0$ unless $d := d(\Pi) \geqslant 1$. We apply Lemma **??** (2) distinguishing cases:
(i) $d = 1$. Necessarily we have $c := c(\Pi) \geqslant 2$, $n - c \leqslant 0$. We get and conclude the stated bounds by

$$P(X)f_+(X) = \overline{a_1}^{c-n}\delta \cdot q^{-\frac{c}{2}}(1-|a_1|^2 q^{-1}) \ll \mathbf{C}(\Pi)^{-\frac{1}{2}}.$$

(ii) $d = 2$. Necessarily $\Pi = \mathrm{St}_\chi \boxplus \chi^{-2}$ for some unramified unitary character χ. We have $c = 1$ and $n - c \leqslant 1$, and may assume $|a_1| = q^{-\frac{1}{2}}$ and $|a_2| = 1$. We get the formula

$$P(X)f_+(X) = \overline{a_1}^{c-n}\delta \cdot q^{-\frac{c}{2}} \frac{(1-|a_1|^2q^{-1})(1-\overline{a_1}a_2q^{-1})}{1-\overline{a_2a_1}^{-1}}(1-\overline{a_2}q^{-\frac{1}{2}}X) +$$
$$\overline{a_2}^{c-n}\delta \cdot q^{-\frac{c}{2}} \frac{(1-|a_2|^2q^{-1})(1-\overline{a_2}a_1q^{-1})}{1-\overline{a_1a_2}^{-1}}(1-\overline{a_1}q^{-\frac{1}{2}}X)$$

and conclude the stated bounds by

$$D^k \left(P(X)f_+(X)\right) \mid_{X=q^{\frac{1}{2}}} \ll_k 1. \tag{4.22}$$

(iii) $d = 3$. Necessarily $\Pi = \mu_1 \boxplus \mu_2 \boxplus \mu_3$ for some unramified unitary characters μ_i. We have $c = 0$ and $n \leqslant 2$, and may assume $|a_j| = 1$ with $a_1a_2a_3 = 1$. It is easy to see $C_{j,k} \ll 1$. Therefore (**??**) still holds and we conclude the stated bounds in the same way. □

Lemma 4.11. *For any $k \in \mathbb{Z}_{\geqslant 0}$ we have the following bounds*

$$\widetilde{H}_n^-(k; \tfrac{1}{2}) \ll_{k,\epsilon} \mathbf{C}(\Pi)^\epsilon, \quad \widetilde{H}_n^-(k; -\tfrac{1}{2}) \ll_{k,\epsilon} \left(\mathbf{C}(\Pi)^2 q^{\frac{n}{2}}\right)^{1+\epsilon}.$$

Consequently if $\mathfrak{c}(\Pi) > 0$ and $n_0 \leqslant A\mathfrak{c}(\Pi)$ for some constant $A \geqslant 1$ then we have

$$\widetilde{H}_c^-(k; \delta\cdot\tfrac{1}{2}) \leqslant \sum_{\substack{n=\frac{2}{e}(n_0-2)+3 \\ 2|en}}^{2n_1-1} \left|\widetilde{H}_n^-(k; \delta\cdot\tfrac{1}{2})\right| \ll_\epsilon \begin{cases} \mathbf{C}(\Pi)^\epsilon & \text{if } \delta = 1 \\ \mathbf{C}(\Pi)^{(2+2Ae^{-1})(1+\epsilon)} & \text{if } \delta = -1 \end{cases}.$$

Proof. With similar argument as in the proof of Lemma **??** we get

$$\widetilde{h}_n^-(|\cdot|_{\mathbf{F}}^s) = q^{-n}\sum_{m=1}^\infty q^{m\left(\frac{1}{2}-s\right)} C_m(n), \quad C_m(n) \ll q^{\frac{n}{2}+\min(m,\frac{n}{2})}\mathbb{1}_{m\leqslant\mathfrak{c}(\Pi)+\frac{n}{2}} + q^{\frac{n}{2}}\mathbb{1}_{m+n\leqslant 2\mathfrak{c}(\Pi)+3}.$$

The stated bounds follow readily. □

5. Dual Weight Functions: Split and Unramified Cases

5.1. First Quadratic Elementary Functions.

Definition 5.1. *For any $n \in \mathbb{Z}_{\geqslant 0}$ we define the* first quadratic elementary functions $F_n \in \mathrm{C}_c^\infty(\mathbf{F}^\times)$ *by*

$$F_n(y^2) := \mathbb{1}_{v(y)=-n} \cdot \sum\nolimits_{\pm} \psi(\pm y),$$

and are supported in the subset of square *elements of* $\mathbf{F}^\times$.

Definition 5.2. *Let η_0 be the character of $\mathbf{F}^\times$ associated with the (ramified) quadratic extension $\mathbf{F}[\sqrt{-\varpi_{\mathbf{F}}}]/\mathbf{F}$. Explicitly it is given by the following formula for all $n \in \mathbb{Z}$*

$$\eta_0(\varpi_{\mathbf{F}}^n u) = \begin{cases} 1 & \text{if } u \in (\mathcal{O}_{\mathbf{F}}^\times)^2 \\ -1 & \text{if } u \in \mathcal{O}_{\mathbf{F}}^\times - (\mathcal{O}_{\mathbf{F}}^\times)^2 \end{cases}.$$

Denote by $\tau_0 = \tau(\eta_0, \psi; \varpi_{\mathbf{F}})$ the quadratic Gauss sum *given by*

$$\tau_0 := \int_{\mathcal{O}_{\mathbf{F}}^\times} \psi\left(\frac{u}{\varpi_{\mathbf{F}}}\right)\eta_0(u)\mathrm{d}u = \int_{\mathcal{O}_{\mathbf{F}}} \psi\left(\frac{u^2}{\varpi_{\mathbf{F}}}\right)\mathrm{d}u.$$

Lemma 5.3. *Let χ be a quasi-character of $\mathbf{F}^\times$. We have*

$$\int_{\mathbf{F}^\times} F_n(y)\chi(y)\mathrm{d}^\times y = \begin{cases} \mathbb{1}_{\mathfrak{c}(\chi^2)=n}\cdot\zeta_{\mathbf{F}}(1)\gamma(1,\chi^{-2},\psi) & \text{if } n\geqslant 2 \\ \mathbb{1}_{\mathfrak{c}(\chi^2)=1}\cdot\zeta_{\mathbf{F}}(1)\gamma(1,\chi^{-2},\psi) - \mathbb{1}_{\mathfrak{c}(\chi^2)=0}\cdot\zeta_{\mathbf{F}}(1)q^{-1}\chi(\varpi_{\mathbf{F}})^{-2} & \text{if } n=1 \\ \mathbb{1}_{\mathfrak{c}(\chi^2)=0} & \text{if } n=0 \end{cases}.$$

Note that we can replace the condition $\mathfrak{c}(\chi^2)=n$ with $\mathfrak{c}(\chi)=n$ if $n\geqslant 1$.

Proof. The case for $n=0$ is simple and omitted. For $n\geqslant 1$ with the change of variables $y\to y^2$

$$\int_{\mathbf{F}^\times} F_n(y)\chi(y)\mathrm{d}^\times y = \int_{\varpi_{\mathbf{F}}^{-n}\mathcal{O}_{\mathbf{F}}^\times}\psi(y)\chi^2(y)\mathrm{d}^\times y,$$

the desired formula follows readily from [? , Exercise 23.5]. □

Corollary 5.4. *Let $n\geqslant a(\Pi)(\geqslant 2)$, the stability barrier of Π (see Proposition **??**). We have*

$$\mathcal{VH}_{\Pi,\psi}(F_n)(y) = \mathbb{1}_{\varpi_{\mathbf{F}}^{-n}\mathcal{O}_{\mathbf{F}}^\times}(y)\cdot\begin{cases} q^{\frac{3n}{2}}\psi(4y) & \text{if } 2\mid n \\ \tau_0 q^{\lceil\frac{3n}{2}\rceil}\psi(4y)\eta_0(4y) & \text{if } 2\nmid n\end{cases}.$$

Proof. By the local functional equation, Lemma **??** and Proposition **??** we have for any $\chi\in\widehat{\mathcal{O}_{\mathbf{F}}^\times}$

$$\begin{aligned}\int_{\mathbf{F}^\times}\mathcal{VH}_{\Pi,\psi}(F_n)(y)\chi(y)^{-1}|y|^{-s}\mathrm{d}^\times y &= \zeta_{\mathbf{F}}(1)\mathbb{1}_{\mathfrak{c}(\chi)=n}\cdot\gamma(s,\chi,\psi)^3\gamma(1-2s,\chi^{-2},\psi) \\ &= \zeta_{\mathbf{F}}(1)\mathbb{1}_{\mathfrak{c}(\chi)=n}\gamma(1,\chi,\psi)^3\gamma(1,\chi^{-2},\psi)\cdot q^{3n-ns}.\end{aligned}$$

Therefore the support of $\mathcal{VH}_{\Pi,\psi}(F_n)$ is contained in $\varpi_{\mathbf{F}}^{-n}\mathcal{O}_{\mathbf{F}}^\times$. Applying [? , Exercise 23.5] we find

$$\zeta_{\mathbf{F}}(1)\mathbb{1}_{\mathfrak{c}(\chi)=n}\gamma(1,\chi,\psi)^3\gamma(1,\chi^{-2},\psi)\cdot q^{3n} = \mathbb{1}_{\mathfrak{c}(\chi)=n}\cdot q^{3n}\int_{(\mathcal{O}_{\mathbf{F}}^\times)^4}\psi\left(\tfrac{t_1+t_2+t_3+t_4}{\varpi_{\mathbf{F}}^n}\right)\chi^{-1}\left(\tfrac{t_1t_2t_3}{\varpi_{\mathbf{F}}^n t_4^2}\right)\mathrm{d}\vec{t},$$

where we have written $\mathrm{d}\vec{t} := \mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4$ for simplicity. The Fourier inversion on $\mathcal{O}_{\mathbf{F}}^\times$ yields for $y\in\mathcal{O}_{\mathbf{F}}^\times$

$$\begin{aligned}\mathcal{VH}_{\Pi,\psi}(F_n)\left(\tfrac{y}{\varpi_{\mathbf{F}}^n}\right) &= \zeta_{\mathbf{F}}(1)q^{3n}\sum_{\substack{\chi\in\widehat{\mathcal{O}_{\mathbf{F}}^\times}\\ \mathfrak{c}(\chi)=n}}\int_{(\mathcal{O}_{\mathbf{F}}^\times)^4}\psi\left(\tfrac{t_1+t_2+t_3+t_4}{\varpi_{\mathbf{F}}^n}\right)\chi\left(\tfrac{t_4^2y}{t_1t_2t_3}\right)\mathrm{d}\vec{t} \\ &= \zeta_{\mathbf{F}}(1)q^{3n}\int_{(\mathcal{O}_{\mathbf{F}}^\times)^4}\psi\left(\tfrac{t_1+t_2+t_3+t_4}{\varpi_{\mathbf{F}}^n}\right)\left(\left|(\mathcal{O}_{\mathbf{F}}/\mathcal{P}_{\mathbf{F}}^n)^\times\right|\mathbb{1}_{1+\mathcal{P}_{\mathbf{F}}^n} - \left|(\mathcal{O}_{\mathbf{F}}/\mathcal{P}_{\mathbf{F}}^{n-1})^\times\right|\mathbb{1}_{1+\mathcal{P}_{\mathbf{F}}^{n-1}}\right)\left(\tfrac{t_4^2y}{t_1t_2t_3}\right)\mathrm{d}\vec{t} \\ &= q^{3n}\int_{(\mathcal{O}_{\mathbf{F}}^\times)^3}\psi\left(\tfrac{t_2+t_3+t_4+t_2^{-1}t_3^{-1}t_4^2y}{\varpi_{\mathbf{F}}^n}\right)\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4 - \\ &\quad q^{4n-1}\int_{(\mathcal{O}_{\mathbf{F}}^\times)^3\times\mathcal{P}_{\mathbf{F}}^{n-1}}\psi\left(\tfrac{t_2+t_3+t_4+t_2^{-1}t_3^{-1}t_4^2y}{\varpi_{\mathbf{F}}^n}\right)\psi\left(\tfrac{t_2^{-1}t_3^{-1}t_4^2yu}{\varpi_{\mathbf{F}}^n}\right)\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4\mathrm{d}u.\end{aligned}$$

The last integral is vanishing because $\mathfrak{c}(\psi)=0$. Re-numbering the variables we get

$$\mathcal{VH}_{\Pi,\psi}(F_n)\left(\tfrac{y}{\varpi_{\mathbf{F}}^n}\right) = q^{3n}\int_{(\mathcal{O}_{\mathbf{F}}^\times)^3}\psi\left(\tfrac{t_1+t_2+t_3+t_1^{-1}t_2^{-1}t_3^2y}{\varpi_{\mathbf{F}}^n}\right)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3.$$

Performing the level $\lceil\frac{n}{2}\rceil$ regularization to $\mathrm{d}t_j$ we see that the non-vanishing of the above integral implies

$$t_1 - \tfrac{t_3^2y}{t_1t_2}\in\mathcal{P}_{\mathbf{F}}^{\lfloor\frac{n}{2}\rfloor},\quad t_2 - \tfrac{t_3^2y}{t_1t_2}\in\mathcal{P}_{\mathbf{F}}^{\lfloor\frac{n}{2}\rfloor},\quad t_3 + \tfrac{2t_3^2y}{t_1t_2}\in\mathcal{P}_{\mathbf{F}}^{\lfloor\frac{n}{2}\rfloor}.$$

Writing $a=\frac{t_3^2y}{t_1t_2}\in\mathcal{O}_{\mathbf{F}}^\times$, we see that the above conditions imply $a\equiv 4y \pmod{\mathcal{P}_{\mathbf{F}}^{\lfloor\frac{n}{2}\rfloor}}$, hence

$$t_1\in 4y\left(1+\mathcal{P}_{\mathbf{F}}^{\lfloor\frac{n}{2}\rfloor}\right),\quad t_2\in 4y\left(1+\mathcal{P}_{\mathbf{F}}^{\lfloor\frac{n}{2}\rfloor}\right),\quad t_3\in -8y\left(1+\mathcal{P}_{\mathbf{F}}^{\lfloor\frac{n}{2}\rfloor}\right).$$

The change of variables $t_1=4y(1+u_1)$, $t_2=4y(1+u_2)$ and $t_3=-8y(1+u_3)$ with $u_j\in\mathcal{P}_{\mathbf{F}}^{\lfloor\frac{n}{2}\rfloor}$ gives

$$\begin{aligned}
\mathcal{VH}_{\Pi,\psi}(F_n)\left(\tfrac{y}{\varpi_{\mathbf{F}}^n}\right) &= q^{3n} \int_{u_j \in \mathcal{P}_{\mathbf{F}}^{\lfloor \frac{n}{2} \rfloor}} \psi\left(\frac{4y(u_1+u_2-2u_3) + \frac{4(1+u_3)^2 y}{(1+u_1)(1+u_2)}}{\varpi_{\mathbf{F}}^n} \right) \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3 \\
&= q^{3n} \psi\left(\tfrac{4y}{\varpi_{\mathbf{F}}^n}\right) \int_{u_j \in \mathcal{P}_{\mathbf{F}}^{\lfloor \frac{n}{2} \rfloor}} \psi\left(\tfrac{4y}{\varpi_{\mathbf{F}}^n} \left(u_1^2 + u_2^2 + u_3^2 + u_1u_2 - 2u_1u_3 - 2u_2u_3 \right) \right) \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3 \\
&= q^{3n} \psi\left(\tfrac{4y}{\varpi_{\mathbf{F}}^n}\right) \int_{u_j \in \mathcal{P}_{\mathbf{F}}^{\lfloor \frac{n}{2} \rfloor}} \psi\left(\tfrac{4y}{\varpi_{\mathbf{F}}^n} \left(u_1^2 + u_2^2 + u_1u_2 - u_1u_3 \right) \right) \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3 \\
&= q^{3n-\lfloor \frac{n}{2} \rfloor} \psi\left(\tfrac{4y}{\varpi_{\mathbf{F}}^n}\right) \int_{u_j \in \mathcal{P}_{\mathbf{F}}^{\lfloor \frac{n}{2} \rfloor}} \psi\left(\tfrac{4y}{\varpi_{\mathbf{F}}^n} \left(u_1^2 + u_2^2 + u_1u_2 \right) \right) \mathbb{1}_{\mathcal{P}_{\mathbf{F}}^{\lceil \frac{n}{2} \rceil}}(u_1) \mathrm{d}u_1 \mathrm{d}u_2 \\
&= q^{2n} \psi\left(\tfrac{4y}{\varpi_{\mathbf{F}}^n}\right) \int_{\mathcal{P}_{\mathbf{F}}^{\lfloor \frac{n}{2} \rfloor}} \psi\left(\tfrac{4y}{\varpi_{\mathbf{F}}^n} u_2^2 \right) \mathrm{d}u_2,
\end{aligned}$$

where we made the change of variables $u_2 \mapsto u_2 + u_3$ in the third line. The desired formula follows. □

Remark 5.5. *If we compute the Mellin transform of $\mathcal{VH}_{\Pi,\psi}(F_n)(y)$ with the given formula in Corollary* **??***, we get in the case $2 \mid n \geqslant 2$ and for $\mathfrak{c}(\chi) = n$ an interesting equation*

$$\chi(4)\gamma\left(\tfrac{1}{2}, \chi^{-1}, \psi\right) = \gamma\left(\tfrac{1}{2}, \chi, \psi\right)^3 \gamma\left(\tfrac{1}{2}, \chi^{-2}, \psi\right).$$

Corollary 5.6. *Suppose $\Pi = \mu_1 \boxplus \mu_2 \boxplus \mu_3$ with $\mathfrak{c}(\mu_j) = 0$ and $\mu_1\mu_2\mu_3 = \mathbb{1}$. Let $E_i(y) := \mu_i^{-1}(y)|y|\mathbb{1}_{\mathcal{O}_{\mathbf{F}}}(y)$ and $f_i := (1 - \mu_i(\varpi_{\mathbf{F}})\mathfrak{t}(\varpi_{\mathbf{F}})).E_i$. For $f, g \in \mathrm{L}^1(\mathbf{F}^\times)$ define $f * g(y) := \int_{\mathbf{F}^\times} f(yt^{-1})g(t)\mathrm{d}^\times t$. We have*

$$\begin{aligned}
\mathcal{VH}_{\Pi,\psi}(F_0)(y) &= (f_1 * f_2 * f_3)(y) + \tau_0 q^2 \cdot \eta_0(-y)\mathbb{1}_{\varpi_{\mathbf{F}}^{-3}\mathcal{O}_{\mathbf{F}}^\times}(y), \\
\mathcal{VH}_{\Pi,\psi}(F_1)(y) &= -q^{-1} \cdot (f_1 * f_2 * f_3)(\varpi_{\mathbf{F}}^{-2} y) - \zeta_{\mathbf{F}}(1) \cdot \mathbb{1}_{\varpi_{\mathbf{F}}^{-1}\mathcal{O}_{\mathbf{F}}^\times}(y) \\
&\quad + \tau_0 q \mathbb{1}_{\varpi_{\mathbf{F}}^{-1}\mathcal{O}_{\mathbf{F}}^\times}(y) \int_{(\varpi_{\mathbf{F}}^{-1}\mathcal{O}_{\mathbf{F}}^\times)^2} \psi\left(t_1 + t_2 - \tfrac{t_1t_2}{4y}\right) \eta_0\left(\tfrac{4y}{t_1t_2}\right) \mathrm{d}t_1 \mathrm{d}t_2.
\end{aligned}$$

Proof. By the local functional equation and Lemma **??** we have for any $\chi \in \widehat{\mathbf{F}^\times}$

$$\int_{\mathbf{F}^\times} \mathcal{VH}_{\Pi,\psi}(F_0)(y)\chi(y)^{-1}|y|^{-s}\mathrm{d}^\times y = \begin{cases} \prod_{i=1}^3 \frac{L(1-s,\mu_i^{-1}\chi^{-1})}{L(s,\mu_i\chi)} & \text{if } \mathfrak{c}(\chi) = 0 \\ q^{3(\frac{1}{2}-s)}\gamma\left(\frac{1}{2}, \chi, \psi\right)^3 & \text{if } \mathfrak{c}(\chi\eta_0) = 0 \\ 0 & \text{otherwise} \end{cases} \cdot$$

Therefore we can write $\mathcal{VH}_{\Pi,\psi}(F_0) = \mathcal{VH}_{\Pi,\psi}(F_0)_0 + \mathcal{VH}_{\Pi,\psi}(F_0)_1$ with the properties

$$\mathcal{VH}_{\Pi,\psi}(F_0)_0(y\delta) = \mathcal{VH}_{\Pi,\psi}(F_0)_0(y), \quad \mathcal{VH}_{\Pi,\psi}(F_0)_1(y\delta) = \mathcal{VH}_{\Pi,\psi}(F_0)_1(y)\eta_0(\delta), \quad \forall y \in \mathbf{F}^\times, \delta \in \mathcal{O}_{\mathbf{F}}^\times;$$

$$\int_{\mathbf{F}^\times} \mathcal{VH}_{\Pi,\psi}(F_0)_0(y)|y|^{-s}\mathrm{d}^\times y = \prod_{i=1}^3 \frac{L(1-s,\mu_i^{-1})}{L(s,\mu_i)} = \prod_{i=1}^3 \int_{\mathbf{F}^\times} f_i(y)|y|^{-s}\mathrm{d}^\times y,$$

$$\int_{\mathbf{F}^\times} \mathcal{VH}_{\Pi,\psi}(F_0)_0(y)\eta_0(y)|y|^{-s}\mathrm{d}^\times y = q^{3(\frac{1}{2}-s)}\gamma\left(\tfrac{1}{2}, \eta_0, \psi\right)^3 = \eta_0(-1)\tau_0 q^2 \cdot \int_{\varpi_{\mathbf{F}}^{-3}\mathcal{O}_{\mathbf{F}}^\times} |y|^{-s}\mathrm{d}^\times y.$$

Hence we identify $\mathcal{VH}_{\Pi,\psi}(F_0)_0 = f_1 * f_2 * f_3$ and $\mathcal{VH}_{\Pi,\psi}(F_0)_1(y) = \tau_0 q^2 \cdot \eta_0(-y)\mathbb{1}_{\varpi_{\mathbf{F}}^{-3}\mathcal{O}_{\mathbf{F}}^\times}(y)$. Similarly we can write $\mathcal{VH}_{\Pi,\psi}(F_1) = \mathcal{VH}_{\Pi,\psi}(F_1)_0 + \mathcal{VH}_{\Pi,\psi}(F_1)_1$ with the properties

$$\mathcal{VH}_{\Pi,\psi}(F_1)_0(y\delta) = \mathcal{VH}_{\Pi,\psi}(F_1)_0(y), \quad \forall \delta \in \mathcal{O}_{\mathbf{F}}^\times;$$

$$\int_{\mathbf{F}^\times} \mathcal{VH}_{\Pi,\psi}(F_1)_0(y)|y|^{-s}\mathrm{d}^\times y = -q^{2s-1} \prod_{i=1}^3 \int_{\mathbf{F}^\times} f_i(y)|y|^{-s}\mathrm{d}^\times y,$$

$$\int_{\mathbf{F}^\times} \mathcal{VH}_{\Pi,\psi}(F_1)_1(y)\chi^{-1}(y)|y|^{-s}\mathrm{d}^\times y = \mathbb{1}_{\mathfrak{c}(\chi)=1} \cdot q^{3-s} \int_{\varpi_{\mathbf{F}}^{-1}\mathcal{O}_{\mathbf{F}}^\times} \psi(y)\chi^2(y)\mathrm{d}^\times y \cdot \left(\int_{\varpi_{\mathbf{F}}^{-1}\mathcal{O}_{\mathbf{F}}^\times} \psi(y)\chi^{-1}(y)\mathrm{d}^\times y \right)^3.$$

We easily identify $\mathcal{VH}_{\Pi,\psi}(F_1)_0(y) = -q^{-1} \cdot (f_1 * f_2 * f_3)(\varpi_{\mathbf{F}}^{-2} y)$. The argument in the proof of Corollary **??** shows that $\mathrm{supp}\left(\mathcal{VH}_{\Pi,\psi}(F_1)_1\right) \subset \varpi_{\mathbf{F}}^{-1}\mathcal{O}_{\mathbf{F}}^\times$, and for $y \in \mathcal{O}_{\mathbf{F}}^\times$ that

$$\mathcal{VH}_{\Pi,\psi}(F_1)_1\left(\tfrac{y}{\varpi_{\mathbf{F}}}\right) = q^3\int_{(\mathcal{O}_{\mathbf{F}}^\times)^3}\psi\left(\tfrac{t_2+t_3+t_4+t_2^{-1}t_3^{-1}t_4^2y}{\varpi_{\mathbf{F}}}\right)\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4 - \zeta_{\mathbf{F}}(1)q^3\int_{(\mathcal{O}_{\mathbf{F}}^\times)^4}\psi\left(\tfrac{t_1+t_2+t_3+t_4}{\varpi_{\mathbf{F}}}\right)\mathrm{d}\vec{t}$$
$$= \tau_0 q^3\int_{(\mathcal{O}_{\mathbf{F}}^\times)^2}\psi\left(\tfrac{t_2+t_3-\frac{t_2t_3}{4y}}{\varpi_{\mathbf{F}}}\right)\eta_0\left(\tfrac{y}{t_2t_3}\right)\mathrm{d}t_2\mathrm{d}t_3 - q^2\int_{(\mathcal{O}_{\mathbf{F}}^\times)^2}\psi\left(\tfrac{t_2+t_3}{\varpi_{\mathbf{F}}}\right)\mathrm{d}t_2\mathrm{d}t_3 - \zeta_{\mathbf{F}}(1)q^{-1}$$
$$= \tau_0 q^3\int_{(\mathcal{O}_{\mathbf{F}}^\times)^2}\psi\left(\tfrac{t_2+t_3-\frac{t_2t_3}{4y}}{\varpi_{\mathbf{F}}}\right)\eta_0\left(\tfrac{y}{t_2t_3}\right)\mathrm{d}t_2\mathrm{d}t_3 - \zeta_{\mathbf{F}}(1).$$

Re-numbering the variables and inserting the formula of $\mathcal{VH}_{\Pi,\psi}(F_0)$ we get the formula of $\mathcal{VH}_{\Pi,\psi}(F_1)$. □

5.2. **Further Reductions.** The functions F_n are "building blocks" of our test functions H_c in (**??**) when $\mathbf{L}/\mathbf{F}$ is not ramified. In fact writing $\varepsilon_{\mathbf{L}} := \eta_{\mathbf{L}/\mathbf{F}}(\varpi_{\mathbf{F}}) \in \{\pm 1\}$ and applying Lemma **??** we can rewrite the summands of H_c as

$$H_{2n+1} = 0, \tag{5.1}$$

$$H_{2n} = \begin{cases} E_n & \text{if } 2n_0 \leqslant n \leqslant n_1 - 1 \\ (\varepsilon_{\mathbf{L}}q)^n \displaystyle\int_{\substack{\mathbf{L}^1\cap\mathcal{O}_{\mathbf{L}} \\ \mathrm{Tr}(\alpha)\in 2(1+\mathcal{P}_{\mathbf{F}}^{2(n_0-1)})}} \beta(\alpha)\cdot\left(\mathfrak{t}\left(\mathrm{Tr}(\alpha)^2\right).F_n\right)\mathrm{d}\alpha & \text{if } n = 2n_0 - 1 \\ (\varepsilon_{\mathbf{L}}q)^n \displaystyle\int_{\substack{\mathbf{L}^1\cap\mathcal{O}_{\mathbf{L}} \\ \mathrm{Tr}(\alpha)\in 2(1+\varpi_{\mathbf{F}}^{2(n-n_0)}\mathcal{O}_{\mathbf{F}}^\times)}} \beta(\alpha)\cdot\left(\mathfrak{t}\left(\mathrm{Tr}(\alpha)^2\right).F_n\right)\mathrm{d}\alpha & \text{if } n_0+1 \leqslant n < 2n_0 - 1 \end{cases}. \tag{5.2}$$

The decomposition of H_{2n_0} is subtler and really goes in the direction of expression in terms of the *quadratic elementary functions*. We shall write (the second numeric parameter m in subscript always indicates the parameter of the relevant quadratic elementary function)

$$H_{2n_0} = H_{2n_0}^a + H_{2n_0}^b,$$
$$H_{2n_0}^a = \tfrac{(\varepsilon_{\mathbf{L}}q)^{n_0}}{2}\mathbb{1}_{\eta_0(-1)=\varepsilon_{\mathbf{L}}}\cdot\sum\nolimits_{m=0}^{n_0-1}H_{2n_0,m}^{a,1} + \tfrac{(\varepsilon_{\mathbf{L}}q)^{n_0}}{2}\mathbb{1}_{\eta_0(-1)=-\varepsilon_{\mathbf{L}}}\cdot\sum\nolimits_{m=0}^{n_0-1}H_{2n_0,m}^{a,\varepsilon}, \tag{5.3}$$
$$H_{2n_0}^b = \tfrac{(\varepsilon_{\mathbf{L}}q)^{n_0}}{2}\cdot\left\{H_{2n_0,n_0}^{b,\varepsilon} + H_{2n_0,n_0}^{b,1}\right\}, \tag{5.4}$$

where the summands are given by (below we assume $m \geqslant 1$ and $\tau \in \{1,\varepsilon\}$)

$$H_{2n_0,0}^{a,\tau} = \int_{\substack{\mathbf{L}^1\cap\mathcal{O}_{\mathbf{L}} \\ \mathrm{Tr}(x_\tau\alpha)\in\mathcal{P}_{\mathbf{F}}^{n_0}}}\beta(x_\tau\alpha)\mathrm{d}\alpha\cdot\left(\mathfrak{t}\left(\tau^{-1}\varpi_{\mathbf{F}}^{2n_0}\right).F_0\right),$$
$$H_{2n_0,m}^{a,\tau} = \int_{\substack{\mathbf{L}^1\cap\mathcal{O}_{\mathbf{L}} \\ \mathrm{Tr}(x_\tau\alpha)\in\varpi_{\mathbf{F}}^{n_0-m}\mathcal{O}_{\mathbf{F}}^\times}}\beta(x_\tau\alpha)\cdot\left(\mathfrak{t}\left(\tau^{-1}\mathrm{Tr}(x_\tau\alpha)^2\right).F_m\right)\mathrm{d}\alpha;$$
$$H_{2n_0,n_0}^{b,\varepsilon} = \int_{\substack{\mathbf{L}^1\cap\mathcal{O}_{\mathbf{L}} \\ \mathrm{Tr}(x_\varepsilon\alpha)\in\mathcal{O}_{\mathbf{F}}^\times}}\beta(x_\varepsilon\alpha)\cdot\left(\mathfrak{t}\left(\varepsilon^{-1}\mathrm{Tr}(x_\varepsilon\alpha)^2\right).F_{n_0}\right)\mathrm{d}\alpha,$$
$$H_{2,1}^{b,1} = \int_{\substack{\mathbf{L}^1\cap\mathcal{O}_{\mathbf{L}} \\ \mathrm{Tr}(\alpha)\in\mathcal{O}_{\mathbf{F}}^\times}}\beta(\alpha)\cdot\left(\mathfrak{t}\left(\mathrm{Tr}(\alpha)^2\right).F_1\right)\mathrm{d}\alpha,$$
$$H_{2n_0,n_0}^{b,1} = \int_{\substack{\mathbf{L}^1\cap\mathcal{O}_{\mathbf{L}} \\ \mathrm{Tr}(\alpha)^2\in\mathcal{O}_{\mathbf{F}}^\times-4(1+\mathcal{P}_{\mathbf{F}})}}\beta(\alpha)\cdot\left(\mathfrak{t}\left(\mathrm{Tr}(\alpha)^2\right).F_{n_0}\right)\mathrm{d}\alpha,\ n_0\geqslant 2.$$

Lemma 5.7. *Let χ be a (unitary) character of $\mathbf{F}^\times$ with $\mathfrak{c}(\chi) = n$. Recall the additive parameter c_β (resp. c_χ) in Lemma **??** (resp. Remark **??**).*
(1) Assume $0 \leqslant n < n_0$. For $\mathbf{L}/\mathbf{F}$ split *resp.* unramified *we have*

$$\int_{\mathcal{O}_{\mathbf{F}}^\times}\chi_0\left(\tfrac{1+\varpi_{\mathbf{F}}^{n_0-n}t}{1-\varpi_{\mathbf{F}}^{n_0-n}t}\right)\chi(t)\mathrm{d}t \quad \text{resp.} \quad \int_{\mathcal{O}_{\mathbf{F}}^\times}\beta(1+\varpi_{\mathbf{F}}^{n_0-n}t\sqrt{\varepsilon})\chi(t)\mathrm{d}t \ll q^{-\frac{n}{2}}.$$

(2) Assume $n = n_0 \geqslant 2$. For $\mathbf{L}/\mathbf{F}$ split *resp.* unramified *we have for any $k \in \{0,1\}$*

$$\int_{\mathcal{O}_{\mathbf{F}}^{\times}-\bigcup_{\pm}(\pm 1+\mathcal{P}_{\mathbf{F}})} \chi_0\left(\tfrac{1+t}{1-t}\right)\chi(t)\eta_0^k(1-t^2)\mathrm{d}t \quad \text{resp.} \quad \int_{\mathcal{O}_{\mathbf{F}}^{\times}} \beta(1+t\sqrt{\varepsilon})\chi(t)\eta_0^k(1-t^2\varepsilon)\mathrm{d}t$$
$$\ll \begin{cases} q^{-\frac{n_0}{2}} & \text{if } \chi \notin \boldsymbol{\mathcal{E}}(\beta) \\ q^{-\frac{n_0}{2}+\frac{1}{2}} & \text{if } \chi \in \boldsymbol{\mathcal{E}}(\beta) \end{cases},$$

where $\boldsymbol{\mathcal{E}}(\beta)\neq\emptyset$ only if $2\nmid n_0$ and $\eta_0(-1)=\varepsilon_{\mathbf{L}}$, under which condition it is given by

$$\boldsymbol{\mathcal{E}}(\beta) = \begin{cases} \left\{\chi \;\middle|\; (c_\chi c_\beta^{-1})^2 \equiv -1 \pmod{\mathcal{P}_{\mathbf{F}}}\right\} & \text{if } \mathbf{L}/\mathbf{F} \text{ split} \\ \left\{\chi \;\middle|\; (c_\chi c_\beta^{-1})^2 \equiv -\varepsilon \pmod{\mathcal{P}_{\mathbf{F}}}\right\} & \text{if } \mathbf{L}/\mathbf{F} \text{ unramified} \end{cases}.$$

Proof. We omit the proof of the split case, which is similar and simpler. The case of $n=0<n_0$ is easy and omitted, since the integrand is constant.
(1) If $n=1$, then $t\mapsto\beta\left(1+\varpi_{\mathbf{F}}^{n_0-n}t\sqrt{\varepsilon}\right)$ is a non-trivial additive character of $\mathcal{O}_{\mathbf{F}}$ and the bound follows from the one for Gauss sums. Assume $n\geqslant 2$. We perform a level $\lceil\frac{n}{2}\rceil$ regularization to $\mathrm{d}t$ and get

$$\int_{\mathcal{O}_{\mathbf{F}}^{\times}} \beta\left(1+\varpi_{\mathbf{F}}^{n_0-n}t\sqrt{\varepsilon}\right)\chi(t)\mathrm{d}t$$
$$= \int_{\mathcal{O}_{\mathbf{F}}^{\times}} \beta\left(1+\varpi_{\mathbf{F}}^{n_0-n}t\sqrt{\varepsilon}\right)\chi(t)\cdot\left(\oint_{\mathcal{O}_{\mathbf{F}}} \beta\left(1+\tfrac{\varpi_{\mathbf{F}}^{n_0-\lfloor\frac{n}{2}\rfloor}}{1+\varpi_{\mathbf{F}}^{n_0-n}t\sqrt{\varepsilon}}tu\sqrt{\varepsilon}\right)\chi(1+\varpi_{\mathbf{F}}^{\lceil\frac{n}{2}\rceil}u)\mathrm{d}u\right)dt.$$

The integrands of the inner integral, denoted by $I(t;n)$, are additive characters of $\mathcal{O}_{\mathbf{F}}$. We have

$$I(t;n) = \oint_{\mathcal{O}_{\mathbf{F}}} \psi\left(\tfrac{1}{\varpi_{\mathbf{F}}^{\lfloor\frac{n}{2}\rfloor}}\cdot\tfrac{2c_\beta tu\varepsilon}{1-\varpi_{\mathbf{F}}^{2(n_0-n)}t^2\varepsilon}\right)\psi\left(\tfrac{c_\chi u}{\varpi_{\mathbf{F}}^{\lfloor\frac{n}{2}\rfloor}}\right)\mathrm{d}u.$$

The non-vanishing of $I(t;n)$ implies the congruence condition

$$\tfrac{2c_\beta t\varepsilon}{1-\varpi_{\mathbf{F}}^{2(n_0-n)}t^2\varepsilon}+c_\chi \in \mathcal{P}_{\mathbf{F}}^{\lfloor\frac{n}{2}\rfloor} \quad\Leftrightarrow\quad 2c_\beta t\varepsilon + c_\chi(1-\varpi_{\mathbf{F}}^{2(n_0-n)}t^2\varepsilon)\in\mathcal{P}_{\mathbf{F}}^{\lfloor\frac{n}{2}\rfloor},$$

which has a unique solution $t\in t_0+\mathcal{P}_{\mathbf{F}}^{\lfloor\frac{n}{2}\rfloor}$ with $t_0\in\mathcal{O}_{\mathbf{F}}^{\times}$ by Hensel's lemma. Consequently we get

$$\int_{\mathcal{O}_{\mathbf{F}}^{\times}} \beta\left(1+\varpi_{\mathbf{F}}^{n_0-n}t\sqrt{\varepsilon}\right)\chi(t)\mathrm{d}t = \int_{t_0+\mathcal{P}_{\mathbf{F}}^{\lfloor\frac{n}{2}\rfloor}} \beta\left(1+\varpi_{\mathbf{F}}^{n_0-n}t\sqrt{\varepsilon}\right)\chi(t)\mathrm{d}t \ll q^{-\lfloor\frac{n}{2}\rfloor}. \tag{5.5}$$

If $2\mid n$ then we are done (for both (1) and (2)). Otherwise let $n=2m+1$. We may assume

$$2c_\beta t_0\varepsilon + c_\chi(1-\varpi_{\mathbf{F}}^{2(n_0-n)}t_0^2\varepsilon)=0,$$

make the change of variables $t=t_0(1+\varpi_{\mathbf{F}}^m u)$, and continue (**??**) to conclude by

$$\int_{\mathcal{O}_{\mathbf{F}}^{\times}} \beta\left(1+\varpi_{\mathbf{F}}^{n_0-n}t\sqrt{\varepsilon}\right)\chi(t)\mathrm{d}t = q^{-m}\beta(1+\varpi_{\mathbf{F}}^{n_0-n}t_0\sqrt{\varepsilon})\chi(t_0)\int_{\mathcal{O}_{\mathbf{F}}}\psi\left(-\tfrac{c_\chi u^2}{2\varpi_{\mathbf{F}}}\right)du \ll q^{-\frac{n}{2}}.$$

(2) Let $n=2m+1$ with $m\geqslant 1$. With the change of variables $t=t_0(1+\varpi_{\mathbf{F}}^m u)$ for $t_0\in\mathcal{O}_{\mathbf{F}}^{\times}$ satisfying

$$2c_\beta t_0\varepsilon + c_\chi(1-t_0^2\varepsilon)=0, \tag{5.6}$$

which is solvable only if $\eta_0(-1)=\varepsilon_{\mathbf{L}}$, we get the equation

$$\int_{\mathcal{O}_{\mathbf{F}}^{\times}} \beta(1+t\sqrt{\varepsilon})\chi(t)\eta_0^k(1-t^2\varepsilon)\mathrm{d}t = \sum\nolimits_{t_0}\beta(1+t_0\sqrt{\varepsilon})\chi(t_0)\eta_0^k(1-t_0^2\varepsilon)\cdot\int_{\mathcal{O}_{\mathbf{F}}}\psi\left(\tfrac{u^2}{\varpi}\left(\tfrac{2c_\beta t_0^3\varepsilon}{(1-t_0^2\varepsilon)^2}-\tfrac{c_\chi}{2}\right)\right)\mathrm{d}u$$
$$= \sum\nolimits_{t_0}\beta(1+t_0\sqrt{\varepsilon})\chi(t_0)\eta_0^k(1-t_0^2\varepsilon)\cdot\int_{\mathcal{O}_{\mathbf{F}}}\psi\left(\tfrac{c_\chi u^2}{2\varpi}\left(\tfrac{c_\chi}{c_\beta}t_0-1\right)\right)\mathrm{d}u.$$

The above integral is $\ll q^{-\frac{1}{2}}$ unless $t_0\equiv c_\chi^{-1}c_\beta \pmod{\mathcal{P}_{\mathbf{F}}}$, in which case (**??**) implies $\chi\in\boldsymbol{\mathcal{E}}(\beta)$. $\square$

Lemma 5.8. *Suppose $n_0+1\leqslant n\leqslant 2n_0-1$ (hence $n_0\geqslant 2$) and $n\geqslant \mathfrak{a}(\Pi)$. Then we have*

$$\widetilde{h}_{2n}^{-}(\chi) \ll q^{-n_0}\mathbb{1}_{2n_0-n}(\mathfrak{c}(\chi\eta_0^n)) + q^{-n_0}\mathbb{1}_{n=2n_0-1}\mathbb{1}_{\mathfrak{c}(\chi\eta_0)=0}.$$

Proof. (1) First consider $n < 2n_0 - 1$. By Corollary **??** we have for any $\delta \in 1 + \varpi_{\mathbf{F}}^{2(n-n_0)} \mathcal{O}_{\mathbf{F}}^{\times}$

$$\int_{\mathbf{F}-\mathcal{O}_{\mathbf{F}}} \mathcal{VH}_{\Pi,\psi} \circ \mathfrak{m}_{-1}\left(\mathfrak{t}(4\delta).F_n\right)(t) \cdot \psi(-t)\chi^{-1}(t)|t|^{-\frac{1}{2}} \mathrm{d}^{\times} t \tag{5.7}$$
$$= q^{-n}\zeta_{\mathbf{F}}(1) \cdot \begin{cases} \chi^{-1}\left(\frac{\delta}{1-\delta}\right) \cdot \gamma(1,\chi,\psi) \mathbb{1}_{2n_0-n}(\mathfrak{c}(\chi)) & \text{if } 2 \mid n \\ \chi^{-1}\eta_0\left(\frac{\delta}{1-\delta}\right) \cdot \tau_0 q^{\frac{1}{2}} \gamma(1,\chi\eta_0,\psi) \mathbb{1}_{2n_0-n}(\mathfrak{c}(\chi\eta_0)) & \text{if } 2 \nmid n \end{cases},$$

where we used $\eta_0(\delta) = 1$. Inserting (**??**) with $\delta = 4^{-1}\mathrm{Tr}(\alpha)^2$ into (**??**) we get

$$\widetilde{h}_{2n}^{-}(\chi) = \int_{\mathbf{F}-\mathcal{O}_{\mathbf{F}}} \mathcal{VH}_{\Pi,\psi} \circ \mathfrak{m}_{-1}(H_{2n})(t) \cdot \psi(-t)\chi^{-1}(t)|t|^{-\frac{1}{2}} \mathrm{d}^{\times} t \ll \tag{5.8}$$
$$\left| \int_{\substack{\mathbf{L}^1 \cap \mathcal{O}_{\mathbf{L}} \\ \mathrm{Tr}(\alpha) \in 2(1+\varpi_{\mathbf{F}}^{2(n-n_0)}\mathcal{O}_{\mathbf{F}}^{\times})}} \beta(\alpha) \cdot \chi^{-1}\eta_0^n \left(\frac{\mathrm{Tr}(\alpha)^2}{4-\mathrm{Tr}(\alpha)^2}\right) \mathrm{d}\alpha \right| \cdot q^{-\frac{2n_0-n}{2}} \mathbb{1}_{2n_0-n}(\mathfrak{c}(\chi\eta_0^n)).$$

In the case of split, resp. unramified $\mathbf{L}/\mathbf{F}$ we apply the change of variables $t = \frac{\alpha-\alpha^{-1}}{\alpha+\alpha^{-1}}$, resp. $t = \frac{\alpha-\alpha^{-1}}{\alpha+\alpha^{-1}} \frac{1}{\sqrt{\varepsilon}}$. The inner integral in (**??**) becomes

$$2\chi\eta_0^n(-1) \int_{\varpi_{\mathbf{F}}^{n-n_0}\mathcal{O}_{\mathbf{F}}^{\times}} \chi_0\left(\frac{1+t}{1-t}\right) \chi^2(t)\mathrm{d}t \quad \text{resp.} \quad \chi\eta_0^n(-\varepsilon) \int_{\varpi_{\mathbf{F}}^{n-n_0}\mathcal{O}_{\mathbf{F}}^{\times}} \beta(1+t\sqrt{\varepsilon})\chi^2(t)\mathrm{d}t.$$

We apply Lemma **??** (1) to bound the above integrals and conclude.
(2) Consider $n = 2n_0 - 1$. We have for any $\delta \in 1 + \mathcal{P}_{\mathbf{F}}^{2(n_0-1)}$

$$\int_{\mathbf{F}-\mathcal{O}_{\mathbf{F}}} \mathcal{VH}_{\Pi,\psi} \circ \mathfrak{m}_{-1}\left(\mathfrak{t}(4\delta).F_n\right)(t) \cdot \psi(-t)\chi^{-1}(t)|t|^{-\frac{1}{2}} \mathrm{d}^{\times} t = q^{-(2n_0-1)} \cdot \tau_0 q^{\frac{1}{2}} \cdot \tag{5.9}$$
$$\begin{cases} \chi\eta(\varpi_{\mathbf{F}})^{2n_0-1} \cdot \mathbb{1}_0(\mathfrak{c}(\chi\eta_0)) & \text{if } \delta \in 1 + \mathcal{P}_{\mathbf{F}}^{2n_0-1} \\ \chi^{-1}\eta_0\left(\frac{\delta}{1-\delta}\right) \cdot \zeta_{\mathbf{F}}(1)\left\{\gamma(1,\chi\eta_0,\psi)\mathbb{1}_1(\mathfrak{c}(\chi\eta_0)) - q^{-1}\mathbb{1}_0(\mathfrak{c}(\chi\eta_0))\right\} & \text{if } \delta \in 1 + \varpi_{\mathbf{F}}^{2(n_0-1)}\mathcal{O}_{\mathbf{F}}^{\times} \end{cases}.$$

Inserting (**??**) with $\delta = 4^{-1}\mathrm{Tr}(\alpha)^2$ into (**??**) we get

$$\widetilde{h}_{2n}^{-}(\chi) = \int_{\mathbf{F}-\mathcal{O}_{\mathbf{F}}} \mathcal{VH}_{\Pi,\psi} \circ \mathfrak{m}_{-1}(H_{2n})(t) \cdot \psi(-t)\chi^{-1}(t)|t|^{-\frac{1}{2}} \mathrm{d}^{\times} t \ll \tag{5.10}$$
$$\left| \int_{\substack{\mathbf{L}^1 \cap \mathcal{O}_{\mathbf{L}} \\ \mathrm{Tr}(\alpha) \in 2(1+\varpi_{\mathbf{F}}^{2(n_0-1)}\mathcal{O}_{\mathbf{F}}^{\times})}} \beta(\alpha) \cdot \chi^{-1}\eta_0 \left(\frac{\mathrm{Tr}(\alpha)^2}{4-\mathrm{Tr}(\alpha)^2}\right) \mathrm{d}\alpha \right| \cdot q^{-\frac{1}{2}} \mathbb{1}_1(\mathfrak{c}(\chi\eta_0)) +$$
$$\left| \int_{\substack{\mathbf{L}^1 \cap \mathcal{O}_{\mathbf{L}} \\ \mathrm{Tr}(\alpha) \in 2(1+\mathcal{P}_{\mathbf{F}}^{2n_0-1})}} \beta(\alpha)\mathrm{d}\alpha - \frac{\chi\eta_0(\varpi_{\mathbf{F}})^{-1}}{q-1} \int_{\substack{\mathbf{L}^1 \cap \mathcal{O}_{\mathbf{L}} \\ \mathrm{Tr}(\alpha) \in 2(1+\varpi_{\mathbf{F}}^{2(n_0-1)}\mathcal{O}_{\mathbf{F}}^{\times})}} \beta(\alpha)\mathrm{d}\alpha \right| \cdot \mathbb{1}_0(\mathfrak{c}(\chi\eta_0)).$$

The first summand is bounded the same way as before. Note that $\mathrm{Tr}(\alpha) \in 2(1+\mathcal{P}_{\mathbf{F}}^{2n_0-1})$ is equivalent to $\alpha \in 1 + \mathcal{P}_{\mathbf{L}}^{n_0}$. With the same change of variables the inner integrals in the second summand become

$$2q^{-n_0} - \frac{2\chi\eta_0(\varpi_{\mathbf{F}})^{-1}}{q-1} \int_{\varpi_{\mathbf{F}}^{n_0-1}\mathcal{O}_{\mathbf{F}}^{\times}} \chi_0\left(\frac{1+t}{1-t}\right)\mathrm{d}t \quad \text{resp.} \quad q^{-n_0} - \frac{\chi\eta_0(\varpi_{\mathbf{F}})^{-1}}{q-1} \int_{\varpi_{\mathbf{F}}^{n_0-1}\mathcal{O}_{\mathbf{F}}^{\times}} \beta(1+t\sqrt{\varepsilon})\mathrm{d}t.$$

They are of size $O(q^{-n_0})$ since the integrands are additive characters of conductor exponent n_0. $\square$

Lemma 5.9. *Suppose* $(2 \leqslant) a(\Pi) \leqslant m < n_0$. *Then we have for unitary* χ *and* $\tau \in \{1, \varepsilon\}$

$$\widetilde{h}_{2n_0,m}^{a,\tau}(\chi) := \int_{\mathbf{F}-\mathcal{O}_{\mathbf{F}}} \mathcal{VH}_{\Pi,\psi} \circ \mathfrak{m}_{-1}\left(H_{2n_0,m}^{a,\tau}\right)(t)\psi(-t)\chi^{-1}(t)|t|^{-\frac{1}{2}} \mathrm{d}^{\times} t \ll q^{-2n_0} \mathbb{1}_{m > \frac{2n_0}{3}} \mathbb{1}_m(\mathfrak{c}(\chi)).$$

Proof. We first use Corollary **??** to obtain for any $\delta \in \varpi_{\mathbf{F}}^{2(n_0-m)}\mathcal{O}_{\mathbf{F}}^{\times}$ (note that $\eta_0(4-\delta)=1$)

$$\begin{aligned}(5.11)\quad &\int_{\mathbf{F}-\mathcal{O}_{\mathbf{F}}} \mathcal{VH}_{\Pi,\psi}\circ \mathfrak{m}_{-1}\left(\mathfrak{t}\left(\delta\right).F_m\right)(t)\cdot\psi(-t)\chi^{-1}(t)|t|^{-\frac{1}{2}}\mathrm{d}^{\times}t \\ &= q^{-n_0}\zeta_{\mathbf{F}}(1)\mathbb{1}_{m>\frac{2n_0}{3}}\cdot\begin{cases}\chi\left(\frac{4-\delta}{\delta}\right)\cdot\gamma(1,\chi,\psi)\mathbb{1}_m(\mathfrak{c}(\chi)) & \text{if } 2\mid m \\ \chi\left(\frac{4-\delta}{\delta}\right)\cdot\tau_0 q^{\frac{1}{2}}\gamma(1,\chi\eta_0,\psi)\mathbb{1}_m(\mathfrak{c}(\chi)) & \text{if } 2\nmid m\end{cases}.\end{aligned}$$

Inserting (**??**) with $\delta=\tau^{-1}\mathrm{Tr}(x_{\tau}\alpha)^2$ into the integral representation of $H^{a,\tau}_{2n_0,m}$ we get

$$(5.12)\qquad \widetilde{h}^{a,\tau}_{2n_0,m}(\chi) \ll \mathbb{1}_{m>\frac{2n_0}{3}}\cdot\left|\int_{\substack{\mathbf{L}^1\cap\mathcal{O}_{\mathbf{L}} \\ \mathrm{Tr}(x_{\tau}\alpha)\in\varpi_{\mathbf{F}}^{n_0-m}\mathcal{O}_{\mathbf{F}}^{\times}}}\beta(x_{\tau}\alpha)\cdot\chi\left(\frac{4\tau-\mathrm{Tr}(x_{\tau}\alpha)^2}{\mathrm{Tr}(x_{\tau}\alpha)^2}\right)\mathrm{d}\alpha\right|\cdot q^{-n_0-\frac{m}{2}}\mathbb{1}_m(\mathfrak{c}(\chi)).$$

In the case of split, resp. unramified $\mathbf{L}/\mathbf{F}$ we apply the change of variables $t=\frac{\alpha+\alpha^{-1}}{\alpha-\alpha^{-1}}$, resp. $t=\frac{\alpha+\alpha^{-1}}{\alpha-\alpha^{-1}}\frac{1}{\sqrt{\varepsilon}}$ for $\widetilde{h}^{a,1}_{2n_0,m}(\chi)$; $t=\frac{\alpha+\varepsilon\alpha^{-1}}{\alpha-\varepsilon\alpha^{-1}}$, resp. $t=\frac{x_{\varepsilon}\alpha+\overline{x_{\varepsilon}\alpha}}{x_{\varepsilon}\alpha-\overline{x_{\varepsilon}\alpha}}\frac{1}{\sqrt{\varepsilon}}$ for $\widetilde{h}^{a,\varepsilon}_{2n_0,m}(\chi)$. The inner integrals in (**??**) become

$$2\chi_0\chi(-1)\int_{\varpi_{\mathbf{F}}^{n_0-m}\mathcal{O}_{\mathbf{F}}^{\times}}\chi_0\left(\frac{1+t}{1-t}\right)\chi^{-2}(t)\mathrm{d}t \quad\text{resp.}\quad 2\beta(\sqrt{\varepsilon})\chi^{-1}(-\varepsilon)\int_{\varpi_{\mathbf{F}}^{n_0-m}\mathcal{O}_{\mathbf{F}}^{\times}}\beta(1+t\sqrt{\varepsilon})\chi^{-2}(t)\mathrm{d}t.$$

We apply Lemma **??** (1) to bound the above integrals and conclude the desired inequalities. □

Lemma 5.10. *Suppose $n_0\geqslant a(\Pi)(\geqslant 2)$. Let $\tau\in\{1,\varepsilon\}$. With $\boldsymbol{\mathcal{E}}(\beta)$ defined in Lemma* **??** *(2) we have*

$$\begin{aligned}\widetilde{h}^{b,\tau}_{2n_0,n_0}(\chi) &:= \int_{\mathbf{F}-\mathcal{O}_{\mathbf{F}}}\mathcal{VH}_{\Pi,\psi}\circ\mathfrak{m}_{-1}\left(H^{b,\tau}_{2n_0,n_0}\right)(t)\psi(-t)\chi^{-1}(t)|t|^{-\frac{1}{2}}\mathrm{d}^{\times}t \\ &\ll q^{-2n_0}\mathbb{1}_{n_0}(\mathfrak{c}(\chi))\left(\mathbb{1}_{\boldsymbol{\mathcal{E}}(\beta)^c}(\chi^2)+q^{\frac{1}{2}}\mathbb{1}_{\boldsymbol{\mathcal{E}}(\beta)}(\chi^2)\right).\end{aligned}$$

Proof. We first use Corollary **??** to obtain for any $\delta\in\mathcal{O}_{\mathbf{F}}^{\times}-4(1+\mathcal{P}_{\mathbf{F}})$

$$\begin{aligned}(5.13)\quad &\int_{\mathbf{F}-\mathcal{O}_{\mathbf{F}}} \mathcal{VH}_{\Pi,\psi}\circ \mathfrak{m}_{-1}\left(\mathfrak{t}\left(\delta\right).F_m\right)(t)\cdot\psi(-t)\chi^{-1}(t)|t|^{-\frac{1}{2}}\mathrm{d}^{\times}t \\ &= q^{-n_0}\zeta_{\mathbf{F}}(1)\cdot\begin{cases}\chi\left(\frac{4-\delta}{\delta}\right)\cdot\gamma(1,\chi,\psi)\mathbb{1}_{n_0}(\mathfrak{c}(\chi)) & \text{if } 2\mid n_0 \\ \eta_0(4-\delta)\chi\left(\frac{4-\delta}{\delta}\right)\cdot\tau_0 q^{\frac{1}{2}}\gamma(1,\chi\eta_0,\psi)\mathbb{1}_{n_0}(\mathfrak{c}(\chi)) & \text{if } 2\nmid n_0\end{cases}.\end{aligned}$$

Inserting (**??**) with $\delta=\tau^{-1}\mathrm{Tr}(x_{\tau}\alpha)^2$ into the integral representation of $H^{b,\tau}_{2n_0,n_0}$ we get

$$(5.14)\qquad \widetilde{h}^{b,1}_{2n_0,n_0}(\chi)\ll\left|\int_{\substack{\mathbf{L}^1\cap\mathcal{O}_{\mathbf{L}} \\ \mathrm{Tr}(\alpha)^2\in\mathcal{O}_{\mathbf{F}}^{\times}-4(1+\mathcal{P}_{\mathbf{F}})}}\beta(\alpha)\cdot\chi\left(\frac{4-\mathrm{Tr}(\alpha)^2}{\mathrm{Tr}(\alpha)^2}\right)\eta_0^{n_0}\left(4-\mathrm{Tr}(\alpha)^2\right)\mathrm{d}\alpha\right|\cdot q^{-\frac{3n_0}{2}}\mathbb{1}_{n_0}(\mathfrak{c}(\chi)).$$

$$(5.15)\quad \widetilde{h}^{b,\varepsilon}_{2n_0,n_0}(\chi)\ll\left|\int_{\substack{\mathbf{L}^1\cap\mathcal{O}_{\mathbf{L}} \\ \mathrm{Tr}(x_{\varepsilon}\alpha)\in\mathcal{O}_{\mathbf{F}}^{\times}}}\beta(x_{\varepsilon}\alpha)\cdot\chi\left(\frac{4\tau-\mathrm{Tr}(x_{\varepsilon}\alpha)^2}{\mathrm{Tr}(x_{\varepsilon}\alpha)^2}\right)\eta_0^{n_0}\left(4-\tau^{-1}\mathrm{Tr}(x_{\varepsilon}\alpha)^2\right)\mathrm{d}\alpha\right|\cdot q^{-\frac{3n_0}{2}}\mathbb{1}_{n_0}(\mathfrak{c}(\chi)).$$

In the case of split, resp. unramified $\mathbf{L}/\mathbf{F}$ we apply the change of variables $t=\frac{\alpha+\alpha^{-1}}{\alpha-\alpha^{-1}}$, resp. $t=\frac{\alpha+\alpha^{-1}}{\alpha-\alpha^{-1}}\frac{1}{\sqrt{\varepsilon}}$ for $\widetilde{h}^{b,1}_{2n_0,n_0}(\chi)$; $t=\frac{\alpha+\varepsilon\alpha^{-1}}{\alpha-\varepsilon\alpha^{-1}}$, resp. $t=\frac{x_{\varepsilon}\alpha+\overline{x_{\varepsilon}\alpha}}{x_{\varepsilon}\alpha-\overline{x_{\varepsilon}\alpha}}\frac{1}{\sqrt{\varepsilon}}$ for $\widetilde{h}^{b,\varepsilon}_{2n_0,n_0}(\chi)$. The inner integrals in (**??**) and (**??**), denoted by I_1 and I_{ε} respectively, become

$$I_1=I_{\varepsilon}=2\chi_0\chi(-1)\int_{\mathcal{O}_{\mathbf{F}}^{\times}-\bigcup_{\pm}(\pm 1+\mathcal{P}_{\mathbf{F}})}\chi_0\left(\frac{1+t}{1-t}\right)\chi^{-2}(t)\eta_0^{n_0}(1-t^2)\mathrm{d}t\quad\text{resp.}$$

$$\begin{cases} I_1 = 2\beta(\sqrt{\varepsilon})\chi^{-1}(-\varepsilon)\eta_0^{n_0}(-\varepsilon)\displaystyle\int_{\substack{\mathcal{O}_{\mathbf{F}}^\times \\ 1-\varepsilon t^2 \in -\varepsilon(\mathcal{O}_{\mathbf{F}}^\times)^2}} \beta(1+t\sqrt{\varepsilon})\chi^{-2}(t)\mathrm{d}t \\ I_\varepsilon = 2\beta(\sqrt{\varepsilon})\chi^{-1}(-\varepsilon)\eta_0^{n_0}(-1)\displaystyle\int_{\substack{\mathcal{O}_{\mathbf{F}}^\times \\ 1-\varepsilon t^2 \in -(\mathcal{O}_{\mathbf{F}}^\times)^2}} \beta(1+t\sqrt{\varepsilon})\chi^{-2}(t)\mathrm{d}t \end{cases}.$$

We apply Lemma **??** (2) to bound the above integrals and conclude the desired inequalities. □

Lemma 5.11. *Let $n_0 \geqslant \max(2\mathfrak{c}(\Pi), \mathfrak{c}(\Pi)+2)$ and $m \leqslant \mathfrak{c}(\Pi)$. Then for any $\delta \in \varpi_{\mathbf{F}}^{2(n_0-m)}\mathcal{O}_{\mathbf{F}}^\times$ we have*

$$\int_{\mathbf{F}-\mathcal{O}_{\mathbf{F}}} \mathcal{VH}_{\Pi,\psi} \circ \mathfrak{m}_{-1}\left(\mathfrak{t}(\delta).F_m\right)(t) \cdot \psi(-t)\chi^{-1}(t)|t|^{-\frac{1}{2}}\mathrm{d}^\times t = 0.$$

Consequently, we get for $\tau \in \{1, \varepsilon\}$ and any unitary χ the vanishing of $\widetilde{h}_{2n_0,m}^{a,\tau}(\chi) = 0$.

Proof. For simplicity we only consider the case $m \geqslant 2$. By the local functional equation and Lemma **??** we have (recall the parameters of $L(s, \Pi \otimes \xi)$ introduced in (**??**))

$$\int_{\mathbf{F}^\times} \mathcal{VH}_{\Pi,\psi}(F_m)(t)\xi^{-1}(t)|t|^{-s}\mathrm{d}^\times t = \mathbb{1}_{\mathfrak{c}(\xi)=m} \cdot \zeta_{\mathbf{F}}(1)\gamma\left(1, \xi^{-2}, \psi\right)\varepsilon\left(\tfrac{1}{2}, \Pi\otimes\xi, \psi\right) q^{\frac{\mathfrak{c}(\Pi\otimes\xi)}{2}}$$
$$\cdot\, q^{(2m-\rho(\Pi\otimes\xi))s} \prod_{j=1}^{d(\Pi\otimes\xi)} \frac{q^s - a_j}{1-\overline{a_j}q^{-1+s}}.$$

As a Laurent series in $X := q^s$, the right hand side contains X^k only for $k \geqslant 2m - \rho(\Pi \otimes \xi)$. We have

$$\rho(\Pi\otimes\xi) \begin{cases} = \mathfrak{c}(\Pi\otimes\xi) \leqslant \mathfrak{c}(\Pi) + 3\mathfrak{c}(\xi) \leqslant 4\mathfrak{c}(\Pi) & \text{if } \xi \notin E(\Pi) \\ \leqslant 2\mathfrak{c}(\Pi) + 3 & \text{if } \xi \in E(\Pi) \end{cases}$$

by [?] and Lemma **??** respectively. Therefore the support of $\mathcal{VH}_{\Pi,\psi} \circ \mathfrak{m}_{-1}\left(\mathfrak{t}(\delta).F_m\right)$ is contained in

$$\delta\mathcal{P}_{\mathbf{F}}^{2m-\max(4\mathfrak{c}(\Pi),2\mathfrak{c}(\Pi)+3)} = \mathcal{P}_{\mathbf{F}}^{2n_0-\max(4\mathfrak{c}(\Pi),2\mathfrak{c}(\Pi)+3)} \subset \mathcal{O}_{\mathbf{F}},$$

proving the desired vanishing. □

5.3. **The Bounds of Dual Weight.**

Lemma 5.12. *Suppose $\mathfrak{c}(\Pi) = 0$ and $n_0 = 1$. Then we have $\widetilde{h}_2^-(\chi) \leqslant q^{-\frac{3}{2}}\mathbb{1}_{\mathfrak{c}(\chi)=0} + q^{-1}\mathbb{1}_{\mathfrak{c}(\chi)=1}$.*

Proof. Necessarily we have $\Pi = \mu_1 \boxplus \mu_2 \boxplus \mu_3$ for unramified μ_j. Note that H_2^a (resp. H_2^b) is related to F_0 (resp. F_1) by the formulae:

$$H_2^a = \varepsilon_{\mathbf{L}}\mathbb{1}_{\eta_0(-1)=\varepsilon_{\mathbf{L}}}\beta(\sqrt{-1}) \cdot \left(\mathfrak{t}(\varpi_{\mathbf{F}}^2).F_0\right) + \varepsilon_{\mathbf{L}}\mathbb{1}_{\eta_0(-1)=-\varepsilon_{\mathbf{L}}}\beta(\sqrt{-\varepsilon}) \cdot \left(\mathfrak{t}(\varepsilon^{-1}\varpi_{\mathbf{F}}^2).F_0\right),$$

$$H_2^b = \tfrac{\varepsilon_{\mathbf{L}}q}{2}\int_{\substack{\mathbf{L}^1\cap\mathcal{O}_{\mathbf{L}} \\ \mathrm{Tr}(\alpha)\in\mathcal{O}_{\mathbf{F}}^\times}} \beta(\alpha)\cdot\left(\mathfrak{t}(\mathrm{Tr}(\alpha)^2).F_1\right)\mathrm{d}\alpha + \tfrac{\varepsilon_{\mathbf{L}}q}{2}\int_{\substack{\mathbf{L}^1\cap\mathcal{O}_{\mathbf{L}} \\ \mathrm{Tr}(x_\varepsilon\alpha)\in\mathcal{O}_{\mathbf{F}}^\times}} \beta(x_\varepsilon\alpha)\cdot\left(\mathfrak{t}(\varepsilon^{-1}\mathrm{Tr}(x_\varepsilon\alpha)^2).F_1\right)\mathrm{d}\alpha.$$

For any $\delta_0 \in \varpi_{\mathbf{F}}^2\mathcal{O}_{\mathbf{F}}^\times$ and $\delta_1 \in \mathcal{O}_{\mathbf{F}}^\times$ we have

$$\int_{\mathbf{F}-\mathcal{O}_{\mathbf{F}}} \mathcal{VH}_{\Pi,\psi}\circ\mathfrak{m}_{-1}\left(\mathfrak{t}(\delta_0).F_0\right)(t)\psi(-t)\chi(t)|t|^{-\frac{1}{2}}\mathrm{d}^\times t = q^{-\frac{5}{2}}\int_{\varpi_{\mathbf{F}}^{-1}\mathcal{O}_{\mathbf{F}}^\times} \mathcal{VH}_{\Pi,\psi}(F_0)(\delta_0^{-1}t)\psi(-t)\chi(t)\mathrm{d}^\times t,$$

$$\int_{\mathbf{F}-\mathcal{O}_{\mathbf{F}}} \mathcal{VH}_{\Pi,\psi}\circ\mathfrak{m}_{-1}\left(\mathfrak{t}(\delta_1).F_1\right)(t)\psi(-t)\chi(t)|t|^{-\frac{1}{2}}\mathrm{d}^\times t = q^{-\frac{5}{2}}\int_{\varpi_{\mathbf{F}}^{-1}\mathcal{O}_{\mathbf{F}}^\times} \mathcal{VH}_{\Pi,\psi}(F_1)(\delta_1^{-1}t)\psi(-t)\chi(t)\mathrm{d}^\times t,$$

by inspecting the supports of $\mathcal{VH}_{\Pi,\psi}(F_0)$ and $\mathcal{VH}_{\Pi,\psi}(F_1)$ given in Corollary **??**. Define

$$K(y) := \tau_0 q\mathbb{1}_{\varpi_{\mathbf{F}}^{-1}\mathcal{O}_{\mathbf{F}}^\times}(y)\int_{(\varpi_{\mathbf{F}}^{-1}\mathcal{O}_{\mathbf{F}}^\times)^2} \psi\left(t_1+t_2-\tfrac{t_1t_2}{4y}\right)\eta_0\left(\tfrac{4y}{t_1t_2}\right)\mathrm{d}t_1\mathrm{d}t_2,$$

$$\widetilde{K}(\delta,\chi) := q^{-\frac{5}{2}}\int_{\varpi_{\mathbf{F}}^{-1}\mathcal{O}_{\mathbf{F}}^\times} K(\delta^{-1}t)\psi(-t)\chi(t)\mathrm{d}^\times t.$$

$$I_1(\chi) := \int_{\substack{\mathbf{L}^1 \cap \mathcal{O}_{\mathbf{L}} \\ \mathrm{Tr}(\alpha) \in \mathcal{O}_{\mathbf{F}}^{\times}}} \beta(\alpha) \widetilde{K}(\mathrm{Tr}(\alpha)^2, \chi) \mathrm{d}\alpha, \quad I_2(\chi) := \int_{\substack{\mathbf{L}^1 \cap \mathcal{O}_{\mathbf{L}} \\ \mathrm{Tr}(x_{\varepsilon}\alpha) \in \mathcal{O}_{\mathbf{F}}^{\times}}} \beta(x_{\varepsilon}\alpha) \widetilde{K}(\varepsilon^{-1}\mathrm{Tr}(x_{\varepsilon}\alpha)^2, \chi) \mathrm{d}\alpha.$$

The formulae of $\mathcal{VH}_{\Pi,\psi}(F_0)$ and $\mathcal{VH}_{\Pi,\psi}(F_1)$ in Corollary **??** easily imply the bound

$$\widetilde{h}_2^-(\chi) \ll q^{-\frac{5}{2}} \mathbb{1}_{\mathfrak{c}(\chi) \leqslant 1} + q \left| I_1(\chi) + I_2(\chi) \right|. \tag{5.16}$$

An easy change of variables gives

$$\begin{aligned} \widetilde{K}(\delta, \chi) &= \tau_0 q^{\frac{1}{2}} \zeta_{\mathbf{F}}(1) \int_{(\mathcal{O}_{\mathbf{F}}^{\times})^3} \psi\left(\frac{t_1 + t_2 - \frac{t_1 t_2 \delta}{4t} - t}{\varpi_{\mathbf{F}}} \right) \eta_0\left(\frac{4t}{\delta t_1 t_2} \right) \chi\left(\frac{t}{\varpi_{\mathbf{F}}} \right) \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}t \\ &= \tau_0 q^{\frac{3}{2}} \zeta_{\mathbf{F}}(1) \int_{\substack{(\mathcal{O}_{\mathbf{F}}^{\times})^4 \\ t_1 t_2 \equiv 4\delta^{-1} t_3 t_4 \pmod{\mathcal{P}_{\mathbf{F}}}}} \psi\left(\frac{t_1 + t_2 - t_3 - t_4}{\varpi_{\mathbf{F}}} \right) \chi\left(\frac{t_3}{\varpi_{\mathbf{F}}} \right) \eta_0(t_4) \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}t_3 \mathrm{d}t_4. \end{aligned} \tag{5.17}$$

Note that the function $t \mapsto K(\delta^{-1}t)\psi(-t)$ is invariant by $1 + \mathcal{P}_{\mathbf{F}}$, hence $\widetilde{K}(\delta, \chi)$ is non-zero only if $\mathfrak{c}(\chi) \leqslant 1$. If $\mathfrak{c}(\chi) = 0$ then one simplifies (**??**) by performing the integrals over t_1, t_3, t_4, t_2 in order

$$\widetilde{K}(\delta, \chi) = \chi(\varpi_{\mathbf{F}})^{-1} \zeta_{\mathbf{F}}(1) \left(q^{-\frac{5}{2}} + q^{-\frac{3}{2}} \cdot \mathbb{1}_{\mathcal{O}_{\mathbf{F}}^{\times}}(\delta - 4) \eta_0\left(\frac{\delta}{\delta - 4} \right) \right). \tag{5.18}$$

Inserting (**??**) into (**??**) we see that the integrands $\widetilde{K}(\cdot)$ are constant, equal to $\chi(\varpi_{\mathbf{F}})^{-1}\zeta_{\mathbf{F}}(1)q^{-\frac{3}{2}}(q^{-1} + \varepsilon_{\mathbf{L}})$, in both integrals. We obtain the bound $\widetilde{h}_2^-(\chi) \ll q^{-\frac{3}{2}}$ by

$$\int_{\substack{\mathbf{L}^1 \cap \mathcal{O}_{\mathbf{L}} \\ \mathrm{Tr}(\alpha) \in \mathcal{O}_{\mathbf{F}}^{\times}}} \beta(\alpha) \mathrm{d}\alpha = -q^{-1} \mathbb{1}_{\eta_0(-1) = \varepsilon_{\mathbf{L}}} \beta(\sqrt{-1}), \quad \int_{\substack{\mathbf{L}^1 \cap \mathcal{O}_{\mathbf{L}} \\ \mathrm{Tr}(x_{\varepsilon}\alpha) \in \mathcal{O}_{\mathbf{F}}^{\times}}} \beta(x_{\varepsilon}\alpha) \mathrm{d}\alpha = -q^{-1} \mathbb{1}_{\eta_0(-1) = -\varepsilon_{\mathbf{L}}} \beta(\sqrt{-\varepsilon}).$$

If $\mathfrak{c}(\chi) = 1$, then regarding χ and η_0 as characters of $\mathbb{F}_q^{\times}$ we rewrite $\widetilde{K}$ as

$$\widetilde{K}(\delta, \chi) = -\tau_0 q^{-1} \zeta_{\mathbf{F}}(1) \cdot H(4\delta^{-1}, q; (\mathbb{1}, \mathbb{1}), (\chi^{-1}, \eta_0)),$$

where $H(\cdot)$ is precisely the hypergeometric sum of Katz appeared in [**?** , §5.4]. In the case of split, resp. unramified $\mathbf{L}/\mathbf{F}$ we apply the change of variables $t = \frac{\alpha - \alpha^{-1}}{\alpha + \alpha^{-1}}$, resp. $t = \frac{\alpha - \alpha^{-1}}{\alpha + \alpha^{-1}} \sqrt{\varepsilon}$ for $I_1(\chi)$; $t = \frac{\alpha - \varepsilon\alpha^{-1}}{\alpha + \varepsilon\alpha^{-1}}$, resp. $t = \frac{x_{\varepsilon}\alpha - \overline{x_{\varepsilon}\alpha}}{x_{\varepsilon}\alpha + \overline{x_{\varepsilon}\alpha}} \sqrt{\varepsilon}$ for $I_2(\chi)$. These integrals become

$$I_1(\chi) = I_2(\chi) = 2 \int_{\mathcal{O}_{\mathbf{F}} - \bigcup_{\pm} (\pm 1 + \mathcal{P}_{\mathbf{F}})} \chi_0\left(\frac{1+t}{1-t} \right) \widetilde{K}\left(\frac{4}{1-t^2}, \chi \right) \mathrm{d}t \quad \text{resp.}$$

$$\begin{cases} I_1(\chi) = 2\beta(\sqrt{\varepsilon})^{-1} \int_{\mathcal{O}_{\mathbf{F}}} \beta(t + \sqrt{\varepsilon}) \widetilde{K}\left(\frac{4}{1 - \varepsilon^{-1}t^2}, \chi \right) \mathbb{1}_{(\mathcal{O}_{\mathbf{F}}^{\times})^2}(1 - \varepsilon^{-1}t^2) \mathrm{d}t \\ I_2(\chi) = 2\beta(\sqrt{\varepsilon})^{-1} \int_{\mathcal{O}_{\mathbf{F}}} \beta(t + \sqrt{\varepsilon}) \widetilde{K}\left(\frac{4}{1 - \varepsilon^{-1}t^2}, \chi \right) \mathbb{1}_{\varepsilon(\mathcal{O}_{\mathbf{F}}^{\times})^2}(1 - \varepsilon^{-1}t^2) \mathrm{d}t \end{cases}.$$

In the case of unramified $\mathbf{L}/\mathbf{F}$ we recognize and get the bound (with $H(\cdot)$ defined in [**?** , §5.3.2])

$$q \left| I_1(\chi) + I_2(\chi) \right| = 2\zeta_{\mathbf{F}}(1) q^{-\frac{3}{2}} \left| \sum\nolimits_{\alpha \in \mathbb{F}_q} \beta(\alpha + \sqrt{\varepsilon}) H(1 - \alpha^2 \varepsilon^{-1}, q; (\mathbb{1}, \mathbb{1}), (\chi^{-1}, \eta_0)) \right| \ll q^{-1}$$

by [**?** , Lemma 5.10]. In the case of split $\mathbf{L}/\mathbf{F}$ we recognize and *claim* a bound of

$$q \left| I_1(\chi) + I_2(\chi) \right| = 4\zeta_{\mathbf{F}}(1) q^{-\frac{3}{2}} \left| \sum\nolimits_{\alpha \in \mathbb{F}_q - \{\pm 1\}} \chi_0\left(\frac{\alpha+1}{\alpha-1} \right) H(1 - \alpha^2, q; (\mathbb{1}, \mathbb{1}), (\chi^{-1}, \eta_0)) \right| \ll q^{-1}. \tag{5.19}$$

The method of the proof of [**?** , Lemma 5.10] works through to bound the above sum. In fact the essence of that proof is to show:

(1) the ℓ-adic sheaf associated with the function $\alpha \mapsto H(1 - \alpha^2\varepsilon^{-1}, q; (\mathbb{1}, \mathbb{1}), (\chi^{-1}, \eta_0))$ has rank 2,
(2) while the sheaf associated with the function $\alpha \mapsto \beta(\alpha + \sqrt{\varepsilon})$ has rank 1.

The analogues for the above split case are

(1') the ℓ-adic sheaf associated with the function $\alpha \mapsto H(1 - \alpha^2, q; (\mathbb{1}, \mathbb{1}), (\chi^{-1}, \eta_0))$ has rank 2,
(2') while the sheaf associated with the function $\alpha \mapsto \chi_0\left(\frac{\alpha+1}{\alpha-1} \right)$ has rank 1.

Note that (1') and (1) follow from the same argument that the sheaf associated with $t \mapsto H(t, q; \cdots)$ has geometric monodromy group SL_2, which does not admit finite index subgroup; while (2') is trivially true because the Kummer sheaf has rank 1. In fact, both (2) and (2') are known to Weil [? , Appendix V]. We conclude the claimed bound (**??**). □

Proposition 5.13. *With the test function H given by (**??**) the dual weight function is bounded as*

$$\widetilde{h}(\chi) \ll_\epsilon \mathbf{C}(\Pi)^{2+\epsilon} q^{-n_0+\epsilon} \left(\mathbb{1}_{\leqslant \max(n_0, 6\mathfrak{c}(\Pi))}(\mathfrak{c}(\chi)) + q^{\frac{1}{2}} \mathbb{1}_{\eta_0(-1)=\varepsilon_{\mathbf{L}}} \mathbb{1}_{2 \nmid n_0 \geqslant 3} \mathbb{1}_{\mathfrak{c}(\chi)=n_0} \mathbb{1}_{\boldsymbol{\mathcal{E}}(\beta)}(\chi^2) \right),$$

*where we recall $\varepsilon_{\mathbf{L}} = \eta_{\mathbf{L}/\mathbf{F}}(\varpi_{\mathbf{F}})$, and $\boldsymbol{\mathcal{E}}(\beta)$ is given in Lemma **??** (2).*

Proof. We distinguish two cases $\mathfrak{c}(\Pi) > 0$ and $\mathfrak{c}(\Pi) = 0$.
(1) Suppose $\mathfrak{c}(\Pi) > 0$. We have $a(\Pi) \leqslant \mathfrak{c}(\Pi)$ by Proposition **??**. If $n_0 \leqslant 2\mathfrak{c}(\Pi)$, then we may take $n_1 = 4\mathfrak{c}(\Pi)$. Applying Lemma **??**, **??** and **??** we deduce

$$\widetilde{h}(\chi) \ll \left| \widetilde{h}_\infty(\chi) \right| + \left| \widetilde{h}_c^+(\chi) \right| + \left| \widetilde{h}_c^-(\chi) \right| \ll \mathfrak{c}(\Pi) \mathbb{1}_{\leqslant 6\mathfrak{c}(\Pi)}(\mathfrak{c}(\chi)),$$

the "worst" bound being offered by Lemma **??**. If $n_0 \geqslant 2\mathfrak{c}(\Pi)$, then we may take $n_1 = 2n_0$. Applying Lemma **??**, **??** and Lemma **??**-**??** we deduce

$$\widetilde{h}(\chi) \ll q^{-n_0} \left(\mathbb{1}_{\leqslant n_0}(\mathfrak{c}(\chi)) + q^{\frac{1}{2}} \mathbb{1}_{\eta_0(-1)=\varepsilon_{\mathbf{L}}} \mathbb{1}_{2 \nmid n_0 \geqslant 3} \mathbb{1}_{\mathfrak{c}(\chi)=n_0} \mathbb{1}_{\boldsymbol{\mathcal{E}}(\beta)}(\chi) \right).$$

The stated bound is simply a common upper bound of the above two.
(2) Suppose $\mathfrak{c}(\Pi) = 0$. Necessarily we have $\Pi = \mu_1 \boxplus \mu_2 \boxplus \mu_3$ for unramified μ_j and $a(\Pi) = 2$. The argument of the second part in (1) works through also for $n_0 \geqslant 2$. It remains to consider the case $n_0 = 1$. Note that $n_1 = 2$, hence the contribution of H_∞ is $\widetilde{h}_\infty(\chi) \ll q^{-1} \mathbb{1}_{\mathfrak{c}(\chi)=0}$ by Lemma **??**. We have

$$\widetilde{h}(\chi) = \widetilde{h}_\infty(\chi) + \widetilde{h}_2^+(\chi) + \widetilde{h}_2^-(\chi)$$

by (**??**) and the condition $2 \mid n$ following (**??**). By Lemma **??** we have $\widetilde{h}_2^+(\chi) \ll_\epsilon q^{-1+\epsilon} \mathbb{1}_{\mathfrak{c}(\chi)=0}$. By Lemma **??** we have $\widetilde{h}_2^-(\chi) \ll q^{-\frac{3}{2}} \mathbb{1}_{\mathfrak{c}(\chi)=0} + q^{-1} \mathbb{1}_{\mathfrak{c}(\chi)=1}$. The stated bound follows. □

5.4. **The Bounds of Unramified Dual Weight.**

Lemma 5.14. *Suppose $n_0 + 1 \leqslant n \leqslant 2n_0 - 1$ and $n \geqslant a(\Pi)$. Then the function $\widetilde{h}_{2n}^-(|\cdot|_{\mathbf{F}}^s) \neq 0$ is non-vanishing only if $n = 2n_0 - 1$. Moreover for any $k \in \mathbb{Z}_{\geqslant 0}$ we have*

$$\widetilde{H}_{2(2n_0-1)}^-(k; \tfrac{1}{2}) \ll_{k,\epsilon} q^{-\frac{1}{2}+(2n_0-1)\epsilon}, \quad \widetilde{H}_{2(2n_0-1)}^-(k; -\tfrac{1}{2}) \ll_{k,\epsilon} q^{\frac{1}{2}-2n_0+(2n_0-1)\epsilon}.$$

Proof. By Corollary **??** and (**??**) the support of $\mathcal{VH}_{\Pi,\psi}(F_n)$, hence also $\mathcal{VH}_{\Pi,\psi} \circ \mathfrak{m}_{-1}(H_{2n})$ are contained in $\varpi_{\mathbf{F}}^{-n} \mathcal{O}_{\mathbf{F}}^\times$. Therefore we get by (**??**) the relation $\widetilde{h}_{2n}^-(|\cdot|_{\mathbf{F}}^s) = q^{ns} \widetilde{h}_{2n}^-(\mathbb{1})$. The stated results follow readily from Lemma **??**. □

Lemma 5.15. *Suppose $\mathfrak{c}(\Pi) = 0$ and $n_0 = 1$. Then we have for any $k \in \mathbb{Z}_{\geqslant 0}$*

$$\widetilde{H}_2^-(k; \tfrac{1}{2}) \ll_{k,\epsilon} q^{-(1-\epsilon)}, \quad \widetilde{H}_2^-(k; -\tfrac{1}{2}) \ll_{k,\epsilon} q^{-2(1-\epsilon)}.$$

Proof. Similarly as the previous lemma, we have $\widetilde{h}_2^-(|\cdot|_{\mathbf{F}}^s) = q^s \cdot \widetilde{h}_2^-(\mathbb{1})$ by the inspection of the supports of $\mathcal{VH}_{\Pi,\psi} \circ \mathfrak{m}_{-1}(H_2)$ via Corollary **??** in the proof of Lemma **??**, from which the stated bounds follow. □

Proposition 5.16. *With the test function H given by (**??**) we have for any $k \in \mathbb{Z}_{\geqslant 0}$*

$$\widetilde{H}(k; \tfrac{1}{2}) \ll_{k,\epsilon} \mathbf{C}(\Pi)^{\frac{1}{2}} \cdot q^{2n_0\epsilon}, \quad \widetilde{H}(k; -\tfrac{1}{2}) \ll_{k,\epsilon} \mathbf{C}(\Pi)^{4+\epsilon} \cdot q^{2n_0\epsilon}.$$

Proof. The argument is quite similar to Proposition **??**. We distinguish two cases $\mathfrak{c}(\Pi) > 0$ and $\mathfrak{c}(\Pi) = 0$.
(1) Suppose $\mathfrak{c}(\Pi) > 0$. We take $n_1 = 4\mathfrak{c}(\Pi)$ if $n_0 \leqslant 2\mathfrak{c}(\Pi)$ and apply Lemma **??** & **??**. If $n_0 \geqslant 2\mathfrak{c}(\Pi)$ we take $n_1 = 2n_0$. The proofs of Lemma **??**-**??** imply $\widetilde{h}_{2n_0}^-(|\cdot|_{\mathbf{F}}^s) = 0$. Together with Lemma **??** & **??** we conclude the stated bounds in this case.
(2) Suppose $\mathfrak{c}(\Pi) = 0$. Necessarily we have $\Pi = \mu_1 \boxplus \mu_2 \boxplus \mu_3$ for unramified μ_j and $a(\Pi) = 2$. The argument of the second part in (1) works through also for $n_0 \geqslant 2$. It remains to consider the case $n_0 = 1$. Note that $n_1 = 2$. We conclude the stated bounds by Lemma **??** & **??**. □

6. Dual Weight Functions: Ramified Cases

6.1. **Second Quadratic Elementary Functions.**

Definition 6.1. *Let $\mathbf{L}=\mathbf{F}[\sqrt{\varpi_{\mathbf{F}}}]$ and recall the associated quadratic character $\eta_{\mathbf{L}/\mathbf{F}}$. For any $n\in\mathbb{Z}_{\geqslant 0}$ we define the* second quadratic elementary functions $G_n\in \mathrm{C}_c^{\infty}(\mathbf{F}^{\times})$ *by*

$$G_n(y^2):=\mathbb{1}_{v(y)=-n}\cdot \sum\nolimits_{\pm}\eta_{\mathbf{L}/\mathbf{F}}(\pm y)\psi(\pm y),$$

and are supported in the subset of square *elements of $\mathbf{F}^{\times}$.*

Lemma 6.2. *Let χ be a quasi-character of $\mathbf{F}^{\times}$. We have*

$$\int_{\mathbf{F}^{\times}} G_n(y)\chi(y)\mathrm{d}^{\times}y = \begin{cases} \mathbb{1}_{\mathfrak{c}(\chi^2)=n}\cdot\zeta_{\mathbf{F}}(1)\gamma(1,\eta_{\mathbf{L}/\mathbf{F}}\chi^{-2},\psi) & \text{if } n\geqslant 2 \\ \mathbb{1}_{\mathfrak{c}(\eta_{\mathbf{L}/\mathbf{F}}\chi^2)=1}\cdot\zeta_{\mathbf{F}}(1)\gamma(1,\eta_{\mathbf{L}/\mathbf{F}}\chi^{-2},\psi)-\mathbb{1}_{\mathfrak{c}(\eta_{\mathbf{L}/\mathbf{F}}\chi^2)=0}\cdot\zeta_{\mathbf{F}}(1)q^{-1}(\eta_{\mathbf{L}/\mathbf{F}}\chi^{-2})(\varpi_{\mathbf{F}}) & \text{if } n=1 \\ \mathbb{1}_{\mathfrak{c}(\eta_{\mathbf{L}/\mathbf{F}}\chi^2)=0} & \text{if } n=0 \end{cases}.$$

Proof. The proof is quite similar to Lemma **??** and is omitted. □

Corollary 6.3. *Let $n\geqslant a(\Pi)(\geqslant 2)$, the stability barrier of Π (see Proposition **??**). We have*

$$\mathcal{VH}_{\Pi,\psi}(G_n)(y)=\eta_0(-1)^n\eta_0(-2)\mathbb{1}_{\varpi_{\mathbf{F}}^{-n}\mathcal{O}_{\mathbf{F}}^{\times}}(y)\cdot\begin{cases} q^{\frac{3n}{2}}\psi(4y)\eta_0(4y) & \text{if } 2\mid n \\ \tau_0 q^{\lceil\frac{3n}{2}\rceil}\psi(4y) & \text{if } 2\nmid n\end{cases}.$$

Proof. The proof is quite similar to Corollary **??**. We only emphasize the difference. We get

$$\mathcal{VH}_{\Pi,\psi}(G_n)\left(\tfrac{y}{\varpi_{\mathbf{F}}^n}\right)=q^{3n}\int_{(\mathcal{O}_{\mathbf{F}}^{\times})^3}\eta_{\mathbf{L}/\mathbf{F}}\left(\tfrac{t_3}{\varpi_{\mathbf{F}}^n}\right)\psi\left(\tfrac{t_1+t_2+t_3+t_1^{-1}t_2^{-1}t_3^2y}{\varpi_{\mathbf{F}}^n}\right)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3.$$

Performing the level $\lceil\frac{n}{2}\rceil$ regularization to $\mathrm{d}t_j$ we see that the non-vanishing of the above integral implies $t_1=4y(1+u_1)$, $t_2=4y(1+u_2)$ and $t_3=-8y(1+u_3)$ with $u_j\in\mathcal{P}_{\mathbf{F}}^{\lfloor\frac{n}{2}\rfloor}$ and obtain

$$\mathcal{VH}_{\Pi,\psi}(G_n)\left(\tfrac{y}{\varpi_{\mathbf{F}}^n}\right)=q^{2n}\eta_{\mathbf{L}/\mathbf{F}}\left(\tfrac{-8y}{\varpi_{\mathbf{F}}^n}\right)\psi\left(\tfrac{4y}{\varpi_{\mathbf{F}}^n}\right)\int_{\mathcal{P}_{\mathbf{F}}^{\lfloor\frac{n}{2}\rfloor}}\psi\left(\tfrac{4y}{\varpi_{\mathbf{F}}^n}u_2^2\right)\mathrm{d}u_2.$$

We conclude by noting $\eta_{\mathbf{L}/\mathbf{F}}\left(\varpi_{\mathbf{F}}^{-n}y\right)=\eta_0(-1)^n\eta_0(y)$ for $y\in\mathcal{O}_{\mathbf{F}}^{\times}$. □

Corollary 6.4. *Suppose $\Pi=\mu_1\boxplus\mu_2\boxplus\mu_3$ with $\mathfrak{c}(\mu_j)=0$ and $\mu_1\mu_2\mu_3=\mathbb{1}$. Let $E_i(y):=\mu_i^{-1}(y)|y|\mathbb{1}_{\mathcal{O}_{\mathbf{F}}}(y)$ and $f_i:=(1-\mu_i(\varpi_{\mathbf{F}})\mathfrak{t}(\varpi_{\mathbf{F}})).E_i$. For $f,g\in\mathrm{L}^1(\mathbf{F}^{\times})$ define $f*g(y):=\int_{\mathbf{F}^{\times}}f(yt^{-1})g(t)\mathrm{d}^{\times}t$.*

1. *We have $G_0=0$ unless $4\mid q-1$, in which case $\left\{\chi\in\widehat{\mathcal{O}_{\mathbf{F}}^{\times}}\;\middle|\;\eta_0=\chi^2\right\}=\{\eta_1,\eta_1^{-1}\}$. Extending η_1 to $\mathbf{F}^{\times}$ by $\eta_1(\varpi_{\mathbf{F}})=1$ we get*
$$\mathcal{VH}_{\Pi,\psi}(G_0)(y)=\mathbb{1}_{\varpi_{\mathbf{F}}^{-3}\mathcal{O}_{\mathbf{F}}^{\times}}(y)\cdot q^{\frac{3}{2}}\cdot\left\{\gamma(\tfrac{1}{2},\eta_1,\psi)^3\eta_1(y)+\gamma(\tfrac{1}{2},\eta_1^{-1},\psi)^3\eta_1^{-1}(y)\right\}.$$
2. *We have the formula*
$$\mathcal{VH}_{\Pi,\psi}(G_1)(y)=\zeta_{\mathbf{F}}(1)\gamma(1,\eta_{\mathbf{L}/\mathbf{F}},\psi)\cdot(f_1*f_2*f_3)(\varpi_{\mathbf{F}}^{-2}y)+$$
$$\eta_{\mathbf{L}/\mathbf{F}}(-1)\mathbb{1}_{\varpi_{\mathbf{F}}^{-1}\mathcal{O}_{\mathbf{F}}^{\times}}(y)\cdot\left\{\int_{(\varpi_{\mathbf{F}}^{-1}\mathcal{O}_{\mathbf{F}}^{\times})^3}\eta_0(t_3)\psi\left(t_1+t_2+t_3+\tfrac{t_3^2y}{t_1t_2}\right)\mathrm{d}\vec{t}+\zeta_{\mathbf{F}}(1)\tau_0\right\}.$$

Proof. (1) From Lemma **??** and the local functional equation we deduce for any $\chi\in\widehat{\mathcal{O}_{\mathbf{F}}^{\times}}$

$$\int_{\mathbf{F}^{\times}}\mathcal{VH}_{\Pi,\psi}(G_0)(y)\chi^{-1}(y)|y|^{-s}\mathrm{d}^{\times}y=\mathbb{1}_{\chi=\eta_1^{\pm1}}\cdot q^{3\left(\frac{1}{2}-s\right)}\gamma\left(\tfrac{1}{2},\chi,\psi\right)^3.$$

We readily deduce the desired formula for $\mathcal{VH}_{\Pi,\psi}(G_0)$.

(2) We can write $\mathcal{VH}_{\Pi,\psi}(G_1) = \mathcal{VH}_{\Pi,\psi}(G_1)_0 + \mathcal{VH}_{\Pi,\psi}(G_1)_1$ with the properties

$$\mathcal{VH}_{\Pi,\psi}(G_1)_0(y\delta) = \mathcal{VH}_{\Pi,\psi}(G_1)_0(y), \quad \forall y \in \mathbf{F}^\times, \delta \in \mathcal{O}_{\mathbf{F}}^\times;$$

$$\int_{\mathbf{F}^\times} \mathcal{VH}_{\Pi,\psi}(G_1)_0(y)|y|^{-s}\mathrm{d}^\times y = \zeta_{\mathbf{F}}(1)\gamma(1,\eta_{\mathbf{L}/\mathbf{F}},\psi)\cdot q^{2s}\cdot \prod\nolimits_{i=1}^3 \int_{\mathbf{F}^\times} f_i(y)|y|^{-s}\mathrm{d}^\times y,$$

$$\int_{\mathbf{F}^\times} \mathcal{VH}_{\Pi,\psi}(G_1)_1(y)\chi^{-1}(y)|y|^{-s}\mathrm{d}^\times y =$$
$$\mathbb{1}_{\mathfrak{c}(\chi)=1}\cdot q^{3-s}\int_{\varpi_{\mathbf{F}}^{-1}\mathcal{O}_{\mathbf{F}}^\times} \psi(y)\eta_{\mathbf{L}/\mathbf{F}}\chi^2(y)\mathrm{d}^\times y \cdot \left(\int_{\varpi_{\mathbf{F}}^{-1}\mathcal{O}_{\mathbf{F}}^\times} \psi(y)\chi^{-1}(y)\mathrm{d}^\times y \right)^3.$$

We easily identify $\mathcal{VH}_{\Pi,\psi}(G_1)_0(y) = \zeta_{\mathbf{F}}(1)\gamma(1,\eta_{\mathbf{L}/\mathbf{F}},\psi)\cdot (f_1 * f_2 * f_3)(\varpi_{\mathbf{F}}^{-2}y)$. We also deduce that $\mathrm{supp}(\mathcal{VH}_{\Pi,\psi}(G_1)_1) \subset \varpi_{\mathbf{F}}^{-1}\mathcal{O}_{\mathbf{F}}^\times$, and for $y \in \mathcal{O}_{\mathbf{F}}^\times$

$$\mathcal{VH}_{\Pi,\psi}(G_1)_1\left(\tfrac{y}{\varpi_{\mathbf{F}}}\right) = q^3 \int_{(\mathcal{O}_{\mathbf{F}}^\times)^3} \eta_{\mathbf{L}/\mathbf{F}}\left(\tfrac{t_4}{\varpi_{\mathbf{F}}}\right)\psi\left(\tfrac{t_2+t_3+t_4+t_2^{-1}t_3^{-1}t_4^2 y}{\varpi_{\mathbf{F}}}\right)\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4$$
$$- \zeta_{\mathbf{F}}(1)q^3\int_{(\mathcal{O}_{\mathbf{F}}^\times)^4} \eta_{\mathbf{L}/\mathbf{F}}\left(\tfrac{t_4}{\varpi_{\mathbf{F}}}\right)\psi\left(\tfrac{t_1+t_2+t_3+t_4}{\varpi_{\mathbf{F}}}\right)\mathrm{d}\vec{t}.$$

The second summand is equal to $\zeta_{\mathbf{F}}(1)\eta_{\mathbf{L}/\mathbf{F}}(-1)\tau_0$. We conclude by re-numbering the variables. □

6.2. **Further Reductions.** The functions G_n are "building blocks" of our test functions H_c in (**??**) when $\mathbf{L}/\mathbf{F}$ is ramified. In fact writing $\lambda_{\mathbf{L}} := \lambda(\mathbf{L}/\mathbf{F},\psi)\eta_{\mathbf{L}/\mathbf{F}}(2)$ and applying Lemma **??** we can rewrite the summands of H_c as

$$H_{2n+1} = 0 \quad \text{if } n \neq n_0/2, \tag{6.1}$$

$$H_{2n} = \begin{cases} E_n & \text{if } n_0+1 \leqslant n \leqslant n_1-1 \\ \lambda_{\mathbf{L}} q^n \int_{\substack{\mathbf{L}^1 \\ \mathrm{Tr}(\alpha)\in 2(1+\mathcal{P}_{\mathbf{F}}^{n_0-1})}} \beta(\alpha)\cdot\left(\mathfrak{t}\left(\mathrm{Tr}(\alpha)^2\right).G_n\right)\mathrm{d}\alpha & \text{if } n = n_0 \\ \lambda_{\mathbf{L}} q^n \int_{\substack{\mathbf{L}^1 \\ \mathrm{Tr}(\alpha)\in 2(1+\varpi_{\mathbf{F}}^{2n-n_0-1}\mathcal{O}_{\mathbf{F}}^\times)}} \beta(\alpha)\cdot\left(\mathfrak{t}\left(\mathrm{Tr}(\alpha)^2\right).G_n\right)\mathrm{d}\alpha & \text{if } \frac{n_0}{2}+1 \leqslant n \leqslant n_0-1 \end{cases}. \tag{6.2}$$

The decomposition of H_{n_0+1} is subtler and really goes in the direction of expression in terms of the *quadratic elementary functions*. We shall write (the second numeric parameter m in subscript always indicates the parameter of the relevant quadratic elementary function)

$$H_{n_0+1} = \tfrac{\lambda(\mathbf{L}/\mathbf{F},\psi)}{2} q^{\frac{n_0+1}{2}} \cdot \sum\nolimits_{m=0}^{\frac{n_0}{2}} H_{n_0+1,m}, \tag{6.3}$$

where the summands are given by (below $m \geqslant 1$)

$$H_{n_0+1,m} = \int_{\substack{\mathbf{L}^1 \\ \mathrm{Tr}(\varpi_{\mathbf{L}}\alpha)\in\varpi_{\mathbf{F}}^{n_0/2+1-m}\mathcal{O}_{\mathbf{F}}^\times}} \beta(\varpi_{\mathbf{L}}\alpha)\cdot\eta_{\mathbf{L}/\mathbf{F}}\left(\mathrm{Tr}(\varpi_{\mathbf{L}}\alpha)\right)\cdot\mathfrak{t}\left(-\varpi_{\mathbf{F}}^{-1}\mathrm{Tr}(\varpi_{\mathbf{L}}\alpha)^2\right).G_m\mathrm{d}\alpha,$$

$$H_{n_0+1,0} = \int_{\substack{\mathbf{L}^1 \\ \mathrm{Tr}(\varpi_{\mathbf{L}}\alpha)\in\mathcal{P}_{\mathbf{F}}^{n_0/2+1}}} \beta(\varpi_{\mathbf{L}}\alpha)\mathrm{d}\alpha\cdot\mathfrak{t}\left((-\varpi_{\mathbf{F}})^{n_0+1}\right).G_0.$$

Lemma 6.5. *Let χ be a (unitary) character of $\mathbf{F}^\times$ with $\mathfrak{c}(\chi) = n \leqslant \frac{n_0}{2}$. Recall the additive parameter c_β (resp. c_χ) in Lemma **??** (resp. Remark **??**). We have*

$$\int_{\mathcal{O}_{\mathbf{F}}^\times} \beta\left(1+\varpi_{\mathbf{F}}^{\frac{n_0}{2}-n}t\varpi_{\mathbf{L}}\right)\chi(t)\mathrm{d}t \ll q^{-\frac{n}{2}}.$$

Proof. Since the proof is quite similar to the one of Lemma **??** (1), we skip some details. The case for $n=0$ is easy. If $n=1$, then $t \mapsto \beta\left(1+\varpi_{\mathbf{F}}^{n_0-n}t\sqrt{\varepsilon}\right)$ is a non-trivial additive character of $\mathcal{O}_{\mathbf{F}}$ and the bound follows from the one for Gauss sums. Assume $n \geqslant 2$. We perform a level $\lceil \frac{n}{2} \rceil$ regularization to $\mathrm{d}t$ and get the non-vanishing condition

$$\frac{2c_\beta t}{1-\varpi_{\mathbf{F}}^{n_0+1-2-n}t^2} + c_\chi \in \mathcal{P}_{\mathbf{F}}^{\lfloor \frac{n}{2} \rfloor} \quad \Leftrightarrow \quad 2c_\beta t + c_\chi(1-\varpi_{\mathbf{F}}^{n_0+1-2n}t^2) \in \mathcal{P}_{\mathbf{F}}^{\lfloor \frac{n}{2} \rfloor},$$

which has a unique solution $t \in t_0 + \mathcal{P}_{\mathbf{F}}^{\lfloor \frac{n}{2} \rfloor}$ with $t_0 \in \mathcal{O}_{\mathbf{F}}^\times$ by Hensel's lemma. Consequently we get

$$\int_{\mathcal{O}_{\mathbf{F}}^\times} \beta\left(1+\varpi_{\mathbf{F}}^{\frac{n_0}{2}-n}t\varpi_{\mathbf{L}}\right)\chi(t)\mathrm{d}t = \int_{t_0+\mathcal{P}_{\mathbf{F}}^{\lfloor \frac{n}{2} \rfloor}} \beta\left(1+\varpi_{\mathbf{F}}^{\frac{n_0}{2}-n}t\varpi_{\mathbf{L}}\right)\chi(t)\mathrm{d}t \ll q^{-\lfloor \frac{n}{2} \rfloor}. \tag{6.4}$$

If $2 \mid n$ then we are done (for both (1) and (2)). Otherwise let $n=2m+1$. We may assume

$$2c_\beta t_0 + c_\chi(1-\varpi_{\mathbf{F}}^{n_0+1-2n}t_0^2\varepsilon)=0,$$

make the change of variables $t=t_0(1+\varpi_{\mathbf{F}}^m u)$, and continue (**??**) as

$$\int_{\mathcal{O}_{\mathbf{F}}^\times} \beta\left(1+\varpi_{\mathbf{F}}^{\frac{n_0}{2}-n}t\varpi_{\mathbf{L}}\right)\chi(t)\mathrm{d}t = q^{-m}\beta\left(1+\varpi_{\mathbf{F}}^{\frac{n_0}{2}-n}t_0\varpi_{\mathbf{L}}\right)\chi(t_0)\int_{\mathcal{O}_{\mathbf{F}}} \psi\left(-\frac{c_\chi u^2}{2\varpi_{\mathbf{F}}}\right)du \ll q^{-\frac{n}{2}}$$

and conclude the proof. ☐

Lemma 6.6. *Suppose $\frac{n_0}{2}+1 \leqslant n \leqslant n_0$ and $n \geqslant a(\Pi)$. Then we have*

$$\widetilde{h}_{2n}^-(\chi) \ll q^{-\frac{n_0+1}{2}}\mathbb{1}_{n_0+1-n}(\mathfrak{c}(\chi\eta_0^{n+1})) + q^{-\frac{n_0+1}{2}}\mathbb{1}_{n=n_0}\mathbb{1}_{\mathfrak{c}(\chi\eta_0)=0}.$$

Proof. (1) First consider $n<n_0$. By Corollary **??** we have for any $\delta \in 1+\varpi_{\mathbf{F}}^{2n-n_0-1}\mathcal{O}_{\mathbf{F}}^\times$

$$\int_{\mathbf{F}-\mathcal{O}_{\mathbf{F}}} \mathcal{VH}_{\Pi,\psi} \circ \mathfrak{m}_{-1}\left(\mathfrak{t}\left(4\delta\right).G_n\right)(t)\cdot\psi(-t)\chi^{-1}(t)|t|^{-\frac{1}{2}}\mathrm{d}^\times t = \tag{6.5}$$

$$q^{-n}\eta_0(2(-1)^{n+1})\zeta_{\mathbf{F}}(1)\cdot\begin{cases} \chi^{-1}\eta_0\left(\frac{\delta}{1-\delta}\right)\cdot\gamma(1,\chi\eta_0,\psi)\mathbb{1}_{n_0+1-n}(\mathfrak{c}(\chi\eta_0)) & \text{if } 2 \mid n \\ \chi^{-1}\left(\frac{\delta}{1-\delta}\right)\cdot\tau_0 q^{\frac{1}{2}}\cdot\gamma(1,\chi,\psi)\mathbb{1}_{n_0+1-n}(\mathfrak{c}(\chi)) & \text{if } 2 \nmid n \end{cases},$$

where we used $\eta_0(\delta)=1$. Inserting (**??**) with $\delta=4^{-1}\mathrm{Tr}(\alpha)^2$ into (**??**) we get

$$\widetilde{h}_{2n}^-(\chi) = \int_{\mathbf{F}-\mathcal{O}_{\mathbf{F}}} \mathcal{VH}_{\Pi,\psi}\circ\mathfrak{m}_{-1}(H_{2n})(t)\cdot\psi(-t)\chi^{-1}(t)|t|^{-\frac{1}{2}}\mathrm{d}^\times t \ll \tag{6.6}$$

$$\left|\int_{\substack{\mathbf{L}^1 \\ \mathrm{Tr}(\alpha)\in 2(1+\varpi_{\mathbf{F}}^{2n-n_0-1}\mathcal{O}_{\mathbf{F}}^\times)}} \beta(\alpha)\cdot\chi^{-1}\eta_0^{n+1}\left(\frac{\mathrm{Tr}(\alpha)^2}{4-\mathrm{Tr}(\alpha)^2}\right)\mathrm{d}\alpha\right|\cdot q^{-\frac{n_0+1-n}{2}}\mathbb{1}_{n_0+1-n}(\mathfrak{c}(\chi\eta_0^{n+1})).$$

Applying the change of variables $t=\frac{\alpha-\alpha^{-1}}{\alpha+\alpha^{-1}}\frac{1}{\varpi_{\mathbf{L}}}$ the inner integral in (**??**) becomes, taking into account *the measure normalization* in Proposition **??** (3)

$$q^{-\frac{1}{2}}\cdot\chi\eta_0^{n+1}(-1)\int_{\varpi_{\mathbf{F}}^{n-\frac{n_0}{2}-1}\mathcal{O}_{\mathbf{F}}^\times} \beta(1+t\varpi_{\mathbf{L}})\chi^2(t)\mathrm{d}t.$$

We apply Lemma **??** (1) to bound the above integrals and conclude.
(2) Consider $n=n_0$. We have for any $\delta \in 1+\mathcal{P}_{\mathbf{F}}^{n_0-1}$

$$\int_{\mathbf{F}-\mathcal{O}_{\mathbf{F}}} \mathcal{VH}_{\Pi,\psi}\circ\mathfrak{m}_{-1}\left(\mathfrak{t}\left(4\delta\right).G_n\right)(t)\cdot\psi(-t)\chi^{-1}(t)|t|^{-\frac{1}{2}}\mathrm{d}^\times t = q^{-n_0}\eta_0(-2)\cdot\tau_0 q^{\frac{1}{2}}\cdot \tag{6.7}$$

$$\begin{cases} \chi^{-1}\eta_0\left(\frac{\delta}{1-\delta}\right)\cdot\zeta_{\mathbf{F}}(1)\left\{\gamma(1,\chi\eta_0,\psi)\mathbb{1}_1(\mathfrak{c}(\chi\eta_0)) - q^{-1}\mathbb{1}_0(\mathfrak{c}(\chi\eta_0))\right\} & \text{if } \delta \in 1+\varpi_{\mathbf{F}}^{n_0-1}\mathcal{O}_{\mathbf{F}}^\times \\ \chi\eta_0(\varpi_{\mathbf{F}})^{n_0}\cdot\mathbb{1}_0(\mathfrak{c}(\chi\eta_0)) & \text{if } \delta \in 1+\mathcal{P}_{\mathbf{F}}^{n_0} \end{cases}.$$

Inserting (**??**) with $\delta=4^{-1}\mathrm{Tr}(\alpha)^2$ into (**??**) we get

$$\widetilde{h}_{2n}^{-}(\chi) = \int_{\mathbf{F}-\mathcal{O}_{\mathbf{F}}} \mathcal{VH}_{\Pi,\psi} \circ \mathfrak{m}_{-1}(H_{2n})(t) \cdot \psi(-t)\chi^{-1}(t)|t|^{-\frac{1}{2}} \mathrm{d}^{\times} t \ll \left| \int_{\substack{\mathbf{L}^1 \\ \mathrm{Tr}(\alpha)\in 2(1+\varpi_{\mathbf{F}}^{n_0-1}\mathcal{O}_{\mathbf{F}}^{\times})}} \beta(\alpha) \cdot \chi^{-1}\eta_0\left(\tfrac{\mathrm{Tr}(\alpha)^2}{4-\mathrm{Tr}(\alpha)^2}\right) \mathrm{d}\alpha \right| \cdot q^{-\frac{1}{2}} \mathbb{1}_1(\mathfrak{c}(\chi\eta_0)) + \left| \int_{\substack{\mathbf{L}^1 \\ \mathrm{Tr}(\alpha)\in 2(1+\mathcal{P}_{\mathbf{F}}^{n_0})}} \beta(\alpha)\mathrm{d}\alpha - \tfrac{\chi\eta_0(\varpi_{\mathbf{F}})^{-1}}{q-1} \int_{\substack{\mathbf{L}^1 \\ \mathrm{Tr}(\alpha)\in 2(1+\varpi_{\mathbf{F}}^{n_0-1}\mathcal{O}_{\mathbf{F}}^{\times})}} \beta(\alpha)\mathrm{d}\alpha \right| \cdot \mathbb{1}_0(\mathfrak{c}(\chi\eta_0)). \tag{6.8}$$

The first summand is bounded the same way as before. With the same change of variables the inner integrals of the second summand become

$$q^{-\frac{1}{2}} \cdot \left(q^{-\frac{n_0}{2}} - \tfrac{\chi\eta_0(\varpi_{\mathbf{F}})^{-1}}{q-1} \int_{\varpi_{\mathbf{F}}^{\frac{n_0}{2}-1}\mathcal{O}_{\mathbf{F}}^{\times}} \beta(1+t\varpi_{\mathbf{L}})\mathrm{d}t \right).$$

It is of size $O(q^{-\frac{n_0+1}{2}})$ since the integrand is an additive character of conductor exponent $n_0/2$. □

Lemma 6.7. *Suppose* $(2 \leqslant)a(\Pi) \leqslant m \leqslant \frac{n_0}{2}$. *Then we have for unitary* χ

$$\widetilde{h}_{n_0+1,m}(\chi) := \int_{\mathbf{F}-\mathcal{O}_{\mathbf{F}}} \mathcal{VH}_{\Pi,\psi} \circ \mathfrak{m}_{-1}\left(H_{n_0+1,m}\right)(t)\psi(-t)\chi^{-1}(t)|t|^{-\frac{1}{2}} \mathrm{d}^{\times} t \ll q^{-n_0-1} \mathbb{1}_{m>\frac{n_0+1}{3}} \mathbb{1}_m(\mathfrak{c}(\chi)).$$

Proof. We first use Corollary **??** to obtain for any $\delta \in \varpi_{\mathbf{F}}^{n_0+1-2m}\mathcal{O}_{\mathbf{F}}^{\times}$ (note that $\eta_0(4-\delta)=1$)

$$\int_{\mathbf{F}-\mathcal{O}_{\mathbf{F}}} \mathcal{VH}_{\Pi,\psi} \circ \mathfrak{m}_{-1}\left(\mathfrak{t}(\delta).G_m\right)(t) \cdot \psi(-t)\chi^{-1}(t)|t|^{-\frac{1}{2}} \mathrm{d}^{\times} t = q^{-\frac{n_0+1}{2}} \eta_0(2(-1)^{m+1})\zeta_{\mathbf{F}}(1)\mathbb{1}_{m>\frac{n_0+1}{3}} \cdot \begin{cases} \chi\left(\frac{4-\delta}{\delta}\right) \cdot \gamma(1,\chi\eta_0,\psi)\mathbb{1}_m(\mathfrak{c}(\chi)) & \text{if } 2 \mid m \\ \chi\left(\frac{4-\delta}{\delta}\right) \cdot \tau_0 q^{\frac{1}{2}}\gamma(1,\chi,\psi)\mathbb{1}_m(\mathfrak{c}(\chi)) & \text{if } 2 \nmid m \end{cases}. \tag{6.9}$$

Inserting (**??**) with $\delta = -\varpi_{\mathbf{F}}^{-1}\mathrm{Tr}(\varpi_{\mathbf{L}}\alpha)^2$ into the integral representation of $H_{n_0+1,m}$ we get

$$\widetilde{h}_{n_0+1,m}(\chi) \ll \mathbb{1}_{m>\frac{n_0+1}{3}} \cdot q^{-\frac{n_0+1+m}{2}} \mathbb{1}_m(\mathfrak{c}(\chi)) \cdot \left| \int_{\substack{\mathbf{L}^1 \\ \mathrm{Tr}(\varpi_{\mathbf{L}}\alpha)\in\varpi_{\mathbf{F}}^{\frac{n_0}{2}+1-m}\mathcal{O}_{\mathbf{F}}^{\times}}} \beta(\varpi_{\mathbf{L}}\alpha)\eta_{\mathbf{L}/\mathbf{F}}\left(\mathrm{Tr}(\varpi_{\mathbf{L}}\alpha)\right) \cdot \chi\left(\tfrac{4\varpi_{\mathbf{F}}+\mathrm{Tr}(\varpi_{\mathbf{L}}\alpha)^2}{-\mathrm{Tr}(\varpi_{\mathbf{L}}\alpha)^2}\right) \mathrm{d}\alpha \right|. \tag{6.10}$$

Applying the change of variables $t = \frac{\varpi_{\mathbf{L}}\alpha+\overline{\varpi_{\mathbf{L}}\alpha}}{\varpi_{\mathbf{L}}\alpha-\overline{\varpi_{\mathbf{L}}\alpha}}\frac{1}{\varpi_{\mathbf{L}}}$ the inner integral in (**??**) becomes, taking into account *the measure normalization* in Proposition **??** (3)

$$q^{-\frac{1}{2}} \cdot 2\beta(\varpi_{\mathbf{L}})\eta_{\mathbf{L}/\mathbf{F}}(-2)\chi^{-1}(-\varpi_{\mathbf{F}}) \int_{\varpi_{\mathbf{F}}^{\frac{n_0}{2}-m}\mathcal{O}_{\mathbf{F}}^{\times}} \beta(1+t\varpi_{\mathbf{L}})\eta_{\mathbf{L}/\mathbf{F}}\chi^{-2}(t)\mathrm{d}t.$$

We apply Lemma **??** (1) to bound the above integrals and conclude the desired inequalities. □

Lemma 6.8. *Let* $n_0 \geqslant \max(4\mathfrak{c}(\Pi), 2\mathfrak{c}(\Pi)+2)$ *and* $m \leqslant \mathfrak{c}(\Pi)$. *Then for any* $\delta \in \varpi_{\mathbf{F}}^{n_0+1-2m}\mathcal{O}_{\mathbf{F}}^{\times}$ *we have*

$$\int_{\mathbf{F}-\mathcal{O}_{\mathbf{F}}} \mathcal{VH}_{\Pi,\psi} \circ \mathfrak{m}_{-1}\left(\mathfrak{t}(\delta).G_m\right)(t) \cdot \psi(-t)\chi^{-1}(t)|t|^{-\frac{1}{2}} \mathrm{d}^{\times} t = 0.$$

Consequently, we get for any unitary χ *the vanishing of* $\widetilde{h}_{n_0+1,m}(\chi)=0$.

Proof. The proof is quite similar to Lemma **??**. One shows that the support of $\mathcal{VH}_{\Pi,\psi} \circ \mathfrak{m}_{-1}\left(\mathfrak{t}(\delta).G_m\right)$ is contained in $\delta\mathcal{P}_{\mathbf{F}}^{2m-\max(4\mathfrak{c}(\Pi),2\mathfrak{c}(\Pi)+3)} \subset \mathcal{O}_{\mathbf{F}}$ under the assumption. □

6.3. **The Bounds of Dual Weight.**

Lemma 6.9. *Suppose $\mathfrak{c}(\Pi) = 0$ and $n_0 = 2$. We have $\widetilde{h}_3^-(\chi) = 0$ for any unitary character χ.*

Proof. Necessarily we have $\Pi = \mu_1 \boxplus \mu_2 \boxplus \mu_3$ for unramified μ_j. Note that $H_{3,0}$ (resp. $H_{3,1}$) is related to G_0 (resp. G_1) by the formulae

$$H_{3,0} = \int_{\substack{\mathbf{L}^1 \\ \mathrm{Tr}(\varpi_{\mathbf{L}}\alpha) \in \mathcal{P}_{\mathbf{F}}^2}} \beta(\varpi_{\mathbf{L}}\alpha) \mathrm{d}\alpha \cdot \mathfrak{t}\left((-\varpi_{\mathbf{F}})^3\right).G_0,$$

$$H_{3,1} = \int_{\substack{\mathbf{L}^1 \\ \mathrm{Tr}(\varpi_{\mathbf{L}}\alpha) \in \varpi_{\mathbf{F}}\mathcal{O}_{\mathbf{F}}^\times}} \beta(\varpi_{\mathbf{L}}\alpha) \cdot \eta_{\mathbf{L}/\mathbf{F}}\left(\mathrm{Tr}(\varpi_{\mathbf{L}}\alpha)\right) \cdot \mathfrak{t}\left(-\varpi_{\mathbf{F}}^{-1}\mathrm{Tr}(\varpi_{\mathbf{L}}\alpha)^2\right).G_1 \mathrm{d}\alpha.$$

Inspecting the supports of G_0 and G_1 given in Corollary **??** we see $\mathrm{supp}(H_{3,m}) \subset \mathcal{O}_{\mathbf{F}}$ for $m \in \{0,1\}$. □

Proposition 6.10. *With the test function H given by (**??**) the dual weight function is bounded as*

$$\widetilde{h}(\chi) \ll_\epsilon \mathbf{C}(\Pi)^{2+\epsilon} q^{-\frac{n_0+1}{2}+\epsilon} \mathbb{1}_{\leqslant \max(\frac{n_0}{2}, 6\mathfrak{c}(\Pi))}(\mathfrak{c}(\chi)).$$

Proof. The argument is quite similar to and simpler than Proposition **??**. In the case $\mathfrak{c}(\Pi) > 0$ we distinguish $n_0 \leqslant 4\mathfrak{c}(\Pi)$, resp. $n_0 \geqslant 4\mathfrak{c}(\Pi)$. We apply Lemma **??**, **??** and **??** (with $A = 4$), resp. Lemma **??**, **??** and Lemma **??**-**??**. In the case $\mathfrak{c}(\Pi) = 0$, we apply Lemma **??**, **??** and **??**. We leave the details to the reader. □

6.4. **The Bounds of Unramified Dual Weight.**

Lemma 6.11. *Suppose $\frac{n_0}{2} + 1 \leqslant n \leqslant n_0$ and $n \geqslant a(\Pi)$. Then the function $\widetilde{h}_{2n}^-(|\cdot|_{\mathbf{F}}^s) \neq 0$ is non-vanishing only if $n = n_0$. Moreover for any $k \in \mathbb{Z}_{\geqslant 0}$ we have*

$$\widetilde{H}_{2n_0}^-(k; \tfrac{1}{2}) \ll_{k,\epsilon} q^{-\frac{1}{2}+n_0\epsilon}, \quad \widetilde{H}_{2n_0}^-(k; -\tfrac{1}{2}) \ll_{k,\epsilon} q^{-n_0-\frac{1}{2}+n_0\epsilon}.$$

Proof. By Corollary **??** and (**??**) the support of $\mathcal{VH}_{\Pi,\psi}(G_n)$, hence also $\mathcal{VH}_{\Pi,\psi} \circ \mathfrak{m}_{-1}(H_{2n})$ are contained in $\varpi_{\mathbf{F}}^{-n}\mathcal{O}_{\mathbf{F}}^\times$. Therefore we get by (**??**) the relation $\widetilde{h}_{2n}^-(|\cdot|_{\mathbf{F}}^s) = q^{ns}\widetilde{h}_{2n}^-(\mathbb{1})$. The stated results follow readily from Lemma **??**. □

Proposition 6.12. *With the test function H given by (**??**) we have for any $k \in \mathbb{Z}_{\geqslant 0}$*

$$\widetilde{H}(k; \tfrac{1}{2}) \ll_{k,\epsilon} \mathbf{C}(\Pi)^{\frac{1}{2}} \cdot q^{(n_0+1)\epsilon}, \quad \widetilde{H}(k; -\tfrac{1}{2}) \ll_{k,\epsilon} \mathbf{C}(\Pi)^{4+\epsilon} \cdot q^{(n_0+1)\epsilon}.$$

Proof. The proof is quite similar to and simpler than the one of Proposition **??**. We simply note that we can take $n_1 = n_0 + 1$, and leave the details to the reader. □

Appendix A. Relation with Petrow–Young's Exponential Sums

For simplicity of notation we shall write $\sum_t$ for $\sum_{t \in \mathbb{F}_q}$, and use the convention $\rho(0) = 0$ for any character ρ of $\mathbb{F}_q^\times$ (even if $\rho = \mathbb{1}$ is the trivial one). In (**??**) we have encountered the following algebraic exponential sum

$$S = S(\chi_0, \chi) := \sum\nolimits_{\alpha \in \mathbb{F}_q - \{\pm 1\}} \chi_0\left(\tfrac{\alpha+1}{\alpha-1}\right) H(1-\alpha^2, q; (\mathbb{1}, \mathbb{1}), (\chi^{-1}, \eta)), \tag{A.1}$$

where χ_0, χ are non-trivial characters of $\mathbb{F}_q^\times$ and η is the unique non-trivial quadratic character of $\mathbb{F}_q^\times$. Note that the setting of the relevant dual weight specializes to the Petrow–Young's [**?**] upon taking $\Pi = \mathbb{1} \boxplus \mathbb{1} \boxplus \mathbb{1}$. It is a natural question to relate S with their algebraic exponential sum

$$T = T(\chi_0, \chi) := \sum\nolimits_{u,v} \chi\left(\tfrac{u(u+1)}{v(v+1)}\right) \chi_0(uv-1). \tag{A.2}$$

The purpose of this appendix is to give such an explicit relation. Our approach will take into account the recent discovery of Xi [**?**], which relates T to a special value of a hypergeometric sum of Katz.

We need to rewrite a *hyper-Kloosterman sum* via the *duplication formula of Gauss sums*.

Definition A.1. *For a character ρ of $\mathbb{F}_q^\times$ and a non-trivial character ψ of $\mathbb{F}_q$ we have the Gauss sum*

$$\tau(\rho) = \tau(\rho, \psi) := \sum\nolimits_t \rho(t)\psi(t).$$

Lemma A.2. *We have the relation $\tau(\rho^2)\tau(\eta) = \rho(4)\tau(\rho)\tau(\rho\eta)$ for any character ρ of $\mathbb{F}_q^\times$.*

Proof. If $\rho = \mathbb{1}$ or η, the stated relation trivially holds true. Assume $\rho \neq \mathbb{1}, \eta$. We have the classical relation of Gauss and Jacobi sums [? , §8.3 Theorem 1]

$$\tfrac{\tau(\rho)^2}{\tau(\rho^2)} = J(\rho,\rho) = \sum\nolimits_t \rho(t(1-t)), \quad \tfrac{\tau(\rho)\tau(\eta)}{\tau(\rho\eta)} = J(\rho,\eta) = \sum\nolimits_t \eta(t)\rho(1-t).$$

Since q is odd, we can rewrite by completing the square

$$\begin{aligned} J(\rho,\rho) = \sum\nolimits_t \rho(t(1-t)) &= \rho(4)^{-1}\sum\nolimits_t \rho(1-t^2) \\ &= \rho(4)^{-1}\sum\nolimits_t (1+\eta(t))\rho(1-t) = \rho(4)^{-1}\sum\nolimits_t \eta(t)\rho(1-t) = \rho(4)^{-1}J(\rho,\eta). \end{aligned}$$

The stated relation follows readily since $\tau(\rho) \neq 0$. ☐

Corollary A.3. *For any $\delta \in \mathbb{F}_q^\times$ we have the equality*

$$\sum\nolimits_{u,t\neq 0} \eta(1-u)\,\psi\left(\tfrac{\delta}{t^2 u} + 2t\right) = \sum\nolimits_{x_1x_2x_3=\delta} \psi(x_1+x_2+x_3).$$

Proof. The right hand side is $\mathrm{Kl}_3(\delta)$, a hyper-Kloosterman sum, whose Mellin transform satisfies

$$\sum\nolimits_{\delta\in\mathbb{F}_q^\times} \mathrm{Kl}_3(\delta)\rho(\delta) = \tau(\rho)^3, \quad \forall \rho \in \widehat{\mathbb{F}_q^\times}.$$

It suffices to identify the Mellin transform of the left hand side as $\tau(\rho)^3$. We have

$$\begin{aligned} \sum\nolimits_\delta \sum\nolimits_{u,t\neq 0} \eta(1-u)\,\psi\left(\tfrac{\delta}{t^2u}+2t\right)\rho(\delta) &= \tau(\rho)\sum\nolimits_{u,t} \eta(1-u)\,\psi(2t)\,\rho(t^2u) \\ &= \rho(4)^{-1}\tau(\rho)\tau(\rho^2)\sum\nolimits_u \eta(1-u)\,\rho(u) = \tfrac{\tau(\rho)^2\tau(\rho^2)\tau(\eta)}{\rho(4)\tau(\rho\eta)}. \end{aligned}$$

Lemma **??** identifies the above right hand side precisely as $\tau(\rho)^3$. ☐

For $S = S(\chi_0,\chi)$, we make the change of variables $u = 1+\alpha$ and $v = 1-\alpha$, and detect the condition $u+v=2$ with the additive character ψ to get

$$\begin{aligned} S &= -q^{-\frac{3}{2}}\chi_0(-1)\sum\nolimits_{u+v=2} \chi_0(u)\overline{\chi_0}(v)\sum\nolimits_{\substack{x_i,y_i \\ x_1x_2=uvy_1y_2}} \chi(y_1)\eta(y_2)\psi(x_1+x_2-y_1-y_2) \\ &= -q^{-\frac{5}{2}}\chi_0(-1)\sum\nolimits_t \sum\nolimits_{\substack{x_i,y_i,u,v \\ x_1x_2=uvy_1y_2}} \chi_0(u)\overline{\chi_0}(v)\chi(y_1)\eta(y_2)\psi(x_1+x_2-y_1-y_2+tu+tv-2t) \\ &= -q^{-\frac{5}{2}}(S_1+S_2). \end{aligned}$$

In the above the sum S_1 is defined and computed as

$$\begin{aligned} S_1 &:= \chi_0(-1)\sum\nolimits_{t\in\mathbb{F}_q^\times} \sum\nolimits_{\substack{x_i,y_i,u,v \\ x_1x_2=uvy_1y_2}} \chi_0(u)\overline{\chi_0}(v)\chi(y_1)\eta(y_2)\psi(x_1+x_2-y_1-y_2-tu-tv+2t) \\ &= \chi_0(-1)\sum\nolimits_{t\in\mathbb{F}_q^\times} \sum\nolimits_{\substack{x_i,y_i,u,v \\ t^2x_1x_2=uvy_1y_2}} \chi_0(u)\overline{\chi_0}(v)\chi(y_1)\eta(y_2)\psi(x_1+x_2-y_1-y_2-u-v+2t) \\ &= \chi_0(-1)\sum\nolimits_{t\in\mathbb{F}_q^\times} \sum\nolimits_{\substack{x_i,y_i,u,v \\ t^2x_1x_2=uvy_1}} \eta(y_2)\psi(y_2(x_2-1))\cdot\chi_0(u)\overline{\chi_0}(v)\chi(y_1)\psi(x_1+2t-u-v-y_1) \\ &= \chi_0(-1)\tau(\eta)\sum\nolimits_{t\in\mathbb{F}_q^\times} \sum\nolimits_{\substack{x_i,y_i \\ t^2x_1x_2=y_1y_2y_3}} \eta(x_2-1)\chi(y_1)\chi_0(y_2)\overline{\chi_0}(y_3)\psi(x_1+2t-y_1-y_2-y_3) \\ &= \chi_0\eta(-1)\tau(\eta)\sum\nolimits_{y_i} \chi(y_1)\chi_0(y_2)\overline{\chi_0}(y_3)\psi(-y_1-y_2-y_3)\sum\nolimits_{x_2,t\neq 0}\eta(1-x_2)\psi\left(\tfrac{y_1y_2y_3}{t^2x_2}+2t\right). \end{aligned}$$

Re-naming the variable $u = x_2$ we identify the inner sum as $\mathrm{Kl}_3(y_1y_2y_3)$ by Corollary **??** so that

$$S_1 = \chi_0\eta(-1)\tau(\eta)\sum\nolimits_{\substack{x_i,y_i \\ x_1x_2x_3=y_1y_2y_3}} \chi(y_1)\chi_0(y_2)\overline{\chi_0}(y_3)\psi(x_1+x_2+x_3-y_1-y_2-y_3)$$

$$= -q^{\frac{5}{2}}\chi_0\eta(-1)\tau(\eta)H(1,q;(\mathbb{1},\mathbb{1},\mathbb{1}),(\overline{\chi},\chi_0,\overline{\chi_0})) = -q^2\cdot T(\chi_0,\chi),$$

where we have applied [**?** , Theorem 1.1] to get the last equality. The sum S_2 is defined as

$$S_2 := \chi_0(-1)\sum\nolimits_{\substack{x_i,y_i,u,v\\ x_1x_2=uvy_1y_2}}\chi_0(u)\overline{\chi_0}(v)\chi(y_1)\eta(y_2)\psi(x_1+x_2-y_1-y_2).$$

Let $t=uv$, so $u=tv^{-1}$, and sum first over $v\in\mathbb{F}_q^\times$. We see that $S_2\neq 0$ only if $\chi_0=\eta$, and

$$\begin{aligned}
S_2 &= \delta_{\chi_0=\eta}\cdot\eta(-1)(q-1)\sum\nolimits_{\substack{t,x_i,y_i\\ x_1x_2=ty_1y_2}}\chi(y_1)\eta(ty_2)\psi(x_1+x_2-y_1-y_2)\\
&= \delta_{\chi_0=\eta}\cdot\eta(-1)(q-1)\sum\nolimits_{\substack{x_i,y_i\\ x_1x_2=y_1y_2}}\chi(y_1)\eta(y_2)\psi(x_1+x_2-y_1)\sum\nolimits_{t\neq 0}\psi(-t^{-1}y_2)\\
&= -\delta_{\chi_0=\eta}\cdot\eta(-1)(q-1)\sum\nolimits_{\substack{x_i,y_i\\ x_1x_2=y_1y_2}}\chi(y_1)\eta(y_2)\psi(x_1+x_2-y_1)\\
&= -\delta_{\chi_0=\eta}\cdot\eta(-1)(q-1)\sum\nolimits_{x_1,x_2,y_1}\chi(y_1)\eta\left(\tfrac{x_1x_2}{y_1}\right)\psi(x_1+x_2-y_1)\\
&= -\delta_{\chi_0=\eta}\cdot\eta(-1)(q-1)\tau(\eta)^2\overline{\tau(\eta\chi^{-1})} = -\delta_{\chi_0=\eta}\cdot q(q-1)\overline{\tau(\eta\chi^{-1})}.
\end{aligned}$$

We summarize the above computation as the following result.

Proposition A.4. *The two algebraic exponential sums $S(\chi_0,\chi)$ in (**??**) and $T(\chi_0,\chi)$ in (**??**) satisfy*

$$S(\chi_0,\chi) = q^{-\frac{1}{2}}\cdot T(\chi_0,\chi) + \delta_{\chi_0=\eta}\cdot q^{-\frac{1}{2}}(1-q^{-1})\overline{\tau(\eta\chi^{-1})}.$$

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