Some New Congruences and Partition-Theoretic Interpretations for the Coefficients of Some Rogers-Ramanujan Type Identities

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Abstract: Ramanujan listed several q-series identities in his lost notebook. The most well known q-series identities are the Rogers-Ramanujan type identities which are first discovered by Rogers and then rediscovered by Ramanujan. In this paper, we give partition-theoretic interpretations of some of the Rogers-Ramanujan type identities using overpartition and colour partition of positive integers, and prove infinite families of congruences modulo powers of 2.

Keywords and phrases: Rogers-Ramanujan type identities; overpartition; colour partition; partition congruences.

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1 Introduction

For any complex numbers A and q with $|q| < 1$, a q-series is a summand containing the expression of the type

$$(A;q)_\infty = \prod_{k=0}^{\infty}(1-Aq^k), \quad where \quad (A;q)_0 = 1, \quad (A;q)_n = \prod_{k=0}^{n-1}(1-Aq^k), \quad n \geq 1.$$

For convenience, one often use the notation

$$(A_1;q)_\infty(A_2;q)_\infty(A_3;q)_\infty....(A_k;q)_\infty = (A_1,A_2,A_3,...,A_k;q)_\infty.$$

Throughout the paper, we write $\ell_n := (q^n;q^n)_\infty$, for any integer $n \geq 1$. Ramanujan defined general theta-function $f(c,d)$ [**?**, p. 34, (18.1)] as

$$f(c,d) = \sum_{m=-\infty}^{\infty} c^{m(m+1)/2} d^{m(m-1)/2}, \quad |cd| < 1. \tag{1.1}$$

The special cases [**?**, p. 35, Entry 18] of $f(c,d)$ are given by

$$\phi(q) := f(q,q) = \sum_{m=-\infty}^{\infty} q^{m^2} = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty} = \frac{\ell_2^5}{\ell_1^2\ell_4^2} \tag{1.2}$$

and

$$\psi(q) := f(q, q^3) = \sum_{m=0}^{\infty} q^{m(m+1)/2} = \frac{\ell_2^2}{\ell_1}. \tag{1.3}$$

The product representations in the special cases (??)-(??) are the consequences of one of the celebrated result in the theory of q-series known as the Jacobi's triple product identity, given by

$$f(c, d) = (-c; cd)_\infty (-d; cd)_\infty (cd; cd)_\infty. \tag{1.4}$$

By using elementary q-operations, it is easily seen that

$$\phi(-q) = \frac{(q; q)_\infty}{(-q; q)_\infty} = \frac{(q; q)_\infty^2}{(q^2; q^2)_\infty} = \frac{\ell_1^2}{\ell_2}. \tag{1.5}$$

Ramanujan [?] listed several q-series identities in his lost notebook. The most well-known q-series identities are the Rogers-Ramanujan identities (RRI) given by

$$\mathbb{S}_1(q) := \prod_{n=0}^{\infty} (1 - q^{5n+1})^{-1} (1 - q^{5n+4})^{-1} \tag{1.6}$$

and

$$\mathbb{S}_2(q) := \prod_{n=0}^{\infty} (1 - q^{5n+2})^{-1} (1 - q^{5n+3})^{-1}. \tag{1.7}$$

The identities (??) and (??) were first discovered by Rogers [?] in 1893 and then rediscovered by Ramanujan in 1913. Partition-theoretic interpretations of (??) and (??) are given by MacMahon [?].

Recently, Afsharijoo [?] established a recurrence relation which gives extended form of these identities where odd and even parts play different roles. Several Rogers-Ramanujan type identities (RRTIs) were also provided by Slater [?] and Chu and Zhang [?]. Gupta and Rana [?] and Gupta et al. [?] offered combinatorial interpretations of many RRTIs by using signed partitions which inspired them to explore more about signed partition and congruence properties of these identities. Gupta and Rana [?] collected severnteen RRTIs from [?] and established some particular congruences modulo powers of $2, 3$ and 6. In this paper, we investigate following RRTIs from [?] (also see [?]) for their partition-theoretic interpretations and new congruence properties:

$$G_k(q) = \sum_{n=0}^{\infty} g_k(n) q^n = \frac{(-q; q)_\infty}{(q; q)_\infty} \phi(q^k), \qquad k = 2. \tag{1.8}$$

$$H(q) = \sum_{n=0}^{\infty} h(n) q^n = \sum_{n=0}^{\infty} \frac{(-q; q)_{2n} q^n}{(q; q)_{2n+1}} = \frac{(-q; q)_\infty}{(q; q)_\infty} (q^4, -q^4, -q^4; q^4)_\infty. \tag{1.9}$$

$$T(q)=\sum_{n=0}^{\infty}t(n)q^n=\sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^n}{(q;q)_{2n+1}}=\frac{(-q;q)_\infty}{(q;q)_\infty}(q^{12},q^3,q^9;q^{12})_\infty. \tag{1.10}$$

$$M(q)=\sum_{n=0}^{\infty}m(n)q^n=\sum_{n=0}^{\infty}\frac{(-q;q)_{2n}q^n}{(q^2;q^2)_n}=\frac{(-q;q)_\infty}{(q;q)_\infty}(q^6,q,q^5;q^6)_\infty. \tag{1.11}$$

$$R(q)=\sum_{n=0}^{\infty}r(n)q^n=\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_nq^{n(n+1)}}{(q;q)_{2n+1}}=\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}(q^6,-q,-q^5;q^6)_\infty. \tag{1.12}$$

$$S(q)=\sum_{n=0}^{\infty}s(n)q^n=\sum_{n=0}^{\infty}\frac{(-1;q^2)_nq^{n(n+1)}}{(q;q)_{2n}}=\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}(q^6,-q^3,-q^3;q^6)_\infty. \tag{1.13}$$

In Section 3, we offer partition-theoretic interpretation of the q-series identities (**??**) and prove their congruence properties. In Section 4, we give partition-theoretic interpretations of (**??**). Some infinite families of congruence for the identities (**??**)-(**??**) modulo power of 2 are also proved.

To end the introduction, we define partition functions and their generating functions which are important in this paper. A partition of an positive integer n can be defined as finite sequence of positive integers $(\beta_1,\beta_2,...,\beta_k)$ such that $\sum_{j=1}^{k}\beta_j=n;\quad \beta_j\geq\beta_{j+1}$, where β_j are called parts or summands of the partition. The number of partitions of n is usually denoted by $p(n)$. As an illustration, $n=3$ has following three partitions: 3, 2+1, 1+1+1.

For positive integer n, an overpartition of n is defined as the partition of n in which the first occurence of each part may be overlined. If $O(n)$ denotes the number of overpartitions of n then

$$\sum_{n=0}^{\infty}O(n)q^n=\frac{(-q;q)_\infty}{(q;q)_\infty}.$$

Ramanujan [**?**] defined the general partition function $p_t(n)$ as

$$\sum_{n=0}^{\infty}p_t(n)q^n=\frac{1}{(q;q)_\infty^t}.$$

For $t>0$, $p_t(n)$ denoted the number of partition of n where each part of the partition is assumed to have t distinct colours. Also, for positive integers r,s and t,

$$\frac{1}{(q^r;q^s)^t}$$

denotes the generating function of the number of partitions of a positive integer such that parts $\equiv r\pmod{s}$ has t colours.

2 Preliminaries

The following lemmas will be used to prove our results.

Lemma 2.1. *([?, Theorem 2.1]). If p is an odd prime, then*

$$\psi(q) = \sum_{t=0}^{(p-3)/2} q^{(t^2+t)/2} f\left(q^{\left(p^2+(2t+1)p\right)/2}, q^{\left(p^2-(2t+1)p\right)/2}\right) + q^{(p^2-1)/8}\psi(q^{p^2}). \tag{2.1}$$

Furthermore, $\frac{(t^2+t)}{2} \not\equiv \frac{(p^2-1)}{8} \pmod{p}$ *for* $0 \leq t \leq (p-3)/2$.

Lemma 2.2. *([?, Theorem 2.2]). If $p \geq 5$ is a prime, then*

$$\ell_1 = \sum_{\substack{t=-(p-1)/2 \\ t\neq(\pm p-1)/6}}^{(p-1)/2} (-1)^t q^{(3t^2+t)/2} f\left(-q^{3p^2+(6t+1)p/2}, -q^{3p^2-(6t+1)p/2}\right) + (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} \ell_{p^2}, \tag{2.2}$$

where

$$\frac{\pm p-1}{6} = \begin{cases} \dfrac{(p-1)}{6} & if\ p \equiv 1 \pmod{6}, \\ \dfrac{(-p-1)}{6} & if\ p \equiv -1 \pmod{6}. \end{cases}$$

Furthermore, if $-\frac{p-1}{2} \leq t \leq \frac{p-1}{2}$ *and* $t \neq \frac{\pm p-1}{6}$, *then* $\frac{3t^2+t}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}$.

Lemma 2.3. *We have,*

$$\frac{\ell_3^3}{\ell_1} = \frac{\ell_4^3\ell_6^2}{\ell_2^2\ell_{12}} + q\frac{\ell_{12}^3}{\ell_4}, \tag{2.3}$$

$$\frac{\ell_2^2}{\ell_1} = \frac{\ell_6\ell_9^2}{\ell_3\ell_{18}} + q\frac{\ell_{18}^2}{\ell_9}, \tag{2.4}$$

$$\frac{\ell_2}{\ell_1^2} = \frac{\ell_6^4\ell_9^6}{\ell_3^8\ell_{18}^3} + 2q\frac{\ell_6^3\ell_9^3}{\ell_3^7} + 4q^2\frac{\ell_6^2\ell_{18}^3}{\ell_3^6}. \tag{2.5}$$

Identity (??) is Equation (22.1.14) in [?]. Identity (??) can be found in [?]. Identity (??) is Equation (14.3.3) of [?].

In addition to above identities, we need the following congruences which is easy consequence of the binomial theorem: For positive integers k and m, we have

$$\ell_{2k}^m \equiv \ell_k^{2m} \pmod{2}, \tag{2.6}$$

$$\ell_{2k}^{2m} \equiv \ell_k^{4m} \pmod{4}. \tag{2.7}$$

In order to state our congruences, we will also use Legendre symbol which is defined as follows:

Let p be any odd prime and ξ be any integer relatively prime to p, then the Legendre symbol $\left(\dfrac{\xi}{p}\right)$ is defined by

$$\left(\frac{\xi}{p}\right) = \begin{cases} 1, & \text{if } \xi \text{ is quadratic residue modulo } p, \\ -1, & \text{if } \xi \text{ is quadratic non-residue modulo } p. \end{cases}$$

3 Congruences for $g_k(n)$

Theorem 3.1. *If $g_k(n)$ is as defined in (**??**), then $g_k(n)$ is the number of overpartitions of a positive integer n into parts such that no part is congruent to 0 modulo $2k$ and parts congruent to $k \pmod{2k}$ have two colours.*

Proof. Employing (**??**) (with q replaced by q^k) in right hand side of (**??**), we obtain

$$\sum_{n=0}^{\infty} g_k(n)q^n = \frac{(-q;q)_\infty(-q^k;q^{2k})_\infty(q^{2k};q^{2k})_\infty}{(q;q)_\infty(q^k;q^{2k})_\infty(-q^{2k};q^{2k})_\infty}. \tag{3.1}$$

The right hand side of (**??**) is the generating function for the number of overpartitions of a positive integer n into parts such that no part is congruent to 0 modulo $2k$ and parts congruent to $k \pmod{2k}$ have two colours. So, the proof is complete. ∎

Theorem 3.2. *For all integers $\alpha \geq 0$,*

(i) Let $p \geq 3$ be any prime and $1 \leq j \leq (p-1)$, we have

$$\sum_{n=0}^{\infty} g_2\left(16\cdot p^{2\alpha}n + 2\cdot p^{2\alpha}\right)q^n \equiv 2\psi(q) \pmod 4, \tag{3.2}$$

$$g_2\left(16\cdot p^{2\alpha+2}n + 16\cdot p^{2\alpha+1}j + 2\cdot p^{2\alpha+2}\right) \equiv 0 \pmod 4. \tag{3.3}$$

(ii) Let $p \geq 5$ be any prime such that $\left(\dfrac{-8}{p}\right) = -1$ and $1 \leq j \leq (p-1)$, we have

$$\sum_{n=0}^{\infty} g_2\left(8\cdot p^{2\alpha}n + 3\cdot p^{2\alpha}\right)q^n \equiv 4\ell_1\ell_8 \pmod 8, \tag{3.4}$$

$$g_2\left(8\cdot p^{2\alpha+2}n + 8\cdot p^{2\alpha+1}j + 3\cdot p^{2\alpha+2}\right) \equiv 0 \pmod 8. \tag{3.5}$$

Proof. Setting $k=2$ in (**??**) and simplifying using (**??**) (with q replaced by q^2) and (**??**), we obtain

$$\sum_{n=0}^{\infty} g_2(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}(q^4,-q^2,-q^2;q^4)_\infty = \frac{\ell_4^5}{\ell_1^2\ell_2\ell_8^2}. \tag{3.6}$$

(i) From [**?**, p. 9 Theorem 4], we note that

$$\sum_{n=0}^{\infty} g_2(8n+2)q^n \equiv 6\frac{\ell_2^7}{\ell_4^2} \pmod 4. \tag{3.7}$$

Extracting the terms involving q^{2n} from (**??**), replacing q^2 by q and using (**??**) and (**??**), we obtain

$$\sum_{n=0}^{\infty} g_2(16n+2)q^n \equiv 2\psi(q) \pmod 4,$$

which is the $\alpha = 0$ case of (??). Assume that (??) is true for some $\alpha \geq 0$. Now, employing (??) in (??) and extracting the terms involving $q^{pn+(p^2-1)/8}$, dividing by $q^{(p^2-1)/8}$ and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} g_2\left(16 \cdot p^{2\alpha+1}n + 2 \cdot p^{2\alpha+2}\right) q^n \equiv 2\psi(q^p) \pmod{4}. \tag{3.8}$$

Extracting the terms involving q^{pn} from (??) and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} g_2\left(16 \cdot p^{2\alpha+2}n + 2 \cdot p^{2\alpha+2}\right) q^n \equiv 2\psi(q) \pmod{4},$$

which is the $\alpha + 1$ case of (??). Hence, by the method of induction, we complete the proof of (??). By extracting the terms involving q^{pn+j}, $1 \leq j \leq (p-1)$ from (??), we arrive at (??).

(ii) From [?, p. 10 Theorem 4], we note that

$$\sum_{n=0}^{\infty} g_2(4n+3)q^n \equiv 4\frac{\ell_4^{11}}{\ell_2^5\ell_8^2} \pmod{8}. \tag{3.9}$$

Extracting the terms involving q^{2n} from (??), replacing q^2 by q and then using (??), we obtain

$$\sum_{n=0}^{\infty} g_2(8n+3)q^n \equiv 4\ell_1\ell_8 \pmod{8},$$

which is the $\alpha = 0$ case of (??). Assume that (??) is true for some $\alpha \geq 0$. Now, substituting (??) in (??), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} g_2\left(8 \cdot p^{2\alpha}n + 3 \cdot p^{2\alpha}\right) q^n \\
&\equiv 4\Big[\sum_{\substack{t=-(p-1)/2 \\ t\neq(\pm p-1)/6}}^{(p-1)/2} (-1)^t q^{(3t^2+t)/2} f\left(-q^{\left(3p^2+(6t+1)p\right)/2}, -q^{\left(3p^2-(6t+1)p\right)/2}\right) \\
&\qquad\qquad + (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} \ell_{p^2}\Big] \\
&\times \Big[\sum_{\substack{m=-(p-1)/2 \\ m\neq(\pm p-1)/6}}^{(p-1)/2} (-1)^m q^{8(3m^2+m)/2} f\left(-q^{8\left(3p^2+(6m+1)p\right)/2}, -q^{8\left(3p^2-(6m+1)p\right)/2}\right) \\
&\qquad\qquad + (-1)^{(\pm p-1)/6} q^{8(p^2-1)/24} \ell_{8p^2}\Big] \pmod{8}.
\end{aligned} \tag{3.10}$$

Consider, the congruence

$$\frac{(3t^2+t)}{2} + 8\left(\frac{m^2+m}{2}\right) \equiv 9\left(\frac{p^2-1}{24}\right) \pmod{p},$$

which is equivalent to

$$(6t+1)^2 + 8(6m+1)^2 \equiv 0 \pmod{p}. \tag{3.11}$$

For $\left(\dfrac{-8}{p}\right) = -1$, the congruence (??) has only solution $t = m = (\pm p - 1)/6$. Therefore, extracting the terms involving $q^{pn+9(p^2-1)/24}$ from (??), dividing by $q^{9(p^2-1)/24}$ and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} g_2\left(8 \cdot p^{2\alpha+1} n + 3 \cdot p^{2\alpha+2}\right) q^n \equiv 4\ell_p \ell_{8p} \pmod{8}. \tag{3.12}$$

Extracting the terms involving q^{pn} from (??) and then replacing q^p by q, we obtain

$$g_2\left(8 \cdot p^{2\alpha+2} n + 3 \cdot p^{2\alpha+2}\right) q^n \equiv 4\ell_1 \ell_8 \pmod{8},$$

which is the $\alpha + 1$ case of (??). Hence, by the method of induction, we complete the proof of (??). The result (??) follows from (??) by extracting the terms involving q^{pn+j}, $1 \leq j \leq (p-1)$. ■

4 Partition-theoretic interpretations and congruences for the identities (??)-(??)

We first give partition-theoretic interpretations of (??). The partition-theoretic interpretations of (??), (??) and (??) can be found in [?].

Theorem 4.1. *If $h(n)$ is as defined in (??), then $h(n)$ is the number of partitions of a positive integer n into parts such that no part $\equiv 0 \pmod{8}$, parts $\equiv 2, 6 \pmod{8}$ have one colour and parts $\equiv 1, 3, 4, 5, 7 \pmod{8}$ have two colours.*

Proof. Simplifying right hand side of (??), we obtain

$$\sum_{n=0}^{\infty} h(n) q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} (q^4, -q^4, -q^4; q^4)_\infty = \frac{(q^2;q^2)_\infty (q^8;q^8)_\infty^2}{(q;q)_\infty^2 (q^4;q^4)_\infty}. \tag{4.1}$$

Changing the base q in $(q^2;q^2)_\infty$, $(q^4;q^4)_\infty$ and $(q;q)_\infty$ to q^8, we obtain

$$(q^2;q^2)_\infty = (q^2, q^4, q^6, q^8; q^8)_\infty, \tag{4.2}$$

$$(q^4;q^4)_\infty = (q^4, q^8; q^8)_\infty, \tag{4.3}$$

$$(q;q)_\infty = (q, q^2, q^3, q^4, q^5, q^6, q^7, q^8; q^8)_\infty. \tag{4.4}$$

Employing (??)-(??) in (??), we arrive at our desired result. ■

Theorem 4.2. *For all integers $\alpha \geq 0$,*

(i) Let $p \geq 5$ be any prime such that $\left(\frac{-2}{p}\right) = -1$ and $1 \leq j \leq (p-1)$, we have

$$\sum_{n=0}^{\infty} h\left(4 \cdot p^{2\alpha} n + \frac{p^{2\alpha}-1}{2}\right) q^n \equiv \ell_1 \ell_2 \pmod{4}, \tag{4.5}$$

$$h\left(4 \cdot p^{2\alpha+2} n + 4 \cdot p^{2\alpha+1} j + \frac{p^{2\alpha+2}-1}{2}\right) \equiv 0 \pmod{4}. \tag{4.6}$$

(ii) Let $p \geq 5$ be any prime such that $\left(\frac{-18}{p}\right) = -1$ and $1 \leq j \leq (p-1)$, we have

$$\sum_{n=0}^{\infty} h\left(12 \cdot p^{2\alpha} n + \frac{19 \cdot p^{2\alpha}-1}{2}\right) q^n \equiv 2\ell_1 \psi(q^6) \pmod{4}, \tag{4.7}$$

$$h\left(12 \cdot p^{2\alpha+2} n + 12 \cdot p^{2\alpha+1} j + \frac{19 \cdot p^{2\alpha+2}-1}{2}\right) \equiv 0 \pmod{4}. \tag{4.8}$$

(iii) Let $p \geq 5$ be any prime such that $\left(\frac{-2}{p}\right) = -1$ and $1 \leq j \leq (p-1)$, we have

$$\sum_{n=0}^{\infty} h\left(12 \cdot p^{2\alpha} n + \frac{11 \cdot p^{2\alpha}-1}{2}\right) q^n \equiv 2\ell_2 \psi(q^3) \pmod{4}, \tag{4.9}$$

$$h\left(12 \cdot p^{2\alpha+2} n + 12 \cdot p^{2\alpha+1} j + \frac{11 \cdot p^{2\alpha+2}-1}{2}\right) \equiv 0 \pmod{4}. \tag{4.10}$$

(iv) Let $p \geq 3$ be any prime such that $\left(\frac{-2}{p}\right) = -1$ and $1 \leq j \leq (p-1)$ we have

$$\sum_{n=0}^{\infty} h\left(4 \cdot p^{2\alpha} n + \frac{3 \cdot p^{2\alpha}-1}{2}\right) q^n \equiv 2\psi(q)\psi(q^2) \pmod{8}, \tag{4.11}$$

$$h\left(4 \cdot p^{2\alpha+2} n + 4 \cdot p^{2\alpha+1} j + \frac{3 \cdot p^{2\alpha+2}-1}{2}\right) \equiv 0 \pmod{8}. \tag{4.12}$$

Proof. (i) From [**?**, p. 10 Theorem 5], we note that

$$\sum_{n=0}^{\infty} h(2n) q^n = \frac{\ell_4^7}{\ell_1^4 \ell_2 \ell_8^2}. \tag{4.13}$$

Using (**??**) in (**??**) and then extracting the terms involving q^{2n}, replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} h(4n) q^n \equiv \ell_1 \ell_2 \pmod{4},$$

which is the $\alpha = 0$ case of (**??**). Now, proceeding in the same way as in (ii) of Theorem **??**, we arrive at (**??**) and (**??**).

(ii) From [**?**, p. 11 Theorem 5], we note that

$$\sum_{n=0}^{\infty} h(6n+3)q^n \equiv 2q\frac{\ell_4\ell_6^2\ell_{24}^2}{\ell_2\ell_{12}^2} \pmod{4}. \tag{4.14}$$

Extracting the terms involving q^{2n+1} from (**??**), dividing by q, replacing q^2 by q and then using (**??**), we obtain

$$\sum_{n=0}^{\infty} h(12n+9)q^n \equiv 2\ell_1\psi(q^6) \pmod{4},$$

which is the $\alpha = 0$ case of (**??**). Now, proceeding in the same way as in (ii) of Theorem **??**, we arrive at the results (**??**) and (**??**).

(iii) From [**?**, p. 11 Theorem 5], we note that

$$\sum_{n=0}^{\infty} h(6n+5)q^n \equiv 2\frac{\ell_2^2\ell_{12}^2}{\ell_6} \pmod{4}. \tag{4.15}$$

Extracting the terms involving q^{2n} from (**??**), replacing q^2 by q and then using (**??**), we obtain

$$\sum_{n=0}^{\infty} h(12n+5)q^n \equiv 2\ell_2\psi(q^3) \pmod{4},$$

which is the $\alpha = 0$ case of (**??**). Now, proceeding in the same way as in (ii) of Theorem **??**, we arrive at (**??**) and (**??**).

(iv) From [**?**, p. 10 Theorem 5], we note that

$$\sum_{n=0}^{\infty} h(2n+1)q^n = 2\frac{\ell_2\ell_4\ell_8^2}{\ell_1^4}. \tag{4.16}$$

Using (**??**) in (**??**) and then extracting the terms involving q^{2n}, replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} h(4n+1)q^n \equiv 2\frac{\ell_2\ell_4^2}{\ell_1} = 2\frac{\ell_2^2\ell_4^2}{\ell_1\ell_2} = 2\psi(q)\psi(q^2) \pmod{8},$$

which is the $\alpha = 0$ case of (**??**). Now, proceeding in the same way as in (ii) of Theorem **??**, we arrive at (**??**) and (**??**). ■

Theorem 4.3. *For all integer $\alpha \geq 0$,*

(i) Let $p \geq 3$ be any prime and $1 \leq j \leq (p-1)$, we have

$$\sum_{n=0}^{\infty} t\left(3 \cdot p^{2\alpha}n + \frac{3 \cdot p^{2\alpha} - 3}{8}\right) q^n \equiv \psi(q) \pmod{2}, \tag{4.17}$$

$$t\left(3\cdot p^{2\alpha+2}n+3\cdot p^{2\alpha+1}j+\frac{3\cdot p^{2\alpha+2}-3}{8}\right)\equiv 0\pmod{2}. \tag{4.18}$$

(ii) Let $p\geq 5$ be any prime such that $\left(\frac{-2}{p}\right)=-1$ and $1\leq j\leq (p-1)$, we have

$$\sum_{n=0}^{\infty} t\left(3\cdot p^{2\alpha}n+\frac{11\cdot p^{2\alpha}-3}{8}\right)q^n\equiv 2\ell_2\psi(q^3)\pmod{4}, \tag{4.19}$$

$$t\left(3\cdot p^{2\alpha+2}n+3\cdot p^{2\alpha+1}j+\frac{11\cdot p^{2\alpha+2}-3}{8}\right)\equiv 0\pmod{4}. \tag{4.20}$$

(iii) Let $p\geq 5$ be any prime such that $\left(\frac{-18}{p}\right)=-1$ and $1\leq j\leq (p-1)$, we have

$$\sum_{n=0}^{\infty} t\left(3\cdot p^{2\alpha}n+\frac{19\cdot p^{2\alpha}-3}{8}\right)q^n\equiv 4\ell_1\psi(q^6)\pmod{8}, \tag{4.21}$$

$$t\left(3\cdot p^{2\alpha+2}n+3\cdot p^{2\alpha+1}j+\frac{19\cdot p^{2\alpha+2}-3}{8}\right)\equiv 0\pmod{8}. \tag{4.22}$$

Proof. We have

$$\sum_{n=0}^{\infty} t(n)q^n=\frac{(-q;q)_\infty}{(q;q)_\infty}(q^{12},q^3,q^9;q^{12})_\infty=\frac{\ell_2\ell_3\ell_{12}}{\ell_1^2\ell_6}. \tag{4.23}$$

Employing (**??**) in (**??**), we obtain

$$\sum_{n=0}^{\infty} t(n)q^n=\frac{\ell_6^3\ell_9^6\ell_{12}}{\ell_3^7\ell_6^3}+2q\frac{\ell_6^2\ell_9^3\ell_{12}}{\ell_3^6}+4q^2\frac{\ell_6\ell_{12}\ell_{18}^3}{\ell_3^5}. \tag{4.24}$$

(i) Extracting the terms involving q^{3n} from (**??**), replacing by q^3 by q and then using (**??**), we obtain

$$\sum_{n=0}^{\infty} t(3n)q^n\equiv\psi(q)\pmod{2},$$

which is the $\alpha=0$ case of (**??**). Now, proceeding in the same way as in (i) of Theorem **??**, we arrive at (**??**) and (**??**).

(ii) Extracting the terms involving q^{3n+1} from (**??**), dividing by q, replacing by q^3 by q and then using (**??**) and (**??**), we obtain

$$\sum_{n=0}^{\infty} t(3n+1)q^n\equiv 2\ell_2\psi(q^3)\pmod{4},$$

which is the $\alpha=0$ case of (**??**). Now, proceeding in the same way as in (ii) of Theorem **??**, we arrive at (**??**) and (**??**).

(iii) Extracting the terms involving q^{3n+2} from (??), dividing by q^2, replacing by q^3 by q and then using (??) and (??), we obtain

$$\sum_{n=0}^{\infty} t(3n+2)q^n \equiv 4\ell_1\psi(q^6) \pmod{8},$$

which is the $\alpha = 0$ case of (??). Now, proceeding in the same way as in (ii) of Theorem ??, we arrive at (??) and (??). ■

Theorem 4.4. *For all integer $\alpha \geq 0$,*

(i) Let $p \geq 5$ be any prime and $1 \leq j \leq (p-1)$, we have

$$m\left(8 \cdot p^2 n + 8 \cdot pj + \frac{p^2+2}{3}\right) \equiv 0 \pmod{2}. \tag{4.25}$$

(ii) Let $p \geq 5$ be any prime such that $\left(\frac{-1}{p}\right) = -1$ and $1 \leq j \leq (p-1)$, we have

$$\sum_{n=0}^{\infty} m\left(16 \cdot p^{2\alpha} n + \frac{10 \cdot p^{2\alpha} - 1}{3}\right) q^n \equiv 4\ell_1\ell_4 \pmod{8}, \tag{4.26}$$

$$m\left(16 \cdot p^{2\alpha+2} n + 16 \cdot p^{2\alpha+1} j + \frac{10 \cdot p^{2\alpha+2} - 1}{3}\right) \equiv 0 \pmod{8}. \tag{4.27}$$

Proof. (i) From [?, p.20 Theorem 15], we note that

$$\sum_{n=0}^{\infty} m(2n+1)q^n \equiv \frac{\ell_4^2\ell_6^4}{\ell_2^2\ell_{12}^2} \pmod{2}. \tag{4.28}$$

Using (??) and then extracting the terms involving q^{4n}, replacing q^4 by q, we obtain

$$\sum_{n=0}^{\infty} m(8n+1)q^n \equiv \ell_1 \pmod{2}. \tag{4.29}$$

Substituting (??) in (??) and then extracting the terms involving $q^{pn+(p^2-1)/24}$, dividing by $q^{(p^2-1)/24}$ and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} m\left(8 \cdot pn + \frac{p^2+2}{3}\right) q^n \equiv (-1)^{(\pm p-1)/6}\ell_p \pmod{2}. \tag{4.30}$$

Hence, the result easily follows from (??) by extracting the terms involving q^{pn+j}, $1 \leq j \leq (p-1)$.

(ii) From [?, p.21 Theorem 15], we note that

$$\sum_{n=0}^{\infty} m(8n+3)q^n \equiv 4\frac{\ell_2\ell_4^2\ell_6^4}{\ell_{12}^2} \pmod{8}. \tag{4.31}$$

Extracting the terms involving q^{2n} from (**??**), replacing q^2 by q and then using (**??**), we obtain

$$\sum_{n=0}^{\infty} m(16n+3)q^n \equiv 4\ell_1\ell_4 \pmod{8},$$

which is the $\alpha = 0$ case of (**??**). Now, proceeding in the same way as in (ii) of Theorem **??**, we arrive at (**??**) and (**??**). ■

Theorem 4.5. *For all integer $\alpha \geq 0$,*

(*i*) *Let $p \geq 5$ be any prime such that $\left(\frac{-6}{p}\right) = -1$ and $1 \leq j \leq (p-1)$, we have*

$$\sum_{n=0}^{\infty} r\left(8 \cdot p^{2\alpha}n + \frac{7 \cdot p^{2\alpha} - 1}{3}\right) q^n \equiv 2\ell_1\psi(q^2) \pmod{4}, \tag{4.32}$$

$$r\left(8 \cdot p^{2\alpha+2}n + 8 \cdot p^{2\alpha+1}j + \frac{7 \cdot p^{2\alpha+2} - 1}{3}\right) \equiv 0 \pmod{4}. \tag{4.33}$$

(*ii*) *Let $p \geq 5$ be any prime such that $\left(\frac{-1}{p}\right) = -1$ and $1 \leq j \leq (p-1)$, we have*

$$\sum_{n=0}^{\infty} r\left(16 \cdot p^{2\alpha}n + \frac{10 \cdot p^{2\alpha} - 1}{3}\right) q^n \equiv 2\ell_1\ell_4 \pmod{4}, \tag{4.34}$$

$$r\left(16 \cdot p^{2\alpha+2}n + 16 \cdot p^{2\alpha+1}j + \frac{10 \cdot p^{2\alpha+2} - 1}{3}\right) \equiv 0 \pmod{4}. \tag{4.35}$$

(*iii*) *Let $p \geq 5$ be any prime such that $\left(\frac{-2}{p}\right) = -1$ and $1 \leq j \leq (p-1)$, we have*

$$\sum_{n=0}^{\infty} r\left(16 \cdot p^{2\alpha}n + \frac{22 \cdot p^{2\alpha} - 1}{3}\right) q^n \equiv 2\ell_2\psi(q^3) \pmod{4}, \tag{4.36}$$

$$r\left(16 \cdot p^{2\alpha+2}n + 16 \cdot p^{2\alpha+1}j + \frac{22 \cdot p^{2\alpha+2} - 1}{3}\right) \equiv 0 \pmod{4}. \tag{4.37}$$

Proof. (i) From [**?**, p.21 Theorem 16], we note that

$$\sum_{n=0}^{\infty} r(4n+2)q^n = 2\frac{\ell_2\ell_6^2\ell_8^2}{\ell_1^4\ell_{12}}. \tag{4.38}$$

Using (**??**) in (**??**) and then extracting the terms involving q^{2n}, replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} r(8n+2)q^n \equiv 2\ell_1\psi(q^2) \pmod{4},$$

which is the $\alpha = 0$ case of (??). Now, proceeding in the same way as in (ii) of Theorem **??**, we arrive at (??) and (??).

(ii) From [?, p.22 Theorem 16], we note that

$$\sum_{n=0}^{\infty} r(8n+3)q^n \equiv 2\frac{\ell_4\ell_8\ell_{12}^2}{\ell_2\ell_{24}} \pmod{8}. \tag{4.39}$$

Using (??) in (??) and then extracting the terms involving q^{2n}, replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} r(16n+3)q^n \equiv 2\ell_1\ell_4 \pmod{4},$$

which is the $\alpha = 0$ case of (??). Now, proceeding in the same way as in (ii) of Theorem **??**, we arrive at (??) and (??).

(iii) From [?, p.22 Theorem 16], we note that

$$\sum_{n=0}^{\infty} r(8n+7)q^n \equiv 2\frac{\ell_4^4\ell_6\ell_{24}}{\ell_2^2\ell_8\ell_{12}} \pmod{8}. \tag{4.40}$$

Using (??) and (??) in (??) and then extracting the terms involving q^{2n}, replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} r(16n+7)q^n \equiv 2\ell_2\psi(q^3) \pmod{4},$$

which is the $\alpha = 0$ case of (??). Now, proceeding in the same way as in (ii) of Theorem **??**, we arrive at (??) and (??). ∎

Theorem 4.6. *We have*

$$s(24n+i) \equiv 0 \pmod{4}, \quad \textit{where} \quad i = 9, 15, 21, \tag{4.41}$$

$$s(24n+23) \equiv 0 \pmod{8}, \tag{4.42}$$

$$s(24n+17) \equiv 0 \pmod{8}, \tag{4.43}$$

$$s(12n+1) \equiv 0 \pmod{16}. \tag{4.44}$$

Proof. From [?, p.12 Theorem 7], we note that

$$\sum_{n=0}^{\infty} s(2n+1)q^n = 2q\frac{\ell_2\ell_{24}^2}{\ell_1^2\ell_{12}}. \tag{4.45}$$

Employing (??) in (??), we obtain

$$\sum_{n=0}^{\infty} s(2n+1)q^n = 2q\frac{\ell_6^4\ell_9^6\ell_{24}^2}{\ell_3^8\ell_{12}\ell_{18}^3} + 4q^2\frac{\ell_6^3\ell_9^3\ell_{24}^2}{\ell_3^7\ell_{12}} + 8q^3\frac{\ell_6^2\ell_{18}^3\ell_{24}^2}{\ell_3^6\ell_{12}}. \tag{4.46}$$

Extracting the terms involving q^{3n} from (??), replacing q^3 by q and then using (??), we obtain

$$\sum_{n=0}^{\infty} s(6n+1)q^n \equiv 8q\frac{\ell_6^3\ell_8^2}{\ell_2\ell_4} \pmod{16}. \tag{4.47}$$

Hence, the result (??) easily follows from (??) by extracting the terms involving q^{2n}. Now, extracting the terms involving q^{3n+2} from (??), dividing by q^2, replacing q^3 by q, using (??) and then employing (??) and (??), we obtain

$$\sum_{n=0}^{\infty} s(6n+5)q^n \equiv 4\frac{\ell_4^2\ell_6^2\ell_8^2}{\ell_2^2\ell_{12}} + 4q\frac{\ell_8^2\ell_{12}^3}{\ell_4^2} \pmod{8}. \tag{4.48}$$

Extracting the terms involving q^{2n} from (??), replacing q^2 by q and then using (??), we obtain

$$\sum_{n=0}^{\infty} s(12n+5)q^n \equiv 4\ell_2\ell_8 \pmod{8}. \tag{4.49}$$

Hence, the result (??) easily follows from (??) by extracting the terms involving q^{2n+1}. Now, extracting the terms involving q^{2n+1} from (??), dividing by q, replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} s(12n+11)q^n \equiv 4\frac{\ell_4^2\ell_6^3}{\ell_2^2} \pmod{8}. \tag{4.50}$$

Hence, the result (??) easily follows from (??) by extracting the terms involving q^{2n+1}.

Now, extracting the terms involving q^{3n+1} from (??), dividing by q, replacing q^3 by q and then using (??) and (??), we obtain

$$\sum_{n=0}^{\infty} s(6n+3)q^n \equiv 2\psi(q^4) \pmod{4}. \tag{4.51}$$

Hence, the result (??) easily follows from (??) by extracting the terms involving q^{4n+i}, $i = 1, 2, 3$. ∎

Theorem 4.7. *For all integer $\alpha \geq 0$,*

(i) Let $p \geq 5$ be any prime such that $\left(\frac{-1}{p}\right) = -1$ and $1 \leq j \leq (p-1)$, we have

$$\sum_{n=0}^{\infty} s\left(24 \cdot p^{2\alpha}n + 3 \cdot p^{2\alpha}\right) q^n \equiv 4\ell_1\ell_4 \pmod{8}, \tag{4.52}$$

$$s\left(24 \cdot p^{2\alpha+2}n + 24 \cdot p^{2\alpha+1}j + 3 \cdot p^{2\alpha+2}\right) \equiv 0 \pmod{8}. \tag{4.53}$$

(ii) Let $p \geq 5$ be any prime such that $\left(\frac{-2}{p}\right) = -1$ and $1 \leq j \leq (p-1)$, we have

$$\sum_{n=0}^{\infty} s\left(24 \cdot p^{2\alpha}n + 11 \cdot p^{2\alpha}\right) q^n \equiv 4\ell_2\psi(q^3) \pmod{8}, \tag{4.54}$$

$$s\left(24\cdot p^{2\alpha+2}n+24\cdot p^{2\alpha+1}j+11\cdot p^{2\alpha+2}\right)\equiv 0\,(\mathrm{mod}\,8). \tag{4.55}$$

(iii) Let $p\geq 5$ be any prime such that $\left(\frac{-6}{p}\right)=-1$ and $1\leq j\leq (p-1)$, we have

$$\sum_{n=0}^{\infty} s\left(24\cdot p^{2\alpha}n+7\cdot p^{2\alpha}\right)q^n\equiv 8\ell_1\psi(q^2)\,(\mathrm{mod}\,16), \tag{4.56}$$

$$s\left(24\cdot p^{2\alpha+2}n+24\cdot p^{2\alpha+1}j+7\cdot p^{2\alpha+2}\right)\equiv 0\,(\mathrm{mod}\,16). \tag{4.57}$$

(iv) Let $p\geq 5$ be any prime such that $\left(\frac{-18}{p}\right)=-1$ and $1\leq j\leq (p-1)$, we have

$$\sum_{n=0}^{\infty} s\left(24\cdot p^{2\alpha}n+19\cdot p^{2\alpha}\right)q^n\equiv 8\ell_1\psi(q^6)\,(\mathrm{mod}\,16), \tag{4.58}$$

$$s\left(24\cdot p^{2\alpha+2}n+24\cdot p^{2\alpha+1}j+19\cdot p^{2\alpha+2}\right)\equiv 0\,(\mathrm{mod}\,16). \tag{4.59}$$

Proof. (i) Extracting the terms involving q^{2n} from (**??**) and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} s(24n+5)q^n\equiv 4\ell_1\ell_4\,(\mathrm{mod}\,8),$$

which is the $\alpha=0$ case of (**??**). Now, proceeding in the same way as in (ii) of Theorem **??**, we arrive at (**??**) and (**??**).

(ii) Extracting the terms involving q^{2n} from (**??**), replacing q^2 by q and then using (**??**) and (**??**), we obtain

$$\sum_{n=0}^{\infty} s(24n+11)q^n\equiv 4\ell_2\psi(q^3)\,(\mathrm{mod}\,8),$$

which is the $\alpha=0$ case of (**??**). Now, proceeding in the same way as in (ii) of Theorem **??**, we arrive at (**??**) and (**??**).

(iii) Extracting the terms involving q^{2n+1} from (**??**), dividing by q, replacing q^2 by q and then employing (**??**), we obtain

$$\sum_{n=0}^{\infty} s(12n+7)q^n\equiv 8\frac{\ell_4^5\ell_6^2}{\ell_2^3\ell_{12}}+8q\frac{\ell_4\ell_{12}^3}{\ell_2}\,(\mathrm{mod}\,16). \tag{4.60}$$

Extracting the terms involving q^{2n} from (**??**), replacing q^2 by q and then using (**??**) and (**??**), we obtain

$$\sum_{n=0}^{\infty} s(24n+7)q^n\equiv 8\ell_1\psi(q^2)\,(\mathrm{mod}\,16),$$

which is the $\alpha=0$ case of (**??**). Now, proceeding in the same way as in (ii) of Theorem **??**, we arrive at (**??**) and (**??**).

(iv) Extracting the terms involving q^{2n+1} from (**??**), dividing by q, replacing q^2 by q and then using (**??**) and (**??**), we obtain

$$\sum_{n=0}^{\infty} s(24n+19)q^n \equiv 8\ell_1\psi(q^6) \pmod{16},$$

which is the $\alpha = 0$ case of (**??**). Now, proceeding in the same way as in (ii) of Theorem **??**, we arrive at (**??**) and (**??**). ■

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Declarations

Conflict of Interest. The authors declare that there is no conflict of interest regarding the publication of this article.
Human and animal rights. The authors declare that there is no research involving human participants or animals in the contained of this paper.
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References

[1] Afsharijoo, P.: Even-odd partition identities of Rogers-Ramanujan type. *Ramanujan J.* **57**, 969-979 (2022).

[2] Agarwal, A. K.: Rogers-Ramanujan identities for n-color partitions. *J. Number Theory.* **28**, 299-305 (1988).

[3] Ahmed, Z., Baruah, N. D.: New congruences for ℓ-regular partitions for $\ell \in \{5, 6, 7, 49\}$. *Ramanujan J.* **40**, 649-668 (2016).

[4] Andrews, G. E., Berndt, B. C.: *Ramanujan's Lost Notebook Part II.* Springer. New York (2009).

[5] Berndt, B. C.: *Ramanujan's Notebooks, Part III.* Springer-Verlag. New York (1991).

[6] Berndt, B. C., Rankin, R. A.: *Ramanujan: letters and commentary.* American Mathematical Society. Providence. RI. London Mathematical Society. London (1995).

[7] Chu, W., Zhang, W.: Bilateral bailey lemma and Rogers-Ramanujan identities. *Adv. Appli. Math.* **42**, 359-391 (2009).

[8] Cui, S. P., Gu, N. S. S.: Arithmetic properties of l-regular paritions. *Adv. Appl. Math.* **51**, 507-523 (2013).

[9] Euler, L.: *Introductio in Analysin Infinitorum.* Marcum-Michaelem Bousquet. Lausannae (1748).

[10] Gupta, V., Rana, M.: On some combinatorics of Rogers-Ramanujan type identities using signed color partitions. In *Current trends in mathematical analysis and its interdisciplinary applications.* Springer. Berlin. Germany. 99-114 (2019).

[11] Gupta, V., Rana, M.: Some congruences for the coefficients of Rogers-Ramanujan type identities. *Mathematics.* **10**, 35-82 (2022).

[12] Gupta, V., Rana, M. and Sharma, S.: On weighted signed color partitions. *Indian Acad. Sci. (Math. Sci.).* DOI: 10.1007/s12044-019-0545-1 (2020).

[13] Hirschhorn, M. D.: *The Power of* q. In Developments in Mathematics. Springer. Berlin. Germany. **49** (2017).

[14] MacMahon, P. A.: *Combinatory Analysis.* American Mathematical Society. Providence. RI. USA. **137** (2001).

[15] Rogers, L. J.: Second memoir on the expansion of certain infinite products. *Proc. Lond. Math. Soc.* **1**, 318-343 (1893).

[16] Saikia, N.: Partition-theoretic interpretations of some q-series identities of Ramanujan. *Kuwait J. Sci.* **48**(2), 1-6 (2021).

[17] Slater, L. J.: Further identities of the Rogers-Ramanujan type. *Proc. Lond. Math. Soc.* **2**, 147-167 (1952).