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Vogel's universality and Macdonald dimensions

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Abstract

We discuss algebraic universality in the sense of P. Vogel for the simplest refined quantity, the Macdonald dimensions. The main known source of universal quantities is given by Chern-Simons theory. Refinement of Chern-Simons theory means introducing additional parameters. At the level of symmetric functions, the refinement is the transition from the Schur functions to the Macdonald polynomials. We consider the Macdonald polynomials associated with the simple Lie algebras, define Macdonald dimensions and dual Macdonald dimensions, and present a universal formula for them that unifies these quantities for algebras associated with simply laced root systems. We also consider mixed Macdonald dimensions that depend on two different root systems.

1 Introduction

For more than two decades, P. Vogel [?, ?] worked on the construction of a Universal Lie algebra which would incorporate all simple Lie algebras. This idea has not found its ultimate realization as an actual algebra; however, a concept of **universal quantity** has been created. We call a quantity universal if it can be described as a rational function of Vogel parameters. Known universal quantities are in some way connected with Chern-Simons theory, which means that various quantities in Chern-Simons theory with a gauge group G associated with structures of a simple Lie algebra g have a universal description. One can hypothesize that universality belongs to gauge theories and quantities within them, rather than algebraic structures of Lie algebras.

Known universal quantities are associated only with the adjoint representation and its descendants. This is quite natural, since the structures of representations of distinct simple Lie algebras are too different. Typical universal quantities are: the Chern-Simons partition function [?, ?, ?, ?], the dimension [?] and quantum dimension [?, ?] of the adjoint representation, eigenvalues of the second and higher Casimir operators [?, ?, ?, ?, ?, ?] in these representations, the volume of simple Lie groups [?], the HOMFLY-PT knot/link polynomial colored with adjoint representation [?, ?, ?] and the Racah matrix involving the adjoint representation and its descendants [?, ?, ?, ?, ?, ?].

Description of universal quantities is based on three Vogel's parameters: $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$, such that one can scale all of them at once with an arbitrary constant. One usually chooses one of the parameters, $\mathfrak{a}$ to be -2. Note that these parameters are usually denoted as α, β and γ, however, we denote them differently, since we use α to denote roots of a root system. Vogel's parameters for simple Lie algebras are listed in Table ??.

Root system	Lie algebra	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{t}=\mathfrak{a}+\mathfrak{b}+\mathfrak{c}$
A_n	sl_{n+1}	-2	2	$n+1$	$n+1$
B_n	so_{2n+1}	-2	4	$2n-3$	$2n-1$
C_n	sp_{2n}	-2	1	$n+2$	$n+1$
D_n	so_{2n}	-2	4	$2n-4$	$2n-2$
G_2	g_2	-2	$\frac{10}{3}$	$\frac{8}{3}$	4
F_4	f_4	-2	5	6	9
E_6	e_6	-2	6	8	12
E_7	e_7	-2	8	12	18
E_8	e_8	-2	12	20	30

Table 1: Vogel's parameters

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There were attempts to extend the notion of universality to the refined Chern-Simons theory [?, ?]. In practice, it was established in [?] for the classical Lie algebras and in [?, ?] for the exceptional algebras that the partition function of the refined Chern-Simons theory is universal only for the simply laced algebras. Note that there is another universal formula, which unifies knot hyperpolynomials for the root systems A_1, A_2, D_4, E_6, E_7, E_8 [?], though it could have also included G_2 and F_4 [?], but does not, because these two root systems are not simply laced.

Further study of the refinement confirmed that algebraic universality manifests itself specifically in the quantities from Chern–Simons theory, which are in fact knot invariants. The analysis has been extended to the case of Macdonald dimensions [?], which we discuss in detail in the present article, and hyperpolynomial of Hopf link [?]. However in refined Chern-Simons theory universality is preserved only for simply laced algebras.

Quantum dimensions	Dual quantum dimensions
$\mathrm{qD}_\lambda^R := \chi_\lambda^R\left(x = q^{2\rho}\right) = \prod\limits_{\alpha \in R_+} \frac{[(\alpha, \lambda+\rho)]_q}{[(\alpha,\rho)]_q}$	${}^\vee\mathrm{qD}_\lambda^R := \chi_\lambda^R\left(x = q^{2r}\right) = \prod\limits_{\alpha \in R_+} \prod\limits_{j=1}^{(\alpha^\vee,\lambda)} \frac{[(\rho,\alpha^\vee)+j]_q}{[(\rho,\alpha^\vee)+j-1]_q}$
Macdonald dimensions	**Dual Macdonald dimensions**
$\mathrm{Md}_\lambda^R := P_\lambda^R\left(x = q^{2\rho_k} \,\vert\, t_\alpha^2 \,\vert\, q^2, t^2\right)$	${}^\vee\mathrm{Md}_\lambda^R = P_\lambda^R\left(x = q^{2r_k} \,\vert\, t_\alpha^2 \,\vert\, q^2, t^2\right) = \prod\limits_{\alpha \in R_+} \prod\limits_{j=1}^{(\alpha^\vee,\lambda)} \frac{\left\{t_\alpha q^{(\rho_k,\alpha^\vee)+j-1}\right\}}{\left\{q^{(\rho_k,\alpha^\vee)+j-1}\right\}}$

Table 2: Quantum and Macdonald dimensions and their counterparts

In Chern-Simons theory quantum dimensions are Wilson loop averages of unknotted loop. In representation theory they are characters of irreducible representations taken at the point of Weyl vector ρ. The nice thing about quantum dimensions is that they factorize and admit universalization in adjoint representation [?, ?] and its descendants. Characters of irreducible representations also factorize at the dual Weyl vector.

The counterpart of quantum dimensions in refined Chern-Simons theory are Macdonald dimensions — Macdonald polynomials at the point of refined Weyl vector. However, they do not factorize. Only dual Macdonald dimensions admit factorization. Dual Macdonald dimensions differ from Macdonald dimensions for non simply laced root systems, which means that Macdonald dimensions for A_n, D_n and E_6, E_7, E_8 root systems do factorize and as it turns out admit universalization. We present the universal formula for Macdonald dimensions in adjoint representation of simply laced root systems (see eqs. (**??**) and (**??**)). We summarize the key elements discussed in this paper in Table 2.

This paper is an extended version of [?]. We go into more details of the definitions of Macdonald polynomials and of Macdonald dimensions. We also consider mixed Macdonald dimensions that depend on two different root systems.

The structure of the paper is as follows. In section **??** we start with the introduction of Macdonald polynomials and of all the components of their definition: admissible pairs, scalar product, dominance order, parameters and basis. We want to emphasize the difference between the Macdonald polynomials for the root system A_n and for other classical root systems. In section **??**, we discuss the Schur functions: a natural basis of Macdonald polynomials. In section **??**, we define quantum and Macdonald dimensions and their dual versions, discuss factorization of these quantities and list them in adjoint representations of simple Lie algebras and admissible pairs (R, R). In section **??**, we consider the Macdonald dimensions depending on two different root systems. Finally in section **??**, we discuss universality in Chern-Simons theory, in particular, universal formulas for the quantum and Macdonald dimensions and possibilities for further universalization.

Notation and comments.

- We use the letter α to denote the roots of the root system R, R_+ is a set of positive roots.

- In variance with the original work by I. Macdonald [?] and subsequent papers on the subject [?, ?, ?, ?], we use symmetric quantum numbers, which allows us to present the results in a shorter and more elegant form: in our notation, the Macdonald polynomials depend on the squares of the original Macdonald parameters:
$$q \to q^2,\ t \to t^2,\ t_\alpha^2 \to t_\alpha^2 \tag{1}$$
and the brackets and quantum numbers are defined to be
$$\{x\} = x - x^{-1}, \quad \{x\}_+ = x + x^{-1}, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_t = \frac{t^n - t^{-n}}{t - t^{-1}}. \tag{2}$$

- When we say that some symmetric polynomial that depends on variables $x_1, \ldots, x_n$ is taken at the point q^ρ, where $\rho = (\rho_1, \ldots, \rho_n)$ is the Weyl vector, we mean that one should make the substitution in this symmetric polynomial
$$x_i = q^{\rho_i}. \tag{3}$$
The number of variables x_i and the length of the Weyl vector coincide and are equal to the dimension of the Euclidean space where the root lattice is embedded.

- We use $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ as well as $\mathfrak{t} = \mathfrak{q} + \mathfrak{b} + \mathfrak{c}$ to denote Vogel parameters.

- Vogel parameters (Table ??) are based on the minimal normalization of the roots, which means that the length of the longest root $(\alpha_l, \alpha_l) = 2$.

2 Macdonald polynomials

Macdonald polynomials $P_\lambda^{(R,S)}(x \,|\, t_\alpha \,|\, q, t)$ associated with a pair of root systems (R, S) were defined by I. Macdonald [?], and are a generalization of famous Macdonald polynomials $M_\lambda(x \,|\, q, t)$ [?, ?], which turn out to be associated with the root system A_n:
$$P_\lambda^{(A_n,A_n)}(x \,|\, q, t) = M_\lambda(x \,|\, q, t). \tag{4}$$
Polynomials $P_\lambda^{(R,S)}(x \,|\, t_\alpha \,|\, q, t)$ are symmetric under action of the Weyl group $W = W_R = W_S$. They are enumerated with dominant weights λ and depend on parameters q and t, and additionally on t_α that are associated with roots of distinct lengths.

2.1 Root systems

It is convenient to use root systems to classify simple Lie algebras. All simple Lie algebras are associated with one of the following root systems:
$$A_n, \quad B_n, \quad C_n, \quad D_n, \quad E_6, \quad E_7, \quad E_8, \quad F_4, \quad G_2. \tag{5}$$
Root systems A_n, B_n, C_n, D_n are called classical and they are associated with algebras sl_{n+1}, so_{2n+1}, sp_{2n} and so_{2n} correspondingly. Root systems E_6, E_7, E_8, F_4, G_2 are called exceptional and are associated with algebras e_6, e_7, e_8, f_4 and g_2.

To define a root system one starts with the Euclidean space V with the **orthogonal basis** ε:
$$(\varepsilon_i, \varepsilon_j) = \delta_{ij}. \tag{6}$$
One can define a reflection on V:
$$s_a(v) := v - 2\frac{(v,a)}{(a,a)}a = v - (v,a)a^\vee, \tag{7}$$
where $v \in V$, $a \in V \setminus \{0\}$, $\alpha^\vee = 2a/(a,a)$.

We call $R \subset V \setminus \{0\}$ a root system if for any α, $\beta \in R$
$$R \text{ spans } V, \qquad s_\alpha(\beta) \in R, \qquad (\alpha^\vee, \beta) \in \mathbb{Z}, \tag{8}$$
α, β are roots of a root system R, $\alpha^\vee$ are coroots $\alpha^\vee = 2\alpha/(\alpha,\alpha)$.

Weyl group W_R of root system R is the subgroup of the orthogonal group $O(V)$ generated by reflections from roots $\alpha \in R$.

In addition to the orthogonal basis there are two other natural bases on the space of a root system: basis of simple roots and basis of fundamental weights.

Simple roots $\{\alpha_i \,|\, i \in I\} \subset R$ are a basis of R if $\forall \alpha \in R$ we can expand $\alpha = c_i\, \alpha_i$ so that all c_i are of the same sign or equal to zero. Other roots can be expressed as

$$\alpha = \sum_{i \in I} c_i\, \alpha_i. \tag{9}$$

Positive roots $\alpha \in R_+$ are the roots with non-negative coefficients c_i:

$$R_+ = \{\alpha \in R \,|\, c_i \geq 0\}. \tag{10}$$

Fundamental weights ω_i are defined as

$$(\omega_i, \alpha_j^\vee) = \delta_{ij}. \tag{11}$$

The combinations of fundamental weights λ enumerate irreducible representations of Lie algebras as well as Macdonald polynomials:

$$\lambda = \sum \lambda_i \omega_i, \quad \lambda_i \in \mathbb{N}_0. \tag{12}$$

They are called dominant weights.

2.1.1 Root system A_n (algebra sl_{n+1})

Roots, positive roots and Weyl group W of the root system A_n are the following:

$$R = \{\varepsilon_i - \varepsilon_j \,|\, 1 \leq i \neq j \leq n+1\}, \qquad W \simeq S_{n+1} \tag{13}$$

$$R^+ = \{\varepsilon_i - \varepsilon_j \,|\, 1 \leq i < j \leq n+1\}. \tag{14}$$

The Schur-Weyl duality establishes correspondence between representations of sl_N and the permutation group S_N. Irreducible representations of S_N are enumerated with partitions Q — sequences of non-negative integers in non-increasing order:

$$Q = [Q_1, Q_2, \ldots, Q_N], \quad Q_1 \geq Q_2 \geq \cdots \geq Q_N, \quad Q_i \in \mathbb{N}_0, \quad |Q| = \sum_{i=1}^{N} Q_i, \tag{15}$$

$l(Q)$ — length of partition — number of non-zero Q_i in Q.

Symmetric functions such as power sum p_Q, monomial m_Q and Schur symmetric functions are enumerated with partitions:

$$p_k = \sum_{i=1}^{N} x_i^k, \quad p_Q = \prod_{i=1}^{l(Q)} p_{Q_i}, \tag{16}$$

$$m_Q = \sum_{\sigma \in S_N} x_{\sigma(1)}^{Q_1} x_{\sigma(2)}^{Q_2} \ldots x_{\sigma(l(Q))}^{Q_{l(Q)}}, \tag{17}$$

and we are going to discuss Schur symmetric functions in detail in section **??**.

Irreducible representations of sl_{n+1} are also labeled with partitions. Partitions can be represented as Young diagrams and are sometimes identified with them. They are connected with dominant weights λ in the following way:

$$\begin{aligned} &\lambda = \sum_{i=1}^{n} \lambda_i \omega_i, \\ &Q_\lambda = [Q_1, Q_2, \ldots, Q_{n+1}], \\ &Q_i = \sum_{j=i}^{n} \lambda_j. \end{aligned} \tag{18}$$

The fundamental weight ω_i in $(n+1)$-dimensional orthogonal basis ε_i is

$$\omega_i = \frac{n+1-i}{n+1}(\varepsilon_1 + \cdots + \varepsilon_i) - \frac{i}{n+1}(\varepsilon_{i+1} + \cdots + \varepsilon_{n+1}). \tag{19}$$

To enumerate irreducible representations of algebras so_N and sp_N one can use the highest weights of representations in the orthogonal basis.

2.1.2 Root system B_n (algebra so_{2n+1})

Roots, positive roots and Weyl group W of the root system B_n are the following:

$$R = \{\pm\varepsilon_i|1 \le i \le n\} \cup \{\pm\varepsilon_i \pm \varepsilon_j|1 \le i < j \le n\}, \qquad W \simeq S_n \ltimes (\{\pm 1\})^n \tag{20}$$

$$R^+ = \{\varepsilon_i|1 \le i \le n\} \cup \{\varepsilon_i \pm \varepsilon_j|1 \le i < j \le n\}. \tag{21}$$

Fundamental weights in the orthogonal n-dimensional basis ε_i are

$$\omega_i = \sum_{k=1}^{i} \varepsilon_k \quad \text{for} \quad i = 1, \dots, n-1, \tag{22}$$

$$\omega_n = \frac{1}{2}\sum_{k=1}^{n} \varepsilon_k. \tag{23}$$

And dominant weight $\lambda = \sum \lambda_i \omega_i$ in the orthogonal basis has the following coordinates:

$$\lambda_\varepsilon^{B_n} = \left[\lambda_1 + \lambda_2 + \dots + \frac{\lambda_n}{2}, \lambda_2 + \dots + \frac{\lambda_n}{2}, \dots, \frac{\lambda_n}{2}\right]. \tag{24}$$

In this paper we treat dominant weights λ in the orthogonal basis as partitions, even though there are weights with half-integer coordinates. The same works for the root system D_n.

2.1.3 Root system C_n (algebra sp_{2n})

Roots, positive roots and Weyl group W of the root system C_n are the following:

$$R = \{\pm\varepsilon_i \pm \varepsilon_j|1 \le i < j \le n\} \cup \{\pm 2\varepsilon_i|1 \le i \le n\}, \qquad W \simeq S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n, \tag{25}$$

$$R_+ = \{\varepsilon_i \pm \varepsilon_j|1 \le i < j \le n\} \cup \{2\varepsilon_i|1 \le i \le n\}. \tag{26}$$

We want to stress that roots (**??**) together with the scalar product $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$ are *not* in the minimal normalization. We use them in this paper in order to be consistent with other works on Macdonald polynomials [**?**].

Fundamental weights in n-dimensional orthogonal basis ε_i are

$$\omega_i = \sum_{k=1}^{i} \varepsilon_k \quad \text{for} \quad i = 1, \dots, n \tag{27}$$

and general weight $\lambda = \sum \lambda_i \omega_i$ in the orthogonal basis is the integer partition:

$$\lambda_\varepsilon^{C_n} = [\lambda_1 + \lambda_2 + \dots + \lambda_n, \lambda_2 + \dots + \lambda_n, \dots, \lambda_n]. \tag{28}$$

2.1.4 Root system D_n (algebra so_{2n})

Roots, positive roots and Weyl group W of the root system D_n are the following:

$$R = \{\pm\varepsilon_i \pm \varepsilon_j|1 \le i < j \le n\}, \qquad W \simeq S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-1}, \tag{29}$$

$$R^+ = \{\varepsilon_i \pm \varepsilon_j|1 \le i < j \le n\}. \tag{30}$$

Weights in the n-dimensional orthogonal basis ε_i are the following:

$$\omega_i = \sum_{k=1}^{i} \varepsilon_k \quad \text{for} \quad i = 1, \dots, n-2, \tag{31}$$

$$\omega_{n-1} = \frac{1}{2}\sum_{k=1}^{n-1} \varepsilon_k - \frac{1}{2}\varepsilon_n, \tag{32}$$

$$\omega_n = \frac{1}{2}\sum_{k=1}^{n} \varepsilon_k \tag{33}$$

and coordinates of the dominant weight $\lambda = \sum \lambda_i \omega_i$ in the orthogonal basis ε_i are the following

$$\lambda_\varepsilon^{D_n} = \left[\lambda_1 + \lambda_2 + \dots + \frac{\lambda_{n-1}}{2} + \frac{\lambda_n}{2}, \lambda_2 + \dots + \frac{\lambda_{n-1}}{2} + \frac{\lambda_n}{2}, \dots, \frac{\lambda_{n-1}}{2} + \frac{\lambda_n}{2}, \frac{\lambda_n}{2} - \frac{\lambda_{n-1}}{2}\right]. \tag{34}$$

They can be half-integer and negative.

Roots of the exceptional roots systems are listed in the Appendix A.

2.2 Definition of Macdonald polynomials associated with root systems

For an admissible pair of root systems (R, S) there exists a unique family of polynomials $P_\lambda^{(R,S)}$ which satisfy the following conditions:

$$P_\lambda^{(R,S)} = m_\lambda^R + \sum_{\mu<\lambda} c_{\lambda\mu}^{(R,S)}\, m_\mu^R, \tag{35}$$

$$\left\langle P_\lambda^{(R,S)}, m_\mu^R \right\rangle = 0 \;\text{ if }\; \mu < \lambda. \tag{36}$$

The resulting polynomials are mutually orthogonal:

$$\left\langle P_\lambda^{(R,S)}, P_\mu^{(R,S)} \right\rangle = 0, \qquad \lambda \neq \mu. \tag{37}$$

In this definition we need to specify the following symbols and operations.

- (R, S) is an **admissible pair** of root systems if they have the same Weyl group $W_R = W_S$, R is irreducible, but not necessarily reduced, S is irreducible and reduced. There are the following possible combinations:

$$(R,R): \;\; R = A_n,\; B_n,\; C_n,\; D_n,\; E_6,\; E_7,\; E_8,\; F_4,\; G_2 \tag{38}$$

$$(R,R^\vee): \;\; R = B_n,\; C_n,\; B_n^\vee = C_n,\; C_n^\vee = B_n \tag{39}$$

$$(BC_n, S): \;\; S = B_n,\; C_n \tag{40}$$

- λ is the **dominant weight** of the root system R, it enumerates polynomials $P_\lambda^{(R,S)}$ and is the combination of the fundamental weights ω_i:

$$\lambda = \sum_i \lambda_i \omega_i, \quad \lambda_i \in \mathbb{N}_0, \quad (\omega_i, \alpha_j^\vee) = \delta_{ij}. \tag{41}$$

- m_λ^R form a **basis** for polynomials $P_\lambda^{(R,S)}$. They are an alternative to monomial symmetric functions (**??**):

$$m_\lambda^R = \frac{1}{|W_R^\lambda|} \sum_{w \in W_R} e^{w\lambda}(v), \tag{42}$$

  where W_R^λ is the stabilizer of λ in Weyl group W_R. In the case of classical Lie algebras one can also use Schur functions (**??**)–(**??**) as a basis for Macdonald polynomials. We discuss them in section **??**.

- The **triangular decomposition** (**??**) is build with the dominance order $\mu < \lambda$ on weights λ and μ:

$$\lambda \geq \mu \iff \lambda - \mu \in \mathbb{N}R_+. \tag{43}$$

  We discuss dominance order for classical root systems in more detail in the section **??**.

- The **scalar product** is defined as follows

$$\langle f, g \rangle = |W|^{-1} \int_T \prod_{\alpha \in R} f(\dot{v}) \overline{g(\dot{v})} \Delta(\dot{v}) d\dot{v}, \tag{44}$$

  where $\int_T d\dot{v} = 1$. In case of finite number of variables x_i, $1 \leq i \leq n$ we can rewrite this formula:

$$\langle f(x), g(x) \rangle = |W|^{-1} [f(x) g(x^{-1}) \Delta]_0, \tag{45}$$

  where the operator $[\dots]_0$ means picking up the constant term of a Laurent polynomial.

  The **Macdonald density** is defined as a product over all roots of the root system R:

$$\Delta(v) := \prod_{\alpha \in R} \frac{\left(t_{2\alpha}^{1/2} e^\alpha(v); q_\alpha\right)_\infty}{\left(t_\alpha t_{2\alpha}^{1/2} e^\alpha(v); q_\alpha\right)_\infty}, \qquad (a;q)_\infty = \prod_{i=0}^\infty (1 - a q^i). \tag{46}$$

  We define the weight function Δ and discuss it in detail in Appendix A. And we discuss the parameters t_α and q_α in detail in section **??**.

- One should treat the exponent in the formulas above as

$$e^{\eta}(v) = e^{(\eta,v)}, \tag{47}$$

where v is a formal vector $v = \sum v_i\varepsilon_i$, ε_i are the elements of the orthogonal basis (**??**) on the Euclidean space V, $\eta \in V$ and variables of Macdonald polynomials x_i emerge as

$$x_i = e^{v_i}. \tag{48}$$

2.3 Dominance order for classical Lie algebras

In this section we are going to formulate the rules of ordering of dominant weights for classical root systems. In the case of root systems A_n we are going to formulate them for partitions Q (**??**), and for B_n, C_n and D_n for weights in orthogonal basis (**??**), (**??**), (**??**). In the case of root system C_n dominant weights in orthogonal basis (**??**) are in fact partitions. In case of root systems B_n and D_n (**??**), (**??**) are not partitions, because their coordinates can be half-integer or negative. However some rules of ordering in this case repeat the ones for partitions.

2.3.1 Dominance order for A_n

Weights λ^{A_n} and μ^{A_n} corresponding to partitions Λ and M satisfy

$$\lambda^{A_n} \geq \mu^{A_n}, \tag{49}$$

when

$$|\Lambda| \equiv |\mathrm{M}|, \tag{50}$$

$$\sum_{i=1}^{k}(\Lambda_i - \mathrm{M}_i) \in \mathbb{N}_0 \quad \text{for} \quad k = 1, \ldots, \max(l(\Lambda), l(\mathrm{M})). \tag{51}$$

It means that in the case of A_n all weights are broken into groups which are enumerated with partitions of integers:

$$\begin{array}{ll} |\Lambda| = 1: & [1] \\ |\Lambda| = 2: & [1,1] < [2] \\ |\Lambda| = 3: & [1,1,1] < [2,1] < [3] \\ |\Lambda| = 4: & [1,1,1,1] < [2,1,1] < [2,2] < [3,1] < [4] \\ |\Lambda| = 5: & [1,1,1,1,1] < [2,1,1,1] < [2,2,1] < [3,1,1] < [3,2] < [4,1] < [5] \end{array} \tag{52}$$

Weights associated with different $|\Lambda|$ are incomparable and do not appear in each other's Macdonald polynomials. If the condition (**??**) is not applicable to two partitions, it means that we can not compare corresponding weights. For example we can't compare $[4,1,1]$ and $[3,3]$:

$$m = 6: \quad \cdots < [3,2,1] < \begin{array}{c} [4,1,1] \\ [3,3] \end{array} < [4,2] < [5,1] < [6]. \tag{53}$$

2.3.2 Dominance order for B_n

The dominance order for two weights $\lambda_\varepsilon^{B_n} = [\Lambda_1, \Lambda_2, \ldots, \Lambda_n]$ and $\mu_\varepsilon^{B_n} = [\mathrm{M}_1, \mathrm{M}_2, \ldots, \mathrm{M}_n]$ (**??**) in the orthogonal basis coincides with (**??**):

$$\lambda_\varepsilon^{B_n} \geq \mu_\varepsilon^{B_n} \quad \longleftrightarrow \quad \sum_{i=1}^{k}(\Lambda_i - \mathrm{M}_i) \in \mathbb{N}_0 \quad \text{for} \quad k = 1, \ldots, n \tag{54}$$

The dominance order divides all dominant weights into just two groups: integer partitions and half-integer partitions. One can get the second group from the first one adding last fundamental weight ω_n (**??**):

$$\begin{array}{ll} \text{integer:} & \varnothing < [1] < [1,1] < \begin{array}{c} [2] \\ [1,1,1] \end{array} < [2,1] < \ldots \\ \text{half-integer} = \text{integer} + \omega_n: & \varnothing + \omega_n < [1] + \omega_n < [1,1] + \omega_n < \begin{array}{c} [2] + \omega_n \\ [1,1,1] + \omega_n \end{array} < [2,1] + \omega_n < \ldots \end{array} \tag{55}$$

Weights from these two groups are incomparable and do not mix in Macdonald polynomials. Inside each group there is the ordering (**??**).

2.3.3 Dominance order for C_n

The dominance order for two weights $\lambda_\varepsilon^{C_n} = [\Lambda_1, \Lambda_2, \ldots, \Lambda_n]$ and $\mu_\varepsilon^{C_n} = [\mathrm{M}_1, \mathrm{M}_2, \ldots, \mathrm{M}_n]$ (**??**) in the orthogonal basis is the following:

$$\lambda_\varepsilon^{C_n} \geq \mu_\varepsilon^{C_n} \quad \longleftrightarrow \quad \sum \Lambda_i \equiv \sum \mathrm{M}_i \,(\mathrm{mod}\ 2) \quad \text{and} \quad \sum_{i=1}^{k} (\Lambda_i - \mathrm{M}_i) \in \mathbb{N}_0 \quad \text{for} \quad k = 1, \ldots, n. \tag{56}$$

Dominant weights are divided into two groups: partitions of even and odd integers:

$$\begin{array}{ll} \text{even integers:} & \varnothing < [1,1] < \begin{array}{c}[2]\\ [1,1,1,1]\end{array} < [2,1,1] < [2,2] < [3,1] < [4] < \ldots \\ \text{odd integers:} & [1] < [1,1,1] < [2,1] < [3] < \ldots \end{array} \tag{57}$$

2.3.4 Dominance order for D_n

The dominance order for two weights $\lambda_\varepsilon^{D_n} = [\Lambda_1, \Lambda_2, \ldots, \Lambda_n]$ and $\mu_\varepsilon^{D_n} = [\mathrm{M}_1, \mathrm{M}_2, \ldots, \mathrm{M}_n]$ (**??**) in the orthogonal basis is the following:

$$\lambda_\varepsilon^{D_n} \geq \lambda_\varepsilon^{D_n} \quad \longleftrightarrow \quad \sum_{i=1}^{n-1} (\Lambda_i - \mathrm{M}_i) \pm (\Lambda_n - \mathrm{M}_n) \in 2\mathbb{N}_0 \quad \text{and} \quad \sum_{i=1}^{k} (\Lambda_i - \mathrm{M}_i) \in \mathbb{N}_0 \quad \text{for} \quad k = 1, \ldots, n. \tag{58}$$

It means that there are four different groups of weights which are mutually incomparable:

1. partitions of even integers
2. partitions of odd integers
3. (partitions of even integers)+ω_n + (partitions of odd integers)+ω_{n-1}
4. (partitions of even integers)+ω_{n-1} + (partitions of odd integers)+ω_n

The first two groups are the same as for C_n:

$$\begin{array}{ll} \text{even integers:} & \varnothing < [1,1] < \begin{array}{c}[2]\\ [1,1,1,1]\end{array} < [2,1,1] < [2,2] < [3,1] < [4] < \ldots \\ \text{odd integers:} & [1] < [1,1,1] < [2,1] < [3] < \ldots \end{array} \tag{59}$$

The third and the fourth groups are mixed combinations of even and odd partitions:

$$\begin{array}{ll} \text{mixed even integers:} & \varnothing + \omega_n < [1] + \omega_{n-1} < [1,1] + \omega_n < \begin{array}{c}[2] + \omega_n\\ [1,1,1] + \omega_{n-1}\end{array} < [2,1] + \omega_{n-1} < [3] + \omega_{n-1} < \ldots \\ \text{mixed odd integers:} & \varnothing + \omega_{n-1} < [1] + \omega_n < [1,1] + \omega_{n-1} < \begin{array}{c}[2] + \omega_{n-1}\\ [1,1,1] + \omega_n\end{array} < [2,1] + \omega_n < [3] + \omega_n < \ldots \end{array} \tag{60}$$

The third and fourth groups can be obtained with the ordering (**??**) of all integer partitions, also one needs to add ω_n to partitions of even integers and ω_{n-1} to partitions of odd integers to third group and vise versa to fourth group. However the picture is in fact more complicated and one has to check all the conditions (**??**) to compare partitions. For example, one can not compare partitions $[1,1,1,1] + \omega_4$ and $[1,1,1] + \omega_3$ for root system D_4, but $[1,1,1,1] + \omega_n > [1,1,1] + \omega_{n-1}$ for other D_n with $n \geq 5$. Also, some partitions from (**??**) can coincide for some root systems. For example $[1,1] + \omega_3 = [1,1,1] + \omega_2$ for D_3.

2.4 Parameters of Macdonald polynomials

In contrast to Macdonald polynomials $M_\lambda(x \,|\, q^2, t^2)$ [**?**, **?**] corresponding to the admissible pair (A_n, A_n) and depending on parameters q and t, in general case we get new parameters: q_α and t_α:

- q_α depends on the correspondence between roots of root systems R and S from an admissible pair:

$$q_\alpha = q^{u_\alpha}, \qquad u_\alpha : \quad \alpha / u_\alpha \in S, \quad \alpha \in R. \tag{61}$$

- $t_\alpha = q_\alpha^{k_\alpha}$, where k_α depends only on a length of a root $\alpha \in R$. For roots from section **??** and Appendix A we use the following notation:

$$\begin{array}{lll} (\alpha,\alpha)=1, & t_\alpha = t_s, & k_\alpha = k_s, \\ (\alpha,\alpha)=2, & t_\alpha = t, & k_\alpha = k, \\ (\alpha,\alpha)=4, & t_\alpha = t_l, & k_\alpha = k_l, \\ (\alpha,\alpha)=6, & t_\alpha = t_3, & k_\alpha = k_3. \end{array} \tag{62}$$

Here are all Macdonald polynomials with corresponding parameters

$$\begin{array}{ll} (R,R): & P_\lambda^{A_n}(x\,|\,q,t)\,, \quad P_\lambda^{B_n}\left(x\,|\,t_s=q^{k_s}\,|\,q,t\right), \quad P_\lambda^{C_n}\left(x\,|\,t_l=q^{k_l}\,|\,q,t\right), \quad P_\lambda^{D_n}(x\,|\,q,t) \\ & P_\lambda^{E_n}(x\,|\,q,t)\,, \quad P_\lambda^{F_4}\left(x\,|\,t_s=q^{k_s}\,|\,q,t\right), \quad P_\lambda^{G_2}\left(x\,|\,t_3=q^{k_3}\,|\,q,t\right) \\ (R,R^\vee): & P_\lambda^{(B_n,C_n)}\left(x\,|\,t_s=q^{k_s}\,|\,q,t\right), \quad P_\lambda^{(C_n,B_n)}\left(x\,|\,t_l=q^{k_l}\,|\,q,t\right) \\ (BC_n,R): & P_\lambda^{(BC_n,B_n)}\left(x\,|\,a=q^{k_s},\,b=q^{2k_l}\,|\,q,t\right), \quad P_\lambda^{(BC_n,B_n)}\left(x\,|\,a=q^{k_s/2},\,b=q^{k_l}\,|\,q,t\right) \end{array} \tag{63}$$

We also want to point out that some Macdonald polynomials can be generalized with Koornwinder polynomial [**?**]. The details are in Appendix B.

2.5 Characters of classical simple Lie algebras

It is natural to use characters of irreducible representations of classical Lie algebras as the basis of Macdonald polynomials instead of (**??**). In this section we want to go into some details about characters of the classical Lie algebras and Schur functions.

2.5.1 Characters of the sl_N

Characters of the sl_N Lie algebra, which corresponds to the root system A_{N-1} are Schur symmetric functions

$$\chi_\lambda^{A_{N-1}} = S_\lambda(x_1,\ldots,x_N), \tag{64}$$

which can be defined with the Cauchy's bialternant formula, also called Weyl's formula:

$$S_\lambda(x_i,\ldots,x_N) = \frac{\det\left(x_i^{\lambda_j+N-j}\right)}{\det\left(x_i^{N-j}\right)}. \tag{65}$$

In fact all dependence on N in Schur functions can be hidden in variables p_k (**??**), which are power sum symmetric functions. It's common to denote Schur functions in these variables as $S_\lambda\{p_k\}$, they depend only on a partition λ.

2.5.2 Characters of the sp_{2n}

The Lie algebra sp_{2n} corresponds to the root system C_n. Characters of sp_{2n} are symplectic Schur functions Sp_λ [**?**, **?**]

$$\chi_\lambda^{C_n}(x_i,\ldots,x_n) = \frac{\det\left(x_i^{\lambda_j+n-j+1} - x_i^{-(\lambda_j+n-j+1)}\right)}{\det\left(x_i^{n-j+1} - x_i^{-(n-j+1)}\right)} = Sp_\lambda(x_i,\ldots,x_n). \tag{66}$$

They also can be expressed via the p_k variables, however in this case these variables are different:

$$p_k = \sum_{i=1}^n \left(x_i^k + x_i^{-k}\right). \tag{67}$$

2.5.3 Characters of the so_n

In the case of Lie algebra so_n, one has to distinguish between the cases of even and odd n. The root system B_n corresponds to so_{2n+1}, and its characters can be obtained with the following formula

$$\chi_\lambda^{B_n}(x_1,\dots,x_n) = \frac{\det\left(x_i^{\lambda_j+n-j+1/2} - x_i^{-(\lambda_j+n-j+1/2)}\right)}{\det\left(x_i^{n-j+1/2} - x_i^{-(n-j+1/2)}\right)}. \tag{68}$$

In the case of so_{2n} and the root system D_n, one gets

$$\chi_\lambda^{D_n}(x_1,\dots,x_n) = \frac{\det\left(x_i^{\lambda_j+n-j} + x_i^{-(\lambda_j+n-j)} - \delta_{j,n}\delta_{\lambda_n,0}\right)}{\det\left(x_i^{n-j} + x_i^{-(n-j)} - \delta_{j,n}\right)}, \quad \text{when} \quad \lambda_n = 0 \tag{69}$$

and

$$\chi_\lambda^{D_n}(x_1,\dots,x_n) = 2\left(\frac{\det\left(x_i^{\lambda_j+n-j} + x_i^{-(\lambda_j+n-j)} - \delta_{j,n}\delta_{\lambda_n,0}\right)}{\det\left(x_i^{n-j} + x_i^{-(n-j)} - \delta_{j,n}\right)} - \frac{\det\left(x_i^{\lambda_j+n-j} + x_i^{-(\lambda_j+n-j)}\right)}{\det\left(x_i^{n-j} + x_i^{-(n-j)}\right)}\right), \tag{70}$$

when $\lambda_n \neq 0$.

These characters can be rewritten with orthogonal Schur functions So_λ. It's also convenient to express them via p_k variables, however the two cases differ in the transition between x_i and p_k variables:

$$\chi_\lambda^{B_n} = So_\lambda\{p_k\}, \quad p_k = \sum(x_i^k + x_i^{-k}) + 1, \tag{71}$$

$$\chi_\lambda^{D_n} = So_\lambda\{p_k\}, \quad p_k = \sum(x_i^k + x_i^{-k}). \tag{72}$$

2.5.4 Schur functions

To conclude we provide some expressions for the symplectic and orthogonal Schur functions via the ordinary Schur functions. They are enumerated with partitions $\Lambda = [\Lambda_1,\dots,\Lambda_n]$, however we omit the brackets $[\dots]$ to shorten the notation.

Schur functions	Orthogonal Schurs functions	Symplectic Schur functions
S_1	S_1	S_1
S_2	$S_2 - 1$	S_2
S_{11}	S_{11}	$S_{11} - 1$
S_3	$S_3 - S_1$	S_3
S_{21}	$S_{21} - S_1$	$S_{21} - S_1$
S_{111}	S_{111}	$S_{111} - S_1$
S_4	$S_4 - S_2$	S_4
S_{31}	$-S_2 - S_{11} + S_{31} + 1$	$S_{31} - S_2$
S_{22}	$S_{22} - S_2$	$S_{22} - S_{11}$
S_{211}	$S_{211} - S_{11}$	$-S_2 - S_{11} + S_{211} + 1$
S_{1111}	S_{1111}	$S_{1111} - S_{11}$
S_5	$S_5 - S_3$	S_5
S_{41}	$S_1 - S_3 - S_{21} + S_{41}$	$S_{41} - S_3$
S_{32}	$S_1 - S_3 - S_{21} + S_{32}$	$S_{32} - S_{21}$
S_{311}	$S_1 - S_{21} - S_{111} + S_{311}$	$S_1 - S_3 - S_{21} + S_{311}$
S_{221}	$S_{221} - S_{21}$	$S_1 - S_{21} - S_{111} + S_{221}$
S_{2111}	$S_{2111} - S_{111}$	$S_1 - S_{21} - S_{111} + S_{2111}$
S_{11111}	S_{11111}	$S_{11111} - S_{111}$

(73)

We stress once again that these relations are written for variables p_k, which differ for different root systems:

$$A_n\,(sl(n+1)): \qquad S\{p_k\}, \qquad p_k = \sum x_i^k, \tag{74}$$

$$B_n\,(so(2n+1)): \qquad So\{p_k\}, \qquad p_k = \sum(x_i^k + x_i^{-k}) + 1, \tag{75}$$

$$C_n\,(sp(2n)): \qquad Sp\{p_k\}, \qquad p_k = \sum(x_i^k + x_i^{-k}), \tag{76}$$

$$D_n\,(so(2n)): \qquad So\{p_k\}, \qquad p_k = \sum(x_i^k + x_i^{-k}). \tag{77}$$

3 Quantum and Macdonald dimensions

Quantum dimensions naturally arise in Chern-Simons theory as Wilson averages for unknotted loop and in knot theory as quantum invariants of the unknot. Since the refinement of Chern-Simons theory is completed with the transition from Schur symmetric functions to Macdonald polynomials, we define Macdonald dimensions with Macdonald polynomials at a special point, analogously to quantum dimensions. In the Table **??** we list the main notions of this section.

3.1 Weyl vectors

Quantum and Macdonald dimensions are defined with the help of Weyl vector and its variations:

$$\rho = \frac{1}{2}\sum_{\alpha>0}\alpha, \tag{78}$$

$$\rho_k = \frac{1}{2}\sum_{\alpha>0}k_\alpha\,\alpha, \tag{79}$$

$$\rho_k^* = \frac{1}{2}\sum_{\alpha>0}k_\alpha\,\alpha_* = \frac{1}{2}\sum_{\alpha>0}k_\alpha\,\alpha/u_\alpha, \tag{80}$$

$$r = \frac{1}{2}\sum_{\alpha>0}\alpha^\vee, \tag{81}$$

$$r_k = \frac{1}{2}\sum_{\alpha>0}k_\alpha\,\alpha^\vee, \tag{82}$$

$$r_k^* = \frac{1}{2}\sum_{\alpha>0}k_\alpha\,\alpha^* = \frac{1}{2}\sum_{\alpha>0}k_\alpha u_\alpha\,\alpha^\vee, \tag{83}$$

where parameter k_α depends only on the length of the root (α,α) and parameter u_α depends on the admissible pair (R,S) and was defined in the previous section (**??**).

3.1.1 A_n

Coordinates of Weyl vectors of root system A_n in the orthogonal basis are the following:

$$\begin{aligned} \left(\rho^{A_n}\right)_i = \left(r^{A_n}\right)_i &= n/2-(i-1), \\ \left(\rho_k^{A_n}\right)_i = \left(r_k^{A_n}\right)_i &= kn/2-k(i-1). \end{aligned} \tag{84}$$

3.1.2 B_n

Coordinates of Weyl vectors of root system B_n in the orthogonal basis are the following:

$$\begin{aligned} \left(\rho^{B_n}\right)_i &= (n-i)+1/2, \\ \left(r^{B_n}\right)_i &= (n-i)+1, \\ \left(\rho_k^{B_n}\right)_i &= k(n-i)+k_s/2, \\ \left(r_k^{B_n}\right)_i &= k(n-i)+k_s. \end{aligned} \tag{85}$$

3.1.3 C_n

Coordinates of Weyl vectors of root system C_n in the orthogonal basis are the following:

$$\begin{aligned} \left(\rho^{C_n}\right)_i &= (n-i)+1, \\ \left(r^{C_n}\right)_i &= (n-i)+1/2, \\ \left(\rho_k^{C_n}\right)_i &= k(n-i)+k_l, \\ \left(r_k^{C_n}\right)_i &= k(n-i)+k_l/2. \end{aligned} \tag{86}$$

3.1.4 D_n

Coordinates of Weyl vectors of root system D_n in the orthogonal basis are the following:

$$\begin{aligned} \left(\rho^{D_n}\right)_i &= \left(r^{D_n}\right)_i = (n-i), \\ \left(\rho_k^{D_n}\right)_i &= \left(r_k^{D_n}\right)_i = k(n-i). \end{aligned} \tag{87}$$

3.2 Quantum and dual quantum dimensions

In representation theory, quantum dimensions are the characters of irreducible representations of Lie algebras at the special point $x = q^{2\rho}$, where ρ is the Weyl vector (**??**). Characters of the Lie algebras χ_λ^R do factorize [**?**, expr. 13.170] at the Weyl vector:

$$\mathrm{qD}_\lambda^R := \chi_\lambda^R \left(x = q^{2\rho}\right) = \prod_{\alpha \in R_+} \frac{[(\alpha, \lambda+\rho)]_q}{[(\alpha,\rho)]_q}, \tag{88}$$

where $(\cdot,\cdot)$ is the scalar product in the Euclidean space where the root system is embedded.

Characters of simple Lie algebras also factorize at another point, which we call the dual Weyl vector r (**??**), which is a half sum over all positive coroots. Hence, we define dual quantum dimension ${}^\vee\mathrm{qD}_\lambda^R$ as a character at the point q^{2r}:

$${}^\vee\mathrm{qD}_\lambda^R := \chi_\lambda^R \left(x = q^{2r}\right) = \prod_{\alpha \in R_+} \prod_{j=1}^{(\alpha^\vee,\lambda)} \frac{[(\rho,\alpha^\vee)+j]_q}{[(\rho,\alpha^\vee)+j-1]_q}. \tag{89}$$

3.3 Macdonald and dual Macdonald dimensions

Analogously to quantum dimensions, we define their refined version based on Macdonald polynomials and call them Macdonald dimensions. Macdonald dimensions are Macdonald polynomials at the refined Weyl vector (**??**):

$$\mathrm{Md}_\lambda^{(R,S)} = P_\lambda^{(R,S)}(x = q^{2\rho_k} \,|\, t_\alpha^2 \,|\, q^2, t^2). \tag{90}$$

Turns out that the Macdonald polynomials do not factorize at refined Weyl vector ρ_k, but they do factorize and the factorization point is the dual refined Weyl vector r_k^* (**??**), that depends on parameters k_α and u_α [**?**] (conj. 12.10):

$$\boxed{P_\lambda^{(R,S)}(x = q^{2r_k^*} \,|\, t_\alpha^2 \,|\, q^2, t^2) = \prod_{\alpha \in R_+} \prod_{j=1}^{(\alpha^\vee,\lambda)} \frac{\left\{t_{\alpha/2}\, t_\alpha\, q_\alpha^{(\rho_k,\alpha^\vee)+j-1}\right\}}{\left\{t_{\alpha/2}\, q_\alpha^{(\rho_k,\alpha^\vee)+j-1}\right\}},} \tag{91}$$

where $t_{\alpha/2} = 1$ if $\alpha/2 \notin R$.

We call Macdonald polynomials in their factorization point dual Macdonald dimensions:

$${}^\vee\mathrm{Md}_\lambda^{(R,S)} = P_\lambda^{(R,S)}(x = q^{2r_k^*} \,|\, t_\alpha^2 \,|\, q^2, t^2). \tag{92}$$

The distinction between r_k and r_k^* is only important when the admissible pair consists of two different root systems $R \neq S$. We discuss such dual Macdonald dimensions in section **??**.

When $R = S$ we get $r_k^* = r_k$ and denote such dual Macdonald dimensions with one root system:

$${}^\vee\mathrm{Md}_\lambda^R = P_\lambda^R(x = q^{2r_k} \,|\, t_\alpha^2 \,|\, q^2, t^2) = \prod_{\alpha \in R_+} \prod_{j=1}^{(\alpha^\vee,\lambda)} \frac{\left\{t_\alpha\, q^{(\rho_k,\alpha^\vee)+j-1}\right\}}{\left\{q^{(\rho_k,\alpha^\vee)+j-1}\right\}}. \tag{93}$$

Refined Weyl vectors ρ_k and r_k do coincide for simply laced root systems, that is why Macdonald dimensions $\mathrm{Md}_\lambda^{A_n}$, $\mathrm{Md}_\lambda^{D_n}$ and $\mathrm{Md}_\lambda^{E_6}$, $\mathrm{Md}_\lambda^{E_7}$, $\mathrm{Md}_\lambda^{E_8}$ coincide with their dual versions, do factorize and can be universalized.

3.4 On the root normalization

We have already mentioned that Vogel parameters (Table ??) are written for the minimal normalization of roots, which means that the length of the longest root of the root system α_l equals to 2:

$$(\alpha_l, \alpha_l) = 2. \tag{94}$$

Roots that we used in calculations satisfy this requirement except for the root systems with one long root: C_n and G_2.

In order to go to minimal normalization one can rescale the scalar product:

$$(\cdot, \cdot) \to a(\cdot, \cdot), \tag{95}$$

then

$$\alpha \to \alpha, \qquad \rho \to \rho, \qquad \lambda \to \lambda, \tag{96}$$

$$\alpha^\vee \to \frac{1}{a}\alpha^\vee, \qquad r \to \frac{1}{a}r \tag{97}$$

We want to point out that dual versions of quantum and Macdonald dimensions ${}^\vee\mathrm{qD}^R_\lambda$ and ${}^\vee\mathrm{Md}^R_\lambda$ do not depend on the choice of normalization of roots:

$${}^\vee\mathrm{qD}^R_\lambda \to {}^\vee\mathrm{qD}^R_\lambda, \qquad {}^\vee\mathrm{Md}^R_\lambda \to {}^\vee\mathrm{Md}^R_\lambda, \tag{98}$$

but ordinary quantum and Macdonald dimensions do depend on the choice of normalization. For quantum dimensions one gets

$$\mathrm{qD}^R_\lambda \to \prod_{\alpha \in R_+} \frac{[a(\alpha, \lambda + \rho)]_q}{[a(\alpha, \rho)]_q}, \tag{99}$$

which can be corrected with the rescaling of q: $q \to q^a$. Expressions X that are not in minimal normalization we denote with overline $\overline{X}$.

3.5 Quantum and Macdonald dimensions in adjoint representation

In this section, we provide explicit expressions for quantum dimensions $\mathrm{qD}^R_{\mathrm{Adj}}$, dual quantum dimensions ${}^\vee\mathrm{qD}^R_{\mathrm{Adj}}$, dual Macdonald dimensions ${}^\vee\mathrm{Md}^R_{\mathrm{Adj}}$ and some Macdonald dimensions $\mathrm{Md}^R_{\mathrm{Adj}}$ corresponding to adjoint representations of simple Lie algebras.

3.5.1 A_n (sl_{n+1})

The highest weight of adjoint representation (in the orthogonal basis)

$$\lambda^{A_n}_{\mathrm{Adj}} = [1, 0, 0, \dots, 0, -1] = \omega_1 + \omega_n, \tag{100}$$

ω_1 and ω_n are the first and the last fundamental weights respectively. It corresponds to the partition $[2, 1^{n-1}]$ ($[2]$ for $sl(2)$, $[2, 1]$ for $sl(3)$, $[2, 1, 1]$ for $sl(4)$ and so on). The quantum dimension in the adjoint representation coincides with the dual quantum dimension:

$$\mathrm{qD}^{A_n}_{\mathrm{Adj}} = [n+2]_q [n]_q = {}^\vee\mathrm{qD}^{A_n}_{\mathrm{Adj}}. \tag{101}$$

The Macdonald dimension and the dual Macdonald dimension also coincide and in the adjoint representation are the following:

$$\boxed{\mathrm{Md}^{A_n}_{\mathrm{Adj}} = \frac{\{t^{n+1}\}\{q\, t^{n+1}\}}{\{t^n\}\{q\, t^n\}} \prod_{j=2}^{n} \frac{\{t^j\}}{\{t^{j-1}\}} \frac{\{t^{n-j+2}\}}{\{t^{n-j+1}\}} = \frac{\{t^n\}\{t^{n+1}\}\{q\, t^{n+1}\}}{\{t\}^2\{q\, t^n\}}.} \tag{102}$$

3.5.2 B_n (so_{2n+1})

The highest weight of adjoint representation (in the orthogonal basis)

$$\lambda_{\text{Adj}}^{B_n} = [1,1,0,0,\dots] = a\,\omega_2, \tag{103}$$

where $a=2$ for root system B_2 and $a=1$ for all other B_n, ω_2 is the second fundamental weight. It corresponds to the partition $[1,1]$. The quantum dimension and the dual quantum dimension in the adjoint representation:

$$\text{qD}_{\text{Adj}}^{B_n} = \frac{[n+1/2]_q[2n]_q[2n-3]_q}{[n-3/2]_q[2]_q}, \quad {}^{\vee}\text{qD}_{\text{Adj}}^{B_n} = \frac{[2n]_q[2n+1]_q}{[2]_q}. \tag{104}$$

The dual Macdonald dimension in the adjoint representation:

$$\boxed{{}^{\vee}\text{Md}_{\text{Adj}}^{B_n} = \frac{[n]_t[n-1]_t}{[2]_t}\frac{\{t^{2(n-2)}\,t_s^2\}\{q\,t^{2(n-2)}\,t_s^2\}\{t^{2(n-1)}\,t_s^2\}\{q\,t^{2(n-1)}\,t_s^2\}}{\{t^{n-2}\,t_s\}\{q\,t^{2(n-2)}\,t_s\}\{t^{n-1}\,t_s\}\{q\,t^{2n-3}\,t_s\}}.} \tag{105}$$

The Macdonald dimension in this case looks more complicated and does not factorize:

$$\begin{aligned}\text{Md}_{\text{Adj}}^{B_n} = \frac{\{t^n\}}{\{t\}}\Bigg(&\frac{\{t_s\}}{\{t\}}\frac{\{t^{n-1}\}}{\{\xi_n\}}\{t_s\,t^{n-1}\}_+\{qt^{n-2}\}_+ - \frac{\{q\,t^{n-2}\}_+}{\{t\}_+}\frac{\{t^{n-1}\}\{t_s^2\,q^{-1}t^{-1}\}}{\{qt^{n-1}\}\{t_s\,t\xi_n\}} + \\ &+\frac{\{t^{n-1}\}\{t_s\}\{t\,t_s\}\{t_s\xi_n\}}{\{t^2\}\{t\xi_n\}\{\xi_n\}\{t_st\xi_n\}}\{q^2\,t^{2(n-1)}\}_+\{q^2\,t^{2(n-2)}\}_+ - \frac{\{q\}}{\{q\,t^{n-1}\}} - \frac{1}{2}\{t_st^{n-1}\}_+^2 - \frac{1}{2}\frac{\{t^n\}_+}{\{t\}_+}\{t_s^2t^{2(n-1)}\}_+\Bigg),\end{aligned}$$

where $\xi_n := qt_st^{2(n-2)}$.

3.5.3 C_n (sp_{2n})

The highest weight of the adjoint representation (in the orthogonal basis)

$$\lambda_{\text{Adj}}^{C_n} = [2,0,\dots] = 2\,\omega_1, \tag{106}$$

where ω_1 is the first fundamental weight. It corresponds to the partition $[2]$.

The quantum dimension and the dual quantum dimension in the adjoint representation:

$$\overline{\text{qD}}_{\text{Adj}}^{C_n} = \frac{[n]_q[2n+1]_q[2n+4]_q}{[n+2]_q[2]_q}, \quad {}^{\vee}\text{qD}_{\text{Adj}}^{C_n} = \frac{[2n]_q[2n+1]_q}{[2]_q}. \tag{107}$$

If one uses minimal normalization, the quantum dimensions of C_n is the following (this formula is included in universal formula for quantum dimensions (??)):

$$\text{qD}_{\text{Adj}}^{C_n} = \frac{[n/2]_q[n+1/2]_q[n+2]_q}{[(n+2)/2]_q[1/2]_q}. \tag{108}$$

The dual Macdonald dimension in adjoint representation:

$$\boxed{{}^{\vee}\text{Md}_{\text{Adj}}^{C_n} = \frac{\{t^n\}\{q\,t^n\}\{t^{2(n-1)}\,t_l^2\}\{q\,t^{2(n-1)}\,t_l^2\}}{\{t\}\{q\,t\}\{t^{n-1}\,t_l\}\{q\,t^{n-1}\,t_l\}}.} \tag{109}$$

And Macdonald dimension (overline means that this answer is not in the minimal normalization):

$$\overline{\text{Md}}_{Adj}^{C_n} = P_{Adj}^{C_n}\left(x = q^{2\rho_k}\,|\,t_l^2\,|\,q^2,t^2\right) = [n]_t\frac{\{qt^n\}}{\{qt\}}\left(\frac{\{t_l^3\xi_n^3\}}{\{t_l\xi_n\}} + \frac{\{q^{-1}\xi_n\}}{\{q\xi_n\}}\right), \quad \xi_n = t^{n-1}t_l. \tag{110}$$

We list more examples of Macdonald dimensions in Appendix C.

3.5.4 D_n (so_{2n})

The highest weight of adjoint representation (in the orthogonal basis)

$$\lambda^{D_n}_{\mathrm{Adj}} = [1,1,0,0,\dots], \tag{111}$$

it corresponds to Young diagram $[1,1]$. Quantum dimension in adjoint representation coincides with dual quantum dimension:

$$\mathrm{qD}^{D_n}_{\mathrm{Adj}} = \frac{[n]_q[2n-1]_q[2n-4]_q}{[n-2]_q[2]_q} = {}^{\vee}\mathrm{qD}^{D_n}_{\mathrm{Adj}}. \tag{112}$$

and the Macdonald dimension in the adjoint representation, which coincides with the dual Macdonald dimension:

$$\boxed{\mathrm{Md}^{D_n}_{Adj} = \frac{\{t^n\}\{t^{2n-4}\}\{t^{2n-2}\}\{q\,t^{2n-2}\}}{\{t\}\{t^2\}\{t^{n-2}\}\{q\,t^{2n-3}\}}.} \tag{113}$$

3.5.5 E_6

$$\mathrm{qD}^{E_6}_{\mathrm{Adj}} = \frac{[8]_q[9]_q[13]_q}{[3]_q[4]_q} = {}^{\vee}\mathrm{qD}^{E_6}_{\mathrm{Adj}}, \tag{114}$$

$$\mathrm{Md}^{E_6}_{\mathrm{Adj}} = \frac{\{t^8\}\{t^9\}\{t^{12}\}\{q\,t^{12}\}}{\{t\}\{t^3\}\{t^4\}\{q\,t^{11}\}}. \tag{115}$$

3.5.6 E_7

$$\mathrm{qD}^{E_7}_{\mathrm{Adj}} = \frac{[12]_q[14]_q[19]_q}{[4]_q[6]_q} = {}^{\vee}\mathrm{qD}^{E_7}_{\mathrm{Adj}}, \tag{116}$$

$$\mathrm{Md}^{E_7}_{\mathrm{Adj}} = \frac{\{t^{12}\}\{t^{14}\}\{t^{18}\}\{q\,t^{18}\}}{\{t\}\{t^4\}\{t^6\}\{q\,t^{17}\}}. \tag{117}$$

3.5.7 E_8

$$\mathrm{qD}^{E_8}_{\mathrm{Adj}} = \frac{[20]_q[24]_q[31]_q}{[6]_q[10]_q} = {}^{\vee}\mathrm{qD}^{E_8}_{\mathrm{Adj}}, \tag{118}$$

$$\mathrm{Md}^{E_8}_{\mathrm{Adj}} = \frac{\{t^{20}\}\{t^{24}\}\{t^{30}\}\{q\,t^{30}\}}{\{t\}\{t^6\}\{t^{10}\}\{q\,t^{29}\}}. \tag{119}$$

3.5.8 F_4

$$\mathrm{qD}^{F_4}_{\mathrm{Adj}} = \frac{[10]_q[13/2]_q[6]_q}{[3]_q[5/2]_q}, \quad {}^{\vee}\mathrm{qD}^{F_4}_{\mathrm{Adj}} = \frac{[8]_q[12]_q[13]_q}{[4]_q[6]_q}, \tag{120}$$

$${}^{\vee}\mathrm{Md}^{F_4}_{\mathrm{Adj}} = \frac{\{t^3\}\{t^4t_s^2\}\{t^4t_s^4\}\{t^6t_s^6\}\{q\,t^4t_s^4\}\{q\,t^6t_s^6\}}{\{t\}\{t^2t_s\}\{t^2t_s^2\}\{t^3t_s^3\}\{q\,t^4t_s\}\{q\,t^5t_s^3\}}. \tag{121}$$

3.5.9 G_2

$$\overline{\mathrm{qD}}^{G_2}_{\mathrm{Adj}} = \frac{[8]_q[7]_q[15]_q}{[3]_q[4]_q[5]_q}, \quad {}^{\vee}\mathrm{qD}^{G_2}_{\mathrm{Adj}} = \frac{[7]_q[8]_q}{[4]_q}. \tag{122}$$

In minimal normalization the quantum dimension of G_2 is the following (this formula is included in universal formula for quantum dimensions (**??**)):

$$\mathrm{qD}^{G_2}_{\mathrm{Adj}} = \frac{[8/3]_q[7/3]_q[5]_q}{[4/3]_q[5/3]_q}. \tag{123}$$

The dual Macdonald dimension:

$${}^{\vee}\mathrm{Md}^{G_2}_{\mathrm{Adj}} = \frac{\{t_3^2\}\{t^3t_3^3\}\{q\,t^3t_3^3\}\{q^2\,t^3t_3^3\}}{\{t_3\}\{t\,t_3\}\{q\,t\,t_3^2\}\{q^2\,t\,t_3^3\}}. \tag{124}$$

4 Mixed Macdonald polynomials and their dimensions

In this section we consider Macdonald polynomials associated with two different root systems (R, S). We discuss parameters of these polynomials and their factorization.

4.1 Factorization formula

Macdonald polynomials depend on a pair of root systems (R, S). When $R \neq S$ we get different q_α:

$$q_\alpha = q^{u_\alpha}, \tag{125}$$

where for each $\alpha \in R$ there exists a unique $u_\alpha \in R_+$ such that

$$\alpha_* := \alpha / u_\alpha \in S,\ \ \alpha^* = (\alpha_*)^\vee = u_\alpha \alpha^\vee. \tag{126}$$

The factorization of Macdonald polynomials associated with root systems in the case of general admissible pair (R, S) occurs at the variation of the Weyl vector that involves parameters k_α and u_α:

$$r_k^* = \frac{1}{2}\sum_{\alpha>0} k_\alpha \alpha^* = \frac{1}{2}\sum_{\alpha>0} k_\alpha u_\alpha \alpha^\vee. \tag{127}$$

The formula for Macdonald factorization is the following:

$$P_\lambda^{(R,S)}\left(x = q^{2r_k^*} \,|\, t_\alpha^2 \,|\, q^2, t^2\right) = \prod_{\alpha \in R_+} \prod_{j=1}^{(\alpha^\vee,\lambda)} \frac{\left\{t_{\alpha/2}\, t_\alpha\, q_\alpha^{(\rho_k,\alpha^\vee)+j-1}\right\}}{\left\{t_{\alpha/2}\, q_\alpha^{(\rho_k,\alpha^\vee)+j-1}\right\}}. \tag{128}$$

4.2 (B_n, C_n)

There are two clusters of positive roots (**??**) in the case of the root system B_n: $|\alpha|^2 = |\varepsilon \pm \varepsilon|^2 = 2$ and $|\alpha|^2 = |\varepsilon|^2 = 1$, hence

$$\begin{array}{llll} u_{|\alpha|^2=2} = 1, & k_{|\alpha|^2=2} = k, & q_{|\alpha|^2=2} = q, & t_{|\alpha|^2=2} = t = q^k, \\ u_{|\alpha|^2=1} = 1/2, & k_{|\alpha|^2=1} = k_s, & q_{|\alpha|^2=1} = q^{1/2}, & t_{|\alpha|^2=1} = t_s = q^{k_s/2}, \\ & & & t_{\alpha/2} = 1 \ \forall\ \alpha \in B_n. \end{array} \tag{129}$$

$$r_k^* = k(n-i) + k_s/2 \tag{130}$$

and factorization occurs at the point

$$q^{2r_k^*}: \quad x_i = t^{2(n-i)}\, t_s^2. \tag{131}$$

In the representation $\mathrm{Adj}^{B_n} = [1,1]$ the dual Macdonald dimension is the following:

$$\begin{gathered} {}^\vee\mathrm{Md}_{\mathrm{Adj}}^{(B_n,C_n)} = P_{\mathrm{Adj}}^{(B_n,C_n)}\left(x = q^{2r_k^*} \,|\, t_s^2 \,|\, q^2, t^2\right) = \\ \frac{[n]_t[n-1]_t}{[2]_t} \frac{\{t^{2(n-2)}\, t_s^2\}\{t^{2(n-1)}\, t_s^2\}\{q\, t^{2(n-1)}\, t_s^2\}\{q^{1/2}\, t^{n-1}\, t_s^2\}\{q^{1/2}\, t^{n-2}\, t_s^2\}}{\{t^{n-2}\, t_s\}\{t^{n-1}\, t_s\}\{q\, t^{2n-3}\, t_s\}\{q^{1/2}\, t^{n-1}\, t_s\}\{q^{1/2}\, t^{n-2}\, t_s\}}. \end{gathered} \tag{132}$$

When $k = k_s = 1$ ($t = q$ and $t_s = q^{1/2}$) ${}^\vee\mathrm{Md}_{\mathrm{Adj}}^{(B_n,C_n)}$ goes to

$${}^\vee\mathrm{Md}_{\mathrm{Adj}}^{(B_n,C_n)} \xrightarrow{t\to q,\, t_s\to q^{1/2}} \frac{[2n]_q[2n-3]_q[n+1/2]_q}{[2]_q[n-3/2]_q}, \tag{133}$$

when $t = q$ and $t_s = q$ ${}^\vee\mathrm{Md}_{\mathrm{Adj}}^{(B_n,C_n)}$ goes to

$${}^\vee\mathrm{Md}_{\mathrm{Adj}}^{(B_n,C_n)} \xrightarrow{t\to q,\, t_s\to q} \frac{[2(n-1)]_q[2n+1]_q[n+3/2]_q}{[2]_q[n-1/2]_q}. \tag{134}$$

4.3 (C_n, B_n)

There are two groups of positive roots (**??**) in the case of the root system C_n: $|\alpha|^2 = |\varepsilon \pm \varepsilon|^2 = 2$ and $|\alpha|^2 = |2\varepsilon|^2 = 4$.

$$\begin{array}{llll} u_{|\alpha|^2=2} = 1, & k_{|\alpha|^2=2} = k, & q_{|\alpha|^2=2} = q, & t_{|\alpha|^2=2} = t = q^k \\ u_{|\alpha|^2=4} = 2, & k_{|\alpha|^2=4} = k_l & q_{|\alpha|^2=4} = q^2 & t_{|\alpha|^2=4} = t_l = q^{2k_l} \\ & & & t^2_{\alpha/2} = 1 \ \forall\ \alpha \in C_n. \end{array} \tag{135}$$

$$r_k^* = k(n-i) + k_l. \tag{136}$$

The factorization occurs at the point (which exactly coincides with factorization point for polynomials $P_\lambda^{C_n}$):

$$q^{2r_k^*}: \quad x_i = t^{2(n-i)}\, t_l. \tag{137}$$

In the adjoint representation $\mathrm{Adj}^{C_n} = [2]$:

$${}^\vee\mathrm{Md}_{\mathrm{Adj}}^{(C_n,B_n)} = P_{\mathrm{Adj}}^{(C_n,B_n)}\left(x = q^{2r_k^*}\,|\,t_l^2\,|\,q^2,t^2\right) = [n]_t \frac{\{q\,t^n\}}{\{q\,t\}} \frac{\{q\,t^{2(n-1)}\,t_l^2\}}{\{q\,t^{(n-1)}\,t_l\}} \frac{\{t^{2(n-1)}\,t_l^2\}}{\{\,t^{n-1}\,t_l\}}, \tag{138}$$

which goes to

$${}^\vee\mathrm{Md}_{\mathrm{Adj}}^{(C_n,B_n)} \xrightarrow{t\to q,\, t_l\to q^2} \frac{[2n+3]_q[2n+2]_q[n]_q}{[2]_q[n+2]_q} \tag{139}$$

and

$${}^\vee\mathrm{Md}_{\mathrm{Adj}}^{(C_n,B_n)} \xrightarrow{t\to q,\, t_l\to q} \frac{[2n]_q[2n+1]_q}{[2]_q}. \tag{140}$$

In representation $[1,1]$ factorization is the following:

$${}^\vee\mathrm{Md}_{[1,1]}^{(C_n,B_n)} = P_{[1,1]}^{(C_n,B_n)}\left(x = q^{2r_k^*}\,|\,t_l^2\,|\,q^2,t^2\right) = \frac{[n]_t[n-1]_t}{[2]_t} \frac{\{t^{2(n-2)}t_l^2\}}{\{t^{n-2}t_l\}} \frac{\{q\,t^{2(n-1)}t_l\}}{\{t^{n-1}t_l\}} \frac{\{t^{2(n-1)}t_l^2\}}{\{q\,t^{2n-3}t_l\}}. \tag{141}$$

When $t = q$ and $t_l = q^2$ ${}^\vee\mathrm{Md}_{[1,1]}^{(C_n,B_n)}$ goes to

$${}^\vee\mathrm{Md}_{[1,1]}^{(C_n,B_n)} \xrightarrow{t\to q,\, t_l\to q^2} \frac{[2n+2]_q[2n+1]_q[n-1]_q}{[2]_q[n+1]_q} \tag{142}$$

and for $t = q$ and $t_l = q$

$${}^\vee\mathrm{Md}_{[1,1]}^{(C_n,B_n)} \xrightarrow{t\to q,\, t_l\to q} \frac{[2n]_q^2[2(n-1)]_q}{[2]_q[2n-1]_q}. \tag{143}$$

4.4 (BC_n, B_n)

The root system BC_n is not reduced and has the following positive roots:

$$2\varepsilon_i, \quad \varepsilon_i, \quad \varepsilon_i + \varepsilon_j, \quad \varepsilon_i - \varepsilon_j, \quad \text{where} \quad i,j = \overline{1,n}, \quad i<j, \tag{144}$$

so it is a combination of roots of the root systems B_n and C_n.

There are three groups of roots in the case of the root system BC_n: $|\alpha|^2 = |\varepsilon \pm \varepsilon|^2 = 2$, $|\alpha|^2 = |2\varepsilon|^2 = 4$ and $|\alpha|^2 = |\varepsilon|^2 = 1$. In the case of the admissible pair (BC_n, B_n) the Macdonald parameters are the following:

$$\begin{array}{lllll} u_{|\alpha|^2=2} = 1, & k_{|\alpha|^2=2} = k, & q_{|\alpha|^2=2} = q, & t_{|\alpha|^2=2} = t = q^k, & t_{(\varepsilon\pm\varepsilon)/2} = 1 \\ u_{|\alpha|^2=4} = 2, & k_{|\alpha|^2=4} = k_l & q_{|\alpha|^2=4} = q^2 & t_{|\alpha|^2=4} = b = q^{2k_l} & t_{2\varepsilon/2} = a, \\ u_{|\alpha|^2=1} = 1, & k_{|\alpha|^2=1} = k_s, & q_{|\alpha|^2=1} = q, & t_{|\alpha|^2=1} = a = q^{k_s}, & t_{\varepsilon/2} = 1. \end{array} \tag{145}$$

We use a and b instead of t_s and t_l as it was done in Koornwinder's original work [**?**].

$$r_k^* = k(n-i) + k_l + k_s \tag{146}$$

The factorization occurs at the point:

$$q^{2r_k^*}: \quad x_i = t^{2(n-i)}\, a^2\, b. \tag{147}$$

In representation $[1,1]$:

$$ {}^{\vee}\mathrm{Md}_{[1,1]}^{(BC_n,B_n)} = P_{[1,1]}^{(BC_n,B_n)}\left(x = q^{2r_k^*}\,|\,a^2,b^2\,|\,q^2,t^2\right) = \\ [n]_t[n-1]_t[2]_t^{-1}\frac{\{a^2\,b^2\,t^{2(n-2)}\}\{a^2\,b^2\,t^{2(n-1)}\}}{\{a\,b\,t^{n-2}\}\{a\,b\,t^{n-1}\}}\frac{\{q\,a^2\,b\,t^{2(n-2)}\}\{q\,a^2\,b\,t^{2(n-1)}\}}{\{q\,a\,b\,t^{2(n-2)}\}\{q\,a\,b\,t^{2n-3}\}}. \tag{148}$$

It can be reduced to the following expressions:

$$ {}^{\vee}\mathrm{Md}_{[1,1]}^{(BC_n,B_n)} \xrightarrow{t\to q,\,a\to q,\,b\to q^2} \frac{[2n+4]_q[2n+3]_q[2n+2]_q[n]_q[n-1]_q}{[2n]_q[n+2]_q[n+1]_q[2]_q}, \tag{149}$$

$$ {}^{\vee}\mathrm{Md}_{[1,1]}^{(BC_n,B_n)} \xrightarrow{t\to q,\,a\to q,\,b\to q} \frac{[2n+2]_q^2[2n]_q[n-1]_q}{[2n-1]_q[n+1]_q[2]_q}. \tag{150}$$

4.5 (BC_n, C_n)

In the case of the admissible pair (BC_n, C_n) the Macdonald parameters are the following:

$$\begin{array}{lllll} u_{|\alpha|^2=2} = 1, & k_{|\alpha|^2=2} = k, & q_{|\alpha|^2=2} = q, & t_{|\alpha|^2=2} = t = q^k, & t_{(\varepsilon\pm\varepsilon)/2} = 1 \\ u_{|\alpha|^2=4} = 1, & k_{|\alpha|^2=4} = k_l & q_{|\alpha|^2=4} = q & t_{|\alpha|^2=4} = b = q^{k_l} & t_{2\varepsilon/2} = a, \\ u_{|\alpha|^2=1} = 1/2, & k_{|\alpha|^2=1} = k_s, & q_{|\alpha|^2=1} = q^{1/2}, & t_{|\alpha|^2=1} = a = q^{k_s/2}, & t_{\varepsilon/2} = 1. \end{array} \tag{151}$$

$$ r_k^* = k(n-i) + k_l/2 + k_s/2. \tag{152}$$

The factorization occurs at the point (which is exactly the same as in the case of (BC_n, B_n)):

$$ q^{2r_k^*}: \quad x_i = t^{2(n-i)}\,a^2\,b. \tag{153}$$

In representation $[1,1]$:

$$ {}^{\vee}\mathrm{Md}_{[1,1]}^{(BC_n,C_n)} = P_{[1,1]}^{(BC_n,C_n)}\left(x = q^{2r_k^*}\,|\,a^2,b^2\,|\,q^2,t^2\right) = \\ [n]_t[n-1]_t[2]_t^{-1}\frac{\{a^2\,b^2\,t^{2(n-2)}\}\{a^2\,b^2\,t^{2(n-1)}\}\{q\,a^2\,b^2\,t^{2(n-1)}\}}{\{a\,b\,t^{n-2}\}\{a\,b\,t^{n-1}\}\{q^{1/2}\,a\,b\,t^{n-1}\}}\frac{\{q^{1/2}\,a^2\,b\,t^{n-2}\}\{q^{1/2}\,a^2\,b\,t^{n-1}\}}{\{q^{1/2}\,a\,b\,t^{n-2}\}\{q\,a^2\,b^2\,t^{2n-3}\}}. \tag{154}$$

It can be reduced to the following expressions:

$$ {}^{\vee}\mathrm{Md}_{[1,1]}^{(BC_n,C_n)} \xrightarrow{t\to q,\,a\to q^{1/2},\,b\to q} \frac{[2n+2]_q[2n-1]_q[n+3/2]_q[n-1]_q}{[n+1]_q[n-1/2]_q[2]_q}, \tag{155}$$

$$ {}^{\vee}\mathrm{Md}_{[1,1]}^{(BC_n,C_n)} \xrightarrow{t\to q,\,a\to q,\,b\to q} \frac{[2n+3]_q[2n]_q[n+5/2]_q[n-1]_q}{[n+1]_q[n+1/2]_q[2]_q}. \tag{156}$$

5 Universality in Chern-Simons theory and its refinement

In this section we start with the simplest universality formula — the universal dimension formula for adjoint representations — and see how it transforms into the universal quantum dimension formula and finally to universal refined quantum dimension formula — universal Macdonald dimension. However, universality in the refined case holds only for Lie algebras associated with simply laced root systems.

Parallel to quantum and Macdonald dimensions exist their dual versions: dual quantum and dual Macdonald dimensions. We discuss the possibility of universalization in case of dual quantum dimensions. Naturally the dual Macdonald dimensions also universalize in the simply-laced case, because in this case they coincide with Macdonald dimensions.

5.1 Universality in Chern-Simons theory. Quantum dimensions and dual quantum dimensions

The first universal formula is the formula for dimensions of adjoint representations [?] for all simple Lie algebras:

$$ \boxed{D_{\mathrm{Adj}}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{(\mathfrak{a}-2\mathfrak{t})(\mathfrak{b}-2\mathfrak{t})(\mathfrak{c}-2\mathfrak{t})}{\mathfrak{a}\mathfrak{b}\mathfrak{c}}.} \tag{157}$$

In Chern-Simons theory the dimension of representations transforms into the **quantum dimension** qD_λ, which is the Wilson average of unknotted loop (colored HOMFLY-PT invariant of the unknot in knot theory). In representation theory, quantum dimension is a character of the Lie algebras at the Weyl vector. Quantum dimensions (**??**) were universalized in the adjoint representation [**?**, **?**]:

$$\boxed{\mathrm{qD}_{\mathrm{Adj}}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{[\mathfrak{a}/2-\mathfrak{t}]_q[\mathfrak{b}/2-\mathfrak{t}]_q[\mathfrak{c}/2-\mathfrak{t}]_q}{[\mathfrak{a}/2]_q[\mathfrak{b}/2]_q[\mathfrak{c}/2]_q}.} \tag{158}$$

Characters of simple Lie algebras also factorize at another point, which we call the dual Weyl vector r. We defined **dual quantum dimension** ${}^\vee\mathrm{qD}_\lambda^R$ as a character at the point q^{2r} (**??**). In the limit of $q\to 1$, both quantum dimensions and dual quantum dimensions certainly reduce to the ordinary dimensions. In the case of simply laced root systems (A_n, D_n, E_6, E_7, E_8), the Weyl vector and the dual Weyl vector coincide $\rho = r$, so the quantum and the dual quantum dimensions coincide as well.

The question is whether or not the dual quantum dimensions can be universalized alongside with the quantum dimensions. It meets, at least, two serious problems. First of all, if there is a universal formula for the dual quantum dimensions, there should exist at once two universal formulas coinciding for the simply laced root systems and differing for the non-simply-laced ones. It is quite an intricate requirement, which looks impossible to satisfy. Indeed, the difference of these two putatively existing universality formulas, $\Delta = {}^\vee\mathrm{qD}_{\mathrm{Adj}}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) - \mathrm{qD}_{\mathrm{Adj}}(\mathfrak{a},\mathfrak{b},\mathfrak{c})$ has to vanish at the whole line associated with the root system of type D. This line is given by $2\mathfrak{a}+\mathfrak{b}=0$, i.e. the difference Δ has to be proportional to $2\mathfrak{a}+\mathfrak{b}$. However, the root system of type B is also associated with this line, where Δ vanishes. At the same time, ${}^\vee\mathrm{qD}_{\mathrm{Adj}}^{B_n} - \mathrm{qD}_{\mathrm{Adj}}^{B_n} \neq 0$![1] Secondly, the dual quantum dimensions for the root systems B_n and C_n coincide in adjoint representation (which is a corollary of duality between the symplectic and orthogonal groups [**?**, **?**, **?**, **?**]) though they are associated with completely distinct Vogel's parameters with even distinct dependence on the rank n. This condition is also not that simple to satisfy.

5.2 Universality in refined Chern-Simons theory. Macdonald dimensions and dual Macdonald dimensions

Now we take the next step and study universal quantities [**?**] in the refined Chern-Simons theory [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**]. In this case, there is a new type of dimension: the refined version of the quantum dimension and the dual quantum dimension, which we call Macdonald dimension and dual Macdonald dimension accordingly. In order to refine Chern-Simons theory with $SU(N)$ gauge group, one should go from the sl_N-characters, the Schur functions, to Macdonald polynomials [**?**, **?**]. Hence, it seems natural to base these refined dimensions on the Macdonald polynomials $P_\lambda^{(R,S)}(x\,|\,t_\alpha^2\,|\,q^2,t^2)$ associated with different root systems R, which we discussed in great detail in the previous sections.

Similarly to quantum dimension and dual quantum dimension, one naturally defines Macdonald dimension Md_λ^R and dual Macdonald dimension ${}^\vee\mathrm{Md}_\lambda^R$ with the refined Weyl vector ρ_k (**??**) and the dual refined Weyl vector r_k (**??**):

$$\mathrm{Md}_\lambda^R := P_\lambda^R\left(x = q^{2\rho_k}\,|\,t_\alpha^2\,|\,q^2,t^2\right), \tag{159}$$

$${}^\vee\mathrm{Md}_\lambda^R := P_\lambda^R\left(x = q^{2r_k}\,|\,t_\alpha^2\,|\,q^2,t^2\right). \tag{160}$$

with the refined Weyl vector ρ_k (**??**). Here we discuss only polynomials $P_\lambda^R(x\,|\,t_\alpha^2\,|\,q^2,t^2)$ that depend on one root system (R,R).

Macdonald polynomials $P_\lambda^R(x\,|\,q^2,t^2)$ associated with the simply laced root systems $R = A_n, D_n, E_6, E_7, E_8$ depend only on two parameters q^2 and t^2 and factorize at the refined Weyl vector ρ_k (**??**), since it coincides with the dual refined Weyl vector r_k (**??**). Hence the Macdonald dimensions (**??**) in this case coincide with the dual Macdonald dimensions (**??**).

In the case of non-simply-laced root systems $R = B_n, C_n, F_4, G_2$, the Macdonald polynomials $P_\lambda^R(x\,|\,t_\alpha^2\,|\,q^2,t^2)$ depend on an additional parameter t_α^2 associated with the root of a distinct length. Since, in the non-simply-laced case, there are several deformation parameters t^2, t_α^2, and, in the simply laced case, there is just one t^2, one does not have to expect that there is a universality in the generic case. However, in the simply-laced case, the universality still persists.

There is another argument against universality of all algebras after the refinement. It is connected to the fact that the quantum dimensions celebrate universality, while the dual quantum dimensions seem to not universalize

[1] One also can find a discussion of uniqueness of the universal formula for the quantum dimensions in [**?**].

as we discussed in the previous section. Hence, it is expected that their refined versions do not universalize as well.

At the same time, the adjoint Macdonald polynomials associated with the simply laced root systems factorize according to formulas (??), (??), (??), (??) and (??) and can be unified with a universal formula which one can call **the simply laced universal Macdonald dimension** (which coincides with the dual one):

$$\boxed{\mathrm{Md}_{\mathrm{Adj}}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = -\frac{\{t^{\mathfrak{a}+\mathfrak{b}/2+\mathfrak{c}}\}\{t^{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}/2}\}\{t^{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}}\}\{q\,t^{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}}\}}{\{t^{\mathfrak{a}/2}\}\{t^{\mathfrak{b}/2}\}\{t^{\mathfrak{c}/2}\}\{q\,t^{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}-1}\}}.} \tag{161}$$

There is also a more symmetric form:

$$\boxed{\mathrm{Md}_{\mathrm{Adj}}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = -\frac{\{t^{\mathfrak{a}/2+\mathfrak{b}+\mathfrak{c}}\}\{t^{\mathfrak{a}+\mathfrak{b}/2+\mathfrak{c}}\}\{t^{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}/2}\}\{t^{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}}\}\{q\,t^{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}}\}}{\{t^{\mathfrak{a}/2}\}\{t^{\mathfrak{b}/2}\}\{t^{\mathfrak{c}/2}\}\{t^{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+1}\}\{q\,t^{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}-1}\}}.} \tag{162}$$

This universality formula for the simply laced algebras is **the main result** of the paper. It is in accordance with earlier claims by authors of [?, ?, ?], who studied the refinement of the Chern-Simons partition function on S^3 and also realized that universality holds for the simply laced algebras.

6 Conclusion

The goal of this paper is two-fold. First of all, we introduce the Macdonald and dual Macdonald dimensions, which are counterparts of the quantum dimensions and dual quantum dimensions. This requires an accurate description of the Macdonald polynomials associated with arbitrary root systems within Macdonald-Cherednik theory. The second goal is to use the introduced dimensions in order to demonstrate that the Vogel's universality can be extended from the adjoint quantum dimensions to the adjoint Macdonald dimensions in the case of simply laced root systems.

The Macdonald dimensions are associated with the refinement of Chern-Simons theory [?, ?], thus, our results expose the fact (which was earlier also observed in [?, ?, ?]) that the Vogel's universality in the refined Chern-Simons theory involves only the simply laced root systems. We explain the origin of this phenomenon: it turns out that only the dual Macdonald dimensions factorize, and the Macdonald dimensions do not (in variance with the quantum dimensions, which factorize), but, in the simply laced case, the both types of Macdonald dimensions coincide, and, hence, factorize, and admit universal description in the adjoint case.

This description was extended to the square of the adjoint polynomials in [?], and, by now, our understanding of the refined Vogel's universality is restricted to these cases. It would be very interesting to study the possibility of universal description of more quantities, and to make the status of the refined Vogel's universality more clear.

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Appendix A. Macdonald densities

In this section we use the notation of I. Macdonald paper [?] and the parameters are q, t, t_α.

The crucial part of Macdonald polynomial definition (??)-(??) is the Macdonald density. It allowed us to systematically connect Macdonald polynomials with the corresponding Koornwinder polynomials and use them to study Macdonald polynomials.

Macdonald density is calculated as a product over all roots of the root system R

$$\Delta(v) := \prod_{\alpha\in R} \frac{\left(t_{2\alpha}^{1/2} e^{\alpha}(v); q_\alpha\right)_\infty}{\left(t_\alpha t_{2\alpha}^{1/2} e^{\alpha}(v); q_\alpha\right)_\infty}, \qquad (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i). \tag{163}$$

where $q_\alpha = q^{u_\alpha}$, u_α is defined from the condition that α/u_α is a root in the root system S, t_α depends only on the length of the root $|\alpha|$ and we use t when $(\alpha,\alpha)=2$, t_l when $(\alpha,\alpha)=4$, t_s when $(\alpha,\alpha)=1$ and t_3 when $(\alpha,\alpha)=6$. $t_{2\alpha}$ equals to 1 if there is no root 2α in the root system R.

In the following sections we calculate Macdonald densities for all possible admissible pairs (R,S) (**??**). We also point out the connection of the resulting density with Koornwinder density Δ_K. We talk about Koornwinder polynomials and their definition in Appendix B.

Roots and positive roots of the root systems are written in orthogonal basis ε_i, with the following scalar product

$$(\varepsilon_i,\varepsilon_j)=\delta_{ij}. \tag{164}$$

A_n

Root system A_n corresponds to the algebra sl_{n+1}. There is only one possible admissible pair for the root system A_n: (A_n,A_n). Roots and Weyl group W of the root system A_n are the following:

$$R=\{\varepsilon_i-\varepsilon_j \,|\, 1\le i\neq j\le n+1\}\,,\qquad W\simeq S_{n+1} \tag{165}$$

$$R^+=\{\varepsilon_i-\varepsilon_j \,|\, 1\le i< j\le n+1\}\,. \tag{166}$$

and the Macdonald density is

$$\Delta_{A_n}(x,q,t)=\prod_{1\le i\neq j\le n+1}\frac{(x_i/x_j;q)_\infty}{(tx_i/x_j;q)_\infty} \tag{167}$$

D_n

Root system D_n corresponds to the algebra so_{2n}. There is only one possible admissible pair for the root system D_n: (D_n,D_n). Roots, positive roots and Weyl group W of the root system D_n are the following:

$$R=\{\pm\varepsilon_i\pm\varepsilon_j|1\le i<j\le n\}\,,\qquad W\simeq S_n\ltimes(\mathbb{Z}/2\mathbb{Z})^{n-1}, \tag{168}$$

$$R^+=\{\varepsilon_i\pm\varepsilon_j|1\le i<j\le n\} \tag{169}$$

and the Macdonald density is

$$\begin{aligned}\Delta_{D_n}(x,q,t)=\prod_{1\le i<j\le n}\frac{(x_ix_j;q)_\infty}{(tx_ix_j;q)_\infty}\prod_{1\le i<j\le n}\frac{(x_i/x_j;q)_\infty}{(tx_i/x_j;q)_\infty}\prod_{1\le i<j\le n}\frac{(x_j/x_i;q)_\infty}{(tx_j/x_i;q)_\infty}\prod_{1\le i<j\le n}\frac{(1/(x_ix_j);q)_\infty}{(t/(x_ix_j);q)_\infty}=\\ \prod_{i<j}\frac{(x_i^{\pm1}x_j^{\pm1};q)_\infty}{(tx_i^{\pm1}x_j^{\pm1};q)_\infty}=\boxed{\Delta_K(x\,|\,1,-1,q^{1/2},-q^{1/2}\,|\,q,t)},\end{aligned} \tag{170}$$

where Δ_K is Koornwinder density, which we discuss in Appendix B.

B_n

Root system B_n corresponds to the algebra so_{2n+1}. There are two possible admissible pairs for the root system B_n: (B_n,B_n) and (B_n,C_n). Roots, positive roots and Weyl group W of the root system B_n are the following:

$$R=\{\pm\varepsilon_i|1\le i\le n\}\cup\{\pm\varepsilon_i\pm\varepsilon_j|1\le i<j\le n\}\,,\qquad W\simeq S_n\ltimes(\{\pm1\})^n \tag{171}$$

$$R^+=\{\varepsilon_i|1\le i\le n\}\cup\{\varepsilon_i\pm\varepsilon_j|1\le i<j\le n\}\,. \tag{172}$$

(B_n,B_n)

$$\begin{matrix} q_\varepsilon=q & q_{\pm\varepsilon\pm\varepsilon}=q\\ t_\varepsilon=t_s & t_{\pm\varepsilon\pm\varepsilon}=t\end{matrix} \tag{173}$$

$$\begin{aligned}\Delta_{B_n}(x,q,t,t_s)=&\Delta_{D_n}(x,q,t)\prod_{i=1}^n\frac{(x_i;q)_\infty}{(t_sx_i;q)_\infty}\frac{(1/x_i;q)_\infty}{(t_s/x_i;q)_\infty}=\Delta_{D_n}(x,q,t)\prod_{i=1}^n\frac{(x_i^{\pm1};q)_\infty}{(t_sx_i;q)_\infty}=\\ &\Delta_{D_n}(x,q,t)\prod_{i=1}^n\frac{(x_i^{\pm2};q)_\infty}{(t_sx_i^{\pm1};q)_\infty(-x_i^{\pm1};q)_\infty(q^{1/2}x_i^{\pm1};q)_\infty(-q^{1/2}x_i^{\pm1};q)_\infty}=\\ &\boxed{\Delta_K(x\,|\,t_s,-1,q^{1/2},-q^{1/2}\,|\,q,t)}.\end{aligned} \tag{174}$$

$(B_n, B_n^\vee)$

$$\begin{array}{ll} q_\varepsilon = q^{1/2} & q_{\pm\varepsilon\pm\varepsilon} = q \\ t_\varepsilon = t_s & t_{\pm\varepsilon\pm\varepsilon} = t \end{array} \tag{175}$$

$$\begin{aligned} \Delta_{(B_n,B_n^\vee)} &= \Delta_{D_n}(x,q,t)\cdot\prod_{i=1}^n \frac{(x_i^{\pm1};q^{1/2})_\infty}{(t_s x_i^{\pm1};q^{1/2})_\infty} = \Delta_{D_n}(x,q,t)\cdot\prod_{i=1}^n \frac{(x_i^{\pm2};q)_\infty}{(-x_i^{\pm1};q^{1/2})_\infty (t_s x_i^{\pm1};q)_\infty (q^{1/2}t_s x_i^{\pm1};q)_\infty} = \\ &\Delta_{D_n}(x,q,t)\cdot\prod_{i=1}^n \frac{(x_i^{\pm2};q)_\infty}{(-x_i^{\pm1};q)_\infty(-q^{1/2}x_i^{\pm1};q)_\infty(t_s x_i^{\pm1};q)_\infty(q^{1/2}t_s x_i^{\pm1};q)_\infty} = \\ &\boxed{\Delta_K(x\,|\,-1,-q^{1/2},t_s,q^{1/2}t_s\,|\,q,t)}. \end{aligned} \tag{176}$$

C_n

Root system C_n corresponds to the algebra sp_{2n}. There are two possible admissible pairs for the root system C_n: (C_n, C_n) and (C_n, B_n). Roots, positive roots and Weyl group W of the root system C_n are the following:

$$R = \{\pm\varepsilon_i \pm \varepsilon_j | 1 \le i < j \le n\} \cup \{\pm 2\varepsilon_i | 1 \le i \le n\}\,, \qquad W \simeq S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n \tag{177}$$

$$R = \{\varepsilon_i \pm \varepsilon_j | 1 \le i < j \le n\} \cup \{2\varepsilon_i | 1 \le i \le n\}\,. \tag{178}$$

(C_n, C_n)

$$\begin{array}{ll} q_{2\varepsilon} = q & q_{\pm\varepsilon\pm\varepsilon} = q \\ t_{2\varepsilon} = t_l & t_{\pm\varepsilon\pm\varepsilon} = t \end{array} \tag{179}$$

$$\begin{aligned} \Delta_{C_n}(x,q,t,t_l) &= \Delta_{D_n}(x,q,t)\prod_{i=1}^n \frac{(x_i^2;q)_\infty}{(t_l x_i^2;q)_\infty}\frac{(1/x_i^2;q)_\infty}{(t_l/x_i^2;q)_\infty} = \Delta_{D_n}(x,q,t)\prod_{i=1}^n \frac{\left(x_i^{\pm2};q\right)_\infty}{\left(t_l x_i^{\pm2};q\right)_\infty} = \\ &\Delta_{D_n}(x,q,t)\prod_{i=1}^n \frac{\left(x_i^{\pm2};q\right)_\infty}{\left(t_l x_i^{\pm2};q^2\right)_\infty\left(q t_l x_i^{\pm2};q^2\right)_\infty} = \\ &\Delta_{D_n}(x,q,t)\prod_{i=1}^n \frac{\left(x_i^{\pm2};q\right)_\infty}{\left(t_l^{1/2}x_i^{\pm1};q\right)_\infty\left(-t_l^{1/2}x_i^{\pm1};q\right)_\infty\left(q^{1/2}t_l^{1/2}x_i^{\pm1};q\right)_\infty\left(-q^{1/2}t_l^{1/2}x_i^{\pm1};q\right)_\infty} = \\ &\boxed{\Delta_K(x\,|\,t_l^{1/2},-t_l^{1/2},q^{1/2}t_l^{1/2},-q^{1/2}t_l^{1/2}\,|\,q,t)} \end{aligned} \tag{180}$$

$(C_n, C_n^\vee)$

$$\begin{array}{ll} q_{2\varepsilon} = q^2 & q_{\pm\varepsilon\pm\varepsilon} = q \\ t_{2\varepsilon} = t_l & t_{\pm\varepsilon\pm\varepsilon} = t \end{array} \tag{181}$$

$$\begin{aligned} \Delta_{(C_n,C_n^\vee)} &= \Delta_{D_n}(x,q,t)\cdot\frac{(x_i^{\pm2};q^2)_\infty}{(t_l x_i^{\pm2};q^2)_\infty} = \Delta_{D_n}(x,q,t)\cdot\frac{(x_i^{\pm2};q)_\infty}{(q x_i^{\pm2};q^2)_\infty(t_l x_i^{\pm2};q^2)_\infty} = \\ &\Delta_{D_n}(x,q,t)\cdot\frac{(x_i^{\pm2};q)_\infty}{(q^{1/2}x_i^{\pm1};q)_\infty(-q^{1/2}x_i^{\pm1};q)_\infty(t_l^{1/2}x_i^{\pm1};q)_\infty(-t_l^{1/2}x_i^{\pm1};q)_\infty} = \\ &\boxed{\Delta_K(x\,|\,t_l^{1/2},-t_l^{1/2},q^{1/2},-q^{1/2}\,|\,q,t)}. \end{aligned} \tag{182}$$

BC_n

Root system BC_n is irreducible, but non-reduced and can only be the first root system R in the admissible pair (R, S). Roots and positive roots of the root system BC_n are the following:

$$R = \{\pm\varepsilon_i | 1 \le i \le n\} \cup \{\pm 2\varepsilon_i | 1 \le i \le n\} \cup \{\pm\varepsilon_i \pm \varepsilon_j | 1 \le i < j \le n\}\,, \tag{183}$$

$$R^+ = \{\varepsilon_i | 1 \le i \le n\} \cup \{2\varepsilon_i | 1 \le i \le n\} \cup \{\varepsilon_i \pm \varepsilon_j | 1 \le i < j \le n\}\,. \tag{184}$$

(BC_n, B_n)

$$\begin{array}{lll} q_{\pm\varepsilon} = q, & q_{\pm 2\varepsilon} = q^2, & q_{\pm\varepsilon\pm\varepsilon} = q \\ t_{\pm\varepsilon} = a, & t_{\pm 2\varepsilon} = b, & t_{\pm\varepsilon\pm\varepsilon} = t \end{array} \tag{185}$$

$$\begin{aligned} \Delta_{(BC_n,B_n)} &= \Delta_{D_n}(x,q,t) \cdot \prod_{1\le i\le n} \frac{(b^{1/2}x_i^{\pm};q)_\infty}{(ab^{1/2}x_i^{\pm};q)_\infty} \prod_{1\le i\le n} \frac{(x_i^{\pm 2};q^2)_\infty}{(bx_i^{\pm 2};q^2)_\infty} = \\ &\Delta_{D_n}(x,q,t) \cdot \prod_{1\le i\le n} \frac{(b^{1/2}x_i^{\pm};q)_\infty (x_i^{\pm 2};q)_\infty}{(qx_i^{\pm 2};q^2)_\infty (ab^{1/2}x_i^{\pm};q)_\infty (b^{1/2}x_i^{\pm};q)_\infty (-b^{1/2}x_i^{\pm};q)_\infty} = \\ &\Delta_{D_n}(x,q,t) \cdot \prod_{1\le i\le n} \frac{(x_i^{\pm 2};q)_\infty}{(q^{1/2}x_i^{\pm};q)_\infty (-q^{1/2}x_i^{\pm};q)_\infty (ab^{1/2}x_i^{\pm};q)_\infty (-b^{1/2}x_i^{\pm};q)_\infty} = \\ &\boxed{\Delta_K(x \,|\, ab^{1/2}, -b^{1/2}, q^{1/2}, -q^{1/2} \,|\, q,t)}. \end{aligned} \tag{186}$$

(BC_n, C_n)

$$\begin{array}{lll} q_{\pm\varepsilon} = q^{1/2}, & q_{\pm 2\varepsilon} = q, & q_{\pm\varepsilon\pm\varepsilon} = q \\ t_{\pm\varepsilon} = a, & t_{\pm 2\varepsilon} = b, & t_{\pm\varepsilon\pm\varepsilon} = t \end{array} \tag{187}$$

$$\begin{aligned} \Delta_{(BC_n,C_n)} &= \Delta_{D_n}(x,q,t) \cdot \prod_{1\le i\le n} \frac{(b^{1/2}x_i^{\pm};q^{1/2})_\infty}{(ab^{1/2}x_i^{\pm};q^{1/2})_\infty} \prod_{1\le i\le n} \frac{(x_i^{\pm 2};q)_\infty}{(bx_i^{\pm 2};q)_\infty} = \\ &\Delta_{D_n}(x,q,t) \cdot \prod_{1\le i\le n} \frac{(b^{1/2}x_i^{\pm};q^{1/2})_\infty (x_i^{\pm 2};q)_\infty}{(ab^{1/2}x_i^{\pm};q^{1/2})_\infty (b^{1/2}x_i^{\pm};q^{1/2})_\infty (-b^{1/2}x_i^{\pm};q^{1/2})_\infty} = \\ &\Delta_{D_n}(x,q,t) \cdot \prod_{1\le i\le n} \frac{(x_i^{\pm 2};q)_\infty}{(ab^{1/2}x_i^{\pm};q)_\infty (ab^{1/2}q^{1/2}x_i^{\pm};q)_\infty (-b^{1/2}x_i^{\pm};q)_\infty (-b^{1/2}q^{1/2}x_i^{\pm};q)_\infty} = \\ &\boxed{\Delta_K(x \,|\, ab^{1/2}, ab^{1/2}q^{1/2}, -b^{1/2}, -b^{1/2}q^{1/2} \,|\, q,t)}. \end{aligned} \tag{188}$$

E_6

There is only one possible admissible pair for the root system E_6: (E_6, E_6). Roots and positive roots of the root system E_6 are the following:

$$R = \{\pm\varepsilon_i \pm \varepsilon_j | 1 \le i < j \le 5\} \cup \{\pm(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 + \sum_{i=1}^{5} \pm\varepsilon_i)/2 \,|\, \text{sum with odd number of } -\}, \tag{189}$$

$$R^+ = \{\pm\varepsilon_i + \varepsilon_j | 1 \le i < j \le 5\} \cup \{(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 + \sum_{i=1}^{5} \pm\varepsilon_i)/2 \,|\, \text{sum with odd number of } -\}. \tag{190}$$

$$\Delta_{E_6} = \Delta_{D_5}(x,q,t) \cdot \prod_{\sigma} \frac{((x_8 x_7^{-1} x_6^{-1} \prod_{i=1}^5 x_i^{\sigma})^{\pm 1/2};q)_\infty}{(t_l (x_8 x_7^{-1} x_6^{-1} \prod_{i=1}^5 x_i^{\sigma})^{\pm 1/2};q)_\infty}, \tag{191}$$

where σ denotes sign choice in the product $\prod_{i=1}^5 x_i^{\sigma}$, but only with an odd number of negative signs.

E_7

There is only one possible admissible pair for the root system E_7: (E_7, E_7). Roots and positive roots of the root system E_7 are the following:

$$R = \{\pm\varepsilon_i \pm \varepsilon_j | 1 \le i < j \le 6\} \cup \{\pm(\varepsilon_7 - \varepsilon_8)\} \cup \{\pm(\varepsilon_7 - \varepsilon_8 + \sum_{i=1}^{5} \pm\varepsilon_i)/2 \,|\, \text{sum with odd number of } -\}, \tag{192}$$

$$R^+ = \{\pm\varepsilon_i + \varepsilon_j | 1 \le i < j \le 6\} \cup \{-(\varepsilon_7 - \varepsilon_8)\} \cup \{-(\varepsilon_7 - \varepsilon_8 + \sum_{i=1}^{5} \pm\varepsilon_i)/2 \,|\, \text{sum with odd number of } -\}. \tag{193}$$

$$\Delta_{E_7} = \Delta_{D_6}(x,q,t) \cdot \frac{((x_7 x_8^{-1})^{\pm 1}; q)_\infty}{(t(x_7 x_8^{-1})^{\pm 1}; q)_\infty} \cdot \prod_\sigma \frac{((x_7 x_8^{-1} \prod_{i=1}^6 x_i^\sigma)^{\pm 1/2}; q)_\infty}{(t_l (x_7 x_8^{-1} \prod_{i=1}^6 x_i^\sigma)^{\pm 1/2}; q)_\infty}, \tag{194}$$

where σ denotes sign choice in the product $\prod_{i=1}^8 x_i^\sigma$, but only with an even number of negative signs.

E_8

There is only one possible admissible pair for the root system E_8: (E_8, E_8). Roots and positive roots of the root system E_8 are the following:

$$R = \{\pm\varepsilon_i \pm \varepsilon_j | 1 \le i < j \le 8\} \cup \{\sum_{i=1}^{8} \pm\varepsilon_i/2 \,|\, \text{sum with even number of } -\}, \tag{195}$$

$$R^+ = \{\pm\varepsilon_i + \varepsilon_j | 1 \le i < j \le 8\} \cup \{\sum_{i=1}^{8} \pm\varepsilon_i/2 \,|\, \text{sum with 0, 2 or 4 } -\}. \tag{196}$$

$$\Delta_{E_8} = \Delta_{D_8}(x,q,t) \cdot \prod_\sigma \frac{((\prod_{i=1}^8 x_i^\sigma)^{1/2}; q)_\infty}{(t_l (\prod_{i=1}^8 x_i^\sigma)^{1/2}; q)_\infty}, \tag{197}$$

where σ denotes sign choice in the product $\prod_{i=1}^6 x_i^\sigma$, but only with an even number of negative signs.

F_4

There is only one possible admissible pair for the root system F_4: (F_4, F_4). Roots and positive roots of the root system F_4 are the following:

$$R = \{\pm\varepsilon_i | 1 \le i \le 4\} \cup \{\pm\varepsilon_i \pm \varepsilon_j | 1 \le i < j \le 4\} \cup \{(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)/2\}, \tag{198}$$

$$R^+ = \{\varepsilon_i | 1 \le i \le 3\} \cup \{-\varepsilon_4\} \cup \{\varepsilon_i \pm \varepsilon_j | 1 \le i < j \le 3\} \cup \{\pm\varepsilon_i - \varepsilon_4 \,|\, 1 \le i \le 3\} \cup \{(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 - \varepsilon_4)/2\}. \tag{199}$$

$$\Delta_{F_4} = \Delta_{D_4}(x,q,t) \cdot \prod_{i=1}^{4} \frac{(x_i^{\pm 1}; q)_\infty}{(t_s x_i^{\pm 1}; q)_\infty} \cdot \frac{((x_1^{\pm 1} x_2^{\pm 1} x_3^{\pm 1} x_4^{\pm 1})^{1/2}; q)_\infty}{(t_l (x_1^{\pm 1} x_2^{\pm 1} x_3^{\pm 1} x_4^{\pm 1})^{1/2}; q)_\infty}. \tag{200}$$

G_2

There is only one possible admissible pair for the root system G_2: (G_2, G_2). Roots and positive roots of the root system G_2 are the following:

$$R = \{\pm(\varepsilon_i - \varepsilon_j) | 1 \le i < j \le 3\} \cup \{\pm(2\varepsilon_i - \varepsilon_j - \varepsilon_k) | \{i,j,k\} = \{1,2,3\}\}, \tag{201}$$

$$R^+ = \{\varepsilon_1 - \varepsilon_2,\ -\varepsilon_1 + \varepsilon_3,\ -\varepsilon_2 + \varepsilon_3,\ -\varepsilon_1 - \varepsilon_2 + 2\varepsilon_3,\ \varepsilon_1 - 2\varepsilon_2 + \varepsilon_2,\ -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3\}. \tag{202}$$

$$\Delta_{G_2} = \prod_{1 \le i < j \le 3} \frac{(x_i^{\pm 1} x_j^{\mp 1}; q)_\infty}{(t x_i^{\pm 1} x_j^{\mp 1}; q)_\infty} \prod_{\{i,j,k\} = \{1,2,3\}} \frac{(x_i^{\pm 2} x_j^{\mp 1} x_k^{\mp 1}; q)_\infty}{(t_l x_i^{\pm 2} x_j^{\mp 1} x_k^{\mp 1}; q)_\infty}. \tag{203}$$

Appendix B. Koornwinder polynomials

Koornwinder polynomials P_λ depend on partition λ, N variables x_i, four new parameters (a,b,c,d) as well as on standard Macdonald parameters q and t :

$$P_\lambda(x\,|\,a,b,c,d\,|\,q,t). \tag{204}$$

In table **??** we listed all types of Macdonald polynomials (we also denote them $P_\lambda^{(R,S)}(x\,|\,t_\alpha\,|\,q,t)$) that are generalized by Koornwinder polynomials. In Appendix A there are details of calculation of Macdonald densities and their connection to density of Koornwinder polynomials.

type of Macdonald polynomials (colored with admissible pair (R,S))	designation of Macdonald polynomials	parameters of corresponding Koornwinder polynomials (a,b,c,d)
(B_n,B_n)	$P_\lambda^{(B_n,B_n)}(x\,|\,t_s\,|\,q,t)$	$(t_s,-1,q^{1/2},-q^{1/2})$
$(B_n,B_n^\vee)=(B_n,C_n)$	$P_\lambda^{(B_n,C_n)}(x\,|\,t_s\,|\,q,t)$	$(-1,-q^{1/2},t_s,q^{1/2}t_s)$
(C_n,C_n)	$P_\lambda^{(C_n,C_n)}(x\,|\,t_l\,|\,q,t)$	$(t_l^{1/2},-t_l^{1/2},q^{1/2}t_l^{1/2},-q^{1/2}t_l^{1/2})$
$(C_n,C_n^\vee)=(C_n,B_n)$	$P_\lambda^{(C_n,B_n)}(x\,|\,t_l\,|\,q,t)$	$(t_l^{1/2},-t_l^{1/2},q^{1/2},-q^{1/2})$
(D_n,D_n)	$P_\lambda^{(D_n,D_n)}(x\,|\,q,t)$	$(1,-1,q^{1/2},-q^{1/2})$
(BC_n,B_n)	$P_\lambda^{(BC_n,B_n)}(x\,|\,a,b\,|\,q,t)$	$(ab^{1/2},-b^{1/2},q^{1/2},-q^{1/2})$
(BC_n,C_n)	$P_\lambda^{(BC_n,C_n)}(x\,|\,a,b\,|\,q,t)$	$(ab^{1/2},ab^{1/2}q^{1/2},-b^{1/2},-b^{1/2}q^{1/2})$

Table 3: Correspondence between Koornwinder and Macdonald polynomials

The Koornwinder polynomials are defined from the triangular decomposition and from the orthogonality w.r.t. the scalar product given by the Koornwinder density

$$\Delta_K := \prod_{i=1}^{n} \frac{(x_i^{\pm 2};q)}{(ax_i^{\pm1},bx_i^{\pm1},cx_i^{\pm1},dx_i^{\pm1};q)} \prod_{i<j} \frac{(x_i^{\pm1}x_j^{\pm1};q)}{(tx_i^{\pm1}x_j^{\pm1};q)} \tag{205}$$

With this density, the scalar product is given by

$$\Big\langle A(x)\Big|B(x)\Big\rangle = \frac{(q;q)^n}{2^n n!}\left[A(x)B(x^{-1})\Delta\right]_0 \tag{206}$$

and the Koornwinder polynomials are defined from the expansion

$$P_\lambda = \xi_\lambda + \sum_{\mu<\lambda} K_{\lambda\mu}\xi_\mu \tag{207}$$

where the condition $\mu<\lambda$ implies the dominant ordering.

In case of root system C_n: $\lambda \geq \mu \iff |\lambda| \equiv |\mu| \,(\mathrm{mod}\ 2)$ and $\sum_{k=1}^{i}\lambda_k \geq \sum_{k=1}^{i}\mu_k$ for all $i=1,2,\ldots$. Partitions of even numbers $|\lambda|=2n$ are dominant to the empty partition $\emptyset$: $\lambda > \emptyset$.

Koornwinder Hamiltonian. Koornwinder polynomials are eigenfunctions

$$\hat{\mathcal{D}}\cdot P_\lambda(x\,|\,a,b,c,d\,|\,q,t) = \Lambda_\lambda P_\lambda(x\,|\,a,b,c,d\,|\,q,t) \tag{208}$$

of the difference operator $\hat{\mathcal{D}}$:

$$\hat{\mathcal{D}} := \sum_{i=1}^{n} t\frac{(1-ax_i)(1-bx_i)(1-cx_i)(1-dx_i)}{\alpha\xi(1-x_i^2)(1-qx_i^2)}\prod_{j\neq i}\frac{(1-tx_ix_j)(1-tx_i/x_j)}{(1-x_ix_j)(1-x_i/x_j)}\left(q^{x_i\partial_i}-1\right)+$$
$$+\sum_{i=1}^{n} t\frac{(1-ax_i^{-1})(1-bx_i^{-1})(1-cx_i^{-1})(1-dx_i^{-1})}{\alpha\xi(1-x_i^{-2})(1-qx_i^{-2})}\prod_{j\neq i}\frac{(1-tx_i^{-1}x_j^{-1})(1-tx_j/x_i)}{(1-x_i^{-1}x_j^{-1})(1-x_j/x_i)}\left(q^{-x_i\partial_i}-1\right),$$
$$\alpha := \sqrt{\frac{abcd}{q}},\ \xi = t^n \tag{209}$$

with the eigenvalues

$$\Lambda_\lambda = \sum_{j=1}^{n}\{\alpha t^{n-j}q^{\lambda_j/2}\}\{q^{\lambda_j/2}\}. \tag{210}$$

Koornwinder polynomials for antisymmetric partitions.
It is very convenient to use explicit expressions for Koornwinder polynomials to get Macdonald polynomials.
For example one of the wonderful formulas was provided in [**?**]

$$\boxed{P_{(1^r)}(x \mid a,b,c,d \mid q,t) = \sum_{k,l,i,j\geq 0}(-1)^{i+j}E_{r-2k-2l-i-j}(x)\widehat{c}_e'(k,l;t^{n-r+1+i+j})\widehat{c}_o(i,j;t^{n-r+1}),} \tag{211}$$

where

$$\widehat{c}_e'(k,l;s) = \frac{(tc^2/\alpha^2;t^2)_k(sc^2t;t^2)_k(s^2c^4/t^2;t^2)_k}{(t^2;t^2)_k(sc^2/t;t^2)_k(s^2a^2c^2/t;t^2)_k}\frac{(1/c^2;t)_l(s/t;t)_{2k+l}}{(t;t)_l(sc^2;t)_{2k+l}}\frac{1-st^{2k+2l-1}}{1-st^{-1}}a^{2k}c^{2l}. \tag{212}$$

and

$$\widehat{c}_o(i,j;s) = \frac{(-a/b;t)_i(scd/t;t)_i}{(t;t)_i(-sac/t;t)_i}\frac{(s;t)_{i+j}(-sac/t;t)_{i+j}(s^2a^2c^2/t^3;t)_{i+j}}{(s^2abcd/t^2;t)_{i+j}(sac/t^{3/2};t)_{i+j}(-sac/t^{3/2};t)_{i+j}}\frac{(-c/d;t)_j(sab/t;t)_j}{(t;t)_j(-sac/t;t)_j}b^id^j. \tag{213}$$

and $E_r(x)$ are defined with the generating function

$$E(x \mid y) = \prod_{i=1}^{n}(1-yx_i)(1-y/x_i) = \sum_{r\geq 0}(-1)^rE_r(x)y^r. \tag{214}$$

$E_r(x)$ are just Schur functions $S_{[1^r]}(p_m)$ expressed with power sum symmetric functions p_m, where one uses $p_m = \sum_{i=1}^n x_i^m + x_i^{-m}$. Some formulas for Macdonald and Koornwinder polynomials can also be found in [**?**,**?**,**?**].

Appendix C. Macdonald polynomials at the point $q^{2\rho_k}$

In this section we return to the notation of this paper where we use squares of Macdonald parameters.
We list some examples how Macdonald polynomials do not factorize at the point $q^{2\rho_k}$, which is a refined version of the point $q^{2\rho}$ in the definition of quantum dimensions (**??**).

Macdonald polynomials for root system C_n at the point $q^{2\rho_k}$

Values of the symplectic Macdonald polynomials at the point $q^{2\rho_k}$ and, to compare, the limit of them to the symplectic characters. Here we use our standard notation $\{x\} = x - x^{-1}$, $[x]_t = \frac{t^n - t^{-n}}{t-t^{-1}}$, $\xi_n = t_l t^{n-1}$.
Power sum symmetric functions $p_m(x_1,\ldots,x_n) := \sum_{i=1}^n x_i^m + x_i^{-m}$ at the point $x = q^{2\rho_k(C_n)} = t_l^2 t^{2(i-1)}$ are the following

$$p_m\left(x = q^{2\rho_k(C_n)}\right) = \frac{[m\,n]_t}{[m]_t}\frac{\{t_l^{4m}t^{2m(n-1)}\}}{\{t_l^{2m}t^{m(n-1)}\}}. \tag{215}$$

And the definition

$$p_\lambda = \prod_{i=1}^{l(\lambda)} p_{\lambda_i} \tag{216}$$

works for power sum symmetric functions at a special point too.

$$P_\lambda^{C_n}\left(x=q^{2\rho_k}\,|\,t_l^2\,|\,q^2,t^2\right)\xrightarrow{t\to q,\,t_l\to q} Sp_\lambda(q^{2\rho}), \tag{217}$$

where Sp_λ are characters of sp_{2n} (symplectic) algebra. We call them symplectic Schur symmetric functions and discuss them in the section **??**.

$$P_{[1]}^{C_n}(q^{\rho_k})=[n]_t\frac{\{t_l^4\,t^{2n-2}\}}{\{t_l^2\,t^{n-1}\}}, \tag{218}$$

$$Sp_{[1]}(q^\rho)=\frac{[n]_q[2n+2]_q}{[n+1]_q}, \tag{219}$$

$$P_{[2]}^{C_n}(q^{\rho_k})=[n]_t\frac{\{q\,t^n\}}{\{q\,t\}}\left(\frac{\{t_l^3\xi_n^3\}}{\{t_l\xi_n\}}+\frac{\{q^{-1}\xi_n\}}{\{q\xi_n\}}\right), \tag{220}$$

$$Sp_{[2]}(q^\rho)=\frac{[n]_q[2n+1]_q[2n+4]_q}{[2]_q[n+2]_q}=\overline{D}_{\mathrm{Adj}}^{C_n}, \tag{221}$$

$$P_{[1,1]}^{C_n}(q^{\rho_k})=\frac{[n]_t}{[2]_t}\frac{\{q\,t\}\{t^{n-1}\}\{q^2\,t_l^2\,t^{2n-4}\}}{\{q\,t_l\,t^{n-1}\}\{q\,t_l\,t^{n-2}\}\{q\,t_l^2\,t^{2n-3}\}}+\frac{1}{2}[n]_t^2\frac{\{t_l^4t^{2n-2}\}^2}{\{t_l^2t^{n-1}\}^2}-\frac{1}{2}\frac{[2n]_t}{[2]_t}\frac{\{t_l^8\,t^{4n-4}\}}{\{t_l^4\,t^{2n-2}\}}, \tag{222}$$

$$Sp_{[1,1]}(q^\rho)=\frac{[n-1]_q[2n+1]_q[2n+2]_q}{[2]_q[n+1]_q}, \tag{223}$$

$$\begin{aligned}P_{[1,1,1]}^{C_n}(q^{\rho_k})&=[n]_t\frac{\{t_l^4t^{2n-2}\}}{\{t_l^2t^{n-1}\}}\left(\frac{[n-2]_t\{t^{n-1}\}\{q\,t\}\{q^2t_l^2t^{2(n-3)}\}}{[2]_t\{q\,t_l\,t^{n-2}\}\{q\,t_l^2t^{2n-5}\}\{q\,t_lt^{n-3}\}}-[n-1]_t\frac{\{q\,t_l\}}{\{q\,t_l\,t^{n-2}\}}\right)+\\&+\frac{1}{3}\frac{[3n]_t}{[3]_t}\frac{\{t_l^{12}\,t^{6(n-1)}\}}{\{t_l^6\,t^{3(n-1)}\}}-\frac{1}{2}\frac{[2n]_t\,[n]_t}{[2]_t}\frac{\{t_l^8\,t^{4(n-1)}\}}{\{t_l^4\,t^{2(n-1)}\}}\frac{\{t_l^4t^{2(n-1)}\}}{\{t_l^2t^{n-1}\}}+\frac{1}{6}[n]_t^3\frac{\{t_l^4t^{2(n-1)}\}^3}{\{t_l^2t^{n-1}\}^3},\end{aligned} \tag{224}$$

$$Sp_{[1,1,1]}(q^\rho)=\frac{[n-2]_q[2n]_q[2n+1]_q[2n+2]_q}{[2]_q[3]_q[n+1]_q} \tag{225}$$

Macdonald polynomials for root system B_n at the point q^{ρ_k}

Power sum symmetric functions $p_m(x_1,\ldots,x_n):=\sum_{i=1}^n x_i^m+x_i^{-m}$ at the point $x=q^{2\rho_k(B_n)}=t^{2(i-1)}t_s^2$ are the following

$$p_m\left(x=q^{2\rho_k(B_n)}\right)=\frac{[m\,n]_t}{[m]_t}\frac{\{t_s^{2m}t^{2m(n-1)}\}}{\{t_s^mt^{m(n-1)}\}}. \tag{226}$$

The limit of B_n-colored Macdonald polynomials at a point $q^{2\rho_k}$ when the parameters t and t_s go to q are the corresponding quantum dimensions: characters of so_{2n+1} Lie algebra — orthogonal Schur functions $So_\lambda(x)$ at the point $x=q^{2\rho}$:

$$P_\lambda^{B_n}\left(x=q^{2\rho_k}\,|\,t_s^2\,|\,q^2,t^2\right)\xrightarrow{t\to q,\,t_s\to q} So_\lambda(q^{2\rho}). \tag{227}$$

We listed some examples of these Schur functions in the section **??**.

$$\begin{aligned}P_{[1,1]}^{B_n}(q^{2\rho_k})=\;&[n]_t[n-1]_t\frac{\{t_s^2\,t^{2(n-1)}\}}{\{t_s\,t^{n-1}\}}\frac{\{t_s\}\{q^2t^{2(n-2)}\}}{\{q\,t^{n-2}\}\{q\,t_st^{2n-4}\}}-\frac{[n]_t\{q\}}{\{q\,t^{n-1}\}}-\frac{[n]_t}{[2]_t}\frac{\{t^{n-1}\}\{q^2\,t^{2n-4}\}\{t_s^2\,q^{-1}t^{-1}\}}{\{qt^{n-1}\}\{qt^{n-2}\}\{qt_s^2\,t^{2n-3}\}}+\\+\frac{[n]_t[n-1]_t}{[2]_t}&\frac{\{t_s\}\{t\,t_s\}\{q\,t_s^2t^{2n-4}\}\{q^2\,t^{2n-2}\}\{q^2\,t^{2n-4}\}}{\{q\,t^{n-1}\}\{q\,t^{n-2}\}\{q\,t_s\,t^{2n-3}\}\{q\,t_s\,t^{2n-4}\}\{q\,t_s^2\,t^{2n-3}\}}+\frac{1}{2}\left([n]_t\frac{\{t_s^2t^{2(n-1)}\}}{\{t_st^{n-1}\}}\right)^2-\frac{1}{2}\frac{[2n]_t}{[2]_t}\frac{\{t_s^4t^{4(n-1)}\}}{\{t_s^2t^{2(n-1)}\}}\end{aligned} \tag{228}$$

$$So_{[1,1]}^{B_n}(q^{2\rho})=\frac{[n+1/2]_q[2n]_q[2n-3]_q}{[n-3/2]_q[2]_q}=D_{\mathrm{Adj}}^{B_n}. \tag{229}$$

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