Nash Equilibria with Irradical Probabilities

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Abstract

We present for every $n \geq 4$ an n-player game in normal form with payoffs in $\{0, 1, 2\}$ that has a unique, fully mixed, Nash equilibrium in which all the probability weights are irradical (i.e., algebraic but not closed form expressible even with m-th roots for any integer m).

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1. Introduction

Every Nash equilibrium (NE) in a 2-player normal form game with rational payoffs is a solution to a linear program with rational weights (**?**). Therefore, if σ is a NE then there is a NE τ with rational probability weights (henceforth we refer to them as *probabilities*) and support $\operatorname{supp} \sigma$. However, this conclusion fails to hold in n-player games when $n \geq 3$. This is because more generally, NEs are solutions of polynomial equations and inequalities in n variables and degree at most $n - 1$, and these might not have solutions in the rationals. A concrete example is found in **?**: a 3-player game with integer payoffs in $\{0, 1, 2, 3\}$, possessing a unique NE, and its probabilities are $((x_1, 1-x_1), (x_2, 1-x_2), (x_3, 1-x_3))$ where $x_1 = \frac{25-\sqrt{409}}{12}$, $x_2 = \frac{13+\sqrt{409}}{60}$, $x_3 = \frac{21-\sqrt{409}}{2}$. These 6 probabilities are all irrational. A different known example is the 3-player poker game in **?**.

Such a result is possible for every $n \geq 4$ as well but there seems to be no previously known game with this property for any such n. In this paper we

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present for every $n \geq 4$ an n-player game G_n that has 2 actions per player and the following properties: (1) The payoffs are in $\{0, 1, 2\}$; (2) G_n has a unique NE; (3) that NE is fully mixed, and (4) the probability weights of this NE are all inexpressible with radicals (henceforth we call such numbers *irradicals*). Inexpressibility with radicals is an even stronger requirement than irrationality – it means that these probabilities do not have closed form expressions using rational arithmetic and m-th roots for any integer m.

To play a mixed NE, a player must randomly select an action from the underlying (discrete) probability distribution. Exact sampling from discrete distributions with rational probabilities is a well understood problem solved with efficient algorithms (**??**). There, the computations are done using integer arithmetic which be done precisely on a modern digital computer. But sampling from distributions with irrational (or irradical) probabilities using rational arithmetic is a less studied problem. Irrational numbers can only be approximated by finite integers. With current known techniques, exact sampling from a distribution with irrational weights can only be done with arbitrary precision arithmetic using a power series expansion of the probabilities (**??**), which is computationally and pragmatically more involved than sampling from distributions with rational probabilities. In our context, all the probability weights are algebraic numbers, so the sampling problem appears to require computing roots of polynomials with potentially very large degrees and coefficients (the polynomials associated with G_5 have degree 26 and 10-digit coefficients). This fact highlights a challenge that players face in this context.

For example, our 4-player game has a unique NE, $((x_1, 1-x_1), (x_2, 1-x_2), (x_3, 1-x_3), (x_4, 1-x_4))$ such that x_1, x_2, x_3, x_4 are these roots of the polynomials

$$
\begin{aligned}
P_1(z) &= 7z^6 - 42z^5 + 89z^4 - 83z^3 + 40z^2 - 10z + 1, \\
P_2(z) &= 4z^6 - 27z^5 + 70z^4 - 79z^3 + 45z^2 - 13z + 1, \\
P_3(z) &= 140z^6 - 511z^5 + 701z^4 - 454z^3 + 141z^2 - 19z + 1, \\
P_4(z) &= 5z^6 - 44z^5 + 143z^4 - 163z^3 + 85z^2 - 21z + 2,
\end{aligned}
$$

that are approximately $x_1 \approx 0.529,\ x_2 \approx 0.846,\ x_3 \approx 0.523,\ x_4 \approx 0.320$. x_1, x_2, x_3, x_4 cannot be expressed with radicals because P_1, P_2, P_3, P_4 are all irreducible polynomials over the rationals whose Galois groups are all S_6 which is not solvable (see **?** for reference).

To obtain the main result we first construct the mentioned 4-player game

(Proposition **??**) and a 5-player game (Proposition **??**) with a unique NE whose probabilities are irradical. We then use these two games to construct such a game for any $n \geq 6$ (Proposition **??**). More specifically, for $n \geq 6$ we juxtapose independent copies of the 4-player and 5-player games, and to one of the game copies possibly attach 2 additional players who force each other to mimic the mixed strategy of one player in the game copy.

We complement this result by showing that every $2 \times 2 \times 2$ game with integer payoffs has a NE whose probabilities can be expressed using only rationals and square roots of rationals (Proposition **??**). We also present a simple 3-player game with integer payoffs in only $\{0, 1, 2\}$ whose unique NE has only irrational probabilities (Proposition **??**). This game is simpler than that of **?**. Its unique NE $((x_1, 1 - x_1), (x_2, 1 - x_2), (x_3, 1 - x_3))$ where $x_1 = \frac{7-\sqrt{13}}{6},\ x_2 = \frac{7+\sqrt{13}}{18}, x_3 = \frac{-3+\sqrt{13}}{2}$. These are approximately $x_1 \approx 0.566,\ x_2 \approx 0.589,\ x_3 \approx 0.303$.

Note that given such a 3-player game it is easy to extend it to an n-player game where *some* of the players need to play mixed actions with irrational probabilities in every NE – simply add players with one dominating action each. But then only the first 3 players will need to play NE strategies with irrational probabilities. We are interested in forcing *all* players to play NE strategies with irrational probabilities. For this reason we are interested only in fully mixed equilibria.

It is also possible to use such a 3-player game to generate n-player games for every $n \geq 5$ where all players need to play NE strategies with irrational probabilities. This is achieved by applying our results in Section **??** to such a 3-player game. However, this approach alone leaves unresolved the case of 4 players, and all the probabilities in the other cases will not be irradical.

There are some related results in the literature. First is the characterization of **?** of (mixed) strategy profiles which can be the unique NEs of a suitable normal form game. In particular, any fully mixed strategy profile for n players with 2 actions each is the unique NE of some n-player game (with 2 actions per player). However, the payoffs in the constructed game will involve the probabilities in that strategy profile. In particular, if the probabilities are irrational then the payoffs found will be irrational. Here we are looking for games with integer payoffs, since these are simpler to store on a computer than irrational payoffs.

Another result is by **?** who constructs for every algebraic number α a 3-player game with integer payoffs such that α is the payoff to some player in the NE of the game. However, in the constructed game player 3 has the fully mixed uniform distribution (which is comprised of rational probabilities) as an equilibrium strategy.

Another related result is the universality of NEs by **?**. This result asserts that every semialgebraic set is isomorphic to the set of NEs of some game. However, an isomorphism in this context only preserves the topological structure of the semialgebraic set rather than its exact members. In particular, in **?**, a semialgebraic set consisting of a single point with irrational entries will be mapped to a NE profile with rational probabilities.

Paper organization. In Section **??** we define the setup and provide algebra background. In Section **??** we present our results for 3-player, $2 \times 2 \times 2$ games. In Section **??** and Section **??** we present the 4-player game and 5-player game. In Section **??** we use the latter two games to construct the remaining n-player games for every $n \geq 6$.

2. Setup and lemmas

The games we define are in the usual normal form. They will be presented as a 2-column table of rows in the format

$$[\text{action tuple} \mid \text{payoffs}].$$

The action tuple and the payoffs in each row are ordered by the players $1, \ldots, n$.

In the games we present each player has 2 actions called 0 and 1. Each mixed action profile is defined by $x = (x_1, \ldots, x_n)$ where $x_i \in [0,1]$ is the probability player i gives to playing action 0. The payoff function of player i is u_i. We then define for every i the polynomial

$$f_i(x_{-i}) = u_i(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) - u_i(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n). \tag{1}$$

This is the expected payoff advantage action 0 has over action 1 for player i given the mixed actions x_{-i} of the other players. We sometimes write $f_i(x)$ or f_i for simplicity.

Given a game G let $\mathrm{NE}(G)$ be the set of NEs of G and $\mathrm{PNE}(G)$ the set of pure NEs of G. For reference we state the following lemma which can be proved by definition.

Lemma 2.1. *If $x \in \mathrm{NE}(G)$ then for every i,*

$$\begin{aligned} f_i(x_{-i}) > 0 \;&\Rightarrow\; x_i = 1, \\ f_i(x_{-i}) < 0 \;&\Rightarrow\; x_i = 0, \\ 0 < x_i < 1 \;&\Rightarrow\; f_i(x_{-i}) = 0. \end{aligned} \tag{2}$$

2.1. Algebra background

Here we cover the necessary algebra background for this paper (see **?** for a detailed reference). Let $\mathbb{Q}[x]$ be the ring (set) of polynomials in the variable x with rational coefficients. $P \in \mathbb{Q}[x]$ is irreducible if it cannot be written as $P_1 P_2$ for $P_1, P_2 \in \mathbb{Q}[x]$ where $0 < \deg P_1, \deg P_2 < \deg P$. A number a is called *algebraic* if there is a $0 \neq P \in \mathbb{Q}[x]$ such that $P(a) = 0$. Every algebraic number has a *minimal polynomial*: a unique, monic polynomial of minimum degree $P_a \in \mathbb{Q}[x]$ such that $P_a(a) = 0$. For example, $P_{\sqrt{2}} = x^2 - 2$. Here we will not insist on monicity in our search of these polynomials.

One way to prove a number a is irrational is by showing that $\deg P_a \geq 2$. To verify that a polynomial at hand P is P_a, it is sufficient to show that $P(a) = 0$ and P is irreducible in $\mathbb{Q}[x]$ (and that P is monic).

An irrational algebraic number can often be represented in a compact way, using *radicals* – m-roots ($\sqrt[m]{\ }$ for an integer m) of rationals. For example $\sqrt{2}$ represents the positive root of $x^2 - 2$. Another example is $\sqrt{2} + 3\sqrt[3]{5}$. Given a polynomial with a root a, we want to know when it is possible to construct such a representation for a. This will be relevant in Section **??** and Section **??**.

Let P be an irreducible polynomial with degree n and roots $a_1, \ldots, a_n$. The field $F = \mathbb{Q}[a_1, \ldots, a_n]$ is defined to be the smallest field that contains $\mathbb{Q}$ and $a_1, \ldots, a_n$.[1] For example, $\mathbb{Q}\left[\sqrt{2}\right]$ is the set of all numbers $a + b\sqrt{2}$ where $a, b \in \mathbb{Q}$. An *automorphism of F that extends* $\mathbb{Q}$ is a function $f : F \to F$ that is a bijection, preserves the field operations (e.g., $f(a+b) = f(a) + f(b)$) and maps every rational to itself. It can be shown that f is uniquely defined by its action on the roots $a_1, \ldots, a_n$. Furthermore, $f(\{a_1, \ldots, a_n\}) = \{a_1, \ldots, a_n\}$, since for every i it holds that $P(f(a_i)) = f(P(a_i)) = f(0) = 0$, meaning $f(a_i)$ is a root of P. The set $Aut(F)$ is called the *Galois group* of P.

[1] $\mathbb{Q}[a_1, \ldots, a_n]$ is a field while $\mathbb{Q}[x]$ and $\mathbb{Q}[x_1, \ldots, x_n]$ are rings because $x_1, \ldots, x_n$ are symbolic variables with no defined inverses, while $a_1, \ldots, a_n$ are real numbers that do have their inverses $\frac{1}{a_1}, \ldots, \frac{1}{a_n}$ in $\mathbb{Q}[a_1, \ldots, a_n]$.

A fundamental result in algebra states that a number a can be represented with radicals if and only if the Galois group of its minimal polynomial P_a is *solvable*. The formal definition of solvability does not matter here. The point is that to determine if a can be written with radicals we only need to determine if the Galois group of P_a is solvable (and wlog we can ignore monicity) A particular type of group that is *unsolvable* is S_n for $n \geq 5$. S_n is the set of all $n!$ permutations of $\{1, \ldots, n\}$ with the group operation of permutation composition.

Take two examples. The first is $\sqrt{2}$. Its minimal polynomial is $x^2 - 2$. Its Galois group is called $\mathbb{Z}_2$, which is simply the set $\{0, 1\}$ equipped with the XOR operation. This group is solvable, and indeed $\sqrt{2}$ is a radical representation of the positive root of $x^2 - 2$. The second example is any root of $P = x^5 - x - 1$. It can be shown that the Galois group of P is S_5. It is unsolvable, and indeed the roots of P cannot be represented with radicals. We call such numbers *irradicals*.

Note that the pair $(x_i, 1 - x_i)$ satisfies $x_i \in \mathbb{Q} \iff 1 - x_i \in \mathbb{Q}$, and similarly for expressbility with radicals. Therefore in our proofs of irrationality and irradicality we only analyze $x_1, \ldots, x_n$.

3. Three players

Define the $2 \times 2 \times 2$ game G_3:

$$\left[\begin{array}{ccc|ccc}
0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 1
\end{array}\right]$$

Figure 1: The game G_3

Proposition 3.1. *G_3 has a unique NE* $((x_1, 1 - x_1), (x_2, 1 - x_2), (x_3, 1 - x_3))$

given by

$$x_1 = \frac{7-\sqrt{13}}{6},\ x_2 = \frac{7+\sqrt{13}}{18},\ x_3 = \frac{-3+\sqrt{13}}{2}.$$

The probabilities in the NE are all irrational.

Proof. First observe that $\mathrm{PNE}(G_3) = \emptyset$ using this table of profitable deviations:

action profile	unsatisfied players
$(0,0,0)$	2
$(0,0,1)$	3
$(0,1,0)$	$1,3$
$(0,1,1)$	1
$(1,0,0)$	$2,3$
$(1,0,1)$	1
$(1,1,0)$	3
$(1,1,1)$	2

Now the polynomials f_1, f_2, f_3 from Equation **??** are

$$f_1(x_2,x_3) = -x_2x_3 + 2x_2 - 1,\ f_2(x_1,x_3) = x_1x_3 - x_1 - 2x_3 + 1,$$
$$f_3(x_1,x_2) = 3x_1x_2 - 1.$$

We show using Lemma **??** that if $x_i \in \{0,1\}$ for some i then $x \notin \mathrm{NE}(G_3)$. There are 6 cases.

Case 1: $x_1 = 0$. then $f_2 = -2x_3 + 1,\ f_3 = -1 < 0$, so $x_3 = 0$ so $f_2 = 1 > 0$ so $x_2 = 1$. Therefore $x = (0,1,0) \notin \mathrm{NE}(G_3)$.

Case 2: $x_1 = 1$. then $f_2 = -x_3,\ f_3 = 3x_2 - 1$. If $x_3 > 0$ then $f_2 < 0$ so $x_2 = 0$ so $f_3 = -1 < 0$ so $x_3 = 0$ – a contradiction. Therefore $x_3 = 0$, so $f_3 \leq 0$ so $x_2 \leq \frac{1}{3}$ so $f_1 \leq -\frac{1}{3} < 0$ so $x_1 = 0$ – a contradiction.

Case 3: $x_2 = 0$. then $f_1 = -1$ so $x_1 = 0$ so by case 1 $x \notin \mathrm{NE}(G_3)$.

Case 4: $x_2 = 1$. then $f_1 = -x_3 + 1,\ f_3 = 3x_1 - 1$. By cases 1 and 2, $x_1 \in (0,1)$ so $f_1 = 0$ so $x_3 = 1$ so $f_2 = -1 < 0$ so $x_2 = 0$ – a contradiction. Therefore $x \notin \mathrm{NE}(G_3)$.

Case 5: $x_3 = 0$. then $f_2 = -x_1 + 1$. By cases 3 and 4, $x_2 \in (0, 1)$ so $f_2 = 0$ so $-x_1 + 1 = 0$ so $x_1 = 1$ so by case 2 we conclude that $x \notin \mathrm{NE}(G_3)$.

Case 6: $x_3 = 1$. then $f_1 = x_2 - 1$. Again $0 = f_1 = x_2 - 1$ so $x_2 = 1$, so by case 4 we conclude that $x \notin \mathrm{NE}(G_3)$.

Therefore the only NEs are fully mixed. This means that $x \in \mathrm{NE}(G_3)$ satisfies $f_i(x) = 0$ for every i. By f_3 we have $x_1 = \frac{1}{3x_2}$. Then by f_2 we have $\frac{x_3}{3x_2} - \frac{1}{3x_2} - 2x_3 + 1 = 0$ so $(6x_3 - 3)x_2 = x_3 - 1$. $x_3 \neq \frac{1}{2}$ because $x_3 \neq 1$ so $x_2 = \frac{x_3-1}{6x_3-3}$. Then by f_1 we have $\frac{(x_3-1)(2-x_3)}{6x_3-3} = 1$ so $-x_3^2 + 3x_3 - 2 = 6x_3 - 3$ so $x_3^2 + 3x_3 - 1 = 0$ so because $x_3 > 0$ we get $x_3 = \frac{-3+\sqrt{13}}{2}$, $x_2 = \frac{-5+\sqrt{13}}{-24+6\sqrt{13}} = \frac{7+\sqrt{13}}{18}$ and $x_1 = \frac{6}{7+\sqrt{13}} = \frac{7-\sqrt{13}}{6}$.

$x_1, x_2, x_3 \notin \mathbb{Q}$ because they are expressions involving only rationals and $\sqrt{13}$ which is irrational. Their minimal polynomials in $\mathbb{Q}[y]$ are

$$P_1(y) = 3y^2 - 7y + 3,\ P_2(y) = 9y^2 - 7y + 1,\ P_3(y) = y^2 + 3y - 1.$$

These are all indeed irreducible by the rational root theorem. □

We conclude with a proof that square roots of rationals (along with rational arithmetic) are always enough to represent a NE of a $2 \times 2 \times 2$ game.

Proposition 3.2. *Let G be a $2 \times 2 \times 2$ game with integer payoffs. There is a NE $((x_1, 1 - x_1), (x_2, 1 - x_2), (x_3, 1 - x_3))$ such that for every $i \in \{1, 2, 3\}$, $x_i = a_i + b_i\sqrt{c_i}$ for some $a_i, b_i, c_i \in \mathbb{Q}$.*

Proof. In full generality, the polynomials f_1, f_2, f_3 have the form

$$\begin{aligned} f_1(x_2, x_3) &= A_1x_2x_3 + B_1x_2 + C_1x_3 + D_1, \\ f_2(x_1, x_3) &= A_2x_1x_3 + B_2x_1 + C_2x_3 + D_2, \\ f_3(x_1, x_2) &= A_3x_1x_2 + B_3x_1 + C_3x_2 + D_3, \end{aligned}$$

where the coefficients A_i, B_i, C_i, D_i are all integers, as the payoffs in the game are integers.

Let $x \in \mathrm{NE}$. There are 3 cases up to permuting actions and payoffs:

Case 1: $x_1 = 0$ *and* $x_2, x_3 \in (0,1)$. Then $f_1 \le 0,\ f_2 = C_2x_3 + D_2 = 0,\ f_3 = C_3x_2 + D_3 = 0$.

From $f_1 \le 0$ we get $(A_1x_3 + B_1)x_2 \le -C_1x_3 - D_1$. If $A_1x_3 + B_1 = 0$ then either $x_3 = -\frac{B_1}{A_1} \in \mathbb{Q}$ or $A_1 = B_1 = 0$, in which case x_2, x_3 are defined by linear equations and inequalities with rational coefficients, so x_2, x_3 can be rational in a NE.

Otherwise, $x_2 \le -\frac{C_1x_3+D_1}{A_1x_3+B_1}$.

If $C_i \neq 0$ for $i \in \{2,3\}$ then $x_{5-i} = -\frac{D_i}{C_i} \in \mathbb{Q}$. If $i = 2$ then $f_1 = \left(-\frac{A_1D_2}{C_2} + B_1\right)x_2 - \frac{C_1D_2}{C_2} + D_1 \le 0$. Since this inequality holds for some value of x_2, the inequality also holds for some rational value of x_2. In total, x_1, x_2, x_3 can all be rational (in a NE).

Otherwise $C_2 = C_3 = 0$. Then the only constraints on x_2, x_3 are being in $(0,1)$ and the inequality $x_2 \le -\frac{C_1x_3+D_1}{A_1x_3+B_1}$. This inequality is in particular satisfied for some rational values of x_2, x_3, meaning that x_1, x_2, x_3 can all be rational.

Case 2: $x_1 = x_2 = 0$ *and* $x_3 \in (0,1)$. Then $f_1 = C_1x_3 + D_1 \le 0,\ f_2 = C_2x_3 + D_2 \le 0,\ f_3 = D_3 = 0$. So x_3 can be rational in a NE with $x_1 = x_2 = 0$.

Case 3: $x_1, x_2, x_3 \in (0,1)$. Then $f_1 = f_2 = f_3 = 0$. Then $(A_1x_3 + B_1)x_2 = -C_1x_3 - D_1$. Suppose that $(*)$ $A_1x_3 + B_1 \neq 0$. Then $x_2 = -\frac{C_1x_3+D_1}{A_1x_3+B_1}$. From $f_3 = 0$ we then get $\left(-\frac{A_3(C_1x_3+D_1)}{A_1x_3+B_1} + B_3\right)x_1 - \frac{C_3(C_1x_3+D_1)}{A_1x_3+B_1} + D_3 = 0$. So

$$((B_3A_1 - A_3C_1)x_3 + B_3B_1 - A_3D_1)x_1 = (C_3C_1 - D_3A_1)x_3 + C_3D_1 - D_3B_1.$$

Suppose that $(**)$ $(B_3A_1 - A_3C_1)x_3 + B_3B_1 - A_3D_1 \neq 0$. Then

$$x_1 = \frac{(C_3C_1 - D_3A_1)x_3 + C_3D_1 - D_3B_1}{(B_3A_1 - A_3C_1)x_3 + B_3B_1 - A_3D_1}.$$

Now $f_2 = 0$ simplifies into a quadratic equation in x_3 involving rational coefficients. So x_3 can be written with rational arithmetic and (non-nested) square roots. The same then holds for x_1, x_2 by their expressions as ratios of linear functions of x_3.

Now suppose that $(*)$ and $(**)$ are not both true. There are 2 cases:

- $(*)$ is false, so $A_1x_3 + B_1 = 0$. Then $C_1x_3 + D_1 = 0$. If $A_1 = C_1 = 0$ then $B_1 = D_1 = 0$ and f_1 is always 0. If f_2 is always 0 then x_3 can take any value in $(0,1)$ (particularly rational ones), and one can verify that x_1, x_2 can take rational values. So f_2, f_3 are not always 0, and because of symmetry we can swap f_1 and f_2 to have one of $A_1 \neq 0$ and $C_1 \neq 0$ being true which implies that $x_3 \in \mathbb{Q}$. It can then be verified that x_1, x_2 can both be rational.

- f_1, f_3 are not always 0 and $A_1 x_3 + B_1 \neq 0$, $(*)$ is true and $(**)$ is false, so $(B_3 A_1 - A_3 C_1)x_3 + B_3 B_1 - A_3 D_1 = 0$. Then $(C_3 C_1 - D_3 A_1)x_3 + C_3 D_1 - D_3 B_1 = 0$. If $B_3 A_1 - A_3 C_1 = C_3 C_1 - D_3 A_1 = 0$ then $C_3 D_1 - D_3 B_1 = B_3 B_1 - A_3 D_1 = 0$ and conditioned on $f_1(x_2, x_3) = 0$, we get that f_3 is always 0 in contradiction to our assumption.

  So one of $B_3 A_1 - A_3 C_1$ and $C_3 C_1 - D_3 A_1$ is not 0. Then $x_3 \in \mathbb{Q}$. It can then be verified that x_1, x_2 can both be rational. $\square$

4. Four players

Define the $2 \times 2 \times 2 \times 2$ game G_4:

$$\left[\begin{array}{cccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1
\end{array}\right]$$

Figure 2: The game G_4

Proposition 4.1. *G_4 has a unique NE $((x_1, 1-x_1), (x_2, 1-x_2), (x_3, 1-x_3), (x_4, 1-x_4))$ given by*

$$x_1 \approx 0.529270752820,\ x_2 \approx 0.846414728986,$$
$$x_3 \approx 0.523440476515,\ x_4 \approx 0.320065197645.$$

The probabilities in the NE are all irrational and irradical.

Proof. First observe that $\mathrm{PNE}(G_4) = \emptyset$ using this table of profitable deviations:

action profile	unsatisfied players
$(0,0,0,0)$	3
$(0,0,0,1)$	2
$(0,0,1,0)$	1
$(0,0,1,1)$	1
$(0,1,0,0)$	$1,3$
$(0,1,0,1)$	4
$(0,1,1,0)$	4
$(0,1,1,1)$	2
$(1,0,0,0)$	$2,3,4$
$(1,0,0,1)$	1
$(1,0,1,0)$	2
$(1,0,1,1)$	3
$(1,1,0,0)$	$3,4$
$(1,1,0,1)$	$1,2$
$(1,1,1,0)$	$1,4$
$(1,1,1,1)$	$1,3$

Now the polynomials f_1, f_2, f_3, f_4 from Equation **??** are

$$\begin{aligned}
f_1(x_2, x_3, x_4) &= x_2x_3x_4 + 2x_2x_3 - 2x_2 - 2x_3x_4 + 1,\\
f_2(x_1, x_3, x_4) &= 3x_1x_3x_4 - 3x_1x_3 + x_1 - x_3x_4 + x_3 - x_4,\\
f_3(x_1, x_2, x_4) &= x_1x_4 - x_1 - 2x_4 + 1,\\
f_4(x_1, x_2, x_3) &= -x_1x_2x_3 + 3x_1x_3 - x_2x_3 + x_2 - 1.
\end{aligned}$$

We show using Lemma **??** that if $x_i \in \{0,1\}$ for some i then $x \notin \mathrm{NE}(G_4)$. There are 8 cases.

Case 1: $x_1 = 0$. then $f_2 = -x_3x_4 + x_3 - x_4,\ f_3 = -2x_4 + 1,\ f_4 = -x_2x_3 + x_2 - 1$. If $x_4 = \frac{1}{2}$ then $f_2 = \frac{x_3-1}{2}$. If $x_3 = 1$ then $f_4 = -1 < 0$ so $x_4 = 0$ – a contradiction. Therefore $x_3 < 1$ so $f_2 < 0$ so $x_2 = 0$. Since $x_4 = \frac{1}{2}$ we get $0 = f_4 = -1$ – a contradiction.

Therefore $x_4 \neq \frac{1}{2}$. If $x_4 < \frac{1}{2}$ then $f_3 > 0$ so $x_3 = 1$ so $f_4 = -1$ so $x_4 = 0$ so $f_2 = 1$ so $x_2 = 1$. Therefore $x = (0, 1, 1, 0) \notin \mathrm{PNE}(G_4)$. If $x_4 > \frac{1}{2}$ then $f_3 < 0$ so $x_3 = 0$ so $f_2 = -x_4 < 0$ so $x_2 = 0$ so $f_4 = -1$ so $x_4 = 0$ – a contradiction.

Overall x_4 cannot take any value in $[0, 1]$, so $x \notin \mathrm{NE}(G_4)$.

Case 2: $x_1 = 1$. then $f_2 = 2x_3x_4 - 2x_3 - x_4 + 1,\ f_3 = -x_4,\ f_4 = -2x_2x_3 + x_2 + 3x_3 - 1$. If $x_4 > 0$ then $f_3 < 0$ so $x_3 = 0$ so $f_2 = -x_4 + 1$ and $f_4 = x_2 - 1$. If $x_4 < 1$ then $f_2 > 0$ so $x_2 = 1$ so $f_1 = -1$ so $x_1 = 0$ – a contradiction. Therefore, still assuming that $x_4 > 0$ we get $x_4 = 1$ so $f_4 = x_2 - 1 \geq 0$ so $x_2 = 1$ so $f_1 = -1$ so $x_1 = 0$ – a contradiction.

Therefore $x_4 = 0$. So $f_2 = -2x_3 + 1$. If $x_3 = \frac{1}{2}$ then $f_4 = \frac{1}{2}$ so $x_4 = 1$ – a contradiction. Therefore $x_3 \neq \frac{1}{2}$. If $x_3 > \frac{1}{2}$ then $f_2 < 0$ so $x_2 = 0$ so $f_4 = 3x_3 - 1 > 0$ so $x_4 = 1$ – a contradiction. Therefore $x_3 < \frac{1}{2}$. Then $f_2 > 0$ so $x_2 = 1$ so $f_1 = 2x_3 - 1 < 0$ so $x_1 = 0$ – a contradiction.

Overall $x_1 \neq 1$.

Case 3: $x_2 = 0$. then $f_1 = -2x_3x_4 + 1,\ f_4 = 3x_1x_3 - 1$. By cases 1 and 2, $x_1 \in (0, 1)$ so $f_1 = 0$ so $x_3x_4 = \frac{1}{2}$ so $x_3, x_4 \neq 0$ so $f_3, f_4 \geq 0$. Therefore $x_3 = \frac{1}{2x_4}$ so $f_4 = \frac{3x_1}{2x_4} - 1 \geq 0$ so $x_1 \geq \frac{2x_4}{3}$. Also, $f_3 = x_1x_4 - x_1 - 2x_4 + 1 \geq 0$ so $(1 - x_4)x_1 \leq 1 - 2x_4$. If $x_4 = 1$ then $f_3 = -1 < 0$ – a contradiction. So $x_4 < 1$, so $\frac{2}{3}x_4 \leq x_1 \leq \frac{1-2x_4}{1-x_4}$. The quadratic inequality is then $2x_4^2 - 8x_4 + 3 \geq 0$, and in particular $x_4 < \frac{1}{2}$ so $x_3 > 1$ – a contradiction to $x_3 \leq 1$.

Overall x_4 cannot take on any value, concluding this case.

Case 4: $x_2 = 1$. then $f_1 = -x_3x_4 + 2x_3 - 1,\ f_4 = 2x_1x_3 - x_3$. Again $f_1 = 0$ so $x_3 = \frac{1}{2-x_4}$. If $x_4 = 1$ then $x_3 = 1$ and $f_3 = -1$ so $x_3 = 0$ – a contradiction. Therefore $x_4 < 1$ so $x_3 < 1$ so $f_3, f_4 \leq 0$. Also $x_3 > 0$ and by $f_4 \leq 0$ we get $x_1 \leq \frac{1}{2}$. By $f_3 \leq 0$ we get $x_1 \geq \frac{1-2x_4}{1-x_4}$, so in total $1 - x_4 \geq 2 - 4x_4$ so $x_4 \geq \frac{1}{3}$. Therefore $x_3 \geq \frac{3}{5}$. This means that $f_3 = f_4 = 0$, so $x_1 = \frac{1}{2}$ so $f_3 = -\frac{3}{2}x_4 + \frac{1}{2} = 0$ so $x_4 = \frac{1}{3}$ so $x_3 = \frac{3}{5}$. Therefore $f_2 = -\frac{1}{30}$ so $x_2 = 0$ – a contradiction to $x_2 = 1$. This establishes case 4.

Case 5: $x_3 = 0$. then $f_1 = -2x_2 + 1$, $f_2 = x_1 - x_4$, $f_4 = x_2 - 1$. By the previous cases $x_1, x_2 \in (0,1)$ so $f_1 = f_2 = 0$ so $x_2 = \frac{1}{2}$ and $x_1 = x_4$. We also get $f_4 < 0$ so $x_4 = 0$ so $x_1 = 0$ – a contradiction.

Case 6: $x_3 = 1$. then $f_1 = x_2x_4 - 2x_4 + 1$, $f_2 = 3x_1x_4 - 2x_1 - 2x_4 + 1$, $f_4 = -x_1x_2 + 3x_1 - 1$. Again $f_1 = f_2 = 0$ so $x_4 = \frac{1}{2-x_2}$ so $f_2 = \frac{3x_1}{2-x_2} - 2x_1 - \frac{2}{2-x_2} + 1 = 0$ so $\frac{2x_2-1}{2-x_2}x_1 = \frac{x_2}{2-x_2}$ so $(2x_1 - 1)x_2 = 1$ so $x_2 > 1$ because $x_1 \in (0,1)$ – a contradiction to $x_2 < 1$.

Case 7: $x_4 = 0$. then $f_3 = -x_1 + 1$. By the previous cases, $x_1, x_2, x_3 \in (0,1)$ so $f_3 = 0$ so $x_1 = 1$ – a contradiction.

Case 8: $x_4 = 1$. then $f_2 = x_1 - 1$. Like in case 7 we get $x_1 \in (0,1)$ and $x_1 = 1$ – a contradiction.

Therefore the only NEs are fully mixed. This means that $x \in \mathrm{NE}(G_4)$ satisfies $f_i(x) = 0$ for every i. We solve this now.

Let $F = \{f_1, f_2, f_3, f_4\}$. It is a set of polynomials in the polynomial ring $\mathbb{Q}[x_1, x_2, x_3, x_4]$. We begin by finding the Gröbner basis $G = \{g_1, g_2, g_3, g_4\}$ of the ideal $\langle F \rangle$, with the monomial ordering generated lexicographically by $x_1 \succ x_2 \succ x_3 \succ x_4$, given by

$$\begin{aligned}
g_1 &= 5x_4^6 - 44x_4^5 + 143x_4^4 - 163x_4^3 + 85x_4^2 - 21x_4 + 2,\\
g_2 &= 28x_3 - 1465x_4^5 + 12302x_4^4 - 36947x_4^3 + 32897x_4^2 - 11699x_4 + 1447,\\
g_3 &= 4x_2 + 255x_4^5 - 2094x_4^4 + 6053x_4^3 - 4687x_4^2 + 1401x_4 - 149,\\
g_4 &= 7x_1 + 5x_4^5 - 39x_4^4 + 104x_4^3 - 59x_4^2 + 26x_4 - 9.
\end{aligned}$$

These were found using Mathematica (**?**).

Now we compute the solutions of $\forall i\ g_i = 0$ to find those of $\forall i\ f_i = 0$. First, we approximate the 6 roots of g_1 numerically:

$$\begin{aligned}
&r_1 \approx 0.3200651976,\ r_2 \approx 0.4231894254,\\
&r_3 \approx 0.4135410939 - 0.09991760306i,\ r_4 \approx 0.4135410939 + 0.09991760306i,\\
&r_5 \approx 3.614831595 - 1.802444362i,\ r_6 \approx 3.614831595 + 1.802444362i.
\end{aligned}$$

$x_4 \in (0,1)$ so $x_4 \in \{r_1, r_2\}$. If $x_4 = r_2$ then for every $r_2' \in [0.42, 0.43]$, $g_2(r_2', x_3) = 0$ iff $x_3 > 1$ – a contradiction to $x_3 < 1$. Therefore $x_4 = r_1$.[2]

[2]See **??** for how to approximate the roots of g_1 using exact arithmetic without relying on the problematic floating point arithmetic.

Using the equalities $g_2 = g_3 = g_4 = 0$ we arrive at the approximations

$$x_1 \approx 0.529270752820,\ x_2 \approx 0.846414728986,$$
$$x_3 \approx 0.523440476515,\ x_4 \approx 0.320065197645.$$

We now show that x_1, x_2, x_3, x_4 are irradical. First, for each of the 3 monomial orderings generated lexicographically by $x_2 \succ x_3 \succ x_4 \succ x_1$, $x_3 \succ x_4 \succ x_1 \succ x_2$ and $x_4 \succ x_1 \succ x_2 \succ x_3$, we find corresponding Gröbner bases, in addition to the one that we found initially. From them we collect the univariate polynomials

$$\begin{aligned}
P_1(y) &= 7y^6 - 42y^5 + 89y^4 - 83y^3 + 40y^2 - 10y + 1,\\
P_2(y) &= 4y^6 - 27y^5 + 70y^4 - 79y^3 + 45y^2 - 13y + 1,\\
P_3(y) &= 140y^6 - 511y^5 + 701y^4 - 454y^3 + 141y^2 - 19y + 1,\\
P_4(y) &= 5y^6 - 44y^5 + 143y^4 - 163y^3 + 85y^2 - 21y + 2.
\end{aligned}$$

which we further know satisfy $P_i(x_i) = 0$. These exist by the elimination theorem for Gröbner bases (**?**).

Now, we use Murty's criterion (**?**) to show that P_1, P_2, P_3, P_4 are all irreducible over $\mathbb{Q}$.

1. P_1 is irreducible because $P_1(18) = 167595301$ is prime and the H value of P_1 is $\frac{89}{7} \approx 12.714$.
2. P_2 is irreducible because $P_2(28) = 1504207909$ is prime and the H value of P_2 is $\frac{79}{4} = 19.75$.
3. P_3 is irreducible because $P_3(11) = 175397399$ is prime and the H value of P_3 is $\frac{701}{140} \approx 5.007$.
4. P_4 is irreducible because $P_4(39) = 13945135583$ is prime and the H value of P_4 is $\frac{163}{5} = 32.6$.

As a result, P_i is the minimal polynomial of x_i (up to normalizing the leading coefficient of P_i), so x_i is irrational for every i. Now, it can be verified that the Galois group of P_i is S_6 for every $i \in \{1, 2, 3, 4\}$ (we used Magma (**?**)). Therefore x_1, x_2, x_3, x_4 are all irradical. This concludes the proof. $\square$

5. Five players

Define the $2 \times 2 \times 2 \times 2 \times 2$ game G_5:

$$\left[\begin{array}{ccccc|ccccc}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 2 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1
\end{array}\right]$$

Figure 3: The game G_5

Proposition 5.1. *G_5 has a unique NE* $((x_1, 1-x_1), (x_2, 1-x_2), (x_3, 1-$

$x_3), (x_4, 1 - x_4), (x_5, 1 - x_5))$ *given by*

$$x_1 \approx 0.350370422,\ x_2 \approx 0.646516479,\ \ x_3 \approx 0.648711818,$$
$$x_4 \approx 0.371748770,\ x_5 \approx 0.368061687.$$

The probabilities in the NE are all irrational and irradical.

Proof. First observe that $\mathrm{PNE}(G_4) = \emptyset$ using this table of profitable deviations:

action profile	unsatisfied players
$(0,0,0,0,0)$	4
$(0,0,0,0,1)$	$3,5$
$(0,0,0,1,0)$	2
$(0,0,0,1,1)$	$1,2$
$(0,0,1,0,0)$	$2,4$
$(0,0,1,0,1)$	$1,5$
$(0,0,1,1,0)$	5
$(0,0,1,1,1)$	3
$(0,1,0,0,0)$	$3,4,5$
$(0,1,0,0,1)$	$1,2$
$(0,1,0,1,0)$	3
$(0,1,0,1,1)$	$1,4$
$(0,1,1,0,0)$	$4,5$
$(0,1,1,0,1)$	2
$(0,1,1,1,0)$	2
$(0,1,1,1,1)$	$1,2$
$(1,0,0,0,0)$	$1,4$
$(1,0,0,0,1)$	$1,3,5$
$(1,0,0,1,0)$	$1,2$
$(1,0,0,1,1)$	$2,4$
$(1,0,1,0,0)$	$1,2,4$
$(1,0,1,0,1)$	5
$(1,0,1,1,0)$	5
$(1,0,1,1,1)$	3
$(1,1,0,0,0)$	$1,3,4,5$
$(1,1,0,0,1)$	2
$(1,1,0,1,0)$	$1,3$
$(1,1,0,1,1)$	4
$(1,1,1,0,0)$	$1,4,5$
$(1,1,1,0,1)$	$2,3$
$(1,1,1,1,0)$	$1,2,5$
$(1,1,1,1,1)$	$2,4$

Now the polynomials f_1, f_2, f_3, f_4, f_5 from Equation **??** are

$$\begin{aligned}
f_1(x_2, x_3, x_4, x_5) =& -5x_2x_3x_4x_5 + 4x_2x_3x_4 + 2x_2x_3x_5 - x_2x_3 + 3x_2x_4x_5 \\
& - 2x_2x_4 - 2x_2x_5 + x_2 + x_3x_4x_5 - x_3x_4 - x_4x_5 + x_4 \\
& + 2x_5 - 1, \\
f_2(x_1, x_3, x_4, x_5) =& \, x_3x_4x_5 + 2x_3x_4 - 2x_3 - 2x_4x_5 + 1, \\
f_3(x_1, x_2, x_4, x_5) =& -2x_1x_2x_4x_5 + x_1x_2x_4 + x_1x_2x_5 + 2x_1x_4x_5 - x_1x_4 - x_1x_5 \\
& + 3x_2x_4x_5 - 3x_2x_4 + x_2 - x_4x_5 + x_4 - x_5, \\
f_4(x_1, x_2, x_3, x_5) =& \, 2x_1x_2x_3x_5 - 2x_1x_2x_3 - x_1x_2x_5 + x_1x_2 - x_1x_3x_5 + x_1x_3 \\
& + x_1x_5 - x_1 - x_2x_3x_5 + x_2x_3 + x_2x_5 - x_2 - 2x_5 + 1, \\
f_5(x_1, x_2, x_3, x_4) =& -x_1x_2x_3x_4 + x_1x_2x_3 + x_1x_2x_4 - x_1x_2 + x_1x_3x_4 - x_1x_3 \\
& - x_1x_4 + x_1 - x_2x_3x_4 + 3x_2x_4 - x_3x_4 + x_3 - 1.
\end{aligned}$$

We show using Lemma **??** that if $x_i \in \{0, 1\}$ for some i then $x \notin \mathrm{NE}(G_5)$. The proof proceeds as follows. For each i in turn, we fix $x_i \in \{0, 1\}$ and consider the subgame between the 4 players obtained by restricting player i to play the pure action corresponding to x_i. We then perform a case analysis to find all profiles x_{-i} for these 4 players which are NEs in this subgame. The case analysis is similar to the case analysis in the proof of Proposition **??**. We then show that for each such NE profile x_{-i}, $(2x_i - 1)f_i(x_{-i}) < 0$, meaning i is not playing a best response to x_{-i} in the full 5-player game G_5. So for player i to play a best response in any NE of the full 5-player game player, i will need to choose $x_i \in (0, 1)$.

There are 10 cases in the proof (one for each $i \in [5]$ and $x_i \in \{0, 1\}$). Here is additional information regarding these 10 cases (here the approximate values were computed using Mathematica):

Case 1: $x_1 = 0$. Here the unique partial NE for players $2, 3, 4, 5$ is given by

$$\begin{aligned}
x_2 \approx 0.650518016106, \;\; x_3 \approx 0.638238319763, \\
x_4 \approx 0.402794248582, \;\; x_5 \approx 0.433321011106.
\end{aligned}$$

And $f_1(x_{-1}) \approx 0.103778882911 > 0$. Therefore $x_1 \neq 0$ (observe that $|\nabla f_1| \leq 1000$ so the value of f_1 at the exact value of x_{-1} will still be > 0 when using 12-digit approximations).

Case 2: $x_1 = 1$. Here the unique partial NE for players $2, 3, 4, 5$ is given by

$$x_2 \approx 0.589697169563, \ x_3 \approx 0.681923203912,$$
$$x_4 \approx 0.338247172171, \ x_5 \approx 0.218624870529.$$

And $f_1(x_{-1}) \approx -0.245843038086 < 0$. Therefore $x_1 \in (0, 1)$.

Case 3: $x_2 = 0$. There is no partial NE where $x_1 \in (0, 1)$, so $x_2 \neq 0$.

Case 4: $x_2 = 1$. Here all partial NEs for players $1, 3, 4, 5$ are given by

$$x_1 \in (0, 1), \ x_3 = 1, \ x_4 = 0, \ x_5 = \frac{1}{2}.$$

And $f_2(x_{-2}) = -1 < 0$. Therefore $x_2 \in (0, 1)$.

Case 5: $x_3 = 0$. There is no partial NE where $x_1, x_2 \in (0, 1)$, so $x_3 \neq 0$.

Case 6: $x_3 = 1$. There is no partial NE where $x_1, x_2 \in (0, 1)$, so $x_3 \in (0, 1)$.

Case 7: $x_4 = 0$. There is no partial NE where $x_1, x_2, x_3 \in (0, 1)$, so $x_4 \neq 0$.

Case 8: $x_4 = 1$. There is no partial NE where $x_1, x_2, x_3 \in (0, 1)$, so $x_4 \in (0, 1)$.

Case 9: $x_5 = 0$. There is no partial NE where $x_1, x_2, x_3, x_4 \in (0, 1)$, so $x_5 \neq 0$.

Case 10: $x_5 = 1$. There is no partial NE where $x_1, x_2, x_3, x_4 \in (0, 1)$, so $x_5 \in (0, 1)$.

Therefore the only NEs are fully mixed. This means that $x \in \mathrm{NE}(G_5)$ satisfies $f_i(x) = 0$ for every i. We solve this now.

Let $F = \{f_1, f_2, f_3, f_4, f_5\}$. It is a set of polynomials in the polynomial ring $\mathbb{Q}[x_1, x_2, x_3, x_4, x_5]$. We begin by finding the Gröbner basis $G = \{g_1, g_2, g_3, g_4, g_5\}$ of the ideal $\langle F \rangle$, with the monomial ordering generated lexicographically by $x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5$, given in full in **??** (using Mathematica).

We now compute the solutions of $\forall i \ g_i = 0$ to find those of $\forall i \ f_i = 0$. The proof proceeds as follows. We observe that like in the proof of Proposition **??**, $g_1 \in \mathbb{Q}[x_5]$ and for $i \geq 2$, $g_i = a_i x_{6-i} + h_i(x_5)$ where $a_i \in \mathbb{Z}$ and $h_i \in \mathbb{Q}[x_5]$. So we first find the roots of g_1, using a similar argument to the one in the proof of Proposition **??** and **??**, and find 5 roots of g_5 in $(0,1)$, meaning 5 possible values for x_5. Then for each of the latter four we show using $g_2,\dots,g_5$ that one of x_1,x_2,x_3,x_4 falls outside $(0,1)$, implying a contradiction. In conclusion, there is a unique possible value for x_5 and by the structure of $g_2,\dots,g_5$ a unique possible value for each of x_1,x_2,x_3,x_4. This solution is given in the approximated decimal form:

$$x_1 \approx 0.3503704221,\ x_2 \approx 0.6465164785,\ \ x_3 \approx 0.6487118183,$$
$$x_4 \approx 0.3717487703,\ x_5 \approx 0.3680616872.$$

To see that x_1,x_2,x_3,x_4,x_5 are irradical, as before, we find their minimal polynomials (up to scaling):

$$\begin{aligned}P_1(y) = {} & 576y^{26} + 7360y^{25} + 4496y^{24} + 250816y^{23} + 355564y^{22}\\ & - 7898280y^{21} + 7658244y^{20} + 75548910y^{19} - 142993432y^{18}\\ & - 189077220y^{17} + 334832314y^{16} + 474658874y^{15} + 1696776519y^{14}\\ & - 7562033350y^{13} + 1512397109y^{12} + 14656660866y^{11}\\ & - 10470778075y^{10} - 3782764895y^{9} - 665928467y^{8} + 10050992181y^{7}\\ & - 5447529632y^{6} - 1586217407y^{5} + 562969676y^{4} + 1493433257y^{3}\\ & - 79516172y^{2} - 522298360y + 132583232,\end{aligned}$$

$$\begin{aligned}P_2(y) = {} & 2057728y^{26} - 47527232y^{25} + 374368064y^{24} - 482543600y^{23}\\ & - 11253727536y^{22} + 90191808584y^{21} - 352161035060y^{20}\\ & + 829456047656y^{19} - 1120735934348y^{18} + 286387243298y^{17}\\ & + 2393470340090y^{16} - 6252418753985y^{15} + 9238553888534y^{14}\\ & - 9587641717941y^{13} + 7391393209142y^{12} - 4277517478697y^{11}\\ & + 1815337752171y^{10} - 515212602860y^{9} + 59595129244y^{8}\\ & + 25514336227y^{7} - 17322169528y^{6} + 5510430025y^{5} - 1139164516y^{4}\\ & + 160112123y^{3} - 14874613y^{2} + 828146y - 20988,\end{aligned}$$

$$
\begin{aligned}
P_3(y) = {} & 5828y^{26} - 80590y^{25} + 471147y^{24} - 1473516y^{23} \\
& + 1995893y^{22} + 3280961y^{21} - 21791522y^{20} + 51425278y^{19} \\
& - 93080861y^{18} + 203283288y^{17} - 444991348y^{16} + 713613468y^{15} \\
& - 837466118y^{14} + 925602099y^{13} - 1210417319y^{12} + 1552957912y^{11} \\
& - 1585613560y^{10} + 1241271492y^{9} - 772369636y^{8} + 401799920y^{7} \\
& - 180281904y^{6} + 69151344y^{5} - 21721184y^{4} + 5260240y^{3} \\
& - 909264y^{2} + 99200y - 5120,
\end{aligned}
$$

$$
\begin{aligned}
P_4(y) = {} & 367535904y^{26} - 4069658144y^{25} + 15863607440y^{24} + 4840925000y^{23} \\
& - 332049318712y^{22} + 1798491222250y^{21} - 5880537317282y^{20} \\
& + 13925726681306y^{19} - 25570689763592y^{18} + 37716767814324y^{17} \\
& - 45641117387319y^{16} + 45922091951685y^{15} - 38746001553780y^{14} \\
& + 27551639828330y^{13} - 16547091697878y^{12} + 8389403526528y^{11} \\
& - 3578413101941y^{10} + 1274916123621y^{9} - 374578292008y^{8} \\
& + 88741617902y^{7} - 16260708632y^{6} + 2104604912y^{5} - 141853168y^{4} \\
& - 7308320y^{3} + 2769472y^{2} - 256128y + 8192,
\end{aligned}
$$

$$
\begin{aligned}
P_5(y) = {} & 25772032y^{26} - 399987456y^{25} + 2634374272y^{24} - 8416506944y^{23} \\
& + 2215910496y^{22} + 110722637568y^{21} - 612619393968y^{20} \\
& + 2025933659884y^{19} - 4916502391844y^{18} + 9383743371458y^{17} \\
& - 14566831934444y^{16} + 18743281388994y^{15} - 20217905397986y^{14} \\
& + 18405783324440y^{13} - 14192403999687y^{12} + 9280251573089y^{11} \\
& - 5141465321619y^{10} + 2406410117193y^{9} - 946553896669y^{8} \\
& + 310404824295y^{7} - 83872244335y^{6} + 18358684229y^{5} \\
& - 3175419088y^{4} + 417903630y^{3} - 39340244y^{2} + 2360724y - 67892.
\end{aligned}
$$

These can all be verified to be irreducible using Mathematica. Then, the Galois groups of each of these 5 polynomials can be verified to be all S_{26} which is unsolvable (using e.g., Magma). Therefore x_1, x_2, x_3, x_4, x_5 are all irradical. This concludes the proof. □

6. Six players and up

Here we use the constructions of Section **??** and Section **??** to construct games G_n for every $n \geq 6$ with the same property as G_4 and G_5. This completes the picture in terms of the (general worst case) inexpressibility of NEs with radicals.

First, we use the notion of *product of games*:

Definition 6.1. *Given are games H^1, H^2 such that H^j has n_j players, the pure action profiles A^j and the payoff functions $u_1^j, \dots, u_{n_j}^j$. Define the* product game $H^1 \times H^2$ *as follows:*

- *there are $n_1 + n_2$ players,*
- *the set of pure action profiles is $A^1 \times A^2$,*
- *the payoff functions are $u_i(a^1, a^2) = u_i^1(a^1)$ if $i \in [n_1]$ and $u_i^2(a^2)$ if $i \in \{n_1 + 1, \dots, n_1 + n_2\}$.*

If $H_1 = H_2$, also write $H_1 \times H_2 = H_1^{\times 2}$, and generalize the notation to a product of q copies of H_1 by $H_1^{\times q}$.

This operation is associative, therefore $H_1^{\times q}$ is well defined. Here is a property of $H^1 \times H^2$ that follows by definition:

Lemma 6.2. $\mathrm{NE}(H^1 \times H^2) = \mathrm{NE}(H^1) \times \mathrm{NE}(H^2)$.

Using this lemma, for every $n = 4n_1 + 5n_2$ where $n_1, n_2 \in \mathbb{N}$ are not both 0, the game $G_4^{\times n_1} \times G_5^{\times n_2}$ has a unique NE and its probabilities are all irradical (see Proposition **??** below). However, these constructions do not cover the cases $n = 6, 7, 11$. To fill these holes, we now show how to add 2 players to each of G_4 and G_5 while maintaining their desired properties (a unique NE, whose probabilities are irradical).

Define the $2 \times 2 \times 2$ game H_3:

$$
\left[\begin{array}{ccc|ccc}
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0
\end{array}\right]
$$

Figure 4: The game H_3

Here is the main property of interest of this game:

Proposition 6.3. *Let* $x = ((x_1, 1-x_1), (x_2, 1-x_2), (x_3, 1-x_3))$ *such that* $x_1 \in (0,1)$. *Then* $x \in \mathrm{NE}(H_3)$ *iff* $x_1 = x_2 = x_3$.

Proof. First, the polynomials f_1, f_2, f_3 from Equation **??** of H_3 are

$$f_1(x_2, x_3) = 0,\ \ f_2(x_1, x_3) = x_1 - x_3,\ \ f_3(x_1, x_2) = x_2 - x_1.$$

By definition, for every $x_1 \in (0,1)$ setting $x_1 = x_2 = x_3$ means that $f_1(x) = f_2(x) = f_3(x) = 0$, meaning that $x \in \mathrm{NE}(H_3)$. Now we show that if $x_1 \in (0,1)$ and $x \in \mathrm{NE}(H_3)$ then $x_1 = x_2 = x_3$.

We use Lemma **??**. $f_1 = 0$ so x_1 can take on any value, regardless of the values of x_2, x_3. Assume that $x_1 \in (0,1)$. Suppose for contradiction that $x_2 = 0$. Then $f_3 = -x_1 < 0$, so $x_3 = 0$ so $f_2 = x_1 > 0$ so $x_2 = 1$ – a contradiction. Similarly, if $x_2 = 1$, then $f_3 = 1 - x_1 > 0$ so $x_3 = 1$ so $f_2 = x_1 - 1 < 0$ so $x_2 = 0$ – again a contradiction. Therefore $x_2 \in (0,1)$ so $f_2 = 0$ so $x_3 = x_1$ so $f_3 = 0$ so $x_1 = x_2 = x_3$. $\square$

Next we define a variation of the product game with a small overlap in the players.

Definition 6.4. *Let* G *be a game such that*

- *there are* n *players,*
- *each player has* 2 *pure actions,*
- *the payoffs are given by functions* $u_1, \ldots, u_n$,
- *there is a unique NE, and this NE is fully mixed.*

Define the game $G \circ H_3$ *as follows:*

- *there are* $n+2$ *players,*
- *each player has* 2 *pure actions,*
- *the payoff functions are given by* $u_i(a, a_{n+1}, a_{n+2}) = u_i(a)$ *for* $i \in [n]$, *and for* $i \in \{n+1, n+2\}$ *the payoff is defined as* $u_i(a, a_{n+1}, a_{n+2}) = u^{H_3}_{i-(n-1)}(a_n, a_{n+1}, a_{n+2})$ *(as defined in Figure* **??***).*

The most important thing about this operation $\circ H_3$ is that it is comptaible with G_4 and G_5. The key property of $G \circ H_3$ is as follows:

Lemma 6.5. *If* $\mathrm{NE}(G) = \{((x_1, 1-x_1), \ldots, (x_n, 1-x_n))\}$ *and* $x_n \in (0,1)$ *then*

$$\mathrm{NE}(G \circ H_3) = \{((x_1, 1-x_1), \ldots, (x_n, 1-x_n), (x_n, 1-x_n), (x_n, 1-x_n))\}.$$

Proof. By definition. Observe that a NE of $G \circ H_3$, restricted to players $n, n+1, n+2$, is a NE to H_3, because the payoff of the first player in H_3 (meaning player n in $G \circ H_3$) is constant. Since player n plays $(x_n, 1-x_n)$ in the only NE of G, player n will need to play it in any NE of $G \circ H_3$. Therefore we can invoke Proposition **??** here to get the desired result. $\square$

Now the games G_n for $n \geq 6$ follow. Let $n \geq 6$. Write $n = 4q + r$ where $1 \leq q \in \mathbb{N}$ and $r \in \{0, 1, 2, 3\}$. Define

$$G_n = \begin{cases} G_4^{\times q} & r = 0, \\ G_4^{\times q-1} \times G_5 & r = 1, \\ G_4^{\times q-1} \times (G_4 \circ H_3) & r = 2, \\ G_4^{\times q-1} \times (G_5 \circ H_3) & r = 3. \end{cases} \tag{3}$$

Proposition 6.6. *For every* $n \geq 6$*, the game* G_n *has a unique NE, and the probabilities in this NE are all irradical.*

Proof. Let $n = 4q + r$. By Proposition **??**, Proposition **??** and Lemma **??** we obtain that $G_4 \circ H_3$ and $G_5 \circ H_3$ both have a unique NE, and that NE is irradical in both games. Now, for any $n \geq 6$, G_n is a product of games which all have a unique NE, and that NE is irradical in all those games. Therefore each G_n has a unique NE, and its probabilities are all irradical. $\square$

Acknowledgements

We thank Teo Collins for the reference to Gröbner bases.

Appendix A. Further details in the proof of Proposition ??

In the proof of Proposition **??** we use the approximations of the roots of g_1 to determine the unique value of x_4. To circumvent possible floating point issues, we can use the Sturm sequence of g_1 to determine using exact arithmetic that (1) there are exactly 2 real roots, and (2) one is in the interval $(0.3, 0.4]$ and one in the interval $(0.4, 0.5]$ (**?**). The Sturm sequence of g_1 can be found using Mathematica:

$$\begin{aligned}
p_0 =& g_1 = 5x_4^6 - 44x_4^5 + 143x_4^4 - 163x_4^3 + 85x_4^2 - 21x_4 + 2,\\
p_1 =& g_1' = 30x_4^5 - 220x_4^4 + 572x_4^3 - 489x_4^2 + 170x_4 - 21,\\
p_2 =& -\operatorname{Remainder}(p_0, p_1) = \frac{55}{9}x_4^4 - \frac{5249}{90}x_4^3 + \frac{943}{15}x_4^2 - \frac{433}{18}x_4 + \frac{47}{15},\\
p_3 =& -\operatorname{Remainder}(p_1, p_2) = -\frac{27110403}{30250}x_4^3 + \frac{15927363}{15125}x_4^2 - \frac{2514591}{6050}x_4\\
&+ \frac{831852}{15125},\\
p_4 =& -\operatorname{Remainder}(p_2, p_3) = \frac{4813052861375}{81663772313601}x_4^2 - \frac{1017283844500}{27221257437867}x_4\\
&+ \frac{417696219875}{81663772313601},\\
p_5 =& -\operatorname{Remainder}(p_3, p_4) = \frac{4666740312733677523326}{1531601841083641320125}x_4\\
&- \frac{1979941124138191228 7163}{1531601841083641320125},\\
p_6 =& -\operatorname{Remainder}(p_4, p_5) = \frac{35060832021368694480186665125}{266676959650248143316770013086 76}.
\end{aligned}$$

where $\operatorname{Remainder}(f, g)$ is the remainder of the polynomial division of f by g.

For every $a \in \mathbb{R} \cup \{-\infty, \infty\}$, let $V(a)$ be the number of sign changes in the sequence $(p_0, p_1, p_2, p_3, p_4, p_5, p_6)$ (and the sign of something like $p_0(\infty)$ is defined to be $+$). Sturm's theorem states that for every $-\infty \leq a < b \leq \infty$, the number of real roots in $(a, b]$ is equal to $V(a) - V(b)$.

So we have $V(-\infty) - V(\infty) = 4 - 2 = 2$, so there are 2 real roots. Note that

$$\begin{aligned}
p_0\left(\frac{3}{10}\right) &= \frac{161}{40000},\\
p_1\left(\frac{3}{10}\right) &= -\frac{2751}{10000},\\
p_2\left(\frac{3}{10}\right) &= \frac{371}{7500},\\
p_3\left(\frac{3}{10}\right) &= \frac{26761959}{30250000},\\
p_4\left(\frac{3}{10}\right) &= -\frac{258737930605}{326655089254404},\\
p_5\left(\frac{3}{10}\right) &= -\frac{28996945737809045150826}{7658009205418206600625},\\
p_6\left(\frac{3}{10}\right) &= \frac{3506083202136869448018665125}{26667695965024814331677001308676},
\end{aligned}$$

and

$$\begin{aligned}
p_0\left(\frac{4}{10}\right) &= -\frac{4}{3125},\\
p_1\left(\frac{4}{10}\right) &= \frac{27}{625},\\
p_2\left(\frac{4}{10}\right) &= -\frac{4}{625},\\
p_3\left(\frac{4}{10}\right) &= -\frac{236727}{1890625},\\
p_4\left(\frac{4}{10}\right) &= -\frac{32955935705}{81663772313601},\\
p_5\left(\frac{4}{10}\right) &= -\frac{5663575581442206389163}{7658009205418206600625},\\
p_6\left(\frac{4}{10}\right) &= \frac{3506083202136869448018665125}{26667695965024814331677001308676},
\end{aligned}$$

and

$$
\begin{aligned}
p_0\left(\frac{5}{10}\right) &= \frac{1}{64}, \\
p_1\left(\frac{5}{10}\right) &= \frac{7}{16}, \\
p_2\left(\frac{5}{10}\right) &= -\frac{31}{360}, \\
p_3\left(\frac{5}{10}\right) &= -\frac{383139}{242000}, \\
p_4\left(\frac{5}{10}\right) &= \frac{380134673875}{326655089254404}, \\
p_5\left(\frac{5}{10}\right) &= \frac{28271671319879411796}{12252814728669130561}, \\
p_6\left(\frac{5}{10}\right) &= \frac{35060832021368694480l8665125}{266676959650248143316770013086 76}.
\end{aligned}
$$

So $V(0.3) - V(0.4) = 4 - 3 = 1$ and $V(0.4) - V(0.5) = 3 - 2 = 1$, so $(0.3, 0.4]$ and $(0.4, 0.5]$ each contain exactly one of the real roots. More specifically, $r_1 \in (0.3, 0.4]$ and $r_2 \in (0.4, 0.5]$.

From here the argument in the proof of Proposition **??** that x_4 cannot be r_2 holds and the rest of the proof goes through.

As for r_1, the method above can be further used to verify that $r_1 \in (0.320065197644, 0.320065197645]$ using exact arithmetic, justifying the approximation for $x_4 = r_1$ given in the proposition statement.

Appendix B. The Gröbner basis in Proposition ??

Here is the Gröbner basis G used in Proposition **??**. It is $\{g_1, g_2, g_3, g_4, g_5\}$ given as follows:

$$
\begin{aligned}
g_1 = {} & 25772032x_5^{26} - 399987456x_5^{25} + 2634374272x_5^{24} - 8416506944x_5^{23} \\
& + 2215910496x_5^{22} + 110722637568x_5^{21} - 612619393968x_5^{20} \\
& + 2025933659884x_5^{19} - 4916502391844x_5^{18} + 9383743371458x_5^{17} \\
& - 14566831934444x_5^{16} + 18743281388994x_5^{15} - 20217905397986x_5^{14} \\
& + 18405783324440x_5^{13} - 14192403999687x_5^{12} + 9280251573089x_5^{11} \\
& - 5141465321619x_5^{10} + 2406410117193x_5^{9} - 946553896669x_5^{8} \\
& + 310404824295x_5^{7} - 83872244335x_5^{6} + 18358684229x_5^{5} - 3175419088x_5^{4} \\
& + 417903630x_5^{3} - 39340244x_5^{2} + 2360724x_5 - 67892,
\end{aligned}
$$

$$
\begin{aligned}
g_2 = {} & -5229757014096166506106706852927054776656409436526688311934217666969182934601238598093813122993784633757481615360x_5^{25} \\
& + 774173483818074381392987127517955248377722562966588994657980715612631081132631681151873373891900070084137706181 12x_5^{24} \\
& - 4790035618654289755558997144360069587660263006580083748192912402227773236122179050038500651271105973180884786417 92x_5^{23} \\
& + 1363516171631648731401937260410517759345140344024728313437486577525612859813544026537686042034050828432977603879232x_5^{22} \\
& + 5335879225878207297080443354664425894792788589271426427028318285824428632506338194289854491186232578989914100371 52x_5^{21} \\
& - 2210014743880930808886181984745277916246922569120706447093220056861206622013911347639645963935637972228087000517 5520x_5^{20} \\
& + 1084549009645354526432806737454818104790682789314804025517332190206543946583316612851360034676950441772903320080 77552x_5^{19} \\
& - 3330737829199574590272788049218699807269972418823910799405349259562289697967257108150194695329919764473835446745 26084x_5^{18} \\
& + 7576242336733383013594405527129045604895027172711129054709499257905329052363344104727372361499401998906861329296 21704x_5^{17} \\
& - 1357384404662525914311341693592403279296839308873698777339142031034705771862285501894438660803278932926999068794 546582x_5^{16} \\
& + 1974990208789200599466586861637319326312916081007911794448182406682496223056735984172167459256026946263774303269 130502x_5^{15} \\
& - 2374297193220515593964694404791585125075717016662628050023610781028665713690408237409030760913488268277385743966 777032x_5^{14} \\
& + 2382354902550652961280745151068702383273085454800813608807850901474781766931357892102898412048158076148480110682 671270x_5^{13} \\
& - 2006547338020399904478992449241580128008830980255928702874144615524529032076799032362502252260795483084720768354 418906x_5^{12} \\
& + 1422303712594438667631688360360711116173856475208264246894179873753516371223301768725492398493859422183929726651 437959x_5^{11} \\
& - 8485918095728074465267373070281160876641968303026774616876793295650880444035839498460814710772080297728457337839 78556x_5^{10} \\
& + 4252567123578404290415848551082591948432599267869498528221998519205027567540066323762926560841881791522653247683 87541x_5^{9} \\
& - 1781901856967982195738881347237581971016576222677164282476874626548050370793929826644821269493373620000593589030 70826x_5^{8} \\
& + 6196879620122294217957789791154531974796634754181209846258527046789000627470501986119247534696330589498180426885 1065x_5^{7} \\
& - 1768691436832142060439559003161993146735311922111270831005899078591743199415757130689348209669426041271597587400 6756x_5^{6} \\
& + 4075259952587509321223128820658917157429499391726104350291091494649891620567629499432147402492820100061680066097 185x_5^{5} \\
& - 7397214546617625618291507790367663860285743195804983944747292423465409978924308997467459783542735734580802786292 82x_5^{4} \\
& + 1018895566270337800677069634051807707310468405511120965277967780963486725201532439359351667395458574502544586650 78x_5^{3} \\
& - 1001496158105932529753408379942322223219317318241527800335634510771505477589075368031578153361542875953189484042 8x_5^{2} \\
& + 626198151376926556516994281746232986281114459167528107589000512836306110760157779105130081244956865692258077872x_5 \\
& + 145627908115322321050302947809087035771277414538673170516253305847464363302676983122017923233931954755 84x_4 \\
& - 18730395737658453954483839445489626955207312148851486189022226778682417235942672754625522902251804284010201836,
\end{aligned}
$$

$$
\begin{aligned}
g_3 = {} & 233454475115507188202672278006134924164259902606867657494624015461952991856180048674891760240983430724994662 4x_5^{25} \\
& - 348263862727326040748206285038937991771944472173750967754836723388601189861038891740630509119271416108103795 20x_5^{24} \\
& + 217633964950393711549186629464978437165300442561564386772405822217081697486573874407123129145535922950706091 648x_5^{23} \\
& - 631004665463527900669117543672267152557837342795954164588602147543470354961220923336218777672617507047396540 736x_5^{22} \\
& - 181187120952388236998386793367411756303633247262581447028534281445658265062871643279630089023039796374318343 072x_5^{21} \\
& + 992541676357092110848092282129061135926179044048373923508802872852485257891187791475766373029379651642453698 2112x_5^{20} \\
& - 495110253626381199890000554446406881807131014943121289221850561302005249690067567982216218317442287177471640 71536x_5^{19} \\
& + 153606450244873998637400182694657307098185015597200929534563721055218708062776648371930100226217096327071566 507812x_5^{18} \\
& - 352426714217615075454859339166993063141216123999988812644511442641474817930073511713523770604296026296813438 396088x_5^{17} \\
& + 636541832234522264020461181180548709485760714008828971077784400214808511794157519019430471954449244547200377 722358x_5^{16} \\
& - 933498979637348381777015122675301004004870810348162357840262569632788867775411763510678948565334319576360783 673102x_5^{15} \\
& + 113107533723436753641825619078103560757145424060520625623075328471141495141489150560286022905057440865015522 6782728x_5^{14} \\
& - 114388852554612538129338126440325880673968019279715481750424462648988728271062721113265777261773845344329207 5022774x_5^{13} \\
& + 971114741160926999426755391516911794308562783195876583953124144282681326795227451967634738501132063379159602 305106x_5^{12} \\
& - 693868399603588601921565117253658200432981408294055991494009684604764816981147591464337021509223488254637528 974415x_5^{11} \\
& + 417307080155928782025068977202423092573461396691977026044607405101837819435313315500107795471975427014871318 266976x_5^{10} \\
& - 210801786743647026314962428699955929372925777966515764581497272997599561283615798792354612095177686041986524 450961x_5^{9} \\
& + 890333638601945206708139569861475296206065152775852981692828019288533968356295227910713597036632759438429565 80202x_5^{8} \\
& - 312072371324275089822737892076298953528933444502925210488423950542129585154602408186693505649894925358689997 25081x_5^{7} \\
& + 897645994233158727927061859812134473570442930084888881587707022925497755113618971707323918683315100786511603 7992x_5^{6} \\
& - 208414907149459906663684090436379873670817643931255131045940802800268736273369290070164627140865826746774309 9101x_5^{5} \\
& + 381158308756241245183778096001699736234731027675341960830001210231765721434485154435182600335913406210638306 866x_5^{4} \\
& - 528894071108805922409350355646524676388603343856759514403772638602273914439550384305658262220210872540006884 446x_5^{3} \\
& + 523630710920529569018320942659998910795867806627666597427120184716105796226914683725013064574471591337181041 2x_5^{2} \\
& - 329731014995559194481139244161781866865235013681345347340507275899274691789786673963774969773751450999559968 x_5 \\
& + 369474411292823674535444822209852670927085163111090115546904181194345490372772487614214890573457740 8x_3 \\
& + 993136957275857724857553657476558176035461875380290770967577155713450750452366926818635429939712583452319 6,
\end{aligned}
$$

$$
\begin{aligned}
g_4 = & -355850075402433708277018015632282949859788628382190556290962687793362743786200407714329372420021900992788896x_5^{25} \\
& + 534595339590801172332381582449118568521321966790582944227525247655823011604715019826071674309740921150944355584x_5^{24} \\
& - 337159203512668656410477904168737985969382049375082645168114655955720623507549883202440822289987871942813580672x_5^{23} \\
& + 994379773429930614855305353972967341994717800292461875312328526647809247215078010238428657356468774616503704256x_5^{22} \\
& + 189245805202958885164717457856217010089940221854661319802127336154003800666496337272960302229722564154334014816x_5^{21} \\
& - 151971251545057369738136891551855654113387695836593011091778973299346273313392074194126351772083890358288278532280x_5^{20} \\
& + 770352006855652326081938568790057296484084689278296481722645738363449516575126358244104736668175865200565924466384x_5^{19} \\
& - 241401429737158706348114011840171158855788925159564563866972792216940735637246580890337940955337976062511379408780x_5^{18} \\
& + 558630449888944438077465724771239908940298471522896091656782046424166365437557890923179651315268131895457834737840x_5^{17} \\
& - 101722459559722813680472842623163692962105401764614802360181970803119666625761675585434123964131266058278666941225 0x_5^{16} \\
& + 150381704529622789247695085351283722162529464903420002474667912564951120712330140468645015480262896525183616315663 8x_5^{15} \\
& - 183689742829361026417737134671138395614629302529037683385386934064619215251938242801520385942998337786632728553444 0x_5^{14} \\
& + 187299578110831506749731246067625792762653342783719747416053550335794037985480962862670717603734223701101885758747 4x_5^{13} \\
& - 160337317090644475739803067622437349981187225472402094134353650044596295935322703424225475466590977909564337341857 8x_5^{12} \\
& + 115528804087850231957260812804997697506774511957464210339302007752906858834743737930772514148211239299378146101492 5x_5^{11} \\
& - 700697376259091308302109508302940157504814446394086929947963510799016007546849283458571446545182585269186009462998x_5^{10} \\
& + 356934874693265870665365221583700170939530876165407760810376569278622878215127192877970425179784984014582661641105x_5^{9} \\
& - 152001087875514448869480222043342566350774439463857717641593193787967622803143243410107426653140398261285409508742x_5^{8} \\
& + 537062960845162389339419393881112670701002983336081641232290655108267512359633939532597450318311458316627225115 11x_5^{7} \\
& - 155668724957602967706362020980106823301536788399797297237497240384844598532649458207243459170346241456008332190 06x_5^{6} \\
& + 364040366750473595590910156063689064714479942118836266142137019363713016881894599127866727396369239804727366645 3x_5^{5} \\
& - 670173581881674007325938393635716663699469008474575049226005766285927286886676689504925496451118751189529451670x_5^{4} \\
& + 935351044532945866154702333391182677358464413455552506923035741785030682965534381025444328001919898010536158 38x_5^{3} \\
& - 930527075636807225455973797782081171594176092343688475332155517570732868330574466019006622120243267668210510 8x_5^{2} \\
& + 588069610334459746429373597750058332341244517659467031174397469825967982566556498337804906924843284145942504x_5 \\
& + 509580528135160723389825923288056926717898171085611247214484524440657009542955603198877709407936 3968x_2 \\
& - 177493058189386032759296123401217224479887532519978413308486732079849913098099883414907160531098666375013 40,
\end{aligned}
$$

$$
\begin{aligned}
g_5 = {} & 9270243108944911855913423923940118860724549686336022978726911310961722458203883119715431043049984039600128x_5^{25} \\
& - 1381758752663371452234345662523550909720501686173336169315566550013695287026064971112830465848970481234961 92x_5^{24} \\
& + 8624923378702863192081071417139161781209149492689395396243118446224983182702203007524532158563159858979874 56x_5^{23} \\
& - 2495102461650519533506960742919876844791681341902817276675711486882107127757546342958357009723667467045595840x_5^{22} \\
& - 7496365541469475439364212596101105689248548707305775692362899360206501635683303065952354072177863645302224 32x_5^{21} \\
& + 3940258015829838461828067319024377336451574458313764839864734652404824663643443539210668085646231540677935 3056x_5^{20} \\
& - 1961221441477003484241574533297178580161196479239077355545344287934760559828543272450961351832813740078380 06224x_5^{19} \\
& + 6075674072207380669506458728235626293564811009523385818378490909425757905181840391476067802645433604047888 05388x_5^{18} \\
& - 1392034559331843275751880128686367355159492660859513756685269123763970049996344394913220744816481337236164 688752x_5^{17} \\
& + 2510558512785233508346113941902852818446593588622592969503358098040942372525809496185640497523813014968232 095706x_5^{16} \\
& - 3675774626200611204927149730882154016887117306216041250003754929446407301397549595440548848847486181249067 607390x_5^{15} \\
& + 4445471379414986634335124565974011524961495924338229912795322982376585158552336715200597575424938671909289 449640x_5^{14} \\
& - 4486118641224030633093582292243068588732437612076418746519037181965028002339068594596667317742675531315640 964002x_5^{13} \\
& + 3798874308555711598482873640101602165951250009581540519626436478636365965265691902241294062878777541961423 998546x_5^{12} \\
& - 2706192488036664027366752238840352044709794830418822829301155231922766756880774252399289499491242862301112 642493x_5^{11} \\
& + 1621786402882167377760236931048484324618716111562222154462408173075352980667480821275949711332943322996801 085942x_5^{10} \\
& - 8157977873949988434964427330116126108012500481642050775342109231836605303624091667660049662478380709091317 17313x_5^{9} \\
& + 3428427547956095414533953895485044555158650069857243257694958989055376720610532943255867959013823107030744 84590x_5^{8} \\
& - 1194642004880484517709693296619053535203591378132361394392078826769128875672749478222904251673283328196741 38255x_5^{7} \\
& + 3412459833800801453608972241927107590527546100060898429548320550302844288100797780421358615761656750276239 5862x_5^{6} \\
& - 7858371647506645013235958461172093630568069977126879574130925982176847618427475770739613780276848717890340 861x_5^{5} \\
& + 1423375266711743719442643469712816450858096992327301545160933803057177533347744363470751224596704554150864 598x_5^{4} \\
& - 1952781037722304752364622018520702325839536160275534651983628429487664198057455720800638988318209107608831 50x_5^{3} \\
& + 1907722911030273239358991830671587449405158270858555946927232633639071850846087752671012348413795280885227 6x_5^{2} \\
& - 1182613115373742448605260562696400057836772355234957546845962252089854311499956589035815582233191712586248x_5 \\
& + 2282271586572006225682226080911924528719125921756905312232077989532676330374023602284100216308249 6x_1 \\
& + 3497011205758160763948020864615676687663904474672495862257755076803448712442213603494233942908014508498 8.
\end{aligned}
$$

References

Artin, M., 2011. Algebra. Pearson Prentice Hall. URL: https://books.google.com/books?id=S6GSAgAAQBAJ.

Bilò, V., Mavronicolas, M., 2014. Complexity of rational and irrational nash equilibria. Theory Comput. Syst. 54, 491–527. doi:10.1007/S00224-013-9523-7.

Bosma, W., Cannon, J., Playoust, C., 1997. The Magma algebra system. I. The user language. J. Symbolic Comput. 24, 235–265. URL: http://dx.doi.org/10.1006/jsco.1996.0125, doi:10.1006/jsco.1996.0125. computational algebra and number theory (London, 1993).

Bubelis, V., 1979. On equilibria in finite games. Int. J. Game Theory 8, 65–79. doi:10.1007/BF01768703.

Cox, D.A., Little, J., O'Shea, D., 2010. Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra. 3rd ed., Springer Publishing Company, Incorporated.

Datta, R.S., 2003. Universality of nash equilibria. Math. Oper. Res. 28, 424–432. doi:10.1287/moor.28.3.424.16397.

Draper, T.L., Saad, F.A., 2025. Efficient rejection sampling in the entropy-optimal range. URL: https://arxiv.org/abs/2504.04267.

Flajolet, P., Pelletier, M., Soria, M., 2011. On buffon machines and numbers, in: Proceedings of the Twenty-Second Annual ACM-SIAM Symposium on Discrete Algorithms, Society for Industrial and Applied Mathematics, USA. p. 172–183.

Knuth, D., 1976. The complexity of nonuniform random number generation. Algorithm and Complexity, New Directions and Results .

Kreps, V.L., 1981. Finite N-person non-cooperative games with unique equilibrium points. Int. J. Game Theory 10, 125–129. doi:10.1007/BF01755957.

Mendo, L., 2025. Simulating a coin with irrational bias using rational arithmetic. Communications in Statistics - Simulation and Computation 54, 302–318. doi:10.1080/03610918.2024.2425702.

Murty, M.R., 2002. Prime numbers and irreducible polynomials. The American Mathematical Monthly 109, 452–458. URL: http://www.jstor.org/stable/2695645.

Nash, J.F., Shapley, L.S., 1950. 10. A simple three-person poker game. Princeton University Press, Princeton. pp. 105–116. doi:doi:10.1515/9781400881727-011.

Nisan, N., Roughgarden, T., Tardos, É., Vazirani, V.V. (Eds.), 2007. Algorithmic Game Theory. Cambridge University Press. doi:10.1017/CBO9780511800481.

Sturm, C., 1835. Mémoire sur la résolution des équations numériques. éditeur non identifié.

Wolfram Research, Inc., . Mathematica, Version 14.2. URL: `https://www.wolfram.com/mathematica`. champaign, IL, 2024.