ADDITIVE SUMSET SIZES WITH TETRAHEDRAL DIFFERENCES

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Abstract. Experimental calculations suggest that the h-fold sumset sizes of 4-element sets of integers are concentrated at h numbers that are differences of tetrahedral numbers. In this paper it is proved that these "popular" sumset sizes exist and explicit h-adically defined sets are constructed for each of these numbers.

1. The sumset size problem

The *h-fold sum* of a set A of integers is the set of all sums of h not necessarily distinct elements of A. The core problem of additive number theory is to understand h-fold sumsets.

If A is a finite set of k integers, then hA is a finite set and

$$h(k-1)+1 \leq |hA| \leq \binom{h+k-1}{k-1}. \tag{1}$$

We have $|hA| = h(k-1)+1$ if and only if A is an arithmetic progression of length k, and $|hA| = \binom{h+k-1}{k-1}$ if and only if A is a B_h-set, that is, a set such that every integer in the sumset hA has a unique representation (up to permutation of the summands) as a sum of h not necessarily distinct elements of A.

The *integer interval* defined by real numbers u and v is the set

$$[u,v] = \{n \in \mathbf{Z} : u \leq n \leq v\}.$$

The *integer part* of the real number u is denoted $[u]$.

Let $\mathcal{R}_{\mathbf{Z}}(h,k)$ be the set of h-fold sumset sizes of sets of size k, that is,

$$\mathcal{R}_{\mathbf{Z}}(h,k) = \{|hA| : A \subseteq \mathbf{Z} \text{ and } |A| = k\}.$$

Inequality (**??**) implies

$$\mathcal{R}_{\mathbf{Z}}(h,k) \subseteq \left[h(k-1)+1, \binom{h+k-1}{k-1}\right]. \tag{2}$$

Not every possible sumset size is actually the size of a sumset. For example, relation (**??**) gives

$$\mathcal{R}_{\mathbf{Z}}(3,3) \subseteq [7,10].$$

We have

$$3\{0,1,2\} = \{0,1,2,3,4,5,6\} \qquad \text{and} \qquad |3\{0,1,2\}| = 7$$

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$$3\{0,1,3\} = \{0,1,2,3,4,5,6,7,9\} \qquad \text{and} \qquad |3\{0,1,3\}| = 9$$
$$3\{0,1,4\} = \{0,1,2,3,4,5,6,8,9,12\} \qquad \text{and} \qquad |3\{0,1,4\}| = 10$$

and so

$$\{7,9,10\} \subseteq \mathcal{R}_{\mathbf{Z}}(3,3).$$

However, there exists no set A of integers with $|A| = 3$ and $|3A| = 8$ (Nathanson [?]). Thus,

$$\mathcal{R}_{\mathbf{Z}}(3,3) = \{7,9,10\}.$$

This example motivates the following problem: For all positive integers h and k, determine the full range of sumset sizes of h-fold sums of sets of k integers, that is, compute the set $\mathcal{R}_{\mathbf{Z}}(h,k)$.[1] For all h and k, we have

$$\mathcal{R}_{\mathbf{Z}}(h,1) = \{1\} \qquad \text{and} \qquad \mathcal{R}_{\mathbf{Z}}(1,k) = \{k\}.$$

Sets A and B are *affinely equivalent* if there exist numbers $\lambda \neq 0$ and μ such that $\lambda * A + B = \{\lambda a + \mu : a \in A\}$. If A and B are affinely equivalent, then $|hA| = |hB|$ for all positive integers h. Every finite set A of integers is affinely equivalent to a set B with $\min B = 0$ and $\gcd(B) = 1$. In particular, every set of size 2 is affinely equivalent to the set $\{0,1\}$. It follows that

$$\mathcal{R}_{\mathbf{Z}}(h,2) = \{h+1\}.$$

Erdős and Szemerédi [?] stated that

$$\mathcal{R}_{\mathbf{Z}}(2,k) = \left[2k-1, \binom{k+1}{2}\right].$$

(A proof is in [?].) Thus, the unsolved problem is to determine $\mathcal{R}_{\mathbf{Z}}(h,k)$ for $h \geq 3$ and $k \geq 3$.

A first step is to fix a positive integer k and find the possible sizes of h-fold sums of sets of size k. Recall that the *jth triangular number* $f_2^j = \binom{j+1}{2}$ is the sum of the first j positive integers. The *jth tetrahedral number* $f_3^j = \binom{j+2}{3}$ is the sum of the first j triangular numbers (Dickson [?]). For $k = 3$, Nathanson [?] proved that

$$\mathcal{R}_{\mathbf{Z}}(h,3) = \left\{ \binom{h+2}{2} - \binom{i_0+1}{2} : i_0 \in [0, h-1] \right\}.$$

Thus, every sumset size of a 3-element set is of the form $f_2^{h+1} - f_2^{i_0}$, that is, a difference of triangular numbers. For $k \geq 4$, the problem is still open. Numerical experiments (Nathanson [?] and O'Bryant [?]) suggest that, for $k = 4$, the "most popular" sumset sizes are the integers

$$f_3^{h+1} - f_3^{i_0} = \binom{h+3}{3} - \binom{i_0+2}{3}$$

for $i_0 \in [0, h-1]$. These are the differences between the tetrahedral number $f_3^{h+1} = \binom{h+3}{3}$, which is also the size of a 4-element B_h-set, and the h consecutive tetrahedral numbers $f_3^0, f_3^1, \ldots, f_3^{h-1}$. It had been an open problem to decide if the integers $f_3^{h+1} - f_3^{i_0}$ are, indeed, sumset sizes for all $h \geq 2$ and $i_0 \in [0, h-1]$. The goal of this paper is to prove that these sumset sizes do exist for all h and i_0, that is,

$$\left\{ \binom{h+3}{3} - \binom{i_0+2}{3} : i_0 \in [0, h-1] \right\} \subseteq \mathcal{R}_{\mathbf{Z}}(h,4)$$

[1] There is the analogous problem in every additive abelian group or semigroup G: Determine the set $\mathcal{R}_G(h,k)$ of the sizes of h-fold sums of k-element subsets of G.

and to construct explicit h-adically defined sets with exactly these sumset sizes.

For related work on sumset size problems in additive number theory, see [?]–[?].

2. A family of h-adic sets

Theorem 1. *For all $h \geq 2$ and $i_0 \in [0, h-1]$, let*

$$c = (h+1-i_0)(h+1).$$

The set

$$A = \{0, 1, h+1, c\}$$

satisfies $|A| = 4$ and

$$|hA| = \binom{h+3}{3} - \binom{i_0+2}{3}.$$

Proof. The set

$$B = \{0, 1, h+1\}$$

is a B_h-set and so B is a B_{h-i}-set for all $i \in [0, h-1]$ and

$$|(h-i)B| = \binom{h-i+2}{2}.$$

Let $0B = \{0\}$.

We have

$$A = B \cup \{c\} = \{0, 1, h+1, (h+1-i_0)(h+1)\}.$$

The inequality $h+1-i_0 \geq 2$ implies $|A| = 4$.

If $i_0 = 0$, then $c = (h+1)^2$ and

$$A = \left\{0, 1, h+1, (h+1)^2\right\}.$$

The uniqueness of $(h+1)$-adic representations implies that A is a B_4-set and so

$$|hA| = \binom{h+3}{3} = \binom{h+3}{3} - \binom{i_0+2}{3}.$$

Let $i_0 \in [1, h-1]$. We decompose the sumset hA as follows:

$$hA = \bigcup_{i=0}^{h} ((h-i)B + ic) = \bigcup_{i=0}^{h} L_i$$

where

$$\begin{aligned} L_i &= (h-i)B + ic \\ &= \bigcup_{j=0}^{h-i} ((h-i-j)(h+1) + [0,j]) + ic \\ &= \bigcup_{j=0}^{h-i} ((h+(h-i_0)i-j)(h+1) + [0,j]). \end{aligned} \tag{3}$$

We have

$$|L_i| = |(h-i)B + ic| = |(h-i)B| = \binom{h-i+2}{2}. \tag{4}$$

The set L_i is the union of $h-i+1$ pairwise disjoint integer intervals whose smallest elements are multiples of $h+1$ and whose lengths are at most h. It follows that if $n \in L_i$ and $n = q(h+1)+r$ with $r \in [0,h]$, then $q = h+(h-i_0)i-j$ for some $j \in [0, h-i]$ and $r \in [0,j]$. Then L_i contains the integer interval $q(h+1)+[0,j]$.

For all $i \in [0, h-1]$, we have

$$\min(L_i) = ic < (i+1)c = \min(L_{i+1})$$

and

$$\begin{aligned}\max(L_i) &= (h-i)(h+1)+ic \\ &< (h-i-1)(h+1)+(i+1)c \\ &= \max(L_{i+1})\end{aligned}$$

and so the sets L_i "move to the right" as i increases from 0 to h. Moreover,

$$\max(L_i) < \min(L_{i+1})$$

if and only if

$$(h-i)(h+1)+ic < (i+1)c$$

if and only if

$$i > h - \frac{c}{h+1}$$

if and only if

$$i \geq 1 + \left[h - \frac{c}{h+1} \right] = i_0.$$

Thus, the sets L_i and L_j are disjoint if $i_0 \leq i < j \leq h$ and

$$\left| \bigcup_{i=i_0+1}^{h} L_i \right| = \sum_{i=i_0+1}^{h} |L_i| = \sum_{i=i_0+1}^{h} \binom{h-i+2}{2}.$$

Because the sets L_i move to the right, we have

$$\left(\bigcup_{i=0}^{i_0} L_i \right) \cap \left(\bigcup_{i=i_0+1}^{h} L_i \right) = \emptyset$$

and

$$\begin{aligned}|hA| &= \left| \bigcup_{i=0}^{h} L_i \right| = \left| \bigcup_{i=0}^{i_0} L_i \right| + \left| \bigcup_{i=i_0+1}^{h} L_i \right| \\ &= \left| \bigcup_{i=0}^{i_0} L_i \right| + \sum_{i=i_0+1}^{h} \binom{h-i+2}{2}. \qquad (5)\end{aligned}$$

We shall compute $L_i \cap L_{i+t}$ for all $i \in [1, h-1]$ and $t \in [1, h-i]$. Relation (**??**) implies

$$L_{i+t} = \bigcup_{j=0}^{h-i-t} \left((h + (h-i_0)(i+t) - j)(h+1) + [0,j] \right).$$

We have $q(h+1) \in L_i \cap L_{i+t}$ if and only if there exist $j_0 \in [0, h-i]$ and $j_t \in [0, h-i-t]$ such that

$$q = h + (h-i_0)i - j_0 = h + (h-i_0)(i+t) - j_t$$

if and only if

$$\begin{aligned} j_0 &= j_t - (h - i_0)t \\ &\in [0, h - i] \cap [-(h - i_0)t, h - i - t - (h - i_0)t] \\ &= [0, h - i - t - (h - i_0)t]. \end{aligned}$$

It follows that

$$L_i \cap L_{i+t} = \bigcup_{j=0}^{h-i-t-(h-i_0)t} ((h + (h - i_0)i - j)(h + 1) + [0, j]) \tag{6}$$

and so

$$\begin{aligned} |L_i \cap L_{i+t}| &= \left| \bigcup_{j=0}^{h-i-t-(h-i_0)t} ((h + (h - i_0)i - j)(h + 1) + [0, j]) \right| \\ &= \sum_{j=0}^{h-i-t-(h-i_0)t} |(h + (h - i_0)i - j)(h + 1) + [0, j]| \\ &= \sum_{j=0}^{h-i-t-(h-i_0)t} (j + 1) \\ &= \binom{h - i - t - (h - i_0)t + 2}{2}. \end{aligned}$$

In particular,

$$|L_i \cap L_{i+1}| = \binom{i_0 + 1 - i}{2}. \tag{7}$$

Relation (**??**) also implies that, for $t \in [1, h - i]$,

$$L_i \setminus L_{i+t} = \bigcup_{j=h-i-t-(h-i_0)t+1}^{h-i} ((h + (h - i_0)i - j)(h + 1) + [0, j])$$

and so

$$L_i \setminus L_{i+1} \subseteq L_i \setminus L_{i+2} \subseteq \cdots \subseteq L_i \setminus L_h.$$

Therefore,

$$\begin{aligned} L_i \setminus \left(\bigcup_{t=1}^{i_0-i} L_{i+t} \right) &= L_i \cap \left(\bigcup_{t=1}^{i_0-i} L_{i+t} \right)^c = L_i \cap \left(\bigcap_{t=1}^{i_0-i} L_{i+t}^c \right) \\ &= \bigcap_{t=1}^{i_0-i} \left(L_i \cap L_{i+t}^c \right) = \bigcap_{t=1}^{i_0-i} (L_i \setminus L_{i+t}) \\ &= L_i \setminus L_{i+1}. \end{aligned}$$

The sets

$$L_i \setminus \left(\bigcup_{t=1}^{i_0-i} L_{i+t} \right)$$

are pairwise disjoint for $i \in [0, i_0]$ and

$$\bigcup_{i=0}^{i_0} L_i = L_{i_0} \cup \bigcup_{i=0}^{i_0-1} \left(L_i \setminus \bigcup_{t=1}^{i_0-i} L_{i+t} \right).$$

From (**??**) and (**??**), we obtain

$$\begin{aligned}
\left|\bigcup_{i=0}^{i_0} L_i\right| &= |L_{i_0}| + \sum_{i=0}^{i_0-1}\left|L_i \setminus \bigcup_{t=1}^{i_0-i} L_{i+t}\right| \\
&= |L_{i_0}| + \sum_{i=0}^{i_0-1} |L_i \setminus L_{i+1}| \\
&= |L_{i_0}| + \sum_{i=0}^{i_0-1} \left(|L_i| - |L_i \cap L_{i+1}|\right) \\
&= \sum_{i=0}^{i_0} |L_i| - \sum_{i=0}^{i_0-1} |L_i \cap L_{i+1}| \\
&= \sum_{i=0}^{i_0} \binom{h-i+2}{2} - \sum_{i=0}^{i_0-1} \binom{i_0+1-i}{2}.
\end{aligned}$$

Relation (**??**) gives

$$\begin{aligned}
|hA| &= \left|\bigcup_{i=0}^{i_0} L_i\right| + \sum_{i=i_0+1}^{h} \binom{h-i+2}{2} \\
&= \sum_{i=0}^{h} \binom{h-i+2}{2} - \sum_{i=0}^{i_0-1} \binom{i_0+1-i}{2} \\
&= \sum_{i=0}^{h} \binom{i+2}{2} - \sum_{i=0}^{i_0-1} \binom{i+2}{2} \\
&= \binom{h+3}{3} - \binom{i_0+2}{3}.
\end{aligned}$$

This completes the proof. □

3. Open problems

Problem 1. *This paper considers the special class of h-adically defined 4-element sets*

$$A = \{0, 1, h+1, h^2+h+1-p\}$$

with

$$p = 1 + (i_0 - 1)(h+1)$$

and $i_0 \in [0, h-1]$. It is of interest to compute, for all $h \geq 3$ and all $p \in [0, h^2-1\}$, the h-fold sumset sizes of the sets

$$A = \{0, 1, h+1, h^2+h+1-p\}.$$

Problem 2. *For all $h \geq 3$, compute the set of h-fold sumset sizes of the sets*

$$A = \{0, 1, a, b\}$$

for $2 \leq a \leq h$ and $a+1 \leq b \leq ha+1$.

Problem 3. *A next step is to determine the popular sumset sizes of 5-element sets of integers. The fundamental problem is to obtain a complete description of the sumset size set $\mathcal{R}_{\mathbf{Z}}(h,k)$ for all positive integers h and k, to explain the distribution* *of sumset sizes for fixed h and k, and to understand why some numbers cannot be sumset sizes. A solution to this problem might be called the* Second Fundamental Theorem of Additive Number Theory.

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