CERTAIN POSITIVE q-SERIES AND INEQUALITIES FOR TWO-COLOR PARTITIONS

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Dedicated to Mourad Ismail for his 80th birthday

ABSTRACT. We consider some q-series which depend on a pair of positive integers (k, m). While positivity of these series holds for the first few values of (k, m), the situation is quite unclear for other values of (k, m). In addition, our series generate the number of certain two-color integer partitions weighted by $(-1)^j$ where j is the number of even parts. Therefore, inequalities involving these partitions will be deduced from the positivity of their generating functions.

1. Introduction

Throughout let q denote a complex number satisfying $|q| < 1$ and let m and n denote nonnegative integers. We adopt the following standard notation from the theory of q-series [?, ?]

$$(a;q)_0 = 1,\ (a;q)_n = \prod_{j=0}^{n-1}(1-aq^j), \quad (a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j),$$

$$(a_1,\ldots,a_k;q)_n = \prod_{j=1}^{k}(a_j;q)_n,\ \text{and}\ (a_1,\ldots,a_k;q)_\infty = \prod_{j=1}^{k}(a_j;q)_\infty.$$

We will need the following basic facts

$$(a;q)_{n+m} = (a;q)_m(aq^m;q)_n,\ \text{and}\ (a;q)_\infty = (a;q)_n(aq^n;q)_\infty. \tag{1.1}$$

A power series $\sum_{n\geq 0} a_n q^n$ is called positive if $a_n \geq 0$ for any nonnegative integer n. Positivity results for q-series have been intensively studied in the past to some extent in connection with Borwein's famous positivity conjecture [?]. For more on this, see for instance [?, ?, ?, ?]. Our main purpose in this note is to make clear that there are interesting problems associated with positivity questions related to the following two q-series.

$$\sum_{n\geq 0} C'(k,m,n)q^n = \sum_{n\geq 0} q^{m(2n+1)} \frac{(q^{2n+2}, q^{2n+2k};q^2)_\infty}{(q^{2n+1};q^2)_\infty^2} \tag{1.2}$$

and

$$\sum_{n\geq 0} D'(k,m,n)q^n = \sum_{n\geq 0} q^{m(2n+2)} \frac{(q^{2n+4}, q^{2n+2+2k};q^2)_\infty}{(q^{2n+3};q^2)_\infty^2}, \tag{1.3}$$

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where k and m are positive integers. For instance, we will see below in Corollary **??** and Theorem **??** that $\sum_{n\geq 0} C'(k,m,n)q^n$ is positive for

$$(k,m) \in \{(1,1),(2,1),(3,1),(2,2),(2,3)\}.$$

Furthermore, it is our conjecture that $\sum_{n\geq 0} C'(2,4,n)q^n$ is positive as well, see Conjecture **??**. On the other hand, $\sum_{n\geq 0} C'(2,5,n)q^n$ is oscillating and it is an open problem whether there exists $m>4$ such that the series $\sum_{n\geq 0} C'(2,m,n)q^n$ is positive.

In addition, inequalities involving integer partitions have received much focus in recent years, see for instance [**?**, **?**]. It turns out that the series we look at in this work are natural generating functions for weighted two-color integer partitions with smallest part. So, we will also deduce inequalities involving two-color partitions through the positivity of their corresponding q-series.

The rest of the paper is organized as follow. In Section **??** we focus on the two-color integer partitions where the smallest part is odd and in Section **??** we discuss the case where the smallest part is even. Sections **??**-**??** are devoted to the proofs of the main theorems. Finally, in Section **??** we give some comments and suggestions for further research.

2. Two-color partitions with odd smallest part

We start with the following natural interpretation for $C'(k,m,n)$ as two-color partitions.

Definition 1. Let k and m be fixed positive integers. For a positive integer n, let $\mathcal{C}(k,m,n)$ denote the set of two-color partitions $\pi(n)$ of n in which:

- the smallest part $s(\pi)$ is odd and occurs at least m times in blue color,
- the even blue parts are at least $2k-1$ more than $s(\pi)$,
- the even parts of the same color are distinct.

By letting $q^{2n+1}\frac{(q^{2n+2k};q^2)_\infty}{(q^{2n+1};q^2)_\infty}$ generate the blue parts and $\frac{(q^{2n+2};q^2)_\infty}{(q^{2n+1};q^2)_\infty}$ generate the green parts, we have

$$\sum_{n\geq 0}\mathcal{C}(k,m,n) = \sum_{n\geq 0} q^{m(2n+1)}\frac{(-q^{2n+2},-q^{2n+2k};q^2)_\infty}{(q^{2n+1};q^2)_\infty^2}.$$

Now let $C_0(k,m,n)$ (resp. $C_1(k,m,n)$) be the number of partitions from $\mathcal{C}(k,m,n)$ wherein the number of even parts is even (resp. odd). Then it is easy to check that

$$\sum_{n\geq 0}\big(C_0(k,m,n)-C_1(k,m,n)\big)q^n = \sum_{n\geq 0} q^{m(2n+1)}\frac{(q^{2n+2},q^{2n+2k};q^2)_\infty}{(q^{2n+1};q^2)_\infty^2}. \tag{2.1}$$

That is,

$$C'(k,m,n) = C_0(k,m,n)-C_1(k,m,n),$$

from which it follows that positivity of the series $\sum_{n\geq 0} C'(k,m,n)q^n$ means the partition inequality $C_1(k,m,n)\leq C_0(k,m,n)$.

We will achieve our main results through the following identity.

Theorem 1. *Let k and m be positive integers. Then*

$$\sum_{n\geq 0} C'(k,m,n)q^n = \begin{cases} \frac{q^m}{(q;q^2)_{k-1}} \sum_{n\geq 0} \frac{q^n(q^{2n+2};q^2)_{m-1}}{(q^{2n+2k-1};q^2)_{m-k+1}} & \text{if } m \geq k \\ \frac{q^m}{(q;q^2)_{k-1}} \sum_{n\geq 0} q^n(q^{2n+2};q^2)_{m-1}(q^{2n+2m+1};q^2)_{k-m-1} & \text{if } m < k. \end{cases}$$

By virtue of Theorem **??** and simplification, for $(k,m) \in \{(1,1),(2,1),(2,2),(3,1)\}$ we get

$$\sum_{n\geq 0} C'(1,1,n)q^n = \sum_{n\geq 0} \frac{q^{n+1}}{1-q^{2n+1}} \tag{2.2}$$

$$\sum_{n\geq 0} C'(2,1,n)q^n = \frac{q}{(1-q)^2} \tag{2.3}$$

$$\sum_{n\geq 0} C'(2,2,n)q^n = \sum_{n\geq 0} \frac{q^{n+2}(1-q^{2n+2})}{(1-q)(1-q^{2n+3})} = \frac{q^2}{(1-q)^2} - \sum_{n\geq 0} \frac{q^{3n+4}}{1-q^{2n+3}}, \tag{2.4}$$

and

$$\sum_{n\geq 0} C'(3,1,n)q^n = \sum_{n\geq 0} \frac{q^{n+1}(1-q^{2n+3})}{(1-q)(1-q^3)} = \frac{q(1+q+q^2-q^3)}{(1-q)(1-q^3)^2}. \tag{2.5}$$

Then it is clear that for these values of (k,m) the series $\sum_{n\geq 0} C'(k,m,n)q^n$ is positive. This leads to the following inequalities involving two-color partitions.

Corollary 1. *(a) For any nonnegative integer n we have $C_1(1,1,n) \leq C_0(1,1,n)$.*
(b) If $m \in \{1,2\}$, then for any positive integer n we have $C_1(2,m,n) \leq C_0(2,m,n)$.
(c) For any nonnegative integer n we have $C_1(3,1,n) \leq C_0(3,1,n)$.

We now deal with the case $(k,m) = (2,3)$.

Theorem 2. *The series $\sum_{n\geq 0} C'(2,3,n)q^n$ is positive.*

Corollary 2. *For any positive integer n we have $C_1(2,3,n) \leq C_0(2,3,n)$.*

We now state our second main result where we handle the case $(k,m) = (4,1)$.

Theorem 3. *The series $\sum_{n\geq 0} C'(4,1,n)q^n$ is positive.*

Corollary 3. *For any positive integer n we have $C_1(4,1,n) \leq C_0(4,1,n)$.*

We close this section by the following positivity conjectures.

Conjecture 1. *For any nonnegative integer k, the series $\sum_{n\geq 0} C'(k,1,n)q^n$ is positive.*

Conjecture 2. *The series $\sum_{n\geq 0} C'(2,4,n)q^n$ is positive.*

3. Two-color partitions with even smallest part

We now focus on the series $\sum_{n\geq 0} D'(k,m,n)q^n$ as in (**??**). We start with the following natural interpretation for $D'(k,m,n)$ as two-color partitions.

Definition 2. Let k and m be fixed positive integers. For a positive integer n, let $\mathcal{D}(k,m,n)$ denote the set of two-color partitions $\pi(n)$ of n in which:

- the smallest part $s(\pi)$ is even, blue, and occurs exactly m times,
- the even blue parts are at least $2k$ more than $s(\pi)$,

- the even parts of the same color are distinct.

By letting $q^{2n+2}\frac{(q^{2n+2+2k};q^2)_\infty}{(q^{2n+3};q^2)_\infty}$ generate the blue parts and $\frac{(q^{2n+4};q^2)_\infty}{(q^{2n+3};q^2)_\infty}$ generate the green parts, we have

$$\sum_{n\geq 0}\mathcal{D}(k,m,n)=\sum_{n\geq 0}q^{m(2n+2)}\frac{(-q^{2n+4},-q^{2n+2+2k};q^2)_\infty}{(q^{2n+3};q^2)_\infty^2}.$$

Let $D_0(k,m,n)$ (resp. $D_1(k,m,n)$) be the number of partitions from $\mathcal{D}(k,m,n)$ wherein the number of even parts that are greater than $s(\pi)$ is even (resp. odd). and let $D'(k,m,n)=D_0(k,m,n)-D_1(k,m,n)$. Then it is easily seen that

$$\sum_{n\geq 0}\big(D_0(k,m,n)-D_1(k,m,n)\big)q^n=\sum_{n\geq 0}q^{m(2n+2)}\frac{(q^{2n+4},q^{2n+2+2k};q^2)_\infty}{(q^{2n+3};q^2)_\infty^2}. \quad (3.1)$$

That is,

$$D'(k,m,n)=D_0(k,m,n)-D_1(k,m,n),$$

from which it follows that positivity of the series $\sum_{n\geq 0}D'(k,m,n)q^n$ means the partition inequality $D_1(k,m,n)\leq D_0(k,m,n)$.

We will establish the following connection between $D'(k,m,n)$ and $C'(k,m,n)$.

Theorem 4. *Let k and m be positive integers. Then*

$$\sum_{n\geq 0}D'(k,m,n)q^n=q^{-m}\sum_{n\geq 0}C'(k,m,n)q^n-\frac{(q^2,q^{2k};q^2)_\infty}{(q;q^2)_\infty^2}.$$

$$=\begin{cases}\frac{1}{(q;q^2)_{k-1}}\sum_{n\geq 0}\frac{q^n(q^{2n+2};q^2)_{m-1}}{(q^{2n+2k-1};q^2)_{m-k+1}}-\frac{(q^2,q^{2k};q^2)_\infty}{(q;q^2)_\infty^2} & \text{if } m\geq k\\ \frac{1}{(q;q^2)_{k-1}}\sum_{n\geq 0}q^n(q^{2n+2};q^2)_{m-1}(q^{2n+2m+1};q^2)_{k-m-1}-\frac{(q^2,q^{2k};q^2)_\infty}{(q;q^2)_\infty^2} & \text{if } m<k.\end{cases}$$

Our first application of Theorem **??** deals with $(k,m)=(2,1)$.

Theorem 5. *The series $\sum_{n\geq 0}D'(2,1,n)q^n$ is positive.*

Corollary 4. *For any positive integer n we have $D_1(2,1,n)\leq D_0(2,1,n)$.*

We cannot prove any other positivity results for the series $\sum_{n\geq 0}D'(k,m,n)q^n$. However, we have the following two conjectures.

Conjecture 3. *The series $\sum_{n\geq 0}D'(2,2,n)q^n$ is positive.*

Conjecture 4. *The only negative coefficients of the series $\sum_{n\geq 0}D'(2,3,n)q^n$ occur at $n=10$ and $n=22$.*

Also, we have the following conjecture.

Conjecture 5. *If $k>m$, then $\sum_{n\geq 0}D'(k,m,n)q^n$ is eventually positive.*

4. Proof of Theorem ?? and Theorem ??

Proof of Theorem **??**. We will require Heine's first transformation [**?**, **?**]

$$ {}_2\phi_1\left[\begin{matrix} a,\ b \\ c \end{matrix}; q,\ z\right] = \frac{(b,az;q)_\infty}{(c,z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix} c/b,\ z \\ az \end{matrix}; q,\ b\right]. \tag{4.1} $$

We have

$$ \begin{aligned} \sum_{n\geq 0} \frac{q^{m(2n+1)}(q^{2n+2},q^{2n+2k};q^2)_\infty}{(q^{2n+1};q^2)_\infty^2} &= q^m \frac{(q^2,q^{2k};q^2)_\infty}{(q;q^2)_\infty^2} \sum_{n\geq 0} \frac{q^{2mn}(q;q^2)_n^2}{(q^2,q^{2k};q^2)_n} \\ &= q^m \frac{(q^2,q^{2k};q^2)_\infty}{(q;q^2)_\infty^2}\,{}_2\phi_1\left[\begin{matrix} q,\ q \\ q^{2k} \end{matrix}; q^2,\ q^{2m}\right] \\ &= q^m \frac{(q^2,q^{2k};q^2)_\infty}{(q;q^2)_\infty^2} \frac{(q,q^{2m+1};q^2)_\infty}{(q^{2k},q^{2m};q^2)_\infty}\,{}_2\phi_1\left[\begin{matrix} q^{2k-1},\ q^{2m} \\ q^{2m+1} \end{matrix}; q^2,\ q\right] \\ &= q^m \frac{(q^2,q^{2k};q^2)_\infty}{(q;q^2)_\infty^2} \frac{(q,q^{2m+1};q^2)_\infty}{(q^{2k},q^{2m};q^2)_\infty} \sum_{n\geq 0} \frac{q^n(q^{2k-1},q^{2m};q^2)_n}{(q^2,q^{2m+1};q^2)_n} \\ &= q^m \frac{(q^{2k-1};q^2)_\infty}{(q;q^2)_\infty} \sum_{n\geq 0} \frac{q^n(q^{2n+2},q^{2n+2m+1};q^2)_\infty}{(q^{2n+2k-1},q^{2n+2m};q^2)_\infty} \\ &= \begin{cases} \frac{q^m}{(q;q^2)_{k-1}} \sum_{n\geq 0} \frac{q^n(q^{2n+2};q^2)_{m-1}}{(q^{2n+2k-1};q^2)_{m-k+1}} & \text{if } m \geq k \\ \frac{q^m}{(q;q^2)_{k-1}} \sum_{n\geq 0} q^n(q^{2n+2};q^2)_{m-1}(q^{2n+2m+1};q^2)_{k-m-1} & \text{if } m < k, \end{cases} \end{aligned} $$

where in the third step we applied (**??**).

Proof of Theorem **??**. By (**??**), we find

$$ \begin{aligned} \sum_{n\geq 0} \frac{q^{m(2n+2)}(q^{2n+4},q^{2n+2+2k};q^2)_\infty}{(q^{2n+3};q^2)_\infty^2} &= q^{2m} \frac{(q^4,q^{2k+2};q^2)_\infty}{(q^3;q^2)_\infty^2} \sum_{n\geq 0} \frac{q^{2mn}(q^3;q^2)_n^2}{(q^4,q^{2k+2};q^2)_n} \\ &= q^{2m} \frac{(q^4,q^{2k+2};q^2)_\infty}{(q^3;q^2)_\infty^2} \frac{(1-q^2)(1-q^{2k})}{(1-q)^2} \sum_{n\geq 0} q^{2mn} \frac{(q;q^2)_{n+1}^2}{(q^2,q^{2k};q^2)_{n+1}} \\ &= q^{2m} \frac{(q^2,q^{2k};q^2)_\infty}{(q;q^2)_\infty^2} \sum_{n\geq 1} q^{2m(n-1)} \frac{(q;q^2)_n^2}{(q^2,q^{2k};q^2)_n} \\ &= \frac{(q^2,q^{2k};q^2)_\infty}{(q;q^2)_\infty^2} \Big(\sum_{n\geq 0} q^{2mn} \frac{(q;q^2)_n^2}{(q^2,q^{2k};q^2)_n} - 1 \Big) \\ &= \sum_{n\geq 0} q^{2mn} \frac{(q^{2n+2},q^{2n+2k};q^2)_\infty}{(q^{2n+1};q^2)_\infty^2} - \frac{(q^2,q^{2k};q^2)_\infty}{(q;q^2)_\infty^2} \\ &= q^{-m} \sum_{n\geq 0} q^{m(2n+1)} \frac{(q^{2n+2},q^{2n+2k};q^2)_\infty}{(q^{2n+1};q^2)_\infty^2} - \frac{(q^2,q^{2k};q^2)_\infty}{(q;q^2)_\infty^2}, \end{aligned} $$

which gives the desired formula by (**??**) and Theorem **??**.

5. Proof of Theorem ??

By Theorem **??** with $(k,m)=(2,3)$ we find after simplification

$$\sum_{n\geq 0} C'(2,3,n)q^n = \sum_{n\geq 0} \frac{q^3(1-q^{2n+2})(1-q^{2n+4})}{(1-q)(1-q^{2n+3})(1-q^{2n+5})}.$$

We want to prove that the coefficient of q^N in

$$\frac{q^3(1-q^{2n+2})(1-q^{2n+4})}{(1-q)(1-q^{2n+3})(1-q^{2n+5})}$$

is nonnegative. To this end, we need some preparation. We start with a lemma.

Lemma 1. *Let for any nonnegative integer n*

$$\frac{1}{(1+q)(1-q^{2n+3})} = \sum_{N\geq 0} a(N)q^N.$$

Then

$$a(N) = \begin{cases} (-1)^N & \text{if } \lfloor \frac{N}{2n+3} \rfloor \equiv 0 \pmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have

$$\frac{1}{(1+q)(1-q^{2n+3})} = \sum_{j=0}^{\infty} (-1)^j q^j \sum_{m=0}^{\infty} q^{m(2n+3)}.$$

Then to find the coefficient of $a(N)$ we need first to solve the equation $N = j + m(2n+3)$ where $j\geq 0$ and $0\leq m \leq \lfloor \frac{N}{2n+3} \rfloor$. The solutions are

$$\begin{cases} j = N & \text{if } m=0, \\ j = N-(2n+3) & \text{if } m=1, \\ j = N-2(2n+3) & \text{if } m=2, \\ \vdots & \vdots \\ j = N - \lfloor \frac{N}{2n+3} \rfloor (2n+3) & \text{if } m = \lfloor \frac{N}{2n+3} \rfloor. \end{cases}$$

As these solutions alternate in sign, we have

$$a(N) = (-1)^N - (1)^N + (-1)^N + \cdots = \begin{cases} (-1)^N & \text{if } \lfloor \frac{N}{2n+3} \rfloor \equiv 0 \pmod 2, \\ 0 & \text{otherwise,} \end{cases}$$

as desired. □

By shifting the coefficient from N to $N-A$, we get the following important consequence of Lemma **??**.

Corollary 5. *Let for any nonnegative integer n and any positive integer A*

$$\frac{q^A}{(1+q)(1-q^{2n+3})} = \sum_{N\geq 0} a(N)q^N.$$

Then

$$a(N) = \begin{cases} (-1)^{N-A} & \text{if } \lfloor \frac{N-A}{2n+3} \rfloor \equiv 0 \pmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

We need another lemma.

Lemma 2. *For any nonnegative integer n we have*

$$\frac{q^3(1-q^{2n+2})(1-q^{2n+4})}{(1-q)(1-q^{2n+3})(1-q^{2n+5})}$$

$$= \frac{q}{1-q} - \frac{q^2}{(1+q)(1-q^{2n+3})} - \frac{q}{1-q^{2n+5}} - \frac{q^3}{(1+q)(1-q^{2n+5})}$$

Proof. Immediate by inspection. □

From Lemma **??** and Corollary **??** we see that the coefficient of q^N in

$$\frac{q^3(1-q^{2n+2})(1-q^{2n+4})}{(1-q)(1-q^{2n+3})(1-q^{2n+5})}$$

is $1 - T_1 - T_2 - T_3$, where

$$T_1 = \begin{cases} (-1)^N & \text{if } \lfloor \frac{N-2}{2n+3} \rfloor \equiv 0 \pmod 2 \\ 0 & \text{otherwise,} \end{cases}$$

$$T_2 = \begin{cases} (-1)^{N-1} & \text{if } \lfloor \frac{N-3}{2n+5} \rfloor \equiv 0 \pmod 2 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$T_3 = \begin{cases} 1 & \text{if } N \equiv 1 \pmod{2n+5} \\ 0 & \text{otherwise.} \end{cases}$$

Note that it is not possible to have $T_1 + T_2 = 2$ because $(-1)^N + (-1)^{N+1} = 0$. Hence, if $N \not\equiv 1 \pmod{2n+5}$, then the coefficient of q^N is nonnegative. So there are only two possible cases where the coefficient of q^N might be -1: $N \equiv 1 \pmod{2n+5}$, N is even, and $\lfloor \frac{N-2}{2n+3} \rfloor$ is even **or** $N \equiv 1 \pmod{2n+5}$, N is odd, and $\lfloor \frac{N-3}{2n+5} \rfloor$ is even. We now deal with each of these two cases.

Case 1. Write $N = k(2n+5)+1$. So for N to be even, k must be odd, say $k = 2j+1$, but then we get $T_2 = 0$. So, $\lfloor \frac{N-3}{2n+5} \rfloor$ must be odd. But

$$\left\lfloor \frac{N-3}{2n+5} \right\rfloor = \left\lfloor \frac{(2j+1)(2n+5)+1-3}{2n+5} \right\rfloor$$

$$\left\lfloor (2j+1) - \frac{2}{2n+5} \right\rfloor = 2j,$$

which gives a contradiction. This yields a nonnegative coefficient for q^N.

Case 2. Writing $N = k(2n+5)+1$ and using the assumption that N is odd we see that k must be even, say $k = 2j$. Thus

$$\left\lfloor \frac{N-3}{2n+5} \right\rfloor = \left\lfloor \frac{(2j)(2n+5)+1-3}{2n+5} \right\rfloor$$

$$\left\lfloor 2j - \frac{2}{2n+5} \right\rfloor = 2j-1,$$

contradicting the fact that $\lfloor \frac{N-3}{2n+5} \rfloor$ is even. This yields a nonnegative coefficient for q^N as well. The proof is complete.

6. Proof of Theorem ??

By Theorem **??** with $(k,m)=(4,1)$, we find

$$\sum_{n\geq 0} C'(4,1,n)q^n = \frac{q}{(1-q)(1-q^3)(1-q^5)} \sum_{n\geq 0} q^n (1-q^{2n+3})(1-q^{2n+5}).$$

So, to establish the desired inequality it is enough to show that $\frac{(1-q^{2n+3})(1-q^{2n+5})}{(1-q)(1-q^3)(1-q^5)}$ has nonnegative coefficients. This clearly holds if $3 \mid 2n+3$ or $3 \mid 2n+5$. Now assume that $3 \nmid 2n+3$ and $3 \nmid 2n+5$. This in particular implies that $3 \mid 2n+1$. Then $3 \nmid 2n$ and $3 \nmid 2n+2$ and therefore that $3 \mid 2n-2$. Thus

$$\begin{aligned}\frac{(1-q^{2n+3})(1-q^{2n+5})}{(1-q)(1-q^3)(1-q^5)} &= \frac{(1-q^{2n+5})\big(1-q^{2n-2}+q^{2n-2}(1-q^5)\big)}{(1-q)(1-q^3)(1-q^5)} \\ &= \frac{1-q^{2n+5}}{(1-q)(1-q^5)} \frac{1-q^{2n-2}}{1-q^3} + q^{2n-2}\frac{1-q^{2n+5}}{(1-q)(1-q^3)}\end{aligned}$$

which has nonnegative coefficients. This completes the proof.

7. Proof of Theorem ??

Recall that a triangular number is an integer of the form $\frac{x(x+1)}{2}$ for a nonnegative integer x. Throughout let $t_2(n)$ denote the number of ways n can be written as a sum of two triangular numbers. By the Gauss sum [**?**]

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \sum_{n\geq 0} q^{\frac{n(n+1)}{2}},$$

we deduce that

$$\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} = \sum_{n\geq 0} t_2(n)q^n. \tag{7.1}$$

By Theorem **??** applied to $(k,m)=(2,1)$, we find

$$\sum_{n\geq 0} D'(2,1,n)q^n = \frac{1}{(1-q)^2} - \frac{(q^2,q^4;q^2)_\infty}{(q;q^2)_\infty^2} = \frac{1}{(1-q)^2} - \frac{1}{1-q^2}\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}. \tag{7.2}$$

As $\frac{1}{(1-q)^2} = \sum_{n\geq 0}(n+1)q^n$, the positivity of the left hand-side of (**??**) means that for every nonnegative integer N, we have

$$t_2(N)+t_2(N-2)+t_2(N-4)+\cdots \leq N+1. \tag{7.3}$$

Now, if

$$\frac{x(x+1)}{2}+\frac{y(y+1)}{2}=z,$$

then

$$(2x+1)^2+(2y+1)^2 = 8z+2. \tag{7.4}$$

Case 1. $z=2N$. Then (**??**) becomes

$$(2x+1)^2+(2y+1)^2 = 16N+2$$

and the sum (**??**) can be viewed as counting integer points (x,y) lying inside the circle

$$X^2+Y^2 = 16N+2 \tag{7.5}$$

such that x and y are odd satisfying $x^2+y^2 \equiv 2 \pmod{16}$. Such points follow a regular pattern in the plane consisting of eight integer points

$$(1,1),(7,7),(1,7),(7,1),(3,3),(5,5),(3,5),(5,5) \tag{7.6}$$

inside the 8×8 square whose vertices are $(0,0),(0,8),(8,0),(8,8)$. Then we may view the plane as covered with such 8×8 squares with exactly eight integer points inside satisfying the above requirements. So, to count the integer points that provide the totality of the sum (**??**) we need count the number of 8×8 squares that cover the first quadrant of the circle (**??**). As the diagonal of the 8×8 square is $8\sqrt{2}$, we have that the circle of radius $\sqrt{16N+2}+8\sqrt{2}$ contains any 8×8 lattice square that intersects the inner circle whose radius is $\sqrt{16N+2}$. Hence, the number of the 8×8 squares required to cover the first quadrant of the circle (**??**) is less than

$$\frac{\pi\left(\sqrt{16N+2}+8\sqrt{2}\right)^2/4}{64}.$$

Combining this with the fact that each 8×8 square has eight admissible integer points, we get

$$\begin{aligned} t_2(2N)+t_2(2N-2)+t_2(2N-4)+\cdots &\leq \frac{\pi\left(\sqrt{16N+2}+8\sqrt{2}\right)^2/4}{64}\cdot 8 \\ &= \frac{\pi\left(16N+130+16\sqrt{2}\sqrt{16N+2}\right)}{32} \\ &= \frac{\pi N}{2}+\frac{65\pi}{16}+2\sqrt{2N+\frac{1}{4}}. \end{aligned} \tag{7.7}$$

Now if

$$f(x)=(2x+1)-\frac{\pi}{2}x-\frac{65\pi}{16}-2\sqrt{2x+\frac{1}{4}},$$

then

$$f'(x)=2-\frac{\pi}{2}-2\left(2x+\frac{1}{4}\right)^{-\frac{1}{2}}$$

and

$$f''(x)=2\left(2x+\frac{1}{4}\right)^{-\frac{3}{2}}.$$

Then clearly $f''(x)>0$ for $x\geq 0$, showing that $f'(x)$ is increasing on the interval $[0,\infty)$. This combined with the fact that

$$2-\frac{\pi}{2}-2\cdot\frac{2}{11}=0.06556\cdots$$

yields that $f'(x)>0$ for $x\geq 15$. Hence since

$$f(90)=181-\frac{90\pi}{2}-\frac{65\pi}{16}-2\sqrt{180+\frac{1}{4}}=0.0141\cdots,$$

we see that

$$(2N+1)-\Big(t_2(2N)+t_2(2N-2)+t_2(2N-4)+\cdots+t_2(0)\Big)>0$$

for any $N\geq 90$. The remaining even cases of $2N\leq 90$ follow by inspection.

Case 2. $z=2N+1$. This works exactly the same as *Case 1* with (**??**) replaced with

$$X^2+Y^2=16N+10 \tag{7.8}$$

and the eight integer points inside the 8×8 square in the pattern (**??**) replaced with

$$(3,1),(5,1),(1,3),(1,5),(7,3),(7,5),(3,7),(5,7). \tag{7.9}$$

8. Concluding remarks

1. Our proofs for the inequalities in Corollaries **??**-**??** went through the theory of q-series. It would be interesting to find combinatorial proofs for these results.

2. It is worth to note that unlike Theorem **??** for $(k,m)=(2,3)$ and Conjecture **??** for $(k,m)=(2,4)$, the series $\sum_{n\geq 0} C'(2,5,n)q^n$ is oscillating with the first few negative values at $n=688,690,692,887,889,891,893,\ldots$. So, it is natural to ask the following question.

Open Problem 1. *Are there any positive integers $m>4$ which for which the series $\sum_{n\geq 0} C'(2,m,n)q^n$ is positive?*

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