THE MODULARITY OF AN ABELIAN VARIETY

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ABSTRACT. We introduce the concept of the modularity of an abelian variety defined over the rational number field extending the modularity of an elliptic curve. We discuss the modularity of an abelian variety over $\mathbb{Q}$. We conjecture that a simple abelian variety over $\mathbb{Q}$ is modular. The detailed article will appear later.

1. The Modularity of an Elliptic Curve

We set $\Gamma_1 := SL(2, \mathbb{Z})$. For a positive integer N, we let $\Gamma(N)$, $\Gamma_1(N)$ and $\Gamma_0(N)$ be the congruence subgroups of Γ_1 such that $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \Gamma_1$. We refer to [**?**, pp. 13–14, p. 21] for the precise definitions and properties of $\Gamma(N)$, $\Gamma_1(N)$ and $\Gamma_0(N)$. Let $\mathbb{H}_1$ be the Poincaré upper half plance. The quotient

$$Y_1(N) := \Gamma_1(N)\backslash\mathbb{H}_1 \text{ (resp. } Y_0(N) := \Gamma_0(N)\backslash\mathbb{H}_1)$$

be the complex manifold which has a natural model $Y_1(N)/\mathbb{Q}$ (resp. $Y_0(N)/\mathbb{Q}$). We let $X_1(N)$ (resp. $X_0(N)$) be the smooth projective curve which contains $Y_1(N)$ (resp. $Y_0(N)$) as a dense Zariski open subset (cf. see [**?**, pp. 45–60]).

Let $S_k(N)$ be the space of cusp forms of weight $k \geq 1$ and level $N \geq 1$. Here k and N be positive integers. We recall that if $f \in S_k(N)$, it satisfies the following properties:

(C1) $f((a\tau + b)(c\tau + d)^{-1}) = (c\tau + d)^k f(\tau)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$ and $\tau \in \mathbb{H}_1$;

(C2) $|f(\tau)|^2 (\mathrm{Im}\,\tau)^k$ is bounded in $\mathbb{H}_1$ and

(C3) the Fourier expansion of $f(\tau)$ is given by

$$f(\tau) = \sum_{n=1}^{\infty} a_n(f)\, q^n, \quad \text{where } q = e^{2\pi i \tau}.$$

We define the L-series of $f \in S_k(N)$ to be

$$L(f, s) := \sum_{n=1}^{\infty} a_n(f)\, n^{-s}.$$

Subject Classification: Primary 14Kxx, 14G35, 11F46, 11F80
Keywords and phrases: modularity, abelian varieties, Siegel modular variety, Siegel modular forms, Galois representation, zeta functions.

For each prime $p \nmid N$, we recall that the Hecke operator $T_p : S_k(N) \longrightarrow S_k(N)$ is defined by

$$(T_p f)(\tau) = p^{-1} \sum_{i=0}^{p-1} f\left(\frac{\tau+i}{p}\right) + p^{k-1} f\left(\frac{ap\tau+b}{cp\tau+d}\right), \quad f \in S_k(N)$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$ with $c \equiv 0 \,(\mathrm{mod}\, N)$ and $d \equiv p \,(\mathrm{mod}\, N)$. We refer to [?, p. 844] or [?, pp. 170–171] for more detail. The Hecke operators $T_p\,(p \nmid N)$ can be simultaneously diagonalized on $S_k(N)$ and a simultaneous eigenvector a *Hecke eigenform* or simply an eigenform.

Let λ be a place of the algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$ lying over a rational integer ℓ and $\bar{\mathbb{Q}}_\lambda$ denote the algebraic closure of $\mathbb{Q}_\ell$ via λ. Let $G_{\mathbb{Q}} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of $\mathbb{Q}$. It is well known that if $f \in S_k(N)$ is a normalized eigenform with $a_1(f) = 1$, then there exists a unique continuous irreducible Galois representation

$$\rho_{f,\lambda} : G_{\mathbb{Q}} \longrightarrow GL(2, \bar{\mathbb{Q}}_\lambda)$$

such that $\rho_{f,\lambda}$ is unramified at p for all primes $p \nmid \ell N$ and

$$\mathrm{Tr}\,(\rho_{f,\lambda}(\mathrm{Frob}_p)) = a_p(f) \quad \text{for any prime } p \nmid \ell N.$$

The existence of $\rho_{f,\lambda}$ is due to Shimura if $k = 2$ [?], due to Deligne if $k > 2$ [?] and due to Deligne and Serre if $k = 1$ [?]. We see that $\rho_{f,\lambda}$ is odd in the sense that $\det \rho_{f,\lambda}$ of complex conjugation is -1. Moreover $\rho_{f,\lambda}$ is potentially semi-stable at ℓ in the sense of Fontaine [?].

We may choose a conjugate of $\rho_{f,\lambda}$ which is valued in $GL(2, \mathcal{O}_{\bar{\mathbb{Q}}_\lambda})$ and reducing modulo the maximal ideal and semi-simplyfing yields an irreducible continuous representation

$$\overline{\rho}_{f,\lambda} : G_{\mathbb{Q}} \longrightarrow GL(2, \overline{\mathbb{F}}_\ell)$$

which, up to isomorphism, does not depend on the choice of conjugate of $\rho_{f,\lambda}$.

Definition 1.1. *Let*

$$\rho : G_{\mathbb{Q}} \longrightarrow GL(2, \bar{\mathbb{Q}}_\ell)$$

be an irreducible continuous Galois representation which is unramified outside finitely many primes and for which the restriction of ρ to a decomposition group at ℓ is potentially semi-stable at ℓ in the sense of Fontaine. Then ρ is called modular *if ρ is equivalent to $\rho_{f,\lambda}$ (denoted $\rho \sim \rho_{f,\lambda}$) for some normalized eigenform f and some place $\lambda | \ell$.*

Definition 1.2. *Let*

$$\overline{\rho} : G_{\mathbb{Q}} \longrightarrow GL(2, \overline{\mathbb{F}}_\ell)$$

be a two-dimensional irreducible continuous representation of $G_{\mathbb{Q}}$. Then $\overline{\rho}$ is called modular *if $\overline{\rho} \sim \overline{\rho}_{f,\lambda}$ for some normalized eigenform f and some place $\lambda | \ell$.*

Let E be an elliptic curve over $\mathbb{Q}$. We define

$$a_p(E) := p + 1 - |E(\mathbb{F}_p)| \quad \text{for a prime } p.$$

The L-function $L(E,s)$ of E is defined by the product of the local L-factors

$$L(E,s) := \prod_{p|D} \left(\frac{1}{1 - a_p(E)p^{-s}}\right) \prod_{p \nmid D} \left(\frac{1}{1 - a_p(E)\, p^{-s} + p^{1-2s}}\right).$$

Then $L(E,s)$ converges absolutely for $\mathrm{Re}\, s > \frac{3}{2}$ and extends to an entire function by [?].

Definition 1.3. *An elliptic curve E over $\mathbb{Q}$ is called* modular *if there exists a Hecke eigenform $f \in S_2(N)$ such that*

$$L(E,s) = L(f,s).$$

Let E be an elliptic curve over $\mathbb{Q}$ with its conductor $N(E)$. Let

$$\rho_{E,\ell} : G_{\mathbb{Q}} \longrightarrow GL(2, \bar{\mathbb{Q}}_\ell)$$

be the ℓ-adic representation of $G_{\mathbb{Q}}$ with the Tate module $T_\ell(E)$ as its representation space. Let

$$J_1(N) := \Omega^1(X_1(N))^\vee / H_1(X_1(N), \mathbb{Z}) \cong S_2(\Gamma_1(N))^\vee / H_1(X_1(N), \mathbb{Z})$$

be the Jacobian variety of the modular curve $X_1(N)$. Here $\Omega^1(X_1(N))$ denotes the complex vector space of holomorphic 1-forms on $X_1(N)$ and $W^\vee$ denotes the dual space of a complex vector space W. It is known that the following statements are equivalent :

(a) E is modular.

(b) There is a non-constant holomorphic mapping $X_1(N) \longrightarrow E(\mathbb{C})$ for some positive integer N.

(c) There is a non-constant holomorphic mapping $J_1(N) \longrightarrow E(\mathbb{C})$ for some positive integer N.

(d) $\rho_{E,\ell}$ is modular for a prime ℓ.

The above statements have been called the Taniyama-Shimura conjecture. We refer to [?] for the historical story of this conjecture. The implication (a) $\Longrightarrow$ (b) follows from a construction of Shimura [?] and a theorem of Faltings [?]. The implication (b) $\Longrightarrow$ (d) is due to Mazur [?]. The implication (d) $\Longrightarrow$ (a) follows from a theorem of Carayol [?]. The implication (c) $\Longrightarrow$ (b) is obvious. Wiles [?, ?] proved that a semistable eliptic curve over $\mathbb{Q}$ is modular by proving the statement (d). Thereafter Breuil, Conrad, Diamond and Taylor [?] proved that every eliptic curve over $\mathbb{Q}$ is modular.

Serre [?] conjectured the following :

Serre's Modularity Conjecture : Let $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow GL(2, \mathbb{F})$ be a two-dimensional absolutely irreducible, continuous, odd representation of $G_{\mathbb{Q}}$. Here $\mathbb{F}$ is a finite field of characteristic p. Then $\overline{\rho}$ is modular, i.e., arises from (with respect to some fixed embedding $\imath : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$) a newform f of some weight $k \geq 2$ and level N prime to p.

In 2009, Khare and Wintenberger [?, ?] proved that Serre's Modularity Conjecture is true.

2. The Modularity of an Abelian Variety

Let $G := Sp(2g, \mathbb{R})$ and $K = U(g)$. Let

$$\mathbb{H}_g := \{\, \Omega \in \mathbb{C}^{(g,g)} \mid \Omega = {}^t\Omega, \ \ \mathrm{Im}\, \Omega > 0 \,\}$$

be the Siegel upper half plane of degree g. Then G acts on $\mathbb{H}_g$ transitively by

$$\alpha \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1},$$

where $\alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ and $\Omega \in \mathbb{H}_g$. The stabilizer of the action (2.1) at iI_g is

$$\left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \,\Big|\, A + iB \in U(g) \right\} \cong U(g).$$

Thus we get the biholomorphic map

$$G/K \longrightarrow \mathbb{H}_g, \qquad \alpha K \mapsto \alpha\cdot iI_g, \quad \alpha \in G.$$

It is known that $\mathbb{H}_g$ is an Einstein-Kähler Hermitian symmetric space.

Let $\Gamma_g := Sp(2g, \mathbb{Z})$ be the Siegel modular group of degree g. For a positive integer N, we let

$$\Gamma_g(N) := \{ \gamma \in \Gamma_g \mid \gamma \equiv I_{2g} \ (\mathrm{mod}\, N) \}$$

be the the principal congruence subgroup of Γ_g of level N. Let

$$\Gamma_{g,0}(N) := \left\{ \gamma \in \Gamma_g \,\Big|\, \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad C \equiv 0 \ (\mathrm{mod}\, N) \right\}$$

and

$$\Gamma_{g,1}(N) := \left\{ \gamma \in \Gamma_g \,\Big|\, \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} I_g & * \\ 0 & I_g \end{pmatrix} \ (\mathrm{mod}\, N) \right\}$$

be the congruence subgroups of Level N. Then we have the relation

$$\Gamma_g(N) \subset \Gamma_{g,1}(N) \subset \Gamma_{g,0}(N) \subset \Gamma_g.$$

Definition 2.1. *Let Γ be a congruence subgroup of Γ_g and let k be a nonnegative integer k. A function $F : \mathbb{H}_g \longrightarrow \mathbb{C}$ is called a* Siegel modular form *of degree g and weight k with respect to Γ if it satisfies the following conditions:*

(S1) *$F(\Omega)$ is holomorphic on $\mathbb{H}_g$;*

(S2) $F(\gamma \cdot \Omega) = (C\Omega + D)^k F(\Omega)$ *for all* $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$ *and* $\Omega \in \mathbb{H}_g$ *;*

(S3) *$F(\Omega)$ is bounded in any domain $Y \geq Y_0 > 0$ in the case $g = 1$.*

We denote the space of all Sigel modular forms of degree g and weight k with respect to Γ by $[\Gamma, k]$.

We define the so-called Siegel operator

$$\Phi_g : [\Gamma_g, k] \longrightarrow [\Gamma_{g-1}, k]$$

by

$$(\Phi_g(F))(\Omega_1) := \lim_{t \longrightarrow \infty} F \begin{pmatrix} \Omega_1 & 0 \\ 0 & i\,t \end{pmatrix}, \quad \Omega_1 \in \mathbb{H}_{g-1}.$$

Then Φ_g is a well-defined linear mapping (cf, [?, pp. 187–189]). A Siegel modular form $F \in [\Gamma_g, k]$ is called a Siegel cusp form if $\Phi_g(F) = 0$ (cf. [?, p. 198]). We denote by $[\Gamma_g, k]_0$ the space of all Siegel cusp forms in $[\Gamma_g, k]$.

Let Γ be a congruence subgroup of Γ_g. If $F \in [\Gamma, k]$, then F has a Fourier expansion

$$F(\Omega) = \sum_T a(T; F)\, e^{2\pi i\, \mathrm{Tr}(T\Omega)},$$

where T runs through all $g \times g$ half-integral semi-positive symmetric matrices. Here $\mathrm{Tr}(M)$ denotes the trace of a square matrix M. Following the Hecke's method, Maass [**?**, pp. 202–210] associated with $F(\Omega)$ the Dirichlet series

$$D(F, s) := \sum_{\{T\}} \frac{a(T; F)}{\varepsilon(T)} (\det T)^{-s},$$

where the summation indicates that T runs through a complete set of representatives of the sets

$$\{\, T[U] \mid U \text{ unimodular} \,\}\,,\ T > 0$$

and $\varepsilon(T)$ denotes the number of unimodular matrices U which satisfy the equation $T[U] = T$. We note that the numbers $\varepsilon(T)$ are finite.

Definition 2.2. *Let F be a nonzero Siegel Hecke eigenform in $[\Gamma_g, k]_0$. Let $\alpha_{p,0}, \alpha_{p,1}, \cdots, \alpha_{p,g}$ be the p-Satake parameters of F at a prime p. We define the* local spinor zeta function $Z_{F,p}(t)$ *of F at p by*

$$Z_{F,p}(t) := (1 - \alpha_{p,0}\, t) \sum_{r=1}^{g} \sum_{1 \le i_1 < \cdots < i_r \le g} (1 - \alpha_{p,0}\alpha_{p,i_1} \cdots \alpha_{p,i_r}\, t).$$

The spinor zeta function $Z_F(s)$ *of F is defined to be the following function*

$$Z_F(s) := \sum_{p:\text{prime}} Z_{F,p}(p^{-s})^{-1}, \quad \mathrm{Re}\, s \gg 0.$$

Secondly one has the so-called standard zeta function $D_F(s)$ *of a Siegel Hecke eigenform F in $[\Gamma_g, k]_0$ defined by*

$$D_F(s) := \sum_{p:\text{prime}} D_{F,p}(p^{-s})^{-1}, \quad \mathrm{Re}\, s \gg 0,$$

where

$$D_{F,p}(t) = (1 - t) \sum_{i=1}^{g} (1 - \alpha_{p,i}\, t)(1 - \alpha_{p,i}^{-1}\, t).$$

We refer to [**?**, p. 249].

Remark 2.3. *(1) If $g = 1$, the spinor zeta function $Z_f(s)$ of a Hecke eigenform f in $S_k(\Gamma_1)$ is nothing but the Hecke L-function $L(f, s)$ of f.*

(2) If $g = 1$, the standard zeta function $D_f(s)$ of a Hecke eigenform $f(\tau) = \sum_{n=1}^{\infty} a(n)\, e^{2\pi i n\tau}$ in $S_k(\Gamma_1)$ has the following equation

$$D_f(s - k + 1) = \sum_{p:\text{prime}} (1 + p^{k-s-1})^{-1} \cdot \sum_{n=1}^{\infty} a(n^2)\, n^{-s}.$$

Let A be a g-dimensional simple abelian variety defined over $\mathbb{Q}$. For a prime ℓ, we set

$$A[\ell^n] := \{\, x \in A(\overline{\mathbb{Q}}) \mid \ell^n \cdot x = 0 \,\}.$$

Then $A[\ell^n] \cong (\mathbb{Z}/\ell^n\mathbb{Z})^g \times (\mathbb{Z}/\ell^n\mathbb{Z})^g$ (cf. [?]). Then the Tate module of A is given by

$$T_\ell(A) := \varprojlim A[\ell^n] \cong \mathbb{Z}_\ell^g \times \mathbb{Z}_\ell^g \cong \mathbb{Z}_\ell^{2g}.$$

Therefore we have the $2g$-dimensional ℓ-adic Galois representation of $G_{\mathbb{Q}}$

$$\rho_{A,\ell} : G_{\mathbb{Q}} \longrightarrow GL(2g, \mathbb{Z}_\ell) \subset GL(2g, \mathbb{Q}_\ell).$$

Definition 2.4. *A $2g$-dimensional ℓ-adic Galois representation ρ of $G_{\mathbb{Q}}$ given by*

$$\rho : G_{\mathbb{Q}} \longrightarrow GL(2g, \mathbb{Z}_\ell) \subset GL(2g, \mathbb{Q}_\ell)$$

is called modular *if there is a Siegel Hecke eigenform $F(\Omega) \in [\Gamma_{g,0}(N), g+1]_0$ of weight $g+1$ with respect to $\Gamma_{g,0}(N)$ such that*

$$\mathrm{Tr}\,(\rho(\mathrm{Frob}_p)) = a(pI_g; F) \quad \text{and} \quad \det\,(\rho(\mathrm{Frob}_p)) = p^g \quad \text{for any prime } p \not| \, \ell N,$$

where

$$F(\Omega) = \sum_T a(T; F)\, e^{2\pi i\, \mathrm{Tr}(T\Omega)}$$

is a Fourier expansion of $F(\Omega)$.

Definition 2.5. *Let A be a g-dimensional simple abelian variety defined over $\mathbb{Q}$ and let ℓ be a prime. For a prime p, we let*

$$L_p(A, s) := \left\{ \det \left(I_{2g} - p^{-s} \cdot \rho_{A,\ell}(\mathrm{Frob}_p)\big|_{T_\ell(A)} \right) \right\}^{-1}$$

be the local L-function of A at p. We define the L-function $L(A, s)$ of A by

$$L(A, s) = \prod_{p:\,\mathrm{prime}} L_p(A, s).$$

Definition 2.6. *Let A be a g-dimensional simple abelian variety defined over $\mathbb{Q}$. A is called* modular *if there exists a Siegel Hecke eigenform $F(\Omega) \in [\Gamma_{g,0}(N), g+1]_0$ of weight $g+1$ with respect to $\Gamma_{g,0}(N)$ such that*

$$L(A, s) = D(F, s),\ \ Z_F(s) \text{ or } D_F(s).$$

For two positive integers g and N, we let

$$\mathcal{A}_{g,0}(N) := \Gamma_{g,0}(N) \backslash \mathbb{H}_g$$

be the Siegel modular variety of level structure N and let $\mathcal{A}_{g,0}^{\mathrm{tor}}(N)$ be a smooth toroidal compactification of $\mathcal{A}_{g,0}(N)$ (cf. [?, ?]). We denote by

$$\Omega^i \left(\mathcal{A}_{g,0}^{\mathrm{tor}}(N) \right), \quad 0 \le i \le \frac{g(g+1)}{2}$$

the complex vector space of holomorphic i-forms on $\mathcal{A}_{g,0}^{\mathrm{tor}}(N)$. The Jacobian variety $\mathrm{Jac}\left(\mathcal{A}_{g,0}^{\mathrm{tor}}(N) \right)$ of $\mathcal{A}_{g,0}^{\mathrm{tor}}(N)$ is defined to be the abelian variety

$$\mathrm{Jac}(\mathcal{A}_{g,0}^{\mathrm{tor}}(N)) := \Omega^\nu \left(\mathcal{A}_{g,0}^{\mathrm{tor}}(N) \right)^\vee / H_\nu \left(\mathcal{A}_{g,0}^{\mathrm{tor}}(N), \mathbb{Z} \right), \qquad \nu = \frac{g(g+1)}{2}. \tag{2.1}$$

The geometric genus of $\mathcal{A}_{g,0}^{\rm tor}(N)$ is the dimension of the Jacobian variety Jac $\left(\mathcal{A}_{g,0}^{\rm tor}(N)\right)$. It is known that the following two vector spaces are isomprphic:

$$[\Gamma_{g,0}(N), g+1]_0 \cong \Omega^\nu\left(\mathcal{A}_{g,0}^{\rm tor}(N)\right), \qquad \nu = \frac{g(g+1)}{2}. \tag{2.2}$$

More precisely, for a coordinate $\Omega = (\omega_{ij}) \in \mathbb{H}_g$, we let

$$\omega_0 := d\omega_{11} \wedge d\omega_{12} \wedge d\omega_{13} \wedge \cdots \wedge d\omega_{gg}$$

be a holomorphic ν-form on $\mathbb{H}_g$. If $\omega = F(\Omega)\,\omega_0$ is a $\Gamma_{g,0}(N)$-invariant holomorphic form on $\mathbb{H}_g$, then

$$F(\gamma\cdot\Omega) = \det(C\Omega + D)^{g+1}F(\Omega)$$

for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{g,0}(N)$ and $\Omega \in \mathbb{H}_g$. Thus $F \in [\Gamma_{g,0}(N), g+1]$. It was shown by Freitag [**?**] that ω can be extended to a holomorphic ν-form on $\mathcal{A}_{g,0}^{\rm tor}(N)$ if and only if F is a cusp form in $[\Gamma_{g,0}(N), g+1]_0$. Indeed, the mapping

$$[\Gamma_{g,0}(N), g+1]_0 \longrightarrow \Omega^\nu\left(\mathcal{A}_{g,0}^{\rm tor}(N)\right), \qquad F \mapsto F\,\omega_0$$

is an isomorphism as complex vector spaces. We observe that if $\omega_k := G(\Omega)\,\omega_0^{\otimes k}$ is a $\Gamma_{g,0}(N)$-invariant holomorphic form on $\mathbb{H}_g$ of degree $k\nu$, then $G(\Omega) \in [\Gamma_{g,0}(N), k(g+1)]_0$ is a cusp form of weight $k(g+1)$.

Therefore according to (**??**) and (**??**), we have

$$\mathrm{Jac}(\mathcal{A}_{g,0}^{\rm tor}(N)) \cong [\Gamma_{g,0}(N), g+1]_0^\vee / H_\nu\left(\mathcal{A}_{g,0}^{\rm tor}(N), \mathbb{Z}\right), \qquad \nu = \frac{g(g+1)}{2}. \tag{2.3}$$

If there is no confusion, we simply set

$$J_{g,0}(N) := \mathrm{Jac}(\mathcal{A}_{g,0}^{\rm tor}(N)).$$

We propose the following conjectures.

Conjecture 2.7. *A simple abelian variety of dimesion g defined over $\mathbb{Q}$ is modular.*

Conjecture 2.8. *Let A be a simple abelian variety of dimesion g defined over $\mathbb{Q}$. The following statements are equivalent:*

(MAV1) *A is modular.*

(MAV2) *There exists a non-constant holomorphic mapping $\mathcal{A}_{g,0}^{\rm tor}(N) \longrightarrow A$ for some positive integer N.*

(MAV3) *There exists a non-constant holomorphic mapping $J_{g,0}(N) \longrightarrow A$ for some positive integer N.*

(MAV4) *$\rho_{A,\ell}$ is modular for any prime ℓ.*

We propose the following problems.

Problem 2.9. *Let $F \in [\Gamma_{g,0}(N), g+1]_0$ be a Siegel Hecke eigenform of weight $g+1$. Associate to F a $2g$-dimensional continuous irreducible Galois representation of $G_{\mathbb{Q}}$.*

Problem 2.10. *Let k be a positive integer. Let $F \in [\Gamma_{g,0}(N), k]_0$ be a Siegel Hecke eigenform of weight k. Associate to F a $2g$-dimensional continuous irreducible Galois representation of $G_{\mathbb{Q}}$.*

Remark 2.11. *As mentioned in Section 1, in the case $g = 1$, to a Hecke eigenform of weight $k \geq 1$, Shimura, Deligne and Serre associated a two-dimensional continuous irreducible Galois representation. For the case $g = 2$, Taylor* [**?**, **?**] *tried to associate the four dimensional continuous Galois representation of $G_{\mathbb{Q}}$ to a Siegel Hecke eigenform of small weight. But he did not specify the precise weight.*

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