PY 251 Exam 2

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1 Problem 1 New Euler Method

This first problem asked us to solve and discretize a simple form of the differential equation for exponential decay using a new Euler method. The equation in question is as follows.

$$\frac{\Delta N}{\Delta t} = -\frac{1}{\tau}N \tag{1}$$

So being asked to solve this according to the new Euler method required us to solve it in terms of N_{i+1} to evaluate the function at the $(n+1)^{th}$ point rather than the n^{th} point. To solve in this manner we take the following steps with the equation (1)

$$N_{i+1} - N_i = -\frac{1}{\tau}N_{i+1} \tag{2}$$

By splitting the ΔN into the difference of position between the points N_{i+1} and N_i we can create a solvable equation for N_{i+1} with the help of algebra that results in the following.

$$N_{i+1} = \frac{N_i}{(1 + \frac{1}{\tau}dt)} \tag{3}$$

By looping this over varying time-steps with the provided initial conditions for τ and N_0 we can graph and analyze the efficiency of both the old and new Euler methods against the analytical solution in question.

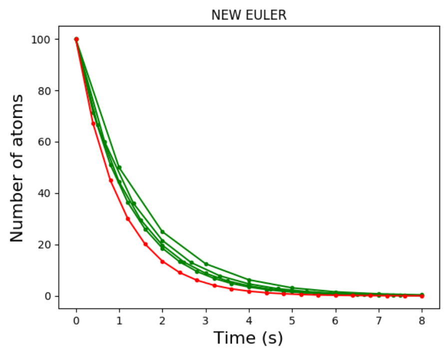

Figure 1: graph of new Euler's approximation for time-steps varying at $nts = [8, 12, 16, 20]$ against the analytical solution

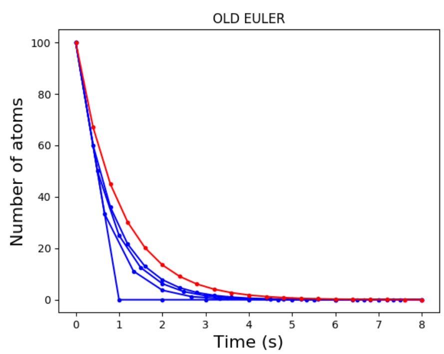

Figure 2: graph of old Euler's approximation for time-steps varying at $nts = [8, 12, 16, 20]$ against the analytical solution

After plotting the approximations against the analytic solution we can see a few noticeable differences between our old and new Euler methods. For one the new approximation creates an over approximation while our original method was an under approximation. In additions to this the new method functions better with larger time-steps where its counter part completely breaks down; however, the older method is more accurate when considering incredibly small time-steps. This became apparent through a comparison of the errors at varying time-steps.

2 Exam Problem 2 Euler Approximation for $sin(x)$

This problem had us use the same method of approximation but with a function that does not have an analytic form. This came in the form of the following equation.

$$\frac{\Delta y}{\Delta t} = sin(t) \tag{4}$$

Here we again rewrite this equation in terms of a difference for the $(n+1)^{th}$ term shown below.

$$y_{i+1} - y_i = sin(y_{i+1})\Delta t \tag{5}$$

Here we encounter said issue where there is no analytic form to the solution, so we instead set the whole equation equal to zero and use Newtons method for root finding to find values of y_{n+1} for each n. So using this process and the provided initial conditions we can create a plot of the approximation that look like so.

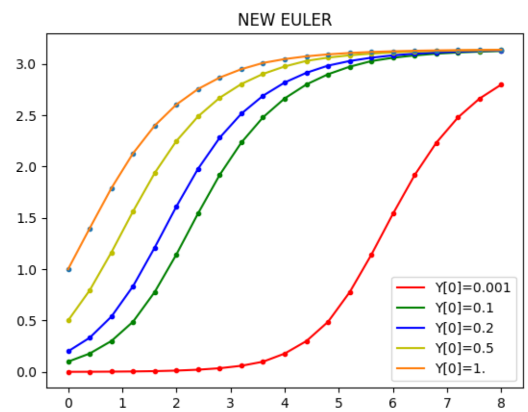

Figure 3: graph of new Euler's approximation from $t = 0$ to $t = 8$ for 20 time-steps varying values for y_0

3 Exam Problem 3 Martingale Betting System

The final question tasked us with finding solutions to the given Eigenvalue problem consisting of a system of 5 spring-bound masses. I was able to do this with the help of the previous problem completed in lesson 12. The first task was to properly create a matrix of motion in terms of the k and dependence of each mass. This process began by determining equations of motion for each mass, and narrowing down what other masses each would depend on.

$$m\ddot{x}_1(t) = -k[x_1(t) - l] + k[x_2(t) - x_1(t) - l] \tag{6}$$

$$m\ddot{x}_2(t) = -k[x_2(t) - x_1(t) - l] + k[x_3(t) - x_2(t) - l] \tag{7}$$

$$m\ddot{x}_3(t) = -k[x_3(t) - x_2(t) - l] + k[x_4(t) - x_3(t) - l] \tag{8}$$

$$m\ddot{x}_4(t) = -k[x_4(t) - x_3(t) - l] + k[x_5(t) - x_4(t) - l] \tag{9}$$

$$m\ddot{x}_5(t) = -k[x_5(t) - x_4(t) - l] + k[x_5(t) - l] \tag{10}$$

Here we can make the substitutions that $\eta_1 = x_1 - l$, $\eta_2 = x_2 - 2l$, etc. to change the coordinates of our equation of motions so that they become the following.

$$m\ddot{\eta}_1 = -2k\eta_1 + k\eta_2 \tag{11}$$

$$m\ddot{\eta}_2 = k\eta_1 - 2k\eta_2 + k\eta_3 \tag{12}$$

$$m\ddot{\eta}_3 = k\eta_2 - 2k\eta_3 + k\eta_4 \tag{13}$$

$$m\ddot{\eta}_4 = k\eta_3 - 2k\eta_4 + k\eta_5 \tag{14}$$

$$m\ddot{\eta}_5 = -2k\eta_5 + k\eta_4 \tag{15}$$

So if we assume normal modes are of the form $\eta_1 = A_1 \cos(\omega)$ with ω and A_1 varying with our Eigenvalues we can also express them as $\eta_1 = C_1 e^{i\omega t}$. Deriving the latter form twice with respect to time we can get a more versatile left hand to equations (11-15) and by making the assertion that we only care about the real values of this system we can approximate $\eta_1 = C_1$ and so forth to generate equations of the following form.

$$m\omega^2 C_1 = 2kC_1 - kC_2 \tag{16}$$

$$m\omega^2 C_2 = -kC_1 + 2kC_2 - kC_3 \tag{17}$$

$$m\omega^2 C_3 = -kC_2 + 2kC_3 - kC_4 \tag{18}$$

$$m\omega^2 C_4 = -kC_3 + 2kC_4 - kC_5 \tag{19}$$

$$m\omega^2 C_5 = 2kC_5 - kC_4 \tag{20}$$

With there we can finally create a matrix and solve for our 5 unique Eigenvalues and their Eigenvectors using pythons linear algebra functions. Using these generated ω values and their respective Eigenvectors we are able to create 5 normal modes of the form $\eta_1 = A_1 \cos(\omega_1)$ where A is one of the 5 vector values associated with the utilized ω multiplied by a constant. With this our equation of motion could be created by any linear combination of the 5 equations generated in this manner. Plotted below are the 5 different modes for each unique ω.

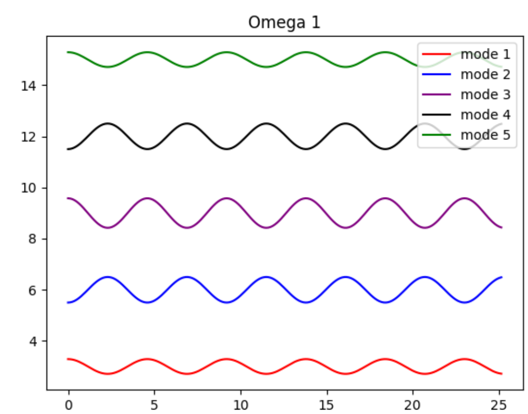

Figure 4: For the first ω value the odd modes have a positive amplitude while the even modes have a negative one, in addition to this it is noticeable that the magnitude of the amplitude grows as you approach the center mass.

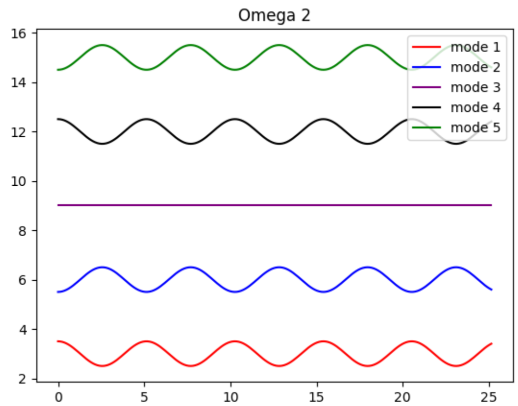

Figure 5: For the second ω value the outer masses oscillate in tandem as they are negative copies of each other while the center mass stays almost completely still.

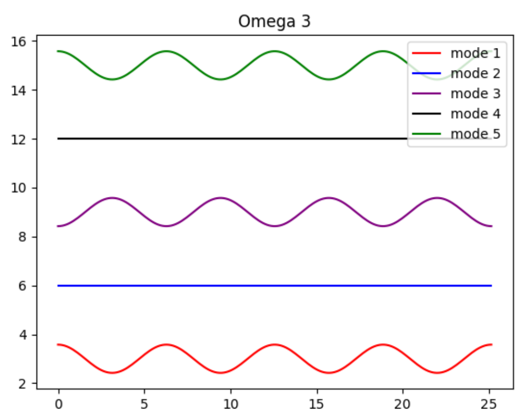

Figure 6: For the third ω value the outer masses oscillate harmoniously, while the center mass does the opposite. In addition to this the middle two masses are nearly completely still.

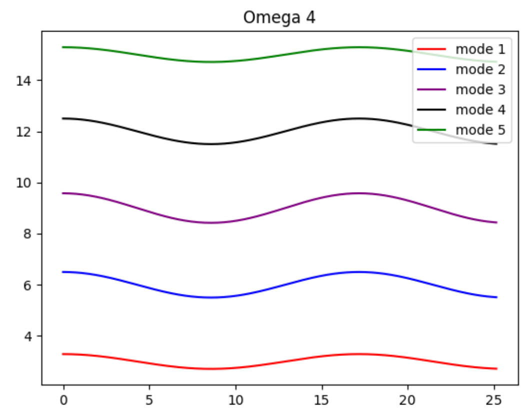

Figure 7: For the fourth ω value each mode has a similar wave function only differing in magnitude as you approach the center.

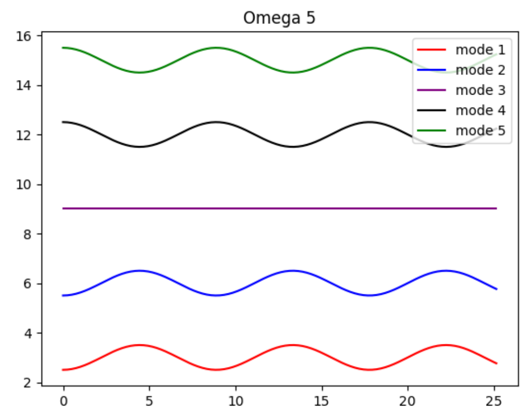

Figure 8: For the fifth ω value features a stable center with masses on either side moving together but opposite to that of their counterpart.