Sapienza Università di Roma

Sapienza University of Rome

Big Data Computing

Homework 3

Student

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Assignment 1

The all assignment was done on Colab in the file `1796575-Lecce_HW3_2023.ipynb`.

Assignment 2

1)

Since A has a spectral decomposition, by definition its corresponding eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are orthonormal, with $\mathbf{U} = (\mathbf{u}_1 \ldots \mathbf{u}_n)$, so $\mathbf{U}^T\mathbf{U} = \mathbf{I}$. We can rewrite $\mathbf{A}$ as $\mathbf{A} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T$.
We would like to prove that

$$\mathbf{A}^k = \mathbf{U}\boldsymbol{\Lambda}^k\mathbf{U}^T$$

For $k = 2$ we have the following:

$$\begin{aligned}\mathbf{A}^2 &= (\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T) \cdot (\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T) \\ &= \mathbf{U}\boldsymbol{\Lambda}\boldsymbol{\Lambda}\mathbf{U}^T \\ &= \mathbf{U}\boldsymbol{\Lambda}^2\mathbf{U}^T\end{aligned}$$

Now consider the case where $k = j + 1$:

$$\begin{aligned}\mathbf{A}^{j+1} &= (\mathbf{U}\boldsymbol{\Lambda}^j\mathbf{U}^T) \cdot (\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T) \\ &= \mathbf{U}\boldsymbol{\Lambda}^j\boldsymbol{\Lambda}\mathbf{U}^T \\ &= \mathbf{U}\boldsymbol{\Lambda}^{j+1}\mathbf{U}^T\end{aligned}$$

Thus, for $k \geq 1$:

$$\mathbf{A}^k = \mathbf{U}\boldsymbol{\Lambda}^k\mathbf{U}^T = \sum_{i=1}^{n} \lambda_i^k \mathbf{u}_i\mathbf{u}_i^T.$$

2)

Assuming that $\mathbf{A}^{-1} = \sum_{i=1}^{n} \lambda_i^{-1}\mathbf{u}_i\mathbf{u}_i^T$, we want to prove that multiplying it by $\mathbf{A}$, we get the identity matrix:

$$\mathbf{A} = \sum_{i=1}^{n} \lambda_i\mathbf{u}_i\mathbf{u}_i^T$$

$$\mathbf{A}\mathbf{A}^{-1} = \left(\sum_{i=1}^{n} \lambda_i\mathbf{u}_i\mathbf{u}_i^T\right) \cdot \left(\sum_{j=1}^{n} \lambda_j^{-1}\mathbf{u}_j\mathbf{u}_j^T\right)$$

$$\delta_{ij} = \mathbf{u}_i^T\mathbf{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Given the property above, we simplify:

$$\mathbf{A}\mathbf{A}^{-1} = \sum_{i=1}^{n} \mathbf{u}_i\mathbf{u}_i^T = \mathbf{I}$$

For $\mathbf{A}^{-1}\mathbf{A}$ we can prove it in the same way.

Assignment 3

1)

First, let's write the expression of $\mathbf{A}\mathbf{A}^T$:

$$\mathbf{A}\mathbf{A}^T = (\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T) \cdot (\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T)^T = (\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T) \cdot (\mathbf{V}\mathbf{\Sigma}^T\mathbf{U}^T)$$

Let's unwrap this expression by saying that $\mathbf{V}^T\mathbf{V} = \mathbf{I}_m$, and $\mathbf{\Sigma} \in \mathbb{R}^{n\times m}$ is a *rectangular diagonal matrix*, so $\mathbf{\Sigma}\mathbf{I}_m\mathbf{\Sigma}^T = \mathbf{\Sigma}\mathbf{\Sigma}^T = \mathbf{\Sigma}^2 \in \mathbb{R}^{n\times n}$:

$$\mathbf{A}\mathbf{A}^T = \mathbf{U}\mathbf{\Sigma}^2\mathbf{U}^T = \sum_{i=1}^{n} \sigma_i^2 \mathbf{u}_i\mathbf{u}_i^T$$

Furthermore, from lecture notes we have that $\lambda_i = \sigma_i^2$, so the above expression is equal to:

$$\mathbf{A}\mathbf{A}^T = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i\mathbf{u}_i^T$$

but this is the same expression as the spectral decomposition in **Assignment 2**, so we can consider $\mathbf{A}\mathbf{A}^T \in \mathbb{R}^{n\times n}$ as a *square symmetric matrix*, therefore:

$$(\mathbf{A}\mathbf{A}^T)^k = \sum_{i=1}^{n} \lambda_i^k \mathbf{u}_i\mathbf{u}_i^T = \sum_{i=1}^{n} \sigma_i^{2k} \mathbf{u}_i\mathbf{u}_i^T$$

2)

$\mathbf{Q} = (\mathbf{q}_1 \dots \mathbf{q}_n)$, where $\mathbf{q}_1, \dots, \mathbf{q}_n$ are orthonormal vectors, so, by definition:

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \Longrightarrow \mathbf{Q}^T = \mathbf{Q}^{-1}$$

and $\det(\mathbf{Q} \neq 0)$. Let's suppose that $\mathbf{T}$ is the right-inverse of $\mathbf{Q}$, *s.t.* $\mathbf{Q}\mathbf{T} = \mathbf{I}$:

$$\mathbf{T} = \mathbf{I}\mathbf{T} = \mathbf{Q}^T\mathbf{Q}\mathbf{T} = \mathbf{Q}^T(\mathbf{Q}\mathbf{T}) = \mathbf{Q}^T\mathbf{I} = \mathbf{Q}^T$$

We can rewrite the equality as the following:

$$\mathbf{Q}^{-1}\mathbf{Q} = \mathbf{I} = \mathbf{Q}\mathbf{T} = \mathbf{Q}\mathbf{Q}^T$$

implying that also the rows are orthonormal.

3)

Since $\mathbf{A}$ is square, $\mathbf{A} = \sum_{i=1}^{n} \sigma_i \mathbf{u}_i\mathbf{v}_i^T = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$, where $\mathbf{U}, \mathbf{\Sigma}, \mathbf{V} \in \mathbb{R}^{n\times n}$ and $\mathbf{\Sigma}$ being a *square diagonal matrix*, and assuming that $\mathbf{A}$ is invertible, to **prove** that $\mathbf{B} = \sum_{i=1}^{n} \frac{1}{\sigma_i}\mathbf{v}_i\mathbf{u}_i^T = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^T$ is its inverse, we can apply the claim of the previous point and also we can proceed very similarly to point 2 of **Assignment 2**, by multiplying both sides of the latter equality with $\mathbf{A}$:

$$\begin{aligned}\mathbf{B}\mathbf{A} &= (\mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^T) \cdot (\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T) \\ &= \mathbf{V}\mathbf{\Sigma}^{-1}(\mathbf{U}^T\mathbf{U})\mathbf{\Sigma}\mathbf{V}^T \\ &= \mathbf{V}(\mathbf{\Sigma}^{-1}\mathbf{\Sigma})\mathbf{V}^T \\ &= \mathbf{V}\mathbf{V}^T = \mathbf{I}\end{aligned}$$

Assignment 4

1)

Let's put $\mathbf{M} := \mathbf{A}^T$, and let's prove that $\mathbf{M}^T\mathbf{M}$ is PSD:

$$\mathbf{x}^T\mathbf{M}^T\mathbf{M}\mathbf{x} = (\mathbf{M}\mathbf{x})^T(\mathbf{M}\mathbf{x})$$

$\mathbf{M}\mathbf{x}$ it's a column vector in $\mathbb{R}^m$, so it's transpose times itself returns $\|\mathbf{M}\mathbf{x}\|_2^2$, which is ≥ 0.

2) BONUS QUESTION

- $\mathbf{A}$ is PSD $\Rightarrow$ all its eigenvalues are non-negative
  Let's define by λ its eigenvalues, and by $\mathbf{v}$ its eigenvectors. By definition, $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, so if $\mathbf{A}$ is PSD, we have:

$$\mathbf{v}^T\mathbf{A}\mathbf{v} = \mathbf{v}^T\lambda\mathbf{v} = \lambda\mathbf{v}^T\mathbf{v} \geq 0,\ \ \textit{where}\ \mathbf{v}^T\mathbf{v} \geq 0,\ \ \textit{so}\ \lambda \geq 0$$

- $\mathbf{A}$ is PSD $\Leftarrow$ all its eigenvalues are non-negative
  To prove this point, let's assume there is $\mathbf{x} \neq 0$ *s.t.* $\mathbf{x}^T\mathbf{A}\mathbf{x} < 0$, and $\mathbf{A}$ has a spectral decomposition in the form $\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T$, with $\mathbf{U}^T\mathbf{U} = \mathbf{I}$.
  Then for $\mathbf{y} \neq 0$ *s.t.* $\mathbf{x} = \mathbf{U}\mathbf{y}$:

$$0 > \mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{y}^T\mathbf{U}^T\mathbf{A}\mathbf{U}\mathbf{y} = \mathbf{y}^T(\mathbf{U}^T\mathbf{U})\mathbf{\Lambda}(\mathbf{U}^T\mathbf{U})\mathbf{y} = \mathbf{y}^T\mathbf{\Lambda}\mathbf{y} = \sum_{i=1}^{n}\lambda_i\mathbf{y}_i^2$$

  But $\mathbf{y}_i^2 \geq 0 \quad \forall i \in \{1, \ldots, n\}$ and by assumption $\lambda_i \geq 0 \quad \forall i \in \{1, \ldots, n\}$, so $\mathbf{x}^T\mathbf{A}\mathbf{x} < 0$ is a contradiction, proving that $\mathbf{A}$ is PSD.