COMP9313: Big Data Management

Mining Data Streams

Source from Dr. Xin Cao

Data Streams

- In many data mining situations, we do not know the entire data set in advance
- Stream Management is important when the input rate is controlled **externally**
  - Google queries
  - Twitter or Facebook status updates
- We can think of the **data** as **infinite** and **non-stationary** (the distribution changes over time)

Characteristics of Data Streams

- Traditional DBMS: data stored in *finite*, *persistent data sets*
- Data Streams: distributed, continuous, unbounded, rapid, time varying, noisy, . . .
- Characteristics
  - Huge volumes of continuous data, possibly infinite
  - Fast changing and requires fast, real-time response
  - Random access is expensive—single scan algorithm (can only have one look)
  - Store only the summary of the data seen thus far

Massive Data Streams

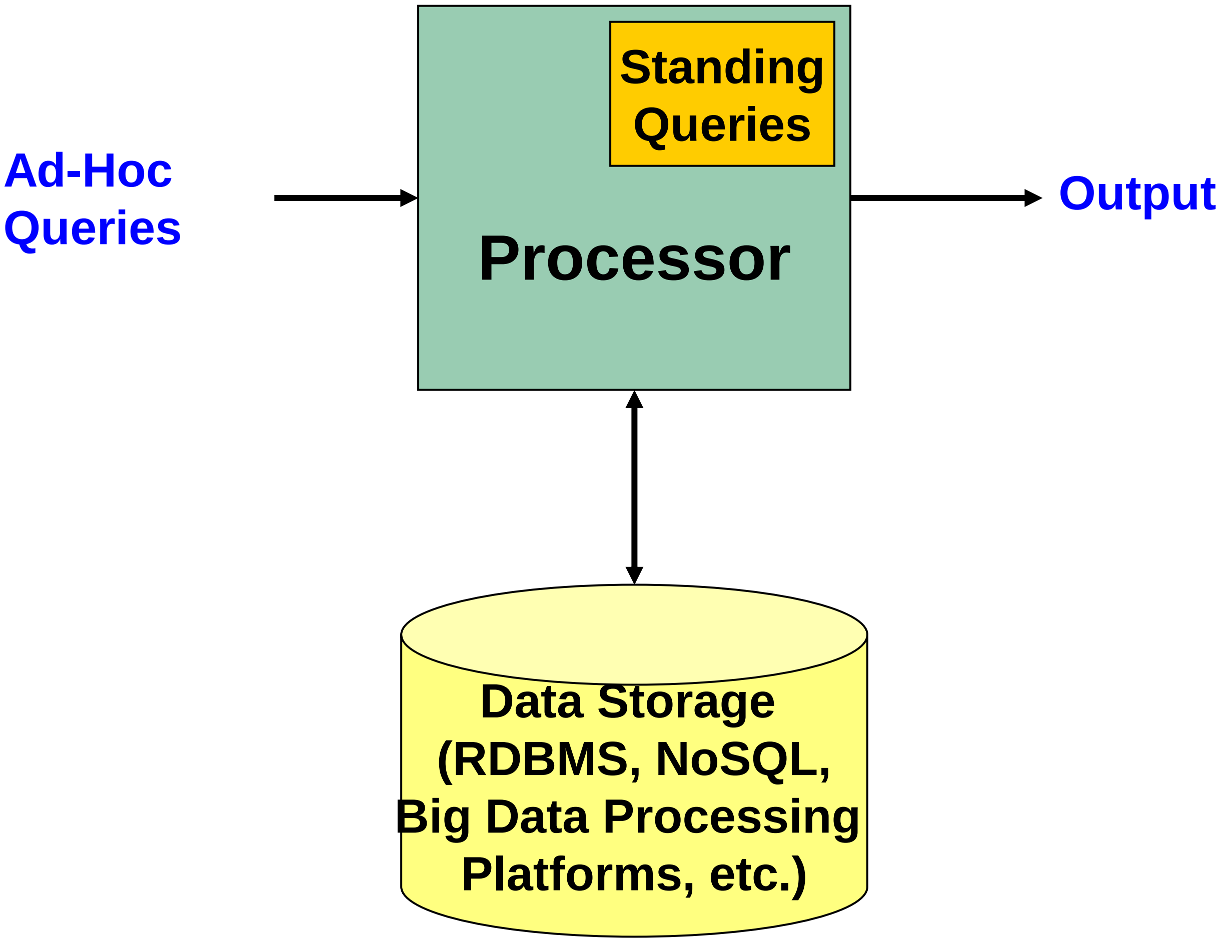
- Data is *continuously growing* faster than our ability to store or index it
- There are 3 Billion Telephone Calls in US each day, 30 Billion emails daily, 1 Billion SMS, IMs
- Scientific data: NASA's observation satellites generate billions of readings each per day
- IP Network Traffic: up to 1 Billion packets per hour per router. Each ISP has many (hundreds) routers!
- Internet of Things
- …

The Stream Model

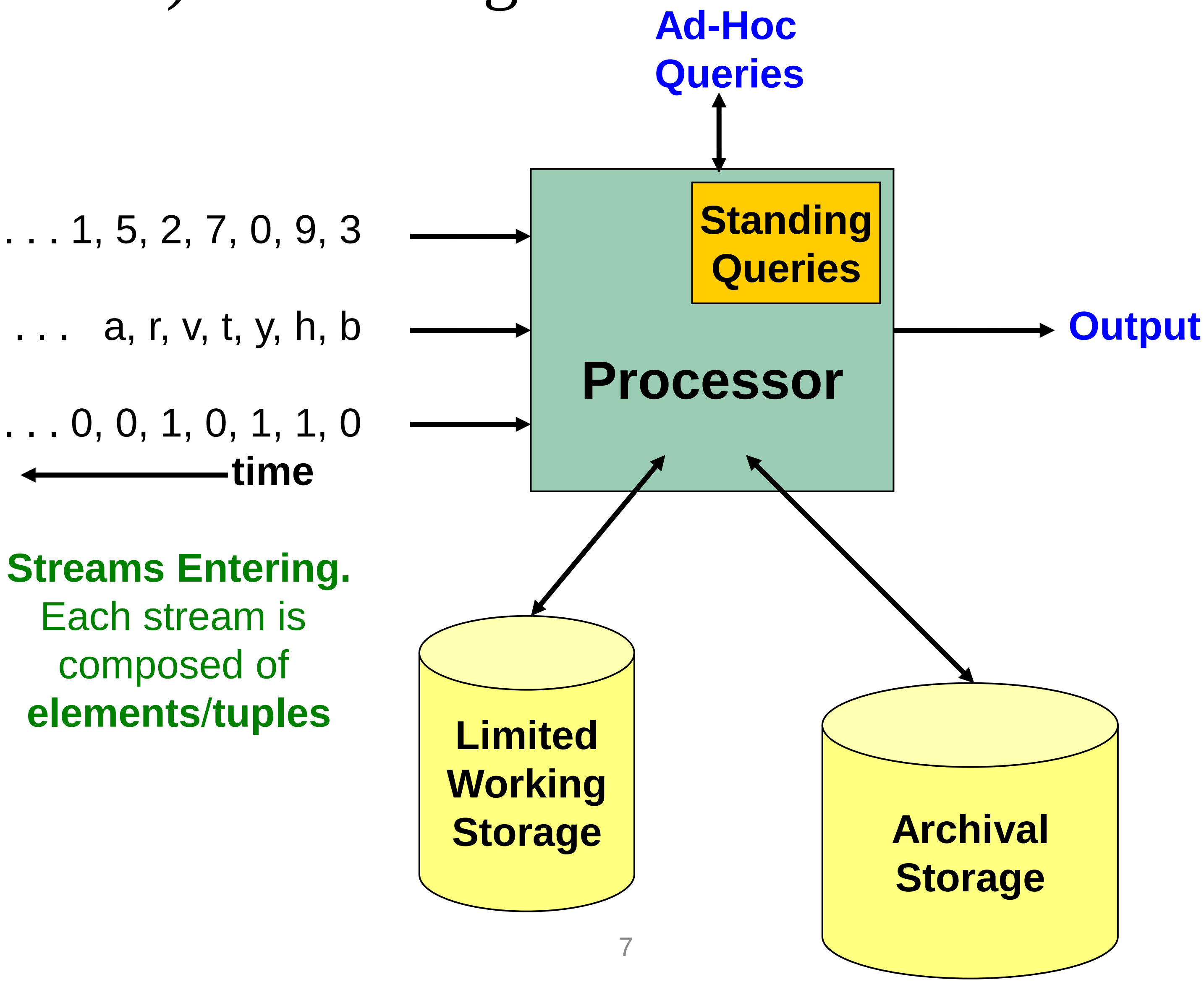
- Input elements enter at a rapid rate, at one or more input ports (i.e., streams)
  - We call elements of the stream tuples
- The system cannot store the entire stream accessibly

- **Q: How do you make critical calculations about the stream using a limited amount of memory?**

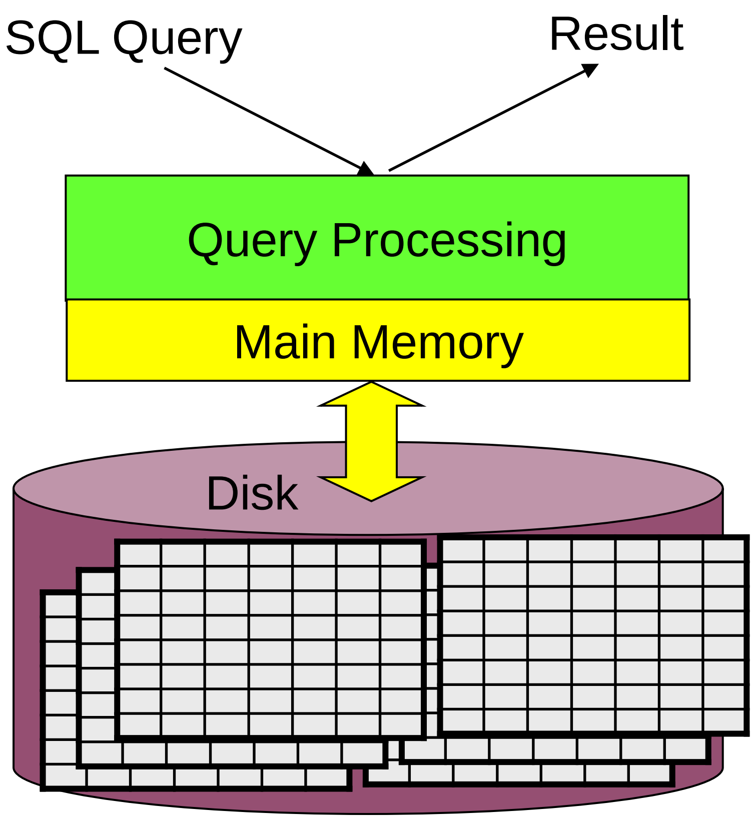
Database Management System (DBMS) Data Processing

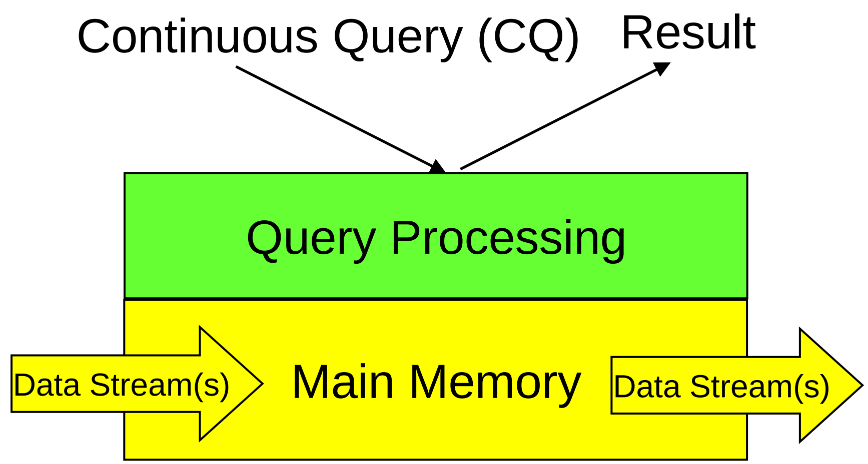


General Data Stream Management System (DSMS) Processing Model



DBMS vs. DSMS #1





DBMS vs. DSMS #2

- Traditional DBMS:
  - stored sets of relatively static records with no pre-defined notion of time
  - good for applications that require persistent data storage and complex querying
- DSMS
  - support on-line analysis of rapidly changing data streams
  - data stream: real-time, continuous, ordered (implicitly by arrival time or explicitly by timestamp) sequence of items, too large to store entirely, no ending
  - continuous queries

DBMS vs. DSMS #3

DBMS

- Persistent relations
  (relatively static, stored)
- One-time queries
- Random access
- “Unbounded” disk store
- Only current state matters
- No real-time services
- Relatively low update rate
- Data at any granularity
- Assume precise data
- Access plan determined by query processor, physical DB design

DSMS

- Transient streams
  (on-line analysis)
- Continuous queries (CQs)
- Sequential access
- Bounded main memory
- Historical data is important
- Real-time requirements
- Possibly multi-GB arrival rate
- Data at fine granularity
- Data stale/imprecise
- Unpredictable/variable data arrival and characteristics

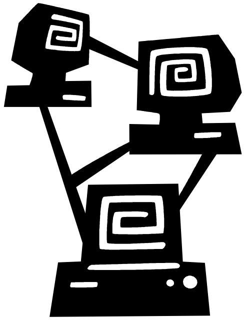
Problems on Data Streams

- Types of queries one wants on answer on a data stream:
  - Sampling data from a stream
    - Construct a random sample
  - Queries over sliding windows
    - Number of items of type x in the last *k* elements of the stream
  - Filtering a data stream
    - Select elements with property x from the stream
  - Counting distinct elements
    - Number of distinct elements in the last *k* elements of the stream

Applications

- Mining query streams
  - Google wants to know what queries are more frequent today than yesterday
- Mining click streams
  - Yahoo wants to know which of its pages are getting an unusual number of hits in the past hour
- Mining social network news feeds
  - E.g., look for trending topics on Twitter, Facebook
- Sensor Networks
  - Many sensors feeding into a central controller
- Telephone call records
  - Data feeds into customer bills as well as settlements between telephone companies
- IP packets monitored at a switch
  - Gather information for optimal routing

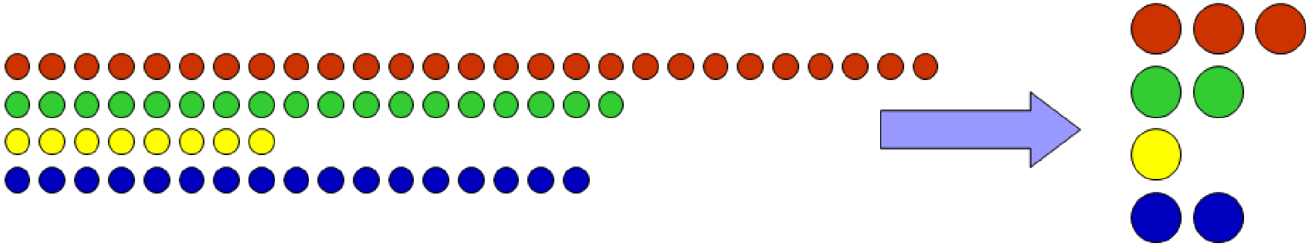
Example: IP Network Data



- Networks are sources of massive data: the metadata per hour per IP router is gigabytes
- Fundamental problem of data stream analysis:
  - Too much information to store or transmit
- So process data as it arrives
  - One pass, small space: the data stream approach
- **Approximate answers** to many questions are OK, if there are guarantees of result quality

Sampling from a Data Stream

- Since we can not store the entire stream, one obvious approach is to store a **sample**



- Two different problems:
  - **(1)** Sample a **fixed proportion** of elements in the stream (say 1 in 10)
    - As the stream grows the sample also gets bigger
  - **(2)** Maintain a **random sample of fixed size** over a potentially infinite stream
    - As the stream grows, the sample is of fixed size
    - At any "time" ***t*** we would like a random sample of ***s*** elements
- **What is the property of the sample we want to maintain?**
  - For all time steps ***t***, each of ***t*** elements seen so far has equal probability of being sampled

Sampling a Fixed Proportion

- Problem 1: Sampling fixed proportion
- Scenario: Search engine query stream
  - **Stream of tuples:** (user, query, time)
  - **Answer questions such as:** **How often did a user run the same query in a single day**
  - Have space to store **1/10$^{\text{th}}$** of query stream
- **Naïve solution:**
  - Generate a random integer in **[0..9]** for each query
  - Store the query if the integer is **0**, otherwise discard

Problem with Naïve Approach

- Simple question: What fraction of queries by an average search engine user are duplicates?
  - Suppose each user issues ***x*** queries once and ***d*** queries twice (total of ***x*+2*d*** queries)
    - **Correct answer: *d*/(*x*+*d*)**
  - **Proposed solution: We keep 10% of the queries**
    - Sample will contain ***x*/10** of the singleton queries and **2*d*/10** of the duplicate queries at least once
    - But only ***d*/100** pairs of duplicates
      - **d/100 = 1/10 · 1/10 · d**
    - Of ***d*** "duplicates" ***18d/100*** appear exactly once
      - **18d/100 = ((1/10 · 9/10)+(9/10 · 1/10)) · d**
  - **So the sample-based answer is** $\frac{\frac{d}{100}}{\frac{x}{10}+\frac{d}{100}+\frac{18d}{100}} = \frac{d}{10x+19d}$

    $\neq d/(x+d)$

Solution: Sample Users

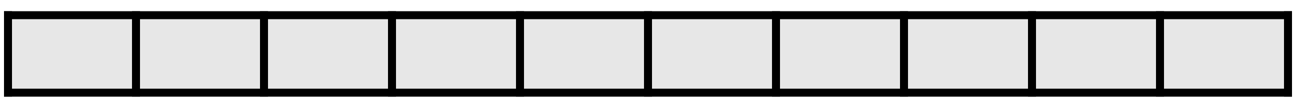
Solution:

- Pick **1/10th** of **users** and take all their searches in the sample

- Use a hash function that hashes the user name or user id uniformly into 10 buckets
  - We hash each user name to one of ten buckets, 0 through 9
  - If the user hashes to bucket 0, then accept this search query for the sample, and if not, then not.

Generalized Problem and Solution

- Problem: Give a data stream, take a sample of fraction a/b.
- Stream of tuples with keys:
  - Key is some subset of each tuple's components
    - e.g., tuple is (user, search, time); key is **user**
  - Choice of key depends on application
- To get a sample of a/b fraction of the stream:
  - Hash each tuple's key uniformly into ***b*** buckets
  - Pick the tuple if its hash value is at most ***a***



How to generate a 30% sample?

Hash into b=10 buckets, take the tuple if it hashes to one of the first 3 buckets

Maintaining a Fixed-size Sample

- **Problem 2**: Fixed-size sample
- Suppose we need to maintain a random sample S of size exactly s tuples
  - E.g., main memory size constraint
- **Why?** Don't know length of stream in advance
- Suppose at time *n* we have seen *n* items
  - Each item is in the sample *S* with equal prob. *s/n*

How to think about the problem: say s = 2
Stream: a x c y z k c d e g…

Note that the same item is treated as different tuples at different timestamps

At **n= 5,** each of the first 5 tuples is included in the sample **S** with equal prob.
At **n= 7,** each of the first 7 tuples is included in the sample **S** with equal prob.

Impractical solution would be to store all the *n* tuples seen so far and out of them pick *s* at random

Solution: Fixed Size Sample

- **Algorithm (a.k.a. Reservoir Sampling)**
  - Store all the first s elements of the stream to S
  - Suppose we have seen n-1 elements, and now the n^{th} element arrives ($n > s$)
    - With probability s/n, keep the n^{th} element, else discard it
    - If we picked the n^{th} element, then it replaces one of the s elements in the sample S, picked uniformly at random
- **Claim:** This algorithm maintains a sample S with the desired property:
  - After n elements, the sample contains each element seen so far with probability s/n

Proof: By Induction

- **We prove this by induction:**
  - Assume that after ***n*** elements, the sample contains each element seen so far with probability ***s/n***
  - We need to show that after seeing element ***n+1*** the sample maintains the property
    - Sample contains each element seen so far with probability ***s/(n+1)***
- Base case:
  - After we see **n=s** elements the sample **S** has the desired property
    - Each out of **n=s** elements is in the sample with probability ***s/s = 1***

Proof: By Induction

- Inductive hypothesis: After ***n*** elements, the sample ***S*** contains each element seen so far with prob. ***s/n***
- **Now element *n+1* arrives**
- **Inductive step:** For elements already in ***S***, probability that the algorithm keeps it in ***S*** is:

$$\left(1-\frac{s}{n+1}\right)+\left(\frac{s}{n+1}\right)\left(\frac{s-1}{s}\right)=\frac{n}{n+1}$$

Element **n+1** discarded

Element **n+1** not discarded

Element in the sample not picked

- So, at time ***n***, tuples in ***S*** were there with prob. **s/n**
- Time ***n→n+1***, tuple stayed in ***S*** with prob. **n/(n+1)**
- So prob. tuple is in ***S*** at time ***n+1*** $= \frac{s}{n} \cdot \frac{n}{n+1} = \frac{s}{n+1}$

Sliding Windows

- A useful model of stream processing is that queries are about a ***window*** of length $\boldsymbol{N}$ – the $\boldsymbol{N}$ most recent elements received
- Interesting case: $\boldsymbol{N}$ is so large that the data cannot be stored in memory, or even on disk
  - Or, there are so many streams that windows for all cannot be stored
- Amazon example:
  - For every product **X** we keep 0/1 stream of whether that product was sold in the $\boldsymbol{n}$-th transaction
  - We want answer queries, how many times have we sold **X** in the last $\boldsymbol{k}$ sales

Sliding Window: 1 Stream

- Sliding window on a single stream:

N = 7

q w e r t y u i o p a s d f g h j k l z x c v b n m

q w e r t y u i o p a s d f g h j k l z x c v b n m

q w e r t y u i o p a s d f g h j k l z x c v b n m

q w e r t y u i o p a s d f g h j k l z x c v b n m

← Past Future →

Counting Bits (1)

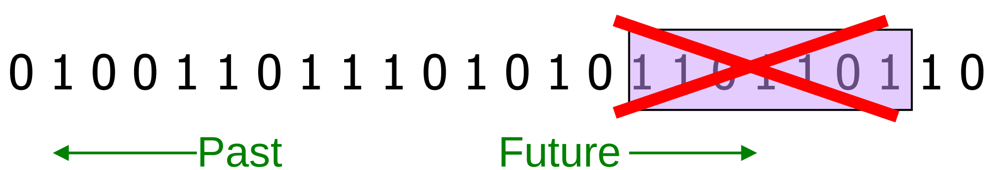
- Problem:
  - Given a stream of **0**s and **1**s
  - Be prepared to answer queries of the form: **How many 1s are in the last *k* bits?** where $k \le N$
- Obvious solution:
  - Store the most recent N bits
    - When new bit comes in, discard the **N+1st** bit

0 1 0 0 1 1 0 1 1 1 0 1 0 1 0 1 1 0 1 1 0 1 1 0   Suppose N=7

Past   Future

Counting Bits (2)

- You can not get an exact answer without storing the entire window
- Real Problem:
  What if we cannot afford to store *N* bits?
  - **E.g.**, we’re processing 1 billion streams and ***N* = 1 billion**
- But we are happy with an approximate answer



An attempt: Simple solution

- Q: How many 1s are in the last N bits?
- A simple solution that does not really solve our problem: **Uniformity Assumption**

N

0 1 0 0 1 1 1 0 0 0 1 0 1 0 0 1 0 0 0 1 0 1 1 0 1 1 0 1 1 1 0 0 1 0 1 0 1 1 0 0 1 1 0 1 0

← Past   Future →

- Maintain 2 counters:
  - S: number of 1s from the beginning of the stream
  - Z: number of 0s from the beginning of the stream
- How many 1s are in the last N bits? $N \cdot \frac{S}{S+Z}$
- **But, what if stream is non-uniform?**
  - What if distribution changes over time?

The Datar-Gionis-Indyk-Motwani (DGIM) Algorithm

- Maintaining Stream Statistics over Sliding Windows (SODA'02)

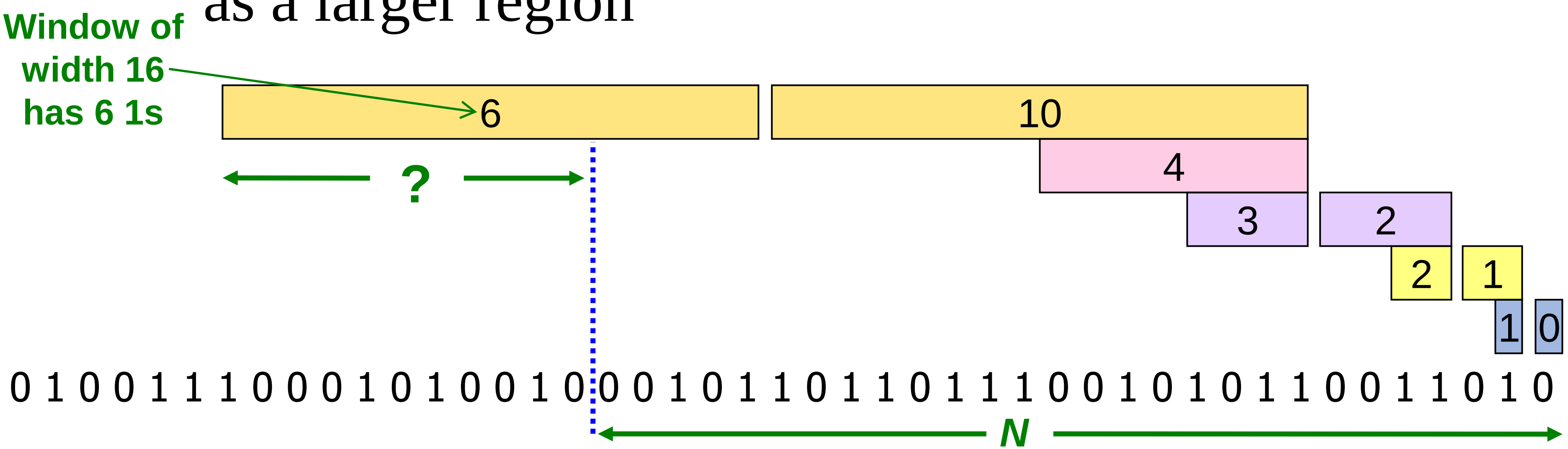
- DGIM solution that does not assume uniformity

- We store $\boldsymbol{O}(\log^{\mathbf{2}} \boldsymbol{N})$ bits per stream

- Solution gives approximate answer, never off by more than 50%
  - Error factor can be reduced to any fraction > 0, with more complicated algorithm and proportionally more stored bits

Idea: Exponential Windows

- Solution that doesn't (quite) work:
  - Summarize **exponentially increasing** regions of the stream, looking backward
  - Drop small regions if they begin at the same point as a larger region



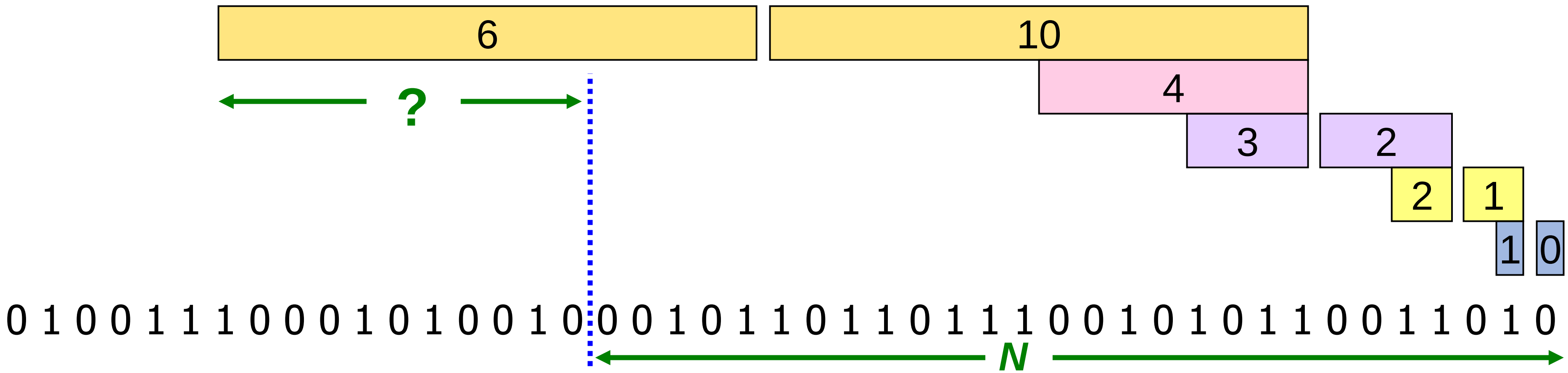
We can reconstruct the count of the last ***N*** bits, except we are not sure how many of the last **6 1s** are included in the ***N***

What's Good?

- Stores only $\boldsymbol{O}(\log^{\mathbf{2}} \boldsymbol{N})$ bits
  - $\boldsymbol{O}(\log \boldsymbol{N})$ counts of $\log_{\mathbf{2}}\boldsymbol{N}$ bits each
- Easy update as more bits enter
- Error in count no greater than the number of **1s** in the "**unknown**" area

What's Not So Good?

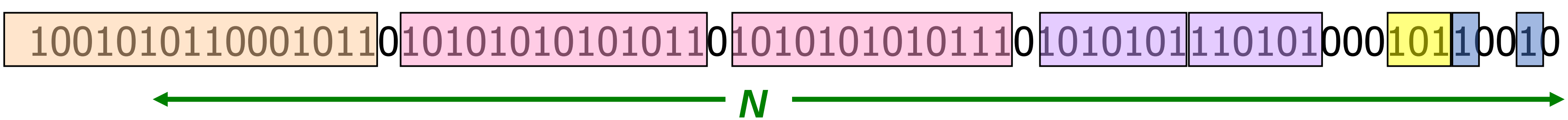
- As long as the **1s** are fairly evenly distributed, the error due to the unknown region is small – **no more than 50%**
- But it could be that all the 1s are in the unknown area at the end
- In that case, the error is unbounded!



Fixup: DGIM Algorithm

- **Idea:** Instead of summarizing fixed-length blocks, summarize blocks with specific number of **1s**:
  - Let the block ***sizes*** (number of **1s**) increase exponentially

- When there are few 1s in the window, block sizes stay small, so errors are small

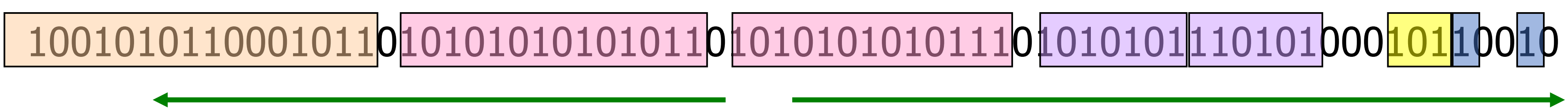


DGIM: Timestamps

- Each bit in the stream has a timestamp, starting from **1**, **2,** …
- Record timestamps modulo $\boldsymbol{N}$ (**the window size**), so we can represent any **relevant** timestamp in $\boldsymbol{O}(\log_{\mathbf{2}}\boldsymbol{N})$ bits
  - E.g., given the windows size 40 ($\boldsymbol{N}$), timestamp 123 will be recorded as 3, and thus the encoding is on 3 rather than 123

DGIM: Buckets

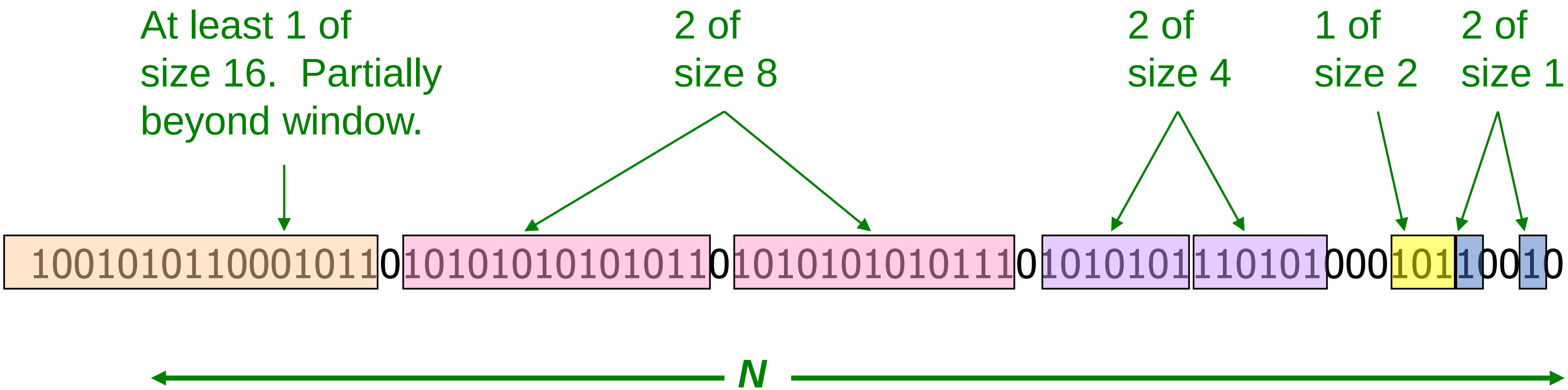
- A bucket in the DGIM method is a record consisting of:
  - (A) The timestamp of its end [$O(\log N)$ bits]
  - (B) The number of 1s between its beginning and end [$O(\log\log N)$ bits]
- Constraint on buckets:
  - Number of 1s must be a power of 2
  - That explains the $O(\log\log N)$ in (B) above



Representing a Stream by Buckets

- The right end of a bucket is always a position with a 1
- Every position with a 1 is in some bucket
- Either **one** or **two** buckets with the same **power-of-2 number** of **1s**
- Buckets do not overlap in timestamps
- **Buckets are sorted by size**
  - Earlier buckets are not smaller than later buckets
- Buckets disappear when their end-time is > N time units in the past

Example: Bucketized Stream



Three properties of buckets that are maintained:

- Either **one** or **two** buckets with the same power-of-2 number of 1s
- Buckets do not overlap in timestamps
- Buckets are sorted by size

Updating Buckets

- When a new bit comes in, drop the last (oldest) bucket if its end-time is prior to N time units before the current time

- 2 cases: Current bit is **0** or **1**

- If the current bit is 0: no other changes are needed
- If the current bit is 1:
  - (1) Create a new bucket of size 1, for just this bit
    - End timestamp = current time
  - (2) If there are now three buckets of size 1, combine the oldest two into a bucket of size 2
  - (3) If there are now three buckets of size 2, combine the oldest two into a bucket of size 4
  - (4) And so on …

Example: Updating Buckets

Current state of the stream:

100101011000101101010101010101101010101010111101010101110101000101100 10

Bit of value 1 arrives

0010101100010110101010101010110101010101011110101010111010100010110010 1

Two white buckets get merged into a yellow bucket

0010101100010110101010101010110101010101011110101010111010100010110010 1

Next bit 1 arrives, new orange white is created, then 0 comes, then 1:

0101100010110101010101010110101010101011110101010111010100010110010 1101

Buckets get merged…

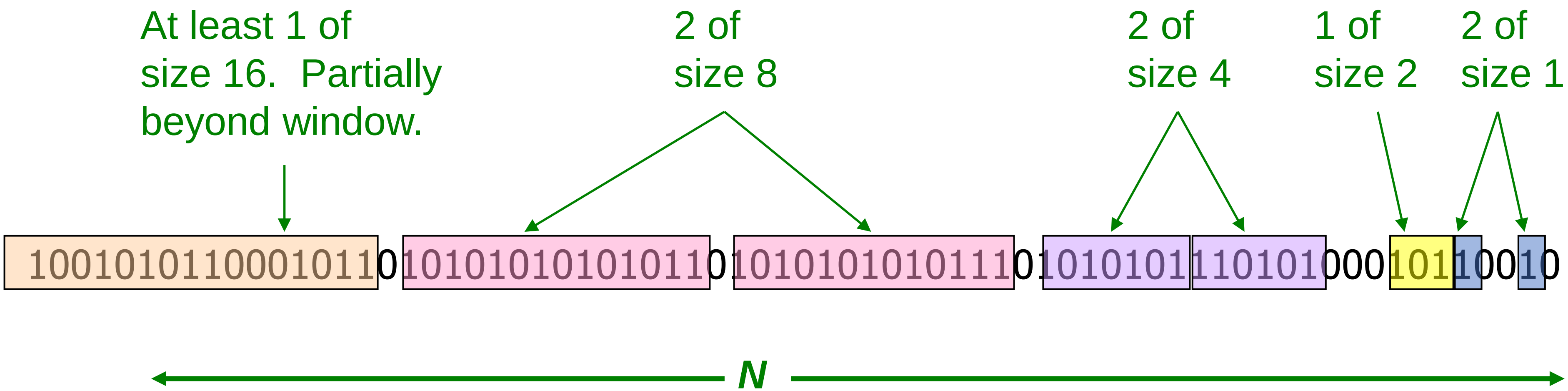
0101100010110101010101010110101010101011110101010111010100010110010 1101

State of the buckets after merging

0101100010110101010101010110101010101011110101010111010100010110010 1101

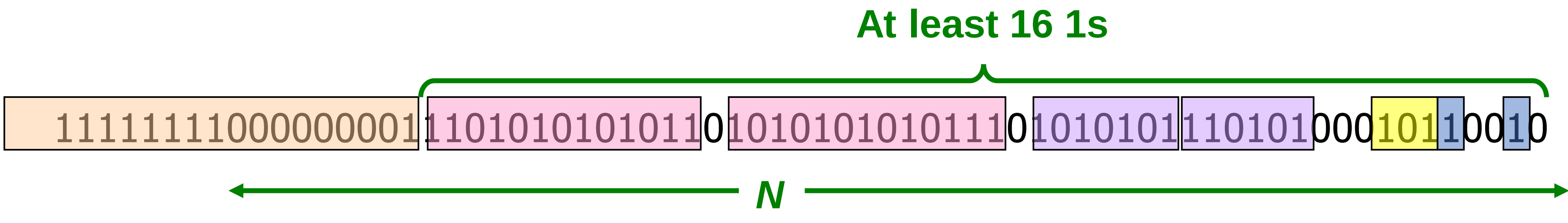
How to Query?

- To estimate the number of 1s in the most recent N bits:
  - Sum the sizes of all buckets but the last
    - (note "size" means the number of 1s in the bucket)
  - Add half the size of the last bucket
- Remember: We do not know how many 1s of the last bucket are still within the wanted window
- Example:



Error Bound: Proof

- **Why is error 50%? Let's prove it!**
- Suppose the last bucket has size $\mathbf{2^{r}}$
- Then by assuming $\mathbf{2^{r-1}}$ (i.e., half) of its **1s** are still within the window, we make an error of at most $\mathbf{2^{r-1}}$
- Since there is at least one bucket of each of the sizes less than $\mathbf{2^{r}}$, the true sum is at least
  $\mathbf{1 + 2 + 4 + .. + 2^{r-1} = 2^{r} - 1}$
- Thus, error at most **50%**

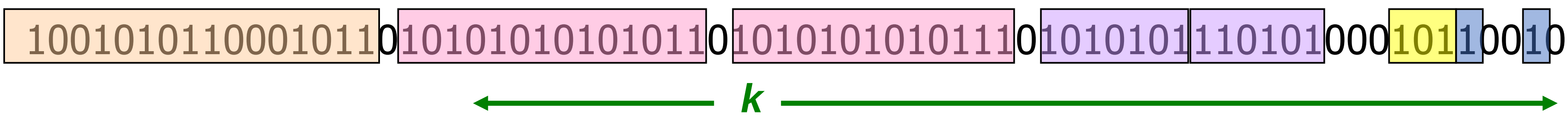


Further Reducing the Error

- Instead of maintaining **1** or **2** of each size bucket, we allow either ***r*-1** or ***r*** buckets (***r* > 2**)
  - Except for the largest size buckets; we can have any number between **1** and ***r*** of those
- **Error is at most *O(*1/*r)***
- By picking ***r*** appropriately, we can tradeoff between number of bits we store and the error

Extensions (optional)

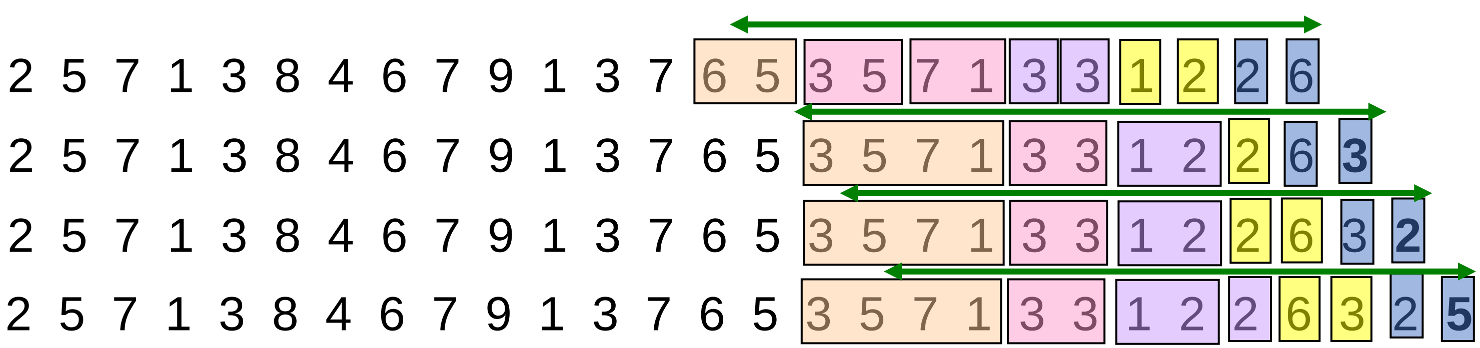
- Can we use the same trick to answer queries **How many 1's in the last *k*?** where ***k*** < ***N***?
  - **A:** Find earliest bucket **B** that at overlaps with ***k***. Number of **1s** is the **sum of sizes of more recent buckets + ½ size of B**



- **Can we handle the case where the stream is not bits, but integers, and we want the sum of the last *k* elements?**

Extensions (optional)

- **Stream of positive integers**
- **We want the sum of the last *k* elements**
  - **Amazon:** Avg. price of last **k** sales
- **Solution:**
  - **(1) If you know all have at most *m* bits**
    - Treat ***m*** bits of each integer as a separate stream
    - Use DGIM to count **1s** in each integer
    - The sum is $= \sum_{i=0}^{m-1} c_i 2^i$ $\quad$ $\boldsymbol{c_i}$ …estimated count for **i-th** bit
  - **(2) Use buckets to keep partial sums**
    - **Sum of elements in size *b* bucket is at most 2^b**



Idea: Sum in each bucket is at most **2^b** (unless bucket has only **1** integer)
Bucket sizes:

16	8	4	2	1

Filtering Data Streams

- Each element of data stream is a tuple
- Given a list of keys **S**
- **Determine which tuples of stream are in *S***

- Obvious solution: Hash table
  - But suppose we **do not have enough memory** to store all of ***S*** in a hash table
    - E.g., we might be processing millions of filters on the same stream

Applications

- Example: Email spam filtering
  - We know 1 billion “good” email addresses
  - If an email comes from one of these, it is **NOT** spam

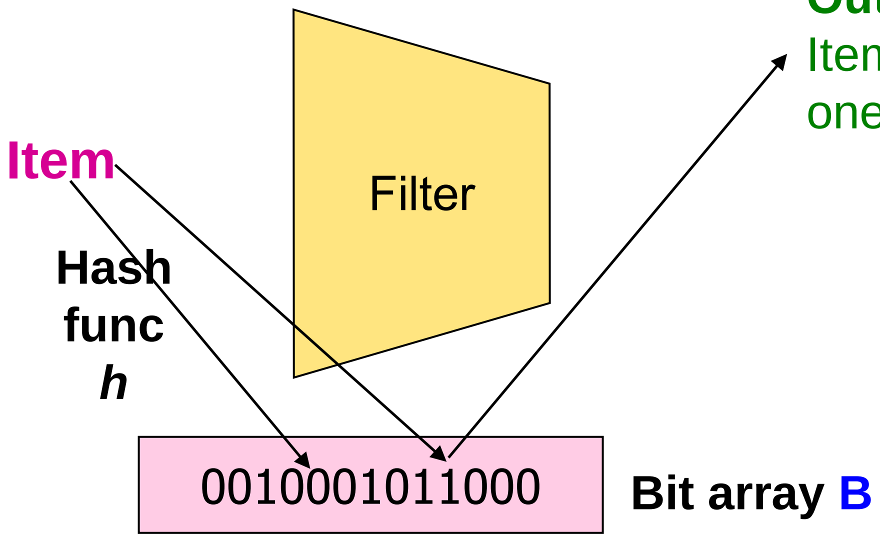
- Publish-subscribe systems
  - You are collecting lots of messages (news articles)
  - People express interest in certain sets of keywords
  - Determine whether each message matches user’s interest

First Cut Solution (1)

- Given a set of keys ***S*** that we want to filter
- Create a **bit array *B*** of ***n*** bits, initially all ***0*s**
- Choose a **hash function *h*** with range **[*0*,*n*)**
- Hash each member of $s \in S$ to one of ***n*** buckets, and set that bit to **1**, i.e., ***B[h(s)]=1***
- Hash each element ***a*** of the stream and output only those that hash to bit that was set to **1**
  - **Output *a* if B[h(a)] == 1**

First Cut Solution (2)

Output the item since it may be in *S*.
Item hashes to a bucket that at least one of the items in **S** hashed to.



Drop the item.
It hashes to a bucket set to **0** so it is surely not in ***S***.

- **Creates false positives but no false negatives**
  - If the item is in ***S*** we surely output it, if not we may still output it

First Cut Solution (3)

- **|S| = 1 billion email addresses**
  |B|= 1GB = 8 billion bits

- If the email address is in ***S***, then it surely hashes to a bucket that has the big set to **1**, so it always gets through (***no false negatives***)
  - False negative: a result indicates that a condition failed, while it actually was successful

- Approximately 1/8 of the bits are set to 1, so about 1/8th of the addresses not in S get through to the output (***false positives***)
  - False positive: a result that indicates a given condition has been fulfilled, when it actually has not been fulfilled
  - Actually, less than 1/8th, because more than one address might hash to the same bit
  - Since the majority of emails are spam, eliminating 7/8th of the spam is a significant benefit

Analysis: Throwing Darts (1)

- More accurate analysis for the number of **false positives**
- **Consider:** If we throw ***m*** darts into ***n*** equally likely targets, **what is the probability that a target gets at least one dart?**
- **In our case:**
  - **Targets** = bits/buckets
  - **Darts** = hash values of items

Analysis: Throwing Darts (2)

- We have ***m*** darts, ***n*** targets
- **What is the probability that a target gets at least one dart?**

Equals **1/e** as **n** $\rightarrow\infty$

Equivalent

$$1 - \boxed{\left(1 - 1/n\right)^{n}}^{\,(m/n)}$$

$$1 - e^{-m/n}$$

Probability some target **X** not hit by a dart

Probability at least one dart hits target **X**

Analysis: Throwing Darts (3)

- **Fraction of 1s in the array B**

  = probability of false positive = 1 – $e^{-m/n}$

- **Example:** **10^9** darts, **$8 \cdot 10^9$** targets
  - Fraction of **1s** in **B = 1 – $e^{-1/8}$ = 0.1175**
    - Compare with our earlier estimate: **1/8 = 0.125**

Bloom Filter

- Consider: |**S**| = $\boldsymbol{m}$, |**B**| = $\boldsymbol{n}$
- Use $\boldsymbol{k}$ independent hash functions $\boldsymbol{h_1, \ldots, h_k}$
- **Initialization:**
  - Set **B** to all **0s**
  - Hash each element $\boldsymbol{s} \in \boldsymbol{S}$ using each hash function $\boldsymbol{h_i}$, set **B[$\boldsymbol{h_i(s)}$] = 1** (for each $\boldsymbol{i = 1, .., k}$)
- **Run-time:**
  - When a stream element with key $\boldsymbol{x}$ arrives
    - If **B[$\boldsymbol{h_i(x)}$] = 1** **for all** $\boldsymbol{i = 1, \ldots, k}$ then declare that $\boldsymbol{x}$ is in $\boldsymbol{S}$
      - That is, $\boldsymbol{x}$ hashes to a bucket set to **1** for every hash function $\boldsymbol{h_i(x)}$
    - Otherwise discard the element $\boldsymbol{x}$

Bloom Filter Example

- Consider a Bloom filter of size m=10 and number of hash functions k=3. Let H(x) denote the result of the three hash functions.
- The 10-bit array is initialized as below

0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0

- Insert x_0 with $H(x_0) = \{1, 4, 9\}$

0	1	2	3	4	5	6	7	8	9
0	1	0	0	1	0	0	0	0	1

- Insert x_1 with $H(x_1) = \{4, 5, 8\}$

0	1	2	3	4	5	6	7	8	9
0	1	0	0	1	1	0	0	1	1

- Query y_0 with $H(y_0) = \{0, 4, 8\}$ => ???
- Query y_1 with $H(y_1) = \{1, 5, 8\}$ => ???   **False positive!**

- Another Example: https://llimllib.github.io/bloomfilter-tutorial/

Bloom Filter – Analysis

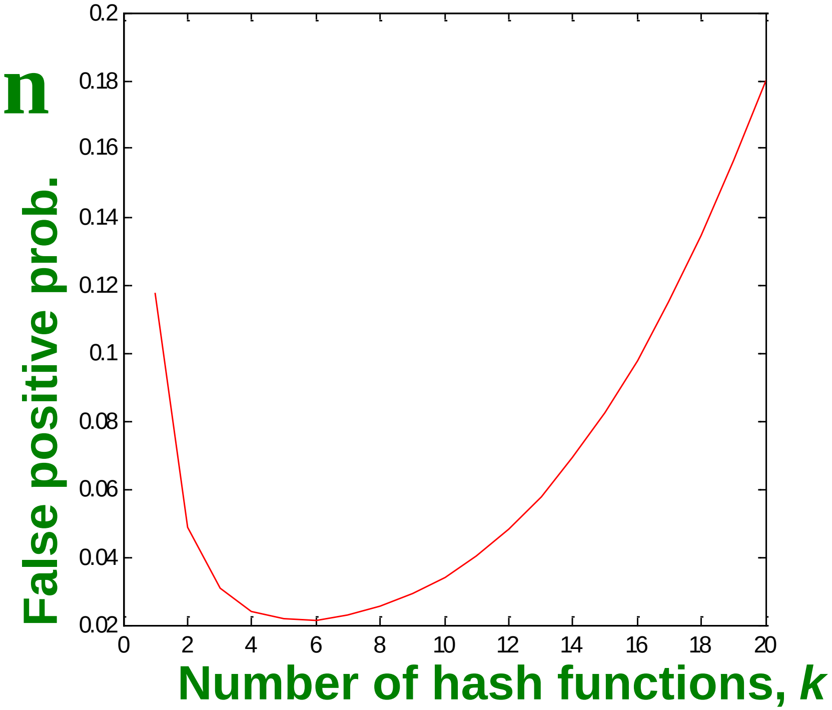
- **What fraction of the bit vector B are 1s?**
  - Throwing $\boldsymbol{k \cdot m}$ darts at $\boldsymbol{n}$ targets
  - So fraction of **1**s is $\boldsymbol{(1 - e^{-km/n})}$

- But we have $\boldsymbol{k}$ independent hash functions and we only let the element $\boldsymbol{x}$ through **if all** $\boldsymbol{k}$ hash element $\boldsymbol{x}$ to a bucket of value **1**

- So, false **positive probability =** $\boldsymbol{(1 - e^{-km/n})^{k}}$

Bloom Filter – Analysis (2)

- m **= 1 billion,** n **= 8 billion**
  - **k = 1**: $(1 - e^{-1/8}) =$ **0.1175**
  - **k = 2**: $(1 - e^{-1/4})^2 =$ **0.0493**



- **What happens as we keep increasing** k**?**

- "Optimal" value of k: $n/m \ln(2)$
  - **In our case:** Optimal **k = 8 ln(2) = 5.54 ≈ 6**
    - **Error at k = 6**: $(1 - e^{-1/6})^2 =$ **0.0235**

Bloom Filter: Wrap-up

- Bloom filters guarantee no false negatives, and use limited memory
  - Great for pre-processing before more expensive checks

- Suitable for hardware implementation
  - Hash function computations can be parallelized

- Is it better to have **1** big **B** or ***k*** small **B**s?
  - **It is the same:** $(1 - e^{-km/n})^k$ vs. $(1 - e^{-m/(n/k)})^k$
  - But keeping **1 big B** is simpler

Counting Distinct Elements

- Problem:
  - Data stream consists of a universe of elements chosen from a set of size N
  - Maintain a count of the number of distinct elements seen so far
- Example:

Data stream: 3 2 5 3 2 1 7 5 1 2 3 7

Number of distinct values: 5

- Obvious approach: Maintain the set of elements seen so far
  - That is, keep a hash table of all the distinct elements seen so far

Applications

- How many different words are found among the Web pages being crawled at a site?
  - Unusually low or high numbers could indicate artificial pages (spam?)
- How many different Web pages does each customer request in a week?
- How many distinct products have we sold in the last week?

Using Small Storage

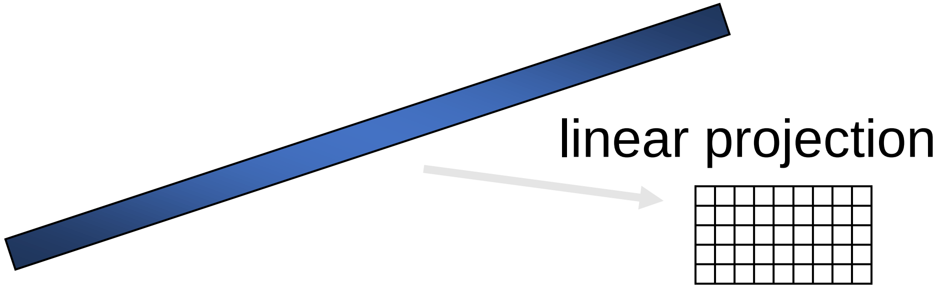
- Real problem: What if we do not have space to maintain the set of elements seen so far?

- Estimate the count in an unbiased way

- Accept that the count may have a little error, but limit the probability that the error is large

Sketches

- Sampling does not work!
  - If a large fraction of items aren't sampled, don't know if they are all same or all different
- Sketch: a technique takes advantage that the algorithm can "see" all the data even if it can't "remember" it all
- Essentially, sketch is a linear transform of the input
  - Model stream as defining a vector, sketch is result of multiplying stream vector by an (implicit) matrix



Flajolet-Martin Sketch

- Probabilistic Counting Algorithms for Data Base Applications. 1985.

- Pick a hash function $\boldsymbol{h}$ that maps each of the $\boldsymbol{N}$ elements to at least $\mathbf{log_2}\, \boldsymbol{N}$ bits

- For each stream element $\boldsymbol{a}$, let $\boldsymbol{r(a)}$ be the number of trailing **0s** in $\boldsymbol{h(a)}$
  - **r(a)** = position of first 1 counting from the right
    - E.g., say ***h(a) = 12***, then ***12*** is ***1100*** in binary, so ***r(a) = 2***

- Record R = the maximum $r(a)$ seen
  - $\mathbf{R = max_a\ r(a)}$, over all the items ***a*** seen so far

- Estimated number of distinct elements = 2^R

Why It Works: Intuition

- Very very rough and heuristic intuition why Flajolet-Martin works:
  - ***h(a)*** hashes ***a*** with **equal prob.** to any of ***N*** values
  - Then ***h(a)*** is a sequence of $\mathbf{log_2 N}$ bits, where 2^{-r} fraction of all ***a***s have a tail of ***r*** zeros
    - About 50% of ***a***s hash to *****0**
    - About 25% of ***a***s hash to ****00**
    - So, if we saw the longest tail of ***r=2*** (i.e., item hash ending ***100**) then we have probably seen **about *4*** distinct items so far
  - So, it takes to hash about 2^r items before we see one with zero-suffix of length *r*

Why It Works: More formally

- Formally, we will show that **probability of finding a tail of $\boldsymbol{r}$ zeros:**
  - **Goes to 1 if $m \gg 2^r$**
  - **Goes to 0 if $m \ll 2^r$**

  where $\boldsymbol{m}$ is the number of distinct elements seen so far in the stream

- Thus, 2^R will almost always be around *m!*

Why It Works: More formally

- **The probability that a given *h*(*a*) ends in at least *r* zeros is 2^{-r}**
  - **h(a)** hashes elements uniformly at random
  - Probability that a random number ends in at least ***r*** zeros is $\mathbf{2^{-r}}$

- Then, the probability of **NOT** seeing a tail of length ***r*** among ***m*** elements:

$$\boxed{\left(\boxed{1-2^{-r}}\right)^{m}}$$

Prob. all end in fewer than ***r*** zeros.

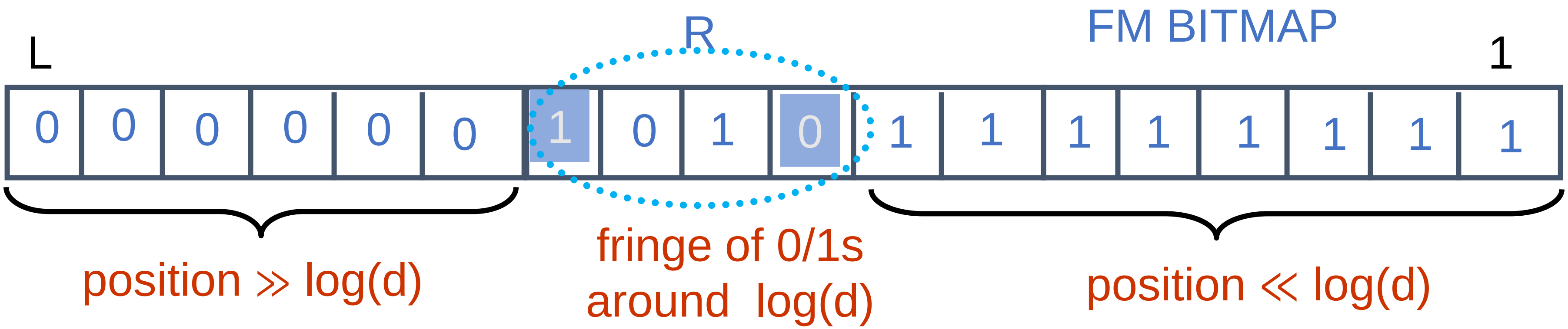
Prob. that given **h(a)** ends in fewer than ***r*** zeros

Why It Works: More formally

- **Note:** $(1-2^{-r})^m = (1-2^{-r})^{2^r (m2^{-r})} \approx e^{-m2^{-r}}$
- **Prob. of NOT finding a tail of length *r* is:**
  - If ***m*** << ***2^r^***, then prob. tends to **1**
    - $(1-2^{-r})^m \approx e^{-m2^{-r}} = 1$ as $\mathbf{m/2^r \to 0}$
    - So, the probability of finding a tail of length ***r*** tends to **0**
  - If ***m*** >> ***2^r^***, then prob. tends to **0**
    - $(1-2^{-r})^m \approx e^{-m2^{-r}} = 0$ as $\mathbf{m/2^r \to \infty}$
    - So, the probability of finding a tail of length ***r*** tends to **1**
- **Thus, 2^R will almost always be around *m*!**

Flajolet-Martin Sketch

- Maintain FM Sketch = bitmap array of L = log ***N*** bits
  - Initialize bitmap to all 0s
  - For each incoming value a, set FM[r(a)] = 1
- If d distinct values, expect d/2 map to FM[1], d/4 to FM[2]…



- Use the leftmost 1: R = $\mathbf{max_a\ r(a)}$
- Use the rightmost 0: also an indicator of log(d)
  - Estimate $d = c2^R$ for scaling constant c ≈ 1.3 (original paper)
- Average many copies (different hash functions) improves accuracy