CIS 580

HOMOGENEOUS

Point (3D)

Finite $(X, Y, Z) \rightarrow (X, Y, Z, 1)$

Infinite $(\infty, 0,0,1) \rightarrow (1,0,0,0)$

$(X, Y, Z) \rightarrow \lambda(x, y, 1)$

Point (2D) $(x, y) \rightarrow (x, y, 1)$

Line (2D) $au + bv + c = 0 \rightarrow (a, b, c)$

CAMERA MODEL

<u>3D-2D PROJECTION</u>

$(u_{ccd}, v_{ccd}) = (f_{mx} X/Z, f_{my} Y/Z)$

$$u_{img} = u_{ccd} \frac{w_{img}}{w_{ccd}} + p_x = f_x \frac{X}{Z} + p_x$$

$$v_{img} = v_{ccd} \frac{h_{img}}{h_{ccd}} + p_y = f_y \frac{Y}{z} + p_y$$

where $f_x = f_{mx} \frac{w_{img}}{w_{ccd}}, f_y = f_{my} \frac{h_{img}}{h_{ccd}}$

$$\lambda \begin{bmatrix} u_{img} \\ v_{img} \\ 1 \end{bmatrix} = \begin{bmatrix} f_x & 0 & p_x \\ 0 & f_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Related topics

- 2d inverse projection
- Dolly zoom: change f and Z at the same time to keep object size on image the same
- Locate center of projection

<u>CAMERA PROJECTION</u>

$$\lambda \begin{bmatrix} u_{img} \\ v_{img} \\ 1 \end{bmatrix} = L\, K[R \quad T] \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

<u>ORTHOGRAPHIC CAMERA</u>

Affine camera

$$P_A = \begin{bmatrix} f_x & 0 & p_x \\ 0 & f_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{x1} & r_{x2} & 0 & 0 \\ r_{y1} & r_{y2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Orthographic camera $K = I$

$$P_o = \begin{bmatrix} r_{x1} & r_{x2} & r_{x3} & 0 \\ r_{y1} & r_{y2} & r_{y3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

PROJECTIVE LINE

<u>VANISHING POINTS & LINES</u>

Information provided by vanishing lines

- *Horizon*
- *Camera pitch angle*
- *Camera roll angle*

Routine to compute vanishing line (2D)

- line through two points: $l = x_1 \times x_2$
- Vanishing point - intersection of two lines which are parallel in 3D space (need rescale): $x = l_1 \times l_2$
- Vanishing line through two points v_1, v_2: $l_v = v_1 \times v_2$

From 2D to 3D

- A 2D line in an image corresponds to a 3D plane passes the camera center of normal: $\lambda \hat{l} = (\lambda_1 K^{-1} x_1) \times (\lambda_2 K^{-1} x_2)$
- The intersection point of two lines corresponds to a 3D ray passes the camera center $\hat{x} = \hat{l}_1 \times \hat{l}_2$

<u>TRANSFORMATION OF POINT AND LINE</u>

$$x' = Hx, l' = (H^{-1})^T l$$

Infinite point/line becomes finite after projective transformation:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = H \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}, \begin{bmatrix} l_1' \\ l_2' \\ l_3' \end{bmatrix} = H^{-T} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

<u>INFINITY</u>

A point at infinity = a direction $x_\infty = (a \quad b \quad 0)^T$

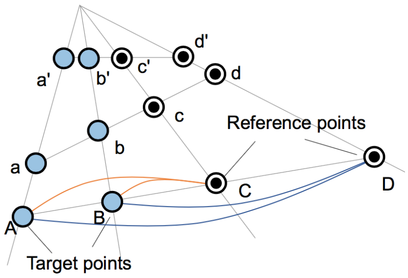
All points at infinity lie in the line at infinity $l_\infty = (0 \quad 0 \quad 1)^T = Z$

SINGLE VIEW METROLOGY

Camera height – where the horizon is

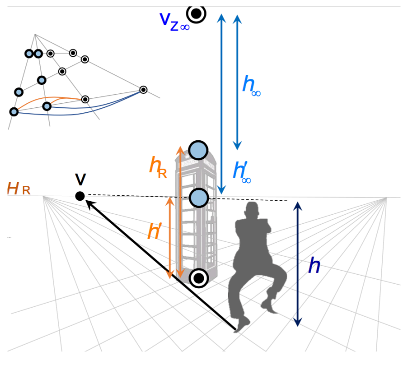
$h = fH/Z$

<u>CROSS RATIO</u>



$$\frac{AC}{BC}\frac{BD}{AD} = \frac{ac}{bc}\frac{bd}{ad} = \frac{a'c'}{b'c'}\frac{b'd'}{a'd'}$$

Reference points: bottom and infinity



$$\frac{h_R}{H'}\frac{H'_\infty}{h_\infty} = \frac{H_R}{H}\frac{H_\infty}{H_\infty} = \frac{H_R}{H}\frac{\infty}{\infty} = \frac{H_R}{H}$$

Given image and $\boldsymbol{H_a}$, estimate $\boldsymbol{H_b}$

Compute vanishing line $\boldsymbol{l_v}$

$\boldsymbol{v} = (\boldsymbol{a_f} \times \boldsymbol{b_f}) \times \boldsymbol{l_v}$

$\boldsymbol{b_h'} = (\boldsymbol{v} \times \boldsymbol{b_h}) \times (\boldsymbol{a_f} \times \boldsymbol{a_h})$

compute vanishing point in vertical $\boldsymbol{v_{Z\infty}}$

$$\boldsymbol{H_b} = \boldsymbol{H_a} \frac{\|\boldsymbol{b_h'} - \boldsymbol{a_f}\|}{\|\boldsymbol{a_h} - \boldsymbol{a_f}\|} \frac{\|\boldsymbol{v_{z\infty}} - \boldsymbol{a_h}\|}{\|\boldsymbol{v_{z\infty}} - \boldsymbol{b_h'}\|}$$

SINGLE VIEW

<u>VANISHING POINT → CAMERA ROT</u>

$r_3 = K^{-1} v_{Z\infty} / \|K^{-1} v_{Z\infty}\|$

$\beta = \tan^{-1}\left(-r_{31}/\sqrt{r_{32}^2 + r_{33}^2}\right)$

$\gamma = \tan^{-1}(r_{32}/r_{33})$

Or

$r_1 = K^{-1} v_{x\infty} / \|K^{-1} v_{x\infty}\|$

$r_2 = K^{-1} v_{y\infty} / \|K^{-1} v_{y\infty}\|$

$r_3 = r_1 \times r_2$

<u>SINGLE-VIEW HOMOGRAPHY</u>

"2D" → 2D: take points on plane $Z = 0$

$$\lambda x = HX = K[r_1 \quad r_2 \quad t]X$$

Homography estimation using 4s points

$$\begin{bmatrix} \mathbf{X}^T & \mathbf{0}_{1\times 3} & -u\mathbf{X}^T \\ \mathbf{0}_{1\times 3} & \mathbf{X}^T & -v\mathbf{X}^T \end{bmatrix} \begin{bmatrix} \mathbf{h}_1^T \\ \mathbf{h}_2^T \\ \mathbf{h}_3^T \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$r_1 = K^{-1} H_1 / \|K^{-1} H_1\|$

$r_2 = K^{-1} H_2 / \|K^{-1} H_1\|$

$r_3 = r_1 \times r_2$

$t = r_2 = K^{-1} H_3 / \|K^{-1} H_1\|$

two configurations, choose the one where points are before the camera

CAMERA CALIBRATION

<u>INTRINSIC BIAS</u>

Scale factor; image center position; skew factor; lens distortion

<u>SINGLE VIEW CALIBRATION</u>

"Guessing"

Change $\boldsymbol{f}$ (***or K***) each time; Repeat:

Compute vanishing point at

$\boldsymbol{X_\infty, Y_\infty}$: $\boldsymbol{V_1, V_2}$

$\boldsymbol{r_1 = K^{-1} V_1 / \|K^{-1} V_1\|}$

$\boldsymbol{r_2 = K^{-1} V_2 / \|K^{-1} V_2\|}$

$\boldsymbol{r_3 = r_1 \times r_2}$

$\boldsymbol{R = [r_1 \quad r_2 \quad r_3]}$

Until $\boldsymbol{R}$ is orthogonal

Calibration using vanishing pts

[Angle between vanishing points]

If we know the angle between two vanishing points in 3D, we can estimate K via:

$$\cos\theta == \frac{\mathbf{x}_1^T (\mathbf{K}^{-T}\mathbf{K}^{-1}) \mathbf{x}_2}{\sqrt{\mathbf{x}_1^T (\mathbf{K}^{-T}\mathbf{K}^{-1}) \mathbf{x}_1} \sqrt{\mathbf{x}_2^T (\mathbf{K}^{-T}\mathbf{K}^{-1}) \mathbf{x}_2}}$$

[Use vanishing points in X, Y, Z]

$$(\mathbf{K}^{-1}\mathbf{v}_{X\infty})^T (\mathbf{K}^{-1}\mathbf{v}_{Y\infty}) = (\mathbf{K}^{-1}\mathbf{v}_{Y\infty})^T (\mathbf{K}^{-1}\mathbf{v}_{Z\infty}) = (\mathbf{K}^{-1}\mathbf{v}_{Z\infty})^T (\mathbf{K}^{-1}\mathbf{v}_{X\infty}) = 0$$

$$(\mathbf{K}^{-1}\mathbf{v}_i)^T (\mathbf{K}^{-1}\mathbf{v}_j) = \mathbf{v}_i^T \mathbf{K}^{-T} \mathbf{K}^{-1} \mathbf{v}_j \qquad \boldsymbol{i, j} = X, Y, Z, \text{ and } \boldsymbol{i \neq j}$$

$$b_1 = \frac{1}{f^2}, \quad b_2 = -\frac{p_x}{f^2}, \quad b_3 = -\frac{p_y}{f^2}, \quad b_4 = \frac{p_x^2}{f^2} + \frac{p_y^2}{f^2} + 1$$

$$\underbrace{\begin{bmatrix} u_1 u_2 + v_1 v_2 & u_1 + u_2 & v_1 + v_2 & 1 \\ u_3 u_2 + v_3 v_2 & u_3 + u_2 & v_3 + v_2 & 1 \\ u_1 u_3 + v_1 v_3 & u_1 + u_3 & v_1 + v_3 & 1 \end{bmatrix}}_{3\text{x}4} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = 0$$

$$p_x = -\frac{b_2}{b_1}, \quad p_y = -\frac{b_3}{b_1}, \quad f = \sqrt{\frac{b_4}{b_1} - (p_x^2 + p_y^2)}$$

<u>SINGLE VIEW CALIBRATION</u>

Checkerboard method (Project 1)

- Compute homography matrices from checkerboard plane to image plane

$H = K[r_1 \quad r_2 \quad t]$

- Recover K from all Hs

$$\mathbf{K}^{-T}\mathbf{K}^{-1} = \begin{bmatrix} \frac{1}{f_x^2} & -\frac{s}{f_x^2 f_y} & \frac{p_y s - p_x f_y}{f_x^2 f_y} \\ -\frac{s}{f_x^2 f_y} & \frac{s^2}{f_x^2 f_y^2} + \frac{1}{f_y^2} & -\frac{s(p_y s - p_x f_y)}{f_x^2 f_y^2} - \frac{p_y}{f_y^2} \\ \frac{p_y s - p_x f_y}{f_x^2 f_y} & -\frac{s(p_y s - p_x f_y)}{f_x^2 f_y^2} - \frac{p_y}{f_y^2} & \frac{(p_y s - p_x f_y)^2}{f_x^2 f_y^2} + \frac{p_y^2}{f_y^2} + 1 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} = \mathbf{B},$$

$$p_y = \frac{B_{12}B_{13} - B_{11}B_{23}}{B_{11}B_{22} - B_{12}^2} \qquad f_x = \sqrt{\frac{c}{B_{11}}}$$

$$c = B_{33} - \frac{B_{13}^2 + p_y(B_{12}B_{13} - B_{11}B_{23})}{B_{11}} \qquad s = -\frac{B_{12}f_x^2 f_y}{c}$$

$$f_y = \sqrt{\frac{cB_{11}}{B_{11}B_{22} - B_{12}^2}} \qquad p_x = \frac{sp_y}{f_y} - \frac{B_{13}f_x^2}{c}.$$

$$\begin{bmatrix} {}^1\mathbf{v}_{11}^T - {}^1\mathbf{v}_{22}^T \\ {}^1\mathbf{v}_{12}^T \\ \vdots \\ {}^N\mathbf{v}_{11}^T - {}^N\mathbf{v}_{22}^T \\ {}^N\mathbf{v}_{12}^T \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{12} \\ B_{22} \\ B_{13} \\ B_{23} \\ B_{33} \end{bmatrix} = \mathbf{0}, \qquad v_{ij} = \begin{bmatrix} h_{i1}h_{j1} \\ h_{i1}h_{j2} + h_{i2}h_{j1} \\ h_{i2}h_{j2} \\ h_{i3}h_{j1} + h_{i1}h_{j3} \\ h_{i3}h_{j2} + h_{i2}h_{j3} \\ h_{i3}h_{j3} \end{bmatrix}$$

- Estimate transformations

$$\mathbf{r}_1 = \frac{1}{z'}\mathbf{K}^{-1}\mathbf{h}_1 \qquad \mathbf{r}_3 = \mathbf{r}_1 \times \mathbf{r}_2$$

$$\mathbf{r}_2 = \frac{1}{z'}\mathbf{K}^{-1}\mathbf{h}_2 \qquad \mathbf{t} = \frac{1}{z'}\mathbf{K}^{-1}\mathbf{h}_3,$$

- Estimate radial distortion

$$\overline{\mathbf{u}}_{\text{distorted}} = \mathbf{K}^{-1}\mathbf{u}_{\text{distorted}}, \quad \overline{\mathbf{u}}_{\text{undistorted}} = \mathbf{K}^{-1}\mathbf{u}_{\text{undistorted}}$$

$$\overline{\mathbf{u}}_{\text{distorted}} = (1 + k_1\rho^2 + k_2\rho^4)\overline{\mathbf{u}}_{\text{undistorted}}$$

$$\begin{bmatrix} \rho^2\overline{\mathbf{u}}^1_{\text{undistorted}} & \rho^4\overline{\mathbf{u}}^1_{\text{undistorted}} \\ \vdots & \vdots \\ \rho^2\overline{\mathbf{u}}^m_{\text{undistorted}} & \rho^4\overline{\mathbf{u}}^m_{\text{undistorted}} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{u}}^1_{\text{distorted}} - \overline{\mathbf{u}}^1_{\text{undistorted}} \\ \vdots \\ \overline{\mathbf{u}}^m_{\text{distorted}} - \overline{\mathbf{u}}^m_{\text{undistorted}} \end{bmatrix}$$

- Define geometric error and run optimization on the estimated initial parameters

TWO VIEW GEOMETRY

In the following topics, define
$P_1 = K_1[I_{3\times3} \quad 0_{3\times1}]$, $P_2 = K_2[R \quad t]$
$C = -R^T t, P_2 = K_2 R[I_{3\times3} \quad -C]$
x^+, X^+ - homogeneous form of x, X

EPIPOLE AND EPIPOLAR LINE

- Epipolar constraints between two images: x_1 in image 1 correspond to an epiploar line in image 2 (vice versa)
- the epipolar line l_{x1} passes corresponding point x_2 in image 2
- any point along the epipolar line can be a candidate of correspondence
- epipolar lines meet at epipole
- pure rotation: no epipolar constraint; pure translation: stereo

Compute epipoles

$$\lambda e_1 = P_1\begin{bmatrix} -R^T t \\ 1 \end{bmatrix} = -K_1 R^T t$$

$$\lambda e_2 = P_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = K_2 t$$

Compute epipolar lines see below

FUNDAMENTAL MATRIX

$F = K_2^{-T}(t \times R)K_1^{-1}, \quad x_2^T F x_1 = 0$

This fundamental matrix transform image 1's space to image 2's space.

$\tilde{F} = K_1^{-T}(R \times t)K_2^{-1} = F^T, \quad x_1^T \tilde{F} x_2 = 0$

rank(F)=2, DOF = 7, requires at least 4 pairs of points to estimate

SVD clean up set the last eigenvalue 0

F vs epipolar line $l_{x1} = Fx_1, l_{x2} = F^T x_2$

$(x_2^T l_{x1} = 0, l_{x2}^T x_1 = 0)$

F vs epipole $Fe_1 = 0, F^T e_2 = 0$

$e_1 = \text{null}(F), e_2 = \text{null}(F^T)$

ESSENTIAL MATRIX

$F = K_2^{-T}(t \times R)K_1^{-1} = K_2^{-T} E K_1^{-1}$

$E = (t \times R) = K_2^T F K_1, \ X_2^T E X_1 = 0$

SVD clean up set the last eigenvalue 0

This essential matrix transform camera 1's space to camera 2's space.

Recover R, t from E

SVD: $E = UDV^T, \ U = [u_1 \quad u_2 \quad u_3]$

$t = \pm null(E) = \pm u_1 \times u_2 = \pm u_3$

$$\mathbf{R} = \mathbf{U}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{V}^T, \text{ or } \mathbf{U}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{V}^T$$

Select the best configuration out of four

LINEAR ESTIMATION

$\underline{Ax = b}$

Minimize $E = \|Ax - b\|^2$

$E = x^T A^T A x - 2x^T A^T b + b^T b$

$\frac{\partial E}{\partial x} = 2(A^T A x - A^T b) = 0$

$x = (A^T A)^{-1} A^T b$

Application: line fitting, circle fitting

Properties: convex, closed form, efficient, no extra parameters

$\underline{Ax = 0}$ using SVD

minimize $E = \|Ax\|^2$

subject to $\|x\| = 1$

$x = null(A)$

SVD $A = UDV^T, \ x = V(:, end)$

TRIANGULATION

Given multiple x, Ps – unknown X

LINEAR TRIANGULATION

$$\begin{bmatrix} \mathbf{x}_1 \\ 1 \end{bmatrix}_\times \mathbf{P}_1 \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = \mathbf{0} \qquad \text{rank}\left(\begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}_\times \mathbf{P}\right) = 2$$

NONLINEAR TRIANGULATION

Rays not intersect at the same 3D point

Reproject the estimated 3D point to each image, consider reprojection error:

$(x_r, y_r) = (P_1 X^+ / P_3 X^+, P_2 X^+ / P_3 X^+)$

minimize E_r:

$$\min_{\mathbf{X}} \sum_{i=1}^{F} \left(u_i - \frac{\mathbf{P}_i^1 \tilde{\mathbf{X}}}{\mathbf{P}_i^3 \tilde{\mathbf{X}}}\right)^2 + \left(v_i - \frac{\mathbf{P}_i^2 \tilde{\mathbf{X}}}{\mathbf{P}_i^3 \tilde{\mathbf{X}}}\right)^2 \qquad \mathbf{f}\left(\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}\right) = \begin{bmatrix} u_1 \\ v_1 \\ \vdots \\ u_F \\ v_F \end{bmatrix}$$

How to solve? – Nonlinear fitting!

STEREO

DEFINITION

Two cameras are in the same orientation; alignment between X axis and baseline (line between two cameras)

Epipoles at infinity

$\frac{dist_{cam12}}{Z} = \frac{dist_{x12}}{f}$

RECTIFICATION

Compute R_{rect}

$R_{rect} = [r_x^T \quad r_y^T \quad r_z^T]^T, \quad r_x = \frac{C}{\|C\|}$

$r_z = \frac{\tilde{r}_z - (\tilde{r}_z \cdot r_x) r_x}{\|\tilde{r}_z - (\tilde{r}_z \cdot r_x) r_x\|}$, where $\tilde{r}_z = [0 \ \ 0 \ \ 1]^T$

$r_y = r_z \times r_x$

Given R_{rect}:

$H_1 = K_1 R_{rect} K_1^{-1}, H_2 = K_2 R_{rect} R^T K_2^{-1}$

PERSPECTIVE-N-POINTS

3D-2D correspondence

Given x, X – unknown P

$$u^x = \frac{p_{11}X + p_{12}Y + p_{13}Z + p_{14}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}} \qquad u^y = \frac{p_{21}X + p_{22}Y + p_{23}Z + p_{24}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}}$$

DOF is 6, at least 3 pairs of points needed

Recover R, t from P

$\lambda R = K^{-1}[p_1 \quad p_2 \quad p_3] = UDV^T$

$R = UV^T$, to make sure $det(R) = 1$

$$R = U\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \det(UV^T) \end{pmatrix}V^T$$

$t = K^{-1}p_4 / d_{11}$

Another solution for P3P

* Collinear points cause ambiguity

$$\begin{cases} d_1^2 + d_2^2 - 2d_1d_2\cos\theta_{12} = p_{12}^2 \\ d_1^2 + d_3^2 - 2d_1d_3\cos\theta_{13} = p_{13}^2 \\ d_2^2 + d_3^2 - 2d_2d_3\cos\theta_{23} = p_{23}^2 \end{cases}, \text{ where}$$

$p_{ij} = d(X_i, X_j), \ \cos\theta_{ij} = \frac{(K^{-1}x_i)^T(K^{-1}x_j)}{\|K^{-1}x_i\|\|K^{-1}x_j\|}$

Four set of possible (d_1, d_2, d_3)

Use another point to verify solution.

[Recover R, t from (d_1, d_2, d_3)]

$[\tilde{X}_1 \quad \tilde{X}_2 \quad \tilde{X}_3] = R[X_1 \quad X_2 \quad X_3]$

where $\tilde{X}_i = d_i K^{-1} x_i / \|K^{-1} x_i\|$

NON-LINEAR FITTING

$\underline{f(x) = b}$

minimize $E = \|f(x) - b\|^2$

= minimize $f(x)^T f(x) - 2b^T f(x)$

$\left.\frac{\partial E}{\partial x}\right|_{x^*} = 2\left(\frac{\partial f}{\partial x}\right)^T f(x) - 2\left(\frac{\partial f}{\partial x}\right)^T b = 0$

where $J = \frac{\partial f}{\partial x}$ is the Jacobian

Use gradient descent to solve iteratively

$f(x + \Delta x) = f(x) + \frac{\partial f}{\partial x}\Delta x$, plug in:

$J^T J \Delta x = J^T(b - f(x)) \Rightarrow \Delta x = (J^T J)^{-1} J^T (b - f(x))$

FEATURE MATCHING

SIFT

Repeatable, discriminative, oriented

RANSAC

Decide number of iterations N

s – number of feature pairs in a sample

e – probability that a point is an outlier

N – number of iterations

p – desired probability of getting good H

$1 - (1 - (1 - e)^s)^N = p$

$N = \log(1 - p) / \log(1 - (1 - e)^s)$

APPENDIX

CROSS PRODUCT

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = [\mathbf{a}]_\times \mathbf{b}$$