TRANSFORMATION

RIGID BODY TRANSFORMATION (REVIEW)

Rotation matrix $SO(3)$; $\det(R) = 1$ (for right hand frames)

Composition post-multiply (intermediate); pre-multiply (fix)

Homogeneous transformation

$$\begin{bmatrix} I & d \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} R & 0 \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} R & d \\ \mathbf{0} & 1 \end{bmatrix}, \boldsymbol{H}^{-1} = \begin{bmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T\boldsymbol{d} \\ \mathbf{0} & 1 \end{bmatrix}$$

Rigid displacement transformation g of points induce action g^* on vectors [Length preserved; Cross product preserved]

ROTATION (REVIEW)

Euler angles (proper and Tait-Bryan)

$$R=\begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix} \text{ (z-y-z)}$$

$$\begin{cases} \phi = Atan2(r_{23}/r_{13}) \\ \theta = Atan2(\sqrt{1-r_{33}^2}/r_{33}) \\ \psi = Atan2(r_{32}/-r_{31}) \end{cases} \text{ or } \begin{cases} \phi = Atan2(-r_{23}/-r_{13}) \\ \theta = Atan2(-\sqrt{1-r_{33}^2}/r_{33}) \\ \psi = Atan2(-r_{32}/r_{31}) \end{cases}$$

Axis-angle

$$R=\begin{bmatrix} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{bmatrix} \quad (v_\theta = 1 - \cos\theta)$$

$$\theta = \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right), \; k = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

Skew Symmetric Matrices $S(\vec{\omega})$

Rodrigues formulas $\exp(\hat{u}\theta) = I + \hat{u}\sin\theta + \hat{u}^2(1-\cos\theta)$, where $\hat{u}^2 = uu^T - I$

(or $R = I\cos\theta + uu^T(1-\cos\theta) + \hat{u}\sin\theta$)

- Exponential of 3x3 Skew Sym. Mat. are rotation matrices
- Exponential of 4x4 $\hat{\xi}$ are homogeneous transform H

Quaternions

Multiplying two quaternions $X = (x_0, \vec{x}), Y = (y_0, \vec{y})$

$$XY = x_0 y_0 - \vec{x}^T\vec{y} + x_0\vec{y} + y_0\vec{x} + \vec{x}\times\vec{y}$$

Vector $\vec{v} = (v_x, v_y, v_z)$ in quaternions: $Q_v = (0, v_x, v_y, v_z)$

Rotated v: $Q_{v'} = QQ_vQ^*$

VELOCITIES

$\|u\| = 1 \Leftrightarrow \exp(\hat{u}\theta)$ is a rotation matrix

$$\exp(\hat{\omega}dt) = I + (\hat{\omega}/\|\omega\|)\sin\omega dt + (\hat{\omega}/\|\omega\|^2)(1-\cos\omega dt)$$

VELOCITY: ROTATION ONLY

$\hat{\omega} = S(\omega(t)) = R\dot{(t)}\, R^T(t),\; \hat{\omega}R(t) = R\dot{(t)}$

For fixed point p in frame 1: $\dot{p}_0 = R_1^0 p_1$

$R_1^{0T}\dot{p}_0 = R_1^{0T}\dot{R}_1^0 p_1$	$\hat{\omega}^b = R^T\dot{R}$
velocity in body frame	angular velocity in body frame
$\dot{p}_0 = \dot{R}_1^0 R_0^1 p_0$	$\hat{\omega}^s = \dot{R}R^T$
velocity in inertial frame	angular velocity in inertial frame

Angular velocity and Euler angles

$\omega^b = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} c_\theta & 0 & -c_\phi s_\theta \\ 0 & 1 & s_\phi \\ s_\theta & 0 & c_\phi c_\theta \end{bmatrix}\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$	Not all angular velocities can be expressed in Φ_e

VELOCITY: GENERAL TRANSFORMATION

$$\dot{\mathbf{H}}(t) = \mathbf{S}_h(t)\mathbf{H}(t)$$

$$\mathbf{S}_h(t) = \begin{bmatrix} \dot{\mathbf{R}}(t)\mathbf{R}(t)^T & \dot{\boldsymbol{t}}(t) - \dot{\mathbf{R}}(t)\mathbf{R}(t)^T\boldsymbol{t}(t) \\ 0 & 0 \end{bmatrix} = \dot{\mathbf{H}}(t)\mathbf{H}(t)^{-1} = \hat{\boldsymbol{\xi}}$$

This is called twist

For any point p: $\dot{p}_0 = \dot{R}_1^0 p_1 + R_1^0\dot{p}_1 + \dot{o}_1^0$

$H^{-1}\dot{p}_0 = H^{-1}\dot{H}p_1$	$\hat{\xi}^b = H^T\dot{H}$
velocity in body frame	velocities in body frame
$\dot{p}_0 = H\dot{H}^{-1}p_0$	$\hat{\xi}^s = \dot{H}H^T\ (s_h)$
velocity in inertial frame	velocities in inertial frame

	Rotation	Pose
Matrix	$\boldsymbol{R} \in R^{3\times3} \mid \mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}, \det\mathbf{R} = 1$	$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t}_{3\times1} \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}$
3D element	$\boldsymbol{p}' = \mathbf{R}\boldsymbol{p}$	$\boldsymbol{P}' = \mathbf{H}\boldsymbol{P}$
Velocities body frame	$\hat{\boldsymbol{\omega}}^b = \boldsymbol{R}^T\dot{\boldsymbol{R}}$	$\hat{\boldsymbol{\xi}}^b = \mathbf{H}\dot{\mathbf{H}}$
Velocities inertial frame	$\hat{\boldsymbol{\omega}}^s = \dot{\boldsymbol{R}}\boldsymbol{R}^T$	$\hat{\boldsymbol{\xi}}^s = \dot{\mathbf{H}}\mathbf{H}^{-1}$

Moving velocities between different moving frames

$$\begin{bmatrix} \dot{p}_2^2 \\ \omega_2^2 \end{bmatrix} = \begin{bmatrix} R_1^2 & -R_1^2 S(r_{12}) \\ 0 & R_1^2 \end{bmatrix}\begin{bmatrix} \dot{p}_1^1 \\ \omega_1^1 \end{bmatrix}$$ This is often called adjoint

DYNAMICS

NEWTON-WULER QUATIONS OF MOTION

Linear and rotational Newton Dynamics

$F = m^A dv^c/dt = {}^A dL/dt,\; d^A H_C^B/dt = M_C^B,\quad {}^A H_C^B = I_C \cdot {}^A\omega^B$

Principle axes

$\mathbf{u^T}\boldsymbol{I}\mathbf{u}$ is called principal moment of inertial

Euler's equations

$$M_c = d^A H_C^B/dt = d^B H_C^B/dt + {}^A\omega^B \times H_C^B$$

where $d^B H_C^B/dt = I_{11}\dot{\omega}_1\mathbf{b_1} + I_{22}\dot{\omega}_2\mathbf{b_2} + I_{33}\dot{\omega}_3\mathbf{b_3}$

$$\begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}\begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}\begin{bmatrix} \overset{p}{\omega_1} \\ \overset{q}{\omega_2} \\ \overset{r}{\omega_3} \end{bmatrix} = \begin{bmatrix} M_{C,1} \\ M_{C,2} \\ M_{C,3} \end{bmatrix}$$

QUADROTOR DYNAMICS

$\boldsymbol{F_i} = k_F\omega_i^2,\; \boldsymbol{F} = \sum_{i=1}^4 \boldsymbol{F_i} - mg\boldsymbol{a_3}$

$\boldsymbol{M_i} = k_m\omega_i^2,\; M = \sum_{i=1}^4 \boldsymbol{r_i}\times\boldsymbol{F_i} + \sum_{i=1}^4 \boldsymbol{M_i}$

In inertial frame:

$$m\ddot{\mathbf{r}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + R\begin{bmatrix} 0 \\ 0 \\ \boxed{F_1 + F_2 + F_3 + F_4} \end{bmatrix}$$

In body frame (where ${}^A w^B = pb_1 + qb_2 + rb_3$)

$$I\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \boxed{\begin{bmatrix} 0 & L & 0 & -L \\ -L & 0 & L & 0 \\ \gamma & -\gamma & \gamma & -\gamma \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}} - \begin{bmatrix} p \\ q \\ r \end{bmatrix}\times I\begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

All in body frame!

STATE-SPACE AND SYSTEM MODELING

$\mathbf{x} = \begin{bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{bmatrix}$, ODE $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$(LTI $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$)

- $y^{(n)} = g(y, \dot{y}, \cdots, y^{(n-1)}, \mathbf{u})$
- Set $x_i = y^{(i-1)}(t)$, state vector $\mathrm{x} = [x_1, \cdots, x_n]^T$
- Rewrite into a system of coupled first-order differential equations ($\dot{x}_i = x_{i+1}$)
- Rewrite in matrix form $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$

Equilibria q_e configuration where $\dot{\mathbf{x}} = f(\mathbf{x}) \equiv 0$

Linearization $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}) \Longrightarrow \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$ about $(\mathbf{x_e}, \mathbf{u_e})$

A, B are the Jacobian of f over x and u respectively.

LINEAR CONTROL

STABILITY

Stability in the sense of Lyapunov/Marginally stable

For any $\epsilon > 0$, $\exists\,\delta(x_0, \epsilon) > 0, s.t.$ if $\|x(t_0, x_0) - x_e\| < \delta$, then $\|x(t; t_0, x_0) - x_e\| < \epsilon$ for all $t > t_0$; It is ***uniformly stable*** if

$\delta(x_0\epsilon) = \delta(\epsilon)$. Elsewise it is not stable!

Asymptotic stability

Marginally stable + convergent ($t \to \infty, x(t; t_0, x_0) \to x_e$)

Convergence along doesn't guarantee AS (e.g. overshoot)

Exponential stability $\|x(t; t_0, x_0) - x_e\| < ce^{\lambda(t-t_0)}\|x(t_0, x_0)\|$

Linear-Time Invariant (LTI) system $\dot{\mathbf{x}} = \boldsymbol{A}\mathbf{x}$

- stable iff. real parts of all eigenvalues of A are negative
- Asymptotic stability = exponential stability
- marginally stable iif. real parts of all eigenvalues are non-positive; and all the Jordan blocks corresponding to eigenvalues with 0 real parts are 1by1 (critical stable)
- unstable iif. exist eigenvalue of positive real part, or 0 real part but the corresponding Jordan block is larger than 1by1

<u>MODEL-BASED CONTROL</u>

$e(t) = x^{des}(t) - x(t)$_

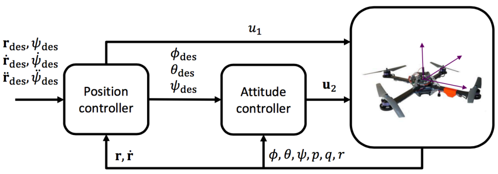
First order system control $\dot{x} = u$
$\dot{e} + K_p e = 0,\ e(t) = \exp\left(-K_p(t-t_0)\right)e(t_0)$ $u(t) = \dot{x}^{des}(t) + K_p e(t)$
Second order system $\ddot{x} = u$
Find u such that $\ddot{e} + K_d\dot{e} + K_p e = 0$ $u(t) = \ddot{x}^{des}(t) + K_d\dot{e}(t) + K_p e(t)$ - **PD control**
Third order system
$u(t) = \ddot{x}^{des}(t) + K_d\dot{e}(t) + K_p e(t) + K_I\int_0^T e(t)dt$ - **PID control** (integral term make steady error goes to 0)

$f(t) = m\left(\ddot{x}^{des}(t) + K_d\dot{e}(t) + K_p e(t)\right) + b\dot{x}(t) + kx(t)$

serve-based model-based

$\ddot{e}^{des} + K_d\dot{e} + K_p e = \left(\frac{m}{\hat{m}} - 1\right)\ddot{x} + \frac{b-\hat{b}}{\hat{m}}\dot{x} + \frac{k-\hat{k}}{\hat{m}}x$

<u>CONTROL OF QUADROTOR</u>



p,q,r are the angular velocity in body-fixed frame

$$\ddot{r}_1 = \ddot{x} = g(\theta\cos\psi + \phi\sin\psi)$$
$$\ddot{r}_2 = \ddot{y} = g(\theta\sin\psi - \phi\cos\psi)$$
$$\ddot{r}_3 = \ddot{z} = -g + \frac{u_1}{m}$$

$$\begin{bmatrix}\dot{p}\\ \dot{q}\\ \dot{r}\end{bmatrix} = I^{-1}\begin{bmatrix}u_2\\ u_3\\ u_4\end{bmatrix} = I^{-1}\mathbf{u}_2$$

PATH PLANNING

<u>CONFIGURATION SPACE</u>

Minkowski Sum

C^{obs} is the *Minkowski difference* of two objects $O \oplus (-A)$

complexity (O – m vertices, A – n vertices, translation only)

<u>MOTION PLANNING</u>

<u>GRAPH SEARCH ALGORITHM</u>

Discretization

What matters: connectivity; resolution; how to deal with partially blocked cells

Dijkstra Algorithm

A* Algorithm

- For each node n in the graph
  - ○ n.f = inf //sum of g and heuristic
  - ○ n.g = inf //distance between n and start
- Create an empty list
- start.g= 0, start.f=H(start), add start to list
- While list not empty:
  - ○ current = node in list with smallest f
    remove current from list
  - ○ if current == goal, report success
  - ○ for each node n that is adjacent to current
    - if n.g > current.g+ length<n,current>
      - n.g= current.g+ length<n,current>
      - n.f= n.g + H(n)
      - n.parent = current
      - add n to list (if it isn't there already)

Heuristic function should be: 1. Admissive 2. Consistent

Jump Point Search

Speed up A* by selectively expanding nodes

<u>PROBABILISTIC PLANNING</u>

Rapidly-exploring Random Trees (RRTs)

function build_RRT(q_0,n)

- V = {q_0}
- E = {}
- For i = 1:n //maximum number of expansion
  - ○q = collision free random configuration
  - ○q_near = closest neighbor in V to q
  - ○N_qnear = randomly sample actions to get a set of neighbors to q_near
  - ○q_new = collision free point in N_qnear that is closest to q
  - ○Add q_new to V
  - ○Add (q,q_new) to E

Return V,E

Probabilistic Roadmap (PRM)

function build_PRM(q_0,n)

- V = {q_0}
- E = {}
- While |V|<n //maximum number of expansion
  - ○ q = collision free random configuration
  - ○ Add q to V
- For each q in V
  - ○ N_q = up to k closest neighbors to q whose distance to q are smaller than center threshold
  - ○ For each q' in N_q
    - If (q,q') is not in E and is collision free
      - Add (q,q_new) to E
- Return V,E

TRAJECTORY PLANNING

<u>CALCULUS OF VARIATION & SMOOTH TRAJECTORY</u>

- $x^*(t) = \text{argmin}_{x(t)}\int_0^T L(x^{(n)}, \cdots, \dot{x}, x, t)dt$,
- Let $x(t) = x^*(t) + \alpha\eta(t)$
- $\frac{d}{d\alpha}\Big|_{\alpha=0}\int_0^T L(\dot{x^*} + \alpha\dot{\eta}(t), x^*(t) + \alpha\eta(t), t)dt = 0$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \cdots + (-1)^n\frac{d^n}{dt^n}\left(\frac{\partial L}{\partial x^{(n)}}\right) = 0$$

N	1	2	3	4	5	6
minimum	vel	acc	jerk	snap	crackle	pop

- $L = \left(x^{(n)}\right)^2 \Longrightarrow x(t) = c_0 + c_1 t + \cdots + c_{2n-1}t^{2n-1}$
- multiple variables – multiple Euler Lagrange equations

<u>MULTI-SEGMENT SMOOTH SPLINE</u>

m segments, nth-order system

2mn coefficients (DoF); 2mn constraints

ends (n-1) order constraint; waypoints 2(n-1) constraints

APPENDIX

<u>CROSS PRODUCT PROPERTIES</u>

Distributive over addition; compatible with scalar multiplication;

Anti-commutative: $a \times b = -b \times a$

Not associative, but: $a\times(b\times c) + b\times(c\times a) + c\times(a\times b) = 0$

$$a\times b + c\times d = (a-c)\times(b-d) + a\times d + c\times b$$
$$a\cdot(b\times c) = b\cdot(c\times a) = c\cdot(a\times b)$$
$$a\times(b\times c) = b(c\cdot a) - c(a\cdot b)$$
$$(a\times b)\times(a\times c) = \left(a\cdot(b\times c)\right)a$$
$$(a\times b)\cdot(c\times d) = (a\cdot c)(b\cdot d) - (a\cdot d)(b\cdot c)$$

CAMERA MODEL

<u>CAMERA PROJECTION MODEL</u>

$$\lambda\begin{bmatrix}u\\v\\1\end{bmatrix}=K[R\quad t]\begin{bmatrix}X\\Y\\Z\\1\end{bmatrix},\qquad \begin{bmatrix}X\\Y\\Z\end{bmatrix}=-R^Tt+\lambda R^TK^{-1}\begin{bmatrix}u\\v\\1\end{bmatrix}$$

<u>UNDISTORTION (IMAGE SPACE)</u>

$$u^{dist}=u(1+k_1r^2+k_2r^4+\cdots),\quad r^2=u_x^2+v_y^2$$

PROJECTIVE GEOMETRY

<u>PROJECTIVE EQUIVALENCE</u>

Projective plane: set of all projective equivalence classes

Injection (one-to-one), surjection(onto) and bijection

<u>PROJECTIVE TRANSFORMATION</u>

$P^2\to P^2$, $p'\sim Ap$, A is an invertible 3×3 matrix ($\det A\neq 0$) also called collineation or homography

estimation DOF=8, need (at least) four pairs of points

properties after projective transformation, any 3 collinear points remains collinear, any 3 concurrent lines remain concurrent;

If $p'\sim Ap$, for line $l'\sim A^{-T}l$

e.g. $\begin{bmatrix}u\\v\\1\end{bmatrix}\sim H\begin{bmatrix}X\\Y\\1\end{bmatrix}$, where $\begin{bmatrix}X\\Y\\1\end{bmatrix}$ is equivalent to real 3D point $\begin{bmatrix}X_w\\Y_w\\Z_w\end{bmatrix}$

<u>VANISHING POINT AND VANISHING LINE</u>

$H=(h_1\quad h_2\quad h_3)$, then $h_1\sim p_{\infty,x}$, $h_2\sim p_{\infty,y}$

horizon line: $h_1\times h_2$, the horizon plane is parallel to ground, $h_1\times h_2$ (as a vector) is normal to the ground

Information provided by horizon: camera pitch and roll

POSE ESTIMATION

- *Given corresponding points and K, estimate R,t*
- *Assume all points are in the world $Z=0$ plane*

$$\begin{bmatrix}u\\v\\1\end{bmatrix}\sim K[r_1\quad r_2\quad t]\begin{bmatrix}X\\Y\\1\end{bmatrix},\quad H\sim K[r_1\quad r_2\quad t]$$

- *homography estimation –* H, $K^{-1}H=[h_1'\quad h_2'\quad h_3']$
- $[h_1'\quad h_2'\quad h_1'\times h_2']=UDV^T$, $R=UD'V^T$, *where* $D'=diag(1,1,\det UV^T)$
- $t=h_3'/\|h_1'\|$

PERSPECTIVE N-POINT

3D-2D CORRESPONDENCE

<u>*SPECIAL SOLUTION FOR P3P*</u>

$$\begin{cases}d_1^2+d_2^2-2d_1d_2\cos\theta_{12}=d_{12}^2\\ d_1^2+d_3^2-2d_1d_3\cos\theta_{13}=d_{13}^2\\ d_2^2+d_3^2-2d_2d_3\cos\theta_{23}=d_{23}^2\end{cases},$$

where $d_{ij}=d(X_i,X_j)$, $\cos\theta_{ij}=\dfrac{(K^{-1}x_i)^T(K^{-1}x_j)}{\|K^{-1}x_i\|\|K^{-1}x_j\|}$

Solve the equation above: four set of possible (d_1,d_2,d_3)

Use another point to verify solution.

[Recover R,t *from* (d_1,d_2,d_3)*]*

$[\tilde X_1\quad \tilde X_2\quad \tilde X_3]=R[X_1\quad X_2\quad X_3]$

where $\tilde X_i=d_iK^{-1}x_i/\|K^{-1}x_i\|$

<u>*DIRECT SOLUTION FOR PNP*</u>

$$u^x=\frac{p_{11}X+p_{12}Y+p_{13}Z+p_{14}}{p_{31}X+p_{32}Y+p_{33}Z+p_{34}}\qquad u^y=\frac{p_{21}X+p_{22}Y+p_{23}Z+p_{24}}{p_{31}X+p_{32}Y+p_{33}Z+p_{34}}$$

Recover R,t from P

3D-3D REGISTRATION

$A_i=RB_i+t$, $i=1,2,3,4$, *points are not coplanar*

3 non-collinear points make an orthogonal basis:

$(A_{21}\quad (A_{21}\times A_{31})\times A_{21}\quad A_{21}\times A_{31})$

problem: $\min_{R,t}\sum_i\|A_i-RB_i-t\|_F$, *let* $A_i=A_i-\bar A, B_i=B_i-\bar B$

problem transformed to $\min_R\sum_i\|A_i-RB_i\|_F$, *which is equivalent to* $\max_R tr(RBA^T)$, $BA^T=UDV^T$, *then*

$R=UD'V^T$, *where* $D'=diag(1,1,\det UV^T)$

3D VELOCITY

All in CAMERA FRAME define $(x,y)=(X/Z,Y/Z)$,

$$p=\begin{bmatrix}\dot x\\\dot y\end{bmatrix}=\frac{1}{Z}\begin{bmatrix}\dot X\\\dot Y\end{bmatrix}-\frac{\dot Z}{Z}\begin{bmatrix}x\\y\end{bmatrix},\qquad \begin{bmatrix}\dot X\\\dot Y\\\dot Z\end{bmatrix}=-\Omega\times\begin{bmatrix}X\\Y\\Z\end{bmatrix}-V$$

where V,Ω *are the instantaneous linear and angular velocity of the camera (w.r.t the camera itself),*

<u>MOTION FLOW GIVEN CAMERA VELOCITIES</u>

$$\dot p=\begin{bmatrix}\dot x\\\dot y\end{bmatrix}=\frac{1}{Z}\begin{bmatrix}xV_z-V_x\\yV_z-V_y\end{bmatrix}+\begin{bmatrix}xy & -(1+x^2) & y\\1+y^2 & -xy & -x\end{bmatrix}\Omega$$

for pure rotation, there is no information about depth Z

intersection of lines spanned by translation flow (the first term in the equation above) provide the focus of expansion (FOE) / epipole

$$FOE=(V_x/V_z,V_y/V_z)$$

the flow vector length of points at the same radial distance from FOE inversely proportional to depth

<u>CAMERA VELOCITIES GIVEN MOTION FLOW</u>

Unknown $\boldsymbol{V}$

$V^T(p\times\dot p_{trans})=0$, image point, flow V are coplanar

Unknown $\boldsymbol{V,\Omega}$

$$\begin{bmatrix}-/Z & 0 & x/Z & xy & -(1+x^2) & y\\0 & -1/Z & y/Z & 1+y^2 & -xy & -x\end{bmatrix}\begin{bmatrix}V\\\Omega\end{bmatrix}=\begin{bmatrix}\dot x\\\dot y\end{bmatrix}$$

<u>OPTICAL FLOW</u>

Feature match: brightness consistency $I_1(x-u,y-v)=I_2(x,y)$

Define optical flow $d=(u,v)^T$, position on image $x=(x,y)^T$,

$E(d)=I_1(x-d)-I_2(x)$, find d that minimize $\iint E(d)^2dxdy$

$$\begin{aligned}E(\vec d)&=I_1(\vec x-\vec d)-I_2(\vec x)\\&=I_1(\vec x)\text{-}\nabla I_1(\vec x)^T\vec d-I_2(\vec x)\\&=-\Delta I(\vec x)\text{-}\nabla I_1(\vec x)^T\vec d\end{aligned}\qquad \Delta I=I_2(x,y)\text{-}I_1(x,y)$$

∇I_1 is the spatial gradient of I_1

GROUPING

<u>LINE FITTING</u>

Single line fitting $x\cos\theta+y\sin\theta=d$

$$\arg\min_{\theta\in[0,2\pi),d\geq 0}\sum_{i=1}^N(x_i\cos\theta+y_i\sin\theta-d)^2\qquad \arg\min_{\theta\in[0,2\pi)}\sum_{i=1}^N((x_i-\bar x)\cos\theta+(y_i-\bar y)\sin\theta)^2$$

$$\arg\min_{\theta\in[0,2\pi)}(\cos\theta\quad\sin\theta)\underbrace{\sum_{i=1}^N\begin{pmatrix}(x_i-\bar x)^2 & (x_i-\bar x)(y_i-\bar y)\\(x_i-\bar x)(y_i-\bar y) & (y_i-\bar y)^2\end{pmatrix}}_{C}\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}$$

Solution is the eigenvector to the minimal eigenvalue of C.

Multiple line fitting

Each point in (x,y) space correspond to a curve in (d,θ) space

Line through two points = intersection of two curves

Hough transform

$$V[\theta_j,d_j]=\sum_i\delta'(d_j-x_i\cos\theta_j+y_i\sin\theta_j)$$

circle & ellipsoid detection

<u>RANSAC</u>

Decide number of iterations N: s – number of feature pairs in a sample; e – probability that a point is an outlier; N – number of iterations; p – desired probability of getting good H

$1-(1-(1-e)^s)^N=p$, $N=\log(1-p)/\log(1-(1-e)^s)$

<u>RANSAC VS HOUGH</u>

- RANSAC can deal only with one model (inliers vs outliers) while Hough detects multiple models
- RANSAC is more efficient when fraction of outliers is low
- RANSAC requires the solution of a minimal set problem,
  - For example, solve of a system of 5 polynomial equations for 5 unknowns
- Hough needs a bounded parameter space
- Hough is intractable for large number of unknowns

TWO VIEW GEOMETRY

P, Q: 3D coordinates
p, q: inhomogeneous representation $p = P/Z_p, q = Q/Z_q$
R, t: transformation from frame P to frame Q, $Q = RP + T$

ESSENTIAL MATRIX

$$q^T E p = 0, \quad p^T E^T q = 0, \quad E = \hat{T} R$$

E is a singular matrix $\det E = 0$. It has two equal non-zero eigenvalues $\|T\|^2$, actually: E is essential iff. $\sigma_1 = \sigma_2 > 0 = \sigma_3$

If $E = UDV^T = U \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T$, there are four groups of T, R

$$(\widehat{T_1}, R_1) = (UR_{z,+\pi/2}\Sigma U^T, UR^T_{z,+\pi/2}V^T)$$
$$(\widehat{T_2}, R_2) = (UR_{z,-\pi/2}\Sigma U^T, UR^T_{z,-\pi/2}V^T)$$
$$(\widehat{T_1}, R_2) = (UR_{z,+\pi/2}\Sigma U^T, UR^T_{z,-\pi/2}V^T)$$
$$(\widehat{T_2}, R_1) = (UR_{z,-\pi/2}\Sigma U^T, UR^T_{z,+\pi/2}V^T)$$

Where

$$R_{z,\frac{\pi}{2}} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, R_{z,-\frac{\pi}{2}} = R^T_{z,\frac{\pi}{2}}$$

(if $\det R < 0, R = -R$)

estimation – 8 points algorithm

$E = (e_1 \quad e_2 \quad e_3)$

$$\underbrace{\begin{pmatrix} p_{1,x}q_1^T & p_{1,y}q_1^T & p_{1,z}q_1^T \\ \vdots & \vdots & \vdots \\ p_{n,x}q_n^T & p_{n,y}q_n^T & p_{n,z}q_n^T \end{pmatrix}}_{A} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = 0$$

Solution - $null(A)$

if $\sigma_8(A) = 0$, give up !

Assume $(e_1' \quad e_2' \quad e_3') = U diag(\sigma_1, \sigma_2, \sigma_3) V^T$, then

$(e_1 \quad e_2 \quad e_3) = U diag\left(\frac{\sigma_1+\sigma_2}{2}, \frac{\sigma_1+\sigma_2}{2}, 0\right) V^T$

recover R, t from E

check all configurations to make sure points are in front of camera

EPIPOLES

$$e_p \sim -R^T t, \qquad e_q \sim T$$
$$E e_p = 0, \qquad E^T e_q = 0$$

EPIPOLAR LINES

$$l_q = F^T q - on\ image\ p$$
$$l_p = Fp - on\ image\ q$$

TRIANGULATION

Given R, T, unknown depth (λ, μ)

$$\underbrace{(q_i - Rp_i)}_{3\times 2} \underbrace{\begin{pmatrix} \mu_i \\ \lambda_i \end{pmatrix}}_{2\times 1} = \underbrace{T}_{3\times 1}$$

$AQ_x + BQ_y + CQ_z = N^T Q = 0$
$\lambda P = RQ + T = RQ + TN^T Q = (R + TN^T)Q$

VISUAL ODOMETRY

CAMERA TRAJECTORY FROM MONOCULAR VIDEO

Given **an estimate** R_k, T_k of the current camera pose as well as the 3D points $\mathbf{X}_p = (X_p, Y_p, Z_p)$ and correspondences to calibrated point projections in frame $(k+1)$ (x_p^{k+1}, y_p^{k+1})

Update to the pose R_{k+1}, T_{k+1}

DETERMINISTIC APPROACH

- find correspondences from view k to view k+1 (RANSAC)
- solve for epipolar geometry between the two frames
- $R_{k+1} = R_k R_{k+1}^k$
- $T_{k+1} = T_k + R_k T_{k+1}^k$
- use the estimated 3D points and their 2D projection on frame k+1 to update R_{k+1}, T_{k+1}
- update estimates of 3D points
- run bundle adjustment over past K frames to re-adjust points and poses

FILTERING APPROACH

$$x_p^k = \frac{R_{11}^k X_p + R_{12}^k Y_p + R_{13}^k Z_p + T_x^k}{R_{31}^k X_p + R_{32}^k Y_p + R_{33}^k Z_p + T_z^k}$$
$$y_p^k = \frac{R_{21}^k X_p + R_{22}^k Y_p + R_{23}^k Z_p + T_y^k}{R_{31}^k X_p + R_{32}^k Y_p + R_{33}^k Z_p + T_z^k}$$

- update estimate of a state vector X_k
  the same point is fixed in global frame

$$R^{k+1} = e^{\hat{\omega}^k} R^k \quad T^{k+1} = T^k + R^k v^k$$
$$\omega^{k+1} = \omega^k \quad v^{k+1} = v^k$$

TWO VIEW METROLOGY

Given two images, necessary point correspondences known
R, t between two cameras and world frame are known

$p \sim H^{-T} l \times l'$

SUMMARY

Problem	Measured	Unknowns
3D-3D Registration	Two sets of 3D points $\{A_i\}$ and $\{B_i\}$	s, R and T that minimize $\min_{R,T} \sum_{i=1}^N \|A_i - RB_i + T\|^2$
PnP	2D-3D correspondences $\{p_j\}$ and $\{P_j\}$	T and R between the camera and the world coordinate system
SFM	2D-2D correspondences between two views $\{p_j\}$ and $\{q_j\}$	T and R between the two views
Triangulation	T, R and 2D point correspondences p_i, q_i	The depth of the point in each camera frame μ_i, λ_i.

APPENDIX

VELOCITY: GENERAL TRANSFORMATION

Rodrigues formulas $R = I + \hat{u}\sin\theta + (1 - \cos\theta)\hat{u}^2 = I\cos\theta + uu^T(1 - \cos\theta) + \hat{u}\sin\theta$, where $\hat{u}^2 = uu^T - I$

$$\hat{\omega} = S(\omega(t)) = \dot{R}(t) R^T(t), \quad \hat{\omega}R(t) = \dot{R}(t)$$

$$\dot{\mathbf{H}}(t) = \mathbf{S}_h(t)\mathbf{H}(t)$$

$$\mathbf{S}_h(t) = \begin{bmatrix} \dot{\mathbf{R}}(t)\mathbf{R}(t)^T & \dot{\boldsymbol{t}}(t) - \dot{\mathbf{R}}(t)\mathbf{R}(t)^T\boldsymbol{t}(t) \\ 0 & 0 \end{bmatrix} = \dot{\mathbf{H}}(t)\mathbf{H}(t)^{-1} = \hat{\boldsymbol{\xi}}$$

This is called twist

$H^{-1}\dot{p}_0 = H^{-1}\dot{H}p_1$	$\hat{\xi}^b = H^T\dot{H}$
velocity in body frame	velocities in body frame
$\dot{p}_0 = H\dot{H}^{-1}p_0$	$\hat{\xi}^s = \dot{H}H^T\ (s_h)$
velocity in inertial frame	velocities in inertial frame

For any point p: $\dot{p}_0 = \dot{R}_1^0 p_1 + R_1^0 \dot{p}_1 + \dot{o}_1^0$

	Rotation	Pose
Matrix	$\boldsymbol{R} \in R^{3\times 3} \mid \mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}, \det\mathbf{R} = 1$	$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$
3D element	$\boldsymbol{p}' = \mathbf{R}\boldsymbol{p}$	$\boldsymbol{P}' = \mathbf{H}\boldsymbol{P}$
Velocities body frame	$\hat{\boldsymbol{\omega}}^b = \boldsymbol{R}^T\dot{\boldsymbol{R}}$	$\hat{\boldsymbol{\xi}}^b = \mathbf{H}\dot{\mathbf{H}}$
Velocities inertial frame	$\hat{\boldsymbol{\omega}}^s = \dot{\boldsymbol{R}}\boldsymbol{R}^T$	$\hat{\boldsymbol{\xi}}^s = \dot{\mathbf{H}}\mathbf{H}^{-1}$