Supplemental Document

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Deep learning estimation of modified Zernike coefficients and recovery of point spread functions in turbulence: supplement

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1. AMBIGUITY ASSOCIATED WITH PREDICTING ZERNIKE COEFFICIENTS FROM INTENSITY IMAGES

In order to explicitly show the ambiguity associated with predicting Zernike coefficients from intensity images, we utilize the symmetry properties of the Fourier transform [1]. We begin by writing an expression for the point spread function $h(x,y)$ in terms of the Fourier transform as

$$h(x,y) \propto \mathcal{F}\left\{p(x,y)\mathrm{e}^{\mathrm{j}\phi(x,y)}\right\}\mathcal{F}^*\left\{p(x,y)\mathrm{e}^{\mathrm{j}\phi(x,y)}\right\}, \tag{S1}$$

and expressing the pupil function $p(x,y)$ and wavefront phase distortion $\phi(x,y)$ in terms of real/imaginary and even/odd parts as

$$p(x,y) = p_{\mathrm{re}}(x,y) + p_{\mathrm{ro}}(x,y) + \mathrm{j}p_{\mathrm{ie}}(x,y) + \mathrm{j}p_{\mathrm{io}}(x,y) \tag{S2}$$

and

$$\phi(x,y) = \phi_{\mathrm{re}}(x,y) + \phi_{\mathrm{ro}}(x,y) + \mathrm{j}\phi_{\mathrm{ie}}(x,y) + \mathrm{j}\phi_{\mathrm{io}}(x,y) \tag{S3}$$

where the subscripts re, ro, ie, and io represent the real-even, real-odd, imaginary-even, and imaginary-odd parts, respectively. Under the assumption that the pupil is a real-even function then only term $p_{\mathrm{re}}(x,y)$ in Eq. (S2) is non-zero. Furthermore, assuming the wavefront aberration phase is purely real, then only terms $\phi_{\mathrm{re}}(x,y)$ and $\phi_{\mathrm{ro}}(x,y)$ in Eq. (S3) are non-zero. Next, we consider the two special cases of particular interest. Specifically, the cases when the Zernike polynomial is either angularly even which corresponds to even radial order, or angularly odd which corresponds to odd radial order.

A. Angularly Even Zernike Polynomial

An angularly even Zernike polynomial $Z_{\mathrm{ae}}(x,y)$ must be a real even symmetric function [2]. Thus, for an angularly even Zernike polynomial $Z_{\mathrm{ae}}(x,y)$ the wavefront phase distortion $\phi_{\mathrm{re}}(x,y)$ has real even symmetry. Therefore, for an angularly even polynomial $Z_{\mathrm{ae}}(x,y)$, we can write

$$\begin{aligned}
\mathcal{F}\left\{p(x,y)\mathrm{e}^{\mathrm{j}\phi(x,y)}\right\} &= \mathcal{F}\left\{p_{\mathrm{re}}(x,y)\mathrm{e}^{\mathrm{j}\phi_{\mathrm{re}}(x,y)}\right\} && \text{(S4)}\\
&= \mathcal{F}\{p_{\mathrm{re}}(x,y)\cos\phi_{\mathrm{re}}(x,y) + \mathrm{j}p_{\mathrm{re}}(x,y)\sin\phi_{\mathrm{re}}(x,y)\} && \text{(S5)}
\end{aligned}$$

For Zernike coefficient $\pm a$, we have $\phi_{\mathrm{re}}(x,y) = \pm aZ_{\mathrm{ae}}(x,y)$ and

$$\begin{aligned}
\mathcal{F}\left\{p(x,y)\mathrm{e}^{\pm\mathrm{j}aZ_{\mathrm{ae}}(x,y)}\right\} &= \mathcal{F}\{p_{\mathrm{re}}(x,y)\cos[\pm aZ_{\mathrm{ae}}(x,y)] + \mathrm{j}p_{\mathrm{re}}(x,y)\sin[\pm aZ_{\mathrm{ae}}(x,y)]\} && \text{(S6)}\\
&= \mathcal{F}\{p_{\mathrm{re}}(x,y)\cos[aZ_{\mathrm{ae}}(x,y)] \pm \mathrm{j}p_{\mathrm{re}}(x,y)\sin[aZ_{\mathrm{ae}}(x,y)]\} && \text{(S7)}\\
&= A(x,y) \pm \mathrm{j}B(x,y) && \text{(S8)}
\end{aligned}$$

where $p_{\mathrm{re}}(x,y)\cos[aZ_{\mathrm{ae}}(x,y)]$ is a real even symmetric function (cosine of a real even symmetric function is also real even symmetric) and $\mathrm{j}p_{\mathrm{re}}(x,y)\sin[aZ_{\mathrm{ae}}(x,y)]$ is an imaginary even symmetric function (sine of a real even symmetric function is also real even symmetric). Using symmetry properties of the Fourier transform, $A(x,y) = \mathcal{F}\{p_{\mathrm{re}}(x,y)\cos[aZ_{\mathrm{ae}}(x,y)]\}$ is real even and $\mathrm{j}B(x,y) = \mathcal{F}\{\mathrm{j}p_{\mathrm{re}}(x,y)\sin[aZ_{\mathrm{ae}}(x,y)]\}$ is imaginary even. This leads to

$$h(x,y) \propto \left|\mathcal{F}\left\{p_{\mathrm{re}}(x,y)\mathrm{e}^{\mathrm{j}\phi_{\mathrm{re}}(x,y)}\right\}\right|^2 = A^2(x,y) + B^2(x,y). \tag{S9}$$

Therefore, as can be seen from Eq. (S9), an angularly even polynomial results in same value for $h(x,y)$ irrespective of the sign of the Zernike coefficient a, thus leading to an intensity image ambiguity.

B. Angularly Odd Zernike Polynomial

Similarly, an angularly odd Zernike polynomial $Z_{\text{ao}}(x,y)$ must be a real odd symmetric function [2]. Thus, for an angularly odd Zernike polynomial $Z_{\text{ao}}(x,y)$ the wavefront phase distortion $\phi_{\text{re}}(x,y)$ has real odd symmetry. Therefore, for an angularly odd polynomial $Z_{\text{ao}}(x,y)$, we can write

$$\begin{aligned}
\mathcal{F}\left\{p(x,y)\mathrm{e}^{\mathrm{j}\phi(x,y)}\right\} &= \mathcal{F}\left\{p_{\text{re}}(x,y)\mathrm{e}^{\mathrm{j}\phi_{\text{ro}}(x,y)}\right\} && \text{(S10)} \\
&= \mathcal{F}\{p_{\text{re}}(x,y)\cos\phi_{\text{ro}}(x,y) + \mathrm{j}p_{\text{re}}(x,y)\sin\phi_{\text{ro}}(x,y)\} && \text{(S11)}
\end{aligned}$$

For Zernike coefficient $\pm a$, we have $\phi_{\text{ro}}(x,y) = \pm aZ_{\text{ao}}(x,y)$ and

$$\begin{aligned}
\mathcal{F}\left\{p(x,y)\mathrm{e}^{\pm\mathrm{j}aZ_{\text{ao}}(x,y)}\right\} &= \mathcal{F}\{p_{\text{re}}(x,y)\cos[\pm aZ_{\text{ao}}(x,y)] + \mathrm{j}p_{\text{re}}(x,y)\sin[\pm aZ_{\text{ao}}(x,y)]\} && \text{(S12)} \\
&= \mathcal{F}\{p_{\text{re}}(x,y)\cos[aZ_{\text{ao}}(x,y)] \pm \mathrm{j}p_{\text{re}}(x,y)\sin[aZ_{\text{ao}}(x,y)]\} && \text{(S13)} \\
&= C(x,y) \pm D(x,y) && \text{(S14)}
\end{aligned}$$

where $p_{\text{re}}(x,y)\cos[aZ_{\text{ao}}(x,y)]$ is a real even symmetric function (cosine of a real odd symmetric function is real even symmetric) and $\mathrm{j}p_{\text{re}}(x,y)\sin[aZ_{\text{ao}}(x,y)]$ is an imaginary odd symmetric function (sine of a real odd symmetric function is also real odd symmetric). Using symmetry properties of the Fourier transform, $C(x,y) = \mathcal{F}\{p_{\text{re}}(x,y)\cos[aZ_{\text{ao}}(x,y)]\}$ is real even and $D(x,y) = \mathcal{F}\{\mathrm{j}p_{\text{re}}(x,y)\sin[aZ_{\text{ao}}(x,y)]\}$ is real odd. This leads to

$$h(x,y) \propto \left|\mathcal{F}\left\{p_{\text{re}}(x,y)\mathrm{e}^{j\phi_{\text{ro}}(x,y)}\right\}\right|^2 = \left[C(x,y) \pm D(x,y)\right]^2. \qquad \text{(S15)}$$

Therefore, as can be seen from Eq. (S15), an angularly odd polynomial results in a different value for $h(x,y)$ respective to the sign of the Zernike coefficient a, thus does not lead to an intensity image ambiguity.

C. Example

As an example, consider Figure S1 which shows the PSF intensity images for a point object modeled using Zernike polynomials Z_{12} and Z_6, respectively, with oppositely signed Zernike coefficients. Observe that for the angularly even polynomial Z_{12} the intensity image is the same in S1(a) and S1(b) regardless of the sign of the Zernike coefficient. On the contrary, for the angularly odd polynomial Z_6, the intensity image is different in S1(c) and S1(d) as a result of the sign of the Zernike coefficient and thus does not result in an ambiguity.

REFERENCES

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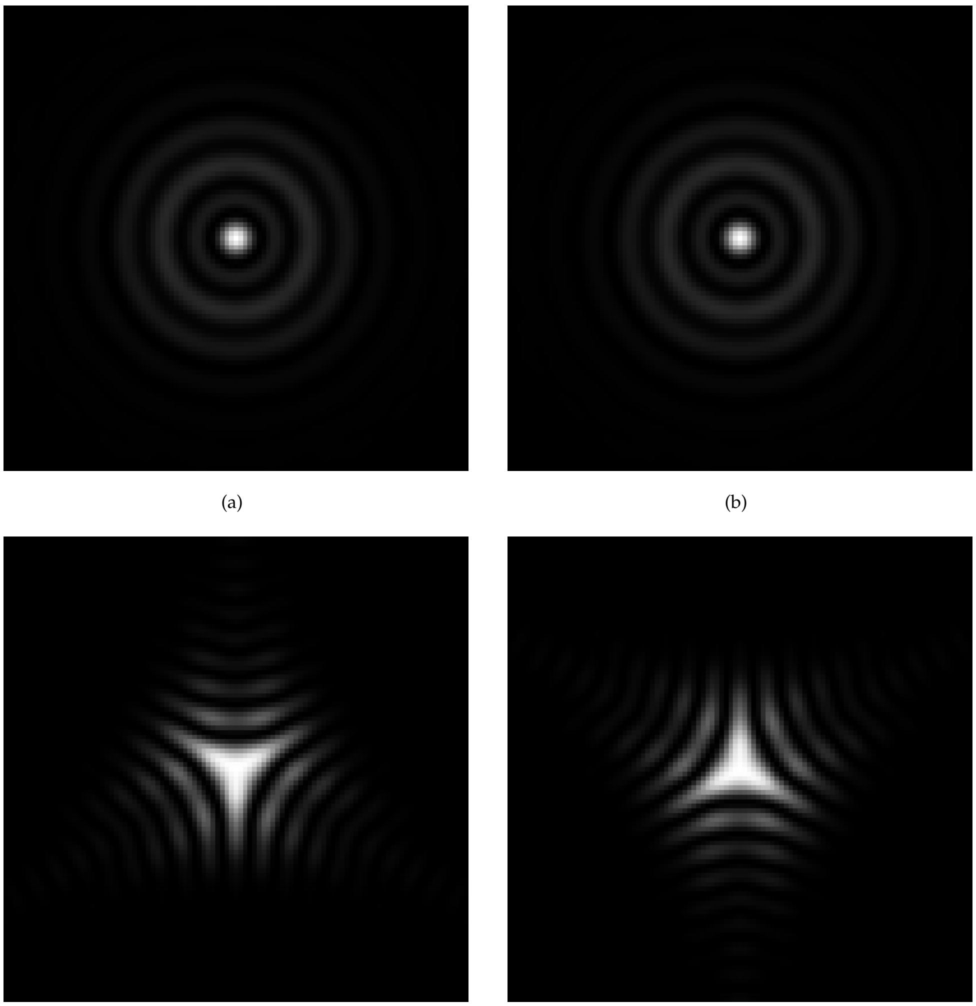

(a) (b) (c) (d)

Fig. S1. The PSF intensity images of a point object with: (a) Zernike coefficient value $+5$ for Z_{12} (b) Zernike coefficient value -5 for Z_{12} (c) Zernike coefficient value $+5$ for Z_6 (d) Zernike coefficient value -5 for Z_6.