ECE 558 HMWK 03

Prob 1: Solve: $f(t) * h(t) = \int_{-\infty}^{\infty} f(z)h(t-z)dz$

(a) $\delta(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} \delta(z)\cdot\delta(t-t_0-z)dz \cdots (1)$

due to $\delta(z) = \delta(-z)$

(1) is equal to $\int_{-\infty}^{\infty} \delta(-z)\delta(t-t_0-z)dz$

let $-z = m$, $dz = -dm$,

we can get $-\int_{\infty}^{-\infty} \delta(m)\delta(m+t-t_0)dm = \int_{-\infty}^{\infty} \delta(m)\delta(m+t-t_0)dm \cdots (2)$

due to the shifting property: $\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$

Therefore (2) is equal to $\delta(t_0-t) = \delta(-t+t_0) = \delta(t-t_0)$

(b) $\delta(t-t_0) * \delta(t+t_0) = \int_{-\infty}^{\infty} \delta(z-t_0)\cdot\delta(t+t_0-z)dz \cdots (3)$

let $z-t_0 = -m$, $dz = -dm$

so (3) is equal to $-\int_{\infty}^{-\infty} \delta(-m)\delta(m+t)dm = \int_{-\infty}^{\infty} \delta(m)\delta(m+t)dm$

$= \delta(-t) = \delta(t)$

Prob2: Solve:

(a) The highest frequency of $f(t)$ is $f_{max} = \frac{1}{T_{min}} = 1/(2\pi/8\pi) = 4Hz$.

(b) The Nyquist rate corresponding to my result in (a) is 8Hz

(c) For perfect recovery of the function from samples,

the rate of my choice is greater than 8Hz, which means $f_{sample} > 8Hz$, maybe 10 Hz.

Prob3: Solve:

(a) $F_1(u) = \int_{-\infty}^{\infty} e^{j2\pi t_0 t} e^{-j2\pi ut} dt = \int_{-\infty}^{\infty} e^{-j2\pi t(u-t_0)} dt \cdots (1)$ t denotes time

due to the fact that $\delta(t-t_0) = \int_{-\infty}^{\infty} e^{-j2\pi ut_0} e^{j2\pi ut} du = \int_{-\infty}^{\infty} e^{-j2\pi u(t_0-t)} du \cdots (2)$ u denotes frequency

So inspired by (2), (1) is equal to $F_1(u) = \delta(u-t_0)$

(b) $F_2(u) = \int_{-\infty}^{\infty} \cos(2\pi u_0 t) e^{-j2\pi ut} dt \cdots (3)$

due to the fact that $\cos(2\pi u_0 t) = \frac{e^{j2\pi u_0 t} + e^{-j2\pi u_0 t}}{2}$

So (3) is equal to $\frac{1}{2}\int_{-\infty}^{\infty} [e^{-j2\pi t(u-u_0)} + e^{-j2\pi t(u+u_0)}]dt \cdots (4)$

From conclusion in part (a),

we know (4) is equal to $F_2(u) = \frac{1}{2}[\delta(u-u_0) + \delta(u+u_0)]$

(c) $F_3(u)=\int_{-\infty}^{\infty}\sin(2\pi u_0 t)e^{-j2\pi ut}\,dt \quad \ldots (5)$

due to the fact that $\sin(2\pi u_0 t) = \dfrac{e^{j2\pi u_0 t} - e^{-j2\pi u_0 t}}{2j}$

So (5) is equal to $\frac{1}{2j}\int_{-\infty}^{\infty}[e^{-j2\pi t(u-u_0)} + e^{-j2\pi t(u+u_0)}]\,dt \quad \ldots (6)$

From conclusion in part (a), we know (6) is equal to $\frac{1}{2j}[\delta(u-u_0) - \delta(u+u_0)]$

Prob 4: Solve:

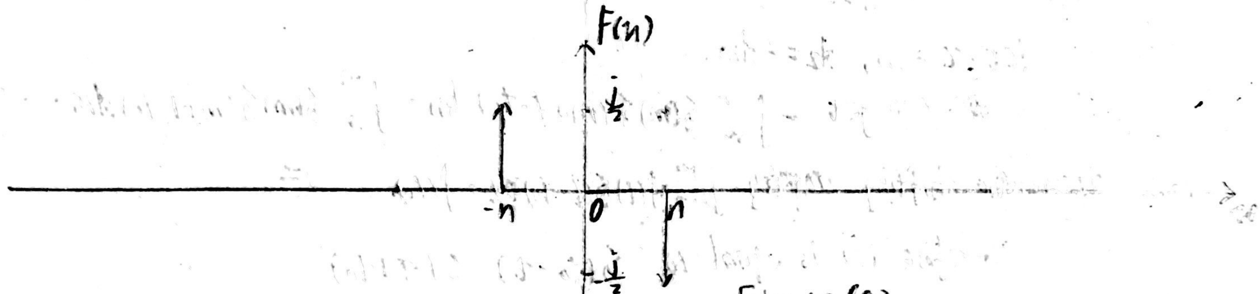

Figure (a)

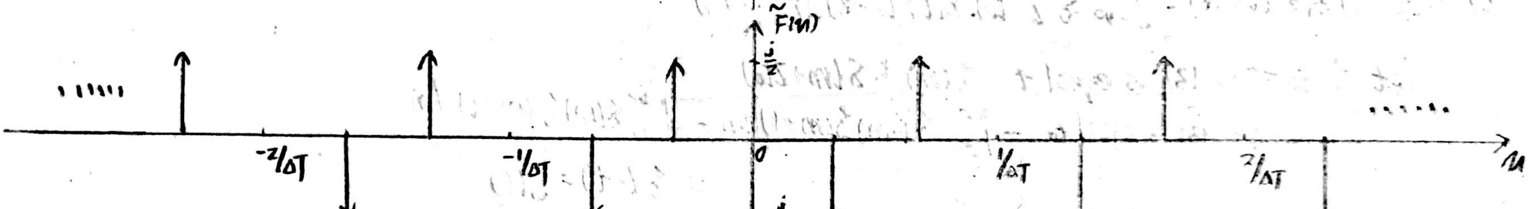

Figure (b)

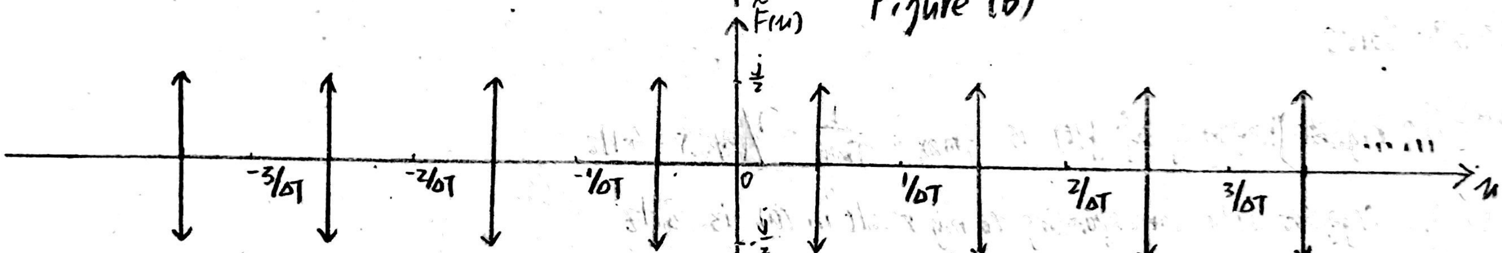

Figure (c)

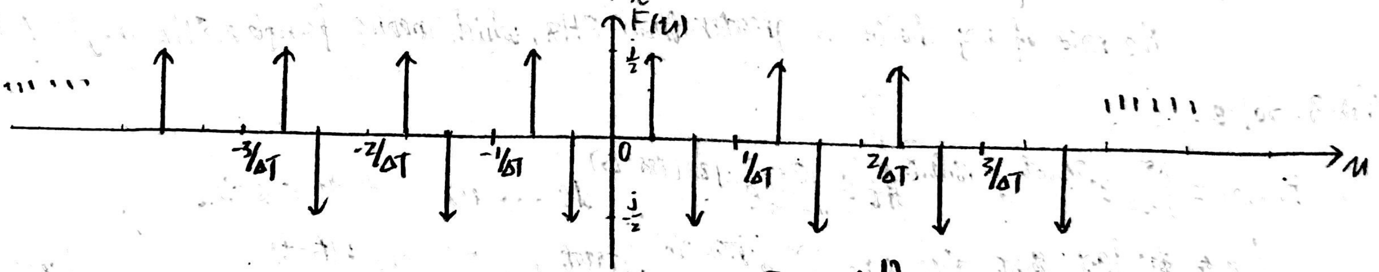

Figure (d)

(a) The period of $f(t)$ is $T = 2\pi/2\pi n = \frac{1}{n}$

(b) The frequency of $f(t)$ is $f = \frac{1}{T} = n$

(c) When higher than the Nyquist rate, it will look like the Figure (b)

(d) When lower than the Nyquist rate, it will look like the Figure (d)

(e) When sampled at the Nyquist rate, it will look like the Figure (c).

Prob 5: Solve:

DTFT: $F(u) = \sum_{x=-\infty}^{\infty} f(x) e^{-j2\pi ux}$ $\quad H(u) = \sum_{x=-\infty}^{\infty} h(x) e^{-j2\pi ux}$

IDTFT: $f(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} F(u) e^{j2\pi ux} du$ $\quad h(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} H(u) e^{j2\pi ux} du$

(a) Let $g(x) = (f * h)(x) = \sum_{m=-\infty}^{\infty} f(m) h(x-m)$

$$G(u) = \sum_{x=-\infty}^{\infty} g(x) e^{-j2\pi ux}$$

$$= \sum_{x=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(m) h(x-m) e^{-j2\pi ux}$$

$$= \sum_{m=-\infty}^{\infty} f(m) e^{-j2\pi um} \sum_{x=-\infty}^{\infty} h(x-m) e^{-j2\pi u(x-m)}$$

$$= F(u) \cdot H(u)$$

So $(f * h)(x) = (F \cdot H)(u)$

(b) $\text{IDTFT}[F(u) * H(u)] = \text{IDTFT}\{\int_{-\infty}^{\infty} F(f) H(u-f) df\}$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} F(f) H(u-f) df\right] e^{j2\pi ux} du$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} F(f) e^{j2\pi fx} df \int_{-\infty}^{\infty} H(u-f) e^{j2\pi x(u-f)} du$$

$= m \cdot h(x) \cdot f(x)$, where m is the length of x suppore

So $(f \cdot h)(x) = \frac{1}{m} (F * H)(u)$

Prob 6: Solve:

(a) In frequency domain:

$$\nabla^2 f(t,z) = \frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial z^2}$$

$$f(t,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ut+vz)} du dv$$

$$\frac{\partial f(t,z)}{\partial t} = j2\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u \cdot F(u,v) e^{j2\pi(ut+vz)} du \cdot dv$$

$$\frac{\partial^2 f(t,z)}{\partial t^2} = j2\pi \cdot j2\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^2 F(u,v) e^{j2\pi(ut+vz)} du dv$$

$$= -4\pi^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^2 F(u,v) e^{j2\pi(ut+vz)} du dv \quad \cdots (1)$$

For the same reason, $\frac{\partial^2 f(t,z)}{\partial z^2} = -4\pi^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v^2 F(u,v) e^{j2\pi(ut+vz)} du dv \quad \cdots (2)$

From (1) and (2): $\frac{\partial^2 f(t,z)}{\partial t^2} + \frac{\partial^2 f(t,z)}{\partial z^2} = -4\pi^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (u^2+v^2) F(u,v) e^{j2\pi(ut+vz)} du dv$

So we can get $\nabla^2 f(t,z) \Leftrightarrow -4\pi^2(u^2+v^2)F(u,v)$

(b) The discrete one: $\nabla^2 f = f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y) \quad \cdots (3)$

due to that $f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v) e^{j2\pi(\frac{ux}{M}+\frac{vy}{N})}$

(3) is equal to $\nabla^2 f = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)\Big[e^{j2\pi(\frac{ux+u}{M}+\frac{vy}{N})} + e^{j2\pi(\frac{ux-u}{M}+\frac{vy}{N})} + e^{j2\pi(\frac{ux}{M}+\frac{vy+v}{N})} + e^{j2\pi(\frac{ux}{M}+\frac{vy-v}{N})} - 4e^{j2\pi(\frac{ux}{M}+\frac{vy}{N})}\Big]$

$$\nabla^2 f = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v) e^{j2\pi(\frac{ux}{M}+\frac{vy}{N})}\left(e^{j2\pi\frac{u}{M}} + e^{-j2\pi\frac{u}{M}} + e^{j2\pi\frac{v}{N}} + e^{-j2\pi\frac{v}{N}} - 4\right)$$

$$= \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\left\{F(u,v) e^{j2\pi(\frac{ux}{M}+\frac{vy}{N})}\left[2\cos(2\pi\tfrac{u}{M}) + 2\cos(2\pi\tfrac{v}{N}) - 4\right]\right\}$$

So the implement is $2\cos(2\pi\frac{u}{M}) + 2\cos(2\pi\frac{v}{N}) - 4$.

$\nabla^2 f(x,y) \Leftrightarrow F(u,v)\left[2\cos(2\pi\frac{u}{M}) + 2\cos(2\pi\frac{v}{N}) - 4\right]$ for discrete variables