VIO第8期第4章作业分享

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作业内容

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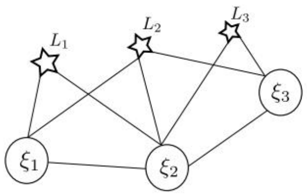
作业

1. 某时刻，SLAM 系统中相机和路标点的观测关系如下图所示，其中 ξ 表示相机姿态，L 表示观测到的路标点。当路标点 L 表示在世界坐标系下时，第 k 个路标被第 i 时刻的相机观测到，重投影误差为 $\mathbf{r}(\xi_i, L_k)$。另外，相邻相机之间存在运动约束，如 IMU 或者轮速计等约束。



   1. 请绘制上述系统的信息矩阵 $\mathbf{\Lambda}$.
   2. 请绘制相机 ξ_1 被 marg 以后的信息矩阵 $\mathbf{\Lambda}'$.

2. 阅读《Relationship between the Hessian and Covariance Matrix for Gaussian Random Variables》. 证明信息矩阵和协方差的逆之间的关系。

3. 请补充作业代码中单目 Bundle Adjustment 信息矩阵的计算，并输出正确的结果。正确的结果为：奇异值最后 7 维接近于 0，表明零空间的维度为 7.

作业1-绘制信息矩阵

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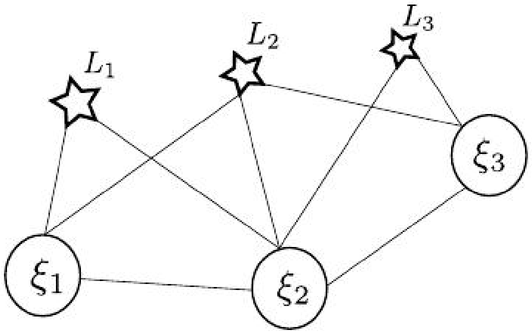


$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \\ \boldsymbol{\xi}_3 \\ \boldsymbol{L}_1 \\ \boldsymbol{L}_2 \\ \boldsymbol{L}_3 \end{bmatrix}$$

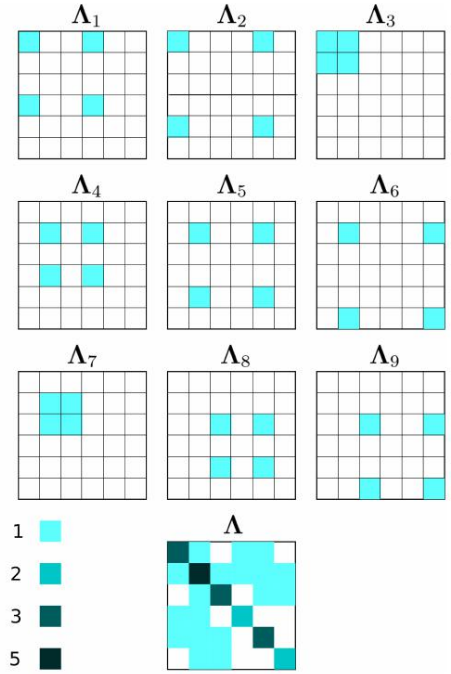
$$\boldsymbol{r} = \begin{bmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \boldsymbol{r}_3 \\ \boldsymbol{r}_4 \\ \boldsymbol{r}_5 \\ \boldsymbol{r}_6 \\ \boldsymbol{r}_7 \\ \boldsymbol{r}_8 \\ \boldsymbol{r}_9 \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}_{\xi_1 L_1} \\ \boldsymbol{r}_{\xi_1 L_2} \\ \boldsymbol{r}_{\xi_1 \xi_2} \\ \boldsymbol{r}_{\xi_2 L_1} \\ \boldsymbol{r}_{\xi_2 L_2} \\ \boldsymbol{r}_{\xi_2 L_3} \\ \boldsymbol{r}_{\xi_2 \xi_3} \\ \boldsymbol{r}_{\xi_3 L_2} \\ \boldsymbol{r}_{\xi_3 L_3} \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_1 \\ \boldsymbol{J}_2 \\ \boldsymbol{J}_3 \\ \boldsymbol{J}_4 \\ \boldsymbol{J}_5 \\ \boldsymbol{J}_6 \\ \boldsymbol{J}_7 \\ \boldsymbol{J}_8 \\ \boldsymbol{J}_9 \end{bmatrix} = \begin{bmatrix} \frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{\xi}} \\ \frac{\partial \boldsymbol{r}_2}{\partial \boldsymbol{\xi}} \\ \frac{\partial \boldsymbol{r}_3}{\partial \boldsymbol{\xi}} \\ \frac{\partial \boldsymbol{r}_4}{\partial \boldsymbol{\xi}} \\ \frac{\partial \boldsymbol{r}_5}{\partial \boldsymbol{\xi}} \\ \frac{\partial \boldsymbol{r}_6}{\partial \boldsymbol{\xi}} \\ \frac{\partial \boldsymbol{r}_7}{\partial \boldsymbol{\xi}} \\ \frac{\partial \boldsymbol{r}_8}{\partial \boldsymbol{\xi}} \\ \frac{\partial \boldsymbol{r}_9}{\partial \boldsymbol{\xi}} \end{bmatrix}$$

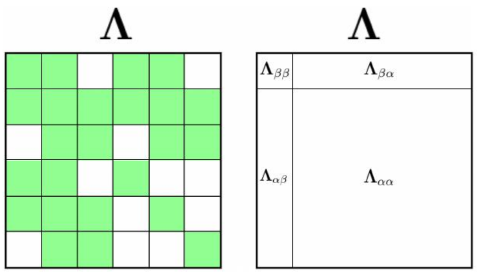
作业1-绘制信息矩阵

$$\boldsymbol{J}_1 = \frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{\xi}} = \left[\begin{array}{cccccc} \frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{\xi}_1} & 0 & 0 & \frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{L}_1} & 0 & 0 \end{array} \right]$$

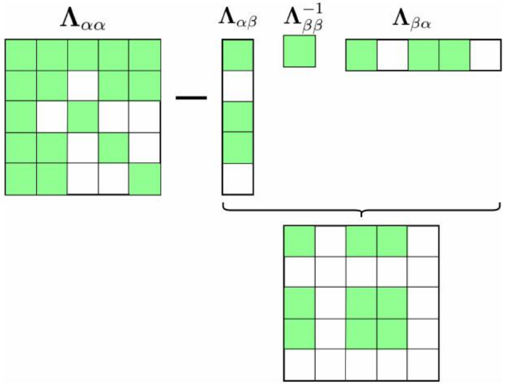
$$\begin{aligned}\boldsymbol{\Lambda}_1 &= \boldsymbol{J}_1^\top \boldsymbol{\Sigma}_1^{-1} \boldsymbol{J}_1 \\ &= \begin{bmatrix} \left(\frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{\xi}_1}\right)^\top \boldsymbol{\Sigma}_1^{-1} \frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{\xi}_1} & 0 & 0 & \left(\frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{\xi}_1}\right)^\top \boldsymbol{\Sigma}_1^{-1} \frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{L}_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \left(\frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{L}_1}\right)^\top \boldsymbol{\Sigma}_1^{-1} \frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{\xi}_1} & 0 & 0 & \left(\frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{L}_1}\right)^\top \boldsymbol{\Sigma}_1^{-1} \frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{L}_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\end{aligned}$$



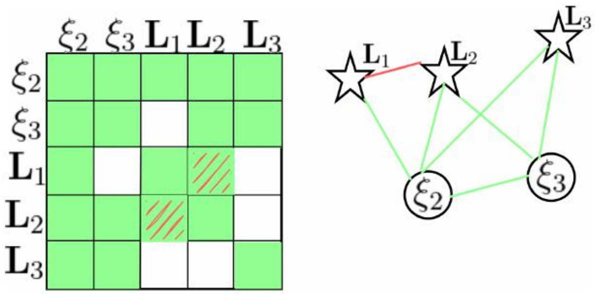
作业1-边缘化后的信息矩阵



（1）



（2）



（3）

作业2-论文阅读

考虑到 SLAM 问题的概率建模，根据贝叶斯法则：

$$\underbrace{p(\boldsymbol{\xi} \mid \boldsymbol{r})}_{\text{Posterior}} = \frac{p(\boldsymbol{r} \mid \boldsymbol{\xi}) p(\boldsymbol{\xi})}{p(\boldsymbol{r})} \propto \underbrace{p(\boldsymbol{r} \mid \boldsymbol{\xi})}_{\text{Likelihood}} \underbrace{p(\boldsymbol{\xi})}_{\text{Prior}} \tag{12}$$

考虑没有先验的 Maximize Likelihood Estimation (MLE)，只有观测函数，且观测属于高斯分布：

$$p(\boldsymbol{r} \mid \boldsymbol{\xi}) = \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \tag{13}$$

对其取 negative logarithm，其实也是对 SLAM 模型取 negative logarithm likelihood。如下所示：

$$-\ln(p(\boldsymbol{r})) = \underbrace{\tfrac{1}{2}((2\pi)^N \det(\Sigma))}_{\text{discarded}} + \frac{1}{2}(\boldsymbol{r} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\boldsymbol{r} - \boldsymbol{\mu}) \tag{14}$$

并选其为 cost function：

$$\boldsymbol{F} = \frac{1}{2}(\boldsymbol{r} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\boldsymbol{r} - \boldsymbol{\mu}) \tag{15}$$

对 cost 进行两次求到 [1]：

$$\boldsymbol{H} = \boldsymbol{\Sigma}^{-1} \tag{16}$$

作业2-论文阅读

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根据 Fisher Information 的定义 [1]，加上 SLAM 的模型（使用高斯分布），根据 log likelihood 二阶求导可得信息矩阵 $\boldsymbol{I}$：

$$
\begin{aligned}
\boldsymbol{I}(\xi) &= -\boldsymbol{E}[\frac{\partial^2}{\partial \xi^2} \ln p(\boldsymbol{r};\boldsymbol{\xi}) \mid \boldsymbol{\xi}] \\
&= \boldsymbol{E}[\frac{\partial^2}{\partial \xi^2}(-\ln p(\boldsymbol{r};\boldsymbol{\xi})) \mid \boldsymbol{\xi}] \\
&= \boldsymbol{H}
\end{aligned}
\tag{17}
$$

根据 Eq. 16 和 Eq.17可得，信息矩阵等于方差的逆。

[1] https://en.wikipedia.org/wiki/Fisher_information

作业3-构建信息矩阵

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本题的节点如图. 3所示，有 10 个相机位姿以及 20 个路标点。

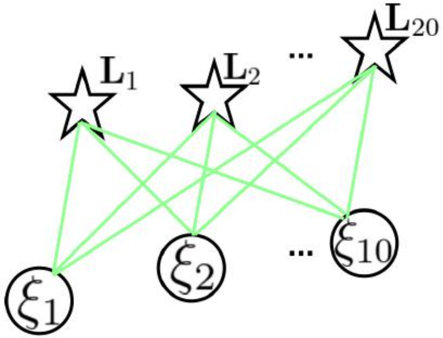

图 3: 节点图

状态向量 $\boldsymbol{\xi}$，残差向量 $\boldsymbol{r}$ 和雅克比向量 $\boldsymbol{J}$ 如下所示：

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\xi}_{1(6\times1)} \\ \vdots \\ \boldsymbol{\xi}_{10} \\ \boldsymbol{L}_{1(3\times1)} \\ \vdots \\ \boldsymbol{L}_{20} \end{bmatrix}_{(120\times1)} \qquad \boldsymbol{r} = \begin{bmatrix} \boldsymbol{r}_{1(2\times1)} \\ \vdots \\ \boldsymbol{r}_{200} \end{bmatrix}_{(400\times1)} = \begin{bmatrix} \boldsymbol{r}_{\xi_1 L_1} \\ \vdots \\ \boldsymbol{r}_{\xi_1 L_{20}} \\ \vdots \\ \boldsymbol{r}_{\xi_{10} L_1} \\ \vdots \\ \boldsymbol{r}_{\xi_{10} L_{20}} \end{bmatrix} \qquad \boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_1 \\ \vdots \\ \boldsymbol{J}_{200} \end{bmatrix} = \begin{bmatrix} \frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{\xi}} \\ \vdots \\ \frac{\partial \boldsymbol{r}_{200}}{\partial \boldsymbol{\xi}} \end{bmatrix}$$

作业3-构建信息矩阵

以第一残差 r_1 为例计算对应的雅克比 $\boldsymbol{J}_1$ 和信息矩阵 $\boldsymbol{\Lambda}_1$，如下所示：

$$\boldsymbol{J}_1 = \frac{\partial \boldsymbol{r}_1}{\partial \xi} = \begin{bmatrix} \underbrace{\frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{\xi}_1}{}_{(2\times 6)} \quad \mathbf{0} \quad \cdots \quad \mathbf{0}}_{10} & \underbrace{\frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{L}_1}{}_{(2\times 3)} \quad \mathbf{0} \quad \cdots \quad \mathbf{0}}_{20} \end{bmatrix} \tag{19}$$

$$\begin{aligned} \boldsymbol{\Lambda}_1 &= \boldsymbol{J}_1^\top \boldsymbol{\Sigma}_1^{-1} \boldsymbol{J}_1 \\ &= \begin{bmatrix} \left(\frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{\xi}_1}\right)^\top \boldsymbol{\Sigma}_1^{-1} \frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{\xi}_1}{}_{(6\times 6)} & \mathbf{0}_{(6\times 54)} & \left(\frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{\xi}_1}\right)^\top \boldsymbol{\Sigma}_1^{-1} \frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{L}_1}{}_{(6\times 3)} & \mathbf{0}_{(6\times 57)} \\ \mathbf{0}_{(54\times 6)} & \mathbf{0}_{(54\times 54)} & \mathbf{0}_{(54\times 3)} & \mathbf{0}_{(54\times 57)} \\ \left(\frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{L}_1}\right)^\top \boldsymbol{\Sigma}_1^{-1} \frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{\xi}_1}{}_{(3\times 6)} & \mathbf{0}_{(3\times 54)} & \left(\frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{L}_1}\right)^\top \boldsymbol{\Sigma}_1^{-1} \frac{\partial \boldsymbol{r}_1}{\partial \boldsymbol{L}_1}{}_{(3\times 3)} & \mathbf{0}_{(3\times 57)} \\ \mathbf{0}_{(57\times 6)} & \mathbf{0}_{(57\times 54)} & \mathbf{0}_{(57\times 3)} & \mathbf{0}_{(57\times 57)} \end{bmatrix} \end{aligned} \tag{20}$$

作业3-求解信息矩阵

一个特征点在坐标系下关系如图. 4, 根据此坐标关系以及相机模型,

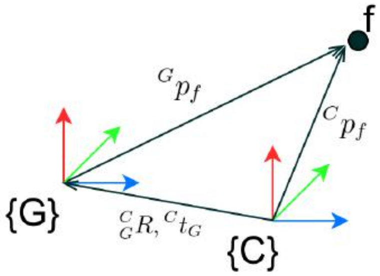

图 4: 坐标图

$$
\frac{1}{z'}\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = K\,{}^{C}p_f = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}
$$

$$
= K\,{}^{C}T_{G}\,{}^{G}p_f
$$

作业3-求解信息矩阵

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残差定义为估计值减去观测值：

$$
\begin{aligned}
\boldsymbol{r} &= \hat{\boldsymbol{z}} - \boldsymbol{z} \\
&= \begin{bmatrix} f_x \frac{x'}{z'} + cx \\ f_y \frac{y'}{z'} + cy \end{bmatrix} - \begin{bmatrix} u \\ v \end{bmatrix}
\end{aligned} \tag{23}
$$

残差对相机位姿（李代数）的导数：

$$
\begin{aligned}
\frac{\partial \boldsymbol{r}}{\partial \delta \boldsymbol{\xi}} &= \frac{\partial \boldsymbol{r}}{\partial {}^C p_f} \frac{\partial {}^C p_f}{\partial \delta \boldsymbol{\xi}} \\
&= \begin{bmatrix} \frac{f_x}{z'} & 0 & \frac{-f_x x'}{z'^2} \\ 0 & \frac{f_y}{z'} & \frac{-f_y y'}{z'^2} \end{bmatrix} \begin{bmatrix} -[{}^C p_f]_\times & \boldsymbol{I}_{(3\times3)} \end{bmatrix} \qquad (\text{Part two: } \delta\boldsymbol{\xi} = [\delta\boldsymbol{\phi}, \delta\boldsymbol{\rho}]^\top, \text{ slambook 4.3.5}) \\
&= \begin{bmatrix} \frac{-f_x x' y'}{z'^2} & f_x + \frac{f_x x'^2}{z'^2} & \frac{-f_x y'}{z'^2} & \frac{f_x}{z'} & 0 & \frac{-f_x x'}{z'^2} \\ -f_y - \frac{f_y y'^2}{z'^2} & \frac{f_y x' y'}{z'2} & \frac{f_y x'}{z'} & 0 & \frac{f_y}{z'} & \frac{-f_y y'}{z'^2} \end{bmatrix}
\end{aligned} \tag{24}
$$

残差对路标点的导数，具体的推到可参考 slam 十四讲第一版 7.7.3（Bundle Adjustment）：

$$
\begin{aligned}
\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{L}} &= \frac{\partial \boldsymbol{r}}{\partial {}^G pf} = \frac{\partial \boldsymbol{r}}{\partial {}^C p_f} \frac{\partial \boldsymbol{r}}{\partial {}^G p_f} \\
&= \begin{bmatrix} \frac{f_x}{z'} & 0 & \frac{-f_x x'}{z'^2} \\ 0 & \frac{f_y}{z'} & \frac{-f_y y'}{z'^2} \end{bmatrix} {}^C R_f
\end{aligned} \tag{25}
$$

将两部分残差带入信息举证即可。

作业3-代码

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```
/// 请补充完整作业信息矩阵块的计算
H.block(i*6, poseNums*6+j*3, 6,3) += jacobian_Ti.transpose() * jacobian_Pj;
H.block(poseNums*6+j*3, i*6, 3, 6) += jacobian_Pj.transpose() * jacobian_Ti;
H.block(poseNums*6+j*3, poseNums*6+j*3, 3, 3) += jacobian_Pj.transpose() * jacobian_Pj;
```

```
 0.00351651
 0.00302963
 0.00253459
 0.00230246
 0.00172459
0.000422374
3.21708e-17
2.06732e-17
1.43188e-17
7.66992e-18
6.08423e-18
6.05715e-18
3.94363e-18
```

在线问答

Q&A

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感谢各位聆听！

Thanks for Listening