
Solutions to Homework 1
for
Subject 101: Introduction to Intermediate

University of Wisconsin-Madison





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TODO LIST

	Include a representative picture here?	3
	Delete this note, or maybe include more detail about alternative method?	5
	Should we cite the Splitting Lemma from the text?	5
	Did this a long time ago, should probably double-check to see if this argument is correct!	7

This is a set of solutions, created by Clark Zinzow, to Homework 1 for Subject 101: Introduction to Intermediate.

NOTATION:

To avoid any possible ambiguity or confusion, we make special note of our use of the following notation:

- Let \mathbb{Z}_+ denote the set of positive integers.
- Given two paths f and g , let $f \sim g$ suggest that f is homotopic to g (and vice versa.)
- Let $\mathbf{1}$ denote the group identity element (unless specified otherwise.)
- For two groups G and H , let $G \cong H$ denote G being isomorphic to H .
- We frequently represent the fact that two groups (spaces) are isomorphic (homeomorphic) by equality.
- For $x \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$, let $x \preceq \gamma$ denote $x_i \leq \gamma$ for $i = 1, \dots, n$, and define \prec , \succ , and \succeq analogously.
- For any set X , let $\text{int}(X)$ denote the interior of the set X .
- For any loop (path) γ , let $\bar{\gamma}$ represent the inverse loop (path). I.e., $\bar{\gamma} = \gamma^{-1}$.

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- Allen Hatcher, whose Algebraic Topology textbook, [1], is awesome.

EXERCISE 1

The mapping torus T_f of a map $f : X \rightarrow X$ is the quotient $X \times I$

$$T_f = \frac{X \times I}{(x, 0) \sim (f(x), 1)}.$$

Let A and B be copies of S^1 , let $X = A \vee B$, and let p be the wedge point of X . Let $f : X \rightarrow X$ be a map that satisfies $f(p) = p$, carries A into A by a degree-3 map, and carries B into B by a degree-5 map.

- (a) Equip T_f with a CW structure by attaching cells to $X \vee S^1$.
- (b) Compute a presentation of $\pi_1(T_f)$.
- (c) Compute $H_1(T_f, K)$.

Proof.

- (a) Let $X = S^1 \vee S^1$. In the typical, explicit formulation of the mapping torus, we would have

$$T_f = \frac{(S^1 \vee S^1) \times I}{(x, 0) \sim (f(x), 1)}.$$

I.e., $S^1 \vee S^1 \times \{0\}$ is identified with $S^1 \vee S^1 \times \{1\}$. Given that f preserves the basepoint p , this identification can be seen as a loop homeomorphic to S^1 attached to $S^1 \vee S^1$ at p , yielding $X \vee S^1 \subset T_f$.

Now, note that $X \vee S^1$ is generated by the loops a, b, c , with c being the loop of the newly wedged copy of S^1 . The mapping torus can be formed by attaching a 2-cell around the path $acf_*(a)\bar{c}$ and attaching a 2-cell around the path $bcf_*(b)\bar{c}$.

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- (b) Given that

$$\pi_1(X \vee S^1) = \pi_1(S^1 \vee S^1 \vee S^1) = \langle a, b, c \rangle,$$

by Proposition 1.26 of [1], we have that

$$\pi_1(T_f) = \langle a, b, c \mid acf_*(a)\bar{c}, bcf_*(b)\bar{c} \rangle.$$

- (c) Let $H_1(T_f)$ denote $H_1(T_f, K)$ in the following argument. Note that we have established that T_f has the following CW structure: one 0-cell (the basepoint wedge point of X , p), three 1-cells $\{a, b, c\}$, and two 2-cells attached by the words $acf_*(a)\bar{c}$ and $bcf_*(b)\bar{c}$. Recall that a cellular chain complex of a CW complex X is given by

$$\cdots \longrightarrow H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \longrightarrow \cdots$$

Moreover, via Lemma 2.34 of [1], we have the following diagram

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & \nearrow & \\
 & & & & H_n(X^{n+1}) \cong H_n(X) & & \\
 & & & \nearrow & & & \\
 0 & & & & & & \\
 & \searrow & & & & & \\
 & & H_n(X^n) & & & & \\
 \nearrow & & \searrow & & & & \\
 \partial_{n+1} & & j_n & & & & \\
 \cdots \longrightarrow H_{n+1}(X^{n+1}, X^n) & \xrightarrow{d_{n+1}} & H_n(X^n, X^{n-1}) & \xrightarrow{d_n} & H_{n-1}(X^{n-1}, X^{n-2}) & \longrightarrow & \cdots \\
 & & \searrow \partial_n & & \nearrow j_{n-1} & & \\
 & & & H_{n-1}(X^{n-1}) & & & \\
 & & \nearrow & & & & \\
 & & 0 & & & &
 \end{array}$$

By this diagram and Theorem 2.35 of [1], we have that

$$H_n(X) \cong \ker d_n / \text{Im } d_{n+1}.$$

This suggests that our particular area of interest in this cellular chain complex is

$$\cdots \longrightarrow H_3(X^3, X^2) \xrightarrow{d_3} H_2(X^2, X^1) \xrightarrow{d_2} H_1(X^1, X^0) \xrightarrow{d_1} H_0(X^0) \xrightarrow{d_0} 0$$

If we are able to compute $\ker d_1$ and $\text{Im } d_2$, we will have $H_1(X)$.

By our construction of a CW complex for T_f , we have that T_f has one 0-cell, three 1-cells, and two 2-cells, and given that $\mathcal{C}_n = H_n(X^n, X^{n-1}) = \mathbb{Z}^{\# \text{ } n\text{-cells}}$, we have that the cellular chain complex of T_f is

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

where $H_3(X^3, X^2) = 0$ because X has an empty k -skeleton for $k > 2$. We clearly have that T_f is connected and only has one 0-cell, which suggests that $d_1 = 0$, hence $\ker d_1 = \mathbb{Z}^3$. Therefore, all that remains is to calculate d_2 .

By the Cellular Boundary Formula,

$$d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}$$

where $d_{\alpha\beta}$ is the degree of the map $S_\alpha^{n-1} \rightarrow X^{n-1} \rightarrow S_\beta^{n-1}$ that is the composition of the attaching map of e_α^n with the quotient map collapsing $X^{n-1} - e_\beta^{n-1}$ to a point. Let e_a^1 , e_b^1 , and e_c^1 denote the 1-cells corresponding to the loop a , b , and c , respectively. Let e_1^2 denote the 2-cell attached around the path $ac\bar{f}_*(a)\bar{c}$ and let e_2^2 denote the 2-cell attached around the path $bc\bar{f}_*(b)\bar{c}$. Then we have that

$$\begin{aligned}
 d_2(e_1^2) &= \sum_\beta d_{1\beta} \cdot e_\beta^1 \\
 &= d_{1a} \cdot e_a^1 + d_{1b} \cdot e_b^1 + d_{1c} \cdot e_c^1 \\
 &= (a - f_*(a)) + (0) + (c - c) \\
 &= a - f_*(a) \\
 &= a - 3 \cdot a \\
 &= -2a
 \end{aligned}$$

by the fact that f carries A into A by a degree-3 map. Proceeding similarly for the other 2-cell, we have that

$$\begin{aligned}
 d_2(e_2^2) &= \sum_{\beta} d_{2\beta} \cdot e_{\beta}^1 \\
 &= d_{2a} \cdot e_a^1 + d_{2b} \cdot e_b^1 + d_{2c} \cdot e_c^1 \\
 &= (0) + (b - f_*(b)) + (c - c) \\
 &= b - f_*(b) \\
 &= b - 5 \cdot b \\
 &= -4b
 \end{aligned}$$

by the fact that f carries B into B by a degree-5 map.

Therefore, we have that

$$d(\mathbf{1}) = \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix}$$

hence

$$\text{Im } d = (-2)\mathbb{Z} \times (-4)\mathbb{Z} \times \{\mathbf{1}\} = 2\mathbb{Z} \times 4\mathbb{Z} \times \{\mathbf{1}\}.$$

Therefore,

$$H_1(T_f) = \mathbb{Z}^3 / 2\mathbb{Z} \times 4\mathbb{Z} \times \{\mathbf{1}\} = \mathbb{Z} / 2\mathbb{Z} \oplus \mathbb{Z} / 4\mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}.$$

Note:

This could also be seen by considering an exact sequence similar to the Mayer-Vietoris sequence, namely

$$\cdots \longrightarrow H_n(X) \xrightarrow{1-f_*} H_n(X) \xrightarrow{\iota_*} H_n(T_f) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$

with $\iota : X \hookrightarrow T_f$ being the inclusion. The derivation of a more general form of this exact sequence is given in Example 2.48 of [1], with this explicit exact sequence given in Exercise 2.2.30 of [1].

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By Corollary 2.25 of [1], we have that

$$H_n(X) = H_n(A \vee B) \cong H_n(A) \oplus H_n(B)$$

for $n > 0$, which, when coupled with the facts that $H_n(S^1) = 0$ for $n > 1$ and $H_2(T_f) = 0$, gives us the exact sequence

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_1(A) \oplus H_1(B) & \xrightarrow{1-f_*} & H_1(A) \oplus H_1(B) & \xrightarrow{\iota_*} & H_1(T_f) \longrightarrow H_0(A \vee B) \xrightarrow{1-f_*} H_0(A \vee B) \longrightarrow 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \downarrow \cong \\
 & & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{1-f_*} & \mathbb{Z} \oplus \mathbb{Z} & & \mathbb{Z} & \xrightarrow{1-f_*} \mathbb{Z}
 \end{array}$$

Since f takes A into A by a degree-3 map and takes B into B by a degree-5 map, we have that

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_1(A) \oplus H_1(B) & \xrightarrow[-4 \text{ on } B]{-2 \text{ on } A} H_1(A) \oplus H_1(B) & \xrightarrow{\iota_*} & H_1(T_f) \longrightarrow H_0(A \vee B) \xrightarrow{0} H_0(A \vee B) \longrightarrow 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \downarrow \cong \\
 & & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow[-4 \text{ on } B]{-2 \text{ on } A} \mathbb{Z} \oplus \mathbb{Z} & & \mathbb{Z} & \xrightarrow{0} \mathbb{Z}
 \end{array}$$

This gives us the exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_4 \longrightarrow H_1(T_f) \longrightarrow \mathbb{Z} \longrightarrow 0$$

Given that \mathbb{Z} is free, we have that this exact sequence splits. Therefore, by the Splitting Lemma, we have that

$$H_1(T_f) = \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}.$$

Should we cite the Splitting Lemma from the text?



EXERCISE 2

Use Mayer-Vietoris to compute the degree of the suspension Σf of a map $f : S^n \rightarrow S^n$.

Proof.

Let

$$A = (S^n \times I)/(S^n \times 1), \quad B = (S^n \times I)/(S^n \times 0).$$

I.e., A is the upper cone of S^{n+1} and B is the lower cone of S^{n+1} . Therefore, $A, B \subset S^{n+1}$, $S^{n+1} = \text{int}(A) \cup \text{int}(B)$, and $S^n = A \cap B$. We therefore have the reduced Mayer-Vietoris sequence

$$\cdots \longrightarrow \tilde{H}_{n+1}(S^n) \xrightarrow{\Phi} \tilde{H}_{n+1}(A) \oplus \tilde{H}_{n+1}(B) \xrightarrow{\Psi} \tilde{H}_{n+1}(S^{n+1}) \xrightarrow{\partial} \tilde{H}_n(S^n) \longrightarrow \cdots$$

Since A and B are both contractible, we have that

$$\tilde{H}_i(A) \oplus \tilde{H}_i(B) \cong 0$$

for $i = 0, \dots, n$. Letting $\partial = \partial_{n+1}$, the above reduced Mayer-Vietoris sequence is equivalent to the exact sequence

$$\cdots \xrightarrow{\Phi_{n+1}} 0 \xrightarrow{\Psi_{n+1}} \tilde{H}_{n+1}(S^{n+1}) \xrightarrow{\partial} \tilde{H}_n(S^n) \xrightarrow{\Phi_n} 0 \xrightarrow{\Psi_n} \cdots$$

Therefore, by (iii) on page 114 of [1], we have that ∂ is an isomorphism hence $\tilde{H}_{n+1}(S^{n+1}) \cong \tilde{H}_n(S^n)$.

Note that f induces a map $Af : (A, S^n) \rightarrow (A, S^n)$ with quotient $\Sigma f : (A, S^n) \rightarrow (A/S^n, S^n/S^n)$. Given that $S^n \subset A$ and there clearly exists a neighborhood U of S^n in A that deformation retracts onto S^n , by Proposition 2.22 of [1] we have that $\tilde{H}_{n+1}(A, S^n) \cong \tilde{H}_{n+1}(A/S^n)$.

We therefore have the following (A, S^n) pair long exact sequence:

$$\cdots \longrightarrow \tilde{H}_{n+1}(S^n) \xrightarrow{\iota_*} \tilde{H}_{n+1}(A) \xrightarrow{(\Sigma f)_*} \tilde{H}_{n+1}(A, S^n) \xrightarrow{\partial} \tilde{H}_n(S^n) \xrightarrow{\iota_*} \tilde{H}_n(A) \longrightarrow \cdots$$

Given that $\tilde{H}_{n+1}(A, S^n) \cong \tilde{H}_{n+1}(A/S^n)$ and by the contractibility of A , this is equivalent to the exact sequence

$$\cdots \xrightarrow{\iota_*} 0 \xrightarrow{(\Sigma f)_*} \tilde{H}_{n+1}(A/S^n) \xrightarrow{\partial} \tilde{H}_n(S^n) \xrightarrow{\iota_*} 0 \longrightarrow \cdots$$

hence $\tilde{H}_{n+1}(A/S^n) \cong \tilde{H}_n(S^n)$ by the boundary map ∂ . Moreover, $A/S^n \cong S^{n+1}$, which suggests that $\tilde{H}_{n+1}(A/S^n) \cong \tilde{H}_{n+1}(S^{n+1})$. By this and the naturality of the boundary maps in the (A, S^n) pair long exact sequence, we have the following commutative diagram:

$$\begin{array}{ccc} \tilde{H}_{n+1}(S^{n+1}) & \xrightarrow[\cong]{\partial} & \tilde{H}_n(S^n) \\ (\Sigma f)_* \downarrow & & \downarrow f_* \\ \tilde{H}_{n+1}(S^{n+1}) & \xrightarrow[\partial]{\cong} & \tilde{H}_n(S^n) \end{array}$$

Hence we have that if f_* is multiplication by d , then $(\Sigma f)_*$ is multiplication by d . Therefore, $\deg \Sigma f = \deg f$.

Did this a long time ago, should probably double-check to see if this argument is correct!

□

REFERENCES

- [1] Allen Hatcher, *Algebraic Topology*, Cambridge University Press, 1st Edition, 2001.