Solutions to Homework 1
for
Subject 101: Introduction to Intermediate

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This is a set of solutions, created by Clark Zinzow, to Homework 1 for Subject 101: Introduction to Intermediate.

Notation:

To avoid any possible ambiguity or confusion, we make special note of our use of the following notation:

- Let $\mathbb{Z}_+$ denote the set of positive integers.
- Given two paths f and g, let $f \sim g$ suggest that f is homotopic to g (and vice versa.)
- Let $\mathbf{1}$ denote the group identity element (unless specified otherwise.)
- For two groups G and H, let $G \cong H$ denote G being isomorphic to H.
- We frequently represent the fact that two groups (spaces) are isomorphic (homeomorphic) by equality.
- For $x \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$, let $x \preceq \gamma$ denote $x_i \leq \gamma$ for $i = 1, \ldots, n$, and define $\prec$, $\succ$, and $\succeq$ analogously.
- For any set X, let $\mathrm{int}(X)$ denote the interior of the set X.
- For any loop (path) γ, let $\overline{\gamma}$ represent the inverse loop (path). I.e., $\overline{\gamma} = \gamma^{-1}$.

Acknowledgments:

Thank you to the following people for helping me, in some way, with these solutions:

- Allen Hatcher, whose Algebraic Topology textbook, [1], is awesome.

Exercise 1

The mapping torus T_f of a map $f : X \to X$ is the quotient $X \times I$

$$T_f = \frac{X \times I}{(x, 0) \sim (f(x), 1)}.$$

Let A and B be copies of S^1, let $X = A \vee B$, and let p be the wedge point of X. Let $f : X \to X$ be a map that satisfies $f(p) = p$, carries A into A by a degree-3 map, and carries B into B by a degree-5 map.

(a) Equip T_f with a CW structure by attaching cells to $X \vee S^1$.

(b) Compute a presentation of $\pi_1(T_f)$.

(c) Compute $H_1(T_f, K)$.

Proof.

(a) Let $X = S^1 \vee S^1$. In the typical, explicit formulation of the mapping torus, we would have

$$T_f = \frac{(S^1 \vee S^1) \times I}{(x, 0) \sim (f(x), 1)}.$$

I.e., $S^1 \vee S^1 \times \{0\}$ is identified with $S^1 \vee S^1 \times \{1\}$. Given that f preserves the basepoint p, this identification can be seen as a loop homeomorphic to S^1 attached to $S^1 \vee S^1$ at p, yielding $X \vee S^1 \subset T_f$.

Now, note that $X \vee S^1$ is generated by the loops a, b, c, with c being the loop of the newly wedged copy of S^1. The mapping torus can be formed by attaching a 2-cell around the path $ac\overline{f}_*(a)\overline{c}$ and attaching a 2-cell around the path $bc\overline{f}_*(b)\overline{c}$.

Include a representative picture here?

(b) Given that

$$\pi_1(X \vee S^1) = \pi_1(S^1 \vee S^1 \vee S^1) = \langle a, b, c \rangle,$$

by Proposition 1.26 of [1], we have that

$$\pi_1(T_f) = \langle a, b, c \mid ac\overline{f}_*(a)\overline{c},\ bc\overline{f}_*(b)\overline{c} \rangle.$$

(c) Let $H_1(T_f)$ denote $H_1(T_f, K)$ in the following argument. Note that we have established that T_f has the following CW structure: one 0-cell (the basepoint wedge point of X, p), three 1-cells $\{a, b, c\}$, and two 2-cells attached by the words $ac\overline{f}_*(a)\overline{c}$ and $bc\overline{f}_*(b)\overline{c}$. Recall that a cellular chain complex of a CW complex X is given by

$$\cdots \longrightarrow H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \longrightarrow \cdots$$

Moreover, via Lemma 2.34 of [1], we have the following diagram

$$\begin{tikzcd}
 & & & & 0 & & \\
 & 0 \arrow[dr] & & H_n(X^{n+1}) \cong H_n(X) \arrow[ur] & & & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"] & & \\
 & & & 0 \arrow[ur] & & &
\end{tikzcd}$$

By this diagram and Theorem 2.35 of [1], we have that

$$H_n(X) \cong \ker d_n / \operatorname{Im} d_{n+1}.$$

This suggests that our particular area of interest in this cellular chain complex is

$$\cdots \longrightarrow H_3(X^3, X^2) \xrightarrow{d_3} H_2(X^2, X^1) \xrightarrow{d_2} H_1(X^1, X^0) \xrightarrow{d_1} H_0(X^0) \xrightarrow{d_0} 0$$

If we are able to compute $\ker d_1$ and $\operatorname{Im} d_2$, we will have $H_1(X)$.

By our construction of a CW complex for T_f, we have that T_f has one 0-cell, three 1-cells, and two 2-cells, and given that $\mathcal{C}_n = H_n(X^n, X^{n-1}) = \mathbb{Z}^{\#\ n\text{-cells}}$, we have that the cellular chain complex of T_f is

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

where $H_3(X^3, X^2) = 0$ because X has an empty k-skeleton for $k > 2$. We clearly have that T_f is connected and only has one 0-cell, which suggests that $d_1 = 0$, hence $\ker d_1 = \mathbb{Z}^3$. Therefore, all that remains is to calculate d_2.

By the Cellular Boundary Formula,

$$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$$

where $d_{\alpha\beta}$ is the degree of the map $S^{n-1}_\alpha \to X^{n-1} \to S^{n-1}_\beta$ that is the composition of the attaching map of e^n_α with the quotient map collapsing $X^{n-1} - e^{n-1}_\beta$ to a point. Let e^1_a, e^1_b, and e^1_c denote the 1-cells corresponding to the loop a, b, and c, respectively. Let e^2_1 denote the 2-cell attached around the path $ac\overline{f}_*(a)\overline{c}$ and let e^2_2 denote the 2-cell attached around the path $bc\overline{f}_*(b)\overline{c}$. Then we have that

$$\begin{aligned}
d_2(e^2_1) &= \sum_\beta d_{1\beta} \cdot e^1_\beta \\
&= d_{1a} \cdot e^1_a + d_{1b} \cdot e^1_b + d_{1c} \cdot e^1_c \\
&= (a - f_*(a)) + (0) + (c - c) \\
&= a - f_*(a) \\
&= a - 3 \cdot a \\
&= -2a
\end{aligned}$$

by the fact that f carries A into A by a degree-3 map. Proceeding similarly for the other 2-cell, we have that

$$\begin{aligned} d_2(e_2^2) &= \sum_\beta d_{2\beta} \cdot e_\beta^1 \\ &= d_{2a} \cdot e_a^1 + d_{2b} \cdot e_b^1 + d_{2c} \cdot e_c^1 \\ &= (0) + (b - f_*(b)) + (c - c) \\ &= b - f_*(b) \\ &= b - 5 \cdot b \\ &= -4b \end{aligned}$$

by the fact that f carries B into B by a degree-5 map.

Therefore, we have that

$$d(\mathbf{1}) = \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix}$$

hence

$$\operatorname{Im} d = (-2)\mathbb{Z} \times (-4)\mathbb{Z} \times \{\mathbf{1}\} = 2\mathbb{Z} \times 4\mathbb{Z} \times \{\mathbf{1}\}.$$

Therefore,

$$H_1(T_f) = \mathbb{Z}^3/2\mathbb{Z} \times 4\mathbb{Z} \times \{\mathbf{1}\} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}.$$

Note:

This could also be seen by considering an exact sequence similar to the Mayer-Vietoris sequence, namely

$$\cdots \longrightarrow H_n(X) \xrightarrow{\mathbf{1}-f_*} H_n(X) \xrightarrow{\iota_*} H_n(T_f) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$

with $\iota : X \hookrightarrow T_f$ being the inclusion. The derivation of a more general form of this exact sequence is given in Example 2.48 of [1], with this explicit exact sequence given in Exercise 2.2.30 of [1].

Delete this note, or maybe include more detail about alternative method?

By Corollary 2.25 of [1], we have that

$$H_n(X) = H_n(A \vee B) \cong H_n(A) \oplus H_n(B)$$

for $n > 0$, which, when coupled with the facts that $H_n(S^1) = 0$ for $n > 1$ and $H_2(T_f) = 0$, gives us the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H_1(A) \oplus H_1(B) \arrow[r, "\mathbf{1}-f_*"] \arrow[d, "\cong"] & H_1(A) \oplus H_1(B) \arrow[r, "\iota_*"] \arrow[d, "\cong"] & H_1(T_f) \arrow[r] & H_0(A \vee B) \arrow[r, "\mathbf{1}-f_*"] \arrow[d, "\cong"] & H_0(A \vee B) \arrow[r] \arrow[d, "\cong"] & 0 \\ & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\mathbf{1}-f_*"] & \mathbb{Z} \oplus \mathbb{Z} & & \mathbb{Z} \arrow[r, "\mathbf{1}-f_*"] & \mathbb{Z} & \end{tikzcd}$$

Since f takes A into A by a degree-3 map and takes B into B by a degree-5 map, we have that

$$\begin{tikzcd} 0 \arrow[r] & H_1(A) \oplus H_1(B) \arrow[r, "-2 \text{ on } A", "-4 \text{ on } B"'] \arrow[d, "\cong"] & H_1(A) \oplus H_1(B) \arrow[r, "\iota_*"] \arrow[d, "\cong"] & H_1(T_f) \arrow[r] & H_0(A \vee B) \arrow[r, "0"] \arrow[d, "\cong"] & H_0(A \vee B) \arrow[r] \arrow[d, "\cong"] & 0 \\ & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "-2 \text{ on } A", "-4 \text{ on } B"'] & \mathbb{Z} \oplus \mathbb{Z} & & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} & \end{tikzcd}$$

This gives us the exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_4 \longrightarrow H_1(T_f) \longrightarrow \mathbb{Z} \longrightarrow 0$$

Given that $\mathbb{Z}$ is free, we have that this exact sequence splits. Therefore, bv the Splitting Lemma, we have that

$$H_1(T_f) = \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}.$$

Should we cite the Splitting Lemma from the text?

$\square$

Exercise 2

Use Mayer-Vietoris to compute the degree of the suspension Σf of a map $f : S^n \to S^n$.

Proof.

Let

$$A = (S^n \times I)/(S^n \times 1), \qquad B = (S^n \times I)/(S^n \times 0).$$

I.e., A is the upper cone of S^{n+1} and B is the lower cone of S^{n+1}. Therefore, $A, B \subset S^{n+1}$, $S^{n+1} = \operatorname{int}(A) \cup \operatorname{int}(B)$, and $S^n = A \cap B$. We therefore have the reduced Mayer-Vietoris sequence

$$\cdots \longrightarrow \widetilde{H}_{n+1}(S^n) \xrightarrow{\Phi} \widetilde{H}_{n+1}(A) \oplus \widetilde{H}_{n+1}(B) \xrightarrow{\Psi} \widetilde{H}_{n+1}(S^{n+1}) \xrightarrow{\partial} \widetilde{H}_n(S^n) \longrightarrow \cdots$$

Since A and B are both contractible, we have that

$$\widetilde{H}_i(A) \oplus \widetilde{H}_i(B) \cong 0$$

for $i = 0, \ldots, n$. Letting $\partial = \partial_{n+1}$, the above reduced Mayer-Vietoris sequence is equivalent to the exact sequence

$$\cdots \xrightarrow{\Phi_{n+1}} 0 \xrightarrow{\Psi_{n+1}} \widetilde{H}_{n+1}(S^{n+1}) \xrightarrow{\partial} \widetilde{H}_n(S^n) \xrightarrow{\Phi_n} 0 \xrightarrow{\Psi_n} \cdots$$

Therefore, by (iii) on page 114 of [1], we have that ∂ is an isomorphism hence $\widetilde{H}_{n+1}(S^{n+1}) \cong \widetilde{H}_n(S^n)$.

Note that f induces a map $Af : (A, S^n) \to (A, S^n)$ with quotient $\Sigma f : (A, S^n) \to (A/S^n, S^n/S^n)$. Given that $S^n \subset A$ and there clearly exists a neighborhood U of S^n in A that deformation retracts onto S^n, by Proposition 2.22 of [1] we have that $\widetilde{H}_{n+1}(A, S^n) \cong \widetilde{H}_{n+1}(A/S^n)$.

We therefore have the following (A, S^n) pair long exact sequence:

$$\cdots \longrightarrow \widetilde{H}_{n+1}(S^n) \xrightarrow{\iota_*} \widetilde{H}_{n+1}(A) \xrightarrow{(\Sigma f)_*} \widetilde{H}_{n+1}(A, S^n) \xrightarrow{\partial} \widetilde{H}_n(S^n) \xrightarrow{\iota_*} \widetilde{H}_n(A) \longrightarrow \cdots$$

Given that $\widetilde{H}_{n+1}(A, S^n) \cong \widetilde{H}_{n+1}(A/S^n)$ and by the contractibility of A, this is equivalent to the exact sequence

$$\cdots \xrightarrow{\iota_*} 0 \xrightarrow{(\Sigma f)_*} \widetilde{H}_{n+1}(A/S^n) \xrightarrow{\partial} \widetilde{H}_n(S^n) \xrightarrow{\iota_*} 0 \longrightarrow \cdots$$

hence $\widetilde{H}_{n+1}(A/S^n) \cong \widetilde{H}_n(S^n)$ by the boundary map ∂. Moreover, $A/S^n \cong S^{n+1}$, which suggests that $\widetilde{H}_{n+1}(A/S^n) \cong \widetilde{H}_{n+1}(S^{n+1})$. By this and the naturality of the boundary maps in the (A, S^n) pair long exact sequence, we have the following commutative diagram:

$$\begin{tikzcd}
\widetilde{H}_{n+1}(S^{n+1}) \arrow[r, "\partial", "\cong"'] \arrow[d, "(\Sigma f)_*"'] & \widetilde{H}_n(S^n) \arrow[d, "f_*"] \\
\widetilde{H}_{n+1}(S^{n+1}) \arrow[r, "\cong", "\partial"'] & \widetilde{H}_n(S^n)
\end{tikzcd}$$

Hence we have that if f_* is multiplication by d, then $(\Sigma f)_*$ is multiplication by d. Therefore, $\deg \Sigma f = \deg f$.

Did this a long time ago, should probably double-check to see if this argument is correct!

$\square$

References

[1] Allen Hatcher, *Algebraic Topology*, Cambridge University Press, 1st Edition, 2001.