GEMP - UFC Quixadá - ICPC Library

Contents

1 Data Structures

1.1 BIT

```cpp
#include <bits/stdc++.h>
using namespace std;
class Bit{
private:
  typedef long long t_bit;
  int nBit;
  int nLog;
  vector<t_bit> bit;
public:
  Bit(int n){
    nBit = n;
    nLog = 20;
    bit.resize(nBit + 1, 0);
  }
  //1-indexed
  t_bit get(int i){
    t_bit s = 0;
    for (; i > 0; i -= (i & -i))
      s += bit[i];
    return s;
  }
  //1-indexed [l, r]
  t_bit get(int l, int r){
    return get(r) - get(l - 1);
  }
  //1-indexed
  void add(int i, t_bit value){
```

```cpp
    assert(i > 0);
    for (; i <= nBit; i += (i & -i))
      bit[i] += value;
  }
  t_bit lower_bound(t_bit value){
    t_bit sum = 0;
    int pos = 0;
    for (int i = nLog; i >= 0; i--){
      if ((pos + (1 << i) <= nBit) and (sum + bit[pos + (1 << i)] <
          value)){
        sum += bit[pos + (1 << i)];
        pos += (1 << i);
      }
    }
    return pos + 1;
  }
};
```

1.2   BIT 2D

```cpp
#include <bits/stdc++.h>
using namespace std;
class Bit2d{
private:
  typedef long long t_bit;
  vector<vector<t_bit>> bit;
  int nBit, mBit;
public:
  Bit2d(int n, int m){
    nBit = n;
    mBit = m;
    bit.resize(nBit + 1, vector<t_bit>(mBit + 1, 0));
  }
  //1-indexed
  t_bit get(int i, int j){
    t_bit sum = 0;
    for (int a = i; a > 0; a -= (a & -a))
      for (int b = j; b > 0; b -= (b & -b))
        sum += bit[a][b];
    return sum;
  }
  //1-indexed
  t_bit get(int a1, int b1, int a2, int b2){
    return get(a2, b2) - get(a2, b1 - 1) - get(a1 - 1, b2) + get(a1 -
        1, b1 - 1);
  }
  //1-indexed [i, j]
  void add(int i, int j, t_bit value){
    for (int a = i; a <= nBit; a += (a & -a))
      for (int b = j; b <= mBit; b += (b & -b))
        bit[a][b] += value;
  }
};
```

1.3   BIT In Range

```cpp
#include <bits/stdc++.h>
using namespace std;
```

```cpp
class BitRange{
private:
  typedef long long t_bit;
  vector<t_bit> bit1, bit2;
  t_bit get(vector<t_bit> &bit, int i){
    t_bit sum = 0;
    for (; i > 0; i -= (i & -i))
      sum += bit[i];
    return sum;
  }
  void add(vector<t_bit> &bit, int i, t_bit value){
    for (; i < (int)bit.size(); i += (i & -i))
      bit[i] += value;
  }
public:
  BitRange(int n){
    bit1.assign(n + 1, 0);
    bit2.assign(n + 1, 0);
  }
  //1-indexed [i, j]
  void add(int i, int j, t_bit v){
    add(bit1, i, v);
    add(bit1, j + 1, -v);
    add(bit2, i, v * (i - 1));
    add(bit2, j + 1, -v * j);
  }
  //1-indexed
  t_bit get(int i){
    return get(bit1, i) * i - get(bit2, i);
  }
  //1-indexed [i,j]
  t_bit get(int i, int j){
    return get(j) - get(i - 1);
  }
};
```

1.4   Dynamic Median

```cpp
#include <bits/stdc++.h>
using namespace std;
class DinamicMedian{
  typedef int t_median;
private:
  priority_queue<t_median> mn;
  priority_queue<t_median, vector<t_median>, greater<t_median>> mx;
public:
  double median(){
    if (mn.size() > mx.size())
      return mn.top();
    else
      return (mn.top() + mx.top()) / 2.0;
  }
  void push(t_median x){
    if (mn.size() <= mx.size())
      mn.push(x);
    else
      mx.push(x);
    if ((!mx.empty()) and (!mn.empty())){
      while (mn.top() > mx.top()){
        t_median a = mx.top();
```

```cpp
          mx.pop();
          t_median b = mn.top();
          mn.pop();
          mx.push(b);
          mn.push(a);
        }
      }
    }
  };
```

1.5   Dynamic Wavelet Tree

```cpp
#include <bits/stdc++.h>
using namespace std;
struct SplayTree{
  struct Node{
    int x, y, s;
    Node *p = 0;
    Node *l = 0;
    Node *r = 0;
    Node(int v){
      x = v;
      y = v;
      s = 1;
    }
    void upd(){
      s = 1;
      y = x;
      if (l){
        y += l->y;
        s += l->s;
      }
      if (r){
        y += r->y;
        s += r->s;
      }
    }
    int left_size(){
      return l ? l->s : 0;
    }
  };
  Node *root = 0;
  void rot(Node *c){
    auto p = c->p;
    auto g = p->p;
    if (g)
      (g->l == p ? g->l : g->r) = c;
    if (p->l == c){
      p->l = c->r;
      c->r = p;
      if (p->l)
        p->l->p = p;
    }
    else{
      p->r = c->l;
      c->l = p;
      if (p->r)
        p->r->p = p;
    }
    p->p = c;
```

```cpp
    c->p = g;
    p->upd();
    c->upd();
  }
  void splay(Node *c){
    while (c->p){
      auto p = c->p;
      auto g = p->p;
      if (g)
        rot((g->l == p) == (p->l == c) ? p : c);
      rot(c);
    }
    c->upd();
    root = c;
  }
  Node *join(Node *l, Node *r){
    if (not l)
      return r;
    if (not r)
      return l;
    while (l->r)
      l = l->r;
    splay(l);
    r->p = l;
    l->r = r;
    l->upd();
    return l;
  }
  pair<Node *, Node *> split(Node *p, int idx){
    if (not p)
      return make_pair(nullptr, nullptr);
    if (idx < 0)
      return make_pair(nullptr, p);
    if (idx >= p->s)
      return make_pair(p, nullptr);
    for (int lf = p->left_size(); idx != lf; lf = p->left_size()){
      if (idx < lf)
        p = p->l;
      else
        p = p->r, idx -= lf + 1;
    }
    splay(p);
    Node *l = p;
    Node *r = p->r;
    if (r){
      l->r = r->p = 0;
      l->upd();
    }
    return make_pair(l, r);
  }
  Node *get(int idx){
    auto p = root;
    for (int lf = p->left_size(); idx != lf; lf = p->left_size()){
      if (idx < lf)
        p = p->l;
      else
        p = p->r, idx -= lf + 1;
    }
    splay(p);
    return p;
  }
}
```

```cpp
int insert(int idx, int x){
  Node *l, *r;
  tie(l, r) = split(root, idx - 1);
  int v = l ? l->y : 0;
  root = join(l, join(new Node(x), r));
  return v;
}
void erase(int idx){
  Node *l, *r;
  tie(l, r) = split(root, idx);
  root = join(l->l, r);
  delete l;
}
int rank(int idx){
  Node *l, *r;
  tie(l, r) = split(root, idx);
  int x = (l && l->l ? l->l->y : 0);
  root = join(l, r);
  return x;
}
int operator[](int idx){
  return rank(idx);
}
~SplayTree(){
  if (!root)
    return;
  vector<Node *> nodes{root};
  while (nodes.size()){
    auto u = nodes.back();
    nodes.pop_back();
    if (u->l)
      nodes.emplace_back(u->l);
    if (u->r)
      nodes.emplace_back(u->r);
    delete u;
  }
}
};
class WaveletTree{
private:
  int lo, hi;
  WaveletTree *l = 0;
  WaveletTree *r = 0;
  SplayTree b;
public:
  WaveletTree(int min_value, int max_value){
    lo = min_value;
    hi = max_value;
    b.insert(0, 0);
  }
  ~WaveletTree(){
    delete l;
    delete r;
  }
  //0-indexed
  void insert(int idx, int x){
    if (lo >= hi)
      return;
    int mid = (lo + hi - 1) / 2;
    if (x <= mid){
      l = l ?: new WaveletTree(lo, mid);
```

```cpp
    l->insert(b.insert(idx, 1), x);
  }else{
    r = r ?: new WaveletTree(mid + 1, hi);
    r->insert(idx - b.insert(idx, 0), x);
  }
}
//0-indexed
void erase(int idx){
  if (lo == hi)
    return;
  auto p = b.get(idx);
  int lf = p->l ? p->l->y : 0;
  int x = p->x;
  b.erase(idx);
  if (x == 1)
    l->erase(lf);
  else
    r->erase(idx - lf);
}
//kth smallest element in range [i, j[
//0-indexed
int kth(int i, int j, int k){
  if (i >= j)
    return 0;
  if (lo == hi)
    return lo;
  int x = b.rank(i);
  int y = b.rank(j);
  if (k <= y - x)
    return l->kth(x, y, k);
  else
    return r->kth(i - x, j - y, k - (y - x));
}
//Amount of numbers in the range [i, j[ Less than or equal to k
//0-indexed
int lte(int i, int j, int k){
  if (i >= j or k < lo)
    return 0;
  if (hi <= k)
    return j - i;
  int x = b.rank(i);
  int y = b.rank(j);
  return l->lte(x, y, k) + r->lte(i - x, j - y, k);
}
//Amount of numbers in the range [i, j[ equal to k
//0-indexed
int count(int i, int j, int k){
  if (i >= j or k < lo or k > hi)
    return 0;
  if (lo == hi)
    return j - i;
  int mid = (lo + hi - 1) / 2;
  int x = b.rank(i);
  int y = b.rank(j);
  if (k <= mid)
    return l->count(x, y, k);
  return r->count(i - x, j - y, k);
}
//0-indexed
int get(int idx){
  return kth(idx, idx + 1, 1);
```

```
      }
    };
```

1.6 Implicit Treap

```cpp
#include <bits/stdc++.h>
using namespace std;
namespace ITreap{
  const int N = 500010;
  typedef long long treap_t;
  treap_t X[N];
  int en = 1, Y[N], sz[N], L[N], R[N], root;

  const treap_t neutral = 0;
  treap_t op_val[N];
  bool rev[N];
  inline treap_t join(treap_t a, treap_t b, treap_t c){
    return a + b + c;
  }
  void calc(int u) { // update node given children info
    sz[u] = sz[L[u]] + 1 + sz[R[u]];
    // code here, no recursion
    op_val[u] = join(op_val[L[u]], X[u], op_val[R[u]]);
  }
  void unlaze(int u) {
    if(!u) return;
    // code here, no recursion
    if (rev[u]){
      if(L[u])
        rev[L[u]] ^= rev[u];
      if(R[u])
        rev[R[u]] ^= rev[u];
      swap(L[u], R[u]);
      rev[u] = false;
    }
  }
  void split(int u, int s, int &l, int &r) { // l gets first s, r gets
        remaining
    unlaze(u);
    if(!u) return (void) (l = r = 0);
    if(sz[L[u]] < s) { split(R[u], s - sz[L[u]] - 1, l, r); R[u] = l;
        l = u; }
    else { split(L[u], s, l, r); L[u] = r; r = u; }
    calc(u);
  }
  int merge(int l, int r) { // els on l <= els on r
    unlaze(l); unlaze(r);
    if(!l || !r) return l + r;
    int u;
    if(Y[l] > Y[r]) { R[l] = merge(R[l], r); u = l; }
    else { L[r] = merge(l, L[r]); u = r; }
    calc(u);
    return u;
  }
  int new_node(treap_t x){
    X[en] = x;
    op_val[en] = x;
    rev[en] = false;
    return en++;
  }
```

```cpp
  int nth(int u, int idx){
    if(!u)
      return 0;
    unlaze(u);
    if(idx <= sz[L[u]])
      return nth(L[u], idx);
    else if(idx == sz[L[u]] + 1)
      return u;
    else
      return nth(R[u], idx - sz[L[u]] - 1);
  }
//Public
  void init(int n=N-1) { // call before using other funcs
    //init position 0
    sz[0] = 0;
    op_val[0] = neutral;
    //init Treap
    root = 0;
    for(int i = en = 1; i <= n; i++) { Y[i] = i; sz[i] = 1; L[i] = R[i
        ] = 0; }
    random_shuffle(Y + 1, Y + n + 1);
  }
  //0-indexed
  void insert(int idx, int val){
    int a, b;
    split(root, idx, a, b);
    root = merge(merge(a, new_node(val)), b);
  }
  //0-indexed
  void erase(int idx){
    int a, b, c, d;
    split(root, idx, a, b);
    split(b, 1, c, d);
    root = merge(a, d);
  }
  //0-indexed
  treap_t nth(int idx){
    int u = nth(root, idx+1);
    return X[u];
  }
  //0-indexed [l, r]
  treap_t query(int l, int r){
    if(l > r) swap(l, r);
    int a, b, c, d;
    split(root, l, a, d);
    split(d, r - l + 1, b, c);
    treap_t ans = op_val[b];
    root = merge(a, merge(b, c));
    return ans;
  }
  //0-indexed [l, r]
  void reverse(int l, int r){
    if (l > r) swap(l, r);
    int a, b, c, d;
    split(root, l, a, d);
    split(d, r - l + 1, b, c);
    if(b)
      rev[b] ^= 1;
    root = merge(a, merge(b, c));
  }
};
```

1.7 LiChao Tree

```cpp
#include <bits/stdc++.h>
using namespace std;
const int INF = 0x3f3f3f3f;
class LiChaoTree{
private:
  typedef int t_line;
  struct Line{
    t_line k, b;
    Line() {}
    Line(t_line k, t_line b) : k(k), b(b) {}
  };
  int n_tree, min_x, max_x;
  vector<Line> li_tree;
  t_line f(Line l, int x){
    return l.k * x + l.b;
  }
  void add(Line nw, int v, int l, int r){
    int m = (l + r) / 2;
    bool lef = f(nw, l) > f(li_tree[v], l);
    bool mid = f(nw, m) > f(li_tree[v], m);
    if (mid)
      swap(li_tree[v], nw);
    if (r - l == 1)
      return;
    else if (lef != mid)
      add(nw, 2 * v, l, m);
    else
      add(nw, 2 * v + 1, m, r);
  }
  int get(int x, int v, int l, int r){
    int m = (l + r) / 2;
    if (r - l == 1)
      return f(li_tree[v], x);
    else if (x < m)
      return max(f(li_tree[v], x), get(x, 2 * v, l, m));
    else
      return max(f(li_tree[v], x), get(x, 2 * v + 1, m, r));
  }
public:
  LiChaoTree(int mn_x, int mx_x){
    min_x = mn_x;
    max_x = mx_x;
    n_tree = max_x - min_x + 5;
    li_tree.resize(4 * n_tree, Line(0, -INF));
  }
  void add(t_line k, t_line b){
    add(Line(k, b), 1, min_x, max_x);
  }
  t_line get(int x){
    return get(x, 1, min_x, max_x);
  }
};
```

1.8 Policy Based Tree

```cpp
#include <bits/stdc++.h>
```

```cpp
#include <ext/pb_ds/tree_policy.hpp>
#include <ext/pb_ds/assoc_container.hpp>
using namespace __gnu_pbds;
using namespace std;
typedef tree<int, null_type, less<int>, rb_tree_tag,
  tree_order_statistics_node_update> OrderedSet;
typedef tree<int, int, less<int>, rb_tree_tag,
  tree_order_statistics_node_update> OrderedMap;
//order_of_key (k) : Number of items strictly smaller than k .
//find_by_order(k) : K-th element in a set (counting from zero).
```

1.9 Queue Query

```cpp
#include <bits/stdc++.h>
using namespace std;
class QueueQuery{
private:
  typedef long long t_queue;
  stack<pair<t_queue, t_queue>> s1, s2;
  t_queue cmp(t_queue a, t_queue b){
    return min(a, b);
  }
  void move(){
    if (s2.empty()){
      while (!s1.empty()){
        t_queue element = s1.top().first;
        s1.pop();
        t_queue result = s2.empty() ? element : cmp(element, s2.top().
            second);
        s2.push({element, result});
      }
    }
  }
public:
  void push(t_queue x){
    t_queue result = s1.empty() ? x : cmp(x, s1.top().second);
    s1.push({x, result});
  }
  void pop(){
    move();
    s2.pop();
  }
  t_queue front(){
    move();
    return s2.top().first;
  }
  t_queue query(){
    if (s1.empty() || s2.empty())
      return s1.empty() ? s2.top().second : s1.top().second;
    else
      return cmp(s1.top().second, s2.top().second);
  }
  t_queue size(){
    return s1.size() + s2.size();
  }
};
```

1.10 Randomized Heap

```cpp
#include <bits/stdc++.h>
using namespace std;

typedef int f_type;
struct Node{
  f_type value;
  Node *l, *r;
  Node(f_type x = 0): value(x){
    l = r = nullptr;
  }
};
inline bool heapMin(f_type a, f_type b){
  return a > b;
}
inline bool heapMax(f_type a, f_type b){
  return a < b;
}
struct RandomizedHeap{
  Node *root;
  int sz;
  RandomizedHeap(){
    srand(time(NULL));
    root = nullptr;
    sz = 0;
  }
  void rdFree(Node *n){
    if(n == nullptr) return;
    rdFree(n->l); rdFree(n->r);
    delete n;
  }
  ~RandomizedHeap(){
    rdFree(root);
  }
  Node* merge(Node *t1, Node *t2) {
    if(!t1 || !t2)
      return t1 ? t1 : t2;
    if(heapMin(t1->value, t2->value))
      swap(t1, t2);
    if(rand() & 1)
      swap(t1->l, t1->r);
    t1->l = merge(t1->l, t2);
    return t1;
  }
  //Can be performed in O(logn) on average.
  void merge(RandomizedHeap &oth){
    root = merge(root, oth.root);
    sz += oth.sz;
    oth.root = nullptr;
  }
  int top(){
    return (root != nullptr) ? root->value : 0;
  }
  void pop(){
    if(root == nullptr) return;
    Node *l = root->l;
    Node *r = root->r;
    delete root;
    root = merge(l, r);
```

```cpp
    sz--;
  }
  void push(int x){
    Node *nw = new Node(x);
    root = merge(root, nw);
    sz++;
  }
  int size(){
    return sz;
  }
};
```

1.11 Range Color

```cpp
#include <bits/stdc++.h>
using namespace std;
class RangeColor{
private:
  typedef long long ll;
  struct Node{
    ll l, r;
    int color;
    Node() {}
    Node(ll l, ll r, int color) : l(l), r(r), color(color) {}
  };
  struct cmp{
    bool operator()(Node a, Node b){
      return a.r < b.r;
    }
  };
  std::set<Node, cmp> st;
  vector<ll> ans;
public:
  RangeColor(ll first, ll last, int maxColor){
    ans.resize(maxColor + 1);
    ans[0] = last - first + 1LL;
    st.insert(Node(first, last, 0));
  }
  //get color in position x
  int get(ll x){
    auto p = st.upper_bound(Node(0, x - 1LL, -1));
    return p->color;
  }
  //set newColor in [a, b]
  void set(ll a, ll b, int newColor){
    auto p = st.upper_bound(Node(0, a - 1LL, -1));
    assert(p != st.end());
    ll l = p->l;
    ll r = p->r;
    int oldColor = p->color;
    ans[oldColor] -= (r - l + 1LL);
    p = st.erase(p);
    if (l < a){
      ans[oldColor] += (a - l);
      st.insert(Node(l, a - 1LL, oldColor));
    }
    if (b < r){
      ans[oldColor] += (r - b);
      st.insert(Node(b + 1LL, r, oldColor));
    }
```

```cpp
    while ((p != st.end()) and (p->l <= b)){
      l = p->l;
      r = p->r;
      oldColor = p->color;
      ans[oldColor] -= (r - l + 1LL);
      if (b < r){
        ans[oldColor] += (r - b);
        st.erase(p);
        st.insert(Node(b + 1LL, r, oldColor));
        break;
      }else{
        p = st.erase(p);
      }
    }
    ans[newColor] += (b - a + 1LL);
    st.insert(Node(a, b, newColor));
  }
  ll countColor(int x){
    return ans[x];
  }
};
```

1.12 Segment Tree

```cpp
#include <bits/stdc++.h>
using namespace std;
class SegTree{
private:
  typedef long long Node;
  Node neutral = 0;
  vector<Node> st;
  vector<int> v;
  int n;
  Node join(Node a, Node b){
    return (a + b);
  }
  void build(int node, int i, int j){
    if (i == j){
      st[node] = v[i];
      return;
    }
    int m = (i + j) / 2;
    int l = (node << 1);
    int r = l + 1;
    build(l, i, m);
    build(r, m + 1, j);
    st[node] = join(st[l], st[r]);
  }
  Node query(int node, int i, int j, int a, int b){
    if ((i > b) or (j < a))
      return neutral;
    if ((a <= i) and (j <= b))
      return st[node];
    int m = (i + j) / 2;
    int l = (node << 1);
    int r = l + 1;
    return join(query(l, i, m, a, b), query(r, m + 1, j, a, b));
  }
  void update(int node, int i, int j, int idx, Node value){
    if (i == j){
```

```cpp
      st[node] = value;
      return;
    }
    int m = (i + j) / 2;
    int l = (node << 1);
    int r = l + 1;
    if (idx <= m)
      update(l, i, m, idx, value);
    else
      update(r, m + 1, j, idx, value);
    st[node] = join(st[l], st[r]);
  }
public:
  template <class MyIterator>
  SegTree(MyIterator begin, MyIterator end){
    n = end - begin;
    v = vector<int>(begin, end);
    st.resize(4 * n + 5);
    build(1, 0, n - 1);
  }
  //0-indexed [a, b]
  Node query(int a, int b){
    return query(1, 0, n - 1, a, b);
  }
  //0-indexed
  void update(int idx, int value){
    update(1, 0, n - 1, idx, value);
  }
};
```

1.13 Segment Tree 2D

```cpp
#include <bits/stdc++.h>
using namespace std;
struct SegTree2D{
private:
  int n, m;
  typedef int Node;
  Node neutral = -0x3f3f3f3f;
  vector<vector<Node>> seg;
  Node join(Node a, Node b){
    return max(a, b);
  }
public:
  SegTree2D(int n1, int m1){
    n = n1, m = m1;
    seg.assign(2 * n, vector<Node>(2 * m, 0));
  }
  void update(int x, int y, int val){
    assert(0 <= x && x < n && 0 <= y && y < m);
    x += n, y += m;
    seg[x][y] = val;
    for (int j = y / 2; j > 0; j /= 2)
      seg[x][j] = join(seg[x][2 * j], seg[x][2 * j + 1]);
    for (x /= 2; x > 0; x /= 2){
      seg[x][y] = join(seg[2 * x][y], seg[2 * x + 1][y]);
      for (int j = y / 2; j > 0; j /= 2){
        seg[x][j] = join(seg[x][2 * j], seg[x][2 * j + 1]);
      }
    }
  }
```

```cpp
  }
  vector<int> getCover(int l, int r, int N){
    l = std::max(0, l);
    r = std::min(N, r);
    vector<int> ans;
    for (l += N, r += N; l < r; l /= 2, r /= 2){
      if (l & 1)
        ans.push_back(l++);
      if (r & 1)
        ans.push_back(--r);
    }
    return ans;
  }
  Node query(int x1, int y1, int x2, int y2){
    auto c1 = getCover(x1, x2 + 1, n);
    auto c2 = getCover(y1, y2 + 1, m);
    Node ans = neutral;
    for (auto i : c1){
      for (auto j : c2){
        ans = join(ans, seg[i][j]);
      }
    }
    return ans;
  }
};
```

1.14   Segment Tree Iterative

```cpp
#include <bits/stdc++.h>
using namespace std;
class SegTreeIterative{
private:
  typedef long long Node;
  Node neutral = 0;
  vector<Node> st;
  int n;
  inline Node join(Node a, Node b){
    return a + b;
  }
public:
  template <class MyIterator>
  SegTreeIterative(MyIterator begin, MyIterator end){
    int sz = end - begin;
    for (n = 1; n < sz; n <<= 1);
    st.assign(n << 1, neutral);
    for (int i = 0; i < sz; i++, begin++)
      st[i + n] = (*begin);
    for (int i = n - 1; i > 1; i--){
      st[i] = join(st[(i << 1)], st[(i << 1) + 1]);
    }
  }
  //0-indexed
  void update(int i, Node x){
    st[i += n] = x;
    for (i >>= 1; i; i >>= 1)
      st[i] = join(st[i << 1], st[(i << 1) + 1]);
  }
  //0-indexed [l, r]
  Node query(int l, int r){
    Node ansL = neutral, ansR = neutral;
```

```cpp
    for (l += n, r += n + 1; l < r; l >>= 1, r >>= 1){
      if (l & 1)
        ansL = join(ansL, st[l++]);
      if (r & 1)
        ansR = join(st[--r], ansR);
    }
    return join(ansL, ansR);
  }
  Node lower_bound(int k){
    int no=1, l=0, r=n-1;
    while(l<r){
      int mid = (l+r)>>1;
      int lo = no<<1;
      if(st[lo] >= k){
        no = lo;
        r = mid;
      }else{
        k -= st[lo];
        no = lo + 1;
        l = mid + 1;
      }
    }
    if(st[no] >= k)
      return l;
    else
      return -1;
  }
};
```

1.15   Segment Tree Lazy

```cpp
#include <bits/stdc++.h>
using namespace std;
class SegTreeLazy{
private:
  typedef long long Node;
  vector<Node> st;
  vector<long long> lazy;
  vector<int> v;
  int n;
  Node neutral = 0;
  inline Node join(Node a, Node b){
    return a + b;
  }
  inline void upLazy(int &node, int &i, int &j){
    if (lazy[node] != 0){
      st[node] += lazy[node] * (j - i + 1);
      //st[node] += lazy[node];
      if (i != j){
        lazy[(node << 1)] += lazy[node];
        lazy[(node << 1) + 1] += lazy[node];
      }
      lazy[node] = 0;
    }
  }
  void build(int node, int i, int j){
    if (i == j){
      st[node] = v[i];
      return;
    }
```

```cpp
      int m = (i + j) / 2;
      int l = (node << 1);
      int r = l + 1;
      build(l, i, m);
      build(r, m + 1, j);
      st[node] = join(st[l], st[r]);
    }
    Node query(int node, int i, int j, int a, int b){
      upLazy(node, i, j);
      if ((i > b) or (j < a))
        return neutral;
      if ((a <= i) and (j <= b)){
        return st[node];
      }
      int m = (i + j) / 2;
      int l = (node << 1);
      int r = l + 1;
      return join(query(l, i, m, a, b), query(r, m + 1, j, a, b));
    }
    void update(int node, int i, int j, int a, int b, int value){
      upLazy(node, i, j);
      if ((i > j) or (i > b) or (j < a))
        return;
      if ((a <= i) and (j <= b)){
        lazy[node] = value;
        upLazy(node, i, j);
      }else{
        int m = (i + j) / 2;
        int l = (node << 1);
        int r = l + 1;
        update(l, i, m, a, b, value);
        update(r, m + 1, j, a, b, value);
        st[node] = join(st[l], st[r]);
      }
    }
  public:
    template <class MyIterator>
    SegTreeLazy(MyIterator begin, MyIterator end){
      n = end - begin;
      v = vector<int>(begin, end);
      st.resize(4 * n + 5);
      lazy.assign(4 * n + 5, 0);
      build(1, 0, n - 1);
    }
    //0-indexed [a, b]
    Node query(int a, int b){
      return query(1, 0, n - 1, a, b);
    }
    //0-indexed [a, b]
    void update(int a, int b, int value){
      update(1, 0, n - 1, a, b, value);
    }
};
```

1.16 Segment Tree Persistent

```cpp
#include <bits/stdc++.h>
using namespace std;
const int MAX = 3e4 + 10, UPD = 2e5 + 10, LOG = 20;
const int MAXS = 4 * MAX + UPD * LOG;
```

```cpp
namespace PerSegTree{
  typedef long long pst_t;
  pst_t seg[MAXS];
  int T[UPD], L[MAXS], R[MAXS], cnt, t;
  int n, *v;
  pst_t neutral = 0;
  pst_t join(pst_t a, pst_t b){
    return a + b;
  }
  pst_t build(int p, int l, int r){
    if (l == r)
      return seg[p] = v[l];
    L[p] = cnt++, R[p] = cnt++;
    int m = (l + r) / 2;
    return seg[p] = join(build(L[p], l, m), build(R[p], m + 1, r));
  }
  pst_t query(int a, int b, int p, int l, int r){
    if (b < l or r < a)
      return 0;
    if (a <= l and r <= b)
      return seg[p];
    int m = (l + r) / 2;
    return join(query(a, b, L[p], l, m), query(a, b, R[p], m + 1, r));
  }
  pst_t update(int a, int x, int lp, int p, int l, int r){
    if (l == r)
      return seg[p] = x;
    int m = (l + r) / 2;
    if (a <= m)
      return seg[p] = join(update(a, x, L[lp], L[p] = cnt++, l, m),
            seg[R[p] = R[lp]]);
    return seg[p] = join(seg[L[p] = L[lp]], update(a, x, R[lp], R[p] =
          cnt++, m + 1, r));
  }
//Public:
  //O(n)
  void build(int n2, int *v2){
    n = n2, v = v2;
    T[0] = cnt++;
    build(0, 0, n - 1);
  }
  //O(log(n))
  pst_t query(int a, int b, int tt){
    return query(a, b, T[tt], 0, n - 1);
  }
  //O(log(n))
  //update: v[idx] = x;
  int update(int idx, int x, int tt = t){
    update(idx, x, T[tt], T[++t] = cnt++, 0, n - 1);
    return t;
  }
}; // namespace perseg
```

1.17 Sparse Table

```cpp
#include <bits/stdc++.h>
using namespace std;
class SparseTable{
private:
  typedef int t_st;
```

```cpp
    vector<vector<t_st>> st;
    vector<int> log2;
    t_st neutral = 0x3f3f3f3f;
    int nLog;
    t_st join(t_st a, t_st b){
        return min(a, b);
    }
public:
    template <class MyIterator>
    SparseTable(MyIterator begin, MyIterator end){
        int n = end - begin;
        nLog = 20;
        log2.resize(n + 1);
        log2[1] = 0;
        for (int i = 2; i <= n; i++)
            log2[i] = log2[i / 2] + 1;
        st.resize(n, vector<t_st>(nLog, neutral));
        for (int i = 0; i < n; i++, begin++)
            st[i][0] = (*begin);
        for (int j = 1; j < nLog; j++)
            for (int i = 0; (i + (1 << (j - 1))) < n; i++)
                st[i][j] = join(st[i][j - 1], st[i + (1 << (j - 1))][j - 1]);
    }
    //0-indexed [a, b]
    t_st query(int a, int b){
        int d = b - a + 1;
        t_st ans = neutral;
        for (int j = nLog - 1; j >= 0; j--){
            if (d & (1 << j)){
                ans = join(ans, st[a][j]);
                a = a + (1 << (j));
            }
        }
        return ans;
    }
    //0-indexed [a, b]
    t_st queryRMQ(int a, int b){
        int j = log2[b - a + 1];
        return join(st[a][j], st[b - (1 << j) + 1][j]);
    }
};
```

1.18   SQRT Decomposition

```cpp
#include <bits/stdc++.h>
using namespace std;
struct SqrtDecomposition{
    typedef long long t_sqrt;
    int sqrtLen;
    vector<t_sqrt> block;
    vector<t_sqrt> v;
    template <class MyIterator>
    SqrtDecomposition(MyIterator begin, MyIterator end){
        int n = end - begin;
        sqrtLen = (int)sqrt(n + .0) + 1;
        v.resize(n);
        block.resize(sqrtLen + 5);
        for (int i = 0; i < n; i++, begin++){
            v[i] = (*begin);
            block[i / sqrtLen] += v[i];
```

```cpp
    }
}
//0-indexed
void update(int idx, t_sqrt new_value){
    t_sqrt d = new_value - v[idx];
    v[idx] += d;
    block[idx / sqrtLen] += d;
}
//0-indexed [l, r]
t_sqrt query(int l, int r){
    t_sqrt sum = 0;
    int c_l = l / sqrtLen, c_r = r / sqrtLen;
    if (c_l == c_r){
        for (int i = l; i <= r; i++)
            sum += v[i];
    }else{
        for (int i = l, end = (c_l + 1) * sqrtLen - 1; i <= end; i++)
            sum += v[i];
        for (int i = c_l + 1; i <= c_r - 1; i++)
            sum += block[i];
        for (int i = c_r * sqrtLen; i <= r; i++)
            sum += v[i];
    }
    return sum;
}
};
```

1.19   SQRT Tree

```cpp
#include <bits/stdc++.h>
using namespace std;
class SqrtTree{
private:
    typedef long long t_sqrt;
    t_sqrt op(const t_sqrt &a, const t_sqrt &b){
        return a | b;
    }
    inline int log2Up(int n){
        int res = 0;
        while ((1 << res) < n)
            res++;
        return res;
    }
    int n, lg, indexSz;
    vector<t_sqrt> v;
    vector<int> clz, layers, onLayer;
    vector<vector<t_sqrt>> pref, suf, between;
    inline void buildBlock(int layer, int l, int r){
        pref[layer][l] = v[l];
        for (int i = l + 1; i < r; i++)
            pref[layer][i] = op(pref[layer][i - 1], v[i]);
        suf[layer][r - 1] = v[r - 1];
        for (int i = r - 2; i >= l; i--)
            suf[layer][i] = op(v[i], suf[layer][i + 1]);
    }
    inline void buildBetween(int layer, int lBound, int rBound, int
        betweenOffs){
        int bSzLog = (layers[layer] + 1) >> 1;
        int bCntLog = layers[layer] >> 1;
        int bSz = 1 << bSzLog;
```

```cpp
    int bCnt = (rBound - lBound + bSz - 1) >> bSzLog;
    for (int i = 0; i < bCnt; i++){
      t_sqrt ans;
      for (int j = i; j < bCnt; j++){
        t_sqrt add = suf[layer][lBound + (j << bSzLog)];
        ans = (i == j) ? add : op(ans, add);
        between[layer - 1][betweenOffs + lBound + (i << bCntLog) + j]
            = ans;
      }
    }
  }
  inline void buildBetweenZero(){
    int bSzLog = (lg + 1) >> 1;
    for (int i = 0; i < indexSz; i++){
      v[n + i] = suf[0][i << bSzLog];
    }
    build(1, n, n + indexSz, (1 << lg) - n);
  }
  inline void updateBetweenZero(int bid){
    int bSzLog = (lg + 1) >> 1;
    v[n + bid] = suf[0][bid << bSzLog];
    update(1, n, n + indexSz, (1 << lg) - n, n + bid);
  }
  void build(int layer, int lBound, int rBound, int betweenOffs){
    if (layer >= (int)layers.size())
      return;
    int bSz = 1 << ((layers[layer] + 1) >> 1);
    for (int l = lBound; l < rBound; l += bSz){
      int r = min(l + bSz, rBound);
      buildBlock(layer, l, r);
      build(layer + 1, l, r, betweenOffs);
    }
    if (layer == 0)
      buildBetweenZero();
    else
      buildBetween(layer, lBound, rBound, betweenOffs);
  }
  void update(int layer, int lBound, int rBound, int betweenOffs, int
      x){
    if (layer >= (int)layers.size())
      return;
    int bSzLog = (layers[layer] + 1) >> 1;
    int bSz = 1 << bSzLog;
    int blockIdx = (x - lBound) >> bSzLog;
    int l = lBound + (blockIdx << bSzLog);
    int r = min(l + bSz, rBound);
    buildBlock(layer, l, r);
    if (layer == 0)
      updateBetweenZero(blockIdx);
    else
      buildBetween(layer, lBound, rBound, betweenOffs);
    update(layer + 1, l, r, betweenOffs, x);
  }
  inline t_sqrt query(int l, int r, int betweenOffs, int base){
    if (l == r)
      return v[l];
    if (l + 1 == r)
      return op(v[l], v[r]);
    int layer = onLayer[clz[(l - base) ^ (r - base)]];
    int bSzLog = (layers[layer] + 1) >> 1;
    int bCntLog = layers[layer] >> 1;
```

```cpp
    int lBound = (((l - base) >> layers[layer]) << layers[layer]) +
        base;
    int lBlock = ((l - lBound) >> bSzLog) + 1;
    int rBlock = ((r - lBound) >> bSzLog) - 1;
    t_sqrt ans = suf[layer][l];
    if (lBlock <= rBlock){
      t_sqrt add;
      if (layer == 0)
        add = query(n + lBlock, n + rBlock, (1 << lg) - n, n);
      else
        add = between[layer - 1][betweenOffs + lBound + (lBlock <<
            bCntLog) + rBlock];
      ans = op(ans, add);
    }
    ans = op(ans, pref[layer][r]);
    return ans;
  }
public:
  template <class MyIterator>
  SqrtTree(MyIterator begin, MyIterator end){
    n = end - begin;
    v.resize(n);
    for (int i = 0; i < n; i++, begin++)
      v[i] = (*begin);
    lg = log2Up(n);
    clz.resize(1 << lg);
    onLayer.resize(lg + 1);
    clz[0] = 0;
    for (int i = 1; i < (int)clz.size(); i++)
      clz[i] = clz[i >> 1] + 1;
    int tlg = lg;
    while (tlg > 1){
      onLayer[tlg] = (int)layers.size();
      layers.push_back(tlg);
      tlg = (tlg + 1) >> 1;
    }
    for (int i = lg - 1; i >= 0; i--)
      onLayer[i] = max(onLayer[i], onLayer[i + 1]);
    int betweenLayers = max(0, (int)layers.size() - 1);
    int bSzLog = (lg + 1) >> 1;
    int bSz = 1 << bSzLog;
    indexSz = (n + bSz - 1) >> bSzLog;
    v.resize(n + indexSz);
    pref.assign(layers.size(), vector<t_sqrt>(n + indexSz));
    suf.assign(layers.size(), vector<t_sqrt>(n + indexSz));
    between.assign(betweenLayers, vector<t_sqrt>((1 << lg) + bSz));
    build(0, 0, n, 0);
  }
  //0-indexed
  inline void update(int x, const t_sqrt &item){
    v[x] = item;
    update(0, 0, n, 0, x);
  }
  //0-indexed [l, r]
  inline t_sqrt query(int l, int r){
    return query(l, r, 0, 0);
  }
};
```

1.20 Stack Query

```cpp
#include <bits/stdc++.h>
using namespace std;
struct StackQuery{
  typedef int t_stack;
  stack<pair<t_stack, t_stack>> st;
  t_stack cmp(t_stack a, t_stack b){
    return min(a, b);
  }
  void push(t_stack x){
    t_stack new_value = st.empty() ? x : cmp(x, st.top().second);
    st.push({x, new_value});
  }
  void pop(){
    st.pop();
  }
  t_stack top(){
    return st.top().first;
  }
  t_stack query(){
    return st.top().second;
  }
  t_stack size(){
    return st.size();
  }
};
```

1.21 Treap

```cpp
#include <bits/stdc++.h>
using namespace std;
namespace Treap{
  const int N = 500010;
  typedef long long treap_t;
  treap_t X[N];
  int en = 1, Y[N], sz[N], L[N], R[N], root;

  const treap_t neutral = 0;
  treap_t op_val[N];
  inline treap_t join(treap_t a, treap_t b, treap_t c){
    return a + b + c;
  }
  void calc(int u) { // update node given children info
    sz[u] = sz[L[u]] + 1 + sz[R[u]];
    // code here, no recursion
    op_val[u] = join(op_val[L[u]], X[u], op_val[R[u]]);
  }
  void unlaze(int u) {
    if(!u) return;
    // code here, no recursion
  }
  void split(int u, treap_t x, int &l, int &r) { // l gets <= x, r
      gets > x
    unlaze(u);
    if(!u) return (void) (l = r = 0);
    if(X[u] <= x) { split(R[u], x, l, r); R[u] = l; l = u; }
    else { split(L[u], x, l, r); L[u] = r; r = u; }
```
```cpp
    calc(u);
  }
  void split_sz(int u, int s, int &l, int &r) { // l gets first s, r
      gets remaining
    unlaze(u);
    if(!u) return (void) (l = r = 0);
    if(sz[L[u]] < s) { split_sz(R[u], s - sz[L[u]] - 1, l, r); R[u] =
        l; l = u; }
    else { split_sz(L[u], s, l, r); L[u] = r; r = u; }
    calc(u);
  }
  int merge(int l, int r) { // els on l <= els on r
    unlaze(l); unlaze(r);
    if(!l || !r) return l + r;
    int u;
    if(Y[l] > Y[r]) { R[l] = merge(R[l], r); u = l; }
    else { L[r] = merge(l, L[r]); u = r; }
    calc(u);
    return u;
  }
  int new_node(treap_t x){
    X[en] = x;
    op_val[en] = x;
    return en++;
  }
  int nth(int u, int idx){
    if(!u)
      return 0;
    unlaze(u);
    if(idx <= sz[L[u]])
      return nth(L[u], idx);
    else if(idx == sz[L[u]] + 1)
      return u;
    else
      return nth(R[u], idx - sz[L[u]] - 1);
  }
//Public
  void init(int n=N-1) { // call before using other funcs
    //init position 0
    sz[0] = 0;
    op_val[0] = neutral;
    //init Treap
    root = 0;
    for(int i = en = 1; i <= n; i++) { Y[i] = i; sz[i] = 1; L[i] = R[i
        ] = 0; }
    random_shuffle(Y + 1, Y + n + 1);
  }
  void insert(treap_t x){
    int a, b;
    split(root, x, a, b);
    root = merge(merge(a, new_node(x)), b);
  }
  void erase(treap_t x){
    int a, b, c, d;
    split(root, x-1, a, b);
    split(b, x, c, d);
    split_sz(c, 1, b, c);
    root = merge(a, merge(c, d));
  }
  int count(treap_t x){
    int a, b, c, d;
```

```
      split(root, x-1, a, b);
      split(b, x, c, d);
      int ans = sz[c];
      root = merge(a, merge(c, d));
      return ans;
    }
    int size(){ return sz[root];}
    //0-indexed
    treap_t nth(int idx){
      int u = nth(root, idx + 1);
      return X[u];
    }
    //Query in k smallest elements
    treap_t query(int k){
      int a, b;
      split_sz(root, k, a, b);
      treap_t ans = op_val[a];
      root = merge(a, b);
      return ans;
    }
};
```

1.22   Union Find

```
#include <bits/stdc++.h>
using namespace std;
class UnionFind{
private:
  vector<int> p, w, sz;
public:
  UnionFind(int n){
    w.resize(n + 1, 1);
    sz.resize(n + 1, 1);
    p.resize(n + 1);
    for (int i = 0; i <= n; i++)
      p[i] = i;
  }
  int find(int x){
    if (p[x] == x)
      return x;
    return p[x] = find(p[x]);
  }
  bool join(int x, int y){
    x = find(x);
    y = find(y);
    if (x == y)
      return false;
    if (w[x] > w[y])
      swap(x, y);
    p[x] = y;
    sz[y] += sz[x];
    if (w[x] == w[y])
      w[y]++;
    return true;
  }
  bool isSame(int x, int y){
    return find(x) == find(y);
  }
  int size(int x){
    return sz[find(x)];
  }
```

```
  }
};
```

1.23   Wavelet Tree

```
#include <bits/stdc++.h>
using namespace std;
namespace WaveletTree{
  const int MAXN = 100010, MAXW = MAXN*30; // MAXN * LOG(maxX-MinX)
  typedef int t_wavelet;
  int last;
  int v[MAXN], aux[MAXN];
  int lo[MAXW], hi[MAXW], l[MAXW], r[MAXW];
  vector<t_wavelet> a[MAXW];
  int stable_partition(int i, int j, t_wavelet mid){
    int pivot=0;
    for(int k=i; k<j; k++)
      aux[k] = v[k], pivot += (v[k]<=mid);
    int i1=i, i2=i+pivot;
    for(int k=i; k<j; k++){
      if(aux[k]<=mid) v[i1++] = aux[k];
      else v[i2++] = aux[k];
    }
    return i1;
  }
  void build(int u, int i, int j, t_wavelet minX, t_wavelet maxX){
    lo[u] = minX, hi[u] = maxX;
    if (lo[u] == hi[u] or i >= j)
      return;
    t_wavelet mid = (minX + maxX - 1)/2;
    a[u].resize(j - i + 1);
    a[u][0] = 0;
    for(int k=i; k<j; k++)
      a[u][k-i+1] = a[u][k-i] + (v[k] <= mid);
    int pivot = stable_partition(i, j, mid);
    l[u] = last++, r[u] = last++;
    build(l[u], i, pivot, minX, mid);
    build(r[u], pivot, j, mid + 1, maxX);
  }
  inline int b(int u, int i){
    return i - a[u][i];
  }
//Public
  template <class MyIterator>
  void init(MyIterator begin, MyIterator end, t_wavelet minX,
      t_wavelet maxX){
    last = 1;
    int n = end-begin;
    for(int i=0; i<n; i++, begin++)
      v[i] = *begin;
    build(last++, 0, n, minX, maxX);
  }
  //kth smallest element in range [i, j]
  //1-indexed
  int kth(int i, int j, int k, int u=1){
    if (i > j)
      return 0;
    if (lo[u] == hi[u])
      return lo[u];
    int inLeft = a[u][j] - a[u][i - 1];
```

```cpp
      int i1 = a[u][i - 1] + 1, j1 = a[u][j];
      int i2 = b(u, i - 1) + 1, j2 = b(u, j);
      if (k <= inLeft)
        return kth(i1, j1, k, l[u]);
      return kth(i2, j2, k - inLeft, r[u]);
    }
    //Amount of numbers in the range [i, j] Less than or equal to k
    //1-indexed
    int lte(int i, int j, int k, int u=1){
      if (i > j or k < lo[u])
        return 0;
      if (hi[u] <= k)
        return j - i + 1;
      int i1 = a[u][i - 1] + 1, j1 = a[u][j];
      int i2 = b(u, i - 1) + 1, j2 = b(u, j);
      return lte(i1, j1, k, l[u]) + lte(i2, j2, k, r[u]);
    }
    //Amount of numbers in the range [i, j] equal to k
    //1-indexed
    int count(int i, int j, int k, int u=1){
      if (i > j or k < lo[u] or k > hi[u])
        return 0;
      if (lo[u] == hi[u])
        return j - i + 1;
      t_wavelet mid = (lo[u] + hi[u] - 1) / 2;
      int i1 = a[u][i - 1] + 1, j1 = a[u][j];
      int i2 = b(u, i - 1) + 1, j2 = b(u, j);
      if (k <= mid)
        return count(i1, j1, k, l[u]);
      return count(i2, j2, k, r[u]);
    }
    //swap v[i] with v[i+1]
    //1-indexed
    void swp(int i, int u=1){
      if (lo[u] == hi[u] or a[u].size() <= 2)
        return;
      if (a[u][i - 1] + 1 == a[u][i] and a[u][i] + 1 == a[u][i + 1])
        swp(a[u][i], l[u]);
      else if (b(u, i - 1) + 1 == b(u, i) and b(u, i) + 1 == b(u, i + 1)
        )
        swp(b(u, i), r[u]);
      else if (a[u][i - 1] + 1 == a[u][i])
        a[u][i]--;
      else
        a[u][i]++;
    }
};
```

2 Graph Algorithms

2.1 2-SAT

```cpp
#include "strongly_connected_component.h"
using namespace std;
struct SAT{
  typedef pair<int, int> pii;
  vector<pii> edges;
  int n;
```

```cpp
  SAT(int size){
    n = 2 * size;
  }
  vector<bool> solve2SAT(){
    vector<bool> vAns(n / 2, false);
    vector<int> comp = SCC::scc(n, edges);
    for (int i = 0; i < n; i += 2){
      if (comp[i] == comp[i + 1])
        return vector<bool>();
      vAns[i / 2] = (comp[i] > comp[i + 1]);
    }
    return vAns;
  }
  int v(int x){
    if (x >= 0)
      return (x << 1);
    x = ~x;
    return (x << 1) ^ 1;
  }
  void add(int a, int b){
    edges.push_back(pii(a, b));
  }
  void addOr(int a, int b){
    add(v(~a), v(b));
    add(v(~b), v(a));
  }
  void addImp(int a, int b){
    addOr(~a, b);
  }
  void addEqual(int a, int b){
    addOr(a, ~b);
    addOr(~a, b);
  }
  void addDiff(int a, int b){
    addEqual(a, ~b);
  }
};
```

2.2 Arborescence

```cpp
#include <bits/stdc++.h>
#include "../data_structures/union_find.h"

using namespace std;
/**
 * Source: https://github.com/spaghetti-source/algorithm/blob/master/
 *     graph/arborescence.cc
 * Description: Edmonds' algorithm for finding the weight of the
 *     minimum spanning
 * tree/arborescence of a directed graph, given a root node. If no MST
 *     exists, returns -1.
 * Time: O(E \log V)
 */
typedef long long ll;
namespace Arborescence{
  struct Edge { int a, b; ll w; };
  struct Heap {
    Edge key;
    Heap *l, *r;
    ll delta;
```

```cpp
  void prop() {
    key.w += delta;
    if (l) l->delta += delta;
    if (r) r->delta += delta;
    delta = 0;
  }
  Edge top() { prop(); return key; }
};
Heap *merge(Heap *a, Heap *b) {
  if (!a || !b) return a ?: b;
  a->prop(), b->prop();
  if (a->key.w > b->key.w) swap(a, b);
  swap(a->l, (a->r = merge(b, a->r)));
  return a;
}
void pop(Heap*& a) { a->prop(); a = merge(a->l, a->r); }

//public
vector<Edge> g;
void init(){
  g.clear();
}
void addEdge(int u, int to, ll w) {
  Edge e;
  e.a = u, e.b = to, e.w = w;
  g.push_back(e);
}
ll dmst(int n, int root) {
  UnionFind uf(n);
  vector<Heap*> heap(n);
  vector<Heap*> vp;
  for (Edge e : g){
    Heap* h = new Heap{e};
    vp.push_back(h);
    heap[e.b] = merge(heap[e.b], h);
  }
  ll res = 0;
  vector<int> seen(n, -1), path(n);
  seen[root] = root;
  for(int s=0; s<n; s++) {
    int u = s, qi = 0, w;
    while (seen[u] < 0) {
      path[qi++] = u, seen[u] = s;
      if (!heap[u]){
        for(Heap *h: vp)
          delete h;
        return -1;
      }
      Edge e = heap[u]->top();
      heap[u]->delta -= e.w, pop(heap[u]);
      res += e.w; u = uf.find(e.a);
      if (seen[u] == s) {
        Heap* cyc = 0;
        do cyc = merge(cyc, heap[w = path[--qi]]);
        while (uf.join(u, w));
        u = uf.find(u);
        heap[u] = cyc, seen[u] = -1;
      }
    }
  }
  for(Heap *h: vp)
```

```cpp
    delete h;
  return res;
}
//Careful with overflow
ll dmstAnyRoot(int n) {
  ll maxEdge = 1000000010;
  ll INF = n*maxEdge;
  for(int i=0; i<n; i++)
    addEdge(n, i, INF);
  ll ans = dmst(n+1, n);
  if(ans >= 0 and ans < 2*INF)
    return ans - INF;
  else
    return -1;
}
};
```

2.3 Articulation Point

```cpp
#include <bits/stdc++.h>

using namespace std;

const int MAXN = 500010;
typedef pair<int, int> pii;
//Articulation Point
namespace AP{
  vector<int> adj[MAXN];
  vector<bool> visited, isAP;
  vector<int> tin, low;
  int timer, n;
  vector<pii> bridges;
  void init(int n1){
    n = n1;
    for(int i=0; i<n; i++) adj[i].clear();
  }
  void addEdge(int a, int b){
    adj[a].push_back(b);
    adj[b].push_back(a);
  }
  void dfs(int u, int p = -1) {
    visited[u] = true;
    tin[u] = low[u] = timer++;
    int children=0;
    for (int to : adj[u]) {
      if (to == p) continue;
      if (visited[to]) {
        low[u] = min(low[u], tin[to]);
      } else {
        dfs(to, u);
        low[u] = min(low[u], low[to]);
        if (low[to] >= tin[u] && p!=-1)
          isAP[u] = true;
        ++children;
      }
    }
    if(p == -1 && children > 1)
      isAP[u] = true;
  }
  vector<bool> findArticulationPoint() {
```

```
    timer = 0;
    visited.assign(n, false);
    tin.assign(n, -1);
    low.assign(n, -1);
    isAP.assign(n, false);
    for (int i = 0; i < n; i++) {
      if (!visited[i])
        dfs(i);
    }
    return isAP;
  }
};
```

2.4 BFS 0-1

```cpp
#include <bits/stdc++.h>
using namespace std;
typedef pair<int, int> pii;
const int N = 500010;
const int INF = 0x3f3f3f3f;
namespace BFS01{
  vector<pii> adj[N];
  int n;
  void init(int n1){
    n = n1;
    for(int i=0; i<n; i++) adj[i].clear();
  }
  //0-indexed
  void addEdge(int u, int to, int w){
    adj[u].emplace_back(to, w);
  }
  vector<int> solve(int s){
    vector<int> d(n, INF);
    d[s] = 0;
    deque<int> q;
    q.push_front(s);
    while (!q.empty()) {
      int u = q.front();
      q.pop_front();
      for (auto edge : adj[u]) {
        int to = edge.first;
        int w = edge.second;
        if (d[u] + w < d[to]) {
          d[to] = d[u] + w;
          if (w == 1)
            q.push_back(to);
          else
            q.push_front(to);
        }
      }
    }
    return d;
  }
};
```

2.5 Bridge

```cpp
#include <bits/stdc++.h>
```

```cpp
using namespace std;
const int MAXN = 500010;
typedef pair<int, int> pii;
namespace Bridge{
  vector<int> adj[MAXN];
  vector<bool> visited;
  vector<int> tin, low;
  int timer, n;
  vector<pii> bridges;
  void init(int n1){
    n = n1;
    for(int i=0; i<n; i++) adj[i].clear();
  }
  void addEdge(int a, int b){
    adj[a].push_back(b);
    adj[b].push_back(a);
  }
  void dfs(int u, int p = -1) {
    visited[u] = true;
    tin[u] = low[u] = timer++;
    for (int to : adj[u]) {
      if (to == p) continue;
      if (visited[to]) {
        low[u] = min(low[u], tin[to]);
      } else {
        dfs(to, u);
        low[u] = min(low[u], low[to]);
        if (low[to] > tin[u])
          bridges.push_back({u, to});
      }
    }
  }
  vector<pii> findBridges() {
    timer = 0;
    visited.assign(n, false);
    tin.assign(n, -1);
    low.assign(n, -1);
    bridges.clear();
    for (int i = 0; i < n; i++) {
      if (!visited[i])
        dfs(i);
    }
    return bridges;
  }
};
```

2.6 Centroid

```cpp
#include <bits/stdc++.h>
using namespace std;
const int MAXN = 500010;
typedef pair<int, int> pii;
namespace Centroid{
  vector<int> adj[MAXN];
  int sub[MAXN];
  int n;
  void init(int n1){
    n = n1;
    for(int i=0; i<n; i++) adj[i].clear();
  }
```

```cpp
void addEdge(int a, int b){
  adj[a].push_back(b);
  adj[b].push_back(a);
}
int dfsS(int u, int p){
  sub[u] = 1;
  for(int to: adj[u]){
    if(to != p)
      sub[u] += dfsS(to, u);
  }
  return sub[u];
}
pii dfsC(int u, int p){
  for(int to : adj[u]){
    if(to != p and sub[to] > n/2)
      return dfsC(to, u);
  }
  for(int to : adj[u]){
    if(to != p and (sub[to]*2) == n)
      return pii(u, to);
  }
  return pii(u, u);
}
pii findCentroid(){
  dfsS(0, -1);
  return dfsC(0, -1);
}
```

2.7 Centroid Decomposition

```cpp
#include <bits/stdc++.h>
using namespace std;
// O(N*log(N))
struct CentroidDecomposition{
  vector<vector<int>> adj;
  vector<int> dad, sub;
  vector<bool> rem;
  int centroidRoot, n;
  void init(int _n){
    n = _n;
    adj.resize(n);
    dad.resize(n);
    sub.resize(n);
    rem.assign(n, false);
  }
  // Return Centroid Decomposition Tree
  vector<vector<int>> build(){
    assert(n > 0);
    centroidRoot = decomp(0, -1);
    vector<vector<int>> ret(n);
    for (int u = 0; u < n; u++){
      if (dad[u] != u)
        ret[dad[u]].push_back(u);
    }
    return ret;
  }
  void addEdge(int a, int b){
    adj[a].push_back(b);
    adj[b].push_back(a);
```

```cpp
}
int decomp(int u, int p){
  int sz = dfs(u, p);
  int c = centroid(u, p, sz);
  if (p == -1)
    p = c;
  dad[c] = p;
  rem[c] = true;
  for (auto to : adj[c]){
    if (!rem[to])
      decomp(to, c);
  }
  return c;
}
int dfs(int u, int p){
  sub[u] = 1;
  for (int to : adj[u]){
    if (!rem[to] and to != p)
      sub[u] += dfs(to, u);
  }
  return sub[u];
}
int centroid(int u, int p, int sz){
  for (auto to : adj[u])
    if (!rem[to] and to != p and sub[to] > sz / 2)
      return centroid(to, u, sz);
  return u;
}
int operator[](int i){
  return dad[i];
}
};
```

2.8 Dinic

```cpp
#include <bits/stdc++.h>
using namespace std;
//O(v^2*E): for generic graph
//O(sqrt(V)*E): for bipartite graph
template <typename flow_t>
struct Dinic{
  struct FlowEdge{
    int from, to;
    flow_t cap, flow = 0;
    FlowEdge(int f, int t, flow_t c) : from(f), to(t), cap(c) {}
  };
  const flow_t flow_inf = numeric_limits<flow_t>::max();
  vector<FlowEdge> edges;
  vector<vector<int>> adj;
  int n, m = 0;
  int s, t;
  vector<int> level, ptr;
  queue<int> q;
  bool bfs(){
    while (!q.empty()){
      int u = q.front();
      q.pop();
      for (int id : adj[u]){
        if (edges[id].cap - edges[id].flow < 1)
          continue;
```

```cpp
      if (level[edges[id].to] != -1)
        continue;
      level[edges[id].to] = level[u] + 1;
      q.push(edges[id].to);
    }
  }
  return level[t] != -1;
}
flow_t dfs(int u, flow_t pushed){
  if (pushed == 0)
    return 0;
  if (u == t)
    return pushed;
  for (int &cid = ptr[u]; cid < (int)adj[u].size(); cid++){
    int id = adj[u][cid];
    int to = edges[id].to;
    if (level[u] + 1 != level[to] || edges[id].cap - edges[id].flow
        < 1)
      continue;
    flow_t tr = dfs(to, min(pushed, edges[id].cap - edges[id].flow))
        ;
    if (tr == 0)
      continue;
    edges[id].flow += tr;
    edges[id ^ 1].flow -= tr;
    return tr;
  }
  return 0;
}
//Public:
Dinic(){}
void init(int _n){
  n = _n;
  adj.resize(n);
  level.resize(n);
  ptr.resize(n);
}
void addEdge(int from, int to, flow_t cap){
  assert(n>0);
  edges.push_back(FlowEdge(from, to, cap));
  edges.push_back(FlowEdge(to, from, 0));
  adj[from].push_back(m);
  adj[to].push_back(m + 1);
  m += 2;
}
flow_t maxFlow(int s1, int t1){
  s = s1, t = t1;
  flow_t f = 0;
  for(int i=0; i<m; i++)
    edges[i].flow = 0;
  while (true){
    level.assign(n, -1);
    level[s] = 0;
    q.push(s);
    if (!bfs())
      break;
    ptr.assign(n, 0);
    while (flow_t pushed = dfs(s, flow_inf))
      f += pushed;
  }
  return f;
```

```cpp
  }
};
typedef pair<int, int> pii;
vector<pii> recoverCut(Dinic<int> &d){
  vector<int> level(d.n, 0);
  vector<pii> rc;
  queue<int> q;
  q.push(d.s);
  level[d.s] = 1;
  while (!q.empty()){
    int u = q.front();
    q.pop();
    for (int id : d.adj[u]){
      if ((id & 1) == 1)
        continue;
      if (d.edges[id].cap == d.edges[id].flow){
        rc.push_back(pii(d.edges[id].from, d.edges[id].to));
      }else{
        if (level[d.edges[id].to] == 0){
          q.push(d.edges[id].to);
          level[d.edges[id].to] = 1;
        }
      }
    }
  }
  vector<pii> ans;
  for (pii p : rc)
    if ((level[p.first] == 0) or (level[p.second] == 0))
      ans.push_back(p);
  return ans;
}
```

2.9 Flow With Demand

```cpp
#include "dinic.h"
using namespace std;
template <typename flow_t>
struct MaxFlowEdgeDemands{
  Dinic<flow_t> mf;
  vector<flow_t> ind, outd;
  flow_t D;
  int n;
  MaxFlowEdgeDemands(int n) : n(n){
    D = 0;
    mf.init(n + 2);
    ind.assign(n, 0);
    outd.assign(n, 0);
  }
  void addEdge(int a, int b, flow_t cap, flow_t demands){
    mf.addEdge(a, b, cap - demands);
    D += demands;
    ind[b] += demands;
    outd[a] += demands;
  }
  bool solve(int s, int t){
    mf.addEdge(t, s, numeric_limits<flow_t>::max());
    for (int i = 0; i < n; i++){
      if (ind[i]) mf.addEdge(n, i, ind[i]);
      if (outd[i]) mf.addEdge(i, n + 1, outd[i]);
    }
```

```
        return mf.maxFlow(n, n + 1) == D;
    }
};
```

2.10   Kruskal

```
#include "../data_structures/union_find.h"
typedef long long ll;
struct Edge{
    int u, v; ll w;
    Edge(){}
    Edge(int u, int v, ll w):u(u), v(v), w(w){}
};
ll kruskal(vector<Edge> v, int nVet){
    ll cost = 0;
    UnionFind uf(nVet);
    sort(v.begin(), v.end(), [&](Edge a, Edge b){
        return a.w < b.w;
    });
    for(Edge &e: v){
        if(!uf.isSame(e.u, e.v)){
            cost += e.w;
            uf.join(e.u, e.v);
        }
    }
    return cost;
}
```

2.11   HLD

```
#include <bits/stdc++.h>
#include "../data_structures/bit_range.h"
using namespace std;
#define F first
#define S second
using hld_t = long long;
using pii = pair<int, hld_t>;
struct HLD{
    vector<vector<pii>> adj;
    vector<int> sz, h, dad, pos;
    vector<hld_t> val, v;
    int t;
    bool edge;
    //Begin Internal Data Structure
    BitRange *bit;
    hld_t neutral = 0;
    inline hld_t join(hld_t a, hld_t b){
        return a+b;
    }
    inline void update(int a, int b, hld_t x){
        bit->add(a+1, b+1, x);
    }
    inline hld_t query(int a, int b){
        return bit->get(a+1, b+1);
    }
    //End Internal Data Structure
    void init(int n){
        dad.resize(n); pos.resize(n); val.resize(n); v.resize(n);
```

```
        adj.resize(n); sz.resize(n); h.resize(n);
        bit = new BitRange(n);
    }
    void dfs(int u, int p = -1){
        sz[u] = 1;
        for(pii &to: adj[u]) if(to.F != p){
            if(edge) val[to.F] = to.S;
            dfs(to.F, u);
            sz[u] += sz[to.F];
            if(sz[to.F] > sz[adj[u][0].F] or adj[u][0].F == p)
                swap(to, adj[u][0]);
        }
    }
    void build_hld(int u, int p=-1){
        dad[u] = p;
        pos[u] = t++;
        v[pos[u]] = val[u];
        for(pii to: adj[u]) if(to.F != p){
            h[to.F] = (to == adj[u][0]) ? h[u] : to.F;
            build_hld(to.F, u);
        }
    }
    void addEdge(int a, int b, hld_t w = 0){
        adj[a].emplace_back(b, w);
        adj[b].emplace_back(a, w);
    }
    void build(int root, bool is_edge){
        assert(!adj.empty());
        edge = is_edge;
        t = 0;
        h[root] = 0;
        dfs(root);
        build_hld(root);
        //Init Internal Data Structure
        for(int i=0; i<t; i++)
            update(i, i, v[i]);
    }
    hld_t query_path(int a, int b) {
        if (edge and a == b) return neutral;
        if (pos[a] < pos[b]) swap(a, b);
        if (h[a] == h[b]) return query(pos[b]+edge, pos[a]);
        return join(query(pos[h[a]], pos[a]), query_path(dad[h[a]], b));
    }
    void update_path(int a, int b, hld_t x) {
        if (edge and a == b) return;
        if (pos[a] < pos[b]) swap(a, b);
        if (h[a] == h[b]) return (void)update(pos[b]+edge, pos[a], x);
        update(pos[h[a]], pos[a], x); update_path(dad[h[a]], b, x);
    }
    hld_t query_subtree(int a) {
        if (edge and sz[a] == 1) return neutral;
        return query(pos[a]+edge, pos[a]+sz[a]-1);
    }
    void update_subtree(int a, hld_t x) {
        if (edge and sz[a] == 1) return;
        update(pos[a] + edge, pos[a]+sz[a]-1, x);
    }
    int lca(int a, int b) {
        if (pos[a] < pos[b]) swap(a, b);
        return h[a] == h[b] ? b : lca(dad[h[a]], b);
    }
}
```

```
};
```

2.12 Min-Cut

```cpp
#include <bits/stdc++.h>
using namespace std;
typedef long long ll;
//This algorithm finds the Global Min-Cut in O(|V|^3)
namespace MinCut{
  const int MAXN = 510;
  bool exist[MAXN], in_a[MAXN];
  ll g[MAXN][MAXN], w[MAXN];
  vector<int> v[MAXN];
  int n;
  void init(int n1){
    n = n1;
    memset(g, 0, sizeof(g));
  }
  void addEdge(int a, int b, int w1){
    if(a == b) return;
    g[a][b] += w1;
    g[b][a] += w1;
  }
  pair<ll, vector<int>> mincut() {
    ll best_cost = 0x3f3f3f3f3f3f3f3fLL;
    vector<int> best_cut;
    for (int i=0; i<n; ++i)
      v[i].assign (1, i);
    memset (exist, true, sizeof(exist));
    for(int ph=0; ph<n-1; ++ph) {
      memset (in_a, false, sizeof in_a);
      memset (w, 0, sizeof w);
      for(int it=0, prev=0; it<n-ph; ++it){
        int sel = -1;
        for(int i=0; i<n; ++i)
          if(exist[i] && !in_a[i] && (sel == -1 || w[i] > w[sel]))
            sel = i;
        if(it == n-ph-1){
          if(w[sel] < best_cost)
            best_cost = w[sel],  best_cut = v[sel];
          v[prev].insert (v[prev].end(), v[sel].begin(), v[sel].end())
            ;
          for(int i=0; i<n; ++i)
            g[prev][i] = g[i][prev] += g[sel][i];
          exist[sel] = false;
        }else{
          in_a[sel] = true;
          for(int i=0; i<n; ++i)
            w[i] += g[sel][i];
          prev = sel;
        }
      }
    }
    return {best_cost, best_cut};
  }
};
```

2.13 Minimum Cost Maximum Flow

```cpp
#include <bits/stdc++.h>
using namespace std;
template <class T = int>
class MCMF{
private:
  struct Edge{
    int to;
    T cap, cost;
    Edge(int a, T b, T c) : to(a), cap(b), cost(c) {}
  };
  int n;
  vector<vector<int>> edges;
  vector<Edge> list;
  vector<int> from;
  vector<T> dist, pot;
  vector<bool> visit;
  pair<T, T> augment(int src, int sink){
    pair<T, T> flow = {list[from[sink]].cap, 0};
    for (int v = sink; v != src; v = list[from[v] ^ 1].to){
      flow.first = std::min(flow.first, list[from[v]].cap);
      flow.second += list[from[v]].cost;
    }
    for (int v = sink; v != src; v = list[from[v] ^ 1].to){
      list[from[v]].cap -= flow.first;
      list[from[v] ^ 1].cap += flow.first;
    }
    return flow;
  }
  queue<int> q;
  bool SPFA(int src, int sink){
    T INF = numeric_limits<T>::max();
    dist.assign(n, INF);
    from.assign(n, -1);
    q.push(src);
    dist[src] = 0;
    while (!q.empty()){
      int on = q.front();
      q.pop();
      visit[on] = false;
      for (auto e : edges[on]){
        auto ed = list[e];
        if (ed.cap == 0)
          continue;
        T toDist = dist[on] + ed.cost + pot[on] - pot[ed.to];
        if (toDist < dist[ed.to]){
          dist[ed.to] = toDist;
          from[ed.to] = e;
          if (!visit[ed.to]){
            visit[ed.to] = true;
            q.push(ed.to);
          }
        }
      }
    }
    return dist[sink] < INF;
  }
  void fixPot(){
    T INF = numeric_limits<T>::max();
    for (int i = 0; i < n; i++){
      if (dist[i] < INF)
        pot[i] += dist[i];
```

```cpp
      }
    }
  public:
    MCMF(int size){
      n = size;
      edges.resize(n);
      pot.assign(n, 0);
      dist.resize(n);
      visit.assign(n, false);
    }
    pair<T, T> solve(int src, int sink){
      pair<T, T> ans(0, 0);
      // Can use dijkstra to speed up depending on the graph
      if (!SPFA(src, sink))
        return ans;
      fixPot();
      // Can use dijkstra to speed up depending on the graph
      while (SPFA(src, sink)){
        auto flow = augment(src, sink);
        ans.first += flow.first;
        ans.second += flow.first * flow.second;
        fixPot();
      }
      return ans;
    }
    void addEdge(int from, int to, T cap, T cost){
      edges[from].push_back(list.size());
      list.push_back(Edge(to, cap, cost));
      edges[to].push_back(list.size());
      list.push_back(Edge(from, 0, -cost));
    }
};
/*bool dij(int src, int sink){
  T INF = numeric_limits<T>::max();
  dist.assign(n, INF);
  from.assign(n, -1);
  visit.assign(n, false);
  dist[src] = 0;
  for(int i = 0; i < n; i++){
    int best = -1;
    for(int j = 0; j < n; j++){
      if(visit[j]) continue;
      if(best == -1 || dist[best] > dist[j]) best = j;
    }
    if(dist[best] >= INF) break;
    visit[best] = true;
    for(auto e : edges[best]){
      auto ed = list[e];
      if(ed.cap == 0) continue;
      T toDist = dist[best] + ed.cost + pot[best] - pot[ed.to];
      assert(toDist >= dist[best]);
      if(toDist < dist[ed.to]){
        dist[ed.to] = toDist;
        from[ed.to] = e;
      }
    }
  }
  return dist[sink] < INF;
}*/
```

2.14 Strongly Connected Component

```cpp
#include "topological_sort.h"
using namespace std;
namespace SCC{
  typedef pair<int, int> pii;
  vector<vector<int>> revAdj;
  vector<int> component;
  void dfs(int u, int c){
    component[u] = c;
    for (int to : revAdj[u]){
      if (component[to] == -1)
        dfs(to, c);
    }
  }
  vector<int> scc(int n, vector<pii> &edges){
    revAdj.assign(n, vector<int>());
    for (pii p : edges)
      revAdj[p.second].push_back(p.first);
    vector<int> tp = TopologicalSort::order(n, edges);
    component.assign(n, -1);
    int comp = 0;
    for (int u : tp){
      if (component[u] == -1)
        dfs(u, comp++);
    }
    return component;
  }
} // namespace SCC
```

2.15 Topological Sort

```cpp
#include <bits/stdc++.h>
using namespace std;
namespace TopologicalSort{
  typedef pair<int, int> pii;
  vector<vector<int>> adj;
  vector<bool> visited;
  vector<int> vAns;
  void dfs(int u){
    visited[u] = true;
    for (int to : adj[u]){
      if (!visited[to])
        dfs(to);
    }
    vAns.push_back(u);
  }
  vector<int> order(int n, vector<pii> &edges){
    adj.assign(n, vector<int>());
    for (pii p : edges)
      adj[p.first].push_back(p.second);
    visited.assign(n, false);
    vAns.clear();
    for (int i = 0; i < n; i++){
      if (!visited[i])
        dfs(i);
    }
    reverse(vAns.begin(), vAns.end());
```

```
      return vAns;
   }
}; // namespace TopologicalSort
```

2.16   Tree ID

```cpp
#include "centroid.h"
#define F first
#define S second
namespace TreeID{
  int id=0;
  map<map<int, int>, int> mpId;
  vector<int> adj[MAXN];
  int treeID(int u, int p){
    map<int, int> mp;
    for(int to: adj[u]){
      if(to != p)
        mp[treeID(to, u)]++;
    }
    if(!mpId.count(mp))
      mpId[mp] = ++id;
    return mpId[mp];
  }
  //Returns a pair of values that represents a tree only. O((N+M)*log(
      M))
  //0-indexed
  pii getTreeID(vector<pii> &edges, int n){
    for(int i=0; i<n; i++)
      adj[i].clear();
    Centroid::init(n);
    for(pii e: edges){
      adj[e.F].push_back(e.S);
      adj[e.S].push_back(e.F);
      Centroid::addEdge(e.F, e.S);
    }
    pii c = Centroid::findCentroid();
    pii ans(treeID(c.F, -1), treeID(c.S, -1));
    if(ans.F > ans.S)
      swap(ans.F, ans.S);
    return ans;
  }
  bool isomorphic(vector<pii> &tree1, vector<pii> &tree2, int n){
    return getTreeID(tree1, n) == getTreeID(tree2, n);
  }
};
```

3   Dynamic Programming

3.1   Divide and Conquer Optimization

Reduces the complexity from $O(n^2 k)$ to $O(nk \log n)$ of PD's in the following ways (and other variants):

$$dp[n][k] = \max_{0 \leq i < n} (dp[i][k-1] + C[i+1][n]), \ base \ case: \ dp[0][j], dp[i][0] \qquad (1)$$

- $C[i][j] =$ the cost only depends on i and j.

- $opt[n][k] = i$ is the optimal value that maximizes $dp[n][k]$.

It is necessary that opt is increasing along each column: $opt[j][k] \leq opt[j+1][k]$.

3.2   Divide and Conquer Optimization Implementation

```cpp
#include <bits/stdc++.h>
using namespace std;
int C(int i, int j);
const int MAXN = 100010;
const int MAXK = 110;
const int INF = 0x3f3f3f3f;
int dp[MAXN][MAXK];
void calculateDP(int l, int r, int k, int opt_l, int opt_r){
  if (l > r)
    return;
  int mid = (l + r) >> 1;
  int ans = -INF, opt = mid;
//  int ans = dp[mid][k-1], opt=mid; //If you accept empty subsegment
  for (int i = opt_l; i <= min(opt_r, mid - 1); i++){
    if (ans < dp[i][k - 1] + C(i + 1, mid)){
      opt = i;
      ans = dp[i][k - 1] + C(i + 1, mid);
    }
  }
  dp[mid][k] = ans;
  calculateDP(l, mid - 1, k, opt_l, opt);
  calculateDP(mid + 1, r, k, opt, opt_r);
}
int solve(int n, int k){
  for (int i = 0; i <= n; i++)
    dp[i][0] = -INF;
  for (int j = 0; j <= k; j++)
    dp[0][j] = -INF;
  dp[0][0] = 0;
  for (int j = 1; j <= k; j++)
    calculateDP(1, n, j, 0, n - 1);
  return dp[n][k];
}
```

3.3   Knuth Optimization

Reduces the complexity from $O(n^3)$ to $O(n^2)$ of PD's in the following ways (and other variants):

$$dp[i][j] = C[i][j] + \min_{i<k<j} (dp[i][k] + dp[k][j]), \ caso \ base: \ dp[i][i] \qquad (2)$$

$$dp[i][j] = \min_{i<k<j} (dp[i][k] + C[i][k]), \ caso \ base: \ dp[i][i] \qquad (3)$$

- $C[i][j] =$ the cost only depends on i and j.
- $opt[i][j] = k$ is the optimal value that maximizes $dp[i][j]$.

The following conditions must be met:

- Foursquare inequality on C: $C[a][c] + C[b][d] \leq C[a][d] + C[b][c]$, $a \leq b \leq c \leq d$.

- Monotonicity on C: $C[b][c] \leq C[a][d]$, $a \leq b \leq c \leq d$.

Or the following condition:

- *opt* increasing in rows and columns: $opt[i][j - 1] \leq opt[i][j] \leq opt[i + 1][j]$.

3.4 Knuth Optimization Implementation

```cpp
#include <bits/stdc++.h>
using namespace std;
typedef long long ll;
const int MAXN = 1009;
const ll INFLL = 0x3f3f3f3f3f3f3f3f;
ll C(int a, int b);
ll dp[MAXN][MAXN];
int opt[MAXN][MAXN];
ll knuth(int n){
  for (int i = 0; i < n; i++){
    dp[i][i] = 0;
    opt[i][i] = i;
  }
  for (int s = 1; s < n; s++){
    for (int i = 0, j; (i + s) < n; i++){
      j = i + s;
      dp[i][j] = INFLL;
      for (int k = opt[i][j - 1]; k < min(j, opt[i + 1][j] + 1); k++){
        ll cur = dp[i][k] + dp[k + 1][j] + C(i, j);
        if (dp[i][j] > cur){
          dp[i][j] = cur;
          opt[i][j] = k;
        }
      }
    }
  }
  return dp[0][n - 1];
}
```

4 Math

4.1 Basic Math

```cpp
#include <bits/stdc++.h>
using namespace std;
typedef long long ll;
typedef unsigned long long ull;

ull fastPow(ull base, ull exp, ull mod){
  base %= mod;
  //exp %= phi(mod) if base and mod are relatively prime
  ull ans = 1LL;
  while (exp > 0){
    if (exp & 1LL)
      ans = (ans * (__int128_t)base) % mod;
    base = (base * (__int128_t)base) % mod;
    exp >>= 1;
```

```cpp
  }
  return ans;
}
int fastPow(int base, string bigExp, int mod){
  int ans = 1;
  for(char c: bigExp){
    ans = fastPow(ans, 10, mod);
    ans = (ans*1LL*fastPow(base, c-'0', mod))%mod;
  }
  return ans;
}
ll gcd(ll a, ll b){ return __gcd(a, b); }
ll lcm(ll a, ll b){ return (a / gcd(a, b)) * b; }
void enumeratingAllSubmasks(int mask){
  for (int s = mask; s; s = (s - 1) & mask)
    cout << s << endl;
}
//MOD to Hash
namespace ModHash{
  const uint64_t MOD = (1ll<<61) - 1;
  uint64_t modmul(uint64_t a, uint64_t b){
    uint64_t l1 = (uint32_t)a, h1 = a>>32, l2 = (uint32_t)b, h2 = b
        >>32;
    uint64_t l = l1*l2, m = l1*h2 + l2*h1, h = h1*h2;
    uint64_t ret = (l&MOD) + (l>>61) + (h << 3) + (m >> 29) + ((m <<
        35) >> 3) + 1;
    ret = (ret & MOD) + (ret>>61);
    ret = (ret & MOD) + (ret>>61);
    return ret-1;
  }
};
```

4.2 BigInt

```cpp
#include <bits/stdc++.h>
using namespace std;
typedef int32_t intB;
typedef int64_t longB;
typedef vector<intB> vib;
class BigInt{
private:
  vib vb;
  bool neg;
  const int BASE_DIGIT = 9;
  const intB base = 1000000LL*1000;//000LL*1000000LL;
  void fromString(string &s){
    if(s[0] == '-'){
      neg = true;
      s = s.substr(1);
    }else{
      neg = false;
    }
    vb.clear();
    vb.reserve((s.size()+BASE_DIGIT-1)/BASE_DIGIT);
    for(int i=(int)s.length(); i>0; i-=BASE_DIGIT){
      if(i < BASE_DIGIT)
        vb.push_back(stol(s.substr(0, i)));
      else
        vb.push_back(stol(s.substr(i-BASE_DIGIT, BASE_DIGIT)));
    }
```

```
    fix(vb);
  }
  void fix(vib &v){
    while(v.size()>1 && v.back()==0)
      v.pop_back();
    if(v.size() == 0)
      neg = false;
  }
  bool comp(vib &a, vib &b){
    fix(a); fix(b);
    if(a.size() != b.size()) return a.size() < b.size();
    for(int i=(int)a.size()-1; i>=0; i--) {
      if(a[i] != b[i]) return a[i] < b[i];
    }
    return false;
  }
  vib sum(vib a, vib b){
    int carry = 0;
    for(size_t i=0; i<max(a.size(), b.size()) or carry; i++){
      if(i == a.size())
        a.push_back(0);
      a[i] += carry + (i<b.size() ? b[i] : 0);
      carry = (a[i] >= base);
      if(carry) a[i] -= base;
    }
    fix(a);
    return a;
  }
  vib sub(vib a, vib b){
    int carry = 0;
    for(size_t i=0; i<b.size() or carry; i++){
      a[i] -= carry + (i<b.size() ? b[i] : 0);
      carry = a[i] < 0;
      if(carry) a[i] += base;
    }
    fix(a);
    return a;
  }

public:
  BigInt(){}
  BigInt(intB n){
    neg = (n<0);
    vb.push_back(abs(n));
    fix(vb);
  }
  BigInt(string s){
    fromString(s);
  }
  BigInt operator =(BigInt oth){
    this->neg = oth.neg;
    this->vb = oth.vb;
    return *this;
  }
  BigInt operator +(BigInt &oth){
    vib &a = vb, &b = oth.vb;
    BigInt ans;
    if(neg == oth.neg){
      ans.vb = sum(vb, oth.vb);
      ans.neg = neg;
    }else{
```

```
      if(comp(a, b)){
        ans.vb = sub(b, a);
        ans.neg = oth.neg;
      }else{
        ans.vb = sub(a, b);
        ans.neg = neg;
      }
    }
    return ans;
  }
  BigInt operator -(BigInt oth){
    oth.neg ^= true;
    return (*this) + oth;
  }
  BigInt operator *(intB b){
    bool negB = false;
    if(b < 0){
      negB = true;
      b = -b;
    }
    BigInt ans = *this;
    auto &a = ans.vb;
    intB carry = 0;
    for(size_t i=0; i<a.size() or carry; i++){
      if(i == a.size()) a.push_back(0);
      longB cur = carry + a[i] *(longB) b;
      a[i] = intB(cur%base);
      carry = intB(cur/base);
    }
    ans.neg ^= negB;
    fix(ans.vb);
    return ans;
  }
  BigInt operator *(BigInt &oth){
    BigInt ans;
    auto a = vb, &b = oth.vb, &c = ans.vb;
    c.assign(a.size() + b.size(), 0);
    for(size_t i=0; i<a.size(); i++){
      intB carry=0;
      for(size_t j=0; j<b.size() or carry; j++){
        longB cur = c[i+j] + a[i]*(longB)(j<b.size() ? b[j] : 0);
        cur += carry;
        c[i+j] = intB(cur%base);
        carry = intB(cur/base);
      }
    }
    ans.neg = neg^oth.neg;
    fix(ans.vb);
    return ans;
  }
  BigInt operator /(intB b){
    bool negB = false;
    if(b < 0){
      negB = true;
      b = -b;
    }
    BigInt ans = *this;
    auto &a = ans.vb;
    intB carry = 0;
    for(int i=(int)a.size()-1; i>=0; i--){
      longB cur = a[i] + (longB)carry * base;
```

```cpp
      a[i] = intB(cur/b);
      carry = intB(cur%b);
    }
    ans.neg ^= negB;
    fix(ans.vb);
    return ans;
  }
  void shiftL(int b){
    vb.resize(vb.size() + b);
    for(int i=(int)vb.size()-1; i>=0; i--) {
      if(i>=b) vb[i] = vb[i-b];
      else vb[i] = 0;
    }
    fix(vb);
  }
  void shiftR(int b) {
    if((int)vb.size() <= b){
      vb.clear();
      vb.push_back(0);
      return;
    }
    for(int i=0; i<((int)vb.size() - b); i++)
      vb[i] = vb[i+b];
    vb.resize((int)vb.size() - b);
    fix(vb);
  }
  void divide(BigInt a, BigInt b, BigInt &q, BigInt &r){
    BigInt z(0), p(1);
    while(b < a) {
      p.shiftL(max(1, int(a.vb.size()-b.vb.size())));
      b.shiftL(max(1, int(a.vb.size()-b.vb.size())));
    }
    while(true) {
      while ((a < b) && (z < p)) {
        p = p/10;
        b = b/10;
      }
      if(!(z < p)) break;
      a = a - b;
      q = q + p;
    }
    r = a;
  }
  BigInt operator /(BigInt &oth){
    BigInt q, r;
    divide(*this, oth, q, r);
    return q;
  }
  BigInt operator %(BigInt &oth){
    BigInt q, r;
    divide(*this, oth, q, r);
    return r;
  }
  bool operator <(BigInt &oth){
    BigInt ans = (*this) - oth;
    return ans.neg;
  }
  bool operator ==(BigInt &oth){
    BigInt ans = (*this) - oth;
    return (ans.vb.size()==1) and (ans.vb.back()==0);
  }
}
```

```cpp
  friend ostream &operator<<(ostream &out, const BigInt &D){
    if(D.neg)
      out << '-';
    out << (D.vb.empty() ? 0 : D.vb.back());
    for(int i=(int)D.vb.size()-2; i>=0; i--)
      out << setfill('0') << setw(D.BASE_DIGIT) << D.vb[i];
    return out;
  }
  string to_string(){
    std::stringstream ss;
    ss << (*this);
    return ss.str();
  }
  friend istream &operator>>(istream  &input, BigInt &D) {
    string s;
    input >> s;
    D.fromString(s);
    return input;
  }
};
```

4.3 Binomial Coefficients

```cpp
#include <bits/stdc++.h>
#include "./basic_math.h"
#include "./modular.h"
using namespace std;
typedef long long ll;
//O(k)
ll C1(int n, int k){
  ll res = 1LL;
  for (int i = 1; i <= k; ++i)
    res = (res * (n - k + i)) / i;
  return res;
}
//O(n^2)
vector<vector<ll>> C2(int maxn, int mod){
  vector<vector<ll>> mat(maxn + 1, vector<ll>(maxn + 1, 0));
  mat[0][0] = 1;
  for (int n = 1; n <= maxn; n++){
    mat[n][0] = mat[n][n] = 1;
    for (int k = 1; k < n; k++)
      mat[n][k] = (mat[n - 1][k - 1] + mat[n - 1][k]) % mod;
  }
  return mat;
}
//O(N)
vector<int> factorial, inv_factorial;
void prevC3(int maxn, int mod){
  factorial.resize(maxn + 1);
  factorial[0] = 1;
  for (int i = 1; i <= maxn; i++)
    factorial[i] = (factorial[i - 1] * 1LL * i) % mod;
  inv_factorial.resize(maxn + 1);
  inv_factorial[maxn] = fastPow(factorial[maxn], mod - 2, mod);
  for (int i = maxn - 1; i >= 0; i--)
    inv_factorial[i] = (inv_factorial[i + 1] * 1LL * (i + 1)) % mod;
}
int C3(int n, int k, int mod){
  if (n < k)
```

```cpp
      return 0;
      return (((factorial[n] * 1LL * inv_factorial[k]) % mod) * 1LL *
          inv_factorial[n - k]) % mod;
}
//O(P*log(P))
//C4(n, k, p) = Comb(n, k)%p
vector<int> changeBase(int n, int p){
  vector<int> v;
  while (n > 0){
    v.push_back(n % p);
    n /= p;
  }
  return v;
}
int C4(int n, int k, int p){
  auto vn = changeBase(n, p);
  auto vk = changeBase(k, p);
  int mx = max(vn.size(), vk.size());
  vn.resize(mx, 0);
  vk.resize(mx, 0);
  prevC3(p - 1, p);
  int ans = 1;
  for (int i = 0; i < mx; i++)
    ans = (ans * 1LL * C3(vn[i], vk[i], p)) % p;
  return ans;
}
//O(P^k)
//C5(n, k, p, pk) = Comb(n, k)%(p^k)
int fat_p(ll n, int p, int pk){
  vector<int> fat1(pk, 1);
    int res = 1;
    for(int i=1; i<pk; i++){
    if(i%p == 0)
      fat1[i] = fat1[i-1];
    else
      fat1[i] = (fat1[i-1]*1LL*i)%pk;
  }
  while(n > 1){
    res = (res*1LL*fastPow(fat1[pk-1], n/pk, pk))%pk;
    res = (res*1LL*fat1[n%pk])%pk;
    n /= p;
  }
  return res;
}
ll cnt(ll n, int p){
  ll ans = 0;
  while(n > 1){
    ans += n/p;
    n/=p;
  }
  return ans;
}
int C5(ll n, ll k, int p, int pk){
  ll exp = cnt(n, p) - cnt(n-k, p) - cnt(k, p);
  int d = (fat_p(n-k, p, pk)*1LL*fat_p(k, p, pk))%pk;
  int ans = (fat_p(n, p, pk)*1LL*inv(d, pk))%pk;
  return (ans*1LL*fastPow(p, exp, pk))%pk;
}
```

4.4   Chinese Remainder Theorem

```cpp
#include <bits/stdc++.h>
#include "extended_euclidean.h"
using namespace std;
typedef long long ll;
namespace CRT{
  inline ll normalize(ll x, ll mod){
    x %= mod;
    if (x < 0)
      x += mod;
    return x;
  }
  ll solve(vector<ll> a, vector<ll> m){
    int n = a.size();
    for (int i = 0; i < n; i++)
      normalize(a[i], m[i]);
    ll ans = a[0];
    ll lcm1 = m[0];
    for (int i = 1; i < n; i++){
      ll x, y;
      ll g = extGcd(lcm1, m[i], x, y);
      if ((a[i] - ans) % g != 0)
        return -1;
      ans = normalize(ans + (((((a[i] - ans) / g) * x) % (m[i] / g)) *
          lcm1, (lcm1 / g) * m[i]);
      lcm1 = (lcm1 / g) * m[i]; //lcm(lcm1, m[i]);
    }
    return ans;
  }
} // namespace CRT
```

4.5   Euler's totient

```cpp
#include <bits/stdc++.h>
using namespace std;
int nthPhi(int n){
  int result = n;
  for (int i = 2; i <= n / i; i++){
    if (n % i == 0){
      while (n % i == 0)
        n /= i;
      result -= result / i;
    }
  }
  if (n > 1)
    result -= result / n;
  return result;
}
vector<int> phiFrom1toN(int n){
  vector<int> vPhi(n + 1);
  vPhi[0] = 0;
  vPhi[1] = 1;
  for (int i = 2; i <= n; i++)
    vPhi[i] = i;
  for (int i = 2; i <= n; i++){
    if (vPhi[i] == i){
      for (int j = i; j <= n; j += i)
```

```
            vPhi[j] -= vPhi[j] / i;
        }
    }
    return vPhi;
}
```

4.6   Extended Euclidean

```cpp
#include <bits/stdc++.h>
using namespace std;
typedef long long ll;
ll extGcd(ll a, ll b, ll &x, ll &y){
    if (b == 0){
        x = 1, y = 0;
        return a;
    }else{
        ll g = extGcd(b, a % b, y, x);
        y -= (a / b) * x;
        return g;
    }
}
//a*x + b*y = g
//a*(x-(b/g)*k) + b*(y+(a/g)*k) = g
bool dioEq(ll a, ll b, ll c, ll &x0, ll &y0, ll &g){
    g = extGcd(abs(a), abs(b), x0, y0);
    if (c % g) return false;
    x0 *= c / g;
    y0 *= c / g;
    if (a < 0) x0 = -x0;
    if (b < 0) y0 = -y0;
    return true;
}
inline void shift_solution(ll &x, ll &y, ll a, ll b, ll cnt){
    x += cnt * b;
    y -= cnt * a;
}
ll findAllSolutions(ll a, ll b, ll c, ll minx, ll maxx, ll miny, ll
    maxy){
    ll x, y, g;
    if(a==0 or b==0){
        if(a==0 and b==0)
            return (c==0)*(maxx-minx+1)*(maxy-miny+1);
        if(a == 0)
            return (c%b == 0)*(maxx-minx+1)*(miny<=c/b and c/b<=maxy);
        return (c%a == 0)*(minx<=c/a and c/a<=maxx)*(maxy-miny+1);
    }
    if (!dioEq(a, b, c, x, y, g))
        return 0;
    a /= g;
    b /= g;
    int sign_a = a > 0 ? +1 : -1;
    int sign_b = b > 0 ? +1 : -1;
    shift_solution(x, y, a, b, (minx - x) / b);
    if (x < minx)
        shift_solution(x, y, a, b, sign_b);
    if (x > maxx)
        return 0;
    ll lx1 = x;
    shift_solution(x, y, a, b, (maxx - x) / b);
    if (x > maxx)
```

```cpp
        shift_solution(x, y, a, b, -sign_b);
    ll rx1 = x;
    shift_solution(x, y, a, b, -(miny - y) / a);
    if (y < miny)
        shift_solution(x, y, a, b, -sign_a);
    if (y > maxy)
        return 0;
    ll lx2 = x;
    shift_solution(x, y, a, b, -(maxy - y) / a);
    if (y > maxy)
        shift_solution(x, y, a, b, sign_a);
    ll rx2 = x;
    if (lx2 > rx2)
        swap(lx2, rx2);
    ll lx = max(lx1, lx2);
    ll rx = min(rx1, rx2);
    if (lx > rx)
        return 0;
    return (rx - lx) / abs(b) + 1;
}
```

4.7   Fraction

```cpp
#include <bits/stdc++.h>
using namespace std;
typedef long long f_type;
//Representation of the a/b
struct Fraction {
    f_type a, b;
    Fraction(f_type _a = 0): a(_a), b(1){}
    Fraction(f_type _a, f_type _b) {
        f_type g = __gcd(_a, _b);
        a = _a/g;
        b = _b/g;
        if(b < 0){
            a = -a;
            b = -b;
        }
    }
    Fraction operator+(Fraction oth) {
        return Fraction(a*oth.b + oth.a*b, b*oth.b);
    }
    Fraction operator-(Fraction oth) {
        return Fraction(a*oth.b - oth.a*b, b*oth.b);
    }
    Fraction operator*(Fraction oth) {
        return Fraction(a*oth.a, b*oth.b);
    }
    Fraction operator/(Fraction oth) {
        return Fraction(a*oth.b, b*oth.a);
    }
    bool operator>=(Fraction oth){
        return ((*this) - oth).a >= 0;
    }
    bool operator==(Fraction oth){
        return a == oth.a and b == oth.b;
    }
    operator f_type() {return a/b;}
    operator double() {return double(a)/b;}
};
```

4.8 Gray Code

```cpp
int grayCode(int nth){
  return nth ^ (nth >> 1);
}
int revGrayCode(int g){
  int nth = 0;
  for (; g > 0; g >>= 1)
    nth ^= g;
  return nth;
}
```

4.9 Matrix

```cpp
#include <bits/stdc++.h>
#include "modular.h"
using namespace std;
const int D = 3;
struct Matrix{
  int m[D][D];
  Matrix(bool identify = false){
    memset(m, 0, sizeof(m));
    for (int i = 0; i < D; i++)
      m[i][i] = identify;
  }
  Matrix(vector<vector<int>> mat){
    for(int i=0; i<D; i++)
      for(int j=0; j<D; j++)
        m[i][j] = mat[i][j];
  }
  int * operator[](int pos){
    return m[pos];
  }
  Matrix operator*(Matrix oth){
    Matrix ans;
    for (int i = 0; i < D; i++){
      for (int j = 0; j < D; j++){
        int &sum = ans[i][j];
        for (int k = 0; k < D; k++)
          sum = modSum(sum, modMul(m[i][k], oth[k][j]));
      }
    }
    return ans;
  }
};
Matrix fastPow(Matrix base, ll exp){
  Matrix ans(true);
  while(exp){
    if(exp&1LL)
      ans = ans * base;
    base = base*base;
    exp>>=1;
  }
  return ans;
}
```

4.10 Modular Arithmetic

```cpp
#include <bits/stdc++.h>
#include "extended_euclidean.h"
using namespace std;
const int MOD = 1000000007;
inline int modSum(int a, int b, int mod = MOD){
  int ans = a+b;
  if(ans > mod) ans -= mod;
  return ans;
}
inline int modSub(int a, int b, int mod = MOD){
  int ans = a-b;
  if(ans < 0) ans += mod;
  return ans;
}
inline int modMul(int a, int b, int mod = MOD){
  return (a*1LL*b)%mod;
}
int inv(int a, int mod=MOD){
  ll inv_x, y;
  extGcd(a, mod, inv_x, y);
  return (inv_x%mod + mod)%mod;
}
int modDiv(int a, int b, int mod = MOD){
  return modMul(a, inv(b, mod));
}
```

4.11 Montgomery Multiplication

```cpp
#include <bits/stdc++.h>
using namespace std;
using u64 = uint64_t;
using u128 = __uint128_t;
using i128 = __int128_t;
struct u256{
  u128 high, low;
  static u256 mult(u128 x, u128 y){
    u64 a = x >> 64, b = x;
    u64 c = y >> 64, d = y;
    u128 ac = (u128)a * c;
    u128 ad = (u128)a * d;
    u128 bc = (u128)b * c;
    u128 bd = (u128)b * d;
    u128 carry = (u128)(u64)ad + (u128)(u64)bc + (bd >> 64u);
    u128 high = ac + (ad >> 64u) + (bc >> 64u) + (carry >> 64u);
    u128 low = (ad << 64u) + (bc << 64u) + bd;
    return {high, low};
  }
};
//x_m := x*r mod n
struct Montgomery{
  u128 mod, inv, r2;
  //the N will be an odd number
  Montgomery(u128 n) : mod(n), inv(1), r2(-n % n){
    for (int i = 0; i < 7; i++)
      inv *= 2 - n * inv;
    for (int i = 0; i < 4; i++){
```

```
      r2 <<= 1;
      if (r2 >= mod)
        r2 -= mod;
    }
    for (int i = 0; i < 5; i++)
      r2 = mult(r2, r2);
  }
  u128 init(u128 x){
    return mult(x, r2);
  }
  u128 reduce(u256 x){
    u128 q = x.low * inv;
    i128 a = x.high - u256::mult(q, mod).high;
    if (a < 0)
      a += mod;
    return a;
  }
  u128 mult(u128 a, u128 b){
    return reduce(u256::mult(a, b));
  }
};
```

4.12   Prime Number

```
#include <bits/stdc++.h>
#include "basic_math.h"
using namespace std;
typedef unsigned long long ull;
ull modMul(ull a, ull b, ull mod){
  return (a * (__uint128_t)b) % mod;
}
bool checkComposite(ull n, ull a, ull d, int s){
  ull x = fastPow(a, d, n);
  if (x == 1 or x == n - 1)
    return false;
  for (int r = 1; r < s; r++){
    x = modMul(x, x, n);
    if (x == n - 1LL)
      return false;
  }
  return true;
};
bool millerRabin(ull n){
  if (n < 2)
    return false;
  int r = 0;
  ull d = n - 1LL;
  while ((d & 1LL) == 0){
    d >>= 1;
    r++;
  }
  for (ull a : {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37}){
    if (n == a)
      return true;
    if (checkComposite(n, a, d, r))
      return false;
  }
  return true;
}
ull pollard(ull n){
```

```
  auto f = [n](ull x) { return modMul(x, x, n) + 1; };
  ull x = 0, y = 0, t = 0, prd = 2, i = 1, q;
  while (t++ % 40 || __gcd(prd, n) == 1){
    if (x == y)
      x = ++i, y = f(x);
    if ((q = modMul(prd, max(x, y) - min(x, y), n)))
      prd = q;
    x = f(x), y = f(f(y));
  }
  return __gcd(prd, n);
}
vector<ull> factor(ull n){
  if (n == 1)
    return {};
  if (millerRabin(n))
    return {n};
  ull x = pollard(n);
  auto l = factor(x), r = factor(n / x);
  l.insert(l.end(), r.begin(), r.end());
  return l;
}
```

5   Geometry

5.1   Basic Geometry

```
#include <bits/stdc++.h>
using namespace std;
#define POINT_DOUBLE
#ifdef POINT_DOUBLE
  typedef double ftype;
  typedef long double ftLong;
  const double EPS = 1e-9;
  #define eq(a, b) (abs(a - b) < EPS)
  #define lt(a, b) ((a + EPS) < b)
  #define gt(a, b) (a > (b + EPS))
  #define le(a, b) (a < (b + EPS))
  #define ge(a, b) ((a + EPS) > b)
#else
  typedef int32_t ftype;
  typedef int64_t ftLong;
  #define eq(a, b) (a == b)
  #define lt(a, b) (a < b)
  #define gt(a, b) (a > b)
  #define le(a, b) (a <= b)
  #define ge(a, b) (a >= b)
#endif
//Begin Point 2D
struct Point2d{
  ftype x, y;
  Point2d() {}
  Point2d(ftype x, ftype y) : x(x), y(y) {}
  Point2d operator+(const Point2d &t){
    return Point2d(x + t.x, y + t.y);
  }
  Point2d operator-(const Point2d &t){
    return Point2d(x - t.x, y - t.y);
  }
```

```cpp
  Point2d operator*(ftype t){
    return Point2d(x * t, y * t);
  }
  Point2d operator/(ftype t){
    return Point2d(x / t, y / t);
  }
  bool operator<(const Point2d &o) const{
    return lt(x, o.x) or (eq(x, o.x) and lt(y, o.y));
  }
  bool operator==(const Point2d &o) const{
    return eq(x, o.x) and eq(y, o.y);
  }
};
ftLong pw2(ftype a){
  return a * (ftLong)a;
}
//Scalar product
ftLong dot(Point2d a, Point2d b){
  return a.x*(ftLong)b.x + a.y*(ftLong)b.y;
}
ftLong norm(Point2d a){
  return dot(a, a);
}
double len(Point2d a){
  return sqrtl(dot(a, a));
}
double dist(Point2d a, Point2d b){
  return len(a - b);
}
//Vector product
ftLong cross(Point2d a, Point2d b){
  return a.x * (ftLong)b.y - a.y * (ftLong)b.x;
}
//Projection size from A to B
double proj(Point2d a, Point2d b){
  return dot(a, b) / len(b);
}
//The angle between A and B
double angle(Point2d a, Point2d b){
  return acos(dot(a, b) / len(a) / len(b));
}
//Left rotation. Angle in radian
Point2d rotateL(Point2d p, double ang){
  return Point2d(p.x * cos(ang) - p.y * sin(ang), p.x * sin(ang) + p.y
      * cos(ang));
}
//90 degree left rotation
Point2d perpL(Point2d a){
  return Point2d(-a.y, a.x);
}
//0-> 1o,2o quadrant, 1-> 3o,4o
int half(Point2d &p){
  if (gt(p.y, 0) or (eq(p.y, 0) and ge(p.x, 0)))
    return 0;
  else
    return 1;
}
//angle(a) < angle(b)
bool cmpByAngle(Point2d a, Point2d b){
  int ha = half(a), hb = half(b);
  if (ha != hb)
```

```cpp
    return ha < hb;
  else
    return gt(cross(a, b), 0);
}
inline int sgn(ftLong x){
  return ge(x, 0) ? (eq(x, 0) ? 0 : 1) : -1;
}
//-1: angle(a, b) < angle(b, c)
// 0: angle(a, b) = angle(b, c)
//+1: angle(a, b) > angle(b, c)
int cmpAngleBetweenVectors(Point2d a, Point2d b, Point2d c){
  ftLong dotAB = dot(a, b), dotBC = dot(b, c);
  int sgnAB = sgn(dotAB), sgnBC = sgn(dotBC);
  if(sgnAB == sgnBC){
    ftLong l = pw2(dotAB)*dot(c, c), r = pw2(dotBC)*dot(a, a);
    if(l == r)
      return 0;
    if(sgnAB == 1)
      return (l > r)? -1 : +1;
    return (l < r)? -1 : +1;
  }else{
    return (sgnAB > sgnBC)? -1 : +1;
  }
}
//Line parameterized: r1 = a1 + d1*t
//This function can be generalized to 3D
Point2d intersect(Point2d a1, Point2d d1, Point2d a2, Point2d d2){
  return a1 + d1 * (cross(a2 - a1, d2) / cross(d1, d2));
}
//distance between the point(a) e line(pl1, pl2)
//This function can be generalized to 3D
double dist(Point2d a, Point2d pl1, Point2d pl2){
  //crs = parallelogram area
  double crs = cross(Point2d(a - pl1), Point2d(pl2 - pl1));
  //h = area/base
  return abs(crs / dist(pl1, pl2));
}
double area(vector<Point2d> p){
  double ret = 0;
  for (int i = 2; i < (int)p.size(); i++)
    ret += cross(p[i] - p[0], p[i - 1] - p[0]) / 2.0;
  return abs(ret);
}
ftLong signed_area_parallelogram(Point2d p1, Point2d p2, Point2d p3){
  return cross(p2 - p1, p3 - p2);
}
double triangle_area(Point2d p1, Point2d p2, Point2d p3){
  return abs(signed_area_parallelogram(p1, p2, p3)) / 2.0;
}
bool pointInTriangle(Point2d a, Point2d b, Point2d c, Point2d p){
  ftLong s1 = abs(cross(b - a, c - a));
  ftLong s2 = abs(cross(a - p, b - p)) + abs(cross(b - p, c - p)) +
      abs(cross(c - p, a - p));
  return eq(s1, s2);
}
bool clockwise(Point2d p1, Point2d p2, Point2d p3){
  return lt(signed_area_parallelogram(p1, p2, p3), 0);
}
bool counter_clockwise(Point2d p1, Point2d p2, Point2d p3){
  return gt(signed_area_parallelogram(p1, p2, p3), 0);
}
```

```cpp
//End Point 2D

//Begin Line
ftLong det(ftype a, ftype b, ftype c, ftype d){
  return a * (ftLong)d - b * (ftLong)c;
}
struct Line{
  ftype a, b, c;
  Line() {}
  Line(ftype a, ftype b, ftype c) : a(a), b(b), c(c){
    normalize();
  }
  Line(Point2d p1, Point2d p2){
    a = p1.y - p2.y;
    b = p2.x - p1.x;
    c = -a * p1.x - b * p1.y;
    normalize();
  }
  void normalize(){
#ifdef POINT_DOUBLE
    ftype z = sqrt(pw2(a) + pw2(b));
#else
    ftype z = __gcd(abs(a), __gcd(abs(b), abs(c)));
#endif
    a /= z;
    b /= z;
    c /= z;
    if (lt(a, 0) or (eq(a, 0) and lt(b, 0))){
      a = -a;
      b = -b;
      c = -c;
    }
  }
};
bool intersect(Line m, Line n, Point2d &res){
  ftype zn = det(m.a, m.b, n.a, n.b);
  if (eq(zn, 0))
    return false;
  res.x = -det(m.c, m.b, n.c, n.b) / zn;
  res.y = -det(m.a, m.c, n.a, n.c) / zn;
  return true;
}
bool parallel(Line m, Line n){
  return eq(det(m.a, m.b, n.a, n.b), 0);
}
bool equivalent(Line m, Line n){
  return eq(det(m.a, m.b, n.a, n.b), 0) &&
         eq(det(m.a, m.c, n.a, n.c), 0) &&
         eq(det(m.b, m.c, n.b, n.c), 0);
}
//Distance from a point(x, y) to a line m
double dist(Line m, ftype x, ftype y){
  return abs(m.a * (ftLong)x + m.b * (ftLong)y + m.c) /
         sqrt(m.a * (ftLong)m.a + m.b * (ftLong)m.b);
}
//End Line

//Begin Segment
struct Segment{
  Point2d a, b;
  Segment() {}
  Segment(Point2d a, Point2d b) : a(a), b(b) {}
};
bool inter1d(ftype a, ftype b, ftype c, ftype d){
  if (a > b)
    swap(a, b);
  if (c > d)
    swap(c, d);
  return le(max(a, c), min(b, d));
}
bool check_intersection(Segment s1, Segment s2){
  Point2d a = s1.a, b = s1.b, c = s2.a, d = s2.b;
  if (cross(a - c, d - c) == 0 && cross(b - c, d - c) == 0)
    return inter1d(a.x, b.x, c.x, d.x) && inter1d(a.y, b.y, c.y, d.y);
  return sgn(cross(b - a, c - a)) != sgn(cross(b - a, d - a)) &&
         sgn(cross(d - c, a - c)) != sgn(cross(d - c, b - c));
}
inline bool betw(ftype l, ftype r, ftype x){
  return le(min(l, r), x) and le(x, max(l, r));
}
bool intersect(Segment s1, Segment s2, Segment &ans){
  Point2d a = s1.a, b = s1.b, c = s2.a, d = s2.b;
  if (!inter1d(a.x, b.x, c.x, d.x) || !inter1d(a.y, b.y, c.y, d.y))
    return false;
  Line m(a, b);
  Line n(c, d);
  if (parallel(m, n)){
    if (!equivalent(m, n))
      return false;
    if (b < a)
      swap(a, b);
    if (d < c)
      swap(c, d);
    ans = Segment(max(a, c), min(b, d));
    return true;
  }else{
    Point2d p(0, 0);
    intersect(m, n, p);
    ans = Segment(p, p);
    return betw(a.x, b.x, p.x) && betw(a.y, b.y, p.y) &&
           betw(c.x, d.x, p.x) && betw(c.y, d.y, p.y);
  }
}
//End Segment

//Begin Circle
struct Circle{
  ftype x, y, r;
  Circle() {}
  Circle(ftype x, ftype y, ftype r) : x(x), y(y), r(r){};
};
bool pointInCircle(Circle c, Point2d p){
  return ge(c.r, dist(Point2d(c.x, c.y), p));
}
//CircumCircle of a triangle is a circle that passes through all the
//    vertices
Circle circumCircle(Point2d a, Point2d b, Point2d c){
  Point2d u((b - a).y, -((b - a).x));
  Point2d v((c - a).y, -((c - a).x));
  Point2d n = (c - b) * 0.5;
  double t = cross(u, n) / cross(v, u);
  Point2d ct = (((a + c) * 0.5) + (v * t));
```

```cpp
  double r = dist(ct, a);
  return Circle(ct.x, ct.y, r);
}
//InCircle is the largest circle contained in the triangle
Circle inCircle(Point2d a, Point2d b, Point2d c){
  double m1 = dist(a, b);
  double m2 = dist(a, c);
  double m3 = dist(b, c);
  Point2d ct = ((c * m1) + (b * m2) + a * (m3)) / (m1 + m2 + m3);
  double sp = 0.5 * (m1 + m2 + m3);
  double r = sqrt(sp * (sp - m1) * (sp - m2) * (sp - m3)) / sp;
  return Circle(ct.x, ct.y, r);
}
//Minimum enclosing circle, O(n)
Circle minimumCircle(vector<Point2d> p){
  random_shuffle(p.begin(), p.end());
  Circle c = Circle(p[0].x, p[0].y, 0.0);
  for (int i = 0; i < (int)p.size(); i++){
    if (pointInCircle(c, p[i]))
      continue;
    c = Circle(p[i].x, p[i].y, 0.0);
    for (int j = 0; j < i; j++){
      if (pointInCircle(c, p[j]))
        continue;
      c = Circle((p[j].x + p[i].x) * 0.5, (p[j].y + p[i].y) * 0.5, 0.5
           * dist(p[j], p[i]));
      for (int k = 0; k < j; k++){
        if (pointInCircle(c, p[k]))
          continue;
        c = circumCircle(p[j], p[i], p[k]);
      }
    }
  }
  return c;
}
//Return the number of the intersection
int circle_line_intersection(Circle circ, Line line, Point2d &p1,
    Point2d &p2){
  ftLong r = circ.r;
  ftLong a = line.a, b = line.b, c = line.c + line.a * circ.x + line.b
       * circ.y; //take a circle to the (0, 0)
  ftLong x0 = -a * c / (pw2(a) + pw2(b)), y0 = -b * c / (pw2(a) + pw2(
      b));          //(x0, y0) is the shortest distance point of the line
                    for (0, 0)
  if (gt(pw2(c), pw2(r) * (pw2(a) + pw2(b)))){
    return 0;
  }
  else if (eq(pw2(c), pw2(r) * (pw2(a) + pw2(b)))){
    p1.x = p2.x = x0 + circ.x;
    p1.y = p2.y = y0 + circ.y;
    return 1;
  }else{
    ftLong d_2 = pw2(r) - pw2(c) / (pw2(a) + pw2(b));
    ftLong mult = sqrt(d_2 / (pw2(a) + pw2(b)));
    p1.x = x0 + b * mult + circ.x;
    p2.x = x0 - b * mult + circ.x;
    p1.y = y0 - a * mult + circ.y;
    p2.y = y0 + a * mult + circ.y;
    return 2;
  }
}
```

```cpp
//Return the number of the intersection
int circle_intersection(Circle c1, Circle c2, Point2d &p1, Point2d &p2
    ){
  if (eq(c1.x, c2.x) and eq(c1.y, c2.y)){
    if (eq(c1.r, c2.r))
      return -1; //INF
    else
      return 0;
  }else{
    Circle circ(0, 0, c1.r);
    Line line;
    line.a = -2 * (c2.x - c1.x);
    line.b = -2 * (c2.y - c1.y);
    line.c = pw2(c2.x - c1.x) + pw2(c2.y - c1.y) + pw2(c1.r) - pw2(c2.
        r);
    int sz = circle_line_intersection(circ, line, p1, p2);
    p1.x += c1.x;
    p2.x += c1.x;
    p1.y += c1.y;
    p2.y += c1.y;
    return sz;
  }
}

bool checkIfTheSegmentIsCompletelyCoveredByCircles(vector<Circle> &vc,
    Segment s){
  vector<Point2d> v = {s.a, s.b};
  Line l(s.a, s.b);
  for (Circle c : vc){
    Point2d p1, p2;
    int inter = circle_line_intersection(c, l, p1, p2);
    if (inter >= 1 and betw(s.a.x, s.b.x, p1.x) and betw(s.a.y, s.b.y,
        p1.y))
      v.push_back(p1);
    if (inter == 2 and betw(s.a.x, s.b.x, p2.x) and betw(s.a.y, s.b.y,
        p2.y))
      v.push_back(p2);
  }
  sort(v.begin(), v.end());
  bool ans = true;
  for (int i = 1; i < (int)v.size(); i++){
    bool has = false;
    for (Circle c : vc){
      if (pointInCircle(c, v[i - 1]) and pointInCircle(c, v[i])){
        has = true;
        break;
      }
    }
    ans = ans && has;
  }
  return ans;
}

void tangents(Point2d c, double r1, double r2, vector<Line> &ans){
  double r = r2 - r1;
  double z = pw2(c.x) + pw2(c.y);
  double d = z - pw2(r);
  if (lt(d, 0))
    return;
  d = sqrt(abs(d));
  Line l;
```

```cpp
    l.a = (c.x * r + c.y * d) / z;
    l.b = (c.y * r - c.x * d) / z;
    l.c = r1;
    ans.push_back(l);
  }
  vector<Line> tangents(Circle a, Circle b){
    vector<Line> ans;
    for (int i = -1; i <= 1; i += 2)
      for (int j = -1; j <= 1; j += 2)
        tangents(Point2d(b.x - a.x, b.y - a.y), a.r * i, b.r * j, ans);
    for (size_t i = 0; i < ans.size(); ++i){
      ans[i].c -= ans[i].a * a.x + ans[i].b * a.y;
      ans[i].normalize();
    }
    return ans;
  }
  //End Circle
```

5.2   Circle Area Union

```cpp
#include "basic_geometry.h"
using namespace std;

const double PI = acos(-1);
pair<double, double> isCC(Circle circ1, Circle circ2){
  Point2d c1(circ1.x, circ1.y), c2(circ2.x, circ2.y);
  double r1 = circ1.r, r2 = circ2.r;
  double d = dist(c1, c2);
  double x1 = c1.x, x2 = c2.x, y1 = c1.y, y2 = c2.y;
  double mid = atan2(y2 - y1, x2 - x1);
  double a = r1, c = r2;
  double t = acos((a * a + d * d - c * c) / (2 * a * d));
  return make_pair(mid - t, mid + t);
}
int testCC(Circle circ1, Circle circ2){
  Point2d c1(circ1.x, circ1.y), c2(circ2.x, circ2.y);
  double r1 = circ1.r, r2 = circ2.r;
  double d = dist(c1, c2);
  if (le(r1 + r2, d))
    return 1; // not intersected or tged
  if (le(r1 + d, r2))
    return 2; // C1 inside C2
  if (le(r2 + d, r1))
    return 3; // C2 inside C1
  return 0;   // intersected
}
struct event_t{
  double theta;
  int delta;
  event_t(double t, int d) : theta(t), delta(d) {}
  bool operator<(const event_t &r) const{
    if (fabs(theta - r.theta) < EPS)
      return delta > r.delta;
    return theta < r.theta;
  }
};
vector<event_t> e;
void add(double begin, double end){
  if (begin <= -PI)
    begin += 2 * PI, end += 2 * PI;
```

```cpp
  if (end > PI){
    e.push_back(event_t(begin, 1));
    e.push_back(event_t(PI, -1));
    e.push_back(event_t(-PI, 1));
    e.push_back(event_t(end - 2 * PI, -1));
  }else{
    e.push_back(event_t(begin, 1));
    e.push_back(event_t(end, -1));
  }
}
double calc(Point2d c, double r, double a1, double a2){
  double da = a2 - a1;
  double aa = r * r * (da - sin(da)) / 2;
  Point2d p1 = Point2d(cos(a1), sin(a1)) * r + c;
  Point2d p2 = Point2d(cos(a2), sin(a2)) * r + c;
  return cross(p1, p2) / 2 + aa;
}
/* O(n^2logn), please remove coincided circles first. */
double circle_union(vector<Circle> &vc){
  int n = vc.size();
  for (int i = n - 1; i >= 0; i--){
    if (eq(vc[i].r, 0)){
      swap(vc[i], vc[n - 1]);
      n--;
      continue;
    }
    for (int j = 0; j < i; j++){
      if (eq(vc[i].x, vc[j].x) and eq(vc[i].y, vc[j].y) and eq(vc[i].r
          , vc[j].r)){
        swap(vc[i], vc[n - 1]);
        n--;
      }
    }
  }
  if (n == 0)
    return 0;
  vc.resize(n);
  vector<double> cntarea(2 * n, 0);
  for (int c = 0; c < n; c++){
    int cvrcnt = 0;
    e.clear();
    for (int i = 0; i < n; i++){
      if (i != c){
        int r = testCC(vc[c], vc[i]);
        if (r == 2){
          cvrcnt++;
        } else if (r == 0){
          auto paa = isCC(vc[c], vc[i]);
          add(paa.first, paa.second);
        }
      }
    }
    if (e.size() == 0){
      double a = PI * vc[c].r * vc[c].r;
      cntarea[cvrcnt] -= a;
      cntarea[cvrcnt + 1] += a;
    } else {
      e.push_back(event_t(-PI, 1));
      e.push_back(event_t(PI, -2));
      sort(e.begin(), e.end());
      for (int i = 0; i < int(e.size()) - 1; i++){
```

```
        cvrcnt += e[i].delta;
        double a = calc(Point2d(vc[c].x, vc[c].y), vc[c].r, e[i].theta
            , e[i + 1].theta);
        cntarea[cvrcnt - 1] -= a;
        cntarea[cvrcnt] += a;
      }
    }
  }
  double ans = 0;
  for(int i=1; i<=n; i++)
    ans += cntarea[i];
  return ans;
}
```

5.3 Circles to Tree

```
#include <bits/stdc++.h>
using namespace std;
typedef pair<int, int> pii;
struct Circle{
  int x, y, r, id;
  Circle(){}
  Circle(int x1, int y1, int r1, int id1): x(x1), y(y1), r(r1), id(id1
      ){}
};
// a^2 + b^2 == c^2
double findB(double a, double c){
  return sqrt(c*c - a*a);
}
//- There is no intersection between the circles
//- The parent of circle i will be the smallest circle that includes i
namespace CirclesToTree{
  int X = 0;
  int n;
  vector<Circle> vc;
  vector<int> p;
  struct SetElement{
    int id;
    int side; //Up:1, Down:-1
    SetElement(int id1, int side1): id(id1), side(side1){};
    double getY(int x = X) const{
      return vc[id].y + side*findB(vc[id].x - x, vc[id].r);
    }
    bool operator <(const SetElement &o) const{
      auto l = getY(), r = o.getY();
      if(abs(l-r)<1e-9)
        return vc[id].r*side < vc[o.id].r*o.side;
      else
        return l < r;
    }
  };
  long long pw2(int a){
    return a*1LL*a;
  }
  bool contains(int big, int small){
    if(big == -1 or small == -1) return false;
    Circle &s = vc[small], &b = vc[big];
    if(s.r > b.r) return false;
    return pw2(s.x-b.x) + pw2(s.y-b.y) <= pw2(b.r-s.r);
  }
```

```
  void updateParent(int id, int par){
    if(par != -1 and p[id] == -1) p[id] = par;
  }
//Public
  vector<vector<int>> solve(vector<Circle> circles){
    vc = circles; n = vc.size();
    p.assign(n, -1);
    vector<vector<int>> adj(n, vector<int>());
    vector<pii> events;
    for(auto c: vc){
      events.emplace_back(c.x-c.r, ~c.id);
      events.emplace_back(c.x+c.r, c.id);
    }
    sort(events.begin(), events.end());
    set<SetElement> st;
    for(auto e: events){
      X = e.first;
      int id = e.second;
      if(id < 0){
        id = ~id;
        auto it = st.lower_bound(SetElement(id, -2));
        if(it != st.end()){
          int id2 = it->id;
          if(contains(id2, id)) updateParent(id, id2);
          if(contains(p[id2], id)) updateParent(id, p[id2]);
        }
        if(it != st.begin()){
          it--;
          int id2 = it->id;
          if(contains(id2, id)) updateParent(id, id2);
          if(contains(p[id2], id)) updateParent(id, p[id2]);
        }
        st.emplace(id, 1);
        st.emplace(id, -1);
        if(p[id] != -1){
          adj[p[id]].push_back(id);
        }
      }else{
        st.erase(SetElement(id, 1));
        st.erase(SetElement(id, -1));
      }
    }
    return adj;
  }
};
```

5.4 Count Lattices

```
#include "../../code/math/fraction.h"
Fraction f_1 = 1;
//Calculates number of integer points (x,y) such for 0<=x<n and 0<y<=
    floor(k*x+b)
//O(log(N)*log(MAXV))
f_type count_lattices(Fraction k, Fraction b, f_type n) {
  auto fk = (f_type)k;
  auto fb = (f_type)b;
  auto cnt = 0LL;

  if (k >= f_1 || b >= f_1) {
    cnt += (fk * (n - 1) + 2 * fb) * n / 2;
```

```cpp
        k = k - Fraction(fk, 1);
        b = b - Fraction(fb, 1);
    }
    auto t = k * Fraction(n, 1) + b;
    auto ft = (f_type)t;
    if (ft >= 1) {
        cnt += count_lattices(f_1 / k, (t - Fraction((f_type)t, 1)) / k, (
            f_type)t);
    }
    return cnt;
}
```

5.5 Convex Hull

```cpp
#include "basic_geometry.h"
using namespace std;
//If accept collinear points then change for <=
bool cw(Point2d a, Point2d b, Point2d c) {
    return lt(cross(b - a, c - b), 0);
}
//If accept collinear points then change for >=
bool ccw(Point2d a, Point2d b, Point2d c) {
    return gt(cross(b - a, c - b), 0);
}
vector<Point2d> convex_hull(vector<Point2d> a){
    if (a.size() == 1)
        return a;
    sort(a.begin(), a.end());
    a.erase(unique(a.begin(), a.end()), a.end());
    vector<Point2d> up, down;
    Point2d p1 = a[0], p2 = a.back();
    up.push_back(p1);
    down.push_back(p1);
    for (int i = 1; i < (int)a.size(); i++){
        if ((i == int(a.size() - 1)) || cw(p1, a[i], p2)){
            while (up.size() >= 2 && !cw(up[up.size() - 2], up[up.size() -
                1], a[i]))
                up.pop_back();
            up.push_back(a[i]);
        }
        if ((i == int(a.size() - 1)) || ccw(p1, a[i], p2)){
            while (down.size() >= 2 && !ccw(down[down.size() - 2], down[down
                .size() - 1], a[i]))
                down.pop_back();
            down.push_back(a[i]);
        }
    }
    a.clear();
    for (int i = 0; i < (int)up.size(); i++)
        a.push_back(up[i]);
    for (int i = down.size() - 2; i > 0; i--)
        a.push_back(down[i]);
    return a;
}
```

5.6 Convex Hull Trick

```cpp
#include "basic_geometry.h"
```

```cpp
using namespace std;
struct LineCHT{
    ftype k, b;
    int id;
    LineCHT() {}
    LineCHT(ftype k, ftype b, int id=-1): k(k), b(b), id(id) {}
};
struct ConvexHullTrick{
    vector<Point2d> hull, vecs;
    ConvexHullTrick(){}
    ConvexHullTrick(vector<LineCHT> v){
        sort(v.begin(), v.end(), [&](LineCHT a, LineCHT b){
            return a.k < a.k;
        });
        for(auto l: v)
            add_line(l.k, l.b);
    }
    //Here we will assume that when linear functions are added, their k
        only increases and we want to find minimum values.
    void add_line(ftype k, ftype b) {
        Point2d nw(k, b);
        while(!vecs.empty() && lt(dot(vecs.back(), nw - hull.back()), 0))
            {
            hull.pop_back();
            vecs.pop_back();
        }
        if(!hull.empty())
            vecs.push_back(perpL(nw - hull.back()));
        hull.push_back(nw);
    }
    //Find minimum value
    ftLong get(ftype x) {
        Point2d query(x, 1);
        auto it = lower_bound(vecs.begin(), vecs.end(), query, [](Point2d
            a, Point2d b) {
            return gt(cross(a, b), 0);
        });
        return dot(query, hull[it - vecs.begin()]);
    }
};
```

5.7 Nearest Pair Of Points

```cpp
#include <bits/stdc++.h>
using namespace std;
struct pt {
    long long x, y, id;
    pt(){}
    pt(int _x, int _y, int _id=-1):x(_x), y(_y), id(_id){}
};
namespace NearestPairOfPoints{
    struct cmp_x {
        bool operator()(const pt & a, const pt & b) const {
            return a.x < b.x || (a.x == b.x && a.y < b.y);
        }
    };
    struct cmp_y {
        bool operator()(const pt & a, const pt & b) const {
            return a.y < b.y;
        }
```

```cpp
    };
    int n;
    vector<pt> v;
    vector<pt> t;
    double mindist;
    pair<int, int> best_pair;
    void upd_ans(const pt & a, const pt & b) {
      double dist = sqrt((a.x - b.x)*(a.x - b.x) + (a.y - b.
          y));
      if (dist < mindist) {
        mindist = dist;
        best_pair = {a.id, b.id};
      }
    }
    void rec(int l, int r) {
      if (r - l <= 3) {
        for (int i = l; i < r; ++i) {
          for (int j = i + 1; j < r; ++j) {
            upd_ans(v[i], v[j]);
          }
        }
        sort(v.begin() + l, v.begin() + r, cmp_y());
        return;
      }
      int m = (l + r) >> 1;
      int midx = v[m].x;
      rec(l, m);
      rec(m, r);
      merge(v.begin() + l, v.begin() + m, v.begin() + m, v.begin() + r,
          t.begin(), cmp_y());
      copy(t.begin(), t.begin() + r - l, v.begin() + l);
      int tsz = 0;
      for (int i = l; i < r; ++i) {
        if (abs(v[i].x - midx) < mindist) {
          for (int j = tsz - 1; j >= 0 && v[i].y - t[j].y < mindist; --j
              )
            upd_ans(v[i], t[j]);
          t[tsz++] = v[i];
        }
      }
    }
    pair<int, int> solve(vector<pt> _v){
      v = _v;
      n = v.size();
      t.resize(n);
      sort(v.begin(), v.end(), cmp_x());
      mindist = 1E20;
      rec(0, n);
      return best_pair;
    }
};
```

5.8   Point 3D

```cpp
#include <bits/stdc++.h>
using namespace std;
//#define POINT_DOUBLE
#ifdef POINT_DOUBLE
  typedef double ftype;
  typedef long double ftLong;
```

```cpp
  const double EPS = 1e-9;
  #define eq(a, b) (abs(a-b)<EPS)
  #define lt(a, b) ((a+EPS)<b)
  #define gt(a, b) (a>(b+EPS))
  #define le(a, b) (a<(b+EPS))
  #define ge(a, b) ((a+EPS)>b)
#else
  typedef int32_t ftype;
  typedef int64_t ftLong;
  #define eq(a, b) (a==b)
  #define lt(a, b) (a<b)
  #define gt(a, b) (a>b)
  #define le(a, b) (a<=b)
  #define ge(a, b) (a>=b)
#endif
//Point3D
struct Point3d{
  ftype x, y, z;
  Point3d() {}
  Point3d(ftype x, ftype y, ftype z) : x(x), y(y), z(z) {}
  Point3d operator+(Point3d t){
    return Point3d(x + t.x, y + t.y, z + t.z);
  }
  Point3d operator-(Point3d t){
    return Point3d(x - t.x, y - t.y, z - t.z);
  }
  Point3d operator*(ftype t){
    return Point3d(x * t, y * t, z * t);
  }
  Point3d operator/(ftype t){
    return Point3d(x / t, y / t, z / t);
  }
};
ftLong dot(Point3d a, Point3d b){
  return a.x * (ftLong)b.x + a.y * (ftLong)b.y + a.z * (ftLong)b.z;
}
double len(Point3d a){
  return sqrt(dot(a, a));
}
double dist(Point3d a, Point3d b){
  return len(a-b);
}
double proj(Point3d a, Point3d b){
  return dot(a, b) / len(b);
}
//theta -> XY; phi -> ZY;
Point3d toVetor(double theta, double phi, double r){
  return Point3d(r*cos(theta)*sin(phi), r*sin(theta)*sin(phi), r*cos(
      phi));
}
double getAngleTheta(Point3d p){
  return atan2(p.y, p.x);
}
double getAnglePhi(Point3d p){
  return acos(p.z/len(p));
}
Point3d rotateX(Point3d p, double ang){
  return Point3d(p.x, p.y*cos(ang)-p.z*sin(ang), p.y*sin(ang)+p.z*cos(
      ang));
}
Point3d rotateY(Point3d p, double ang){
```

```
  return Point3d(p.x*cos(ang)+p.z*sin(ang), p.y, -p.x*sin(ang)+p.z*cos
      (ang));
}
Point3d rotateZ(Point3d p, double ang){
  return Point3d(p.x*cos(ang)-p.y*sin(ang), p.x*sin(ang)+p.y*cos(ang),
      p.z);
}
//Rotation in relation to the normal axis
Point3d rotateNormal(Point3d v, Point3d n, double ang){
  double theta = getAngleTheta(n);
  double phi = getAnglePhi(n);
  v = rotateZ(v, -theta);
  v = rotateY(v, -phi);
  v = rotateZ(v, ang);
  v = rotateY(v, phi);
  v = rotateZ(v, theta);
  return v;
}
Point3d cross(Point3d a, Point3d b){
  return Point3d(a.y * b.z - a.z * b.y,
                 a.z * b.x - a.x * b.z,
                 a.x * b.y - a.y * b.x);
}
ftLong triple(Point3d a, Point3d b, Point3d c){
  return dot(a, cross(b, c));
}
Point3d planeIntersect(Point3d a1, Point3d n1, Point3d a2, Point3d n2,
    Point3d a3, Point3d n3){
  Point3d x(n1.x, n2.x, n3.x);
  Point3d y(n1.y, n2.y, n3.y);
  Point3d z(n1.z, n2.z, n3.z);
  Point3d d(dot(a1, n1), dot(a2, n2), dot(a3, n3));
  return Point3d(triple(d, y, z),
                 triple(x, d, z),
                 triple(x, y, d)) / triple(n1, n2, n3);
}
struct Sphere{
  ftype x, y, z, r;
  Sphere(){}
  Sphere(ftype x, ftype y, ftype z,  ftype r):x(x), y(y), z(z), r(r){}
};
//Minimum enclosing Sphere, O(n*70000)
//It is also possible to do with ternary search in the 3 dimensions
Sphere minimumSphere(vector<Point3d> vp){
  Point3d ans(0, 0, 0);
  int n = vp.size();
  for(Point3d p: vp)
    ans = ans + p;
  ans = ans/n;
  double P = 0.1;
  double d = 0, e = 0;
  for(int i = 0; i < 70000; i++){
    int f = 0;
    d = dist(ans, vp[0]);
    for (int j = 1; j < n; j++) {
      e = dist(ans, vp[j]);
      if (d < e) {
        d = e;
        f = j;
      }
    }
```

```
    ans = ans + (vp[f]-ans)*P;
    P *= 0.998;
  }
  return Sphere(ans.x, ans.y, ans.z, d);
}
```

6 String Algorithms

6.1 Aho Corasick

```
#include <bits/stdc++.h>
#define F first
#define S second
using namespace std;
const int K = 26;
inline int getID(char c){
  return c-'a';
}
namespace Aho{
  struct Vertex {
    int next[K], go[K];
    int leaf = -1; // CAUTION with repeated strings!
    int p = -1, sz, match=-1;
    char pch;
    int suff_link = -1;
    int end_link = -1;
    Vertex(int p1=-1, char ch1='$', int sz1=0) : p(p1), pch(ch1){
      fill(begin(next), end(next), -1);
      fill(begin(go), end(go), -1);
      sz = sz1;
    }
  };
  vector<Vertex> trie;
  void init(){
    trie.clear();
    trie.emplace_back();
  }
  int add_string(string const& s, int id=1) {
    int v = 0;
    for (char ch : s) {
      int c = getID(ch);
      if (trie[v].next[c] == -1) {
        trie[v].next[c] = trie.size();
        trie.emplace_back(v, ch, trie[v].sz+1);
      }
      v = trie[v].next[c];
    }
    trie[v].leaf = id;
    return v;
  }
  int go(int v, char ch);
  int get_suff_link(int v) {
    if (trie[v].suff_link == -1) {
      if (v == 0 || trie[v].p == 0)
        trie[v].suff_link = 0;
      else
        trie[v].suff_link = go(get_suff_link(trie[v].p), trie[v].pch);
    }
```

```
      return trie[v].suff_link;
    }
    int get_end_link(int v) {
      if (trie[v].end_link == -1) {
        if (v == 0 || trie[v].p == 0){
          trie[v].end_link = 0;
        }else{
          int suff_link = get_suff_link(v);
          if(trie[suff_link].leaf != -1)
            trie[v].end_link = suff_link;
          else
            trie[v].end_link = get_end_link(suff_link);
        }
      }
      return trie[v].end_link;
    }
    int go(int v, char ch) {
      int c = getID(ch);
      if (trie[v].go[c] == -1) {
        if (trie[v].next[c] != -1)
          trie[v].go[c] = trie[v].next[c];
        else
          trie[v].go[c] = (v == 0) ? 0 : go(get_suff_link(v), ch);
      }
      return trie[v].go[c];
    }
};
//Aplication:
typedef pair<int, int> pii;
void addMatch(vector<pii> &ans, int v, int i){
    // This runs at most sqrt(N) times:1+2+3+4+..+sqrt(N)=N
    while(v != 0){
        // The string id is Aho::trie[v].leaf
        ans.emplace_back(i - Aho::trie[v].sz + 1, i);
        v = Aho::get_end_link(v);
    }
}
//Get match positions: O(answer) = O(N * sqrt(N))
vector<pii> whatMatch(string t){
    int state = 0;
    int i=0;
    vector<pii> ans;
    for(char c : t){
        state = Aho::go(state, c);
        if(Aho::trie[state].leaf != -1)
            addMatch(ans, state, i);
        else
            addMatch(ans, Aho::get_end_link(state), i);
        i++;
    }
    sort(ans.begin(), ans.end());
    return ans;
}

int countMatch(int v){
    if(Aho::trie[v].match == -1) {
        if (v == 0 || Aho::trie[v].p == 0){
            if(Aho::trie[v].leaf != -1)
                Aho::trie[v].match = 1;
            else
                Aho::trie[v].match = 0;
```

```
        }else{
            if(Aho::trie[v].leaf != -1)
                Aho::trie[v].match = 1 + countMatch(Aho::get_end_link(v));
            else
                Aho::trie[v].match = countMatch(Aho::get_end_link(v));
        }
    }
    return Aho::trie[v].match;
}
//Get match amount: O(t)
long long matchAmount(string t){
    int state = 0;
    long long ans = 0;
    for(char c : t){
        state = Aho::go(state, c);
        ans += countMatch(state);
    }
    return ans;
}
```

6.2 Min Cyclic String

```
#include <bits/stdc++.h>
using namespace std;
string min_cyclic_string(string s){
    s += s;
    int n = s.size();
    int i = 0, ans = 0;
    while (i < n / 2){
        ans = i;
        int j = i + 1, k = i;
        while (j < n && s[k] <= s[j]){
            if (s[k] < s[j])
                k = i;
            else
                k++;
            j++;
        }
        while (i <= k)
            i += j - k;
    }
    return s.substr(ans, n / 2);
}
```

6.3 String Hashing

```
#include <bits/stdc++.h>
using namespace std;
struct StringHashing{
    const uint64_t MOD = (1ll<<61) - 1;
    const int base = 31;
    const uint64_t invBase = 2231460976658413501uLL; //31^-1 % MOD
    uint64_t modMul(uint64_t a, uint64_t b){
        uint64_t l1 = (uint32_t)a, h1 = a>>32, l2 = (uint32_t)b, h2 = b
            >>32;
        uint64_t l = l1*l2, m = l1*h2 + l2*h1, h = h1*h2;
        uint64_t ret = (l&MOD) + (l>>61) + (h << 3) + (m >> 29) + ((m <<
            35) >> 3) + 1;
```

```cpp
      ret = (ret & MOD) + (ret>>61);
      ret = (ret & MOD) + (ret>>61);
      return ret-1;
    }
    int getInt(char c){
      return c-'a'+1;
    }
    vector<uint64_t> pre, p, inv;
    //Public:
    StringHashing(string s){
      int n = s.size();
      pre.resize(n);
      p.resize(n);
      inv.resize(n);
      p[0] = 1;
      inv[0] = 1;
      pre[0] = getInt(s[0]);
      for(int i=1; i<n; i++){
        p[i] = modMul(p[i-1], base);
        inv[i] = modMul(inv[i-1], invBase);
        pre[i] = (pre[i-1] + modMul(p[i], getInt(s[i])))%MOD;
      }
    }
    uint64_t getValue(int i, int j){
      return modMul(inv[i], (pre[j] - ((i>0)?pre[i-1]:0) + MOD)%MOD);
    }
};
```

6.4   Suffix Automaton

```cpp
#include <bits/stdc++.h>
using namespace std;
typedef long long ll;
struct SuffixAutomaton{
  struct state{
    int len, link, first_pos;
    bool is_clone = false;
    map<char, int> next;
  };
  vector<state> st;
  int sz, last;
  SuffixAutomaton(string s){
    st.resize(2 * s.size() + 10);
    st[0].len = 0;
    st[0].link = -1;
    st[0].is_clone = false;
    sz = 1;
    last = 0;
    for (char c : s)
      insert(c);
    preCompute();
  }
  void insert(char c){
    int cur = sz++;
    st[cur].len = st[last].len + 1;
    st[cur].first_pos = st[cur].len - 1;
    st[cur].is_clone = false;
    int p = last;
    while (p != -1 && !st[p].next.count(c)){
      st[p].next[c] = cur;
```

```cpp
      p = st[p].link;
    }
    if (p == -1){
      st[cur].link = 0;
    }else{
      int q = st[p].next[c];
      if (st[p].len + 1 == st[q].len){
        st[cur].link = q;
      }else{
        int clone = sz++;
        st[clone].len = st[p].len + 1;
        st[clone].next = st[q].next;
        st[clone].link = st[q].link;
        st[clone].first_pos = st[q].first_pos;
        st[clone].is_clone = true;
        while (p != -1 && st[p].next[c] == q){
          st[p].next[c] = clone;
          p = st[p].link;
        }
        st[q].link = st[cur].link = clone;
      }
    }
    last = cur;
  }
  string lcs(string s){
    int v = 0, l = 0, best = 0, bestpos = 0;
    for (int i = 0; i < (int)s.size(); i++){
      while (v and !st[v].next.count(s[i])){
        v = st[v].link;
        l = st[v].len;
      }
      if (st[v].next.count(s[i])){
        v = st[v].next[s[i]];
        l++;
      }
      if (l > best){
        best = l;
        bestpos = i;
      }
    }
    return s.substr(bestpos - best + 1, best);
  }
  vector<ll> dp;
  vector<int> cnt;
  ll dfsPre(int s){
    if (dp[s] != -1)
      return dp[s];
    dp[s] = cnt[s]; //Accepts repeated substrings
    //dp[s] = 1; //Does not accept repeated substrings
    for (auto p : st[s].next)
      dp[s] += dfsPre(p.second);
    return dp[s];
  }
  void preCompute(){
    cnt.assign(sz, 0);
    vector<pair<int, int>> v(sz);
    for (int i = 0; i < sz; i++){
      cnt[i] = !st[i].is_clone;
      v[i] = make_pair(st[i].len, i);
    }
    sort(v.begin(), v.end(), greater<pair<int, int>>());
```

```
    for (int i = 0; i < sz - 1; i++)
      cnt[st[v[i].second].link] += cnt[v[i].second];
    dp.assign(sz, -1);
    dfsPre(0);
  }
};
```

6.5 Suffix Tree

```cpp
#include <bits/stdc++.h>
typedef long long ll;
using namespace std;
namespace SuffixTree {
const int NS = 60; //Number of strings
const int MAXN = 100010; //Number of letters
int cn, cd, ns, en = 1, lst;
string S[NS]; int lastS = -1;
/* sufn[si][i] no do sufixo S[si][i...] */
vector<int> sufn[NS];
struct Node {
  int l, r, si=0;
  int p, suf=0;
  map<char, int> adj;
  Node() : l(0), r(-1){ suf = p = 0; }
  Node(int l1, int r1, int s1, int p1) : l(l1), r(r1), si(s1), p(p1)
      {}
  inline int len() { return r - l + 1; }
  inline int operator[](int i) { return S[si][l + i]; }
  inline int& operator()(char c) { return adj[c]; }
};
Node t[2*MAXN];
inline int new_node(int l, int r, int s, int p) {
  t[en] = Node(l, r, s, p);
  return en++;
}
void init(){
  t[0] = Node();
  cn=0, cd=0, ns=0, en = 1, lst=0;
  lastS = -1;
}
//The strings are inserted independently
void add_string(string s, char id='$') {
  assert(id < 'A');
  s += id;
  S[++lastS] = s;
  sufn[lastS].resize(s.size() + 1);
  cn = cd = 0;
  int i = 0; const int n = s.size();
  for(int j = 0; j < n; j++){
    for(; i <= j; i++) {
      if(cd == t[cn].len() && t[cn](s[j]))
        cn = t[cn](s[j]), cd = 0;
      if(cd < t[cn].len() && t[cn][cd] == s[j]) {
        cd++;
        if(j < (int)s.size() - 1) break;
        else {
          if(i) t[lst].suf = cn;
          for(; i <= j; i++) {
            sufn[lastS][i] = cn;
            cn = t[cn].suf;
```

```cpp
          }
        }
      } else if(cd == t[cn].len()) {
        sufn[lastS][i] = en;
        if(i) t[lst].suf = en;
        lst = en;
        t[cn](s[j]) = new_node(j, n - 1, lastS, cn);
        cn = t[cn].suf;
        cd = t[cn].len();
      } else {
        int mid = new_node(t[cn].l, t[cn].l + cd - 1, t[cn].si, t[cn].
            p);
        t[t[cn].p](t[cn][0]) = mid;
        if(ns) t[ns].suf = mid;
        if(i) t[lst].suf = en;
        lst = en;
        sufn[lastS][i] = en;
        t[mid](s[j]) = new_node(j, n - 1, lastS, mid);
        t[mid](t[cn][cd]) = cn;
        t[cn].p = mid; t[cn].l += cd;
        cn = t[mid].p;
        int g = cn? j - cd : i + 1;
        cn = t[cn].suf;
        while(g < j && g + t[t[cn](S[lastS][g])].len() <= j)
          cn = t[cn](S[lastS][g]), g += t[cn].len();
        if(g == j)
          ns = 0, t[mid].suf = cn, cd = t[cn].len();
        else
          ns = mid, cn = t[cn](S[lastS][g]), cd = j - g;
      }
    }
  }
}
bool match(string &s, int i=0, int no=0, int iEdge=0){
  if(i == (int)s.size())
    return true;
  if(iEdge == t[no].len()){ //I arrived at the Node
    if(t[no].adj.count(s[i]))
      return match(s, i+1, t[no].adj[s[i]], 1);
    else
      return false;
  }
  if(t[no][iEdge] == s[i])
    return match(s, i+1, no, iEdge+1);
  return false;
}
};
```

6.6 Trie

```cpp
#include <bits/stdc++.h>
using namespace std;
const int K = 26;
inline int getId(char c){
  return c - 'a';
}
struct Vertex {
  int next[K];
  int leaf;
  int count;
```

```cpp
  Vertex() {
    fill(begin(next), end(next), -1);
    leaf = 0;
    count = 0;
  }
};
struct Trie{
  vector<Vertex> trie;
  Trie(){
    trie.emplace_back();
  }
  void add(string const& s) {
    int v = 0;
    trie[v].count++;
    for(char ch: s) {
      int c = getId(ch);
      if (trie[v].next[c] == -1) {
        trie[v].next[c] = trie.size();
        trie.emplace_back();
      }
      v = trie[v].next[c];
      trie[v].count++;
    }
    trie[v].leaf++;
  }
  int countStr(string const& s) {
    int v = 0;
    for (char ch : s) {
      int c = getId(ch);
      if (trie[v].next[c] == -1)
        return 0;
      v = trie[v].next[c];
    }
    return trie[v].leaf;
  }
  int countPre(string const& s) {
    int v = 0;
    for (char ch : s) {
      int c = getId(ch);
      if (trie[v].next[c] == -1)
        return 0;
      v = trie[v].next[c];
    }
    return trie[v].count;
  }
  bool remove(string const& s) {
    vector<int> rm;
    int v = 0;
    rm.push_back(v);
    for(char ch: s) {
      int c = getId(ch);
      if (trie[v].next[c] == -1)
        return false;
      v = trie[v].next[c];
      rm.push_back(v);
    }
    if(trie[v].leaf > 0){
      trie[v].leaf--;
      for(int x: rm)
        trie[x].count--;
      return true;
```

```cpp
    }
    return false;
  }
};
```

7 Miscellaneous

7.1 Longest Increasing Subsequence

```cpp
#include <bits/stdc++.h>
using namespace std;
vector<int> lis(vector<int> &v){
  vector<int> st, ans;
  vector<int> pos(v.size()+1), dad(v.size()+1);
  for(int i=0; i < (int)v.size(); i++){
    auto it = lower_bound(st.begin(), st.end(), v[i]); // Do not
        accept repeated values
    //auto it = upper_bound(st.begin(), st.end(), v[i]); //Accept
        repeated values
    int p = it-st.begin();
    if(it==st.end())
      st.push_back(v[i]);
    else
      *it = v[i];
    pos[p] = i;
    dad[i] = (p==0)? -1 : pos[p-1];
  }
  int p = pos[st.size() - 1];
  while(p >= 0){
    ans.push_back(v[p]);
    p=dad[p];
  }
  reverse(ans.begin(), ans.end());
  return ans;
}
```

7.2 Mo Algorithm

```cpp
#include <bits/stdc++.h>
using namespace std;
const int BLOCK_SIZE = 700;
void remove(int idx);
void add(int idx);
void clearAnswer();
int getAnswer();
struct Query{
  int l, r, idx;
  bool operator<(Query other) const{
    if (l / BLOCK_SIZE != other.l / BLOCK_SIZE)
      return l < other.l;
    return (l / BLOCK_SIZE & 1) ? (r < other.r) : (r > other.r);
  }
};
vector<int> mo_s_algorithm(vector<Query> queries){
  vector<int> answers(queries.size());
  sort(queries.begin(), queries.end());
  clearAnswer();
```

```cpp
  int L = 0, R = 0;
  add(0);
  for(Query q : queries){
    while(q.l < L) add(--L);
    while(R < q.r) add(++R);
    while(L < q.l) remove(L++);
    while(q.r < R) remove(R--);
    answers[q.idx] = getAnswer();
  }
  return answers;
}
```

7.3 Polyominoes

```cpp
#include <bits/stdc++.h>
#define F first
#define S second
using namespace std;
const int MAXP = 10;
typedef pair<int, int> pii;
//This implementation considers the rotations as distinct
//              0, 10, 10+9, 10+9+8...
int pos[11] = {0, 10, 19, 27, 34, 40, 45, 49, 52, 54, 55};
struct Polyominoes{
  pii v[MAXP];
  int64_t id;
  int n;
  Polyominoes(){
    n = 1;
    v[0] = {0, 0};
    normalize();
  }
  pii& operator[](int i){
    return v[i];
  }
  bool add(int a, int b){
    for(int i=0; i<n; i++)
      if(v[i].F == a and v[i].S == b)
        return false;
    v[n++] = pii(a, b);
    normalize();
    return true;
  }
  void normalize(){
    int mnx=100, mny=100;
    for(int i=0; i<n; i++)
      mnx = min(mnx, v[i].F), mny = min(mny, v[i].S);
    id = 0;
    for(int i=0; i<n; i++){
      v[i].F -= mnx, v[i].S -= mny;
      id |= (1LL<<(pos[v[i].F] + v[i].S));
    }
  }
};
vector<Polyominoes> polyominoes[MAXP+1];
int dx[] = {0, 0, -1, 1};
int dy[] = {-1, 1, 0, 0};
void buildPolyominoes(int mxN=10){
  for(int i=0; i<=mxN; i++)
    polyominoes[i].clear();
```

```cpp
  Polyominoes init;
  queue<Polyominoes> q;
  unordered_set<int64_t> used;
  q.push(init);
  used.insert(init.id);
  while(!q.empty()){
    Polyominoes u = q.front(); q.pop();
    polyominoes[u.n].push_back(u);
    if(u.n == mxN)
      continue;
    for(int i=0; i<u.n; i++){
      for(int j=0; j<4; j++){
        Polyominoes to = u;
        bool ok = to.add(to[i].F + dx[j], to[i].S + dy[j]);
        if(ok and !used.count(to.id)){
          q.push(to);
          used.insert(to.id);
        }
      }
    }
  }
}
```

8 Theorems and Formulas

8.1 Binomial Coefficients

$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{k}a^{n-k}b^k + \cdots + \binom{n}{n}b^n$

Pascal's Triangle: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Symmetry rule: $\binom{n}{k} = \binom{n}{n-k}$

Factoring in: $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$

Sum over k: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Sum over n: $\sum_{m=0}^{n} \binom{m}{k} = \binom{n+1}{k+1}$

Sum over n and k: $\sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}$

Sum of the squares: $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$

Weighted sum: $1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n2^{n-1}$

Connection with the Fibonacci numbers: $\binom{n}{0} + \binom{n-1}{1} + \cdots + \binom{n-k}{k} + \cdots + \binom{0}{n} = F_{n+1}$

More formulas: $\sum_{k=0}^{m}(-1)^k \cdot \binom{n}{k} = (-1)^m \cdot \binom{n-1}{m}$ ⎯⎯⎯⎯⎯⎯⎯⎯

8.2 Catalan Number

Recursive formula: $C_0 = C_1 = 1$

$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}, n \geq 2$

Analytical formula: $C_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}, n \geq 0$

The first few numbers Catalan numbers, C_n (starting from zero): $1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots$

The Catalan number C_n is the solution for:

- Number of correct bracket sequence consisting of n opening and n closing brackets.

- The number of rooted full binary trees with $n+1$ leaves (vertices are not numbered). A rooted binary tree is full if every vertex has either two children or no children.

- The number of ways to completely parenthesize $n+1$ factors.

- The number of triangulations of a convex polygon with $n+2$ sides (i.e. the number of partitions of polygon into disjoint triangles by using the diagonals).

- The number of ways to connect the $2n$ points on a circle to form n disjoint chords.

- The number of non-isomorphic full binary trees with n internal nodes (i.e. nodes having at least one son).

- The number of monotonic lattice paths from point $(0,0)$ to point (n,n) in a square lattice of size $n \times n$, which do not pass above the main diagonal (i.e. connecting $(0,0)$ to (n,n)).

- Number of permutations of length n that can be stack sorted (i.e. it can be shown that the rearrangement is stack sorted if and only if there is no such index $i < j < k$, such that $a_k < a_i < a_j$).

- The number of non-crossing partitions of a set of n elements.

- The number of ways to cover the ladder $1 \ldots n$ using n rectangles (The ladder consists of n columns, where i^{th} column has a height i).

8.3 Euler's Totient

If p is a prime number: $\phi(p) = p - 1$ and $\phi(p^k) = p^k - p^{k-1}$

If a and b are relatively prime, then: $\phi(ab) = \phi(a) \cdot \phi(b)$

In general: $\phi(ab) = \phi(a) \cdot \phi(b) \cdot \dfrac{gcd(a,b)}{\phi(gcd(a,b))}$

This interesting property was established by Gauss: $\sum_{d|n} \phi(d) = n$, Here the sum is over all positive divisors d of n.

Euler's theorem: $a^{\phi(m)} \equiv 1 \pmod{m}$, if a and m are relatively prime.

Generalization: $a^n \equiv a^{\phi(m)+[n \bmod \phi(m)]} \bmod m$, for arbitrary a, m and n $\geq log_2(m)$.

8.4 Formulas

Count the number of ways to partition a set of n labelled objects into k nonempty labelled subsets.

$$f(n,k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i}(k-i)^n$$

Stirling Number 2nd: Partitions of an n element set into k not-empty set. Or count the number of ways to partition a set of n labelled objects into k nonempty unlabelled subsets.

$$S_{2nd}(n,k) = \left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i}(k-i)^n$$

Euler's formula: $f = e - v + 2$

Euler's formula to n Lines or Segment if there is no three lines/segments that contains the same point: $R = intersects + component - n$

Number of regions in a planar graph: $R = E - V + C + 1$ where C is the number of connected components

Given a and b co-prime, $n = a \cdot x + b \cdot y$ where $x \geq 0$ and $y \geq 0$. You are required to find the least value of n, such that all currency values greater than or equal to n can be made using any number of coins of denomination a and b: $n = (a-1)*(b-1)$

generalization of the above problem, n is multiple of $gcd(a,b)$: $n = lcm(a,b) - a - b + gcd(a,b)$

8.5 Graph

8.6 Manhattan Distance

Transformation of the manhattan distance to 2 dimensions between $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$:

$|x_1 - x_2| + |y_1 - y_2| = max(|A_1 - B_1|, |A_2 - B_2|)$ where $A = (x_1 + y_1, x_1 - y_1)$ e $B = (x_2 + y_2, x_2 - y_2)$

Transformation of the manhattan distance to 3 dimensions between $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$:

$|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2| = max(|A_1 - B_1|, |A_2 - B_2|, |A_3 - B_3|, |A_4 - B_4|)$ where $A = (x_1 + y_1 + z_1, x_1 + y_1 - z_1, x_1 - y_1 + z_1, -x_1 + y_1 + z_1)$ e $B = (x_2 + y_2 + z_2, x_2 + y_2 - z_2, x_2 - y_2 + z_2, -x_2 + y_2 + z_2)$

Transformation of the manhattan distance to D dimensions between P_1 and P_2:

$isSet(i, x) = 1$ if the i-th bit is setted in x and 0 otherwise.

$A[i] = \sum_{j=0}^{d-1} (-1)^{isSet(j,i)} P_1[j]$

$B[i] = \sum_{j=0}^{d-1} (-1)^{isSet(j,i)} P_2[j]$

$$\sum_{i=0}^{d-1} |P_1[i] - P_2[i]| = \max_{i=0}^{2^d-1} |A_i - B_i|$$

8.7 Primes

If $n = p_1^{e_1} \cdot p_2^{e_2} \cdots p_k^{e_k}$, then:

Number of divisors is $d(n) = (e_1 + 1) \cdot (e_2 + 1) \cdots (e_k + 1)$.

Sum of divisors is $\sigma(n) = \frac{p_1^{e_1+1}-1}{p_1-1} \cdot \frac{p_2^{e_2+1}-1}{p_2-1} \cdots \frac{p_k^{e_k+1}-1}{p_k-1}$