

GEMP - UFC Quixadá - ICPC Library

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1 Data Structures

1.1 BIT

```

#include <bits/stdc++.h>
using namespace std;
class Bit{
private:
    typedef long long t_bit;
    int nBit;
    int nLog;
    vector<t_bit> bit;
public:
    Bit(int n){
        nBit = n;
        nLog = 20;
        bit.resize(nBit + 1, 0);
    }
    //1-indexed
    t_bit get(int i){
        t_bit s = 0;
        for (; i > 0; i -= (i & -i))
            s += bit[i];
        return s;
    }
    //1-indexed [l, r]
    t_bit get(int l, int r){
        return get(r) - get(l - 1);
    }
    //1-indexed
    void add(int i, t_bit value){
        assert(i > 0);
        for (; i <= nBit; i += (i & -i))
            bit[i] += value;
    }
    t_bit lower_bound(t_bit value){
        t_bit sum = 0;
        int pos = 0;
        for (int i = nLog; i >= 0; i--){
            if ((pos + (1 << i) <= nBit) and (sum + bit[pos + (1 << i)] <
                value)){
                sum += bit[pos + (1 << i)];
                pos += (1 << i);
            }
        }
        return pos + 1;
    }
};

```

```

    }
};

```

1.2 BIT 2D

```

#include <bits/stdc++.h>
using namespace std;
class Bit2d{
private:
    typedef long long t_bit;
    vector<vector<t_bit>> bit;
    int nBit, mBit;
public:
    Bit2d(int n, int m){
        nBit = n;
        mBit = m;
        bit.resize(nBit + 1, vector<t_bit>(mBit + 1, 0));
    }
    //1-indexed
    t_bit get(int i, int j){
        t_bit sum = 0;
        for (int a = i; a > 0; a -= (a & -a))
            for (int b = j; b > 0; b -= (b & -b))
                sum += bit[a][b];
        return sum;
    }
    //1-indexed
    t_bit get(int a1, int b1, int a2, int b2){
        return get(a2, b2) - get(a2, b1 - 1) - get(a1 - 1, b2) + get(a1 - 1, b1 - 1);
    }
    //1-indexed [i, j]
    void add(int i, int j, t_bit value){
        for (int a = i; a <= nBit; a += (a & -a))
            for (int b = j; b <= mBit; b += (b & -b))
                bit[a][b] += value;
    }
};

```

1.3 BIT In Range

```

#include <bits/stdc++.h>
using namespace std;
class BitRange{
private:
    typedef long long t_bit;
    vector<t_bit> bit1, bit2;
    t_bit get(vector<t_bit> &bit, int i){
        t_bit sum = 0;
        for (; i > 0; i -= (i & -i))
            sum += bit[i];
        return sum;
    }
    void add(vector<t_bit> &bit, int i, t_bit value){
        for (; i < (int)bit.size(); i += (i & -i))
            bit[i] += value;
    }
public:

```

```

    BitRange(int n){
        bit1.assign(n + 1, 0);
        bit2.assign(n + 1, 0);
    }
    //1-indexed [i, j]
    void add(int i, int j, t_bit v){
        add(bit1, i, v);
        add(bit1, j + 1, -v);
        add(bit2, i, v * (i - 1));
        add(bit2, j + 1, -v * j);
    }
    //1-indexed
    t_bit get(int i){
        return get(bit1, i) * i - get(bit2, i);
    }
    //1-indexed [i, j]
    t_bit get(int i, int j){
        return get(j) - get(i - 1);
    }
};

```

1.4 Dynamic Median

```

#include <bits/stdc++.h>
using namespace std;
class DinamicMedian{
    typedef int t_median;
private:
    priority_queue<t_median> mn;
    priority_queue<t_median, vector<t_median>, greater<t_median>> mx;
public:
    double median(){
        if (mn.size() > mx.size())
            return mn.top();
        else
            return (mn.top() + mx.top()) / 2.0;
    }
    void push(t_median x){
        if (mn.size() <= mx.size())
            mn.push(x);
        else
            mx.push(x);
        if ((!mx.empty()) and (!mn.empty())){
            while (mn.top() > mx.top()){
                t_median a = mx.top();
                mx.pop();
                t_median b = mn.top();
                mn.pop();
                mx.push(b);
                mn.push(a);
            }
        }
    }
};

```

1.5 Dynamic Wavelet Tree

```

#include <bits/stdc++.h>

```

```

using namespace std;
struct SplayTree{
    struct Node{
        int x, y, s;
        Node *p = 0;
        Node *l = 0;
        Node *r = 0;
        Node(int v){
            x = v;
            y = v;
            s = 1;
        }
        void upd(){
            s = 1;
            y = x;
            if (l){
                y += l->y;
                s += l->s;
            }
            if (r){
                y += r->y;
                s += r->s;
            }
        }
        int left_size(){
            return l ? l->s : 0;
        }
    };
    Node *root = 0;
    void rot(Node *c){
        auto p = c->p;
        auto g = p->p;
        if (g)
            (g->l == p ? g->l : g->r) = c;
        if (p->l == c){
            p->l = c->r;
            c->r = p;
            if (p->l)
                p->l->p = p;
        }
        else{
            p->r = c->l;
            c->l = p;
            if (p->r)
                p->r->p = p;
        }
        p->p = c;
        c->p = g;
        p->upd();
        c->upd();
    }
    void splay(Node *c){
        while (c->p){
            auto p = c->p;
            auto g = p->p;
            if (g)
                rot((g->l == p) == (p->l == c) ? p : c);
            rot(c);
        }
        c->upd();
        root = c;
    }
};

```

```

}
Node *join(Node *l, Node *r){
    if (not l)
        return r;
    if (not r)
        return l;
    while (l->r)
        l = l->r;
    splay(l);
    r->p = l;
    l->r = r;
    l->upd();
    return l;
}
pair<Node *, Node *> split(Node *p, int idx){
    if (not p)
        return make_pair(nullptr, nullptr);
    if (idx < 0)
        return make_pair(nullptr, p);
    if (idx >= p->s)
        return make_pair(p, nullptr);
    for (int lf = p->left_size(); idx != lf; lf = p->left_size()){
        if (idx < lf)
            p = p->l;
        else
            p = p->r, idx -= lf + 1;
    }
    splay(p);
    Node *l = p;
    Node *r = p->r;
    if (r){
        l->r = r->p = 0;
        l->upd();
    }
    return make_pair(l, r);
}
Node *get(int idx){
    auto p = root;
    for (int lf = p->left_size(); idx != lf; lf = p->left_size()){
        if (idx < lf)
            p = p->l;
        else
            p = p->r, idx -= lf + 1;
    }
    splay(p);
    return p;
}
int insert(int idx, int x){
    Node *l, *r;
    tie(l, r) = split(root, idx - 1);
    int v = l ? l->y : 0;
    root = join(l, join(new Node(x), r));
    return v;
}
void erase(int idx){
    Node *l, *r;
    tie(l, r) = split(root, idx);
    root = join(l->l, r);
    delete l;
}
int rank(int idx){

```

```

Node *l, *r;
tie(l, r) = split(root, idx);
int x = (l && l->l ? l->l->y : 0);
root = join(l, r);
return x;
}
int operator[](int idx){
    return rank(idx);
}
~SplayTree(){
    if (!root)
        return;
    vector<Node*> nodes{root};
    while (nodes.size()){
        auto u = nodes.back();
        nodes.pop_back();
        if (u->l)
            nodes.emplace_back(u->l);
        if (u->r)
            nodes.emplace_back(u->r);
        delete u;
    }
};
class WaveletTree{
private:
    int lo, hi;
    WaveletTree *l = 0;
    WaveletTree *r = 0;
    SplayTree b;
public:
    WaveletTree(int min_value, int max_value){
        lo = min_value;
        hi = max_value;
        b.insert(0, 0);
    }
    ~WaveletTree(){
        delete l;
        delete r;
    }
    //0-indexed
    void insert(int idx, int x){
        if (lo >= hi)
            return;
        int mid = (lo + hi - 1) / 2;
        if (x <= mid){
            l = l ? new WaveletTree(lo, mid);
            l->insert(b.insert(idx, 1), x);
        } else {
            r = r ? new WaveletTree(mid + 1, hi);
            r->insert(b.insert(idx, 0), x);
        }
    }
    //0-indexed
    void erase(int idx){
        if (lo == hi)
            return;
        auto p = b.get(idx);
        int lf = p->l ? p->l->y : 0;
        int x = p->x;
        b.erase(idx);

```

```

        if (x == 1)
            l->erase(lf);
        else
            r->erase(idx - lf);
    }
    //kth smallest element in range [i, j]
    //0-indexed
    int kth(int i, int j, int k){
        if (i >= j)
            return 0;
        if (lo == hi)
            return lo;
        int x = b.rank(i);
        int y = b.rank(j);
        if (k <= y - x)
            return l->kth(x, y, k);
        else
            return r->kth(i - x, j - y, k - (y - x));
    }
    //Amount of numbers in the range [i, j] Less than or equal to k
    //0-indexed
    int lte(int i, int j, int k){
        if (i >= j or k < lo)
            return 0;
        if (hi <= k)
            return j - i;
        int x = b.rank(i);
        int y = b.rank(j);
        return l->lte(x, y, k) + r->lte(i - x, j - y, k);
    }
    //Amount of numbers in the range [i, j] equal to k
    //0-indexed
    int count(int i, int j, int k){
        if (i >= j or k < lo or k > hi)
            return 0;
        if (lo == hi)
            return j - i;
        int mid = (lo + hi - 1) / 2;
        int x = b.rank(i);
        int y = b.rank(j);
        if (k <= mid)
            return l->count(x, y, k);
        return r->count(i - x, j - y, k);
    }
    //0-indexed
    int get(int idx){
        return kth(idx, idx + 1, 1);
    }
};

```

1.6 Implicit Treap

```

#include <bits/stdc++.h>
using namespace std;
namespace ITreap{
    const int N = 500010;
    typedef long long treap_t;
    treap_t X[N];
    int en = 1, Y[N], sz[N], L[N], R[N], root;

```

```

const treap_t neutral = 0;
treap_t op_val[N];
bool rev[N];
inline treap_t join(treap_t a, treap_t b, treap_t c){
    return a + b + c;
}
void calc(int u) { // update node given children info
    sz[u] = sz[L[u]] + 1 + sz[R[u]];
    // code here, no recursion
    op_val[u] = join(op_val[L[u]], X[u], op_val[R[u]]);
}
void unlaze(int u) {
    if(!u) return;
    // code here, no recursion
    if (rev[u]){
        if(L[u])
            rev[L[u]] ^= rev[u];
        if(R[u])
            rev[R[u]] ^= rev[u];
        swap(L[u], R[u]);
        rev[u] = false;
    }
}
void split(int u, int s, int &l, int &r) { // l gets first s, r gets
    remaining
    unlaze(u);
    if(!u) return (void) (l = r = 0);
    if(sz[L[u]] < s) { split(R[u], s - sz[L[u]] - 1, l, r); R[u] = l;
        l = u; }
    else { split(L[u], s, l, r); L[u] = r; r = u; }
    calc(u);
}
int merge(int l, int r) { // els on l <= els on r
    unlaze(l); unlaze(r);
    if(!l || !r) return l + r;
    int u;
    if(Y[l] > Y[r]) { R[l] = merge(R[l], r); u = l; }
    else { L[r] = merge(l, L[r]); u = r; }
    calc(u);
    return u;
}
int new_node(treap_t x){
    X[en] = x;
    op_val[en] = x;
    rev[en] = false;
    return en++;
}
int nth(int u, int idx){
    if(!u)
        return 0;
    unlaze(u);
    if(idx <= sz[L[u]])
        return nth(L[u], idx);
    else if(idx == sz[L[u]] + 1)
        return u;
    else
        return nth(R[u], idx - sz[L[u]] - 1);
}
}
//Public
void init(int n=N-1) { // call before using other funcs
    //init position 0

```

```

    sz[0] = 0;
    op_val[0] = neutral;
    //init Treap
    root = 0;
    for(int i = en = 1; i <= n; i++) { Y[i] = i; sz[i] = 1; L[i] = R[i]
        ] = 0; }
    random_shuffle(Y + 1, Y + n + 1);
}
//0-indexed
void insert(int idx, int val){
    int a, b;
    split(root, idx, a, b);
    root = merge(merge(a, new_node(val)), b);
}
//0-indexed
void erase(int idx){
    int a, b, c, d;
    split(root, idx, a, b);
    split(b, 1, c, d);
    root = merge(a, d);
}
//0-indexed
treap_t nth(int idx){
    int u = nth(root, idx+1);
    return X[u];
}
//0-indexed [l, r]
treap_t query(int l, int r){
    if(l > r) swap(l, r);
    int a, b, c, d;
    split(root, l, a, d);
    split(d, r - l + 1, b, c);
    treap_t ans = op_val[b];
    root = merge(a, merge(b, c));
    return ans;
}
//0-indexed [l, r]
void reverse(int l, int r){
    if (l > r) swap(l, r);
    int a, b, c, d;
    split(root, l, a, d);
    split(d, r - l + 1, b, c);
    if(b)
        rev[b] ^= 1;
    root = merge(a, merge(b, c));
}
};

```

1.7 LiChao Tree

```

#include <bits/stdc++.h>
using namespace std;
const int INF = 0x3f3f3f3f;
class LiChaoTree{
private:
    typedef int t_line;
    struct Line{
        t_line k, b;
        Line() {}
        Line(t_line k, t_line b) : k(k), b(b) {}

```

```

};
int n_tree, min_x, max_x;
vector<Line> li_tree;
t_line f(Line l, int x){
    return l.k * x + l.b;
}
void add(Line nw, int v, int l, int r){
    int m = (l + r) / 2;
    bool lef = f(nw, l) > f(li_tree[v], l);
    bool mid = f(nw, m) > f(li_tree[v], m);
    if (mid)
        swap(li_tree[v], nw);
    if (r - l == 1)
        return;
    else if (lef != mid)
        add(nw, 2 * v, l, m);
    else
        add(nw, 2 * v + 1, m, r);
}
int get(int x, int v, int l, int r){
    int m = (l + r) / 2;
    if (r - l == 1)
        return f(li_tree[v], x);
    else if (x < m)
        return max(f(li_tree[v], x), get(x, 2 * v, l, m));
    else
        return max(f(li_tree[v], x), get(x, 2 * v + 1, m, r));
}
public:
LiChaoTree(int mn_x, int mx_x){
    min_x = mn_x;
    max_x = mx_x;
    n_tree = max_x - min_x + 5;
    li_tree.resize(4 * n_tree, Line(0, -INF));
}
void add(t_line k, t_line b){
    add(Line(k, b), 1, min_x, max_x);
}
t_line get(int x){
    return get(x, 1, min_x, max_x);
}
};

```

1.8 Policy Based Tree

```

#include <bits/stdc++.h>
#include <ext/pb_ds/tree_policy.hpp>
#include <ext/pb_ds/assoc_container.hpp>
using namespace __gnu_pbds;
using namespace std;
typedef tree<int, null_type, less<int>, rb_tree_tag,
tree_order_statistics_node_update> OrderedSet;
typedef tree<int, int, less<int>, rb_tree_tag,
tree_order_statistics_node_update> OrderedMap;
//order_of_key(k) : Number of items strictly smaller than k .
//find_by_order(k) : K-th element in a set (counting from zero).

```

1.9 Queue Query

```

#include <bits/stdc++.h>
using namespace std;
class QueueQuery{
private:
    typedef long long t_queue;
    stack<pair<t_queue, t_queue>> s1, s2;
    t_queue cmp(t_queue a, t_queue b){
        return min(a, b);
    }
    void move(){
        if (s2.empty()){
            while (!s1.empty()){
                t_queue element = s1.top().first;
                s1.pop();
                t_queue result = s2.empty() ? element : cmp(element, s2.top().second);
                s2.push({element, result});
            }
        }
    }
public:
    void push(t_queue x){
        t_queue result = s1.empty() ? x : cmp(x, s1.top().second);
        s1.push({x, result});
    }
    void pop(){
        move();
        s2.pop();
    }
    t_queue front(){
        move();
        return s2.top().first;
    }
    t_queue query(){
        if (s1.empty() || s2.empty())
            return s1.empty() ? s2.top().second : s1.top().second;
        else
            return cmp(s1.top().second, s2.top().second);
    }
    t_queue size(){
        return s1.size() + s2.size();
    }
};

```

1.10 Range Color

```

#include <bits/stdc++.h>
using namespace std;
class RangeColor{
private:
    typedef long long ll;
    struct Node{
        ll l, r;
        int color;
        Node() {}
        Node(ll l, ll r, int color) : l(l), r(r), color(color) {}
    }
};

```

```

};
struct cmp{
    bool operator() (Node a, Node b){
        return a.r < b.r;
    }
};
std::set<Node, cmp> st;
vector<ll> ans;
public:
    RangeColor(ll first, ll last, int maxColor){
        ans.resize(maxColor + 1);
        ans[0] = last - first + 1LL;
        st.insert(Node(first, last, 0));
    }
    //get color in position x
    int get(ll x){
        auto p = st.upper_bound(Node(0, x - 1LL, -1));
        return p->color;
    }
    //set newColor in [a, b]
    void set(ll a, ll b, int newColor){
        auto p = st.upper_bound(Node(0, a - 1LL, -1));
        assert(p != st.end());
        ll l = p->l;
        ll r = p->r;
        int oldColor = p->color;
        ans[oldColor] -= (r - l + 1LL);
        p = st.erase(p);
        if (l < a){
            ans[oldColor] += (a - l);
            st.insert(Node(l, a - 1LL, oldColor));
        }
        if (b < r){
            ans[oldColor] += (r - b);
            st.insert(Node(b + 1LL, r, oldColor));
        }
        while ((p != st.end()) and (p->l <= b)){
            l = p->l;
            r = p->r;
            oldColor = p->color;
            ans[oldColor] -= (r - l + 1LL);
            if (b < r){
                ans[oldColor] += (r - b);
                st.erase(p);
                st.insert(Node(b + 1LL, r, oldColor));
                break;
            }else{
                p = st.erase(p);
            }
        }
        ans[newColor] += (b - a + 1LL);
        st.insert(Node(a, b, newColor));
    }
    ll countColor(int x){
        return ans[x];
    }
};

```

```

#include <bits/stdc++.h>
using namespace std;
class SegTree{
private:
    typedef long long Node;
    Node neutral = 0;
    vector<Node> st;
    vector<int> v;
    int n;
    Node join(Node a, Node b){
        return (a + b);
    }
    void build(int node, int i, int j){
        if (i == j){
            st[node] = v[i];
            return;
        }
        int m = (i + j) / 2;
        int l = (node << 1);
        int r = l + 1;
        build(l, i, m);
        build(r, m + 1, j);
        st[node] = join(st[l], st[r]);
    }
    Node query(int node, int i, int j, int a, int b){
        if ((i > b) or (j < a))
            return neutral;
        if ((a <= i) and (j <= b))
            return st[node];
        int m = (i + j) / 2;
        int l = (node << 1);
        int r = l + 1;
        return join(query(l, i, m, a, b), query(r, m + 1, j, a, b));
    }
    void update(int node, int i, int j, int idx, Node value){
        if (i == j){
            st[node] = value;
            return;
        }
        int m = (i + j) / 2;
        int l = (node << 1);
        int r = l + 1;
        if (idx <= m)
            update(l, i, m, idx, value);
        else
            update(r, m + 1, j, idx, value);
        st[node] = join(st[l], st[r]);
    }
public:
    template <class MyIterator>
    SegTree(MyIterator begin, MyIterator end){
        n = end - begin;
        v = vector<int>(begin, end);
        st.resize(4 * n + 5);
        build(1, 0, n - 1);
    }
    //0-indexed [a, b]
    Node query(int a, int b){
        return query(1, 0, n - 1, a, b);
    }
    //0-indexed

```

1.11 Segment Tree

```

void update(int idx, int value){
    update(1, 0, n - 1, idx, value);
}
};

```

1.12 Segment Tree 2D

```

#include <bits/stdc++.h>
using namespace std;
struct SegTree2D{
private:
    int n, m;
    typedef int Node;
    Node neutral = -0x3f3f3f3f;
    vector<vector<Node>> seg;
    Node join(Node a, Node b){
        return max(a, b);
    }
public:
    SegTree2D(int n1, int m1){
        n = n1, m = m1;
        seg.assign(2 * n, vector<Node>(2 * m, 0));
    }
    void update(int x, int y, int val){
        assert(0 <= x && x < n && 0 <= y && y < m);
        x += n, y += m;
        seg[x][y] = val;
        for (int j = y / 2; j > 0; j /= 2)
            seg[x][j] = join(seg[x][2 * j], seg[x][2 * j + 1]);
        for (x /= 2; x > 0; x /= 2){
            seg[x][y] = join(seg[2 * x][y], seg[2 * x + 1][y]);
            for (int j = y / 2; j > 0; j /= 2){
                seg[x][j] = join(seg[x][2 * j], seg[x][2 * j + 1]);
            }
        }
    }
    vector<int> getCover(int l, int r, int N){
        l = std::max(0, l);
        r = std::min(N, r);
        vector<int> ans;
        for (l += N, r += N; l < r; l /= 2, r /= 2){
            if (l & 1)
                ans.push_back(l++);
            if (r & 1)
                ans.push_back(--r);
        }
        return ans;
    }
    Node query(int x1, int y1, int x2, int y2){
        auto c1 = getCover(x1, x2 + 1, n);
        auto c2 = getCover(y1, y2 + 1, m);
        Node ans = neutral;
        for (auto i : c1){
            for (auto j : c2){
                ans = join(ans, seg[i][j]);
            }
        }
        return ans;
    }
};

```

1.13 Segment Tree Iterative

```

#include <bits/stdc++.h>
using namespace std;
class SegTreeIterative{
private:
    typedef long long Node;
    Node neutral = 0;
    vector<Node> st;
    int n;
    inline Node join(Node a, Node b){
        return a + b;
    }
public:
    template <class MyIterator>
    SegTreeIterative(MyIterator begin, MyIterator end){
        int sz = end - begin;
        for (n = 1; n < sz; n <= 1);
        st.assign(n < 1, neutral);
        for (int i = 0; i < sz; i++, begin++)
            st[i + n] = (*begin);
        for (int i = n + sz - 1; i > 1; i--)
            st[i >> 1] = join(st[i >> 1], st[i]);
    }
    //0-indexed
    void update(int i, Node x){
        st[i += n] = x;
        for (i >= 1; i; i >= 1)
            st[i] = join(st[i << 1], st[1 + (i << 1)]);
    }
    //0-indexed [l, r]
    Node query(int l, int r){
        Node ans = neutral;
        for (l += n, r += n + 1; l < r; l >= 1, r >= 1){
            if (l & 1)
                ans = join(ans, st[l++]);
            if (r & 1)
                ans = join(ans, st[--r]);
        }
        return ans;
    }
};

```

1.14 Segment Tree Lazy

```

#include <bits/stdc++.h>
using namespace std;
class SegTreeLazy{
private:
    typedef long long Node;
    vector<Node> st;
    vector<long long> lazy;
    vector<int> v;
    int n;
    Node neutral = 0;
    inline Node join(Node a, Node b){
        return a + b;
    }
};

```



```

inline void upLazy(int &node, int &i, int &j){
    if (lazy[node] != 0){
        st[node] += lazy[node] * (j - i + 1);
        //st[node] += lazy[node];
        if (i != j){
            lazy[(node << 1)] += lazy[node];
            lazy[(node << 1) + 1] += lazy[node];
        }
        lazy[node] = 0;
    }
}

void build(int node, int i, int j){
    if (i == j){
        st[node] = v[i];
        return;
    }
    int m = (i + j) / 2;
    int l = (node << 1);
    int r = l + 1;
    build(l, i, m);
    build(r, m + 1, j);
    st[node] = join(st[l], st[r]);
}

Node query(int node, int i, int j, int a, int b){
    upLazy(node, i, j);
    if ((i > b) or (j < a))
        return neutral;
    if ((a <= i) and (j <= b)){
        return st[node];
    }
    int m = (i + j) / 2;
    int l = (node << 1);
    int r = l + 1;
    return join(query(l, i, m, a, b), query(r, m + 1, j, a, b));
}

void update(int node, int i, int j, int a, int b, int value){
    upLazy(node, i, j);
    if ((i > j) or (i > b) or (j < a))
        return;
    if ((a <= i) and (j <= b)){
        lazy[node] = value;
        upLazy(node, i, j);
    } else {
        int m = (i + j) / 2;
        int l = (node << 1);
        int r = l + 1;
        update(l, i, m, a, b, value);
        update(r, m + 1, j, a, b, value);
        st[node] = join(st[l], st[r]);
    }
}

public:
template <class MyIterator>
SegTreeLazy(MyIterator begin, MyIterator end){
    n = end - begin;
    v = vector<int>(begin, end);
    st.resize(4 * n + 5);
    lazy.assign(4 * n + 5, 0);
    build(1, 0, n - 1);
}
//0-indexed [a, b]

```

```

Node query(int a, int b){
    return query(1, 0, n - 1, a, b);
}
//0-indexed [a, b]
void update(int a, int b, int value){
    update(1, 0, n - 1, a, b, value);
}
};

```

1.15 Segment Tree Persistent

```

#include <bits/stdc++.h>
using namespace std;
const int MAX = 3e4 + 10, UPD = 2e5 + 10, LOG = 20;
const int MAXS = 4 * MAX + UPD * LOG;
namespace PerSegTree{
    typedef long long pst_t;
    pst_t seg[MAXS];
    int T[UPD], L[MAXS], R[MAXS], cnt, t;
    int n, *v;
    pst_t neutral = 0;
    pst_t join(pst_t a, pst_t b){
        return a + b;
    }
    pst_t build(int p, int l, int r){
        if (l == r)
            return seg[p] = v[l];
        L[p] = cnt++, R[p] = cnt++;
        int m = (l + r) / 2;
        return seg[p] = join(build(L[p], l, m), build(R[p], m + 1, r));
    }
    pst_t query(int a, int b, int p, int l, int r){
        if (b < l or r < a)
            return 0;
        if (a <= l and r <= b)
            return seg[p];
        int m = (l + r) / 2;
        return join(query(a, b, L[p], l, m), query(a, b, R[p], m + 1, r));
    }
    pst_t update(int a, int x, int lp, int p, int l, int r){
        if (l == r)
            return seg[p] = x;
        int m = (l + r) / 2;
        if (a <= m)
            return seg[p] = join(update(a, x, L[lp], L[p] = cnt++, l, m),
                                   seg[R[p] = R[lp]]);
        return seg[p] = join(seg[L[p] = L[lp]], update(a, x, R[lp], R[p] =
                                                         cnt++, m + 1, r));
    }
}
//Public:
//O(n)
void build(int n2, int *v2){
    n = n2, v = v2;
    T[0] = cnt++;
    build(0, 0, n - 1);
}
//O(log(n))
pst_t query(int a, int b, int tt){
    return query(a, b, T[tt], 0, n - 1);
}

```

```
//O(log(n))
//update: v[idx] = x;
int update(int idx, int x, int tt = t){
    update(idx, x, T[tt], T[+tt] = cnt++, 0, n - 1);
    return t;
}
}; // namespace perseg
```

1.16 Sparse Table

```
#include <bits/stdc++.h>
using namespace std;
class SparseTable{
private:
    typedef int t_st;
    vector<vector<t_st>> st;
    vector<int> log2;
    t_st neutral = 0x3f3f3f3f;
    int nLog;
    t_st join(t_st a, t_st b){
        return min(a, b);
    }
public:
    template <class MyIterator>
    SparseTable(MyIterator begin, MyIterator end){
        int n = end - begin;
        nLog = 20;
        log2.resize(n + 1);
        log2[1] = 0;
        for (int i = 2; i <= n; i++)
            log2[i] = log2[i / 2] + 1;
        st.resize(n, vector<t_st>(nLog, neutral));
        for (int i = 0; i < n; i++, begin++){
            st[i][0] = (*begin);
            for (int j = 1; j < nLog; j++)
                for (int i = 0; (i + (1 << (j - 1))) < n; i++)
                    st[i][j] = join(st[i][j - 1], st[i + (1 << (j - 1))][j - 1]);
        }
        //0-indexed [a, b]
        t_st query(int a, int b){
            int d = b - a + 1;
            t_st ans = neutral;
            for (int j = nLog - 1; j >= 0; j--){
                if (d & (1 << j)){
                    ans = join(ans, st[a][j]);
                    a = a + (1 << j);
                }
            }
            return ans;
        }
        //0-indexed [a, b]
        t_st queryRMQ(int a, int b){
            int j = log2[b - a + 1];
            return join(st[a][j], st[b - (1 << j) + 1][j]);
        }
    };
};
```

1.17 Sqrt Decomposition

```
#include <bits/stdc++.h>
using namespace std;
struct SqrtDecomposition{
    typedef long long t_sqrt;
    int sqrtLen;
    vector<t_sqrt> block;
    vector<t_sqrt> v;
    template <class MyIterator>
    SqrtDecomposition(MyIterator begin, MyIterator end){
        int n = end - begin;
        sqrtLen = (int)sqrt(n + .0) + 1;
        v.resize(n);
        block.resize(sqrtLen + 5);
        for (int i = 0; i < n; i++, begin++){
            v[i] = (*begin);
            block[i / sqrtLen] += v[i];
        }
    }
    //0-indexed
    void update(int idx, t_sqrt new_value){
        t_sqrt d = new_value - v[idx];
        v[idx] += d;
        block[idx / sqrtLen] += d;
    }
    //0-indexed [l, r]
    t_sqrt query(int l, int r){
        t_sqrt sum = 0;
        int c_l = l / sqrtLen, c_r = r / sqrtLen;
        if (c_l == c_r){
            for (int i = l; i <= r; i++)
                sum += v[i];
        }else{
            for (int i = l, end = (c_l + 1) * sqrtLen - 1; i <= end; i++)
                sum += v[i];
            for (int i = c_l + 1; i <= c_r - 1; i++)
                sum += block[i];
            for (int i = c_r * sqrtLen; i <= r; i++)
                sum += v[i];
        }
        return sum;
    }
};
```

1.18 Sqrt Tree

```
#include <bits/stdc++.h>
using namespace std;
class SqrtTree{
private:
    typedef long long t_sqrt;
    t_sqrt op(const t_sqrt &a, const t_sqrt &b){
        return a | b;
    }
    inline int log2Up(int n){
        int res = 0;
        while ((1 << res) < n)
            res++;
    }
};
```

```

    res++;
    return res;
}
int n, lg, indexSz;
vector<t_sqrt> v;
vector<int> clz, layers, onLayer;
vector<vector<t_sqrt>> pref, suf, between;
inline void buildBlock(int layer, int l, int r){
    pref[layer][l] = v[l];
    for (int i = l + 1; i < r; i++){
        pref[layer][i] = op(pref[layer][i - 1], v[i]);
        suf[layer][r - 1] = v[r - 1];
        for (int i = r - 2; i >= l; i--){
            suf[layer][i] = op(v[i], suf[layer][i + 1]);
        }
    }
    inline void buildBetween(int layer, int lBound, int rBound, int
        betweenOffs){
        int bSzLog = (layers[layer] + 1) >> 1;
        int bCntLog = layers[layer] >> 1;
        int bSz = 1 << bSzLog;
        int bCnt = (rBound - lBound + bSz - 1) >> bSzLog;
        for (int i = 0; i < bCnt; i++){
            t_sqrt ans;
            for (int j = i; j < bCnt; j++){
                t_sqrt add = suf[layer][lBound + (j << bSzLog)];
                ans = (i == j) ? add : op(ans, add);
                between[layer - 1][betweenOffs + lBound + (i << bCntLog) + j]
                    = ans;
            }
        }
    }
    inline void buildBetweenZero(){
        int bSzLog = (lg + 1) >> 1;
        for (int i = 0; i < indexSz; i++){
            v[n + i] = suf[0][i << bSzLog];
        }
        build(1, n, n + indexSz, (1 << lg) - n);
    }
    inline void updateBetweenZero(int bid){
        int bSzLog = (lg + 1) >> 1;
        v[n + bid] = suf[0][bid << bSzLog];
        update(1, n, n + indexSz, (1 << lg) - n, n + bid);
    }
    void build(int layer, int lBound, int rBound, int betweenOffs){
        if (layer >= (int)layers.size())
            return;
        int bSz = 1 << ((layers[layer] + 1) >> 1);
        for (int l = lBound; l < rBound; l += bSz){
            int r = min(l + bSz, rBound);
            buildBlock(layer, l, r);
            build(layer + 1, l, r, betweenOffs);
        }
        if (layer == 0)
            buildBetweenZero();
        else
            buildBetween(layer, lBound, rBound, betweenOffs);
    }
    void update(int layer, int lBound, int rBound, int betweenOffs, int
        x){
        if (layer >= (int)layers.size())
            return;

```

```

        int bSzLog = (layers[layer] + 1) >> 1;
        int bSz = 1 << bSzLog;
        int blockIdx = (x - lBound) >> bSzLog;
        int l = lBound + (blockIdx << bSzLog);
        int r = min(l + bSz, rBound);
        buildBlock(layer, l, r);
        if (layer == 0)
            updateBetweenZero(blockIdx);
        else
            buildBetween(layer, lBound, rBound, betweenOffs);
        update(layer + 1, l, r, betweenOffs, x);
    }
    inline t_sqrt query(int l, int r, int betweenOffs, int base){
        if (l == r)
            return v[l];
        if (l + 1 == r)
            return op(v[l], v[r]);
        int layer = onLayer[clz[(l - base) ^ (r - base)]];
        int bSzLog = (layers[layer] + 1) >> 1;
        int bCntLog = layers[layer] >> 1;
        int lBlock = ((l - lBound) >> layers[layer]) << layers[layer] +
            base;
        int rBlock = ((r - lBound) >> layers[layer]) - 1;
        t_sqrt ans = suf[layer][l];
        if (lBlock <= rBlock){
            t_sqrt add;
            if (layer == 0)
                add = query(n + lBlock, n + rBlock, (1 << lg) - n, n);
            else
                add = between[layer - 1][betweenOffs + lBlock + (lBlock <<
                    bCntLog) + rBlock];
            ans = op(ans, add);
        }
        ans = op(ans, pref[layer][r]);
        return ans;
    }
}
public:
    template <class MyIterator>
    SqrtTree(MyIterator begin, MyIterator end){
        n = end - begin;
        v.resize(n);
        for (int i = 0; i < n; i++, begin++){
            v[i] = (*begin);
            lg = log2Up(n);
            clz.resize(1 << lg);
            onLayer.resize(lg + 1);
            clz[0] = 0;
            for (int i = 1; i < (int)clz.size(); i++){
                clz[i] = clz[i >> 1] + 1;
            }
            int tl = lg;
            while (tl > 1){
                onLayer[tl] = (int)layers.size();
                layers.push_back(tl);
                tl = (tl + 1) >> 1;
            }
            for (int i = lg - 1; i >= 0; i--){
                onLayer[i] = max(onLayer[i], onLayer[i + 1]);
            }
            int betweenLayers = max(0, (int)layers.size() - 1);
            int bSzLog = (lg + 1) >> 1;
            int bSz = 1 << bSzLog;

```

```

    indexSz = (n + bSz - 1) >> bSzLog;
    v.resize(n + indexSz);
    pref.assign(layers.size(), vector<t_sqrt>(n + indexSz));
    suf.assign(layers.size(), vector<t_sqrt>(n + indexSz));
    between.assign(betweenLayers, vector<t_sqrt>((1 << lg) + bSz));
    build(0, 0, n, 0);
}
//0-indexed
inline void update(int x, const t_sqrt &item){
    v[x] = item;
    update(0, 0, n, 0, x);
}
//0-indexed [l, r]
inline t_sqrt query(int l, int r){
    return query(l, r, 0, 0);
}
};

```

1.19 Stack Query

```

#include <bits/stdc++.h>
using namespace std;
struct StackQuery{
    typedef int t_stack;
    stack<pair<t_stack, t_stack>> st;
    t_stack cmp(t_stack a, t_stack b){
        return min(a, b);
    }
    void push(t_stack x){
        t_stack new_value = st.empty() ? x : cmp(x, st.top().second);
        st.push({x, new_value});
    }
    void pop(){
        st.pop();
    }
    t_stack top(){
        return st.top().first;
    }
    t_stack query(){
        return st.top().second;
    }
    t_stack size(){
        return st.size();
    }
};

```

1.20 Treap

```

#include <bits/stdc++.h>
using namespace std;
namespace Treap{
    const int N = 500010;
    typedef long long treap_t;
    treap_t X[N];
    int en = 1, Y[N], sz[N], L[N], R[N], root;

    const treap_t neutral = 0;
    treap_t op_val[N];

```

```

    inline treap_t join(treap_t a, treap_t b, treap_t c){
        return a + b + c;
    }
    void calc(int u) { // update node given children info
        sz[u] = sz[L[u]] + 1 + sz[R[u]];
        // code here, no recursion
        op_val[u] = join(op_val[L[u]], X[u], op_val[R[u]]);
    }
    void unlaze(int u) {
        if(!u) return;
        // code here, no recursion
    }
    void split(int u, treap_t x, int &l, int &r) { // l gets <= x, r
        gets > x
        unlaze(u);
        if(!u) return (void) (l = r = 0);
        if(X[u] <= x) { split(R[u], x, l, r); R[u] = l; l = u; }
        else { split(L[u], x, l, r); L[u] = r; r = u; }
        calc(u);
    }
    int merge(int l, int r) { // els on l <= els on r
        unlaze(l); unlaze(r);
        if(!l || !r) return l + r;
        int u;
        if(Y[l] > Y[r]) { R[l] = merge(R[l], r); u = l; }
        else { L[r] = merge(l, L[r]); u = r; }
        calc(u);
        return u;
    }
    int new_node(treap_t x){
        X[en] = x;
        op_val[en] = x;
        return en++;
    }
    int nth(int u, int idx){
        if(!u)
            return 0;
        unlaze(u);
        if(idx <= sz[L[u]])
            return nth(L[u], idx);
        else if(idx == sz[L[u]] + 1)
            return u;
        else
            return nth(R[u], idx - sz[L[u]] - 1);
    }
}
//Public
void init(int n=N-1) { // call before using other funcs
    //init position 0
    sz[0] = 0;
    op_val[0] = neutral;
    //init Treap
    root = 0;
    for(int i = en = 1; i <= n; i++) { Y[i] = i; sz[i] = 1; L[i] = R[i]
        ] = 0; }
    random_shuffle(Y + 1, Y + n + 1);
}
void insert(treap_t x){
    int a, b;
    split(root, x, a, b);
    root = merge(merge(a, new_node(x)), b);
}

```

```

void erase(treap_t x){
    int a, b, c, d;
    split(root, x-1, a, b);
    split(b, x, c, d);
    root = merge(a, d);
}
int count(treap_t x){
    int a, b, c, d;
    split(root, x-1, a, b);
    split(b, x, c, d);
    int ans = sz[c];
    root = merge(a, merge(c, d));
    return ans;
}
int size(){ return sz[root];}
//0-indexed
treap_t nth(int idx){
    int u = nth(root, idx + 1);
    return X[u];
}
};

```

1.21 Union Find

```

#include <bits/stdc++.h>
using namespace std;
class UnionFind{
private:
    vector<int> p, w, sz;
public:
    UnionFind(int n){
        w.resize(n + 1, 1);
        sz.resize(n + 1, 1);
        p.resize(n + 1);
        for (int i = 0; i <= n; i++)
            p[i] = i;
    }
    int find(int x){
        if (p[x] == x)
            return x;
        return p[x] = find(p[x]);
    }
    void join(int x, int y){
        x = find(x);
        y = find(y);
        if (x == y)
            return;
        if (w[x] > w[y])
            swap(x, y);
        p[x] = y;
        sz[y] += sz[x];
        if (w[x] == w[y])
            w[y]++;
    }
    bool isSame(int x, int y){
        return find(x) == find(y);
    }
    int size(int x){
        return sz[find(x)];
    }
}

```

```
};
```

1.22 Wavelet Tree

```

#include <bits/stdc++.h>
using namespace std;
struct WaveletTree{
private:
    typedef int t_wavelet;
    t_wavelet lo, hi;
    WaveletTree *l = nullptr, *r = nullptr;
    vector<t_wavelet> a;
public:
    template <class MyIterator>
    WaveletTree(MyIterator begin, MyIterator end, t_wavelet minX,
                t_wavelet maxX){
        lo = minX, hi = maxX;
        if (lo == hi or begin >= end)
            return;
        t_wavelet mid = (lo + hi - 1) / 2;
        auto f = [mid](int x) {
            return x <= mid;
        };
        a.reserve(end - begin + 2);
        a.push_back(0);
        for (auto it = begin; it != end; it++)
            a.push_back(a.back() + f(*it));
        auto pivot = stable_partition(begin, end, f);
        l = new WaveletTree(begin, pivot, lo, mid);
        r = new WaveletTree(pivot, end, mid + 1, hi);
    }
    inline int b(int i){
        return i - a[i];
    }
}
//kth smallest element in range [i, j]
//1-indexed
int kth(int i, int j, int k){
    if (i > j)
        return 0;
    if (lo == hi)
        return lo;
    int inLeft = a[j] - a[i - 1];
    int i1 = a[i - 1] + 1, j1 = a[j];
    int i2 = b(i - 1) + 1, j2 = b(j);
    if (k <= inLeft)
        return l->kth(i1, j1, k);
    return r->kth(i2, j2, k - inLeft);
}
//Amount of numbers in the range [i, j] Less than or equal to k
//1-indexed
int lte(int i, int j, int k){
    if (i > j or k < lo)
        return 0;
    if (hi <= k)
        return j - i + 1;
    int i1 = a[i - 1] + 1, j1 = a[j];
    int i2 = b(i - 1) + 1, j2 = b(j);
    return l->lte(i1, j1, k) + r->lte(i2, j2, k);
}
//Amount of numbers in the range [i, j] equal to k

```

```

//1-indexed
int count(int i, int j, int k){
    if (i > j or k < lo or k > hi)
        return 0;
    if (lo == hi)
        return j - i + 1;
    t_wavelet mid = (lo + hi - 1) / 2;
    int i1 = a[i - 1] + 1, j1 = a[j];
    int i2 = b(i - 1) + 1, j2 = b(j);
    if (k <= mid)
        return l->count(i1, j1, k);
    return r->count(i2, j2, k);
}
//swap v[i] with v[i+1]
//1-indexed
void swap(int i){
    if (lo == hi or a.size() <= 2)
        return;
    if (a[i - 1] + 1 == a[i] and a[i] + 1 == a[i + 1])
        l->swap(a[i]);
    else if (b(i - 1) + 1 == b(i) and b(i) + 1 == b(i + 1))
        r->swap(b(i));
    else if (a[i - 1] + 1 == a[i])
        a[i]--;
    else
        a[i]++;
}
WaveletTree(){
    if (l) delete l;
    if (r) delete r;
}
};

```

2 Graph Algorithms

2.1 2-SAT

```

#include "strongly_connected_component.h"
using namespace std;
struct SAT{
    typedef pair<int, int> pii;
    vector<pii> edges;
    int n;
    SAT(int size){
        n = 2 * size;
    }
    vector<bool> solve2SAT(){
        vector<bool> vAns(n / 2, false);
        vector<int> comp = SCC::scc(n, edges);
        for (int i = 0; i < n; i += 2){
            if (comp[i] == comp[i + 1])
                return vector<bool>();
            vAns[i / 2] = (comp[i] > comp[i + 1]);
        }
        return vAns;
    }
    int v(int x){
        if (x >= 0)

```

```

        return (x << 1);
        x = ~x;
        return (x << 1) ^ 1;
    }
    void add(int a, int b){
        edges.push_back(pii(a, b));
    }
    void addOr(int a, int b){
        add(v(~a), v(b));
        add(v(~b), v(a));
    }
    void addImp(int a, int b){
        addOr(~a, b);
    }
    void addEqual(int a, int b){
        addOr(a, ~b);
        addOr(~a, b);
    }
    void addDiff(int a, int b){
        addEqual(a, ~b);
    }
};

```

2.2 Centroid Decomposition

```

#include <bits/stdc++.h>
using namespace std;
// O(N*log(N))
struct CentroidDecomposition{
    vector<vector<int>> adj;
    vector<int> dad, sub;
    vector<bool> rem;
    int centroidRoot, n;
    void init(int _n){
        n = _n;
        adj.resize(n);
        dad.resize(n);
        sub.resize(n);
        rem.assign(n, false);
    }
    // Return Centroid Decomposition Tree
    vector<vector<int>> build(){
        assert(n > 0);
        centroidRoot = decomp(0, -1);
        vector<vector<int>> ret(n);
        for (int u = 0; u < n; u++){
            if (dad[u] != u)
                ret[dad[u]].push_back(u);
        }
        return ret;
    }
    void addEdge(int a, int b){
        adj[a].push_back(b);
        adj[b].push_back(a);
    }
    int decomp(int u, int p){
        int sz = dfs(u, p);
        int c = centroid(u, p, sz);
        if (p == -1)
            p = c;

```

```

    dad[c] = p;
    rem[c] = true;
    for (auto to : adj[c]){
        if (!rem[to])
            decomp(to, c);
    }
    return c;
}
int dfs(int u, int p){
    sub[u] = 1;
    for (int to : adj[u]){
        if (!rem[to] and to != p)
            sub[u] += dfs(to, u);
    }
    return sub[u];
}
int centroid(int u, int p, int sz){
    for (auto to : adj[u])
        if (!rem[to] and to != p and sub[to] > sz / 2)
            return centroid(to, u, sz);
    return u;
}
int operator[] (int i){
    return dad[i];
}
};

```

2.3 Dinic

```

#include <bits/stdc++.h>
using namespace std;
//O(V^2*E): for generic graph
//O(sqrt(V)*E): for bipartite graph
template <typename flow_t>
struct Dinic{
    struct FlowEdge{
        int from, to;
        flow_t cap, flow = 0;
        FlowEdge(int from, int to, flow_t cap) : from(from), to(to), cap(
            cap) {}
    };
    const flow_t flow_inf = numeric_limits<flow_t>::max();
    vector<FlowEdge> edges;
    vector<vector<int>> adj;
    int n, m = 0;
    int s, t;
    vector<int> level, ptr;
    queue<int> q;
    bool bfs(){
        while (!q.empty()){
            int u = q.front();
            q.pop();
            for (int id : adj[u]){
                if (edges[id].cap - edges[id].flow < 1)
                    continue;
                if (level[edges[id].to] != -1)
                    continue;
                level[edges[id].to] = level[u] + 1;
                q.push(edges[id].to);
            }
        }
    }
};

```

```

    }
    return level[t] != -1;
}
flow_t dfs(int u, flow_t pushed){
    if (pushed == 0)
        return 0;
    if (u == t)
        return pushed;
    for (int &cid = ptr[u]; cid < (int)adj[u].size(); cid++){
        int id = adj[u][cid];
        int to = edges[id].to;
        if (level[u] + 1 != level[to] || edges[id].cap - edges[id].flow
            < 1)
            continue;
        flow_t tr = dfs(to, min(pushed, edges[id].cap - edges[id].flow));
        ;
        if (tr == 0)
            continue;
        edges[id].flow += tr;
        edges[id ^ 1].flow -= tr;
        return tr;
    }
    return 0;
}
Dinic(){}
void init(int _n){
    n = _n;
    adj.resize(n);
    level.resize(n);
    ptr.resize(n);
}
void addEdge(int from, int to, flow_t cap){
    assert(n>0);
    edges.push_back(FlowEdge(from, to, cap));
    edges.push_back(FlowEdge(to, from, 0));
    adj[from].push_back(m);
    adj[to].push_back(m + 1);
    m += 2;
}
flow_t maxFlow(int s1, int t1){
    s = s1, t = t1;
    flow_t f = 0;
    for(int i=0; i<m; i++)
        edges[i].flow = 0;
    while (true){
        level.assign(n, -1);
        level[s] = 0;
        q.push(s);
        if (!bfs())
            break;
        ptr.assign(n, 0);
        while (flow_t pushed = dfs(s, flow_inf))
            f += pushed;
    }
    return f;
}
};
typedef pair<int, int> pii;
vector<pii> recoverCut(Dinic<int> &d){
    vector<int> level(d.n, 0);
    vector<pii> rc;
}

```

```

queue<int> q;
q.push(d.s);
level[d.s] = 1;
while (!q.empty()){
    int u = q.front();
    q.pop();
    for (int id : d.adj[u]){
        if ((id & 1) == 1)
            continue;
        if (d.edges[id].cap == d.edges[id].flow){
            rc.push_back(pii(d.edges[id].from, d.edges[id].to));
        }else{
            if (level[d.edges[id].to] == 0){
                q.push(d.edges[id].to);
                level[d.edges[id].to] = 1;
            }
        }
    }
}
vector<pii> ans;
for (pii p : rc)
    if ((level[p.first] == 0) or (level[p.second] == 0))
        ans.push_back(p);
return ans;
}

```

2.4 Flow With Demand

```

#include "dinic.h"
using namespace std;
template <typename flow_t>
struct MaxFlowEdgeDemands{
    Dinic<flow_t> mf;
    vector<flow_t> ind, outd;
    flow_t D;
    int n;
    MaxFlowEdgeDemands(int n) : n(n){
        D = 0;
        mf.init(n + 2);
        ind.assign(n, 0);
        outd.assign(n, 0);
    }
    void addEdge(int a, int b, flow_t cap, flow_t demands){
        mf.addEdge(a, b, cap - demands);
        D += demands;
        ind[b] += demands;
        outd[a] += demands;
    }
    bool solve(int s, int t){
        mf.addEdge(t, s, numeric_limits<flow_t>::max());
        for (int i = 0; i < n; i++){
            if (ind[i]) mf.addEdge(n, i, ind[i]);
            if (outd[i]) mf.addEdge(i, n + 1, outd[i]);
        }
        return mf.maxFlow(n, n + 1) == D;
    }
};

```

2.5 Kruskal

```

#include "../data_structures/union_find.h"
typedef long long ll;
struct Edge{
    int u, v; ll w;
    Edge(){}
    Edge(int u, int v, ll w):u(u), v(v), w(w){}
};
ll kruskal(vector<Edge> v, int nVet){
    ll cost = 0;
    UnionFind uf(nVet);
    sort(v.begin(), v.end(), [&](Edge a, Edge b){
        return a.w < b.w;
    });
    for(Edge &e: v){
        if(!uf.isSame(e.u, e.v)){
            cost += e.w;
            uf.join(e.u, e.v);
        }
    }
    return cost;
}

```

2.6 HLD

```

#include <bits/stdc++.h>
#include "../data_structures/bit_range.h"
using namespace std;
#define F first
#define S second
using hld_t = long long;
using pii = pair<int, hld_t>;
struct HLD{
    vector<vector<pii>> adj;
    vector<int> sz, h, dad, pos;
    vector<hld_t> val, v;
    int t;
    bool edge;
    //Begin Internal Data Structure
    BitRange *bit;
    hld_t neutral = 0;
    inline hld_t join(hld_t a, hld_t b){
        return a+b;
    }
    inline void update(int a, int b, hld_t x){
        bit->add(a+1, b+1, x);
    }
    inline hld_t query(int a, int b){
        return bit->get(a+1, b+1);
    }
    //End Internal Data Structure
    void init(int n){
        dad.resize(n); pos.resize(n); val.resize(n); v.resize(n);
        adj.resize(n); sz.resize(n); h.resize(n);
        bit = new BitRange(n);
    }
    void dfs(int u, int p = -1){

```


2.7 Minimum Cost Maximum Flow

```
#include <bits/stdc++.h>
using namespace std;
template <class T = int>
class MCMF{
private:
    struct Edge{
        int to;
        T cap, cost;
        Edge(int a, T b, T c) : to(a), cap(b), cost(c) {}
    };
    int n;
    vector<vector<int>>> edges;
    vector<Edge> list;
    vector<int> from;
    vector<T> dist, pot;
    vector<bool> visit;
    pair<T, T> augment(int src, int sink){
        pair<T, T> flow = {list[from[sink]].cap, 0};
        for (int v = sink; v != src; v = list[from[v] ^ 1].to){
            flow.first = std::min(flow.first, list[from[v]].cap);
            flow.second += list[from[v]].cost;
        }
        for (int v = sink; v != src; v = list[from[v] ^ 1].to){
            list[from[v]].cap -= flow.first;
            list[from[v] ^ 1].cap += flow.first;
        }
        return flow;
    }
    queue<int> q;
    bool SPFA(int src, int sink){
        T INF = numeric_limits<T>::max();
        dist.assign(n, INF);
        from.assign(n, -1);
        q.push(src);
        dist[src] = 0;
        while (!q.empty()){
            int on = q.front();
            q.pop();
            visit[on] = false;
            for (auto e : edges[on]){
                auto ed = list[e];
                if (ed.cap == 0)
                    continue;
                T toDist = dist[on] + ed.cost + pot[on] - pot[ed.to];
                if (toDist < dist[ed.to]){
                    dist[ed.to] = toDist;
                    from[ed.to] = e;
                    if (!visit[ed.to]){
                        visit[ed.to] = true;
                        q.push(ed.to);
                    }
                }
            }
        }
        return dist[sink] < INF;
    }
    void fixPot(){
        T INF = numeric_limits<T>::max();
```

```
sz[u] = 1;
for(pii &to: adj[u]) if(to.F != p){
    if(edge) val[to.F] = to.S;
    dfs(to.F, u);
    sz[u] += sz[to.F];
    if(sz[to.F] > sz[adj[u][0].F] or adj[u][0].F == p)
        swap(to, adj[u][0]);
}
}
void build_hld(int u, int p=-1){
    dad[u] = p;
    pos[u] = t++;
    v[pos[u]] = val[u];
    for(pii to: adj[u]) if(to.F != p){
        h[to.F] = (to == adj[u][0]) ? h[u] : to.F;
        build_hld(to.F, u);
    }
}
void addEdge(int a, int b, hld_t w = 0){
    adj[a].emplace_back(b, w);
    adj[b].emplace_back(a, w);
}
void build(int root, bool is_edge){
    assert(!adj.empty());
    edge = is_edge;
    t = 0;
    h[root] = 0;
    dfs(root);
    build_hld(root);
    //Init Internal Data Structure
    for(int i=0; i<t; i++)
        update(i, i, v[i]);
}
hld_t query_path(int a, int b) {
    if (edge and a == b) return neutral;
    if (pos[a] < pos[b]) swap(a, b);
    if (h[a] == h[b]) return query(pos[b]+edge, pos[a]);
    return join(query(pos[h[a]], pos[a]), query_path(dad[h[a]], b));
}
void update_path(int a, int b, hld_t x) {
    if (edge and a == b) return;
    if (pos[a] < pos[b]) swap(a, b);
    if (h[a] == h[b]) return (void)update(pos[b]+edge, pos[a], x);
    update(pos[h[a]], pos[a], x); update_path(dad[h[a]], b, x);
}
hld_t query_subtree(int a) {
    if (edge and sz[a] == 1) return neutral;
    return query(pos[a]+edge, pos[a]+sz[a]-1);
}
void update_subtree(int a, hld_t x) {
    if (edge and sz[a] == 1) return;
    update(pos[a] + edge, pos[a]+sz[a]-1, x);
}
int lca(int a, int b) {
    if (pos[a] < pos[b]) swap(a, b);
    return h[a] == h[b] ? b : lca(dad[h[a]], b);
}
};
```

```

    for (int i = 0; i < n; i++){
        if (dist[i] < INF)
            pot[i] += dist[i];
    }
}
public:
    MCMF(int size){
        n = size;
        edges.resize(n);
        pot.assign(n, 0);
        dist.resize(n);
        visit.assign(n, false);
    }
    pair<T, T> solve(int src, int sink){
        pair<T, T> ans(0, 0);
        // Can use dijkstra to speed up depending on the graph
        if (!SPFA(src, sink))
            return ans;
        fixPot();
        // Can use dijkstra to speed up depending on the graph
        while (SPFA(src, sink)){
            auto flow = augment(src, sink);
            ans.first += flow.first;
            ans.second += flow.first * flow.second;
            fixPot();
        }
        return ans;
    }
    void addEdge(int from, int to, T cap, T cost){
        edges[from].push_back(list.size());
        list.push_back(Edge(to, cap, cost));
        edges[to].push_back(list.size());
        list.push_back(Edge(from, 0, -cost));
    }
};

/*bool dij(int src, int sink){
    T INF = numeric_limits<T>::max();
    dist.assign(n, INF);
    from.assign(n, -1);
    visit.assign(n, false);
    dist[src] = 0;
    for(int i = 0; i < n; i++){
        int best = -1;
        for(int j = 0; j < n; j++){
            if(visit[j]) continue;
            if(best == -1 || dist[best] > dist[j]) best = j;
        }
        if(dist[best] >= INF) break;
        visit[best] = true;
        for(auto e : edges[best]){
            auto ed = list[e];
            if(ed.cap == 0) continue;
            T toDist = dist[best] + ed.cost + pot[best] - pot[ed.to];
            assert(toDist >= dist[best]);
            if(toDist < dist[ed.to]){
                dist[ed.to] = toDist;
                from[ed.to] = e;
            }
        }
    }
    return dist[sink] < INF;
}

```

```

}*/

```

2.8 Strongly Connected Component

```

#include "topological_sort.h"
using namespace std;
namespace SCC{
    typedef pair<int, int> pii;
    vector<vector<int>> revAdj;
    vector<int> component;
    void dfs(int u, int c){
        component[u] = c;
        for (int to : revAdj[u]){
            if (component[to] == -1)
                dfs(to, c);
        }
    }
    vector<int> scc(int n, vector<pii> &edges){
        revAdj.assign(n, vector<int>());
        for (pii p : edges)
            revAdj[p.second].push_back(p.first);
        vector<int> tp = TopologicalSort::order(n, edges);
        component.assign(n, -1);
        int comp = 0;
        for (int u : tp){
            if (component[u] == -1)
                dfs(u, comp++);
        }
        return component;
    }
} // namespace SCC

```

2.9 Topological Sort

```

#include <bits/stdc++.h>
using namespace std;
namespace TopologicalSort{
    typedef pair<int, int> pii;
    vector<vector<int>> adj;
    vector<bool> visited;
    vector<int> vAns;
    void dfs(int u){
        visited[u] = true;
        for (int to : adj[u]){
            if (!visited[to])
                dfs(to);
        }
        vAns.push_back(u);
    }
    vector<int> order(int n, vector<pii> &edges){
        adj.assign(n, vector<int>());
        for (pii p : edges)
            adj[p.first].push_back(p.second);
        visited.assign(n, false);
        vAns.clear();
        for (int i = 0; i < n; i++){
            if (!visited[i])
                dfs(i);
        }
    }
}

```

```

    }
    reverse(vAns.begin(), vAns.end());
    return vAns;
}
}; // namespace TopologicalSort

```

3 Dynamic Programming

3.1 Divide and Conquer Optimization

Reduces the complexity from $O(n^2k)$ to $O(nk \log n)$ of PD's in the following ways (and other variants):

$$dp[n][k] = \max_{0 \leq i < n} (dp[i][k-1] + C[i+1][n]), \text{ base case : } dp[0][j], dp[i][0] \quad (1)$$

- $C[i][j]$ = the cost only depends on i and j .
- $opt[n][k] = i$ is the optimal value that maximizes $dp[n][k]$.

It is necessary that opt is increasing along each column: $opt[j][k] \leq opt[j+1][k]$.

3.2 Divide and Conquer Optimization Implementation

```

#include <bits/stdc++.h>
using namespace std;
int C(int i, int j);
const int MAXN = 100010;
const int MAXK = 110;
const int INF = 0x3f3f3f3f;
int dp[MAXN][MAXK];
void calculateDP(int l, int r, int k, int opt_l, int opt_r){
    if (l > r)
        return;
    int mid = (l + r) >> 1;
    int ans = -INF, opt = mid;
    // int ans = dp[mid][k-1], opt=mid; //If you accept empty subsegment
    for (int i = opt_l; i <= min(opt_r, mid - 1); i++){
        if (ans < dp[i][k-1] + C(i+1, mid)){
            opt = i;
            ans = dp[i][k-1] + C(i+1, mid);
        }
    }
    dp[mid][k] = ans;
    calculateDP(l, mid - 1, k, opt_l, opt);
    calculateDP(mid + 1, r, k, opt, opt_r);
}
int solve(int n, int k){
    for (int i = 0; i <= n; i++)
        dp[i][0] = -INF;
    for (int j = 0; j <= k; j++)
        dp[0][j] = -INF;
    dp[0][0] = 0;
    for (int j = 1; j <= k; j++)
        calculateDP(1, n, j, 0, n - 1);
    return dp[n][k];
}

```

3.3 Knuth Optimization

Reduces the complexity from $O(n^3)$ to $O(n^2)$ of PD's in the following ways (and other variants):

$$dp[i][j] = C[i][j] + \min_{i < k < j} (dp[i][k] + dp[k][j]), \text{ caso base : } dp[i][i] \quad (2)$$

$$dp[i][j] = \min_{i < k < j} (dp[i][k] + C[i][k]), \text{ caso base : } dp[i][i] \quad (3)$$

- $C[i][j]$ = the cost only depends on i and j .
- $opt[i][j] = k$ is the optimal value that maximizes $dp[i][j]$.

The following conditions must be met:

- Foursquare inequality on C : $C[a][c] + C[b][d] \leq C[a][d] + C[b][c]$, $a \leq b \leq c \leq d$.
- Monotonicity on C : $C[b][c] \leq C[a][d]$, $a \leq b \leq c \leq d$.

Or the following condition:

- opt increasing in rows and columns: $opt[i][j-1] \leq opt[i][j] \leq opt[i+1][j]$.

3.4 Knuth Optimization Implementation

```

#include <bits/stdc++.h>
using namespace std;
typedef long long ll;
const int MAXN = 1009;
const ll INFLL = 0x3f3f3f3f3f3f3f3f;
ll C(int a, int b);
ll dp[MAXN][MAXN];
int opt[MAXN][MAXN];
ll knuth(int n){
    for (int i = 0; i < n; i++){
        dp[i][i] = 0;
        opt[i][i] = i;
    }
    for (int s = 1; s < n; s++){
        for (int i = 0, j; (i + s) < n; i++){
            j = i + s;
            dp[i][j] = INFLL;
            for (int k = opt[i][j-1]; k < min(j, opt[i+1][j]+1); k++){
                ll cur = dp[i][k] + dp[k+1][j] + C(i, j);
                if (dp[i][j] > cur){
                    dp[i][j] = cur;
                    opt[i][j] = k;
                }
            }
        }
    }
    return dp[0][n-1];
}

```

4 Math

4.1 Basic Math

```
#include <bits/stdc++.h>
using namespace std;
typedef long long ll;
typedef unsigned long long ull;

ull fastPow(ull base, ull exp, ull mod){
    base %= mod;
    //exp %= phi(mod) if base and mod are relatively prime
    ull ans = 1LL;
    while (exp > 0){
        if (exp & 1LL)
            ans = (ans * (__int128_t)base) % mod;
        base = (base * (__int128_t)base) % mod;
        exp >>= 1;
    }
    return ans;
}

ll gcd(ll a, ll b){ return __gcd(a, b); }
ll lcm(ll a, ll b){ return (a / gcd(a, b)) * b; }
void enumeratingAllSubmasks(int mask){
    for (int s = mask; s; s = (s - 1) & mask)
        cout << s << endl;
}

//MOD to Hash
namespace ModHash{
    const uint64_t MOD = (1ll<<61) - 1;
    uint64_t modmul(uint64_t a, uint64_t b){
        uint64_t l1 = (uint32_t)a, h1 = a>>32, l2 = (uint32_t)b, h2 = b
            >>32;
        uint64_t l = l1*l2, m = l1*h2 + l2*h1, h = h1*h2;
        uint64_t ret = (l&MOD) + (l>>61) + (h << 3) + (m >> 29) + ((m <<
            35) >> 3) + 1;
        ret = (ret & MOD) + (ret>>61);
        ret = (ret & MOD) + (ret>>61);
        return ret-1;
    }
};
```

4.2 BigInt

```
#include <bits/stdc++.h>
using namespace std;
typedef int32_t intB;
typedef int64_t longB;
typedef vector<intB> vib;
class BigInt{
private:
    vib vb;
    bool neg;
    const int BASE_DIGIT = 9;
    const intB base = 1000000LL*1000; //000LL*1000000LL;
    void fromString(string &s){
        if(s[0] == '-'){
```

```
            neg = true;
            s = s.substr(1);
        }else{
            neg = false;
        }
        vb.clear();
        vb.reserve((s.size()+BASE_DIGIT-1)/BASE_DIGIT);
        for(int i=(int)s.length(); i>0; i-=BASE_DIGIT){
            if(i < BASE_DIGIT)
                vb.push_back(stol(s.substr(0, i)));
            else
                vb.push_back(stol(s.substr(i-BASE_DIGIT, BASE_DIGIT)));
        }
        fix(vb);
    }

    void fix(vib &v){
        while(v.size()>1 && v.back()==0)
            v.pop_back();
        if(v.size() == 0)
            neg = false;
    }

    bool comp(vib &a, vib &b){
        fix(a); fix(b);
        if(a.size() != b.size()) return a.size() < b.size();
        for(int i=(int)a.size()-1; i>=0; i--) {
            if(a[i] != b[i]) return a[i] < b[i];
        }
        return false;
    }

    vib sum(vib a, vib b){
        int carry = 0;
        for(size_t i=0; i<max(a.size(), b.size()) or carry; i++){
            if(i == a.size())
                a.push_back(0);
            a[i] += carry + (i<b.size() ? b[i] : 0);
            carry = (a[i] >= base);
            if(carry) a[i] -= base;
        }
        fix(a);
        return a;
    }

    vib sub(vib a, vib b){
        int carry = 0;
        for(size_t i=0; i<b.size() or carry; i++){
            a[i] -= carry + (i<b.size() ? b[i] : 0);
            carry = a[i] < 0;
            if(carry) a[i] += base;
        }
        fix(a);
        return a;
    }
};
```

```
public:
    BigInt(){}
    BigInt(intB n){
        neg = (n<0);
        vb.push_back(abs(n));
        fix(vb);
    }
    BigInt(string s){
        fromString(s);
```

```

}
BigInt operator =(BigInt oth){
    this->neg = oth.neg;
    this->vb = oth.vb;
    return *this;
}
BigInt operator +(BigInt &oth){
    vib &a = vb, &b = oth.vb;
    BigInt ans;
    if(neg == oth.neg){
        ans.vb = sum(vb, oth.vb);
        ans.neg = neg;
    }else{
        if(comp(a, b)){
            ans.vb = sub(b, a);
            ans.neg = oth.neg;
        }else{
            ans.vb = sub(a, b);
            ans.neg = neg;
        }
    }
    return ans;
}
BigInt operator -(BigInt oth){
    oth.neg ^= true;
    return (*this) + oth;
}
BigInt operator *(intB b){
    bool negB = false;
    if(b < 0){
        negB = true;
        b = -b;
    }
    BigInt ans = *this;
    auto &a = ans.vb;
    intB carry = 0;
    for(size_t i=0; i<a.size() or carry; i++){
        if(i == a.size()) a.push_back(0);
        longB cur = carry + a[i] *(longB) b;
        a[i] = intB(cur%base);
        carry = intB(cur/base);
    }
    ans.neg ^= negB;
    fix(ans.vb);
    return ans;
}
BigInt operator *(BigInt &oth){
    BigInt ans;
    auto a = vb, &b = oth.vb, &c = ans.vb;
    c.assign(a.size() + b.size(), 0);
    for(size_t i=0; i<a.size(); i++){
        intB carry=0;
        for(size_t j=0; j<b.size() or carry; j++){
            longB cur = c[i+j] + a[i]*(longB) (j<b.size() ? b[j] : 0);
            cur += carry;
            c[i+j] = intB(cur%base);
            carry = intB(cur/base);
        }
    }
    ans.neg = neg^oth.neg;
    fix(ans.vb);
}

```

```

    return ans;
}
BigInt operator /(intB b){
    bool negB = false;
    if(b < 0){
        negB = true;
        b = -b;
    }
    BigInt ans = *this;
    auto &a = ans.vb;
    intB carry = 0;
    for(int i=(int)a.size()-1; i>=0; i--){
        longB cur = a[i] + (longB)carry * base;
        a[i] = intB(cur/b);
        carry = intB(cur%b);
    }
    ans.neg ^= negB;
    fix(ans.vb);
    return ans;
}
void shiftL(int b){
    vb.resize(vb.size() + b);
    for(int i=(int)vb.size()-1; i>=0; i--){
        if(i>=b) vb[i] = vb[i-b];
        else vb[i] = 0;
    }
    fix(vb);
}
void shiftR(int b){
    if((int)vb.size() <= b){
        vb.clear();
        vb.push_back(0);
        return;
    }
    for(int i=0; i<((int)vb.size() - b); i++){
        vb[i] = vb[i+b];
    }
    vb.resize((int)vb.size() - b);
    fix(vb);
}
void divide(BigInt a, BigInt b, BigInt &q, BigInt &r){
    BigInt z(0), p(1);
    while(b < a) {
        p.shiftL(max(1, (int)(a.vb.size()-b.vb.size())));
        b.shiftL(max(1, (int)(a.vb.size()-b.vb.size())));
    }
    while(true) {
        while ((a < b) && (z < p)) {
            p = p/10;
            b = b/10;
        }
        if(!(z < p)) break;
        a = a - b;
        q = q + p;
    }
    r = a;
}
BigInt operator /(BigInt &oth){
    BigInt q, r;
    divide(*this, oth, q, r);
    return q;
}

```

```

BigInt operator %(BigInt &oth){
    BigInt q, r;
    divide(*this, oth, q, r);
    return r;
}
bool operator <(BigInt &oth){
    BigInt ans = (*this) - oth;
    return ans.neg;
}
bool operator ==(BigInt &oth){
    BigInt ans = (*this) - oth;
    return (ans.vb.size()==1) and (ans.vb.back()==0);
}
friend ostream &operator<<(ostream &out, const BigInt &D){
    if(D.neg)
        out << '-';
    out << (D.vb.empty() ? 0 : D.vb.back());
    for(int i=(int)D.vb.size()-2; i>=0; i--)
        out << setfill('0') << setw(D.BASE_DIGIT) << D.vb[i];
    return out;
}
string to_string(){
    std::stringstream ss;
    ss << (*this);
    return ss.str();
}
friend istream &operator>>(istream &input, BigInt &D) {
    string s;
    input >> s;
    D.fromString(s);
    return input;
}
};

```

4.3 Binomial Coefficients

```

#include <bits/stdc++.h>
#include "../basic_math.h"
#include "../modular.h"
using namespace std;
typedef long long ll;
//O(k)
ll C1(int n, int k){
    ll res = 1LL;
    for (int i = 1; i <= k; ++i)
        res = (res * (n - k + i)) / i;
    return res;
}
//O(n^2)
vector<vector<ll>> C2(int maxn, int mod){
    vector<vector<ll>> mat(maxn + 1, vector<ll>(maxn + 1, 0));
    mat[0][0] = 1;
    for (int n = 1; n <= maxn; n++){
        mat[n][0] = mat[n][n] = 1;
        for (int k = 1; k < n; k++)
            mat[n][k] = (mat[n - 1][k - 1] + mat[n - 1][k]) % mod;
    }
    return mat;
}
//O(N)

```

```

vector<int> factorial, inv_factorial;
void prevC3(int maxn, int mod){
    factorial.resize(maxn + 1);
    factorial[0] = 1;
    for (int i = 1; i <= maxn; i++)
        factorial[i] = (factorial[i - 1] * 1LL * i) % mod;
    inv_factorial.resize(maxn + 1);
    inv_factorial[maxn] = fastPow(factorial[maxn], mod - 2, mod);
    for (int i = maxn - 1; i >= 0; i--)
        inv_factorial[i] = (inv_factorial[i + 1] * 1LL * (i + 1)) % mod;
}
int C3(int n, int k, int mod){
    if (n < k)
        return 0;
    return ((factorial[n] * 1LL * inv_factorial[k]) % mod) * 1LL *
        inv_factorial[n - k]) % mod;
}
//O(P*log(P))
//C4(n, k, p) = Comb(n, k)%p
vector<int> changeBase(int n, int p){
    vector<int> v;
    while (n > 0){
        v.push_back(n % p);
        n /= p;
    }
    return v;
}
int C4(int n, int k, int p){
    auto vn = changeBase(n, p);
    auto vk = changeBase(k, p);
    int mx = max(vn.size(), vk.size());
    vn.resize(mx, 0);
    vk.resize(mx, 0);
    prevC3(p - 1, p);
    int ans = 1;
    for (int i = 0; i < mx; i++)
        ans = (ans * 1LL * C3(vn[i], vk[i], p)) % p;
    return ans;
}
//O(P^k)
//C5(n, k, p, pk) = Comb(n, k)%(p^k)
int fat_p(ll n, int p, int pk){
    vector<int> fat1(pk, 1);
    int res = 1;
    for(int i=1; i<pk; i++){
        if(i%p == 0)
            fat1[i] = fat1[i-1];
        else
            fat1[i] = (fat1[i-1]*1LL*i)%pk;
    }
    while(n > 1){
        res = (res*1LL*fastPow(fat1[pk-1], n/pk, pk))%pk;
        res = (res*1LL*fat1[n%pk])%pk;
        n /= p;
    }
    return res;
}
ll cnt(ll n, int p){
    ll ans = 0;
    while(n > 1){
        ans += n/p;

```

```

    n/=p;
}
return ans;
}
int C5(ll n, ll k, int p, int pk){
    ll exp = cnt(n, p) - cnt(n-k, p) - cnt(k, p);
    int d = (fat_p(n-k, p, pk)*1LL*fat_p(k, p, pk))%pk;
    int ans = (fat_p(n, p, pk)*1LL*inv(d, pk))%pk;
    return (ans*1LL*fastPow(p, exp, pk))%pk;
}

```

4.4 Chinese Remainder Theorem

```

#include <bits/stdc++.h>
#include "extended_euclidean.h"
using namespace std;
typedef long long ll;
namespace CRT{
    inline ll normalize(ll x, ll mod){
        x %= mod;
        if (x < 0)
            x += mod;
        return x;
    }
    ll solve(vector<ll> a, vector<ll> m){
        int n = a.size();
        for (int i = 0; i < n; i++)
            normalize(a[i], m[i]);
        ll ans = a[0];
        ll lcm1 = m[0];
        for (int i = 1; i < n; i++){
            ll x, y;
            ll g = extGcd(lcm1, m[i], x, y);
            if ((a[i] - ans) % g != 0)
                return -1;
            ans = normalize(ans + (((a[i] - ans) / g) * x) % (m[i] / g)) *
                        lcm1, (lcm1 / g) * m[i]);
            lcm1 = (lcm1 / g) * m[i]; //lcm(lcm1, m[i]);
        }
        return ans;
    }
} // namespace CRT

```

4.5 Euler's totient

```

#include <bits/stdc++.h>
using namespace std;
int nthPhi(int n){
    int result = n;
    for (int i = 2; i <= n / i; i++){
        if (n % i == 0){
            while (n % i == 0)
                n /= i;
            result -= result / i;
        }
    }
    if (n > 1)
        result -= result / n;
}

```

```

    return result;
}
vector<int> phiFrom1toN(int n){
    vector<int> vPhi(n + 1);
    vPhi[0] = 0;
    vPhi[1] = 1;
    for (int i = 2; i <= n; i++)
        vPhi[i] = i;
    for (int i = 2; i <= n; i++){
        if (vPhi[i] == i){
            for (int j = i; j <= n; j += i)
                vPhi[j] -= vPhi[j] / i;
        }
    }
    return vPhi;
}

```

4.6 Extended Euclidean

```

#include <bits/stdc++.h>
using namespace std;
typedef long long ll;
ll extGcd(ll a, ll b, ll &x, ll &y){
    if (b == 0){
        x = 1, y = 0;
        return a;
    }else{
        ll g = extGcd(b, a % b, y, x);
        y -= (a / b) * x;
        return g;
    }
}
//a*x + b*y = g
//a*(x-(b/g)*k) + b*(y+(a/g)*k) = g
bool dioEq(ll a, ll b, ll c, ll &x0, ll &y0, ll &g){
    g = extGcd(abs(a), abs(b), x0, y0);
    if (c % g) return false;
    x0 *= c / g;
    y0 *= c / g;
    if (a < 0) x0 = -x0;
    if (b < 0) y0 = -y0;
    return true;
}
inline void shift_solution(ll &x, ll &y, ll a, ll b, ll cnt){
    x += cnt * b;
    y -= cnt * a;
}
ll findAllSolutions(ll a, ll b, ll c, ll minx, ll maxx, ll miny, ll
maxy){
    ll x, y, g;
    if (a==0 or b==0){
        if (a==0 and b==0)
            return (c==0)*(maxx-minx+1)*(maxy-miny+1);
        if (a == 0)
            return (c%b == 0)*(maxx-minx+1)*(miny<=c/b and c/b<=maxy);
        return (c%a == 0)*(minx<=c/a and c/a<=maxx)*(maxy-miny+1);
    }
    if (!dioEq(a, b, c, x, y, g))
        return 0;
    a /= g;
}

```

```

b /= g;
int sign_a = a > 0 ? +1 : -1;
int sign_b = b > 0 ? +1 : -1;
shift_solution(x, y, a, b, (minx - x) / b);
if (x < minx)
    shift_solution(x, y, a, b, sign_b);
if (x > maxx)
    return 0;
ll lx1 = x;
shift_solution(x, y, a, b, (maxx - x) / b);
if (x > maxx)
    shift_solution(x, y, a, b, -sign_b);
ll rx1 = x;
shift_solution(x, y, a, b, -(miny - y) / a);
if (y < miny)
    shift_solution(x, y, a, b, -sign_a);
if (y > maxy)
    return 0;
ll lx2 = x;
shift_solution(x, y, a, b, -(maxy - y) / a);
if (y > maxy)
    shift_solution(x, y, a, b, sign_a);
ll rx2 = x;
if (lx2 > rx2)
    swap(lx2, rx2);
ll lx = max(lx1, lx2);
ll rx = min(rx1, rx2);
if (lx > rx)
    return 0;
return (rx - lx) / abs(b) + 1;
}

```

4.7 Fraction

```

#include <bits/stdc++.h>
using namespace std;
typedef long long f_type;
//Representation of the a/b
struct Fraction {
    f_type a, b;
    Fraction(f_type _a = 0): a(_a), b(1){}
    Fraction(f_type _a, f_type _b) {
        f_type g = __gcd(_a, _b);
        a = _a/g;
        b = _b/g;
        if(b < 0){
            a = -a;
            b = -b;
        }
    }
    Fraction operator+(Fraction oth) {
        return Fraction(a*oth.b + oth.a*b, b*oth.b);
    }
    Fraction operator-(Fraction oth) {
        return Fraction(a*oth.b - oth.a*b, b*oth.b);
    }
    Fraction operator*(Fraction oth) {
        return Fraction(a*oth.a, b*oth.b);
    }
    Fraction operator/(Fraction oth) {

```

```

        return Fraction(a*oth.b, b*oth.a);
    }
    bool operator>=(Fraction oth){
        return ((*this) - oth).a >= 0;
    }
    bool operator==(Fraction oth){
        return a == oth.a and b == oth.b;
    }
    operator f_type() {return a/b;}
    operator double() {return double(a)/b;}
};

```

4.8 Gray Code

```

int grayCode(int nth){
    return nth ^ (nth >> 1);
}
int revGrayCode(int g){
    int nth = 0;
    for (; g > 0; g >>= 1)
        nth ^= g;
    return nth;
}

```

4.9 Matrix

```

#include <bits/stdc++.h>
#include "modular.h"
using namespace std;
const int D = 3;
struct Matrix{
    int m[D][D];
    Matrix(bool identify = false){
        memset(m, 0, sizeof(m));
        for (int i = 0; i < D; i++)
            m[i][i] = identify;
    }
    Matrix(vector<vector<int>> mat){
        for(int i=0; i<D; i++){
            for(int j=0; j<D; j++){
                m[i][j] = mat[i][j];
            }
        }
    }
    int * operator[](int pos){
        return m[pos];
    }
    Matrix operator*(Matrix oth){
        Matrix ans;
        for (int i = 0; i < D; i++){
            for (int j = 0; j < D; j++){
                int &sum = ans[i][j];
                for (int k = 0; k < D; k++){
                    sum = modSum(sum, modMul(m[i][k], oth[k][j]));
                }
            }
        }
        return ans;
    }
};

```


4.10 Modular Arithmetic

```
#include <bits/stdc++.h>
#include "extended_euclidean.h"
using namespace std;
const int MOD = 1000000007;
inline int modSum(int a, int b, int mod = MOD){
    int ans = a+b;
    if(ans > mod) ans -= mod;
    return ans;
}
inline int modSub(int a, int b, int mod = MOD){
    int ans = a-b;
    if(ans < 0) ans += mod;
    return ans;
}
inline int modMul(int a, int b, int mod = MOD){
    return (a*1LL*b)%mod;
}
int inv(int a, int mod=MOD){
    ll inv_x, y;
    extGcd(a, mod, inv_x, y);
    return (inv_x%mod + mod)%mod;
}
int modDiv(int a, int b, int mod = MOD){
    return modMul(a, inv(b, mod));
}
```

4.11 Montgomery Multiplication

```
#include <bits/stdc++.h>
using namespace std;
using u64 = uint64_t;
using u128 = __uint128_t;
using i128 = __int128_t;
struct u256{
    u128 high, low;
    static u256 mult(u128 x, u128 y){
        u64 a = x >> 64, b = x;
        u64 c = y >> 64, d = y;
        u128 ac = (u128)a * c;
        u128 ad = (u128)a * d;
        u128 bc = (u128)b * c;
        u128 bd = (u128)b * d;
        u128 carry = (u128)(u64)ad + (u128)(u64)bc + (bd >> 64u);
        u128 high = ac + (ad >> 64u) + (bc >> 64u) + (carry >> 64u);
        u128 low = (ad << 64u) + (bc << 64u) + bd;
        return {high, low};
    }
};
//x_m := x*r mod n
struct Montgomery{
    u128 mod, inv, r2;
    //the N will be an odd number
    Montgomery(u128 n) : mod(n), inv(1), r2(-n % n){
        for (int i = 0; i < 7; i++){
            inv *= 2 - n * inv;
        }
        for (int i = 0; i < 4; i++){
```

```
            r2 <= 1;
            if (r2 >= mod)
                r2 -= mod;
        }
        for (int i = 0; i < 5; i++){
            r2 = mult(r2, r2);
        }
    }
    u128 init(u128 x){
        return mult(x, r2);
    }
    u128 reduce(u256 x){
        u128 q = x.low * inv;
        i128 a = x.high - u256::mult(q, mod).high;
        if (a < 0)
            a += mod;
        return a;
    }
    u128 mult(u128 a, u128 b){
        return reduce(u256::mult(a, b));
    }
};
```

4.12 Prime Number

```
#include <bits/stdc++.h>
#include "basic_math.h"
using namespace std;
typedef unsigned long long ull;
ull modMul(ull a, ull b, ull mod){
    return (a * (__uint128_t)b) % mod;
}
bool checkComposite(ull n, ull a, ull d, int s){
    ull x = fastPow(a, d, n);
    if (x == 1 or x == n - 1)
        return false;
    for (int r = 1; r < s; r++){
        x = modMul(x, x, n);
        if (x == n - 1LL)
            return false;
    }
    return true;
};
bool millerRabin(ull n){
    if (n < 2)
        return false;
    int r = 0;
    ull d = n - 1LL;
    while ((d & 1LL) == 0){
        d >>= 1;
        r++;
    }
    for (ull a : {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37}){
        if (n == a)
            return true;
        if (checkComposite(n, a, d, r))
            return false;
    }
    return true;
}
ull pollard(ull n){
```

```

auto f = [n](ull x) { return modMul(x, x, n) + 1; };
ull x = 0, y = 0, t = 0, prd = 2, i = 1, q;
while (t++ % 40 || __gcd(prd, n) == 1){
    if (x == y)
        x = ++i, y = f(x);
    if ((q = modMul(prd, max(x, y) - min(x, y), n))
        prd = q;
    x = f(x), y = f(f(y));
}
return __gcd(prd, n);
}
vector<ull> factor(ull n){
    if (n == 1)
        return {};
    if (millerRabin(n))
        return {n};
    ull x = pollard(n);
    auto l = factor(x), r = factor(n / x);
    l.insert(l.end(), r.begin(), r.end());
    return l;
}

```

5 Geometry

5.1 Basic Geometry

```

#include <bits/stdc++.h>
using namespace std;
#define POINT_DOUBLE
#ifdef POINT_DOUBLE
    typedef double ftype;
    typedef long double ftLong;
    const double EPS = 1e-9;
    #define eq(a, b) (abs(a - b) < EPS)
    #define lt(a, b) ((a + EPS) < b)
    #define gt(a, b) (a > (b + EPS))
    #define le(a, b) (a < (b + EPS))
    #define ge(a, b) ((a + EPS) > b)
#else
    typedef int32_t ftype;
    typedef int64_t ftLong;
    #define eq(a, b) (a == b)
    #define lt(a, b) (a < b)
    #define gt(a, b) (a > b)
    #define le(a, b) (a <= b)
    #define ge(a, b) (a >= b)
#endif
//Begin Point 2D
struct Point2d{
    ftype x, y;
    Point2d() {}
    Point2d(ftype x, ftype y) : x(x), y(y) {}
    Point2d operator+(const Point2d &t){
        return Point2d(x + t.x, y + t.y);
    }
    Point2d operator-(const Point2d &t){
        return Point2d(x - t.x, y - t.y);
    }
}

```

```

Point2d operator*(ftype t){
    return Point2d(x * t, y * t);
}
Point2d operator/(ftype t){
    return Point2d(x / t, y / t);
}
bool operator<(const Point2d &o) const{
    return lt(x, o.x) or (eq(x, o.x) and lt(y, o.y));
}
bool operator==(const Point2d &o) const{
    return eq(x, o.x) and eq(y, o.y);
}
};
ftLong pw2(ftype a){
    return a * (ftLong)a;
}
//Scalar product
ftLong dot(Point2d a, Point2d b){
    return a.x*(ftLong)b.x + a.y*(ftLong)b.y;
}
ftLong norm(Point2d a){
    return dot(a, a);
}
double len(Point2d a){
    return sqrtl(dot(a, a));
}
double dist(Point2d a, Point2d b){
    return len(a - b);
}
//Vector product
ftLong cross(Point2d a, Point2d b){
    return a.x * (ftLong)b.y - a.y * (ftLong)b.x;
}
//Projection size from A to B
double proj(Point2d a, Point2d b){
    return dot(a, b) / len(b);
}
//The angle between A and B
double angle(Point2d a, Point2d b){
    return acos(dot(a, b) / len(a) / len(b));
}
//Left rotation. Angle in radian
Point2d rotateL(Point2d p, double ang){
    return Point2d(p.x * cos(ang) - p.y * sin(ang), p.x * sin(ang) + p.y * cos(ang));
}
//90 degree left rotation
Point2d perpL(Point2d a){
    return Point2d(-a.y, a.x);
}
//0-> 1o,2o quadrant, 1-> 3o,4o
int half(Point2d &p){
    if (gt(p.y, 0) or (eq(p.y, 0) and ge(p.x, 0)))
        return 0;
    else
        return 1;
}
//angle(a) < angle(b)
bool cmpByAngle(Point2d a, Point2d b){
    int ha = half(a), hb = half(b);
    if (ha != hb)

```

```

    return ha < hb;
else
    return gt(cross(a, b), 0);
}
inline int sgn(ftLong x){
    return ge(x, 0) ? (eq(x, 0) ? 0 : 1) : -1;
}
// -1: angle(a, b) < angle(b, c)
// 0: angle(a, b) = angle(b, c)
// +1: angle(a, b) > angle(b, c)
int cmpAngleBetweenVectors(Point2d a, Point2d b, Point2d c){
    ftLong dotAB = dot(a, b), dotBC = dot(b, c);
    int sgnAB = sgn(dotAB), sgnBC = sgn(dotBC);
    if(sgnAB == sgnBC){
        ftLong l = pw2(dotAB)*dot(c, c), r = pw2(dotBC)*dot(a, a);
        if(l == r)
            return 0;
        if(sgnAB == 1)
            return (l > r)? -1 : +1;
        return (l < r)? -1 : +1;
    }else{
        return (sgnAB > sgnBC)? -1 : +1;
    }
}
// Line parameterized: r1 = a1 + d1*t
// This function can be generalized to 3D
Point2d intersect(Point2d a1, Point2d d1, Point2d a2, Point2d d2){
    return a1 + d1 * (cross(a2 - a1, d2) / cross(d1, d2));
}
// distance between the point(a) e line(pl1, pl2)
// This function can be generalized to 3D
double dist(Point2d a, Point2d pl1, Point2d pl2){
    // crs = parallelogram area
    double crs = cross(Point2d(a - pl1), Point2d(pl2 - pl1));
    // h = area/base
    return abs(crs / dist(pl1, pl2));
}
double area(vector<Point2d> p){
    double ret = 0;
    for (int i = 2; i < (int)p.size(); i++){
        ret += cross(p[i] - p[0], p[i - 1] - p[0]) / 2.0;
    }
    return abs(ret);
}
ftLong signed_area_parallelogram(Point2d p1, Point2d p2, Point2d p3){
    return cross(p2 - p1, p3 - p1);
}
double triangle_area(Point2d p1, Point2d p2, Point2d p3){
    return abs(signed_area_parallelogram(p1, p2, p3)) / 2.0;
}
bool pointInTriangle(Point2d a, Point2d b, Point2d c, Point2d p){
    ftLong s1 = abs(cross(b - a, c - a));
    ftLong s2 = abs(cross(a - p, b - p)) + abs(cross(b - p, c - p)) +
        abs(cross(c - p, a - p));
    return eq(s1, s2);
}
bool clockwise(Point2d p1, Point2d p2, Point2d p3){
    return lt(signed_area_parallelogram(p1, p2, p3), 0);
}
bool counter_clockwise(Point2d p1, Point2d p2, Point2d p3){
    return gt(signed_area_parallelogram(p1, p2, p3), 0);
}

```

```

// End Point 2D

// Begin Line
ftLong det(ftype a, ftype b, ftype c, ftype d){
    return a * (ftLong)d - b * (ftLong)c;
}
struct Line{
    ftype a, b, c;
    Line() {}
    Line(ftype a, ftype b, ftype c) : a(a), b(b), c(c){
        normalize();
    }
    Line(Point2d p1, Point2d p2){
        a = p1.y - p2.y;
        b = p2.x - p1.x;
        c = -a * p1.x - b * p1.y;
        normalize();
    }
    void normalize(){
#ifdef POINT_DOUBLE
        ftype z = sqrt(pw2(a) + pw2(b));
#else
        ftype z = __gcd(abs(a), __gcd(abs(b), abs(c)));
#endif
        a /= z;
        b /= z;
        c /= z;
        if (lt(a, 0) or (eq(a, 0) and lt(b, 0))){
            a = -a;
            b = -b;
            c = -c;
        }
    }
};
bool intersect(Line m, Line n, Point2d &res){
    ftype zn = det(m.a, m.b, n.a, n.b);
    if (eq(zn, 0))
        return false;
    res.x = -det(m.c, m.b, n.c, n.b) / zn;
    res.y = -det(m.a, m.c, n.a, n.c) / zn;
    return true;
}
bool parallel(Line m, Line n){
    return eq(det(m.a, m.b, n.a, n.b), 0);
}
bool equivalent(Line m, Line n){
    return eq(det(m.a, m.b, n.a, n.b), 0) &&
        eq(det(m.a, m.c, n.a, n.c), 0) &&
        eq(det(m.b, m.c, n.b, n.c), 0);
}
// Distance from a point(x, y) to a line m
double dist(Line m, ftype x, ftype y){
    return abs(m.a * (ftLong)x + m.b * (ftLong)y + m.c) /
        sqrt(m.a * (ftLong)m.a + m.b * (ftLong)m.b);
}
// End Line

// Begin Segment
struct Segment{
    Point2d a, b;
    Segment() {}

```

```

    Segment(Point2d a, Point2d b) : a(a), b(b) {}
};
bool interld(ftype a, ftype b, ftype c, ftype d){
    if (a > b)
        swap(a, b);
    if (c > d)
        swap(c, d);
    return le(max(a, c), min(b, d));
}
bool check_intersection(Segment s1, Segment s2){
    Point2d a = s1.a, b = s1.b, c = s2.a, d = s2.b;
    if (cross(a - c, d - c) == 0 && cross(b - c, d - c) == 0)
        return interld(a.x, b.x, c.x, d.x) && interld(a.y, b.y, c.y, d.y);
    return sgn(cross(b - a, c - a)) != sgn(cross(b - a, d - a)) &&
        sgn(cross(d - c, a - c)) != sgn(cross(d - c, b - c));
}
inline bool betw(ftype l, ftype r, ftype x){
    return le(min(l, r), x) and le(x, max(l, r));
}
bool intersect(Segment s1, Segment s2, Segment &ans){
    Point2d a = s1.a, b = s1.b, c = s2.a, d = s2.b;
    if (!interld(a.x, b.x, c.x, d.x) || !interld(a.y, b.y, c.y, d.y))
        return false;
    Line m(a, b);
    Line n(c, d);
    if (parallel(m, n)){
        if (!equivalent(m, n))
            return false;
        if (b < a)
            swap(a, b);
        if (d < c)
            swap(c, d);
        ans = Segment(max(a, c), min(b, d));
        return true;
    } else {
        Point2d p(0, 0);
        intersect(m, n, p);
        ans = Segment(p, p);
        return betw(a.x, b.x, p.x) && betw(a.y, b.y, p.y) &&
            betw(c.x, d.x, p.x) && betw(c.y, d.y, p.y);
    }
}
//End Segment

//Begin Circle
struct Circle{
    ftype x, y, r;
    Circle() {}
    Circle(ftype x, ftype y, ftype r) : x(x), y(y), r(r){};
};
bool pointInCircle(Circle c, Point2d p){
    return ge(c.r, dist(Point2d(c.x, c.y), p));
}
//CircumCircle of a triangle is a circle that passes through all the
//vertices
Circle circumCircle(Point2d a, Point2d b, Point2d c){
    Point2d u((b - a).y, -((b - a).x));
    Point2d v((c - a).y, -((c - a).x));
    Point2d n = (c - b) * 0.5;
    double t = cross(u, n) / cross(v, u);
    Point2d ct = (((a + c) * 0.5) + (v * t));

```

```

    double r = dist(ct, a);
    return Circle(ct.x, ct.y, r);
}
//InCircle is the largest circle contained in the triangle
Circle inCircle(Point2d a, Point2d b, Point2d c){
    double m1 = dist(a, b);
    double m2 = dist(a, c);
    double m3 = dist(b, c);
    Point2d ct = ((c * m1) + (b * m2) + a * (m3)) / (m1 + m2 + m3);
    double sp = 0.5 * (m1 + m2 + m3);
    double r = sqrt(sp * (sp - m1) * (sp - m2) * (sp - m3)) / sp;
    return Circle(ct.x, ct.y, r);
}
//Minimum enclosing circle, O(n)
Circle minimumCircle(vector<Point2d> p){
    random_shuffle(p.begin(), p.end());
    Circle c = Circle(p[0].x, p[0].y, 0.0);
    for (int i = 0; i < (int)p.size(); i++){
        if (pointInCircle(c, p[i]))
            continue;
        c = Circle(p[i].x, p[i].y, 0.0);
        for (int j = 0; j < i; j++){
            if (pointInCircle(c, p[j]))
                continue;
            c = Circle((p[j].x + p[i].x) * 0.5, (p[j].y + p[i].y) * 0.5, 0.5
                * dist(p[j], p[i]));
            for (int k = 0; k < j; k++){
                if (pointInCircle(c, p[k]))
                    continue;
                c = circumCircle(p[j], p[i], p[k]);
            }
        }
    }
    return c;
}
//Return the number of the intersection
int circle_line_intersection(Circle circ, Line line, Point2d &p1,
    Point2d &p2){
    ftLong r = circ.r;
    ftLong a = line.a, b = line.b, c = line.c + line.a * circ.x + line.b
        * circ.y; //take a circle to the (0, 0)
    ftLong x0 = -a * c / (pw2(a) + pw2(b)), y0 = -b * c / (pw2(a) + pw2(
        b)); // (x0, y0) is the shortest distance point of the line
        for (0, 0)
    if (gt(pw2(c), pw2(r) * (pw2(a) + pw2(b)))){
        return 0;
    }
    else if (eq(pw2(c), pw2(r) * (pw2(a) + pw2(b)))){
        p1.x = p2.x = x0 + circ.x;
        p1.y = p2.y = y0 + circ.y;
        return 1;
    } else {
        ftLong d_2 = pw2(r) - pw2(c) / (pw2(a) + pw2(b));
        ftLong mult = sqrt(d_2 / (pw2(a) + pw2(b)));
        p1.x = x0 + b * mult + circ.x;
        p2.x = x0 - b * mult + circ.x;
        p1.y = y0 - a * mult + circ.y;
        p2.y = y0 + a * mult + circ.y;
        return 2;
    }
}

```

```

//Return the number of the intersection
int circle_intersection(Circle c1, Circle c2, Point2d &p1, Point2d &p2)
{
    if (eq(c1.x, c2.x) and eq(c1.y, c2.y)){
        if (eq(c1.r, c2.r))
            return -1; //INF
        else
            return 0;
    }else{
        Circle circ(0, 0, c1.r);
        Line line;
        line.a = -2 * (c2.x - c1.x);
        line.b = -2 * (c2.y - c1.y);
        line.c = pw2(c2.x - c1.x) + pw2(c2.y - c1.y) + pw2(c1.r) - pw2(c2.r);
        int sz = circle_line_intersection(circ, line, p1, p2);
        p1.x += c1.x;
        p2.x += c1.x;
        p1.y += c1.y;
        p2.y += c1.y;
        return sz;
    }
}

bool checkIfTheSegmentIsCompletelyCoveredByCircles(vector<Circle> &vc,
    Segment s){
    vector<Point2d> v = {s.a, s.b};
    Line l(s.a, s.b);
    for (Circle c : vc){
        Point2d p1, p2;
        int inter = circle_line_intersection(c, l, p1, p2);
        if (inter >= 1 and betw(s.a.x, s.b.x, p1.x) and betw(s.a.y, s.b.y, p1.y))
            v.push_back(p1);
        if (inter == 2 and betw(s.a.x, s.b.x, p2.x) and betw(s.a.y, s.b.y, p2.y))
            v.push_back(p2);
    }
    sort(v.begin(), v.end());
    bool ans = true;
    for (int i = 1; i < (int)v.size(); i++){
        bool has = false;
        for (Circle c : vc){
            if (pointInCircle(c, v[i - 1]) and pointInCircle(c, v[i])){
                has = true;
                break;
            }
        }
        ans = ans && has;
    }
    return ans;
}

void tangents(Point2d c, double r1, double r2, vector<Line> &ans){
    double r = r2 - r1;
    double z = pw2(c.x) + pw2(c.y);
    double d = z - pw2(r);
    if (lt(d, 0))
        return;
    d = sqrt(abs(d));
    Line l;

```

```

    l.a = (c.x * r + c.y * d) / z;
    l.b = (c.y * r - c.x * d) / z;
    l.c = r1;
    ans.push_back(l);
}

vector<Line> tangents(Circle a, Circle b){
    vector<Line> ans;
    for (int i = -1; i <= 1; i += 2)
        for (int j = -1; j <= 1; j += 2)
            tangents(Point2d(b.x - a.x, b.y - a.y), a.r * i, b.r * j, ans);
    for (size_t i = 0; i < ans.size(); ++i){
        ans[i].c -= ans[i].a * a.x + ans[i].b * a.y;
        ans[i].normalize();
    }
    return ans;
}
//End Circle

```

5.2 Circle Area Union

```

#include "basic_geometry.h"
using namespace std;

const double PI = acos(-1);
pair<double, double> isCC(Circle circ1, Circle circ2){
    Point2d c1(circ1.x, circ1.y), c2(circ2.x, circ2.y);
    double r1 = circ1.r, r2 = circ2.r;
    double d = dist(c1, c2);
    double x1 = c1.x, x2 = c2.x, y1 = c1.y, y2 = c2.y;
    double mid = atan2(y2 - y1, x2 - x1);
    double a = r1, c = r2;
    double t = acos((a * a + d * d - c * c) / (2 * a * d));
    return make_pair(mid - t, mid + t);
}

int testCC(Circle circ1, Circle circ2){
    Point2d c1(circ1.x, circ1.y), c2(circ2.x, circ2.y);
    double r1 = circ1.r, r2 = circ2.r;
    double d = dist(c1, c2);
    if (le(r1 + r2, d))
        return 1; // not intersected or tged
    if (le(r1 + d, r2))
        return 2; // C1 inside C2
    if (le(r2 + d, r1))
        return 3; // C2 inside C1
    return 0; // intersected
}

struct event_t{
    double theta;
    int delta;
    event_t(double t, int d) : theta(t), delta(d) {}
    bool operator<(const event_t &r) const{
        if (fabs(theta - r.theta) < EPS)
            return delta > r.delta;
        return theta < r.theta;
    }
};

vector<event_t> e;
void add(double begin, double end){
    if (begin <= -PI)
        begin += 2 * PI, end += 2 * PI;

```

```

if (end > PI){
    e.push_back(event_t(begin, 1));
    e.push_back(event_t(PI, -1));
    e.push_back(event_t(-PI, 1));
    e.push_back(event_t(end - 2 * PI, -1));
} else {
    e.push_back(event_t(begin, 1));
    e.push_back(event_t(end, -1));
}
}
double calc(Point2d c, double r, double a1, double a2){
    double da = a2 - a1;
    double aa = r * r * (da - sin(da)) / 2;
    Point2d p1 = Point2d(cos(a1), sin(a1)) * r + c;
    Point2d p2 = Point2d(cos(a2), sin(a2)) * r + c;
    return cross(p1, p2) / 2 + aa;
}
/* O(n^2 log n), please remove coincided circles first. */
double circle_union(vector<Circle> &vc){
    int n = vc.size();
    for (int i = n - 1; i >= 0; i--){
        if (eq(vc[i].r, 0)){
            swap(vc[i], vc[n - 1]);
            n--;
            continue;
        }
        for (int j = 0; j < i; j++){
            if (eq(vc[i].x, vc[j].x) and eq(vc[i].y, vc[j].y) and eq(vc[i].r,
                vc[j].r)){
                swap(vc[i], vc[n - 1]);
                n--;
            }
        }
    }
    if (n == 0)
        return 0;
    vc.resize(n);
    vector<double> cntarea(2 * n, 0);
    for (int c = 0; c < n; c++){
        int cvrcnt = 0;
        e.clear();
        for (int i = 0; i < n; i++){
            if (i != c){
                int r = testCC(vc[c], vc[i]);
                if (r == 2){
                    cvrcnt++;
                } else if (r == 0){
                    auto paa = isCC(vc[c], vc[i]);
                    add(paa.first, paa.second);
                }
            }
        }
        if (e.size() == 0){
            double a = PI * vc[c].r * vc[c].r;
            cntarea[cvrcnt] -= a;
            cntarea[cvrcnt + 1] += a;
        } else {
            e.push_back(event_t(-PI, 1));
            e.push_back(event_t(PI, -2));
            sort(e.begin(), e.end());
            for (int i = 0; i < int(e.size()) - 1; i++){

```

```

                cvrcnt += e[i].delta;
                double a = calc(Point2d(vc[c].x, vc[c].y), vc[c].r, e[i].theta,
                    e[i + 1].theta);
                cntarea[cvrcnt - 1] -= a;
                cntarea[cvrcnt] += a;
            }
        }
    }
    double ans = 0;
    for (int i = 1; i <= n; i++){
        ans += cntarea[i];
    }
    return ans;
}

```

5.3 Count Lattices

```

#include "../code/math/fraction.h"
Fraction f_1 = 1;
//Calculates number of integer points (x,y) such for 0<=x<n and 0<y<=
    floor(k*x+b)
//O(log(N)*log(MAXV))
f_type count_lattices(Fraction k, Fraction b, f_type n) {
    auto fk = (f_type)k;
    auto fb = (f_type)b;
    auto cnt = 0LL;

    if (k >= f_1 || b >= f_1) {
        cnt += (fk * (n - 1) + 2 * fb) * n / 2;
        k = k - Fraction(fk, 1);
        b = b - Fraction(fb, 1);
    }
    auto t = k * Fraction(n, 1) + b;
    auto ft = (f_type)t;
    if (ft >= 1) {
        cnt += count_lattices(f_1 / k, (t - Fraction((f_type)t, 1)) / k, (
            f_type)t);
    }
    return cnt;
}

```

5.4 Point 3D

```

#include <bits/stdc++.h>
using namespace std;
//#define POINT_DOUBLE
#ifdef POINT_DOUBLE
    typedef double ftype;
    typedef long double ftLong;
    const double EPS = 1e-9;
    #define eq(a, b) (abs(a-b)<EPS)
    #define lt(a, b) ((a+EPS)<b)
    #define gt(a, b) (a>(b+EPS))
    #define le(a, b) (a<(b+EPS))
    #define ge(a, b) ((a+EPS)>b)
#else
    typedef int32_t ftype;
    typedef int64_t ftLong;
    #define eq(a, b) (a==b)

```

```

#define lt(a, b) (a<b)
#define gt(a, b) (a>b)
#define le(a, b) (a<=b)
#define ge(a, b) (a>=b)
#endif
//Point3D
struct Point3d{
    ftype x, y, z;
    Point3d() {}
    Point3d(ftype x, ftype y, ftype z) : x(x), y(y), z(z) {}
    Point3d operator+(Point3d t){
        return Point3d(x + t.x, y + t.y, z + t.z);
    }
    Point3d operator-(Point3d t){
        return Point3d(x - t.x, y - t.y, z - t.z);
    }
    Point3d operator*(ftype t){
        return Point3d(x * t, y * t, z * t);
    }
    Point3d operator/(ftype t){
        return Point3d(x / t, y / t, z / t);
    }
};
ftLong dot(Point3d a, Point3d b){
    return a.x * (ftLong)b.x + a.y * (ftLong)b.y + a.z * (ftLong)b.z;
}
double len(Point3d a){
    return sqrt(dot(a, a));
}
double dist(Point3d a, Point3d b){
    return len(a-b);
}
double proj(Point3d a, Point3d b){
    return dot(a, b) / len(b);
}
double angle(Point3d a, Point3d b){
    return acos(dot(a, b) / len(a) / len(b));
}
Point3d cross(Point3d a, Point3d b){
    return Point3d(a.y * b.z - a.z * b.y,
        a.z * b.x - a.x * b.z,
        a.x * b.y - a.y * b.x);
}
ftLong triple(Point3d a, Point3d b, Point3d c){
    return dot(a, cross(b, c));
}
Point3d planeIntersect(Point3d a1, Point3d n1, Point3d a2, Point3d n2,
    Point3d a3, Point3d n3){
    Point3d x(n1.x, n2.x, n3.x);
    Point3d y(n1.y, n2.y, n3.y);
    Point3d z(n1.z, n2.z, n3.z);
    Point3d d(dot(a1, n1), dot(a2, n2), dot(a3, n3));
    return Point3d(triple(d, y, z),
        triple(x, d, z),
        triple(x, y, d)) / triple(n1, n2, n3);
}
struct Sphere{
    ftype x, y, z, r;
    Sphere() {}
    Sphere(ftype x, ftype y, ftype z, ftype r):x(x), y(y), z(z), r(r){}
};

```

```

//Minimum enclosing Sphere, O(n*70000)
//It is also possible to do with ternary search in the 3 dimensions
Sphere minimumSphere(vector<Point3d> vp){
    Point3d ans(0, 0, 0);
    int n = vp.size();
    for(Point3d p: vp)
        ans = ans + p;
    ans = ans/n;
    double P = 0.1;
    double d = 0, e = 0;
    for(int i = 0; i < 70000; i++){
        int f = 0;
        d = dist(ans, vp[0]);
        for (int j = 1; j < n; j++) {
            e = dist(ans, vp[j]);
            if (d < e) {
                d = e;
                f = j;
            }
        }
        ans = ans + (vp[f]-ans)*P;
        P *= 0.998;
    }
    return Sphere(ans.x, ans.y, ans.z, d);
}

```

6 String Algorithms

6.1 Min Cyclic String

```

#include <bits/stdc++.h>
using namespace std;
string min_cyclic_string(string s){
    s += s;
    int n = s.size();
    int i = 0, ans = 0;
    while (i < n / 2){
        ans = i;
        int j = i + 1, k = i;
        while (j < n && s[k] <= s[j]){
            if (s[k] < s[j]){
                k = i;
            }
            else
                k++;
            j++;
        }
        while (i <= k)
            i += j - k;
    }
    return s.substr(ans, n / 2);
}

```

6.2 Suffix Automaton

```

#include <bits/stdc++.h>
using namespace std;
typedef long long ll;

```

```

struct SuffixAutomaton{
    struct state{
        int len, link, first_pos;
        bool is_clone = false;
        map<char, int> next;
    };
    vector<state> st;
    int sz, last;
    SuffixAutomaton(string s){
        st.resize(2 * s.size() + 10);
        st[0].len = 0;
        st[0].link = -1;
        st[0].is_clone = false;
        sz = 1;
        last = 0;
        for (char c : s)
            insert(c);
        preCompute();
    }
    void insert(char c){
        int cur = sz++;
        st[cur].len = st[last].len + 1;
        st[cur].first_pos = st[cur].len - 1;
        st[cur].is_clone = false;
        int p = last;
        while (p != -1 && !st[p].next.count(c)){
            st[p].next[c] = cur;
            p = st[p].link;
        }
        if (p == -1){
            st[cur].link = 0;
        }else{
            int q = st[p].next[c];
            if (st[p].len + 1 == st[q].len){
                st[cur].link = q;
            }else{
                int clone = sz++;
                st[clone].len = st[p].len + 1;
                st[clone].next = st[q].next;
                st[clone].link = st[q].link;
                st[clone].first_pos = st[q].first_pos;
                st[clone].is_clone = true;
                while (p != -1 && st[p].next[c] == q){
                    st[p].next[c] = clone;
                    p = st[p].link;
                }
                st[q].link = st[cur].link = clone;
            }
        }
        last = cur;
    }
    string lcs(string s){
        int v = 0, l = 0, best = 0, bestpos = 0;
        for (int i = 0; i < (int)s.size(); i++){
            while (v && !st[v].next.count(s[i])){
                v = st[v].link;
                l = st[v].len;
            }
            if (st[v].next.count(s[i])){
                v = st[v].next[s[i]];
                l++;
            }
        }
    }
};

```

```

    }
    if (l > best){
        best = l;
        bestpos = i;
    }
}
return s.substr(bestpos - best + 1, best);
}
vector<ll> dp;
vector<int> cnt;
ll dfsPre(int s){
    if (dp[s] != -1)
        return dp[s];
    dp[s] = cnt[s]; //Accepts repeated substrings
    //dp[s] = 1; //Does not accept repeated substrings
    for (auto p : st[s].next)
        dp[s] += dfsPre(p.second);
    return dp[s];
}
void preCompute(){
    cnt.assign(sz, 0);
    vector<pair<int, int>> v(sz);
    for (int i = 0; i < sz; i++){
        cnt[i] = !st[i].is_clone;
        v[i] = make_pair(st[i].len, i);
    }
    sort(v.begin(), v.end(), greater<pair<int, int>>());
    for (int i = 0; i < sz - 1; i++)
        cnt[st[v[i].second].link] += cnt[v[i].second];
    dp.assign(sz, -1);
    dfsPre(0);
}
};

```

7 Miscellaneous

7.1 Longest Increasing Subsequence

```

#include <bits/stdc++.h>
using namespace std;
vector<int> lis(vector<int> &v){
    vector<int> st, ans;
    vector<int> pos(v.size()+1), dad(v.size()+1);
    for(int i=0; i < (int)v.size(); i++){
        auto it = lower_bound(st.begin(), st.end(), v[i]); // Do not
        accept repeated values
        //auto it = upper_bound(st.begin(), st.end(), v[i]); //Accept
        repeated values
        int p = it-st.begin();
        if(it==st.end())
            st.push_back(v[i]);
        else
            *it = v[i];
        pos[p] = i;
        dad[i] = (p==0)? -1 : pos[p-1];
    }
    int p = pos[st.size() - 1];
    while(p >= 0){

```



```

    ans.push_back(v[p]);
    p=dad[p];
}
reverse(ans.begin(), ans.end());
return ans;
}

```

7.2 Mo Algorithm

```

#include <bits/stdc++.h>
using namespace std;
const int BLOCK_SIZE = 700;
void remove(int idx);
void add(int idx);
void clearAnswer();
int getAnswer();
struct Query{
    int l, r, idx;
    bool operator<(Query other) const{
        if (l / BLOCK_SIZE != other.l / BLOCK_SIZE)
            return l < other.l;
        return (l / BLOCK_SIZE & 1) ? (r < other.r) : (r > other.r);
    }
};
vector<int> mo_s_algorithm(vector<Query> queries){
    vector<int> answers(queries.size());
    sort(queries.begin(), queries.end());
    clearAnswer();
    int L = 0, R = 0;
    add(0);
    for(Query q : queries){
        while(q.l < L) add(--L);
        while(R < q.r) add(++R);
        while(L < q.l) remove(L++);
        while(q.r < R) remove(R--);
        answers[q.idx] = getAnswer();
    }
    return answers;
}

```

8 Theorems and Formulas

8.1 Binomial Coefficients

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{k}a^{n-k}b^k + \dots + \binom{n}{n}b^n$$

$$\text{Pascal's Triangle: } \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\text{Symmetry rule: } \binom{n}{k} = \binom{n}{n-k}$$

$$\text{Factoring in: } \binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}$$

$$\text{Sum over } k: \sum_{k=0}^n \binom{n}{k} = 2^n$$

$$\text{Sum over } n: \sum_{m=0}^n \binom{m}{k} = \binom{n+1}{k+1}$$

$$\text{Sum over } n \text{ and } k: \sum_{k=0}^m \binom{n+k}{k} = \binom{n+m+1}{m}$$

$$\text{Sum of the squares: } \binom{n}{0}^2 + \binom{n}{1}^2 + \dots + \binom{n}{n}^2 = \binom{2n}{n}$$

$$\text{Weighted sum: } 1\binom{n}{1} + 2\binom{n}{2} + \dots + n\binom{n}{n} = n2^{n-1}$$

Connection with the Fibonacci numbers: $\binom{n}{0} + \binom{n-1}{1} + \dots + \binom{n-k}{k} + \dots + \binom{0}{n} = F_{n+1}$

More formulas: $\sum_{k=0}^m (-1)^k \cdot \binom{n}{k} = (-1)^m \cdot \binom{n-1}{m}$ _____

8.2 Catalan Number

Recursive formula: $C_0 = C_1 = 1$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}, n \geq 2$$

$$\text{Analytical formula: } C_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n}, n \geq 0$$

The first few numbers Catalan numbers, C_n (starting from zero):
1, 1, 2, 5, 14, 42, 132, 429, 1430, ...

The Catalan number C_n is the solution for:

- Number of correct bracket sequence consisting of n opening and n closing brackets.
- The number of rooted full binary trees with $n+1$ leaves (vertices are not numbered). A rooted binary tree is full if every vertex has either two children or no children.
- The number of ways to completely parenthesize $n+1$ factors.
- The number of triangulations of a convex polygon with $n+2$ sides (i.e. the number of partitions of polygon into disjoint triangles by using the diagonals).
- The number of ways to connect the $2n$ points on a circle to form n disjoint chords.
- The number of non-isomorphic full binary trees with n internal nodes (i.e. nodes having at least one son).
- The number of monotonic lattice paths from point $(0,0)$ to point (n,n) in a square lattice of size $n \times n$, which do not pass above the main diagonal (i.e. connecting $(0,0)$ to (n,n)).
- Number of permutations of length n that can be stack sorted (i.e. it can be shown that the rearrangement is stack sorted if and only if there is no such index $i < j < k$, such that $a_k < a_i < a_j$).
- The number of non-crossing partitions of a set of n elements.
- The number of ways to cover the ladder $1 \dots n$ using n rectangles (The ladder consists of n columns, where i^{th} column has a height i).

8.3 Euler's Totient

If p is a prime number: $\phi(p) = p - 1$ and $\phi(p^k) = p^k - p^{k-1}$

If a and b are relatively prime, then: $\phi(ab) = \phi(a) \cdot \phi(b)$

In general: $\phi(ab) = \phi(a) \cdot \phi(b) \cdot \frac{gcd(a, b)}{\phi(gcd(a, b))}$

This interesting property was established by Gauss: $\sum_{d|n} \phi(d) = n$, Here the sum is over all positive divisors d of n .

Euler's theorem: $a^{\phi(m)} \equiv 1 \pmod{m}$, if a and m are relatively prime.

Generalization: $a^n \equiv a^{\phi(m) + [n \bmod \phi(m)]} \pmod{m}$, for arbitrary a , m and $n \geq \log_2(m)$.

8.4 Formulas

Count the number of ways to partition a set of n labelled objects into k nonempty labelled subsets.

$$f(n, k) = \sum_{i=0}^k (-1)^i \binom{k}{i} (k-i)^n$$

Stirling Number 2nd: Partitions of an n element set into k not-empty set. Or count the number of ways to partition a set of n labelled objects into k nonempty unlabelled subsets.

$$S_{2nd}(n, k) = \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{1}{k!} \sum_{i=0}^k (-1)^i \binom{k}{i} (k-i)^n$$

Euler's formula: $f = e - v + 2$

Euler's formula to n Lines or Segment if there is no three lines/segments that contains the same point: $R = intersects + component - n$

Number of regions in a planar graph: $R = E - V + C + 1$ where C is the number of connected components

Given a and b co-prime, $n = a \cdot x + b \cdot y$ where $x \geq 0$ and $y \geq 0$. You are required to find the least value of n , such that all currency values greater than

or equal to n can be made using any number of coins of denomination a and b :

$$n = (a - 1) * (b - 1)$$

generalization of the above problem, n is multiple of $gcd(a, b)$: $n = lcm(a, b) - a - b + gcd(a, b)$

8.5 Manhattan Distance

Transformation of the manhattan distance to 2 dimensions between $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$:

$|x_1 - x_2| + |y_1 - y_2| = \max(|A_1 - B_1|, |A_2 - B_2|)$ where $A = (x_1 + y_1, x_1 - y_1)$ e $B = (x_2 + y_2, x_2 - y_2)$

Transformation of the manhattan distance to 3 dimensions between $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$:

$|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2| = \max(|A_1 - B_1|, |A_2 - B_2|, |A_3 - B_3|, |A_4 - B_4|)$ where $A = (x_1 + y_1 + z_1, x_1 + y_1 - z_1, x_1 - y_1 + z_1, -x_1 + y_1 + z_1)$ e $B = (x_2 + y_2 + z_2, x_2 + y_2 - z_2, x_2 - y_2 + z_2, -x_2 + y_2 + z_2)$

Transformation of the manhattan distance to D dimensions between P_1 and P_2 :

$isSet(i, x) = 1$ if the i -th bit is setted in x and 0 otherwise.

$$A[i] = \sum_{j=0}^{d-1} (-1)^{isSet(j, i)} P_1[j]$$

$$B[i] = \sum_{j=0}^{d-1} (-1)^{isSet(j, i)} P_2[j]$$

$$\sum_{i=0}^{d-1} |P_1[i] - P_2[i]| = \max_{i=0}^{2^d-1} |A_i - B_i|$$

8.6 Primes

If $n = p_1^{e_1} \cdot p_2^{e_2} \cdot \dots \cdot p_k^{e_k}$, then:

Number of divisors is $d(n) = (e_1 + 1) \cdot (e_2 + 1) \cdot \dots \cdot (e_k + 1)$.

Sum of divisors is $\sigma(n) = \frac{p_1^{e_1+1}-1}{p_1-1} \cdot \frac{p_2^{e_2+1}-1}{p_2-1} \cdot \dots \cdot \frac{p_k^{e_k+1}-1}{p_k-1}$