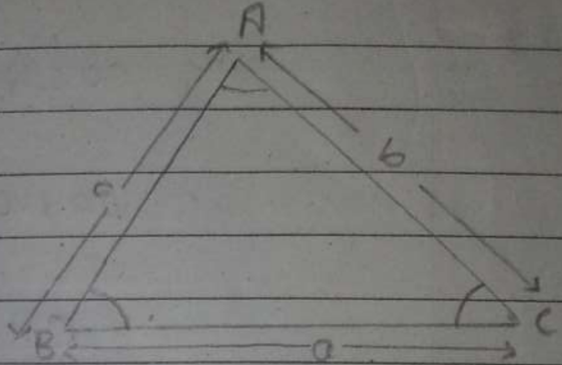
Trigonometry

(i) Properties of triangle (~~6~~ 2·1)

~~(ii) Solution of a triangle (2·2)~~

Sine Law

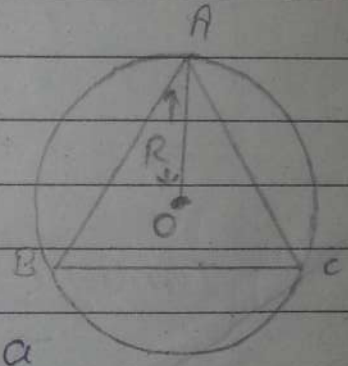


→ It states that,

In any ΔABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

In expanded form,



$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R}$$

then,

• $a = 2R\sin A$	• $\sin A = \cancel{a/2R}\ \frac{a}{2R}$
• $b = 2R\sin B$	• $\sin B = \frac{b}{2R}$
• $c = 2R\sin C$	• $\sin C = \frac{c}{2R}$

Similarly,

- $a\sin B = b\sin A$
- $b\sin C = c\sin B$
- $a\sin C = c\sin A$

Cosine law

→ Cosine law states that,
In any ΔABC,

- $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$
- $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$
- $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

OR,

- $a^2 = b^2+c^2-2bc\cos A$
- $b^2 = a^2+b^2-2ac\cos B$
- $c^2 = a^2+b^2-2ab\cos C$

Tangent law

→ Tangent law states that,
In any ΔABC,

i) $\tan\dfrac{B-C}{2} = \dfrac{b-c}{b+c}\cdot\cot A/2$

ii) $\tan\dfrac{C-A}{2} = \dfrac{c-a}{c+a}\cdot\cot B/2$

iii) $\tan\dfrac{A-B}{2} = \dfrac{a-b}{a+b}\cdot\cot C/2$

Proof :

9) $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cdot \cot A/2$

Here,

RHS : $\frac{b-c}{b+c} \cdot \cot A/2$

$= \frac{2R\sin B - 2R\sin C}{2R\sin B + 2R\sin C} \cdot \cot A/2$

$= \frac{\cancel{2R}(\sin B - \sin C)}{\cancel{2R}(\sin B + \sin C)} \cdot \cot A/2$

$= \frac{\cancel{2}\cos\left(\frac{B+C}{2}\right) \cdot \sin\left(\frac{B-C}{2}\right)}{\cancel{2}\sin\left(\frac{B+C}{2}\right) \cdot \cos\left(\frac{B-C}{2}\right)} \times \cot A/2$

$= \cot\left(\frac{B+C}{2}\right) \times \tan\left(\frac{B-C}{2}\right) \times \cot A/2$

$= \cot\left(\frac{180^\circ - A}{2}\right) \times \tan\left(\frac{B-C}{2}\right) \times \cot A/2$

[$\because$ In any ΔABC, $A+B+C = 180^\circ$]

$= \cot\left(90^\circ - A/2\right) \times \tan\left(\frac{B-C}{2}\right) \times \cot A/2$

$= \tan\frac{A}{2} \times \tan\left(\frac{B-C}{2}\right) \times \frac{1}{\tan\frac{A}{2}}$

$= \tan\frac{B-C}{2}$

$=$ LHS. ~~proved~~.

11) $\tan\frac{C-A}{2} = \frac{c-a}{c+a} \cdot \cot\frac{B}{2}$

Here,

RHS:- $\frac{c-a}{c+a} \cdot \cot\frac{B}{2}$

$= \frac{2R\sin C - 2R\sin A}{2R\sin C + 2R\sin A} \cdot \cot\frac{B}{2}$

$= \frac{\cancel{2R}(\sin C - \sin A)}{\cancel{2R}(\sin C + \sin A)} \cdot \cot\frac{B}{2}$

$= \frac{\cancel{2}\cos\left(\frac{C+A}{2}\right) \cdot \sin\left(\frac{C-A}{2}\right)}{\cancel{2}\sin\left(\frac{C+A}{2}\right) \cdot \cos\left(\frac{C-A}{2}\right)} \times \cot\frac{B}{2}$

$= \cot\left(\frac{C+A}{2}\right) \cdot \tan\left(\frac{C-A}{2}\right) \times \cot\frac{B}{2}$

$= \cot\left(\frac{180° - B}{2}\right) \cdot \tan\left(\frac{C-A}{2}\right) \times \cot\frac{B}{2}$ $[\because A+B+C = 180°]$

$= \cot\left(90° - \frac{B}{2}\right) \cdot \tan\left(\frac{C-A}{2}\right) \cot \frac{B}{2}$

$= \tan\frac{B}{2} \times \tan\left(\frac{C-A}{2}\right) \times \frac{1}{\tan\frac{B}{2}}$

$= \tan\left(\frac{C-A}{2}\right)$

$=$ LHS. proved.

iii) $\tan\frac{A-B}{2} = \frac{a-b}{a+b} \cdot \cot\frac{C}{2}$

Here,

RHS: $\frac{a-b}{a+b} \cdot \cot\frac{C}{2}$

$= \frac{2R\sin A - 2R\sin B}{2R\sin A + 2R\sin B} \cdot \cot\frac{C}{2}$

$= \frac{2R(\sin A - \sin B)}{2R(\sin A + \sin B)} \cdot \cot\frac{C}{2}$

$= \frac{2\cos\left(\frac{A+B}{2}\right) \cdot \sin\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right)} \times \cot\frac{C}{2}$

$= \cot\left(\frac{\cancel{C+A}\ A+B}{2}\right) \cdot \tan\left(\frac{\cancel{C-A}\ A-B}{2}\right) \times \cot\frac{C}{2}$

$$= \cot\left(\frac{180° - C}{2}\right) \times \tan\left(\frac{A-B}{2}\right) \times \cot C/2$$

$$= \cot\left(90° - C/2\right) \times \tan\left(\frac{A-B}{2}\right) \times \cot C/2$$

$$= \cancel{\tan C/2} \times \tan\left(\frac{A-B}{2}\right) \times \frac{1}{\cancel{\tan C/2}}$$

$$= \tan \frac{A-B}{2}$$

$= LHS.$ Proved.

Projection law

→ Projection law states that,
In any ΔABC,

i) $a = b\cos C + c\cos B$
ii) $b = a\cos C + c\cos A$
iii) $c = a\cos B + b\cos A$

Proof:

1) $a = b\cos C + c\cos B$
Here,
RHS: $b\cos C + c\cos B$

$$= b \times \frac{a^2+b^2-c^2}{2ab} + c \times \frac{c^2+a^2-b^2}{2ac}$$

$$= \frac{a^2+b^2-c^2}{2a} + \frac{c^2+a^2-b^2}{2a}$$

$$= \frac{a^2+\cancel{b^2}-\cancel{c^2}+\cancel{c^2}+a^2-\cancel{b^2}}{2a}$$

$$= \frac{2a^2}{2a}$$

$$= a$$

= LHS proved

11) $b = a\cos C + c\cos A$

Here,

RHS: $a\cos C + c\cos A$

$$= a \times \frac{a^2+b^2-c^2}{2ab} + c \times \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{a^2+b^2-c^2}{2b} + \frac{b^2+c^2-a^2}{2b}$$

$$= \frac{a^2+b^2-c^2+b^2+c^2-a^2}{2b}$$

$$= \frac{2b^2}{2b}$$

$$= b$$

$= LHS$ Proved.

iii) $c = b\cos A + a\cos B$

Here,

RHS: $b\cos A + a\cos B$

$$= b \times \frac{b^2+c^2-a^2}{2bc} + a \times \frac{a^2+c^2-b^2}{2ac}$$

$$= \frac{b^2+c^2-a^2}{2c} + \frac{a^2+c^2-b^2}{2c}$$

$$= \frac{b^2+c^2-a^2+a^2+c^2-b^2}{2c}$$

$$= \frac{2c^2}{2c}$$

$$= c$$

$= LHS.$ Proved.

$\therefore s =$ semi-perimeter

i.e. $s = \frac{a+b+c}{2}$

Half Angle Formula

i) $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

ii) $\sin \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}$

iii) $\sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

iv) $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

v) $\cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}$

vi) $\cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

vii) $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

viii) $\tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

ix) $\tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

x) $\cot \frac{A}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$

xi) $\cot \frac{B}{2} = \sqrt{\frac{s(s-b)}{s(s-a)(s-c)}}$

xii) $\cot \frac{C}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$

Proof:

1) $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

Here,
We know, from cosine law,

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

or, $1 - 2\sin^2 \frac{A}{2} = \frac{b^2+c^2-a^2}{2bc}$

or, $1 - \frac{b^2+c^2-a^2}{2bc} = 2\sin^2 \frac{A}{2}$

or, $\frac{2bc-b^2-c^2+a^2}{2bc \times 2} = 2\sin^2 \frac{A}{2}$

$$\text{or, } \frac{a^2-(b-c)^2}{4bc} = \sin^2 A/2$$

$$\text{or, } \frac{(a+b-c)(a-b+c)}{4bc} = \sin^2 A/2$$

$$\text{or, } \frac{(2s-c-c)(2s-b-b)}{4bc} = \sin^2 A/2$$

$$\text{or, } \sin^2 A/2 = \frac{(2s-2c)(2s-2b)}{4bc}$$

$$\text{or, } \sin^2 A/2 = \frac{2(s-c)\times 2(s-b)}{4bc}$$

$$\therefore \sin A/2 = \sqrt{\frac{(s-c)(s-b)}{bc}}$$ proved

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11) $\sin B/2 = \sqrt{\frac{(s-a)(s-c)}{ac}}$

Here,
We know, from cosine law,

$$\cos B = \frac{a^2+c^2-b^2}{2ac}$$

$$\text{or, } 1-2\sin^2 B/2 = \frac{a^2+c^2-b^2}{2ac}$$

$$or,\ 1 - \frac{a^2+c^2-b^2}{2ac} = 2\sin^2 B/2$$

$$or,\ 2\sin^2 B/2 = \frac{2ac-a^2-c^2+b^2}{2ac}$$

$$or,\ \sin^2 B/2 = \frac{b^2-(a-c)^2}{2\times 2ac}$$

$$or,\ \sin^2 B/2 = \frac{(b+a-c)(b-a+c)}{4ac}$$

$$or,\ \sin^2 B/2 = \frac{(2s-c-c)(2s-a-a)}{4ac}$$

$$or,\ \sin^2 B/2 = \frac{(2s-2c)(2s-2a)}{4ac}$$

$$or,\ \sin^2 B/2 = \frac{2(s-c)\times 2(s-a)}{4ac}$$

$$\therefore\ \sin B/2 = \sqrt{\frac{(s-c)(s-a)}{ac}}$$ proved

iii) $\sin C/2 = \sqrt{\frac{(s-a)(s-b)}{ab}}$

Here,

We know, from cosine law,

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

or, $1 - 2\sin^2 C/_2 = \frac{a^2+b^2-c^2}{2ab}$

or, $2\sin^2 C/_2 = 1 - \frac{a^2+b^2-c^2}{2ab}$

or, $2\sin^2 C/_2 = \frac{2ab - a^2 - b^2 + c^2}{2ab}$

or, $\sin^2 C/_2 = \frac{c^2 - (a-b)^2}{4ab}$

or, $\sin^2 C/_2 = \frac{(c+a-b)(c-a+b)}{4ab}$

or, $\sin^2 C/_2 = \frac{(2s-b-b)(2s-\cancel{b}a-a)}{4ab}$

or, $\sin^2 C/_2 = \frac{(2s-2b)(2s-2a)}{4ab}$

or $\sin^2 C/_2 = \frac{2(s-b) \times 2(s-a)}{4ab}$

$\therefore \sin C/_2 = \sqrt{\frac{(s-b)(s-a)}{ab}}$ Proved

iv) $\cos A/2 = \sqrt{\frac{s(s-a)}{bc}}$

Here,
we know, from cosine law,

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\text{or, } 2\cos^2 A - 1 = \frac{b^2+c^2-a^2}{2bc}$$

$$\text{or, } 2\cos^2 A = \frac{b^2+c^2-a^2}{2bc} + 1$$

$$\text{or, } 2\cos^2 A = \frac{b^2+c^2-a^2+2bc}{2bc}$$

$$\text{or, } \cos^2 A = \frac{(b+c)^2-a^2}{4bc}$$

$$\text{or, } \cos^2 A = \frac{(b+c+a)(b+c-a)}{4bc}$$

$$\text{or, } \cos^2 A = \frac{2s}{} \times \frac{(2s-a-a)}{4bc}$$

$$\text{or, } \cos^2 A = \frac{2s(2s-2a)}{4bc}$$

$$\text{or, } \cos^2 A/2 = \frac{4\cancel{8}\, s(s-a)}{4bc}$$

$$\therefore \cos A/2 = \sqrt{\frac{s(s-a)}{bc}}$$ Proved.

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ii) $\cos B/2 = \sqrt{\frac{s(s-b)}{ac}}$

Here,

We know, from cosine law,

$$\cos B = \frac{a^2+c^2-b^2}{2ac}$$

$$\text{or, } 2\cos^2 B/2 - 1 = \frac{a^2+c^2-b^2}{2ac}$$

$$\text{or, } 2\cos^2 B/2 = \frac{a^2+c^2-b^2}{2ac} + 1$$

$$\text{or, } 2\cos^2 B/2 = \frac{a^2+c^2-b^2+2ac}{2ac}$$

$$\text{or, } \cos^2 B/2 = \frac{(a+c)^2-b^2}{4ac}$$

$$\text{or, } \cos^2 B/2 = \frac{(a+c+b)(a+c-b)}{4ac}$$

or, $\cos^2 \frac{B}{2} = \frac{2s(2s-b-b)}{4ac}$

or, $\cos^2 \frac{B}{2} = \frac{2s(2s-2b)}{4ac}$

or, $\cos^2 \frac{B}{2} = \frac{2s \times 2(s-b)}{4ac}$

$\therefore \cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}$ proved.

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vi) $\cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

Here,
We knows from cosine law,

$\cos C = \frac{a^2+b^2-c^2}{2ab}$

or, $2\cos^2 \frac{C}{2} - 1 = \frac{a^2+b^2-c^2}{2ab}$

or, $2\cos^2 \frac{C}{2} = \frac{a^2+b^2-c^2}{2ab} + 1$

or, $2\cos^2 \frac{C}{2} = \frac{a^2+b^2-c^2+2ab}{2ab}$

or, $\cos^2 \frac{C}{2} = \frac{(a+b)^2 - c^2}{4ab}$

or $\cos^2 \frac{C}{2} = \frac{(a+b+c)(a+b-c)}{4ab}$

or, $\cos^2 \frac{C}{2} = \frac{2s(2s-c-c)}{4ab}$

or, $\cos^2 \frac{C}{2} = \frac{2s(2s-2c)}{4ab}$

or, $\cos^2 \frac{C}{2} = \frac{2\times 2s(s-c)}{4ab}$

$\therefore \cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$ Proved.

Other formula:

We know,

Δ = Area of ΔABC

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

then,

$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)} \times \frac{s(s-a)}{s(s-a)}} = \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2(s-a)^2}}$

Note:-
$\sin^2 A - \sin^2 B = \sin(A+B) \cdot \sin(A-B)$

i) $\tan \frac{A}{2} = \frac{\Delta}{s(s-a)}$

ii) $\tan \frac{B}{2} = \frac{\Delta}{s(s-b)}$

iii) $\tan \frac{C}{2} = \frac{\Delta}{s(s-c)}$

iv) $\cot \frac{A}{2} = \frac{s(s-a)}{\Delta}$

v) $\cot \frac{B}{2} = \frac{s(s-b)}{\Delta}$

vi) $\cot \frac{C}{2} = \frac{s(s-c)}{\Delta}$

Also, we know,
Area of $\Delta ABC = \Delta = \frac{1}{2} bc \sin A$
then,

vii) $\sin A = \frac{2\Delta}{bc}$

viii) $\sin B = \frac{2\Delta}{ac}$

ix) $\sin C = \frac{2\Delta}{ab}$

exercise-2.1

1) In any ΔABC, prove that:

a) $a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B) = 0$

soln:-

LHS:- $a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B)$

$= a\left(\frac{b}{2R} - \frac{c}{2R}\right) + b\left(\frac{c}{2R} - \frac{a}{2R}\right) + c\left(\frac{a}{2R} - \frac{b}{2R}\right)$

$= \frac{a(b-c)}{2R} + \frac{b(c-a)}{2R} + \frac{c(a-b)}{2R}$

$= \frac{\cancel{ab} - \cancel{ac} + \cancel{bc} - \cancel{ba} + \cancel{ca} - \cancel{cb}}{2R}$

$= 0$

$=$ RHS proved.

b) $a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$.

soln:-

LHS:- $a\sin(B-C) + b\sin(C-A) + c\sin(A-B)$

$= 2R\sin A\cdot\sin(B-C) + 2R\sin B\cdot\sin(C-A) + 2R\sin C\cdot\sin(A-B)$

$= 2R[\sin\{180-(B+C)\}\cdot\sin(B-C) + \sin\{180-(C+A)\}\cdot\sin(C-A) +$
$\sin\{180-(A+B)\}\cdot\sin(A-B)]$

$= 2R[\sin(B+C)\cdot\sin(B-C) + \sin(C+A)\cdot\sin(C-A) + \sin(A+B)\cdot$
$\sin(A-B)]$

$= 2R[\sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A + \sin^2 A - \sin^2 B]$

$= 0$

$=$ RHS. Proved.

c) $\dfrac{\sin(B-C)}{\sin(B+C)} = \dfrac{(b^2-c^2)}{a^2}$

Soln:

LHS: $\dfrac{\sin(B-C)}{\sin(B+C)}$

$= \dfrac{\sin(B-C)\cdot\sin(B+C)}{\sin(B+C)\cdot\sin(B+C)}$

$= \dfrac{\sin^2 B - \sin^2 C}{\sin^2(B+C)}$

$= \dfrac{(2R\sin B)^2 - (2R\sin C)^2}{\sin^2(180-A)}$ [$\because A+B+C = 180°$]

$= \dfrac{\cancel{4R^2 \sin}\left(\frac{b}{2R}\right)^2 - \left(\frac{c}{2R}\right)^2}{\sin^2(180-A)}$ [$\because A+B+C = 180°$]

$= \dfrac{\frac{b^2}{4R^2} - \frac{c^2}{4R^2}}{\sin^2 A}$

$$= \frac{\frac{b^2-c^2}{4R^2}}{\frac{a^2}{4R^2}}$$

$$= \frac{b^2-c^2}{a^2}$$

$= RHS$. proved

d) $\frac{(a-b)}{c} \cos \frac{C}{2} = \sin \frac{(A-B)}{2}$

soln:

LHS: $\frac{(a-b)}{c} \cdot \cos \frac{C}{2}$

$$= \frac{2R\sin A - 2R\sin B}{2R\sin C} \cdot \cos \frac{C}{2}$$

$$= \frac{\cancel{2R}\left[\cancel{2}\cos\left(\frac{A+B}{2}\right)\cdot \sin\left(\frac{A-B}{2}\right)\right]}{\cancel{2R}\left[\cancel{2}\sin \frac{C}{2} \cdot \cancel{\cos \frac{C}{2}}\right]} \times \cancel{\cos \frac{C}{2}}$$

$$= \frac{\cos\left(\frac{180-C}{2}\right) \cdot \sin\left(\frac{A-B}{2}\right)}{\sin \frac{C}{2}}$$

$$= \frac{\cos(90 - C/2) \cdot \sin\left(\frac{A-B}{2}\right)}{\sin C/2}$$

$$= \frac{\cancel{\sin C/2} \cdot \sin\left(\frac{A-B}{2}\right)}{\cancel{\sin C/2}}$$

$$= \sin\frac{(A-B)}{2}$$

$= RHS$: Proved.

e) $\frac{(b-c)}{(b+c)} = \frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)}$

soln :-

$$RHS:- \frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)}$$

$$= \frac{\frac{b-c}{b+c} \times \cot A/2}{\frac{\sin \frac{1}{2}(B+C)}{\cos \frac{1}{2}(B+C)}}$$

$$= \frac{b-c}{b+c} \times \cot A/2 \times \cancel{\cos \frac{1}{2}(B+C)} \ \frac{\cos \frac{1}{2}(B+C)}{\sin \frac{1}{2}(B+C)}$$

$$= \frac{b-c}{b+c} \times \cot \frac{A}{2} \times \frac{\cos\left(\frac{180-A}{2}\right)}{\sin\left(\frac{180-A}{2}\right)}$$

$$= \frac{b-c}{b+c} \times \cot \frac{A}{2} \times \frac{\cos(90-A/2)}{\sin(90-A/2)}$$

$$= \frac{b-c}{b+c} \times \cot A/2 \times \frac{\sin A/2}{\cos A/2}$$

$$= \frac{b-c}{b+c} \times \cot A/2 \times \frac{1}{\cot A/2}$$

$$= \frac{b-c}{b+c}$$

= LHS <u>proved</u>

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f) $a(\cos C - \cos B) = 2(b-c)\cos^2 A/2$

Soln:-

LHS:- $a(\cos C - \cos B)$

$= 2R\sin A\left[2\sin\left(\frac{B+C}{2}\right)\cdot \sin\left(\frac{B-C}{2}\right)\right]$

$$= 4R\sin A \cdot \sin\left(\frac{180-A}{2}\right) \cdot \sin\left(\frac{B-C}{2}\right)$$

$$= 4R\sin A \cdot \sin\left(90 - A/2\right) \cdot \sin\left(\frac{B-C}{2}\right)$$

$$= 4R\sin A \cdot \cos A/2 \cdot \sin\left(\frac{B-C}{2}\right)$$

Again,

RHS:- $2(b-c)\cos^2 A/2$

$$= 2(2R\sin B - 2R\sin C)\cdot\cos^2 A/2$$

$$= 2\times 2R\left[2\cos\left(\frac{B+C}{2}\right)\cdot\sin\left(\frac{B-C}{2}\right)\right]\cdot\cos^2 A/2$$

$$= 8R\left[\cos\left(\frac{180-A}{2}\right)\cdot\sin\left(\frac{B-C}{2}\right)\right]\cdot\cos^2 A/2$$

$$= 8R\left[\cos\left(90 - A/2\right)\cdot\sin\left(\frac{B-C}{2}\right)\right]\cdot\cos^2 A/2$$

$$= 4R\times 2\sin A/2 \cdot \cos A/2 \cdot \sin\left(\frac{B-C}{2}\right)\cdot\cos A/2$$

$$= 4R\sin A \cdot \cos A/2 \cdot \sin\left(\frac{B-C}{2}\right)$$

$\therefore$ LHS = RHS <u>Proved</u>

g) $a\cos A + b\cos B + c\cos C = 2a\sin B\cdot\sin C$.

soln:-

LHS:- $a\cos A + b\cos B + c\cos C$

$= 2R\sin A\cdot\cos A + 2R\sin B\cdot\cos B + 2R\sin C\cdot\cos C$

$= R(\sin 2A + \sin 2B + \sin 2C)$

$= R\left[2\sin\left(\frac{2A+2B}{2}\right)\cdot\cos\left(\frac{2A-2B}{2}\right) + \sin 2C\right]$

$= 2R\left[\sin(A+B)\cdot\cos(A-B) + \sin C\cdot\cos C\right]$

$= 2R\left[\sin(180-C)\cdot\cos(A-B) + \sin C\cdot\cos C\right]$

$= 2R\left[\sin C\cdot\cos(A-B) + \sin C\cdot\cos C\right]$

$= 2R\sin C\left[\cos(A-B) + \cos C\right]$

$= 2R\sin C\left[\cos(A-B) + \cos(180-(A+B))\right]$

$= 2R\sin C\left[\cos(A-B) - \cos(A+B)\right]$

$= 2R\sin C\times\left[2\sin\left(\frac{A-B+A+B}{2}\right)\cdot\sin\left(\frac{A+B-A+B}{2}\right)\right]$

$= 4R\sin C\cdot\sin\frac{2A}{2}\cdot\sin\frac{2B}{2}$

$= 4R\sin A\cdot\sin B\cdot\sin C$

$= 2\times 2R\sin A\cdot\sin B\cdot\sin C$

$= 2a\sin B\cdot\sin C$

$=$ RHS. proved.

h) $(b^2-c^2)\cot A + (c^2-a^2)\cot B + (a^2-b^2)\cot C = 0$

soln:-

LHS :- $(b^2-c^2)\cot A + (c^2-a^2)\cot B + (a^2-b^2)\cot C$

$$= (b^2-c^2)\cdot\frac{\cos A}{\sin A} + (c^2-a^2)\cdot\frac{\cos B}{\sin B} + (a^2-b^2)\cdot\frac{\cos C}{\sin C}$$

$$= \frac{(b^2-c^2)\left(\frac{b^2+c^2-a^2}{2bc}\right)}{\frac{a}{2R}} + \frac{(c^2-a^2)\cdot\left(\frac{a^2+c^2-b^2}{2ac}\right)}{\frac{b}{2R}} + \frac{(a^2-b^2)\left(\frac{a^2+b^2-c^2}{2ab}\right)}{\frac{c}{2R}}$$

$$= \frac{\cancel{2}R(b^2-c^2)(b^2+c^2-a^2)}{\cancel{2}abc} + \frac{(c^2-a^2)\cdot\cancel{2}R(a^2+c^2-b^2)}{\cancel{2}abc} + \frac{\cancel{2}R(a^2-b^2)(a^2+b^2-c^2)}{\cancel{2}abc}$$

$$= \frac{R\left[(b^2-c^2)(b^2+c^2-a^2) + (c^2-a^2)(a^2+c^2-b^2) + (a^2-b^2)(a^2+b^2-c^2)\right]}{abc}$$

$$= R\times\frac{\left[\cancel{b^4}+\cancel{b^2c^2}-\cancel{a^2b^2}-\cancel{b^2c^2}-\cancel{c^4}+\cancel{a^2c^2}+a^2c^2+\cancel{c^4}-b^2c^2-\cancel{a^4}-\cancel{a^2c^2}+ \cancel{a^2b^2}+\cancel{a^4}+\cancel{a^2b^2}-a^2c^2-\cancel{a^2b^2}-\cancel{b^4}+b^2c^2\right]}{abc}$$

$$= \frac{R}{abc}\times 0$$

$= 0$

$=$ RHS proved.

9) $(a-b)^2\cdot\cos^2\frac{C}{2} + (a+b)^2\cdot\sin^2\frac{C}{2} = c^2$

Soln :-

LHS :- $(a-b)^2 \cdot \cos^2 C/2 + (a+b)^2 \cdot \sin^2 C/2$

$= (a^2-2ab+b^2) \cdot \cos^2 C/2 + (a^2+2ab+b^2) \cdot \sin^2 C/2$

$= a^2 \cdot \cos^2 C/2 - 2ab\cos^2 C/2 + b^2\cos^2 C/2 + a^2\sin^2 C/2 + 2ab\sin^2 C/2$
$+ b^2 \sin^2 C/2$

$= a^2(\cos^2 C/2 + \sin^2 C/2) + b^2(\cos^2 C/2 + \sin^2 C/2) - 2ab(\cos^2 C/2 - \sin^2 C/2)$

$= a^2 + b^2 - 2ab\cos C$

$= c^2$

= RHS. proved.

---

j) $(b-c)\cot A/2 + (c-a)\cot B/2 + (a-b)\cot C/2 = 0$

soln :-

LHS :- $(b-c)\cot A/2 + (c-a) \cdot \cot B/2 + (a-b) \cdot \cot C/2$

$= (b-c) \cdot \frac{s(s-a)}{\Delta} + (c-a) \frac{s(s-b)}{\Delta} + (a-b) \frac{s(s-c)}{\Delta}$

$= \frac{s}{\Delta}\left[(b-c)(s-a) + (c-a)(s-b) + (a-b)(s-c)\right]$

$= \frac{s}{\Delta}\left[\cancel{bs} - \cancel{ba} - \cancel{cs} + \cancel{ca} + \cancel{cs} - \cancel{cb} - \cancel{as} + \cancel{ab} + \cancel{as} - \cancel{ac} - \cancel{bs} + \cancel{bc}\right]$

$= \frac{s}{\Delta} \times 0$

$= 0$

= RHS proved.

k) $\sin A + \sin B + \sin C = s/R$

Soln:-

LHS:- $\sin A + \sin B + \sin C$

$= \frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R}$

$= \frac{a+b+c}{2 \times R}$

$= \frac{s}{R} \quad \left(\because \frac{a+b+c}{2} = s\right)$

= RHS. proved

l) $\frac{b^2-c^2}{a^2} \sin 2A + \frac{c^2-a^2}{b^2} \sin 2B + \frac{a^2-b^2}{c^2} \sin 2C = 0$

Soln:-

LHS:- $\frac{b^2-c^2}{a^2} \sin 2A + \frac{c^2-a^2}{b^2} \sin 2B + \frac{a^2-b^2}{c^2} \sin 2C$

$= \frac{(2R\sin B)^2 - (2R\sin C)^2}{(2R\sin A)^2} \cdot \sin 2A + \frac{(2R\sin C)^2 - (2R\sin A)^2}{(2R\sin B)^2} \cdot \sin 2B$
$+ \frac{(2R\sin A)^2 - (2R\sin B)^2}{(2R\sin C)^2} \cdot \sin 2C$

$= \frac{4R^2(\sin^2 B - \sin^2 C)}{4R^2 \sin^2 A} \cdot \sin 2A + \frac{4R^2(\sin^2 C - \sin^2 A)}{4R^2 \sin^2 B} \cdot \sin 2B + \frac{4R^2(\sin^2 A - \sin^2 B)}{4R^2 \sin^2 C} \times \sin 2C$

$$= \frac{\sin(B+C)\cdot\sin(B-C)}{\sin^2 A}\cdot 2\sin A\cdot\cos A + \frac{\sin(C+A)\cdot\sin(C-A)}{\sin^2 B}\cdot 2\sin B\cos B$$

$$+ \frac{\sin(A+B)\cdot\sin(A-B)}{\sin^2 C}\cdot 2\sin C\cdot\cos C$$

$$= \frac{\sin(\cancel{180}\,180-A)\cdot\sin(B-C)\cdot 2\sin A\cdot\cos A}{\sin^2 A} + \frac{\sin(180-B)\cdot\sin(C-A)\cdot 2\sin B\cos B}{\sin^2 B}$$

$$+ \frac{\sin(180-C)\cdot\sin(A-B)\cdot 2\sin C\cdot\cos C}{\sin^2 C}$$

$$= \frac{2\sin^2 A\cdot\sin(B-C)\cdot\cos A}{\sin^2 A} + \frac{2\sin^2 B\cdot\sin(C-A)\cdot\cos B}{\sin^2 B} + \frac{2\sin^2 C\cdot\sin(A-B)\cdot\cos C}{\sin^2 C}$$

$$= 2\cos A\cdot\sin(B-C) + 2\cos B\cdot\sin(C-A) + 2\cos C\cdot\sin(A-B)$$

$$= -2\cos(B+C)\cdot\sin(B-C) - 2\cos(C+A)\cdot\sin(C-A) - 2\cos(A+B)\sin(A-B)$$

$$= -2\cos\left(\frac{2B+2C}{2}\right)\cdot\sin\left(\frac{2B-2C}{2}\right) - 2\cos\left(\frac{2C+2A}{2}\right)\cdot\sin\left(\frac{2C-2A}{2}\right)$$

$$-2\cos\left(\frac{2A+2B}{2}\right)\sin\left(\frac{2A-2B}{2}\right)$$

$$= \cancel{-2\cos} - [\sin 2B - \sin 2C + \sin 2C - \sin 2A + \sin 2A - \sin 2B]$$

$= 0$

$=$ R.H.S. Proved

m) $$\frac{a^2\sin(B-C)}{\sin B+\sin C} + \frac{b^2\sin(C-A)}{\sin C+\sin A} + \frac{c^2\sin(A-B)}{\sin A+\sin B} = 0$$

Soln:

LHS: $\frac{a^2\sin(B-C)}{\sin B+\sin C}+\frac{b^2\sin(C-A)}{\sin C+\sin A}+\frac{c^2\sin(A-B)}{\sin A+\sin B}$

$=\frac{a\cdot 2R\sin A\cdot\sin(B-C)}{\sin B+\sin C}+\frac{b\cdot 2R\sin B\cdot\sin(C-A)}{\sin C+\sin A}+\frac{c\cdot 2R\sin C\cdot\sin(A-B)}{\sin A+\sin B}$

$=\frac{2Ra\cdot\sin(B+C)\cdot\sin(B-C)}{\sin B+\sin C}+\frac{2Rb\sin(C+A)\cdot\sin(C-A)}{\sin C+\sin A}+\frac{2Rc\sin(A+B)\cdot\sin(A-B)}{\sin A+\sin B}$

$=\frac{2Ra(\sin^2B-\sin^2C)}{\sin B+\sin C}+\frac{2Rb(\sin^2C-\sin^2A)}{\sin C+\sin A}+\frac{2Rc(\sin^2A-\sin^2B)}{\sin A+\sin B}$

$= 2Ra(\sin B-\sin C)+2Rb(\sin C-\sin A)+2Rc(\sin A-\sin B)$

$= a(2R\sin B-2R\sin C)+b(2R\sin C-2R\sin A)+c(2R\sin A-2R\sin B)$

$= a(b-c)+b(c-a)+c(a-b)$

$= \cancel{ab}-\cancel{ac}+\cancel{bc}-\cancel{ab}+\cancel{ac}-\cancel{bc}$

$= 0$

$=$ RHS proved

2) In a ΔABC, if $\cos A=\frac{\sin B}{2\sin C}$, then show that triangle is isosceles.

Soln:

~~given~~:

given: $\cos A=\frac{\sin B}{2\sin C}$

or, $2\sin C \cdot \cos A = \sin B$

or, $\sin(C+A) + \sin(C-A) = \sin(180 - (C+A))$

or, $\sin(C+A) + \sin(C-A) = \sin(C+A)$

or, $\sin(C-A) = 0$

or, $\sin(C-A) = \sin 0$

or, $C - A = 0$

or, $A = C$

Since, two angles A and C are equals.
The triangle is isosceles. Proved.

3) In a ΔABC, if $a\cos A = b\cos B$, show that the triangle is either isosceles or right angled

Soln:-

given:- $a\cos A = b\cos B$

or, $2R\sin A \cdot \cos A = 2R\sin B \cdot \cos B$

or, $\sin 2A = \sin 2B$

or, $\sin 2A - \sin 2B = 0$

or, $2\cos\left(\frac{2A+2B}{2}\right) \cdot \sin\left(\frac{2A-2B}{2}\right) = 0$

or, $\cos(A+B) \cdot \sin(A-B) = 0$

or, $-\cos C \cdot \sin(A-B) = 0$

Either, OR,

$\cos C = 0$ $\quad$ $\sin(A-B) = \sin 0$

$\therefore C = \pi/2$ $\quad$ $\therefore A - B = 0 \Rightarrow A = B$

Hence, ΔABC is either right angled or isosceles triangle. proved.

4) The angles of a triangle ABC are in A.P and if it is being given that $b:c=\sqrt{3}:\sqrt{2}$, find $\angle A$, $\angle B$ and $\angle C$.

Soln:-

Let, the angles of triangle in A.P is given as.

$A = (a-d)^\circ$, $B = a^\circ$, $C = (a+d)^\circ$

We know,

$$A + B + C = 180^\circ$$

or, $a - d + a + a + d = 180^\circ$

or, $3a = 180^\circ$

$\therefore a = 60^\circ$

then,

By question,

$$\frac{b}{c} = \frac{\sqrt{3}}{\sqrt{2}}$$

or, $\dfrac{\cancel{2R} \sin B}{\cancel{2R} \sin C} = \dfrac{\sqrt{3}}{\sqrt{2}}$

or, $\dfrac{\sin a^\circ}{\sin(a+d)^\circ} = \dfrac{\sqrt{3}}{\sqrt{2}}$

or, $\dfrac{\sqrt{3}/2}{\sin(a+d)^\circ} = \dfrac{\sqrt{3}}{\sqrt{2}}$

or, $\dfrac{\sqrt{2}}{2} = \sin(a+d)^\circ$.

or, $\sin 45 = \sin(a+d)^\circ$

or, $a + d = 45°$

~~ans~~ ie. $\sin C = \sin 45°$

$\therefore C = 45°$

Lastly,

$A + B + C = 180°$

or $60° + B + 45° = 180°$

$\therefore B = 75°$

Hence, $\angle A = 60°$; $\angle B = 75°$ and $\angle C = 45°$ //

5) In any ΔABC, prove that:

a) $a(b\cos C - c\cos B) = (b^2 - c^2)$

Soln:

LHS: $a(b\cos C - c\cos B)$

$= a\left(b \times \frac{a^2+b^2-c^2}{2ab} - c \times \frac{a^2+c^2-b^2}{2ac}\right)$

$= \frac{a(a^2+b^2-c^2-a^2-c^2+b^2)}{a}$

$= 0$

$=$ RHS. Proved.

b) $ac\cos B - bc\cos A = (a^2-b^2)$

Soln:-

LHS:- $ac\cos B - bc\cos A$

$= ac \times \frac{a^2+c^2-b^2}{2ac} - bc \times \frac{b^2+c^2-a^2}{2bc}$

$= \frac{a^2+c^2-b^2-b^2-c^2+a^2}{2}$

$= \frac{2a^2-2b^2}{2}$

$= a^2-b^2$

$=$ RHS. Proved.

c) $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{(a^2+b^2+c^2)}{2abc}$

Soln:-

LHS:- $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$

$= \frac{\frac{b^2+c^2-a^2}{2bc}}{a} + \frac{\frac{a^2+c^2-b^2}{2ac}}{b} + \frac{\frac{a^2+b^2-c^2}{2ab}}{c}$

$= \frac{b^2+c^2-a^2}{2abc} + \frac{a^2+c^2-b^2}{2abc} + \frac{a^2+b^2-c^2}{2abc}$

$$= \frac{\cancel{b^2}+c^2-\cancel{a^2}+a^2+c^2-\cancel{b^2}+a^2+b^2-\cancel{c^2}}{2abc}$$

$$= \frac{a^2+b^2+c^2}{2abc}$$

$= RHS.$ Proved.

e) $2(bc\cos A + ca\cos B + ab\cos C) = (a^2+b^2+c^2)$

Soln-:

LHS-: $2(bc\cos A + ca\cos B + ab\cos C)$

$$= 2\left[bc \times \frac{b^2+c^2-a^2}{2bc} + ca \times \frac{a^2+c^2-b^2}{2ac} + ab \times \frac{a^2+b^2-c^2}{2ab}\right]$$

$$= \frac{\cancel{2}(b^2+c^2-a^2+a^2+c^2-b^2+a^2+b^2-c^2)}{\cancel{2}}$$

$= a^2+b^2+c^2$

$= RHS.$ Proved

f) $4(bc\cos^2 A/2 + ca\cos^2 B/2 + ab\cos^2 C/2) = (a+b+c)^2$

Soln-:

LHS-: $4(bc\cos^2 A/2 + ca\cos^2 B/2 + ab\cos^2 C/2)$

$$= 4\left[bc \times \left(\sqrt{\frac{s(s-a)}{bc}}\right)^2 + ca \times \left(\sqrt{\frac{s(s-b)}{ac}}\right)^2 + ab \times \left(\sqrt{\frac{s(s-c)}{ab}}\right)^2\right]$$

$= 4\left[bc \times \frac{s(s-a)}{bc} + ca \times \frac{s(s-b)}{ac} + ab \times \frac{s(s-c)}{ab}\right]$

$= 4s(s-a+s-b+s-c)$

$= 4s[3s-(a+b+c)]$

$= 4s(3s-2s) \quad \{\because a+b+c=2s\}$

$= 4s \times s$

$= 4s^2$

$= (a+b+c)^2$

$=$ RHS. Proved.

g) $a\sin A - b\sin B = c\sin(A-B)$

Soln:-

LHS:- $a\sin A - b\sin B$

$= 2R\sin A \cdot \sin A - 2R\sin B \cdot \sin B$

$= 2R(\sin^2 A - \sin^2 B)$

$= 2R[\sin(A+B) \cdot \sin(A-B)]$

$= 2R[\sin(180-C) \cdot \sin(A-B)]$

$= 2R\sin C \cdot \sin(A-B)$

$= c\sin(A-B)$

$=$ RHS. Proved.

h) $a^2\sin(B-C) = (b^2-c^2)\sin A$

Soln:-

RHS:- $(b^2-c^2) \cdot \sin A$

$= [(2R\sin B)^2 - (2R\sin C)^2] \cdot \sin A$

$= [4R^2\sin^2 B - 4R^2\sin^2 C] \cdot \sin A$

$= 4R^2\sin A(\sin^2 B - \sin^2 C)$

$= 4R^2\sin A \cdot [\sin(B+C) \cdot \sin(B-C)]$

$= 4R^2\sin A \cdot [\sin(180 - A) \cdot \sin(B-C)]$

$= 4R^2\sin A \cdot \sin A \cdot \sin(B-C)$

$= (2R\sin A)^2 \cdot \sin(B-C)$

$= a^2 \cdot \sin(B-C)$

$=$ LHS. Proved

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i) $\dfrac{\sin(A-B)}{\sin(A+B)} = \dfrac{(a^2-b^2)}{c^2}$

Soln:-

LHS:- $\dfrac{\sin(A-B)}{\sin(A+B)}$

$= \dfrac{\sin(A-B) \times \sin(A+B)}{\sin(A+B) \times \sin(A+B)}$

$= \dfrac{\sin^2 A - \sin^2 B}{\sin^2(A+B)}$

$= \dfrac{\cancel{2\,4R^2}\left(\frac{a}{2R}\right)^2 - \left(\frac{b}{2R}\right)^2}{\sin^2(180-C)}$

$= \dfrac{\frac{a^2-b^2}{4R^2}}{\sin^2 C}$

$$= \frac{\frac{a^2-b^2}{4\cancel{R^2}}}{\frac{c^2}{4\cancel{R^2}}}$$

$$= \frac{a^2-b^2}{c^2}$$

$= RHS$ ~~Proved~~

j) $\frac{(b-c)}{a} \cdot \cos A/2 = \sin \frac{(B-C)}{2}$

Soln:

LHS: $\frac{(b-c)}{a} \cdot \cos A/2$

$$= \frac{2R\sin B - 2R\sin C}{2R\sin A} \cdot \cos A/2$$

$$= \frac{\cancel{2R}(\sin B - \sin C)}{\cancel{2R}\sin A} \cdot \cos A/2$$

$$= \frac{\cancel{2}\cos\left(\frac{B+C}{2}\right) \cdot \sin\left(\frac{B-C}{2}\right) \times \cancel{\cos A/2}}{\cancel{2}\sin A/2 \cdot \cancel{\cos A/2}}$$

$$= \frac{\cos\left(\frac{180-A}{2}\right) \cdot \sin\left(\frac{B-C}{2}\right)}{\sin A/2}$$

$$= \frac{\cos\left(90-A/2\right) \cdot \sin\left(\frac{B-C}{2}\right)}{\sin A/2}$$

$$= \frac{\cancel{\sin A/2} \cdot \sin\frac{(B-C)}{2}}{\cancel{\sin A/2}}$$

$$= \sin\frac{(B-C)}{2}$$

$= RHS$. Proved.

k) $\frac{(a+b)}{c} \cdot \sin C/2 = \cos\frac{(A-B)}{2}$

Soln:

LHS: $\frac{(a+b)}{c} \cdot \sin C/2$

$$= \frac{2R\sin A + 2R\sin B}{2R\sin C} \cdot \sin C/2$$

$$= \frac{\cancel{2R}(\sin A + \sin B)}{\cancel{2R}\sin C} \cdot \sin C/_2$$

$$= \frac{\cancel{2}\sin\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right)}{\cancel{2}\,\cancel{\sin C/_2} \cdot \cos C/_2} \times \sin C/_2$$

$$= \frac{\sin\left(\frac{180 - C}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right)}{\cos C/_2}$$

$$= \frac{\sin\left(90 - \cancel{B}\,{}^{C}/_2\right) \cdot \cos\left(\frac{A-B}{2}\right)}{\cos C/_2}$$

$$= \frac{\cos C/_2}{\cos C/_2} \times \overset{\cos}{\cancel{\sin}}\left(\frac{A-B}{2}\right)$$

$$= \cos \cancel{\sin}\left(\frac{A-B}{2}\right)$$

= RHS. ~~Proved~~

1) $\frac{(b+c)}{a} \cdot \cos\frac{(B+C)}{2} = \cos\frac{(B-C)}{2}$

soln:-

$LHS = \frac{(b+c)}{a} \cdot \cos\frac{(B+C)}{2}$

$= \frac{2R\sin B + 2R\sin C}{2R\sin A} \cdot \cos\left(\frac{B+C}{2}\right)$

$= \frac{\cancel{2R}(\sin B + \sin C)}{\cancel{2R}\sin A} \cdot \cos\left(\frac{B+C}{2}\right)$

$= \frac{2\sin\left(\frac{B+C}{2}\right) \cdot \cos\left(\frac{B-C}{2}\right) \cdot \cos\left(\frac{B+C}{2}\right)}{2\sin A/2 \cdot \cos A/2}$

$= \frac{\sin(90 - A/2) \cdot \cos\left(\frac{B-C}{2}\right) \cdot \cos(90 - A/2)}{\sin A/2 \cdot \cos A/2}$

$= \frac{\cos A/2 \cdot \cancel{s} \cos\left(\frac{B-C}{2}\right) \cdot \sin A/2}{\cancel{\sin A/2} \cdot \cancel{\cos A/2}}$

$= \cos\left(\frac{B-C}{2}\right) = RHS$. proved

m) $a^2(\cos^2 B - \cos^2 C) + b^2(\cos^2 C - \cos^2 A) + c^2(\cos^2 A - \cos^2 B) = 0$

Soln:-

LHS:- $a^2(\cos^2 B - \cos^2 C) + b^2(\cos^2 C - \cos^2 A) + c^2(\cos^2 A - \cos^2 B)$

$= a^2(1 - \sin^2 B - 1 + \sin^2 C) + b^2(1 - \sin^2 C - 1 + \sin^2 A) + c^2(1 - \sin^2 A - 1 + \sin^2 B)$

$= a^2(\sin^2 C - \sin^2 B) + b^2(\sin^2 A - \sin^2 C) + c^2(\sin^2 B - \sin^2 A)$

$= a^2\sin^2 C - a^2\sin^2 B + b^2\sin^2 A - b^2\sin^2 C + c^2\sin^2 B - c^2\sin^2 A$

$= \sin^2 A(b^2 - c^2) + \sin^2 B(c^2 - a^2) + \sin^2 C(a^2 - b^2)$

$= \left(\frac{a}{2R}\right)^2(b^2 - c^2) + \left(\frac{b}{2R}\right)^2(c^2 - a^2) + \left(\frac{c}{2R}\right)^2(a^2 - b^2)$

$= \frac{a^2(b^2 - c^2)}{4R^2} + \frac{b^2(c^2 - a^2)}{4R^2} + \frac{c^2(a^2 - b^2)}{4R^2}$

$= \frac{\cancel{a^2b^2} - \cancel{a^2c^2} + \cancel{b^2c^2} - \cancel{a^2b^2} + \cancel{a^2c^2} - \cancel{b^2c^2}}{4R^2}$

$= 0$

$=$ RHS Proved

n) $\frac{\cos^2 B - \cos^2 C}{b+c} + \frac{\cos^2 C - \cos^2 A}{c+a} + \frac{\cos^2 A - \cos^2 B}{a+b} = 0$

Soln:-

LHS:- $\frac{\cos^2 B - \cos^2 C}{b+c} + \frac{\cos^2 C - \cos^2 A}{c+a} + \frac{\cos^2 A - \cos^2 B}{a+b}$

$$= \frac{1-\sin^2 B - 1 + \sin^2 C}{b+c} + \frac{1-\sin^2 C - 1 + \sin^2 A}{c+a} + \frac{1-\sin^2 A - 1 + \sin^2 B}{a+b}$$

$$= \frac{\sin^2 C - \sin^2 B}{b+c} + \frac{\sin^2 A - \sin^2 C}{c+a} + \frac{\sin^2 B - \sin^2 A}{a+b}$$

$$= \frac{\left(\frac{c}{2R}\right)^2 - \left(\frac{b}{2R}\right)^2}{b+c} + \frac{\left(\frac{a}{2R}\right)^2 - \left(\frac{c}{2R}\right)^2}{c+a} + \frac{\left(\frac{b}{2R}\right)^2 - \left(\frac{a}{2R}\right)^2}{a+b}$$

$$= \frac{c^2-b^2}{4R^2(b+c)} + \frac{a^2-c^2}{4R^2(a+c)} + \frac{b^2-a^2}{4R^2(a+b)}$$

$$= \frac{\cancel{(c+b)}(c-b)}{4R^2\cancel{(b+c)}} + \frac{\cancel{(a+c)}(a-c)}{4R^2\cancel{(a+c)}} + \frac{\cancel{(b+a)}(b-a)}{4R^2\cancel{(a+b)}}$$

$$= \frac{c-b}{4R^2} + \frac{a-c}{4R^2} + \frac{b-a}{4R^2}$$

$$= \frac{\cancel{c}-\cancel{b}+\cancel{a}-\cancel{c}+\cancel{b}-\cancel{a}}{4R^2}$$

$$= \frac{0}{4R^2}$$

$= 0$

$=$ RHS. Proved.

Q) $\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2} = \left(\frac{1}{a^2} - \frac{1}{b^2}\right)$

Soln:-

LHS:- $\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2}$

$= \frac{1 - 2\sin^2 A}{a^2} - \frac{1 - 2\sin^2 B}{b^2}$

$= \frac{b^2(1-2\sin^2 A) - a^2(1-2\sin^2 B)}{a^2 b^2}$

$= \frac{b^2 - 2b^2\sin^2 A - a^2 + 2a^2\sin^2 B}{a^2b^2}$

$= \frac{b^2 - a^2 + 2(a^2\sin^2 B - b^2\sin^2 A)}{a^2b^2}$

$= \frac{b^2 - a^2 + 2\left[a^2 \times \frac{b^2}{4R^2} - b^2 \times \frac{a^2}{4R^2}\right]}{a^2b^2}$ $\left(\because \sin A = \frac{a}{2R}\right)$

$= \frac{b^2 - a^2 + 2\left[\frac{a^2b^2 - b^2a^2}{4R^2}\right]}{a^2b^2}$

$= \frac{b^2 - a^2 + 2 \times 0}{a^2b^2}$

$$= \frac{b^2}{a^2b^2} - \frac{a^2}{a^2b^2}$$

$$= \frac{1}{a^2} - \frac{1}{b^2}$$

$= RHS$ proved.

p) $(c^2-a^2+b^2)\tan A = (a^2-b^2+c^2)\tan B = (b^2-c^2+a^2)\tan C$

Soln:-

LHS:- $(c^2-a^2+b^2)\cdot\tan A$

$$= (c^2-a^2+b^2)\cdot\frac{\sin A}{\cos A}$$

$$= (c^2-a^2+b^2)\cdot\frac{\frac{a}{2R}}{\frac{b^2+c^2-a^2}{2bc}}$$

$$= \cancel{(b^2+c^2-a^2)} \times \frac{2abc}{2R\cancel{(b^2+c^2-a^2)}}$$

$$= \frac{abc}{R}$$

again:-

$\Longrightarrow$

M.S :- $(a^2-b^2+c^2)\tan B$

$= (a^2-b^2+c^2)\cdot\dfrac{\sin B}{\cos B}$

$= \cancel{(a^2+c^2-b^2)}\cdot\dfrac{\dfrac{b}{2R}}{\dfrac{\cancel{a^2+c^2-b^2}}{2ac}}$

$= \dfrac{2abc}{2R}$

$= \dfrac{abc}{R}$.

lastly,

RHS :- $(b^2-c^2+a^2)\cdot\tan C$

$=(b^2-c^2+a^2)\cdot\dfrac{\sin C}{\cos C}$

$=\cancel{(a^2+b^2-c^2)}\times\dfrac{\dfrac{c}{2R}}{\dfrac{\cancel{a^2+b^2-c^2}}{\cancel{2}ab}}$

$= \dfrac{abc}{R}$

Hence, L.H.S = M.S = RHS. Proved.

6) In a ΔABC, if $\frac{\cos A}{a} = \frac{\cos B}{b}$, show that the triangle is isosceles.

soln:-

given:- $\frac{\cos A}{a} = \frac{\cos B}{b}$

or, $b\cos A = a\cos B$

or, $b \times \frac{(b^2+c^2-a^2)}{2bc} = a \times \frac{(a^2+c^2-b^2)}{2ac}$

or, $b^2+c^2-a^2 = a^2+c^2-b^2$

or, $2b^2 = 2a^2$

$\therefore a = b$

Since, two sides a and b are equal.

Hence, the triangle is isosceles. Proved

7) In a ΔABC, if $\sin^2 A + \sin^2 B = \sin^2 C$, show that the triangle is right-angled.

soln:-

given:- $\sin^2 A + \sin^2 B = \sin^2 C$

or, $\left(\frac{a}{2R}\right)^2 + \left(\frac{b}{2R}\right)^2 = \left(\frac{c}{2R}\right)^2$

or, $\frac{a^2}{4R^2} + \frac{b^2}{4R^2} = \frac{c^2}{4R^2}$

$\therefore a^2 + b^2 = c^2$.

which is right angled triangled equation.

Hence, ΔABC is right angled triangle. Proved.

Imp

8) If $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$, then show that $\angle C = 60°$

soln:-

given:- $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$

or, $\frac{b+c+a+c}{(a+c)(b+c)} = \frac{3}{a+b+c}$

or, $(a+b+c)(a+b+2c) = 3(ab+ac+bc+c^2)$

or, $a^2+ab+2ac+ab+b^2+2bc+ac+bc+2c^2$

$= 3ab+3ac+3bc+3c^2$

or, $a^2+b^2+2c^2-3c^2 = 3ab+\cancel{3ac}+\cancel{3bc}-2ab-\cancel{3ac}-\cancel{3bc}$

or, $a^2+b^2-c^2 = ab$

or, $\frac{a^2+b^2-c^2}{ab} = 1$

or, $\frac{2}{2} \times \frac{a^2+b^2-c^2}{ab} = 1$

or, $\frac{a^2+b^2-c^2}{2ab} = 1/2$

or, $\cos C = \cos 60^\circ$

$\therefore C = 60^\circ$

Hence, $\angle C = 60^\circ$ proved.

Imp
9) If $(\sin A + \sin B + \sin C)(\sin A + \sin B - \sin C) = 3 \sin A \cdot \sin B$, then show that $\angle C = 60^\circ$.

Soln:-

given:- $(\sin A + \sin B + \sin C)(\sin A + \sin B - \sin C) = 3 \sin A \cdot \sin B$

or, $\left(\frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R}\right)\left(\frac{a}{2R} + \frac{b}{2R} - \frac{c}{2R}\right) = 3 \times \frac{a}{2R} \times \frac{b}{2R}$

or, $\frac{(a+b+c)(a+b-c)}{2R \times 2R} = \frac{3ab}{\cancel{2R} \times \cancel{2R}}$

or, $a^2 + ab - \cancel{ac} + ab + b^2 - \cancel{bc} + \cancel{ac} + \cancel{bc} - c^2 = 3ab$

or, $a^2 + b^2 - c^2 + 2ab \cancel{-3ab} = 3ab$

or, $a^2 + b^2 - c^2 = 3ab - 2ab$

or, $a^2 + b^2 - c^2 = ab$

or, $\frac{2 \times a^2 + b^2 - c^2}{2 \quad ab} = 1$

or, $\frac{a^2 + b^2 - c^2}{2ab} = \frac{1}{2}$

or, $\cos C = \cos 60^\circ$

$\therefore C = 60^\circ$

Hence, $\angle C = 60^\circ$ proved.

10) If $\frac{\cos A+2\cos C}{\cos A+2\cos B} = \frac{\sin B}{\sin C}$, the prove that the Δ is either isosceles or right angled.

soln :-

given:- $\frac{\cos A+2\cos C}{\cos A+2\cos B} = \frac{\sin B}{\sin C}$

or, $\sin C\cdot\cos A + 2\cos C\cdot\sin C = \sin B\cdot\cos A + 2\sin B\cdot\cos B$

or, $\sin C\cdot\cos A - \sin B\cdot\cos A + \sin 2C = \sin 2B$

or, $\cos A(\sin C - \sin B) + \sin 2C - \sin 2B = 0$

or, $\cos A\left[2\cos\left(\frac{B+C}{2}\right)\cdot\sin\left(\frac{C-B}{2}\right)\right] + 2\cos\left(\frac{2C+2B}{2}\right)\cdot\sin\left(\frac{2C-2B}{2}\right) = 0$

or, $\cos A\cdot\left[2\cos\left(90-A/2\right)\cdot\sin\left(\frac{C-B}{2}\right)\right] + 2\cos(B+C)\cdot\sin(C-B) = 0$

or, $2\cos A\cdot\sin A/2\cdot\sin\left(\frac{C-B}{2}\right) + 2\cos(180-A)\cdot\sin(C-B) = 0$

or, $2\cos A\cdot\sin A/2\cdot\sin\left(\frac{C-B}{2}\right) - 2\cos A\cdot 2\sin\left(\frac{C-B}{2}\right)\cdot\cos\left(\frac{C-B}{2}\right) = 0$

or, $\cos A\cdot\sin\left(\frac{C-B}{2}\right)\left[2\cos A - 4\sin\left(\frac{C-B}{2}\right)\right] = 0$

or, $\cos A\cdot\sin\left(\frac{C-B}{2}\right) = 0$

Either,

$\cos A = 0$

or, $\cos A = \cos \pi/2$

$\therefore A = \pi/2$ or $90°$

OR,

$\sin\left(\frac{C-B}{2}\right) = 0$

or $\sin\left(\frac{C-B}{2}\right) = \sin 0$

or $\frac{C-B}{2} = 0$

$\therefore C = B$

Since, $A = 90°$ ~~or, and~~ or $B = C$,

Hence, Δ is isosceles or right angled ~~proved~~.

11) Derive projection formula from (i) law of sines, ~~(ii)~~ ii) law of cosines.

i) → ~~Soln:~~ Here,

Derivation of projection formula from law of sines

a) $a = b\cos C + c\cos B$

Soln:-

from sine law,

RHS:- $b\cos C + c\cos B$

$= 2R\sin B \cdot \cos C + 2R\sin C \cdot \cos B$

$= 2R(\sin B \cdot \cos C + \sin C \cdot \cos B)$

$= 2R \sin(B+C)$

$= 2R\sin(180 - A)$

$= 2R\sin A$

$= a =$ RHS proved

b) $b = a\cos C + c\cos A$

soln:

from sine law,

RHS: $a\cos C + c\cos A$

$= 2R\sin A.\cos C + 2R\sin C.\cos A$

$= 2R(\sin A.\cos C + \sin C.\cos A)$

$= 2R\sin(A+C)$

$= 2R\sin(180 - B)$

$= 2R\sin B$

$= b$

= LHS. proved

c) $c = a\cos B + b\cos A$

soln:

from sine law,

RHS: $a\cos B + b\cos A$

$= 2R\sin A.\cos B + 2R\sin B.\cos A$

$= 2R(\sin A.\cos B + \sin B.\cos A)$

$= 2R\sin(A+B)$

$= 2R\sin(180 - C)$

$= 2R\sin C$

$= c$

= LHS. proved.

ii) → Here,

Derivation of projection formula from cosine law,

a) $a = \cancel{a\cos}\ b\cos C + c\cos B$

Soln:
from cosine law,
RHS: $b\cos C + c\cos B$

$= b \times \frac{a^2+b^2-c^2}{2ab} + c \times \frac{a^2+c^2-b^2}{2ac}$

$= \frac{a^2+b^2-c^2}{2a} + \frac{a^2+c^2-b^2}{2a}$

$= \frac{a^2+\cancel{b^2}-\cancel{c^2}+a^2+\cancel{c^2}-\cancel{b^2}}{2a}$

$= \frac{2a^2}{2a}$

$= a$

$=$ LHS proved.

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b) $b = a\cos C + c\cos A$

Soln:
from cosine law,
RHS: $a\cos C + c\cos A$

$= a \times \frac{a^2+b^2-c^2}{2ab} + c \times \frac{b^2+c^2-a^2}{2bc}$

$= \frac{a^2+b^2-\cancel{c^2}+b^2+\cancel{c^2}-a^2}{2b}$

$= \frac{2b^2}{2b}$

$= b$

$=$ LHS. Proved

c) $c = a\cos B + b\cos A$

Soln:

from cosine law,

RHS: $a\cos B + b\cos A$

$= a \times \frac{a^2+c^2-b^2}{2ac} + b \times \frac{b^2+c^2-a^2}{2bc}$

$= \frac{\cancel{a^2}+c^2\cancel{-b^2}+\cancel{b^2}+c^2\cancel{-a^2}}{2c}$

$= \frac{2c^2}{2c}$

$= c$

$=$ LHS Proved.

Thank you!