Homework 5 - Gradient-Free Optimization

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1 Improvements to Our Code

Our project is to optimize the trajectory of a rocket as it flies to orbit through an atmosphere. The primary issue that we were having before the previous homework was that the "optimal" trajectory was just to go straight up and turn almost 90° at the last moment to reach the desired orbit. Although this does conserve fuel, it's not practical. The following improvements were made to our code and the trajectories now look great:

- Instead of using a large number of inputs to the optimizer that define the trajectory, we used only a few points and performed an akima spline interpolation between them. This let our trajectory use over 2,000 points in the calculations, but the optimizer only had to use about 10 points. This also sped the optimization up so we could debug faster.
- The angle between the first two interpolated values (ie. at liftoff) has to be less than five degrees. This constrains the rocket to take off nearly-vertically.
- The angle between the final interpolated value relative to the liftoff angle (ie. at orbit insertion) is also set by the user. For example, this can be set at 80°, which makes the rocket enter orbit at nearly a flat trajectory relative to the ground.
- The maximum angle between any two interpolated points is just 1°. This prevents the rocket from making any sudden turns late (or early) in the trajectory.
- The successive distances between spline points must get larger during the trajectory. This helps speed up the optimization by providing some guidance that just kind of "makes sense".

Results after applying these improvements are shown in Figure 1. Note that the initial starting point was a straight line between the launch pad and the orbital insertion point. Also note that the rocket only achieves a little more than half of orbital velocity. The final velocity can be changed by modifying the total distance downrange that the rocket flies before orbital insertion, and is something to be looked at after this is submitted.

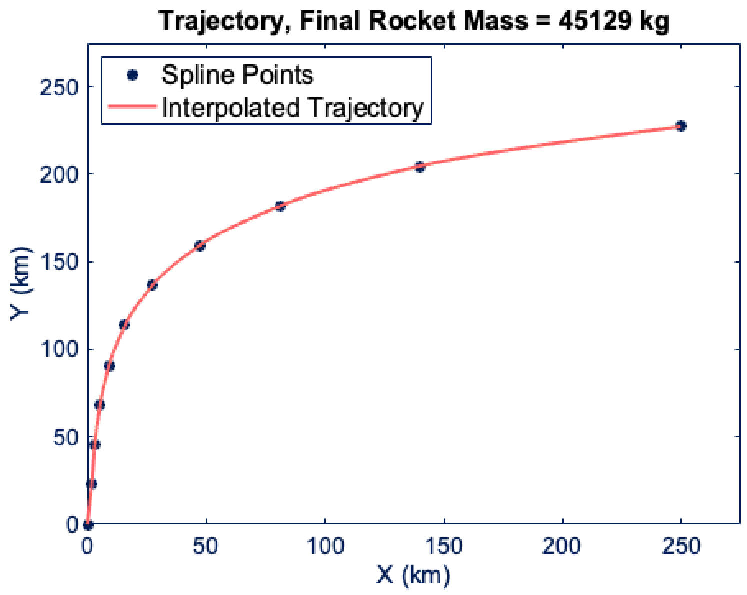


Figure 1: The trajectory as calculated using fmincon after starting with a linear trajectory between the start and end points.

The next steps for this project are to improve the final velocity to become a true orbital velocity. This would likely entail enabling the final point in the trajectory to move. Alternatively, we could re-market our model as modeling that of a first stage, where the leftover mass could be used to construct a second stage that would then hypothetically get into a true orbit. The current progress, especially the use of spline interpolation, is good for now and feels like a large accomplishment relative to where we were a few weeks ago.

2 Optimizing Our Problem Using a Gradient-Free Method

The method of choice for optimizing our problem using a gradient-free approach is the pattern search method. This was chosen solely because it uses the same input format as fmincon and was reasonably easy to just switch out in our code. The results are shown in 2. Note that the solver failed, giving errors about constraint dependency in the problem at the current iterate, if the initial guess was a linear fit between the start and end points. Therefore, a logarithmic guess was provided as the initial estimate. Also note that the optimizer terminated when the mesh size and constraint tolerance were below their thresholds. This took five iterations and 1,712 function evaluations of the objcon() function.

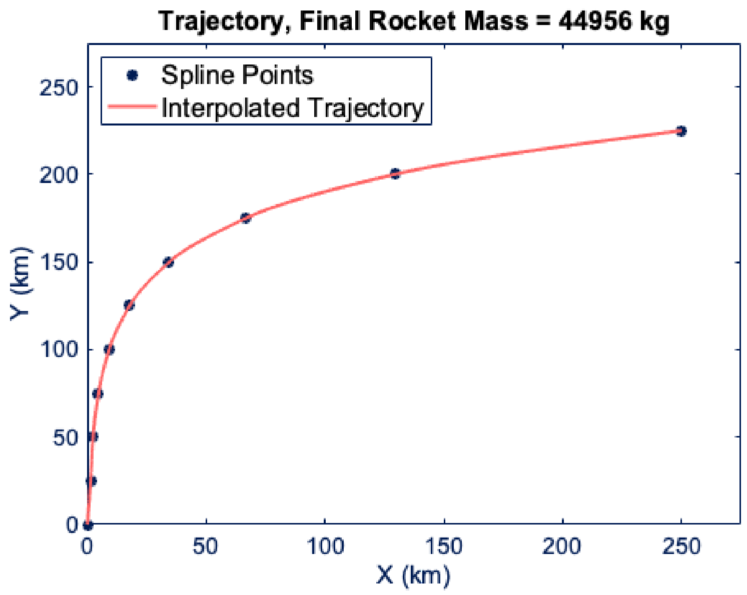


Figure 2: The trajectory as calculated using the "patternsearch" function after starting with a logarithmic trajectory.

3 My Own Gradient-Free Algorithm

Included below is the saved Jupyter Notebook LaTeX PDF of my own Nelder-Mead algorithm. At the bottom are two examples:

1. A Rosenbrock problem in four dimensions
2. A Brachistochrone problem with 30 points. (Yes, that took ages to run, but it was a cool test)

The results work well, and I'm very satisfied with the algorithm.

My Nelder-Mead Algorithm

March 19, 2021

[25]:
```python
import numpy as np
import scipy.optimize as optimize
import copy
import matplotlib.pyplot as plt
```

[89]:
```python
# The final product function
# Inputs:
#    - func: a function handle to the function that you want to minimize
#    - x0: an initial guess
#    - tol: the tolerance level that you want to converge to
#
# Outputs:
#    - xstar: the optimal x-values
#    - fstar: the optimal function value

def NelderMead(obj,x0,tol = 1e-6,maxIter = 1000,commentary = False,showxstar =␣
↪False,TriangleOpsShow = False,startingPointsStep = 1):

    # Define the starting points (tetrahedron)
    points = defineStartingPoints(x0,startingPointsStep)

    for i in range(0,maxIter):

        # Order points
        orderedPoints, orderedValues = orderPoints(obj,points)

        # Get centroid
        centroid = getCentroid(orderedPoints)

        # Get next triangle
        newPoints, method =␣
↪getNextTriangle(obj,orderedPoints,orderedValues,centroid,TriangleOpsShow)

        # Check convergence criteria
        converged,norm = checkConvergence(newPoints,orderedPoints,tol)

        if commentary:
            print("Iteration:",i+1,"\tf(x) =","{:.5e}".format(obj(newPoints[:
↪,0])),"\tNorm:","{:.5e}".format(norm),"\tMethod:",method)
            if showxstar:
                print("\tCurrent x* value: ",newPoints[:,0])
                print(" ")

        if converged:
            print(" ")
            print("Optimization comlete.")

            break
        else:
            points = newPoints

    xstar = newPoints[:,0]
    fstar = obj(xstar)
    print("Iterations:",i + 1)
    print("Norm:",norm)
    print("Function Value:",fstar)
    print("Optimal Inputs:",xstar)

    return xstar, fstar


def defineStartingPoints(x0,step):

    n = len(x0)

    startingPoints = np.zeros((n,n+1))
    startingPoints[:,0] = x0

    # Change each vector by adding a value to only one dimension at a time
    for i in range(1,n+1):

        difference = np.zeros(n)
        difference[i-1] = step
        startingPoints[:,i] = x0 + difference

    return startingPoints

def orderPoints(obj,points):

    numDims = len(points)
    numPoints = len(np.transpose(points))

    values = np.zeros(numPoints)
    sortedPoints = np.zeros((numDims,numPoints))
    # Evaluate points
    for i in range(0,numPoints):
        xi = points[:,i]
        values[i] = obj(xi)

    sortedValues = np.sort(values)

    # Perform the sorting
    for i in range(0,numPoints):
        # Take a value
        currentValue = values[i]

        # Find the corresponding value in sortedValues
        for j in range(0,numPoints):
            if sortedValues[j] == currentValue:
                index = j
                break

        # Place the point associated with the value in the proper location in␣
 ↪the sortedPoints array
        sortedPoints[:,j] = points[:,i]

    return sortedPoints, sortedValues

def getCentroid(points):

    numPoints = len(np.transpose(points))
    numDims = len(points)

    centroid = np.zeros(numDims)
    for i in range(0,numPoints - 1):
        centroid = centroid + points[:,i]

    centroid = centroid / (numPoints - 1)

    return centroid

def getNextTriangle(obj,points,values,centroid,commentary = False):

    worstPoint = points[:,-1]

    points = copy.copy(points)

    improvedPoint, method =␣
 ↪performTriangleOperations(obj,points,values,centroid,commentary)
```

```
    if np.array_equal(improvedPoint,worstPoint):
        if commentary:
            print("None of these worked, performing a shrink")
        return performShrink(points), method
    else:
        points[:,-1] = improvedPoint

    if commentary:
        print(" ")

    return points, method

def performTriangleOperations(obj,points,values,centroid,commentary = False):

    numPoints = len(np.transpose(points))
    numDims = len(points)

    bestPoint = points[:,0]
    bestValue = values[0]

    secondWorstPoint = points[:,-2]
    secondWorstValue = values[-2]

    worstPoint = points[:,-1]
    worstValue = values[-1]

    if commentary:
        print("Worst Value: ",worstValue,"\tSecond Worst Value:␣
 ↪",secondWorstValue,"\t Best Value:",bestValue)

    if commentary:
        print("Performing Reflection.")

    # Try reflection
    alpha = 1
    newPoint,newValue = generateNewPoint(worstPoint,centroid,alpha,obj)
    if commentary:
        print("New value = ",newValue)

    if newValue <= bestValue: # If the new value has an objective less than the␣
 ↪best point,
                              # try doubling alpha to get an even better point

        if commentary:
            print("This new value is less than the best value.")
            print("Performing Expansion.")
        alpha = 2
        newPoint2,newValue2 = generateNewPoint(worstPoint,centroid,alpha,obj)
        if commentary:
            print("New value = ",newValue2)

        if newValue2 < newValue: # If that works, we're done here
            if commentary:
                print("New value is even smaller than the previous value after␣
↪simple reflection. Returning...")
            return newPoint2, "Expansion"
        else:
            if commentary:
                print("The other new value was better, let's use that one.␣
↪Returning...")
            return newPoint, "Reflection" # If not, return the point from when␣
↪alpha = 1

    elif newValue < secondWorstValue: # If the new value is at least better␣
↪than the second worst value, keep it
        if commentary:
            print("New value is more than the best, but less than the second␣
↪worst. Let's use it. Returning...")
        return newPoint, "Reflection, better than second worst"

    else: # If the new point is worse than all other points, try a smaller␣
↪alpha value
        alpha = 0.5
        newPoint2,newValue2 = generateNewPoint(worstPoint,centroid,alpha,obj)

        if commentary:
            print("Performing Outside Contraction.")
            print("New value = ", newValue2)

        if newValue2 <= bestValue: # If that works, return
            if commentary:
                print("New value is less than the best value. Returning...")
            return newPoint2, "Outside Contraction"
        elif newValue2 <= secondWorstValue: # We'll take it if it's better than␣
↪another point
            if commentary:
                print("New value is more than the best, but less than the␣
↪second worst. Let's use it. Returning...")
            return newPoint2, "Outside Contraction, better than second worst"

        else: # If reflections didn't work, try an inside contraction
            alpha = -0.5
            newPoint2,newValue2 =␣
↪generateNewPoint(worstPoint,centroid,alpha,obj)

            if commentary:
                print("Performing Inside Contraction")
                print("New value = ",newValue)

            if newValue2 <= bestValue: # If that works, return
                if commentary:
                    print("New value is less than the best value. Returning...")
                return newPoint2, "Inside Contraction"
            elif newValue2 <= secondWorstValue: # We'll take it if it's better␣
↪than another point
                if commentary:
                    print("New value is more than the best, but less than the␣
↪second worst. Let's use it. Returning...")
                return newPoint2, "Inside Contraction, better than second worst"

            else:
                return worstPoint, "Shrink"

def generateNewPoint(currentPoint,centroid,alpha,obj):
    newPoint = centroid + alpha * (centroid - currentPoint)
    newValue = obj(newPoint)
    return newPoint,newValue

def performShrink(points):

    numPoints = len(np.transpose(points))
    numDims = len(points)
    newPoints = np.zeros((numDims,numPoints))

    bestPoint = points[:,0]

    gamma = 0.5

    for i in range(0,numPoints):
        newPoints[:,i] = bestPoint + gamma * (points[:,i] - bestPoint)

    return newPoints

def checkConvergence(newPoints,oldPoints,tol):
    difference = np.subtract(newPoints,oldPoints)
    norm = np.linalg.norm(difference,np.inf)
    if norm < tol:
        return True, norm
    else:
        return False, norm
```

1 Examples

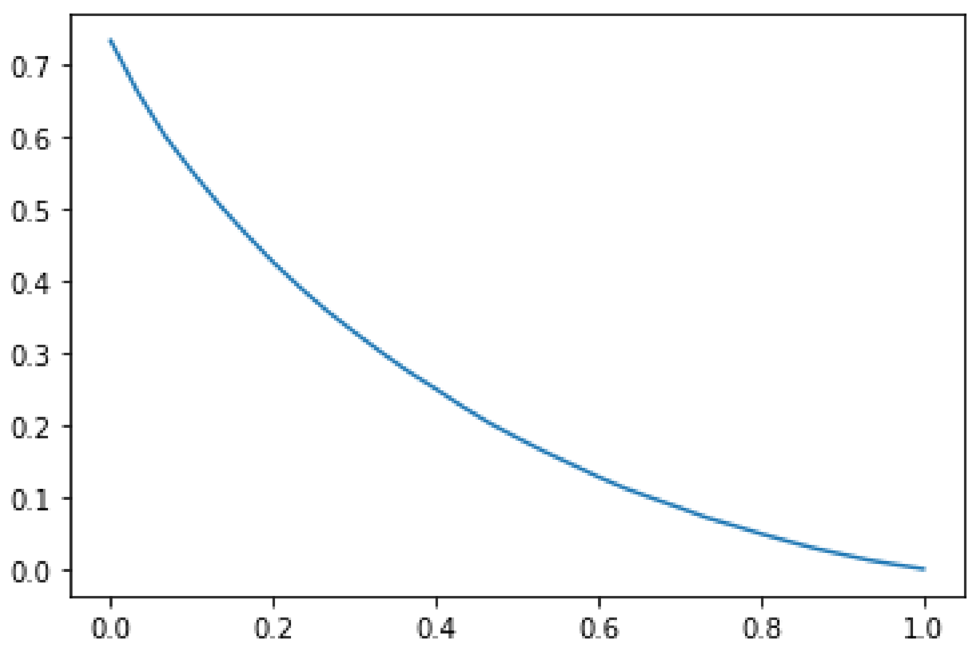
[90]:
```
def rosenbrock(x):

    n = len(x)

    total = 0

    for i in range(0,n-1):

        total = total + (100*(x[i+1] - x[i]**2)**2 + (1 - x[i])**2)

    return total

NelderMead(rosenbrock,[-1,1,-12,0])

# Using the keyword arguments
NelderMead(rosenbrock,[-1,1,-12,0],tol = 1e-9,maxIter = 10000,commentary =␣
 ↪False,showxstar = False);
```

```
Optimization comlete.
Iterations: 371
Norm: 9.54768101246728e-07
Function Value: 5.886575859739628e-11
Optimal Inputs: [0.99999932 0.99999827 0.99999603 0.99999196]

Optimization comlete.
Iterations: 469
Norm: 7.216303110624267e-10
Function Value: 1.2881928908454321e-17
Optimal Inputs: [1. 1. 1. 1.]
```

[95]:
```
def brachistochrone(y):
    startingPoint = np.array([0,1])
    endingPoint = np.array([1,0])

    y = np.insert(y,0,startingPoint[1])
    y = np.append(y,endingPoint[1])

    n = len(y)

    deltaX = (endingPoint[0] - startingPoint[0]) / n
    x = np.linspace(startingPoint[0],endingPoint[0],n)
    h = startingPoint[1] - endingPoint[1]

    mu = 0.3
    f = 0

    for i in range(0,n-1):
        deltaY = y[i+1] - y[i]

        f = f + np.sqrt(deltaX**2 + deltaY**2) / (np.sqrt(h - y[i+1] -␣
↪mu*x[i+1]) + np.sqrt(h - y[i] - mu*x[i]))

    return f

ystar, fstar = NelderMead(brachistochrone,np.linspace(0.9,0.1,30),
                          tol = 1e-6,
                          maxIter = 100000,
                          commentary = False,
                          TriangleOpsShow = False,
                          startingPointsStep = 0.05)

y = np.insert(ystar,0,1)
y = np.append(ystar,0)
n = len(y)
x = np.linspace(0,1,n)

plt.figure()
plt.plot(x,y)
```

```
Optimization comlete.
Iterations: 85518
Norm: 7.775747827287738e-07
Function Value: 1.369379517374197
Optimal Inputs: [0.73344399 0.66221295 0.6018669  0.55246441 0.50672203
0.46492364
 0.42609129 0.39074932 0.35815887 0.32889924 0.30073935 0.273981
 0.25009798 0.22516854 0.2023967  0.18254645 0.16342031 0.14576469
 0.12799066 0.1119736  0.09832487 0.08515514 0.07135624 0.06033338
 0.04894744 0.03855896 0.02856756 0.02031814 0.01222314 0.00590317]
```

```
[95]: [<matplotlib.lines.Line2D at 0x7f957e16d340>]
```


0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.0
0.2
0.4
0.6
0.8
1.0

```
[ ]:
```