Foundations of Computing 2 - 2022 Exam 15/05/2024

1ai. Find the number of distinct words that can be made up using all the letters from the word EXAMINATION

EXAMINATION : 11 letters : 11!

E = 1, X = 1, A = 2, M = 1, I = 2, N = 2, T = 1, O = 1

A = 2, I = 2, N = 2 : 2! x 2! x 2!

$$\frac{11!}{2!2!2!} = 4{,}989{,}600 \text{ distinct words}$$

ii. How many words can be made when AA must not occur?

(AA)EXMINTION → 11 letters to 10 : 10!

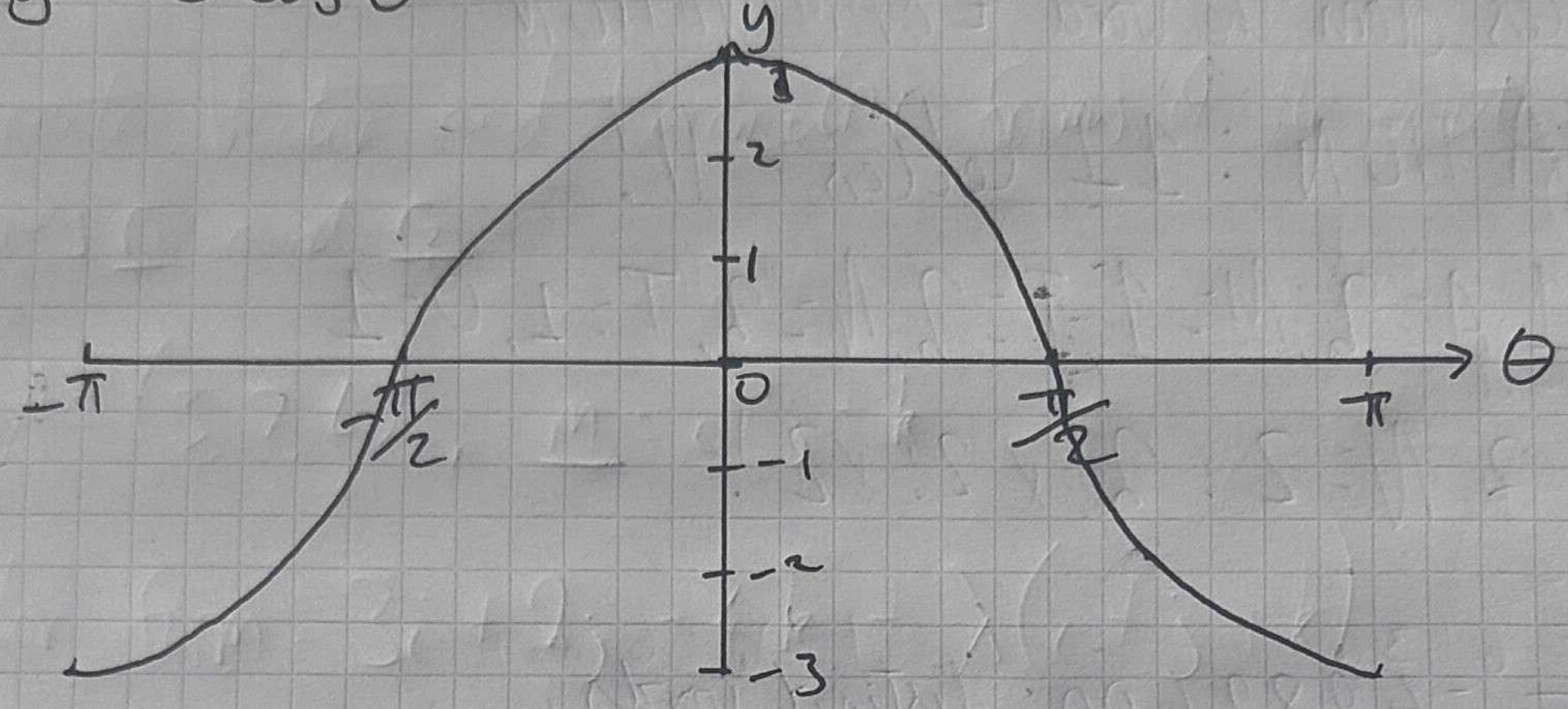
I = 2, N = 2 : 2! x 2!

$$\frac{10!}{2!2!} = 907{,}200$$

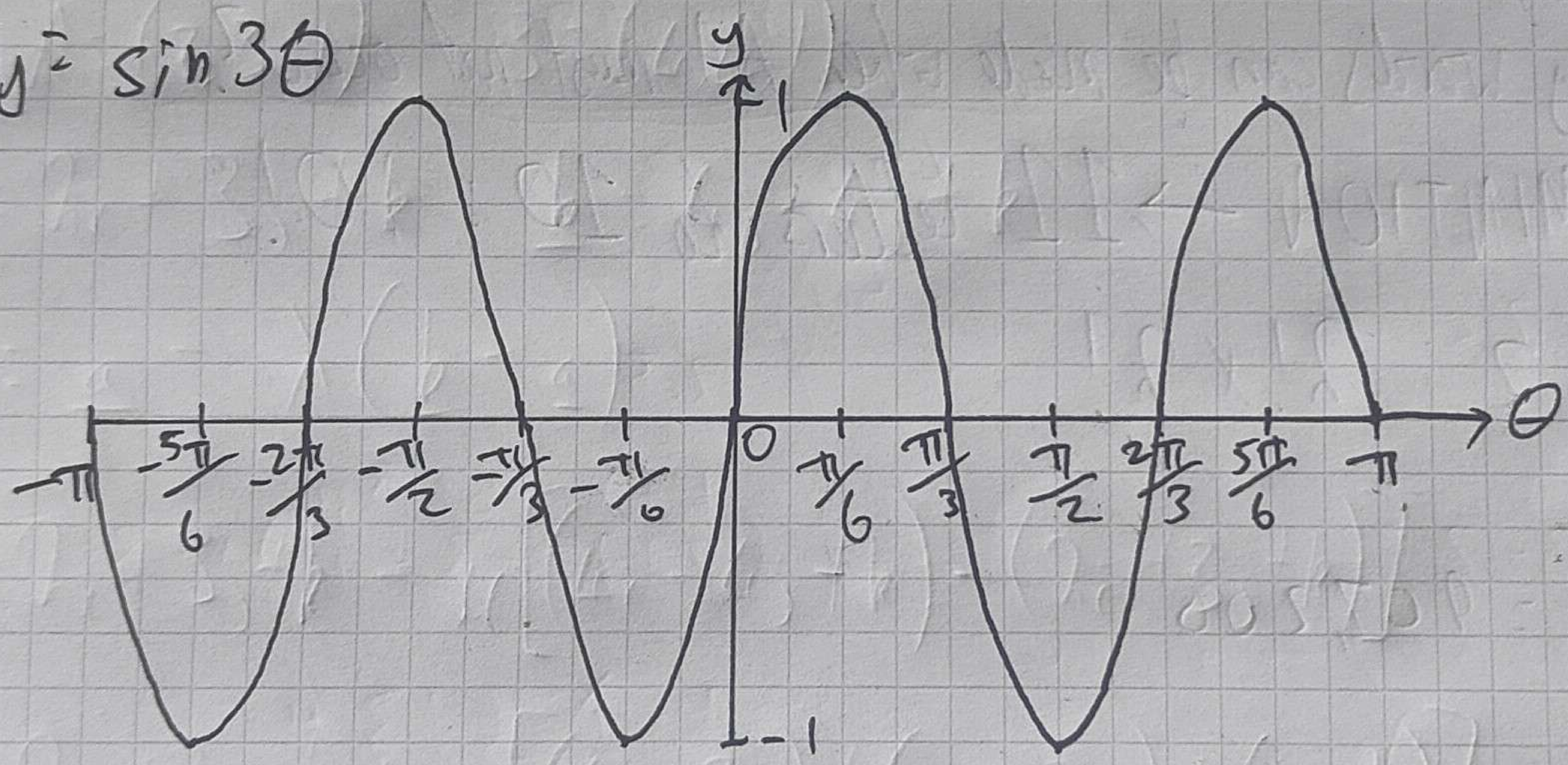
4,989,600 - 907,200 = 4,082,400

b. Sketch the following graphs for $-\pi < \theta < \pi$

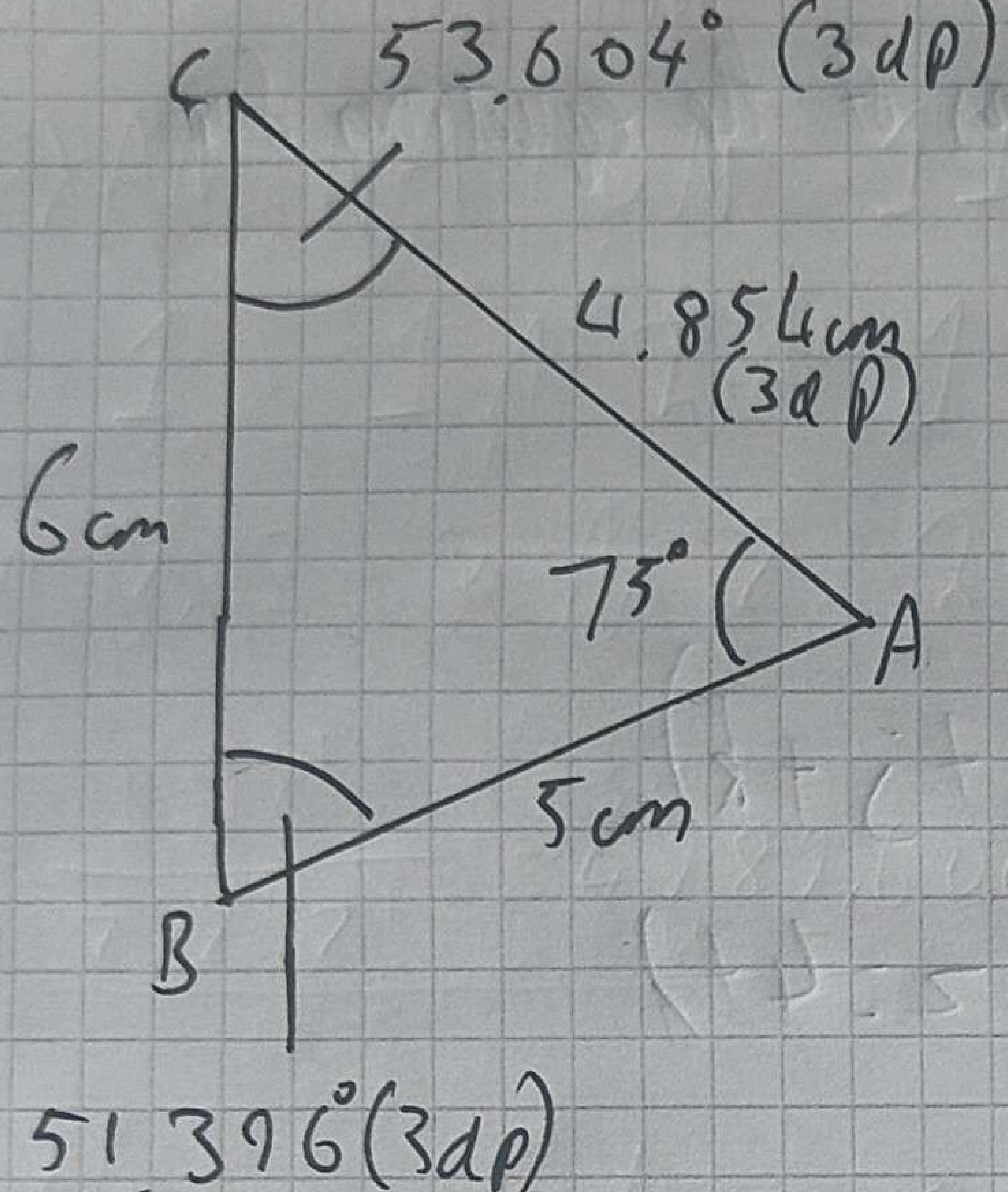
i. $y = 3\cos\theta$



ii. $y = \sin 3\theta$



c. Solve (find all missing lengths and angles) the triangle ABC where AB = 5 cm, BC = 6 cm, and angle A = 75°



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{6}{\sin(75)} = \frac{5}{\sin(B)} \equiv \frac{\sin 75}{6} = \frac{\sin(B)}{5}$$

$$\frac{5\sin(75)}{6} = \sin(B)$$

$$B = \sin^{-1}\left(\frac{5 \times \sin(75)}{6}\right)$$

$$B = 53.604 \text{ (3dp)}$$

Angles of a triangle = 180°

$180 - 75 - 53.604$

$= 51.396°$ (3dp)

$$\frac{a}{\sin(51.396)} = \frac{6}{\sin(75)} \longrightarrow a = \frac{6 \times \sin(51.396)}{\sin(75)}$$

$a = 4.854$ cm (3dp)

2. $\underline{r_1}$: $A = (3, 2, 4)$, $\underline{m} = i + j + k$

$\underline{r_2}$: $A = (2, 3, 1)$, $B = (4, 4, 1)$

a. Create Vector and Parametric forms of the equations for lines $\underline{r_1}$ and $\underline{r_2}$

$\underline{r_1}$: $A = (3, 2, 4)$, $\underline{m} = i + j + k$

Vector form: $3i + 2j + 4k + \lambda(i + j + k)$

Parametric form: $x = 3 + \lambda$, $y = 2 + \lambda$, $z = 4 + \lambda$

$\underline{r_2}$: $A = (2, 3, 1)$, $B = (4, 4, 1)$

$\underline{a} = 2i + 3j + k$, $\underline{b} = 4i + 4j + k$

$\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a})$

$\underline{r_2} = 2i + 3j + k + \mu[(4i + 4j + k) - (2i + 3j + k)]$

$= 2i + 3j + k + \mu[(4i - 2i) + (4j - 3j) + (k - k)]$

Vector form: $2i + 3j + k + \mu(2i + j)$

Parametric form: $x = 2 + 2\mu$, $y = 3 + \mu$, $z = 1$

b. Find the points of intersection between the two lines.

Not possible - The lines do not intersect

c. Find the size of the angle between the two lines.

Not possible - The lines do not intersect.

3a. Differentiate:

i. $7x^2 + 14$

$\frac{dy}{dx} = 14x$

ii. $e^x(4x^2 + 3)$

$f(x) = e^x$ $\quad g(x) = 4x^2 + 3$

$f'(x) = e^x$ $\quad g'(x) = 8x$

Sum rule: $f'(x)g(x) + f(x) + g'(x)$

$\frac{dy}{dx} = e^x(4x^2 + 3) + e^x(8x)$

$= e^x(4x^2 + 8x + 3)$

b. Integrate

i. $\int (4x^3 - 3x^2 + 2x - 1)\, dx$

$= \frac{4x^4}{4} - \frac{3x^3}{3} + \frac{2x^2}{2} - x + c$

$= x^4 - x^3 + x^2 - x + c$

ii. $\int_{z=-1}^{4} \int_{y=0}^{3} \int_{x=1}^{2} (2xy+z)\,dx\,dy\,dz$

$$\int_{x=1}^{2} (2xy+z)\,dx = \frac{2x^2 y}{2} + xz \Big|_1^2 = x^2 y + xz \Big|_1^2$$

$$= \left[2^2 y + 2z\right] - \left[1^2 y + z\right] = \left[4y + 2z\right] - \left[y+z\right] = 3y + z$$

$$\int_{y=0}^{3} (3y+z)\,dy = \frac{3y^2}{2} + yz \Big|_0^3$$

$$= \left[\frac{3\times 3^2}{2} + 3z\right] - \left[\frac{3\times 0^2}{2} + 0z\right] = \frac{27}{2} + 3z$$

$$\int_{z=-1}^{4} \left(\frac{27}{2} + 3z\right) dz = \frac{27}{2}z + \frac{3z^2}{2} \Big|_{-1}^{4}$$

$$= \left[\frac{27\times 4}{2} + \frac{3\times 4^2}{2}\right] - \left[\frac{27\times -1}{2} + \frac{3\times(-1)^2}{2}\right]$$

$$= \left[54 + 24\right] - \left[\frac{-27}{2} + \frac{3}{2}\right] = 78 + 12 = 90$$

4a. Explain in words the meaning of $P(A|B)$

State the relationship between A and B when:

i. $P(A|B) = P(A)$

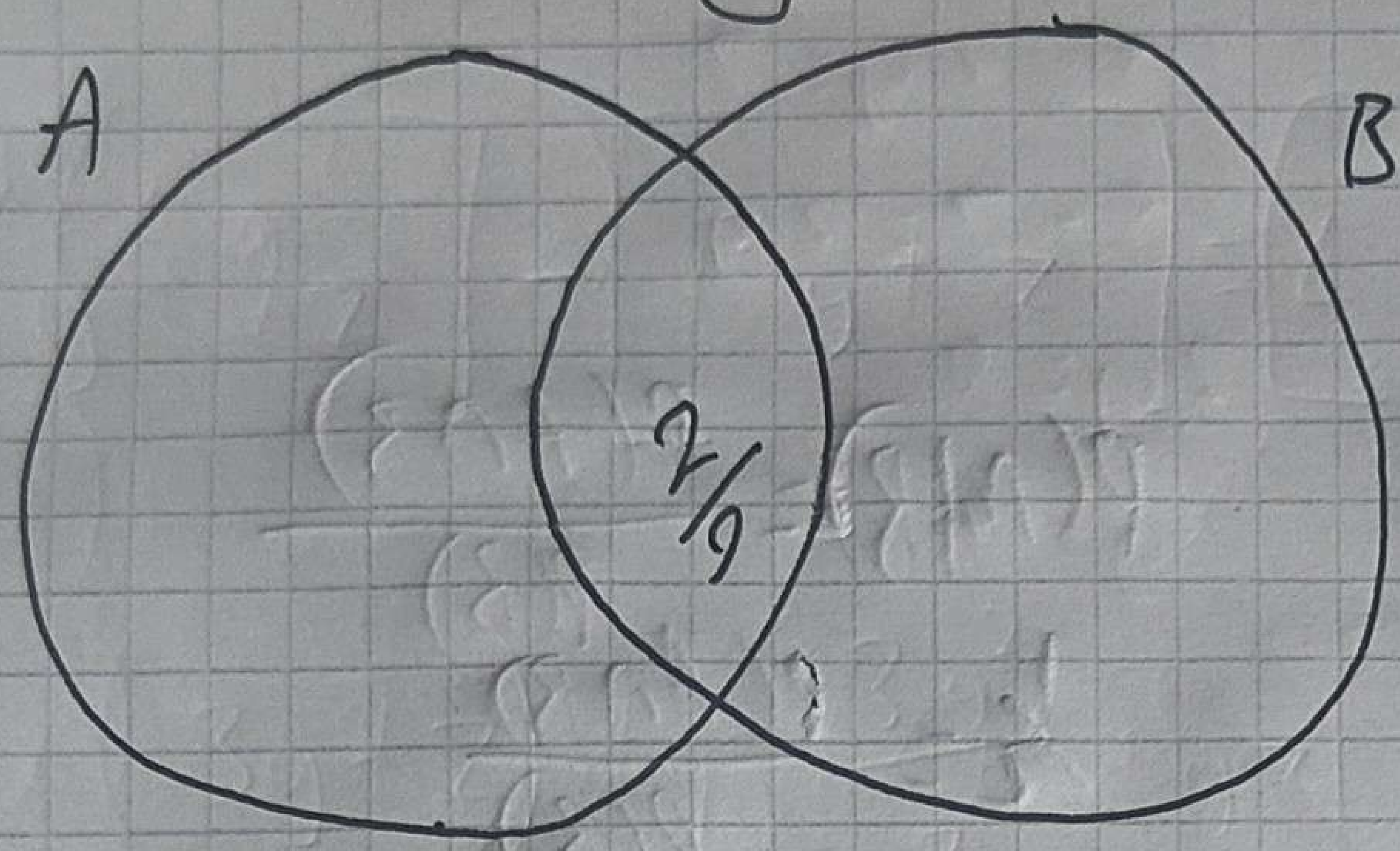
Event A is independent of event B - the occurance of B does not affect the probability of A.

ii. $P(A|B) = 0$

The occurance of event B makes the occurance of event A impossible.

b. The events A and B are such that $P(A) = 4/9$, $P(A|B) = 1/2$ and $P(A \cup B) = 2/3$.

i. Draw a Venn diagram to represent the information



$P(B) = 4/9$

$P(A) = 4/9$

$P(A \cup B) = 2/3$ $P(\bar{A} \cap \bar{B}) = 1/3$

$P(A) = 4/9$

$P(B) = 4/9$

$P(A \cap B) = 2/9$

$P(A \cup B) = 2/3$

$P(A \cap \bar{B}) = 2/9$

$P(\bar{A} \cap B) = 2/9$

Calculate:

i. $P(B) = 4/9$

ii. $P(B|A) = 1/2$

iii. $P(A \cap B) = 2/9$

iv. $P(A \cup \overline{B}) = 7/9$

v. $P(\overline{A}|\overline{B}) = 3/5$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$2/3 = 4/9 + P(B) - P(A \cap B)$$
$$2/9 = P(B) - P(A \cap B)$$
$$2/9 = P(B) - \tfrac{1}{2}P(B)$$
$$2/9 = \tfrac{1}{2}P(B) \rightarrow 4/9 = P(B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$
$$\frac{1}{2} = \frac{P(A \cap B)}{P(B)}$$
$$\tfrac{1}{2}P(B) = P(A \cap B)$$
$$\tfrac{1}{2} \times \tfrac{4}{9} = P(A \cap B)$$
$$P(A \cap B) = 2/9$$

$$P(A \cap \overline{B}) = P(A) - P(A \cap B)$$
$$= 4/9 - 2/9 = 2/9$$

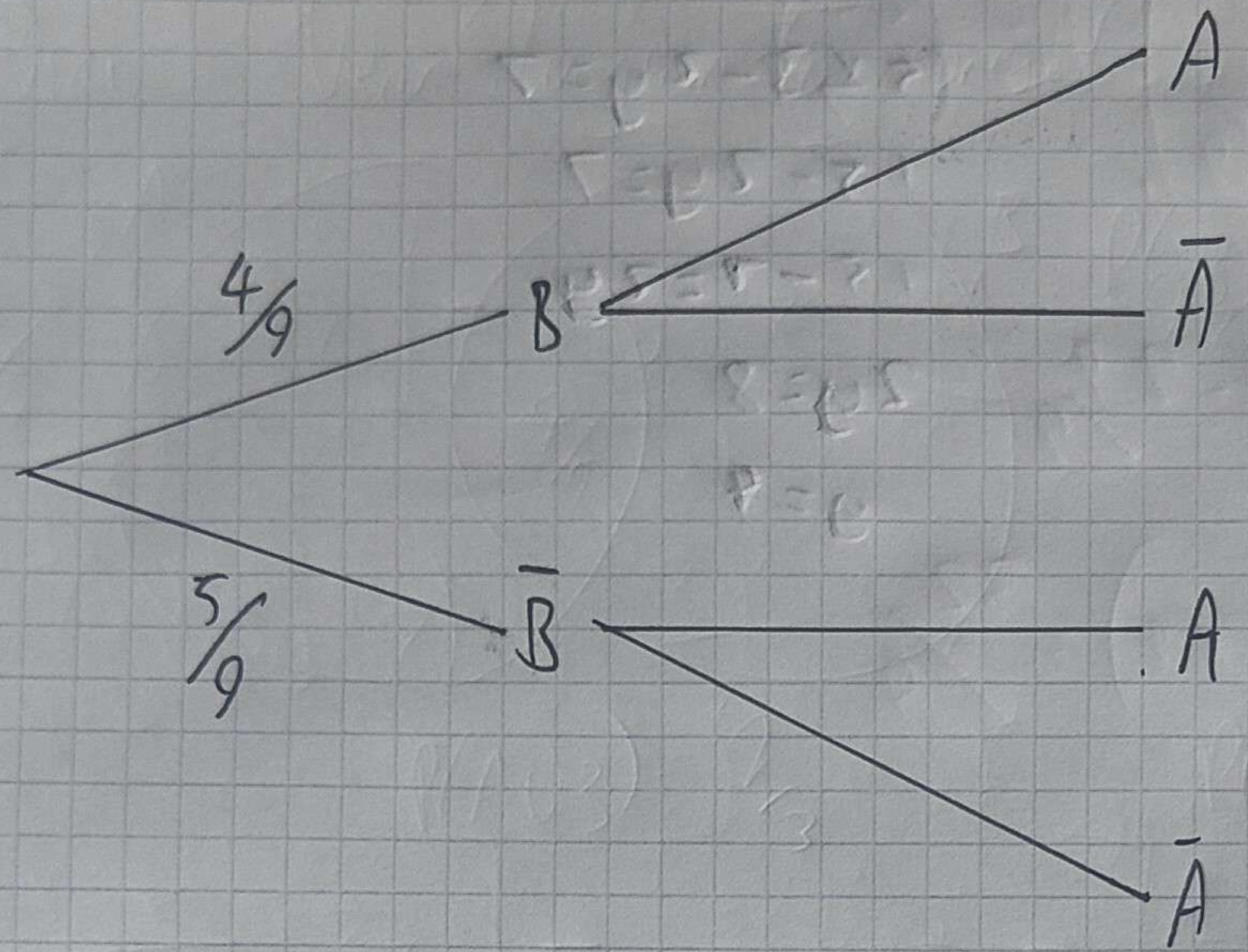
$$P(\overline{A} \cap B) = P(B) - P(A \cap B)$$
$$= 4/9 - 2/9 = 2/9$$

$$P(A \cup \overline{B}) = P(A) + P(\overline{B}) - P(A \cap \overline{B})$$
$$= 4/9 + (1 - 4/9) - 2/9$$
$$= 7/9$$

$$P(B|A) = \frac{P(A \cap B)}{P(B)} = \frac{2/9}{4/9} = 1/2$$

$$P(\overline{A}|\overline{B}) = \frac{P(\overline{A} \cap \overline{B})}{P(\overline{B})}$$
$$= \frac{1 - P(A \cup B)}{1 - 4/9} = \frac{1 - 2/3}{5/9} = \frac{1/3}{5/9} = 3/5$$

vi. Represent the information as a tree diagram with B preceding A.



5a. Solve the simultaneous equations:

$4x+2y=20$

$5x-2y=7$

$$\begin{array}{l} 4x+2y=20 \\ \underline{5x-2y=7 \quad +} \\ 9x \qquad = 27 \end{array}$$

$9x=27$

$x=3$

$5\times 3-2y=7$

$15-2y=7$

$15-7=2y$

$2y=8$

$y=4$

b. $M = \begin{pmatrix} 4 & 2 \\ 5 & -2 \end{pmatrix}$, $N = \begin{pmatrix} 20 \\ 7 \end{pmatrix}$

Find:

i. M^{-1}

Determinant

$M = \begin{pmatrix} 4 & 2 \\ 5 & -2 \end{pmatrix} : (4 \times -2) - (5 \times 2) = -8 - 10 = -18$

$\begin{pmatrix} 4 & 2 \\ 5 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & 2 \\ 5 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & -2 \\ -5 & 4 \end{pmatrix}$

$$\begin{pmatrix} -2/-18 & -2/-18 \\ -5/18 & 4/-18 \end{pmatrix}$$

$$= \begin{pmatrix} 1/9 & 1/9 \\ 5/18 & -2/9 \end{pmatrix}$$

ii. $M^{-1}N$

$$M^{-1} = \begin{pmatrix} 1/9 & 1/9 \\ 5/18 & -2/9 \end{pmatrix}, \quad N = \begin{pmatrix} 20 \\ 7 \end{pmatrix}$$

$$\left(\tfrac{1}{9} \times 20\right) + \left(\tfrac{1}{9} \times 9\right) = \tfrac{29}{9}$$

$$\left(\tfrac{5}{18} \times 20\right) + \left(\tfrac{-2}{9} \times 9\right) = \tfrac{32}{9}$$

$$M^{-1}N = \begin{pmatrix} 29/9 \\ 32/9 \end{pmatrix}$$

c. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Find AB and explain why BA cannot be found.

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$\because (1 \times 1) + (2 \times 3) = 7$ $\quad (1 \times 2) + (2 \times 4) = 10$

$(3 \times 1) + (4 \times 3) = 15$ $\quad (3 \times 2) + (4 + 4) = 22$

$(5 \times 1) + (6 \times 3) = 23$ $\quad (5 \times 2) + (6 \times 4) = 34$

$$AB = \begin{pmatrix} 7 & 10 \\ 15 & 22 \\ 23 & 34 \end{pmatrix}$$

BA is not possible because matrix multiplication requires that the number of columns in the first matrix be equivalent to the number of rows in the second matrix.

If we attempted BA, then B would have 2 columns, and A would have three rows, therefore making the matrix multiplication impossible.