COMP3250

Exam Revision Session (1)

(2021) Question 1

(a) Solve the simultaneous equations:

$$\begin{cases} 3x + 2y = 19 & (1) \times 2 \\ 2x + 5y = -2 & (2) \times 3 \end{cases}$$

$$\begin{cases} 6x + 4y = 38 \\ 6x + 15y = -6 \end{cases} \quad (-)$$

$$-11y = 44$$

$$\boldsymbol{y = -4} \quad (3)$$

Substitute (3) into (2)

$$2x + 5(-4) = -2$$

$$2x - 20 = -2$$

$$2x = 18$$

$$\boldsymbol{x = 9}$$

(b) $M = \begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix}$ $\quad N = \begin{pmatrix} 19 \\ -2 \end{pmatrix}$

Find:

(i) M^{-1}

$$|M| = (3)(5) - (2)(2) = 15 - 4 = 11$$

$$M^{-1} = \frac{1}{11}\begin{pmatrix} 5 & -2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} \mathbf{5/11} & \mathbf{-2/11} \\ \mathbf{-2/11} & \mathbf{3/11} \end{pmatrix}$$

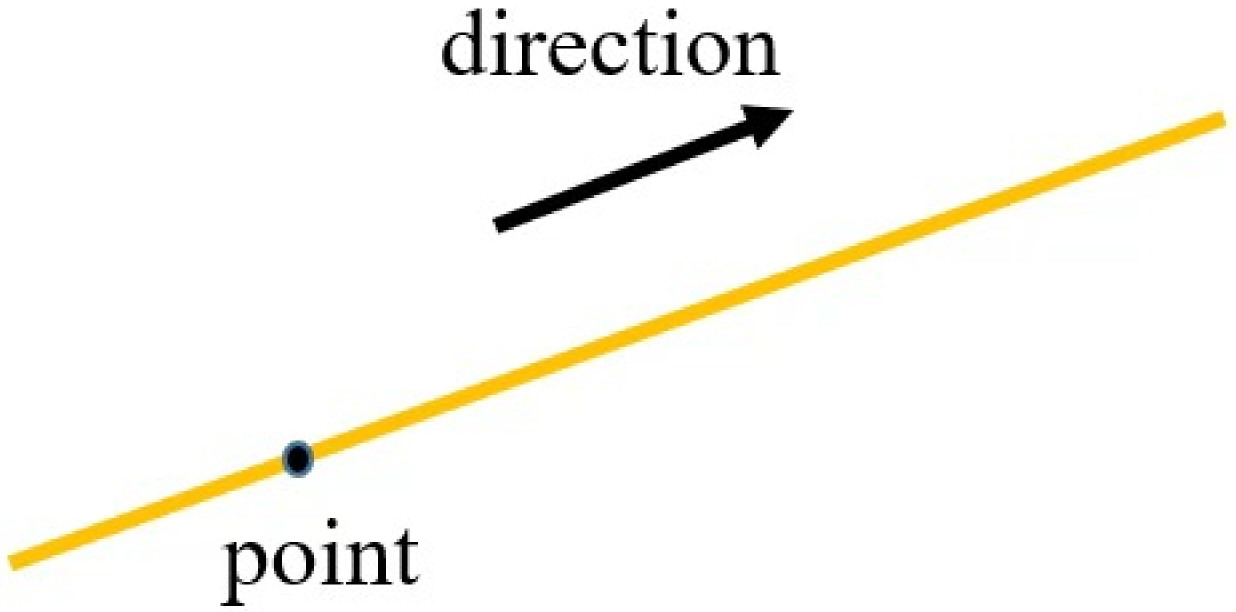
(ii) $M^{-1}N$

$$M^{-1}N = \frac{1}{11}\begin{pmatrix} 5 & -2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 19 \\ -2 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 95 + 4 \\ -38 - 6 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 99 \\ -44 \end{pmatrix} = \begin{pmatrix} \mathbf{9} \\ \mathbf{-4} \end{pmatrix}$$

(c) $A = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$ $B = \begin{pmatrix} 3 & 5 \\ 7 & 9 \end{pmatrix}$ Find AB and BA.

$$AB = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ 7 & 9 \end{pmatrix} = \begin{pmatrix} 6+28 & 10+36 \\ 18+56 & 30+72 \end{pmatrix} = \begin{pmatrix} \mathbf{34} & \mathbf{46} \\ \mathbf{74} & \mathbf{102} \end{pmatrix}$$

$$BA = \begin{pmatrix} 3 & 5 \\ 7 & 9 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 6+30 & 12+40 \\ 14+54 & 28+72 \end{pmatrix} = \begin{pmatrix} \mathbf{36} & \mathbf{52} \\ \mathbf{68} & \mathbf{100} \end{pmatrix}$$



A line is determined by a point and a direction

Vector form

$$\underline{r} = \underbrace{i - 3j + 8k}_{\text{point}} + \lambda(\underbrace{2i + 7j + 5k}_{\text{direction}})$$

Parametric form

$$\begin{aligned} x &= 1 + 2\lambda \\ y &= -3 + 7\lambda \\ z &= 8 + 5\lambda \end{aligned}$$

Cartesian form

$$\frac{x - 1}{2} = \frac{y - (-3)}{7} = \frac{z - 8}{5}$$

(2021) Question 2

$$L_1: \quad \frac{x+2}{4} = \frac{y-2}{3} = z - 1$$

$$L_2: \quad \underline{r} = 3i + 4j - 3k + \mu(i + j + k)$$

(a) Convert both equations to their parametric form.

L_1: $\boldsymbol{x = -2 + 4\lambda}$
$\boldsymbol{y = 2 + 3\lambda}$
$\boldsymbol{z = 1 + \lambda}$

L_2: $\boldsymbol{x = 3 + \mu}$
$\boldsymbol{y = 4 + \mu}$
$\boldsymbol{z = -3 + \mu}$

(b) Find the point of intersection of the two lines.

L_1: $x = -2 + 4\lambda$
$y = 2 + 3\lambda$
$z = 1 + \lambda$

L_2: $x = 3 + \mu$
$y = 4 + \mu$
$z = -3 + \mu$

$$\begin{cases} -2 + 4\lambda = 3 + \mu \\ 2 + 3\lambda = 4 + \mu \end{cases} \quad (-)$$

$$-4 + \lambda = -1$$

$$\lambda = 3$$

$x = -2 + 4(3) = 10$
$y = 2 + 3(3) = 11$
$z = 1 + 3 = 4$

the point of intersection is **(10, 11, 4)**

(c) Find the size of the angle between the two lines.

$$L_1: \quad \frac{x+2}{4} = \frac{y-2}{3} = z-1 \qquad \underline{a} = 4i + 3j + k$$

$$L_2: \quad \underline{r} = 3i + 4j - 3k + \mu(i + j + k) \qquad \underline{b} = i + j + k$$

$$\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}} = |\underline{\boldsymbol{a}}||\underline{\boldsymbol{b}}| \boldsymbol{cos\theta}$$

$$4 \times 1 + 3 \times 1 + 1 \times 1 = \sqrt{4^2 + 3^2 + 1^2}\sqrt{1^2 + 1^2 + 1^2}\, cos\theta$$

$$8 = \sqrt{26}\sqrt{3}\, cos\theta$$

$$cos\theta = \frac{8}{\sqrt{26}\sqrt{3}}$$

$$\theta = cos^{-1}\left(\frac{8}{\sqrt{26}\sqrt{3}}\right) = \mathbf{25.07°}$$

Conditional Probability

$P(A|B)$: probability of event A happening, given that event B has already happened

condition

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \longleftarrow \textbf{condition}$$

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

events A, B both happen — event B happens first, then event A happens — event A happens first, then event B happens

Conditional Probability

Why do we study conditional probability?

Because events affect each other: when one event happens, it makes another event more likely to happen or less likely to happen

Example 1: computer plays chess with a human player

event A: computer wins

event B: computer learned your machine learning algorithm to play chess

$\boldsymbol{P(A|B)}$**:** the chance that computer wins, given that computer has learned your machine learning algorithm to play chess

Your algorithm **increases** the chance that computer wins $P(A|B) > P(A)$

Example 2:

event A: my computer is infected by computer virus

event B: I installed an anti-virus software on my computer

$\boldsymbol{P(A|B)}$**:** the chance that my computer is infected by virus, given that I have installed an anti-virus software on my computer

The anti-virus software **decreases** the chance that my computer is infected by virus $P(A|B) < P(A)$

Example 3:

event A: I flip coin #2 and get a head

event B: I flipped coin #1 and got a head

$\boldsymbol{P(A|B)}$**:** the chance that I get a head on coin #2, given that I have already got a head on coin #1

$P(A|B) = P(A)$ **events A, B are independent**

events A, B are independent

$$P(A|B) = P(A)$$

$$P(B|A) = P(B)$$

$$P(A \cap B) = P(A)P(B)$$

$$P(A|\boldsymbol{B}) = \frac{P(A \cap B)}{P(\boldsymbol{B})}$$

$$P(A \cap B) = P(A|B)P(B) = P(A)P(B)$$

events A, B are mutually exclusive

Example **event A:** I take train to go to Edinburgh

event B: I take airplane to go to Edinburgh

$$P(A|B) = 0$$

$$P(B|A) = 0$$

$$P(A \cap B) = 0$$

$$P(A \cup B) = P(A) + P(B)$$

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - 0 \\ &= P(A) + P(B) \end{aligned}$$

If events A, B are mutually exclusive, can they also be independent?

mutually exclusive $P(A|B) = 0$

event B completely blocks event A from happening

the extreme case of one event making another event less likely to happen

independent $P(A|B) = P(A)$

event B has no effect on event A

No, events A, B can not be both mutually exclusive and independent

Events A, B can be either mutually exclusive, or independent, or neither, but can *not* be both.

(2021) Question 3

(a) Explain in words the meaning of $P(A|B)$

$P(A|B)$: the chance of event A happening, given that event B has already happened

State the relationship between A and B when:

(i) $P(A|B) = P(A)$

events A and B are independent

(ii) $P(A|B) = 0$

events A and B are mutually exclusive

(b) The events A and B are such that $P(A) = 5/8$; $P(A|B) = 1/2$,

$P(A \cup B) = 7/8$

To draw a Venn diagram, we need to know $P(A)$, $P(B)$, $P(A \cap B)$, $P(A \cup B)$. $P(A)$ and $P(A \cup B)$ are already given.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{2}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

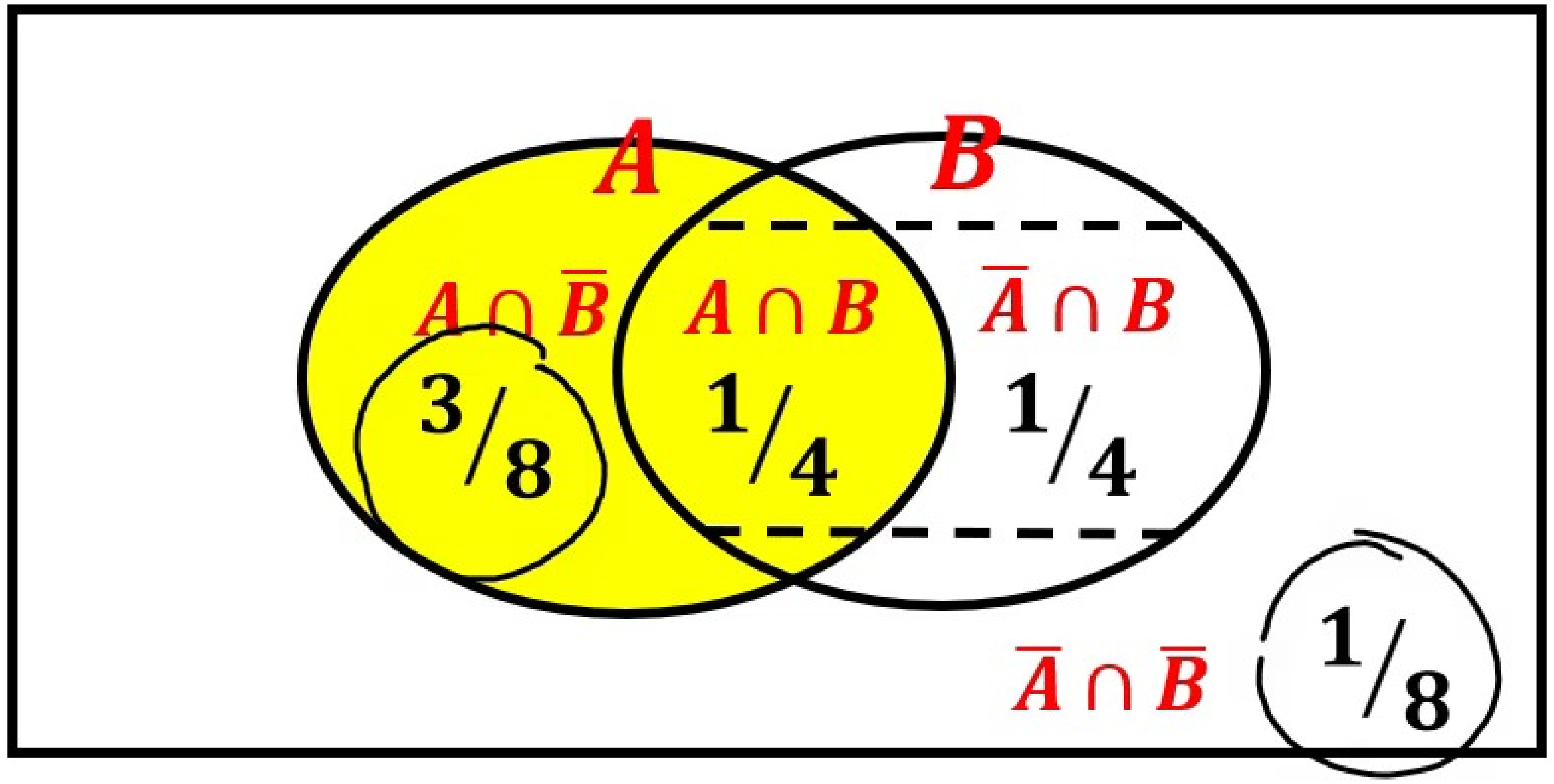
$$P(A \cap B) = \frac{1}{2} P(B)$$

$$\frac{7}{8} = \frac{5}{8} + P(B) - \frac{1}{2} P(B)$$

$$P(A \cap B) = \frac{1}{2} \times \frac{1}{2} = \mathbf{\frac{1}{4}}$$

$$P(B) = \mathbf{\frac{1}{2}}$$

$P(A) = {}^5/_8 \quad P(B) = {}^1/_2 \quad P(A \cap B) = {}^1/_4 \quad P(A \cup B) = {}^7/_8$



$$P(A \cap \bar{B}) = P(A) - P(A \cap B) = {}^3/_8$$

$$P(\bar{A} \cap B) = P(B) - P(A \cap B) = {}^1/_4$$

$$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) = {}^1/_8$$

By drawing a Venn diagram, or otherwise, calculate

(iii) $P\left(A \cup \overline{B}\right) = P(A) + P\left(\overline{B}\right) - P(A \cap \overline{B})$

$$= \frac{5}{8} + \left(1 - \frac{1}{2}\right) - \frac{3}{8} = \mathbf{\frac{3}{4}}$$

(iv) $P(\bar{A} \cap \bar{B}) = \mathbf{\frac{1}{8}}$

(v) Are events A and B independent?

$$P(A) = {}^{5}/_{8} \qquad P(A|B) = {}^{1}/_{2}$$

$P(A) \neq P(A|B)$ $\Longrightarrow$ events A and B are ***not*** independent.

OR

$$P(A)P(B) = \left(\frac{5}{8}\right)\left(\frac{1}{2}\right) = \frac{5}{16}$$

$$P(A \cap B) = \frac{1}{4} = \frac{4}{16}$$

$P(A \cap B) \neq P(A)P(B)$ $\Longrightarrow$ events A and B are ***not*** independent.

(c) Event C is independent of event A and $P(A \cap C) = {}^{1}/_{4}$

Find $P(C|A)$ and $P(C)$

$P(A) = {}^{5}/_{8}$

events C and A are independent, so

$$P(A \cap C) = P(A)P(C)$$

$$\frac{1}{4} = \frac{5}{8}P(C) \quad \Rightarrow \quad P(C) = \mathbf{\frac{2}{5}}$$

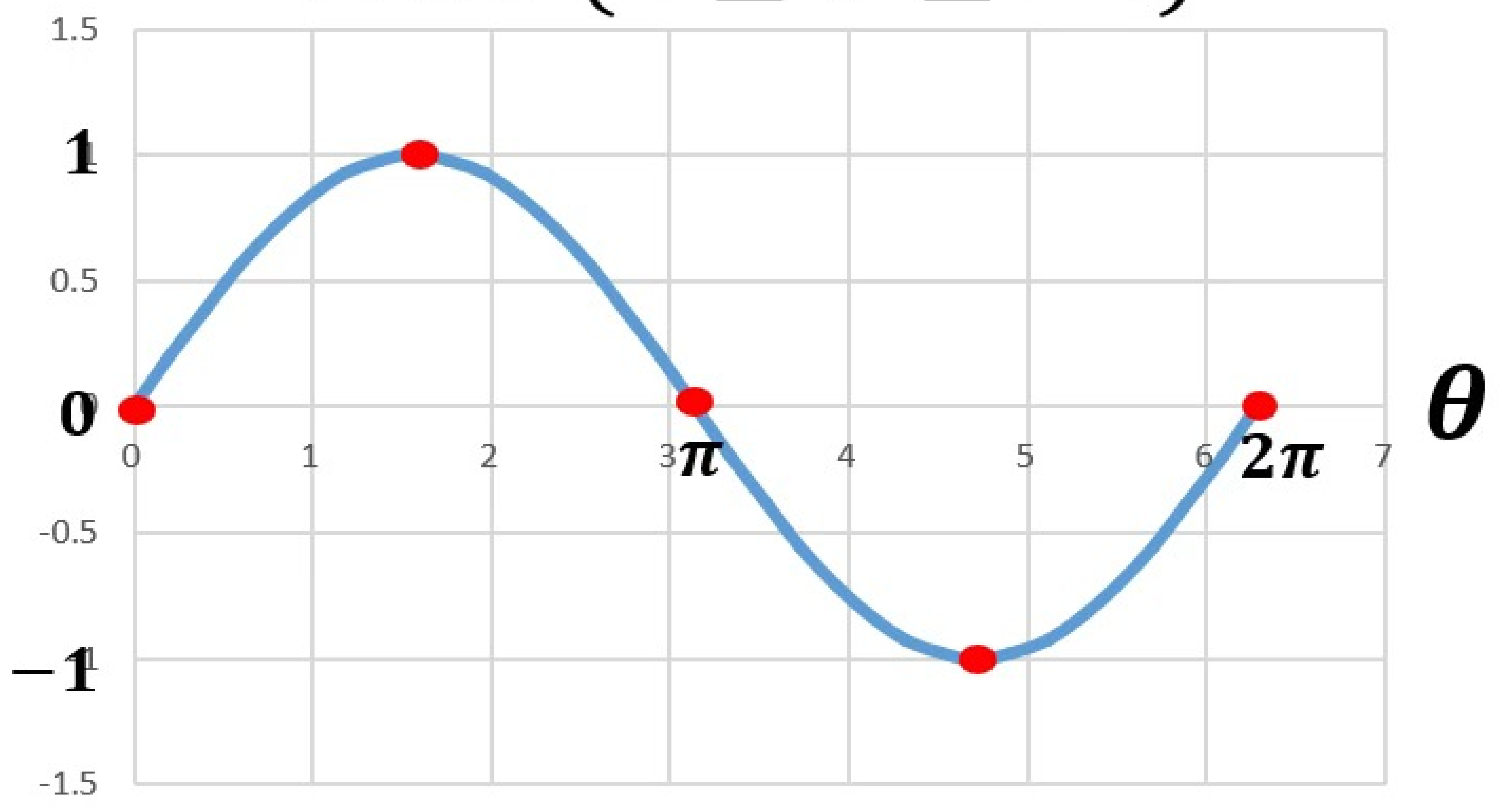
events C and A are independent, so

$$P(C|A) = P(C) = \mathbf{\frac{2}{5}}$$

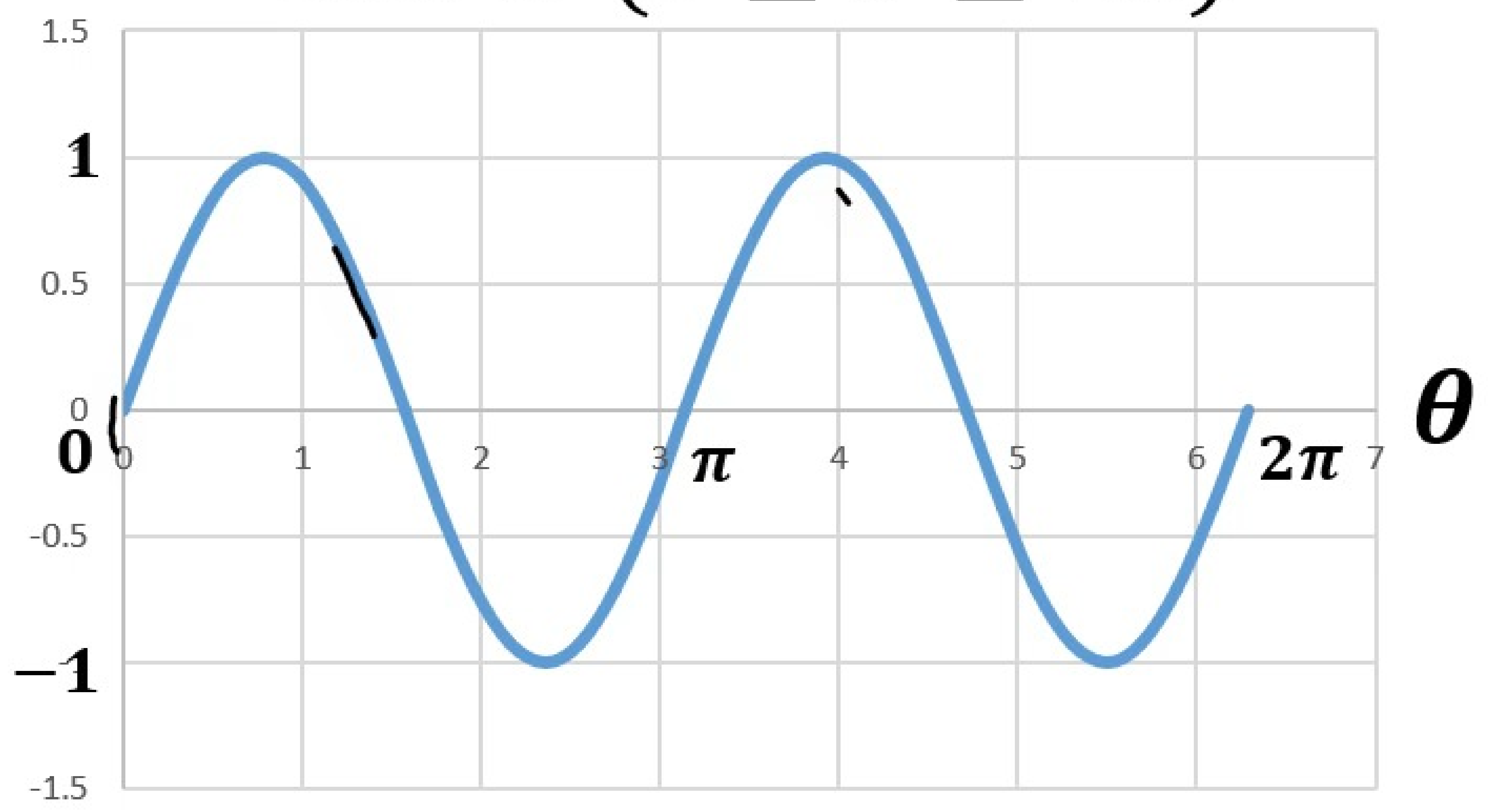
(2021) Question 4

(a) Sketch the following graphs for values of θ, where $\underline{-\pi \leq \theta \leq \pi}$

$\boldsymbol{sin\theta}$ $(0 \leq \theta \leq 2\pi)$

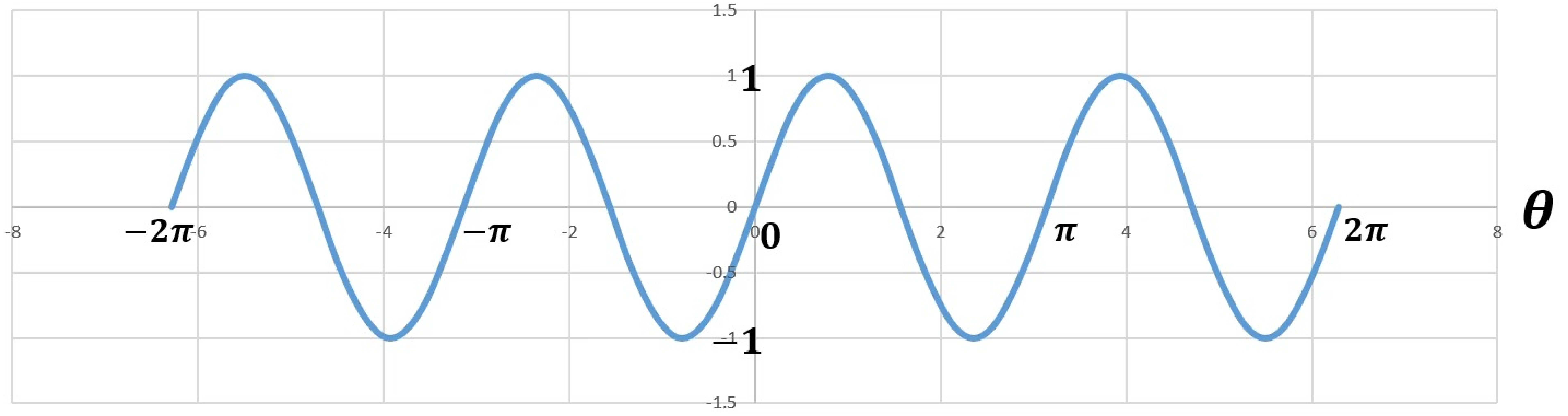


$\boldsymbol{sin2\theta}$ $(0 \leq \theta \leq 2\pi)$



$\boldsymbol{sin2\theta}$

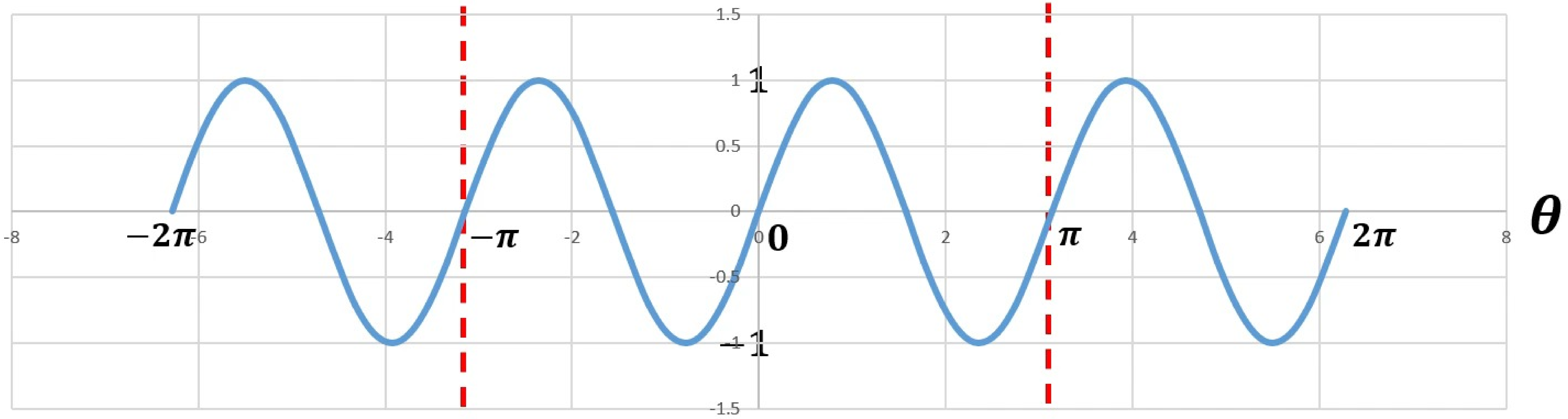
$(\underline{-2\pi \leq} \theta \underline{\leq 2\pi})$



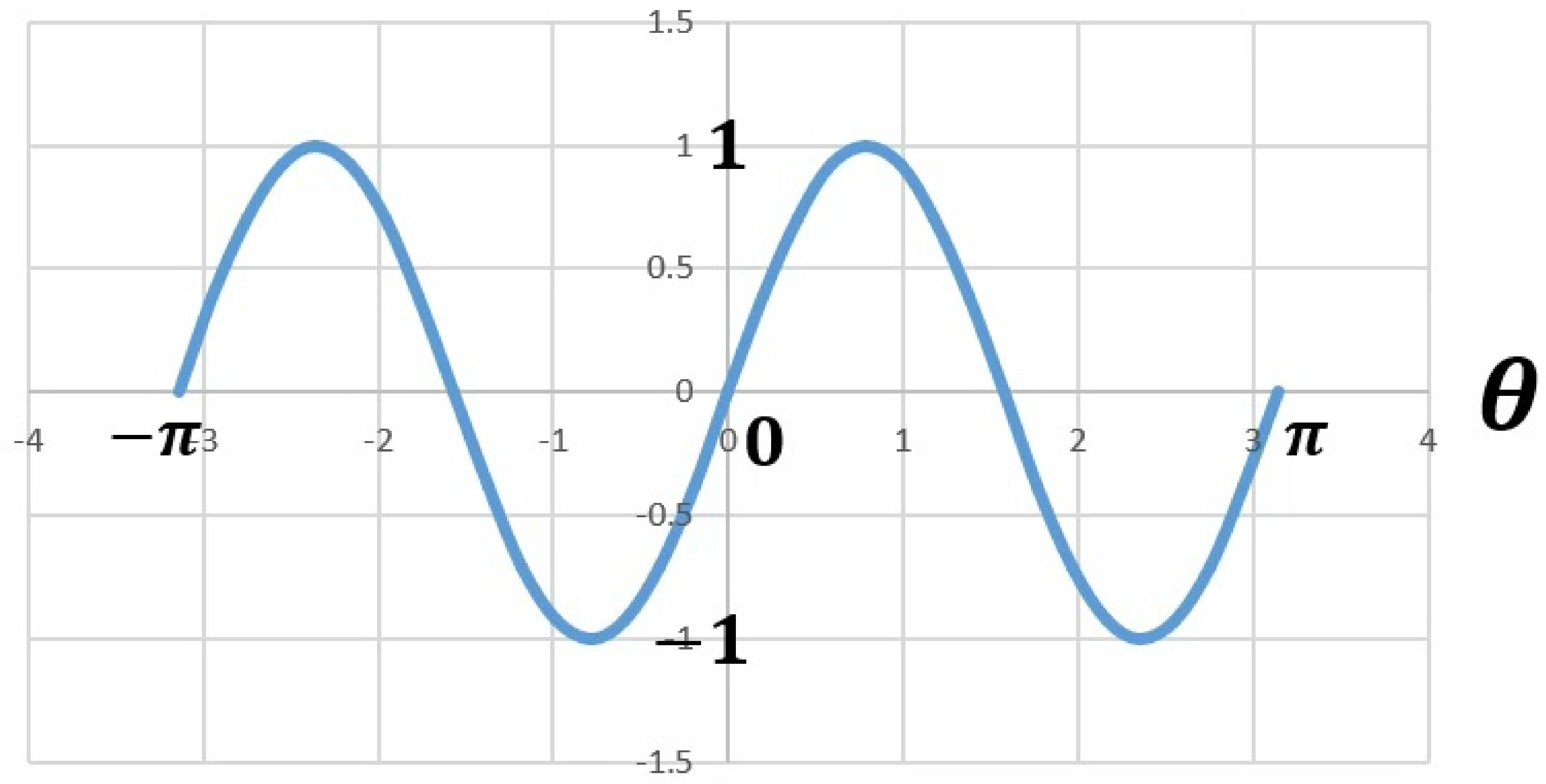
(a) Sketch the following graphs for values of θ, where $-\pi \leq \theta \leq \pi$

$\boldsymbol{sin2\theta}$

$(-2\pi \leq \theta \leq 2\pi)$

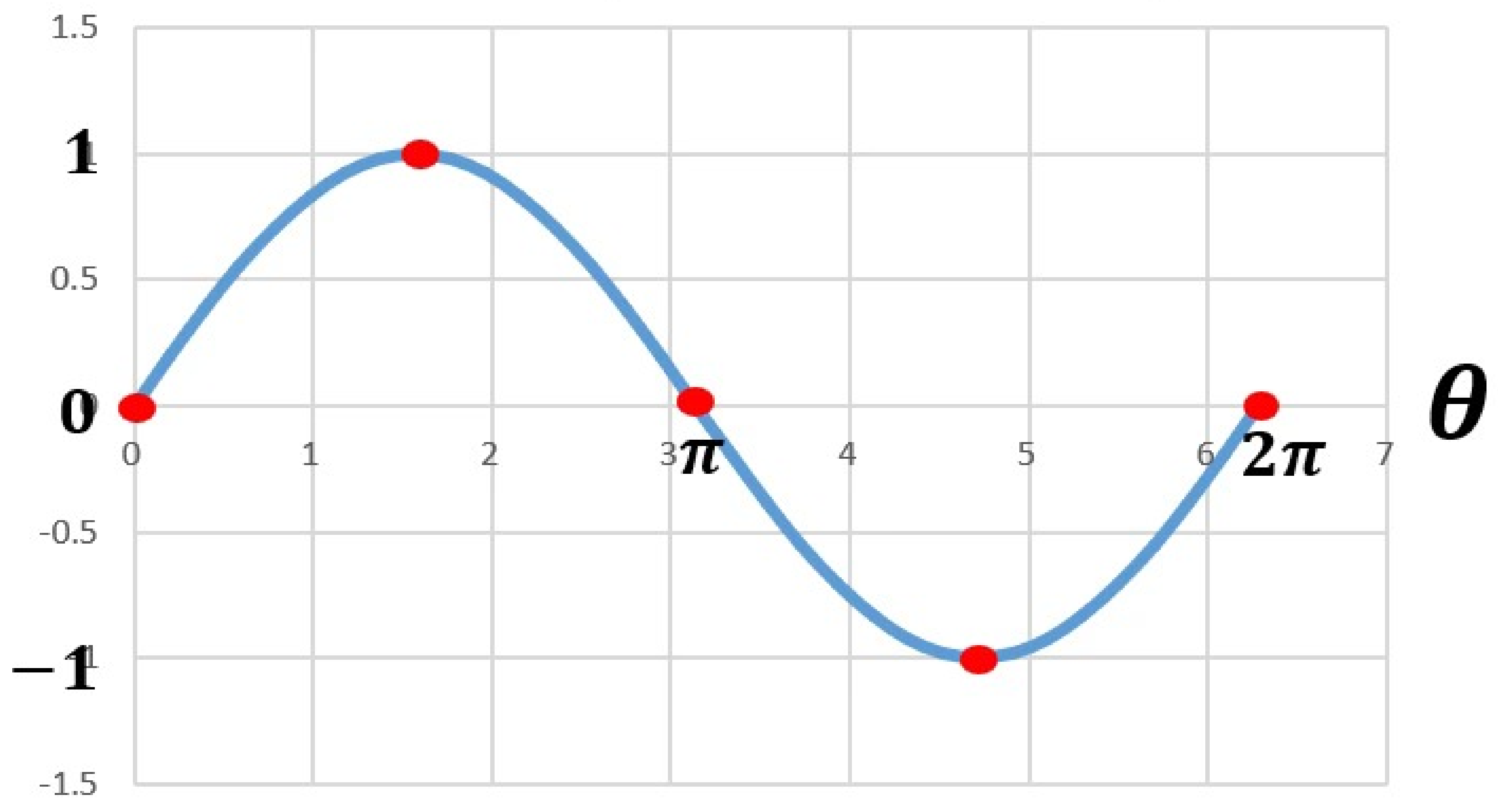


(i) $y = sin2\theta \ (-\pi \leq \theta \leq \pi)$

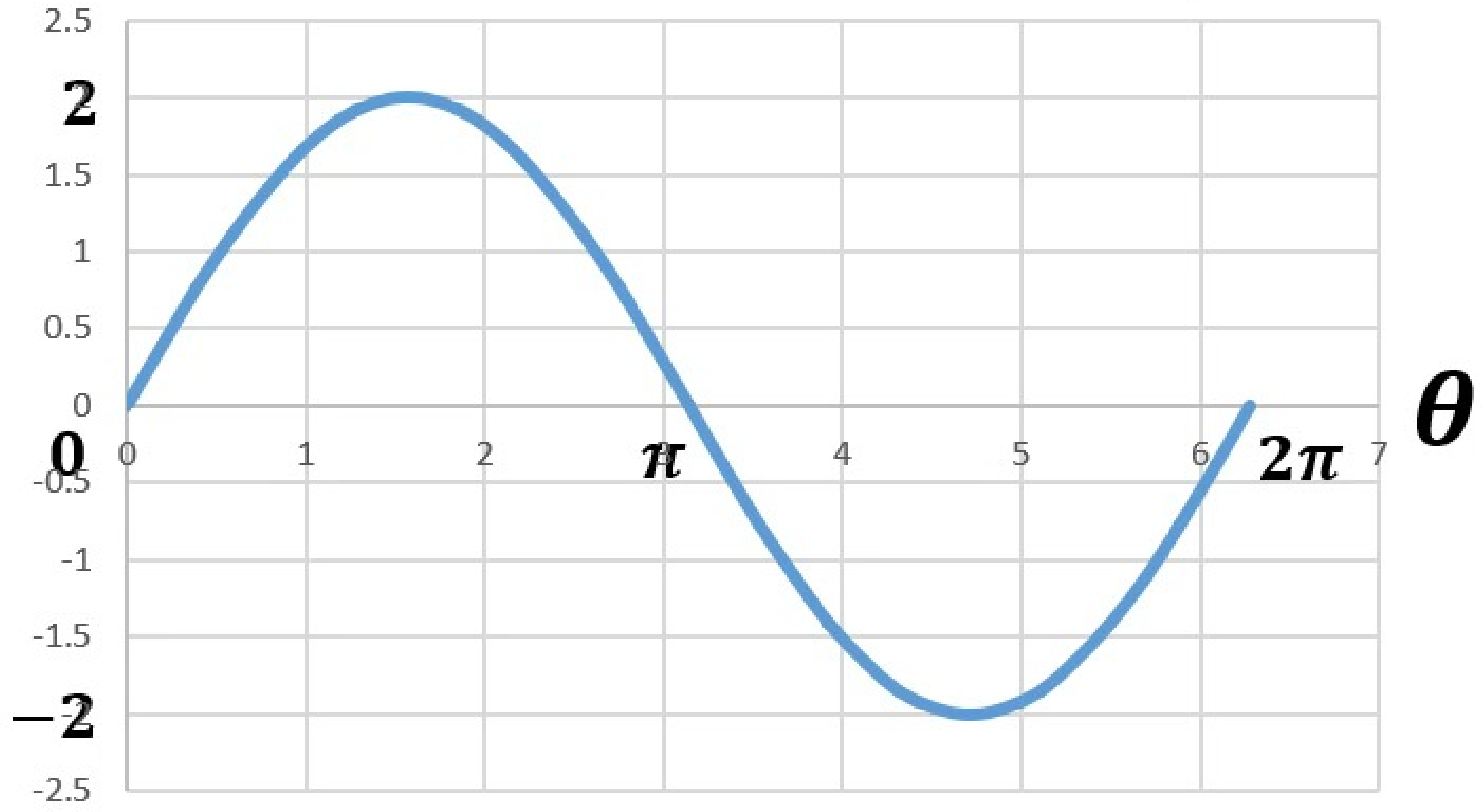


(a) Sketch the following graphs for values of θ, where $-\pi \leq \theta \leq \pi$

$\boldsymbol{sin\theta}$ $(0 \leq \theta \leq 2\pi)$

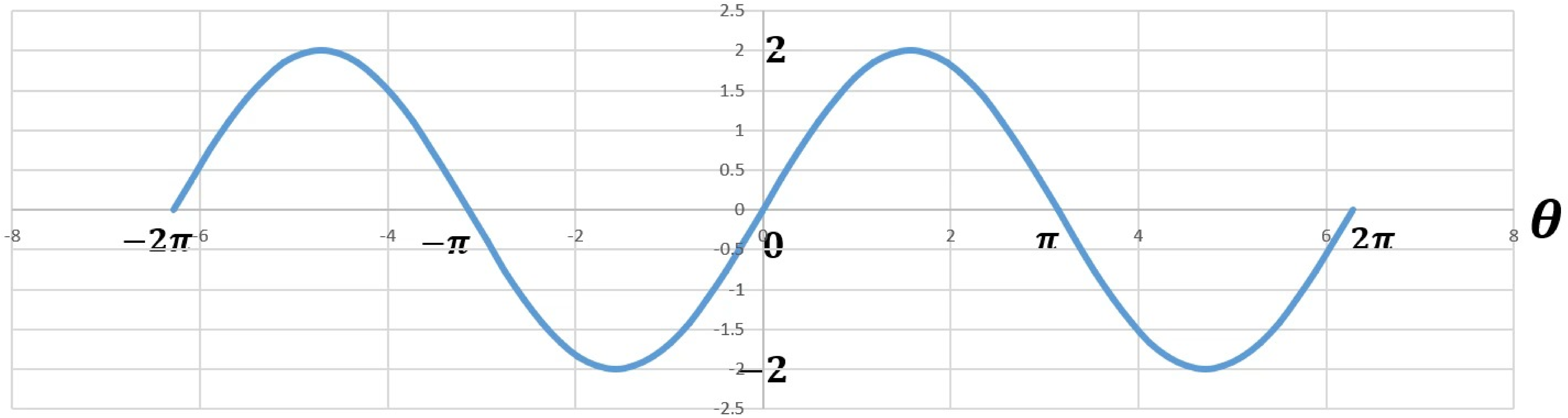


$\boldsymbol{2sin\theta}$ $(0 \leq \theta \leq 2\pi)$



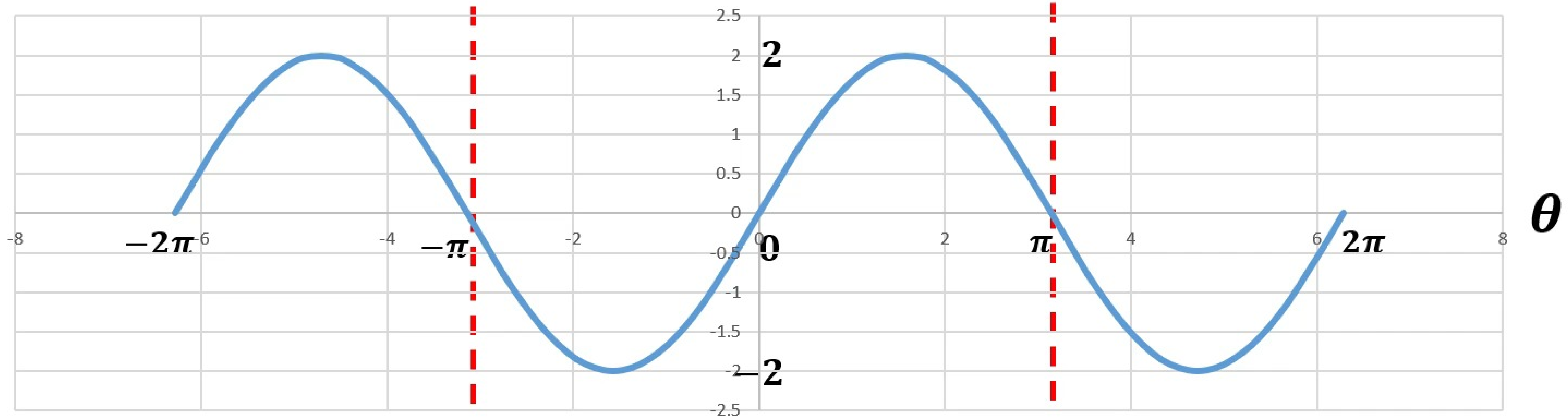
$\boldsymbol{2sin\theta}$

$(-2\pi \leq \theta \leq 2\pi)$

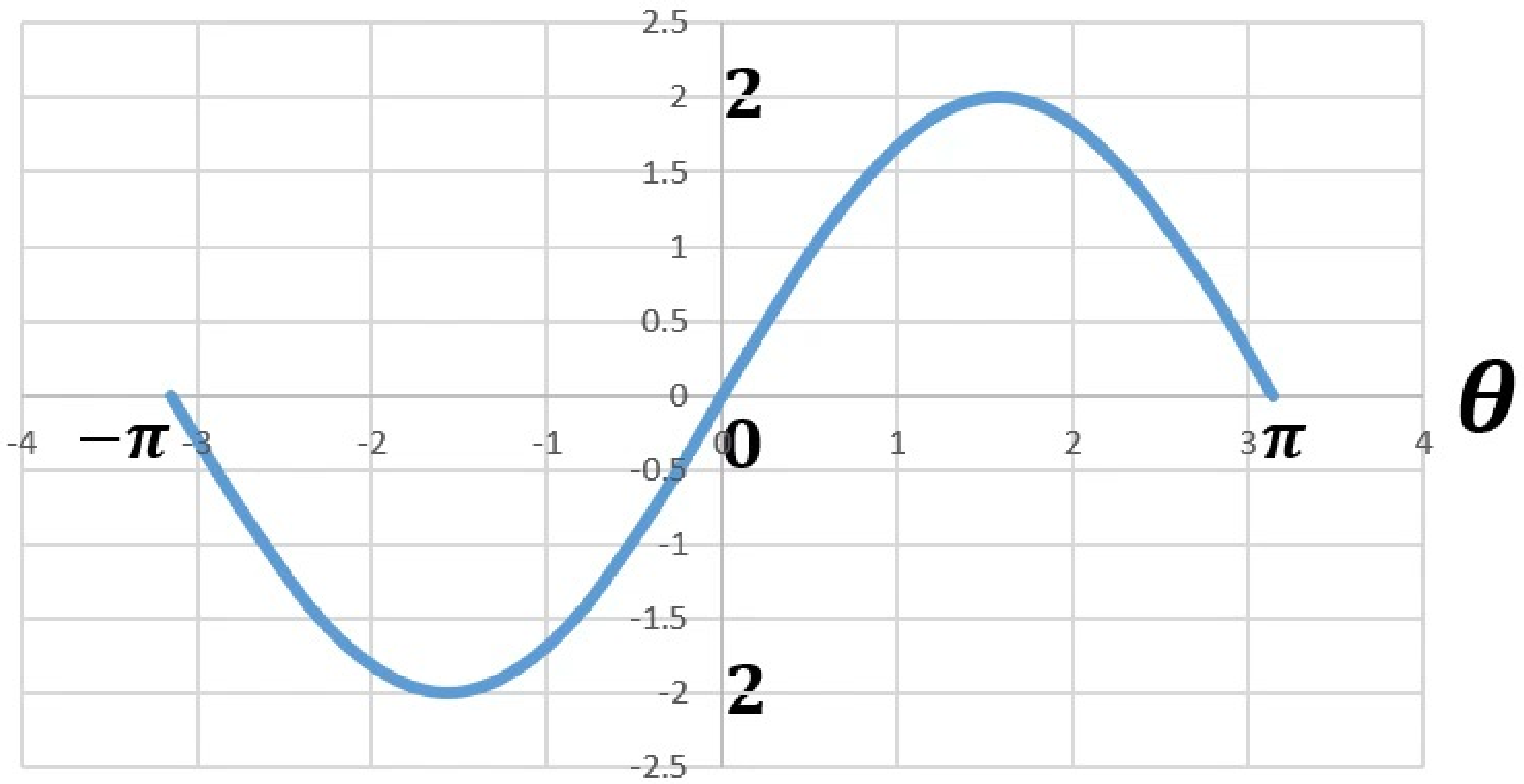


(a) Sketch the following graphs for values of θ, where $-\pi \leq \theta \leq \pi$

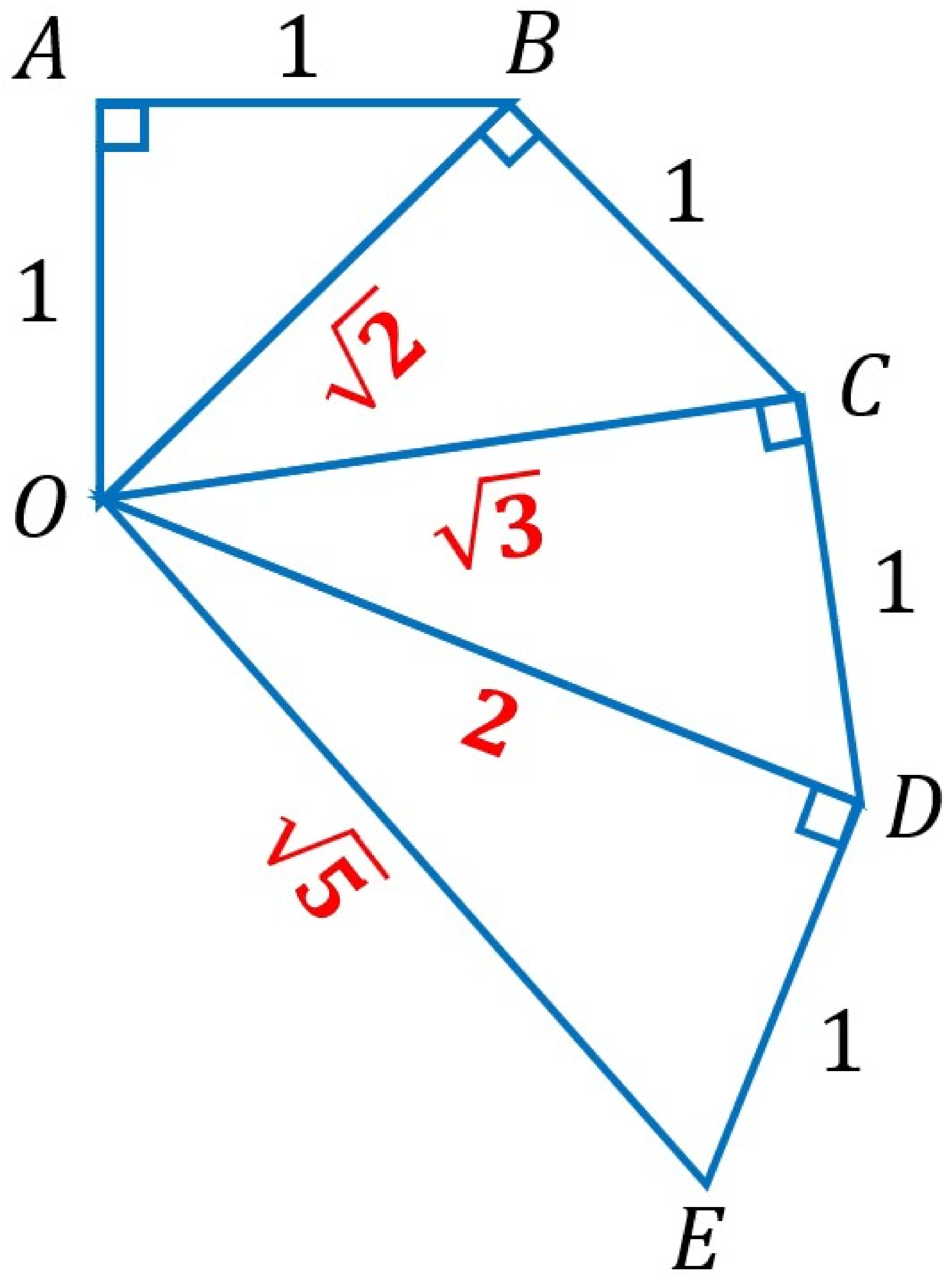
$\boldsymbol{2sin\theta}$
$(-2\pi \leq \theta \leq 2\pi)$



(ii) $y = 2sin\theta$ $(-\pi \leq \theta \leq \pi)$



(b) $OABCDE$ is a hexagon with $OA = AB = BC = CD = DE = 1cm$.



(i) Calculate the lengths OB, OC, OD, OE, correct to $1\ d.p.$

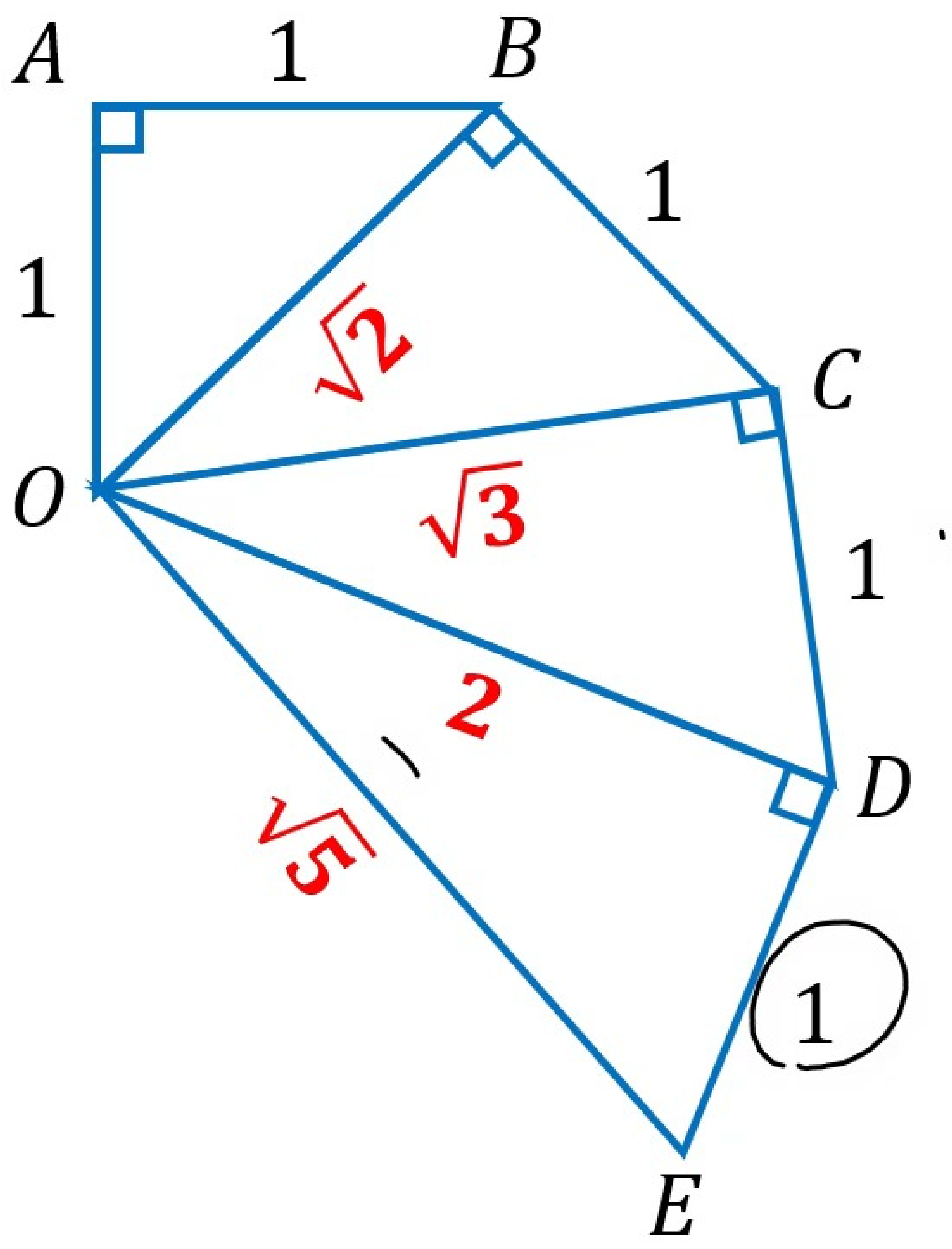
$$OB = \sqrt{1^2 + 1^2} = \sqrt{2} = \mathbf{1.4\ (cm)}$$

$$OC = \sqrt{1^2 + (\sqrt{2})^2} = \sqrt{3} = \mathbf{1.7\ (cm)}$$

$$OD = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = \mathbf{2.0\ (cm)}$$

$$OE = \sqrt{1^2 + 2^2} = \sqrt{5} = \mathbf{2.2\ (cm)}$$

(b) $OABCDE$ is a hexagon with $OA = AB = BC = CD = DE = 1cm$.



(ii) Calculate the perimeter of $OABCDE$

$$OA + AB + BC + CD + DE + EO$$

$$= 1 + 1 + 1 + 1 + 1 + \sqrt{5}$$

$$= 5 + \sqrt{5}$$

$$= \mathbf{7.24\ (cm)}$$

(2021) Question 5

(a) Find the equation of the tangent to the curve $y = x^2 + x + 1$ when it passes through the point $(1, 3)$

$y = mx + c$

find m: $m = f'(x_0) = f'(1)$

$y = f(x) = x^2 + x + 1$

$f'(x) = 2x + 1$

$f'(1) = 2(1) + 1 = 3$

use $(x_0, y_0) = (1, 3)$ to find c

$y = 3x + c$

$3 = 3(1) + c \longrightarrow c = 0$

$\boldsymbol{y = 3x}$

(b) Evaluate $\int_{y=3}^{4} \boxed{\int_{x=1}^{2} (2x+4y)\, dx}dy$

$$\int_{x=1}^{2} (2x+4y)\, dx = 2\left(\frac{x^2}{2}\right) + 4xy = x^2 + 4xy \,\Big|_{\boldsymbol{x=1}}^{\mathbf{2}}$$

$$= (2^2 + 4(2)y) - (1^2 + 4(1)y) = \boxed{3+4y}$$

$$\int_{y=3}^{4} (3+4y)\, dy = 3y + 4\left(\frac{y^2}{2}\right) = 3y + 2y^2 \,\Big|_{\boldsymbol{y=3}}^{\mathbf{4}}$$

$$= (3(4) + 2(4^2)) - (3(3) + 2(3^2)) = \mathbf{17}$$