

COMP3250

Exam Revision Session (1)

(2021) Question 1

(a) Solve the simultaneous equations:

$$\begin{cases} 3x + 2y = 19 & (1) \times 2 \\ 2x + 5y = -2 & (2) \times 3 \end{cases}$$

$$\begin{cases} 6x + 4y = 38 \\ 6x + 15y = -6 \end{cases} \quad (-)$$

$$-11y = 44$$

$$\mathbf{y = -4} \quad (3)$$

Substitute (3) into (2)

$$2x + 5(-4) = -2$$

$$2x - 20 = -2$$

$$2x = 18$$

$$\mathbf{x = 9}$$

$$(b) M = \begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix} \quad N = \begin{pmatrix} 19 \\ -2 \end{pmatrix}$$

Find:

$$(i) M^{-1}$$

$$|M| = (3)(5) - (2)(2) = 15 - 4 = 11$$

$$M^{-1} = \frac{1}{11} \begin{pmatrix} 5 & -2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} \mathbf{5/11} & \mathbf{-2/11} \\ \mathbf{-2/11} & \mathbf{3/11} \end{pmatrix}$$

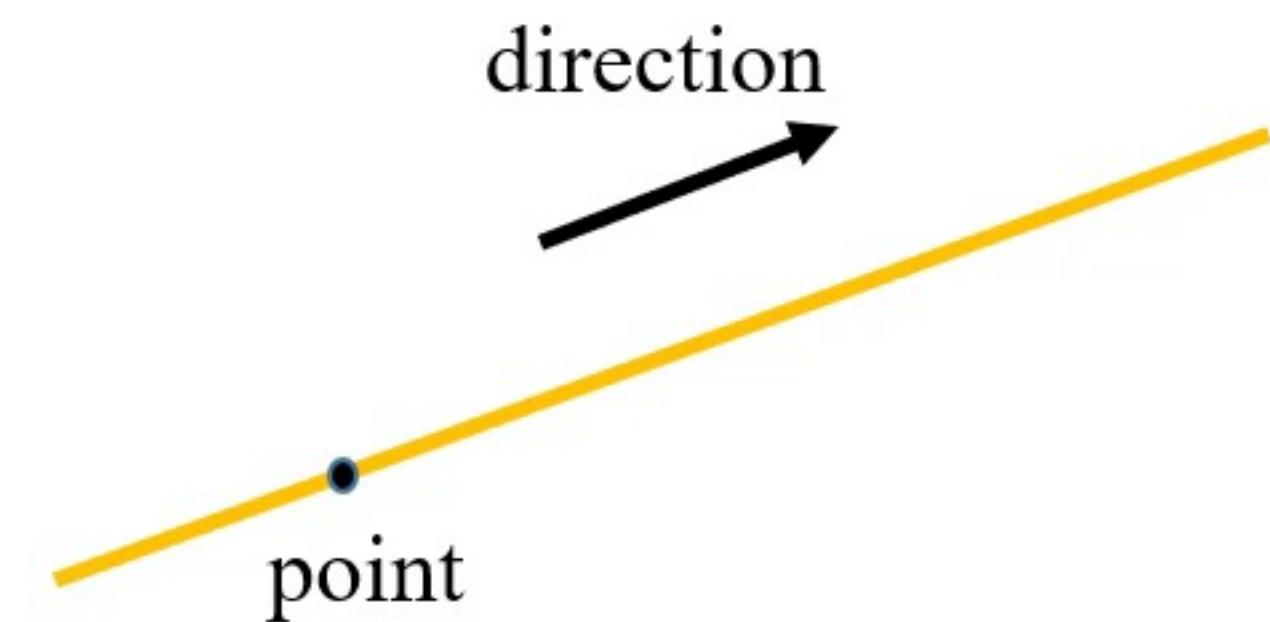
$$(ii) M^{-1}N$$

$$\text{[calculator icon]} \quad M^{-1}N = \frac{1}{11} \begin{pmatrix} 5 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 19 \\ -2 \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 95 + 4 \\ -38 - 6 \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 99 \\ -44 \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \end{pmatrix}$$

(c) $A = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$ $B = \begin{pmatrix} 3 & 5 \\ 7 & 9 \end{pmatrix}$ Find AB and \underline{BA} .

$$AB = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 7 & 9 \end{pmatrix} = \begin{pmatrix} 6 + 28 & 10 + 36 \\ 18 + 56 & 30 + 72 \end{pmatrix} = \begin{pmatrix} 34 & 46 \\ 74 & 102 \end{pmatrix}$$

$$BA = \begin{pmatrix} 3 & 5 \\ 7 & 9 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 6 + 30 & 12 + 40 \\ 14 + 54 & 28 + 72 \end{pmatrix} = \begin{pmatrix} 36 & 52 \\ 68 & 100 \end{pmatrix}$$



A line is determined by a point and a direction

Vector
form

$$\underline{r} = i - 3j + 8k + \lambda(2i + 7j + 5k)$$

Parametric form

$$\begin{aligned}x &= 1 + 2\lambda \\y &= -3 + 7\lambda \\z &= 8 + 5\lambda\end{aligned}$$



Cartesian form

$$\frac{x - 1}{2} = \frac{y - (-3)}{7} = \frac{z - 8}{5}$$

(2021) Question 2

$$L_1: \frac{x+2}{4} = \frac{y-2}{3} = z-1$$

$$L_2: \underline{r} = 3i + 4j - 3k + \mu(i + j + k)$$

(a) Convert both equations to their parametric form.

$$L_1: \mathbf{x} = -2 + 4\lambda$$

$$\mathbf{y} = 2 + 3\lambda$$

$$\mathbf{z} = 1 + \lambda$$

$$L_2: \mathbf{x} = 3 + \mu$$

$$\mathbf{y} = 4 + \mu$$

$$\mathbf{z} = -3 + \mu$$

(b) Find the point of intersection of the two lines.

$$L_1: \begin{aligned} x &= -2 + 4\lambda \\ y &= 2 + 3\lambda \\ z &= 1 + \lambda \end{aligned}$$

$$L_2: \begin{aligned} x &= 3 + \mu \\ y &= 4 + \mu \\ z &= -3 + \mu \end{aligned}$$

$$\left\{ \begin{array}{l} -2 + 4\lambda = 3 + \mu \\ 2 + 3\lambda = 4 + \mu \\ -4 + \lambda = -1 \end{array} \right. \quad (-)$$

$\lambda = 3$

$$\begin{aligned} x &= -2 + 4(3) = 10 \\ y &= 2 + 3(3) = 11 \\ z &= 1 + 3 = 4 \end{aligned}$$

the point of intersection
is **(10, 11, 4)**

(c) Find the size of the angle between the two lines.

$$L_1: \frac{x+2}{4} = \frac{y-2}{3} = z-1 \quad \underline{\underline{a}} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$L_2: \underline{\underline{r}} = 3\mathbf{i} + 4\mathbf{j} - 3\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k}) \quad \underline{\underline{b}} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\underline{\underline{a}} \cdot \underline{\underline{b}} = |\underline{\underline{a}}| |\underline{\underline{b}}| \cos\theta$$

$$4 \times 1 + 3 \times 1 + 1 \times 1 = \sqrt{4^2 + 3^2 + 1^2} \sqrt{1^2 + 1^2 + 1^2} \cos\theta$$

$$8 = \sqrt{26}\sqrt{3} \cos\theta$$

$$\cos\theta = \frac{8}{\sqrt{26}\sqrt{3}}$$

$$\theta = \cos^{-1} \left(\frac{8}{\sqrt{26}\sqrt{3}} \right) = \mathbf{25.07^\circ}$$

Conditional Probability

$P(A|B)$: probability of event A happening, given that event B has already happened
condition

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{← condition}$$

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

events A, B
both happen

event B happens first,
then event A happens

event A happens first,
then event B happens

Conditional Probability

Why do we study conditional probability?

Because events affect each other: when one event happens, it makes another event more likely to happen or less likely to happen

Example 1: computer plays chess with a human player

{ **event A:** computer wins
event B: computer learned your machine learning algorithm to play chess

$P(A|B)$: the chance that computer wins, given that computer has learned your machine learning algorithm to play chess



Your algorithm increases the chance that computer wins $P(A|B) > P(A)$

Example 2:

event A: my computer is infected by computer virus

event B: I installed an anti-virus software on my computer

$P(A|B)$: the chance that my computer is infected by virus, given that I have installed an anti-virus software on my computer

The anti-virus software **decreases** the chance that my computer is infected by virus $P(A|B) < P(A)$

Example 3:

event A: I flip coin #2 and get a head

event B: I flipped coin #1 and got a head

$P(A|B)$: the chance that I get a head on coin #2, given that I have already got a head on coin #1

$$P(A|B) = P(A)$$

events A, B are independent

events A, B are independent

$$P(A|B) = P(A)$$

$$P(B|A) = P(B)$$

$$P(A \cap B) = P(A)P(B)$$

$$P(A|\textcolor{red}{B}) = \frac{P(A \cap B)}{P(\textcolor{red}{B})}$$

$$P(A \cap B) = P(A|B)P(B) = P(A)P(B)$$

events A, B are mutually exclusive

Example event A: I take train to go to Edinburgh

event B: I take airplane to go to Edinburgh

$$P(A|B) = 0$$

$$P(B|A) = 0$$

$$P(A \cap B) = 0$$

$$P(A \cup B) = P(A) + P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - 0$$

$$= P(A) + P(B)$$



If events A, B are mutually exclusive, can they also be independent?

mutually exclusive $P(A|B) = 0$

event B completely blocks event A from happening

the extreme case of one event making another event less likely to happen

independent

$$P(A|B) = P(A)$$

event B has no effect on event A

No, events A, B can not be both mutually exclusive and independent

Events A, B can be either mutually exclusive, or independent, or neither, but can *not* be both.

(2021) Question 3

(a) Explain in words the meaning of $P(A|B)$

$P(A|B)$: the chance of event A happening, given that event B has already happened

State the relationship between A and B when:

(i) $P(A|B) = P(A)$

events A and B are independent

(ii) $P(A|B) = 0$

events A and B are mutually exclusive

(b) The events A and B are such that $P(A) = \underline{5/8}$; $P(A|B) = \underline{1/2}$,
 $P(A \cup B) = \underline{7/8}$

To draw a Venn diagram, we need to know $P(A)$, $P(B)$,
 $P(A \cap B)$, $P(A \cup B)$. **$P(A)$ and $P(A \cup B)$ are already given.**

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{2}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

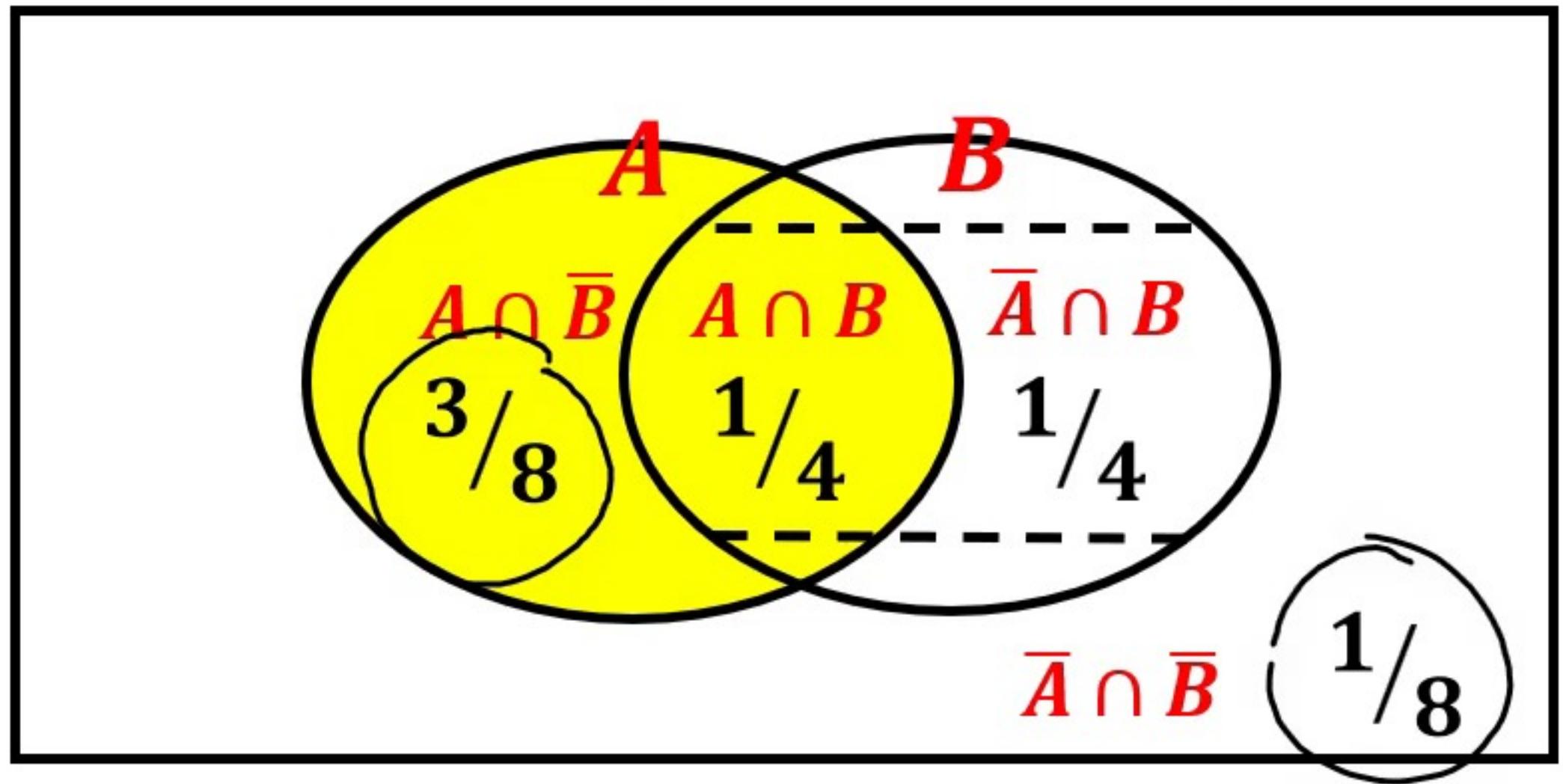
$$P(A \cap B) = \frac{1}{2}P(B)$$

$$\text{P}\left(\begin{array}{l} \text{16/28} \\ \text{P}(A \cap B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \end{array}\right)$$

$$\frac{7}{8} = \frac{5}{8} + P(B) - \frac{1}{2}P(B)$$

$$P(B) = \frac{1}{2}$$

$$P(A) = \underbrace{5/8}_{1/2} \quad P(B) = \underbrace{1/2}_{1/4} \quad P(A \cap B) = 1/4 \quad P(A \cup B) = 7/8$$



$$P(A \cap \bar{B}) = P(A) - P(A \cap B) = 3/8$$

$$P(\bar{A} \cap B) = P(B) - P(A \cap B) = 1/4$$

$$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) = 1/8$$

By drawing a Venn diagram, or otherwise, calculate

$$(iii) P(A \cup \bar{B}) = P(A) + P(\bar{B}) - P(A \cap \bar{B})$$

$$(iv) P(\bar{A} \cap \bar{B}) = \frac{1}{8}$$

$$= \frac{5}{8} + \left(1 - \frac{1}{2}\right) - \frac{3}{8} = \frac{3}{4}$$

(v) Are events A and B independent?

$$P(A) = \underline{5/8} \quad P(A|B) = 1/2$$

$P(A) \neq P(A|B)$ \longrightarrow events A and B are **not** independent.

OR

$$P(A)P(B) = \left(\frac{5}{8}\right)\left(\frac{1}{2}\right) = \frac{5}{16}$$

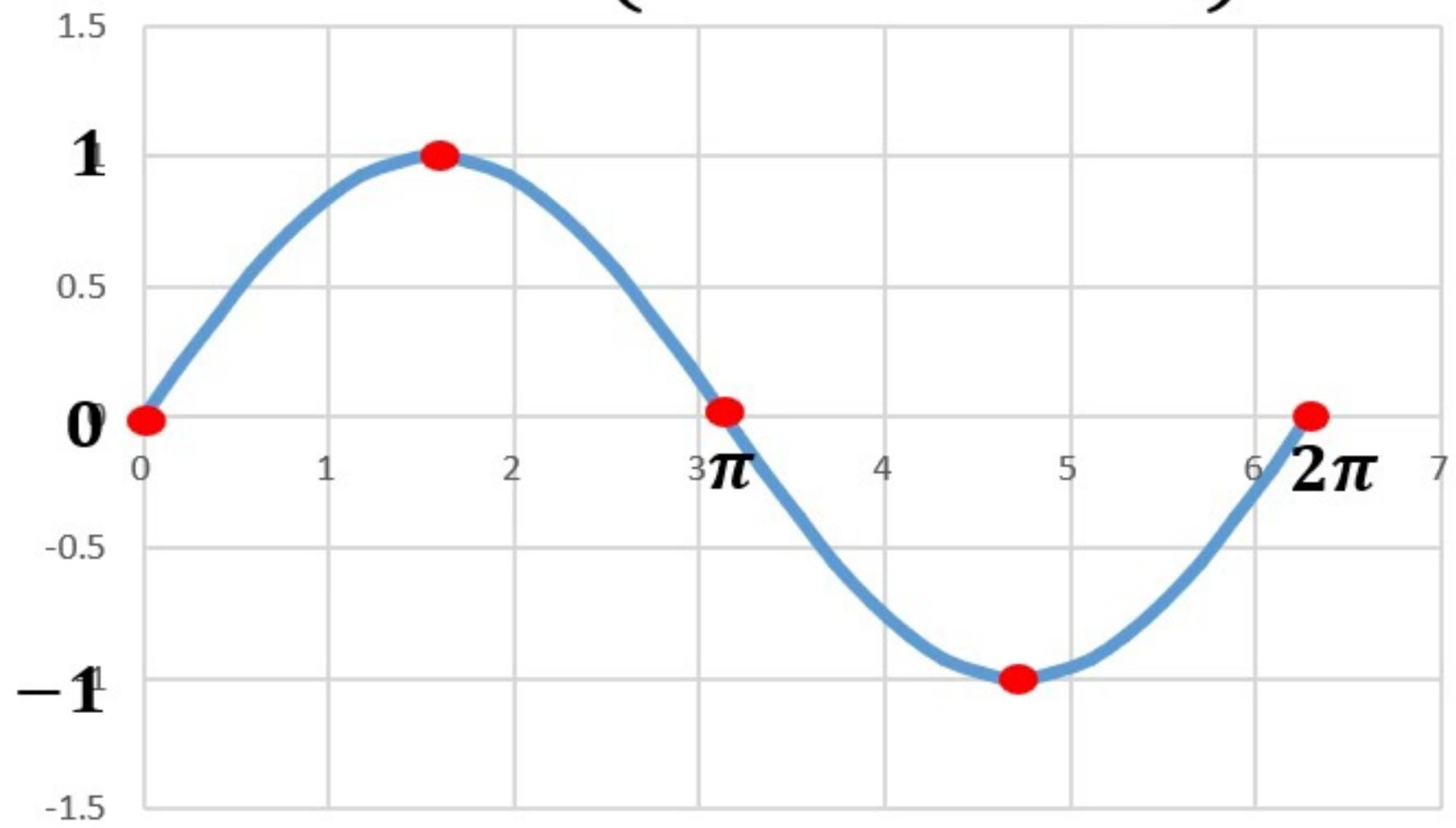
$$P(A \cap B) = \frac{1}{4} = \frac{4}{16}$$

 $P(A \cap B) \neq P(A)P(B)$ \longrightarrow events A and B are **not** independent.

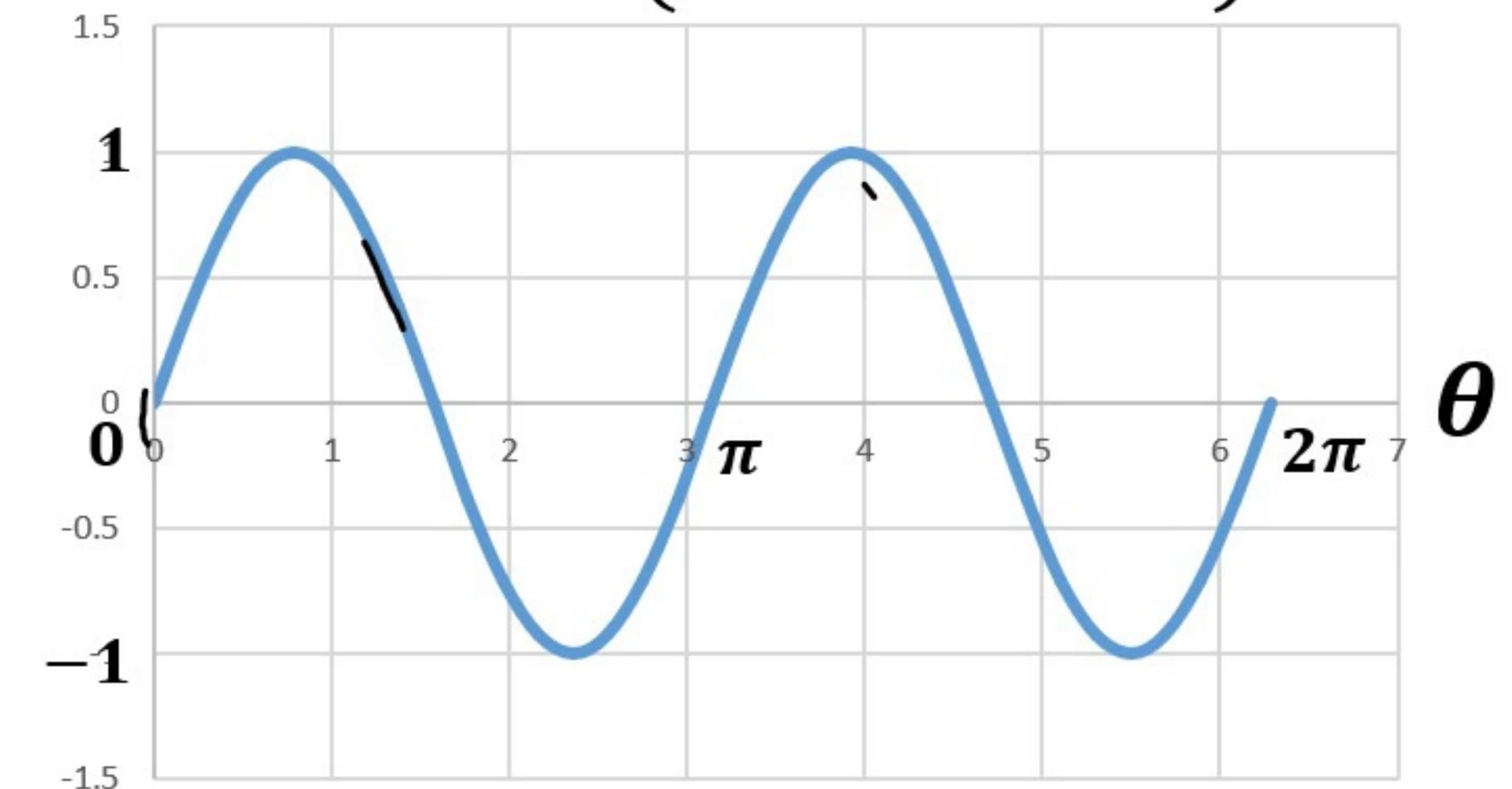
(2021) Question 4

(a) Sketch the following graphs for values of θ , where $-\pi \leq \theta \leq \pi$

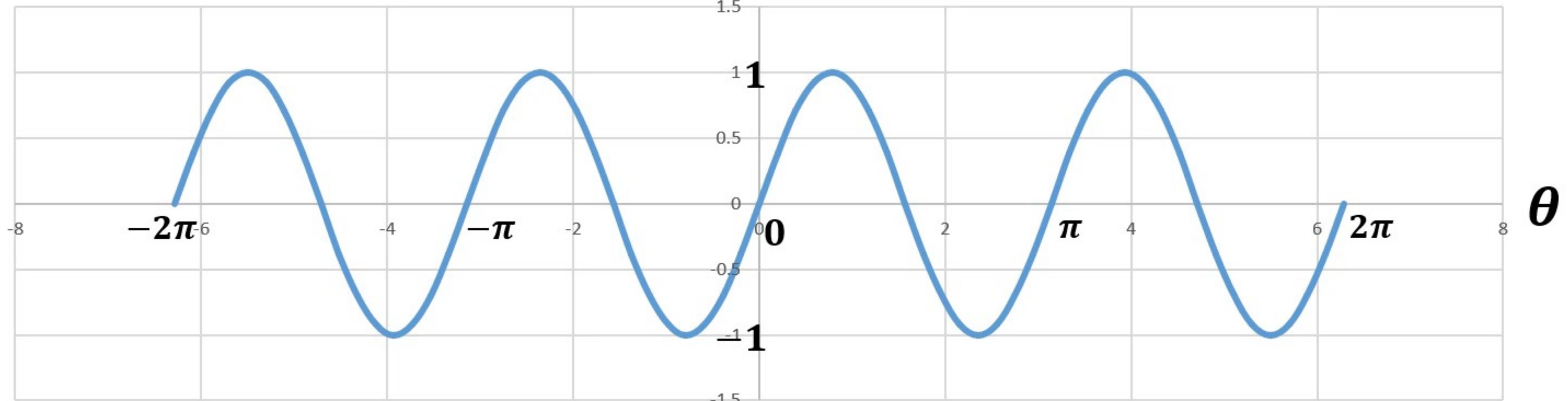
$$\sin\theta \quad (0 \leq \theta \leq 2\pi)$$



$$\sin 2\theta \quad (0 \leq \theta \leq 2\pi)$$

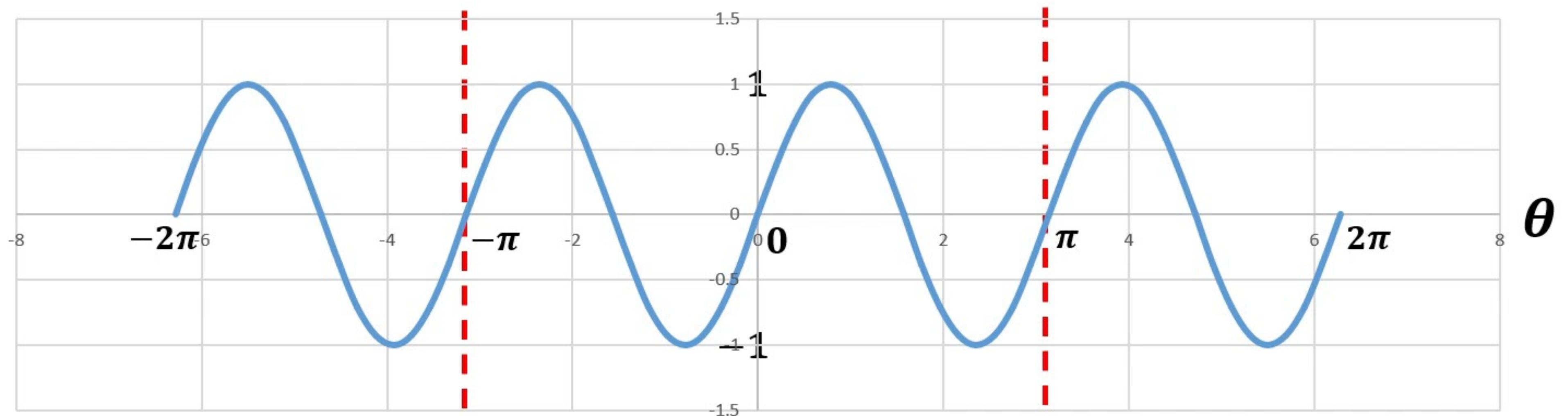


$$\sin 2\theta \quad (-2\pi \leq \theta \leq 2\pi)$$

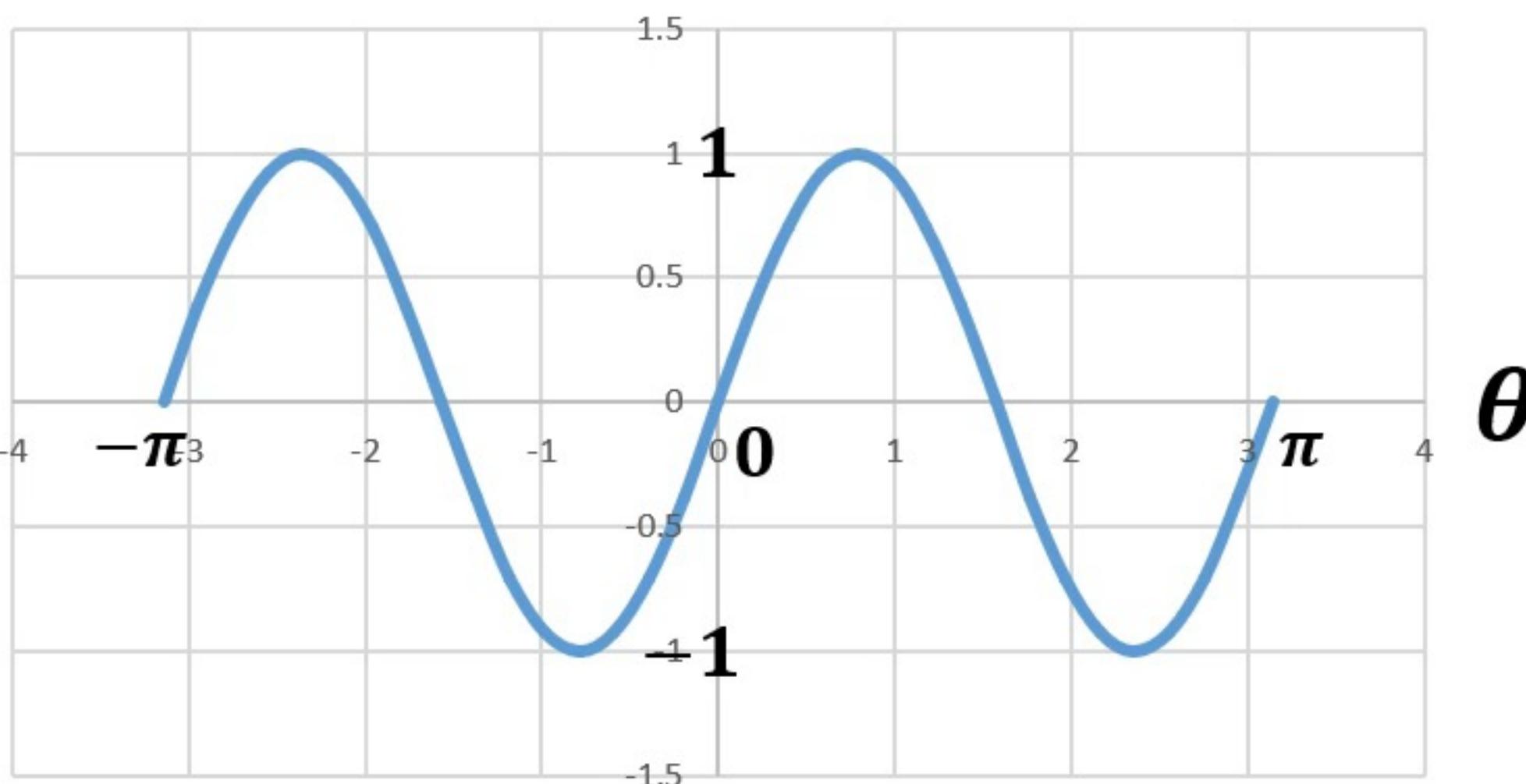


(a) Sketch the following graphs for values of θ , where $-\pi \leq \theta \leq \pi$

$\sin 2\theta$
 $(-2\pi \leq \theta \leq 2\pi)$

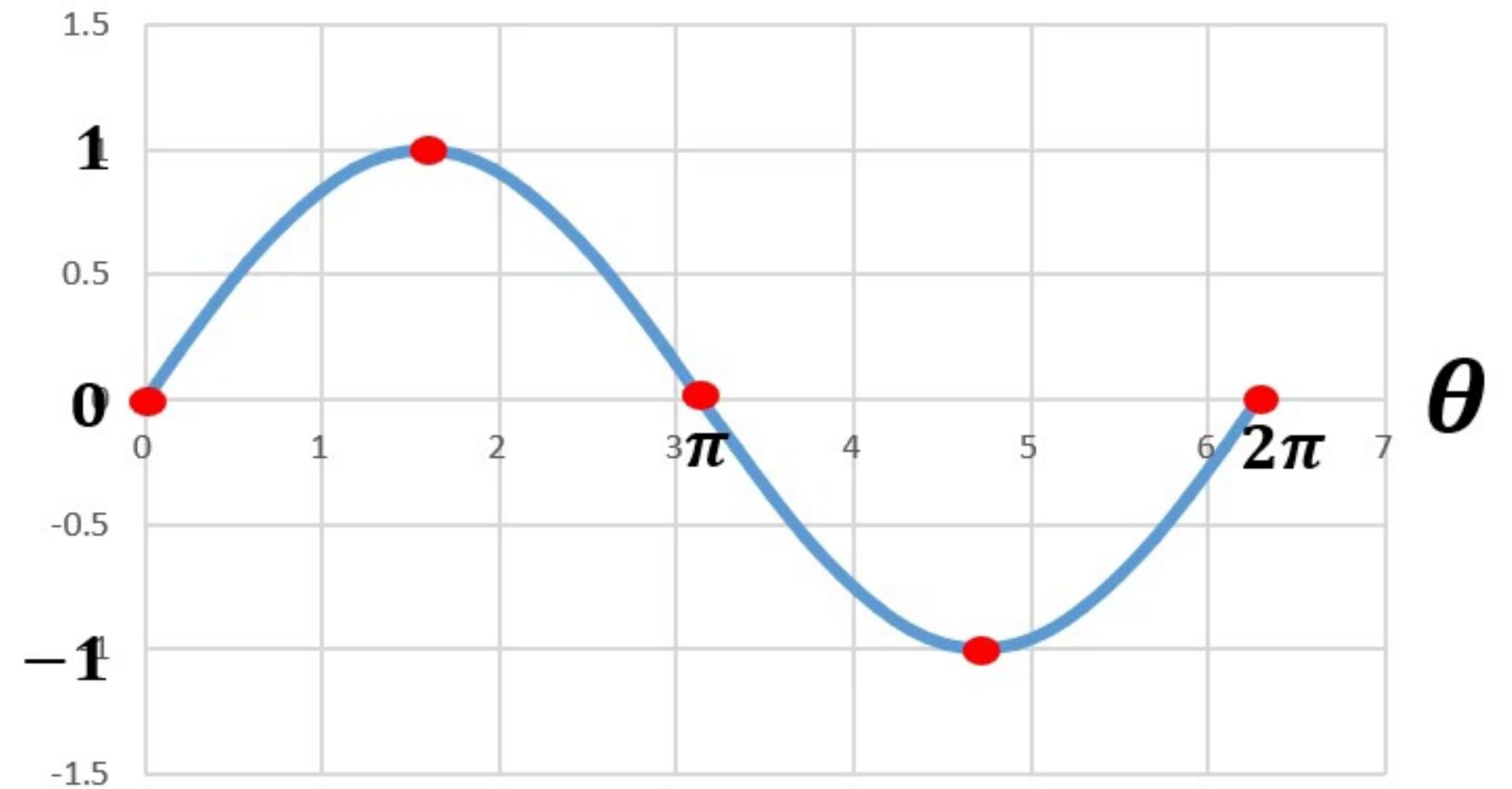


(i) $y = \sin 2\theta$ ($-\pi \leq \theta \leq \pi$)

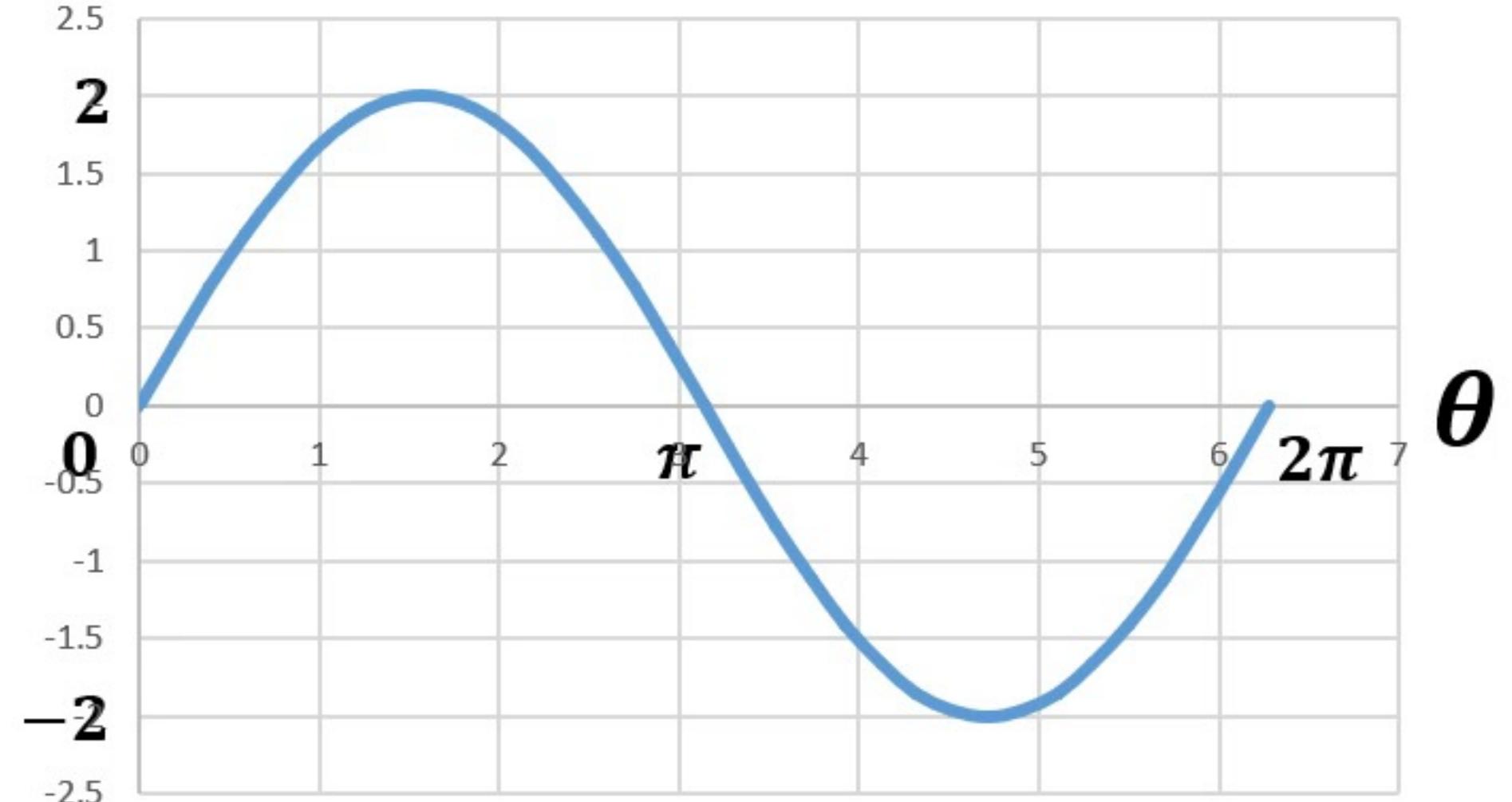


(a) Sketch the following graphs for values of θ , where $-\pi \leq \theta \leq \pi$

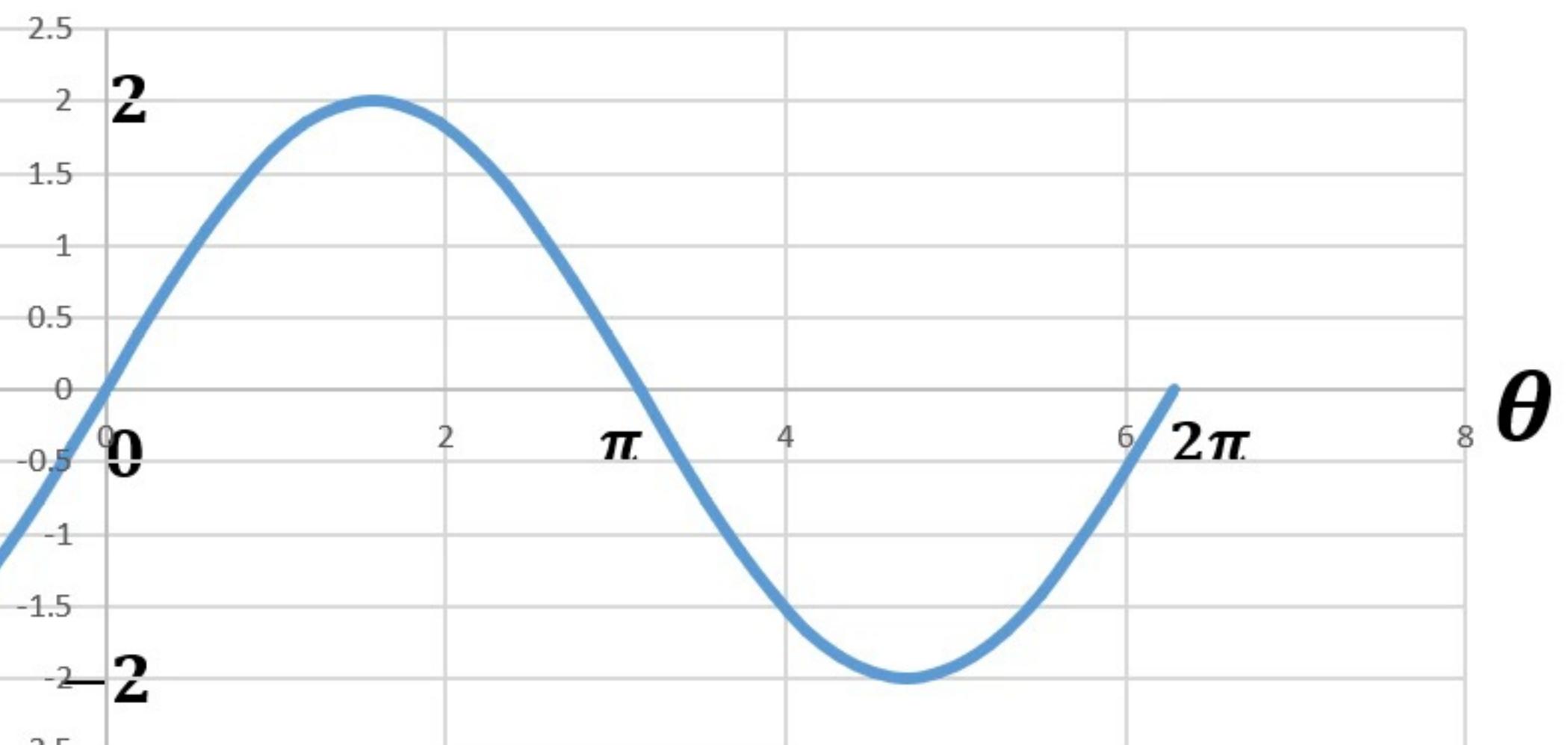
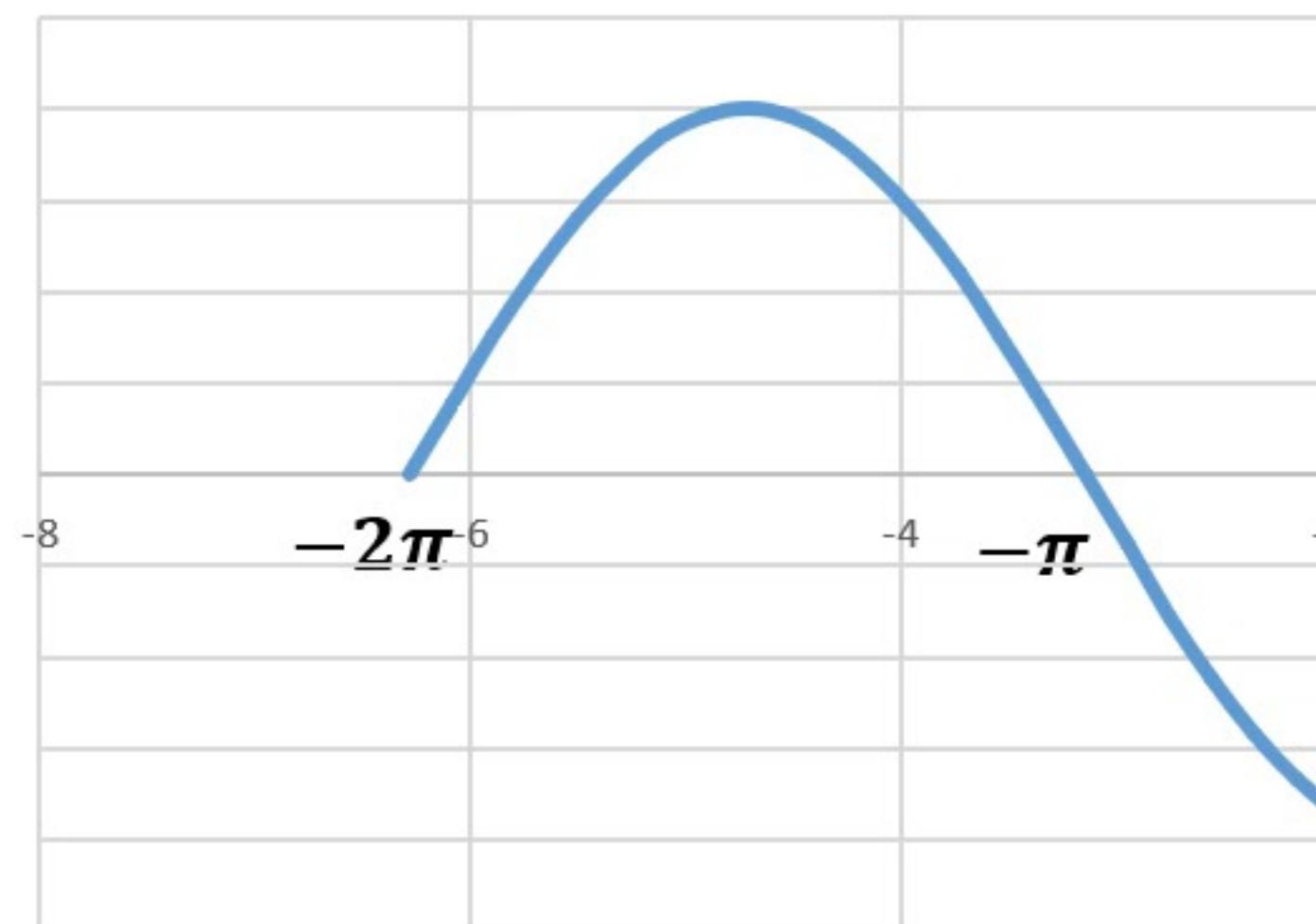
$$\sin\theta \quad (0 \leq \theta \leq 2\pi)$$



$$2\sin\theta \quad (0 \leq \theta \leq 2\pi)$$

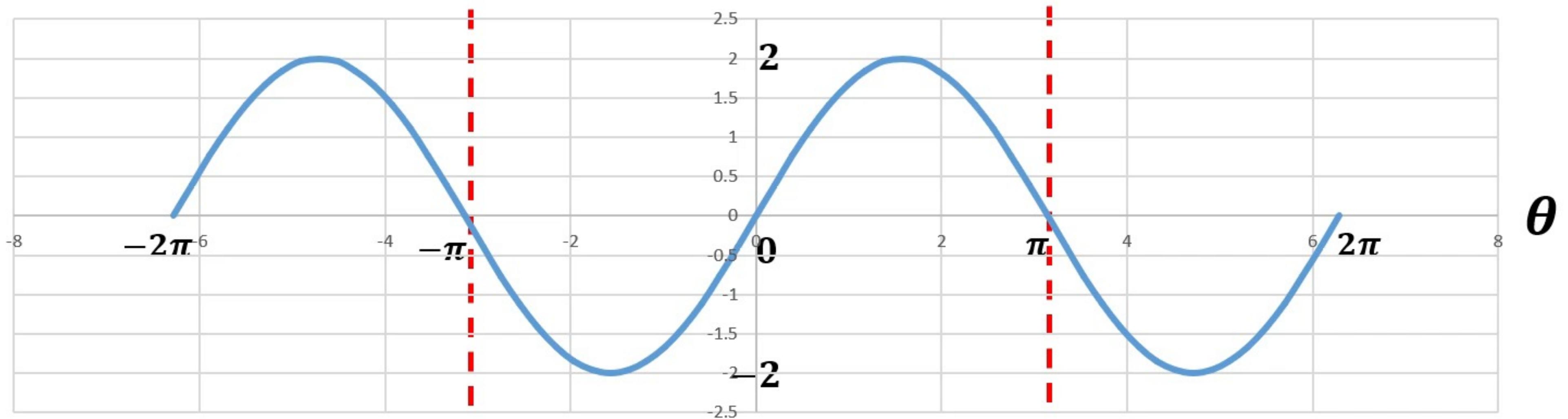


$$2\sin\theta \quad (-2\pi \leq \theta \leq 2\pi)$$

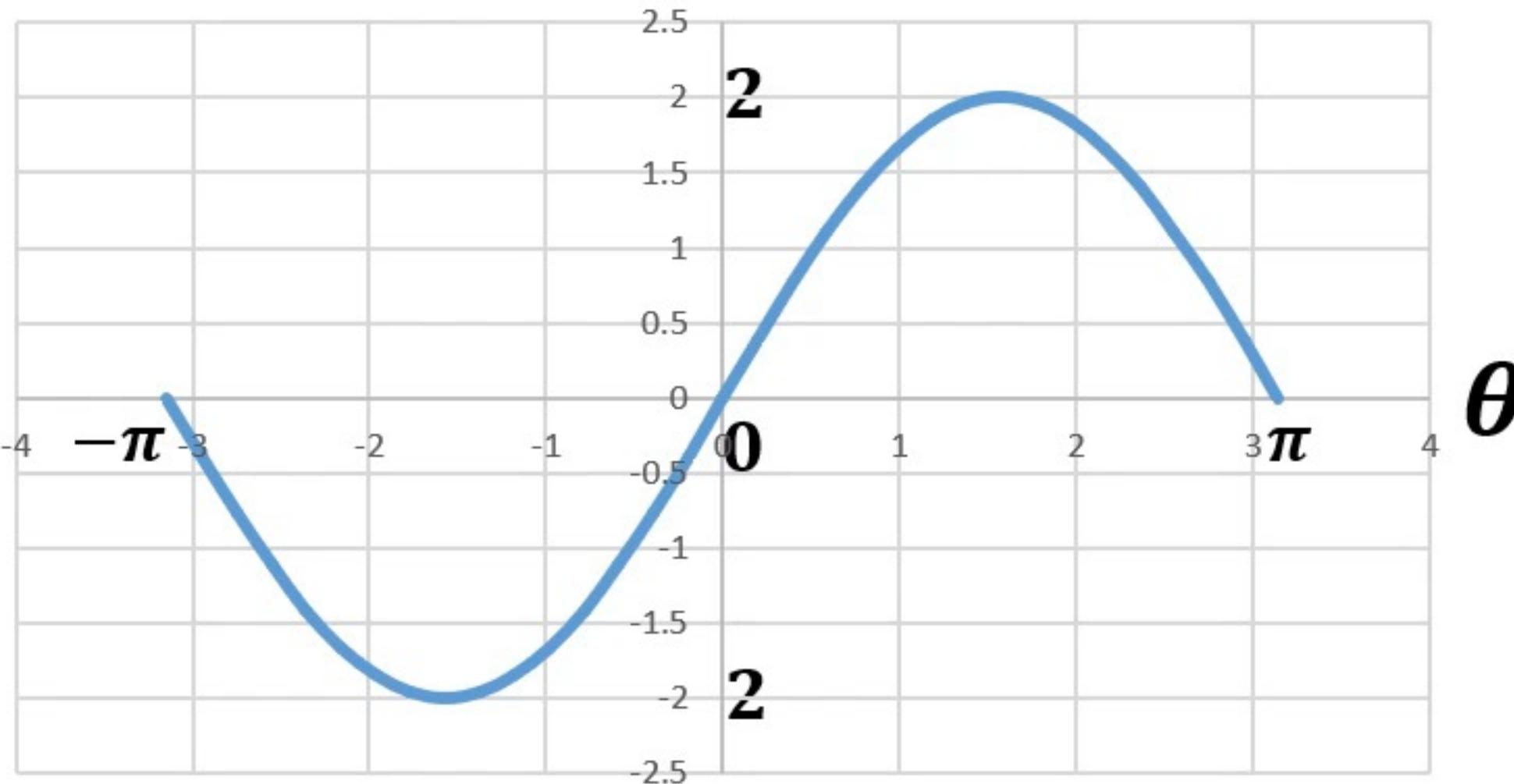


(a) Sketch the following graphs for values of θ , where $-\pi \leq \theta \leq \pi$

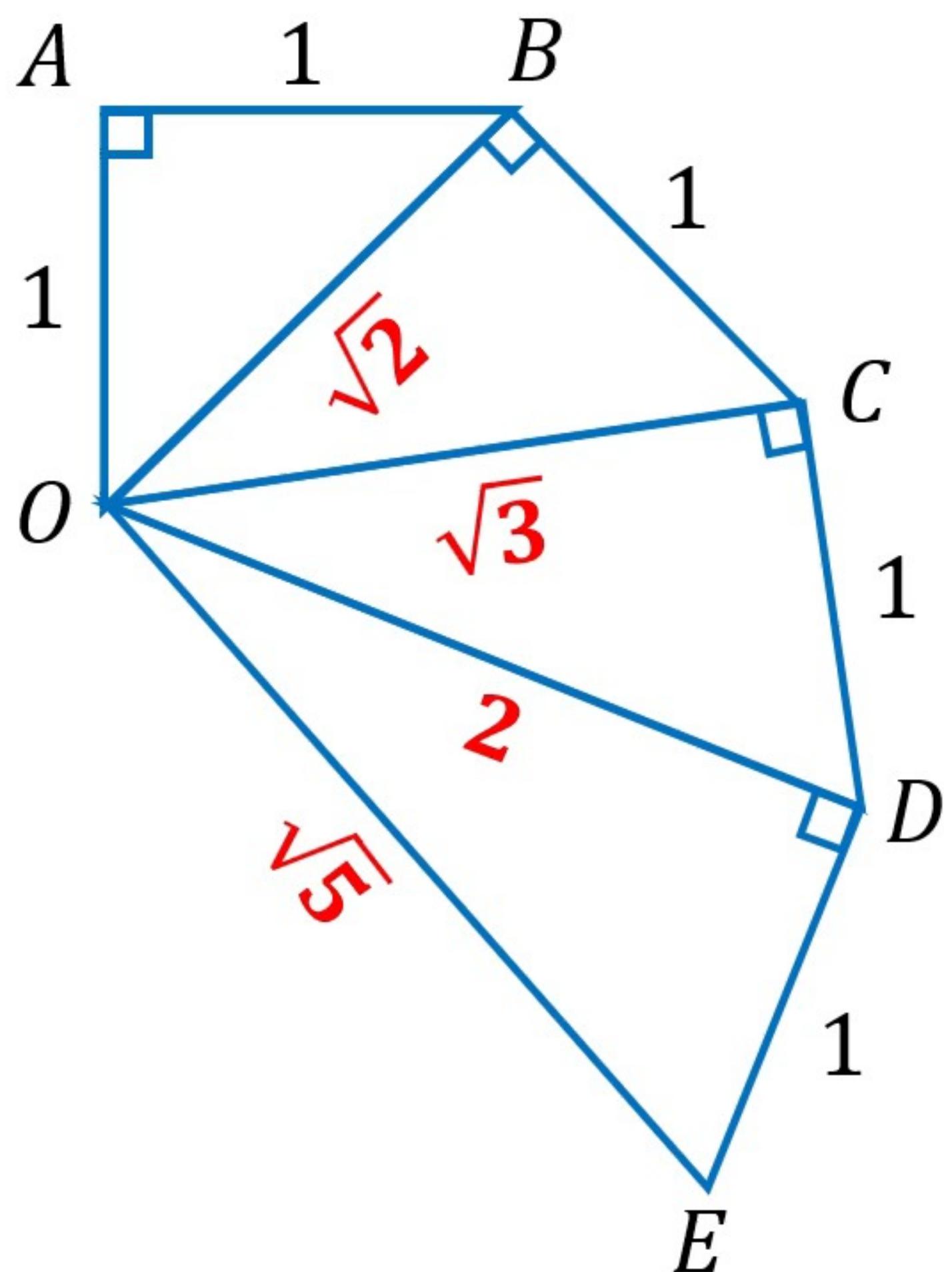
$$2\sin\theta$$
$$(-2\pi \leq \theta \leq 2\pi)$$



(ii) $y = 2\sin\theta$ ($-\pi \leq \theta \leq \pi$)



(b) $OABCDE$ is a hexagon with $OA = AB = BC = CD = DE = 1\text{cm}$.



(i) Calculate the lengths OB, OC, OD, OE , correct to 1 d.p.

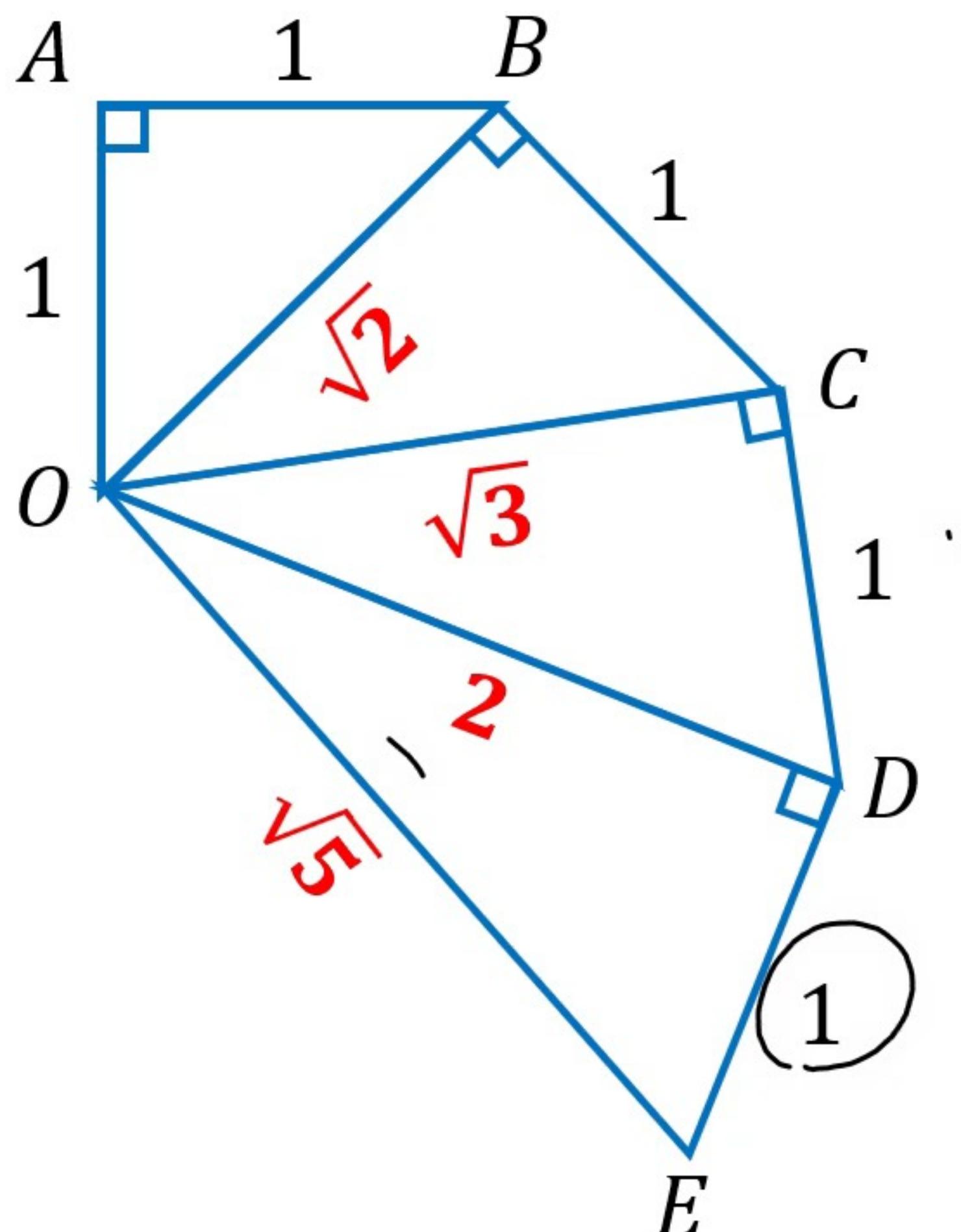
$$OB = \sqrt{1^2 + 1^2} = \sqrt{2} = \mathbf{1.4\text{ (cm)}}$$

$$OC = \sqrt{1^2 + (\sqrt{2})^2} = \sqrt{3} = \mathbf{1.7\text{ (cm)}}$$

$$OD = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = \mathbf{2.0\text{ (cm)}}$$

$$OE = \sqrt{1^2 + 2^2} = \sqrt{5} = \mathbf{2.2\text{ (cm)}}$$

(b) $OABCDE$ is a hexagon with $OA = AB = BC = CD = DE = 1\text{cm}$.



(ii) Calculate the perimeter of $OABCDE$

$$\begin{aligned} & OA + AB + BC + CD + DE + EO \\ &= 1 + 1 + 1 + 1 + 1 + \sqrt{5} \\ &= 5 + \sqrt{5} \\ &= \mathbf{7.24\text{ (cm)}} \end{aligned}$$

(2021) Question 5

- (a) Find the equation of the tangent to the curve $y = x^2 + x + 1$ when it passes through the point $(1, 3)$

$$y = mx + c$$

find m : $m = f'(x_0) = f'(1)$

$$y = f(x) = x^2 + x + 1$$

$$f'(x) = 2x + 1$$

$$f'(1) = 2(1) + 1 = 3$$

use $(x_0, y_0) = (1, 3)$ to find c

$$y = 3x + c$$

$$3 = 3(1) + c \rightarrow c = 0$$

$$\mathbf{y = 3x}$$

(b) Evaluate $\int_{y=3}^4 \int_{x=1}^2 (2x + 4y) dx dy$

$$\begin{aligned}\int_{x=1}^2 (2x + 4y) dx &= 2\left(\frac{x^2}{2}\right) + 4xy = x^2 + 4xy \Big|_{x=1}^2 \\ &= (2^2 + 4(2)y) - (1^2 + 4(1)y) = 3 + 4y\end{aligned}$$

$$\begin{aligned}\int_{y=3}^4 (3 + 4y) dy &= 3y + 4\left(\frac{y^2}{2}\right) = 3\underline{y} + 2\underline{y^2} \Big|_{y=3}^4 \\ &= (3(4) + 2(4^2)) - (3(3) + 2(3^2)) = \mathbf{17}\end{aligned}$$