(2023) Question 1

(a) Perform the following Octal calculations without changing base. Show your working.

(i) $47_8 + 63_8$

$$47_8 + 63_8 = \mathbf{132_8}$$

$$\begin{array}{rr} & 47_8 \\ + & 63_8 \\ \hline & 132 \end{array}$$

$$7 + 3 = 10_{10} = 12_8$$
$$4 + 6 + \mathbf{1} = 11_{10} = 13_8$$

(b) Solve the following

(i) $4x + 7 = 51$

(b) Solve the following

(iii) $5x^2 + 3x - 6 = 0$

$$a = 5, \qquad b = 3, \qquad c = -6$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(5)(-6)}}{2(5)} = \mathbf{0.84,\ -1.44}$$

(2023) Question 2

(a) (i) Find the mean and standard deviation for the following set of data

x	10	20	30	40	50
f	2	3	3	1	1

Mean $\bar{x} = \mathbf{26}$

Standard deviation $\sigma = \mathbf{12}$

Number of items $n = 2 + 3 + 3 + 1 + 1 = 10$

(ii) A further 20 items with a mean of 25 and standard deviation of 10 are added to the initial set of data. Find the combined mean and standard deviation for all 30 items.

group 1: $\bar{x}_1 = 26,\ \sigma_1 = 12,\ n_1 = 10$

group 2: $\bar{x}_2 = 25,\ \sigma_2 = 10,\ n_2 = 20$

combined mean of 30 items

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} = \frac{(10)(26) + (20)(25)}{10 + 20} = \mathbf{25.33}$$

(ii) A further 20 items with a mean of 25 and standard deviation of 10 are added to the initial set of data. Find the combined mean and standard deviation for all 30 items.

group 1: $\bar{x}_1 = 26, \sigma_1 = 12, n_1 = 10$

group 2: $\bar{x}_2 = 25, \sigma_2 = 10, n_2 = 20$

combined standard deviation of 30 items

$$\sum x^2 = n(\sigma^2 + \overline{x}^2) \qquad \sigma = \sqrt{\frac{\sum x^2}{n} - \overline{x}^2}$$

group 1: $\sum x_1^2 = 10(12^2 + 26^2) = 8200$

group 2: $\sum x_2^2 = 20(10^2 + 25^2) = 14500$

$$= \sqrt{\frac{8200 + 14500}{10 + 20} - (25.33)^2}$$

$$= \mathbf{10.73}$$

(b) Widgets are packed in boxes ready for shipping, with 10 widgets per full box. The mean weight of an individual widget is 12g and the standard deviation is 3g. The corresponding figures for the boxes are $\mu = 15g$ and $\sigma = 4g$. Find the combined mean and standard deviation of the weight of a full box of widgets.

One widget: $\mu_{widget} = 12,\quad \sigma_{widget} = 3$

An empty box: $\mu_{empty\ box} = 15,\quad \sigma_{empty\ box} = 4$

A full box of widgets: $\mu_{full\ box} = 12 \times 10 + 15 = \mathbf{135\ (g)}$

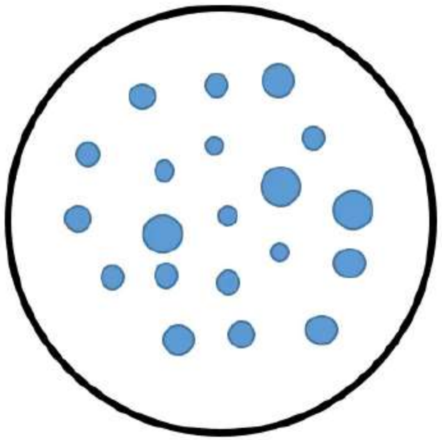
$${\sigma^2}_{full\ box} = 3^2 \times 10 + 4^2 = 106$$

$$\sigma_{full\ box} = \sqrt{106} = \mathbf{10.30\ (g)}$$

Find the combined mean and standard deviation

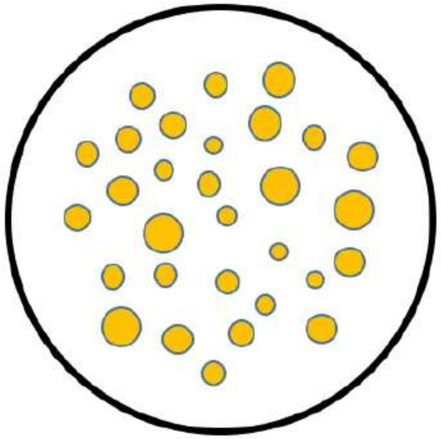
Case 1:

group 1 (20 items)



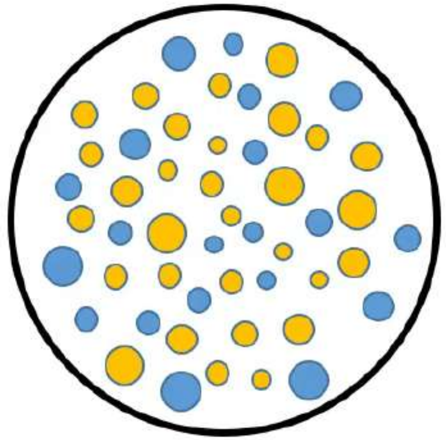
$\bar{x}_1, \sigma_1, n_1$

group 2 (30 items)



$\bar{x}_2, \sigma_2, n_2$

Mix group 1 and group 2 (50 items)



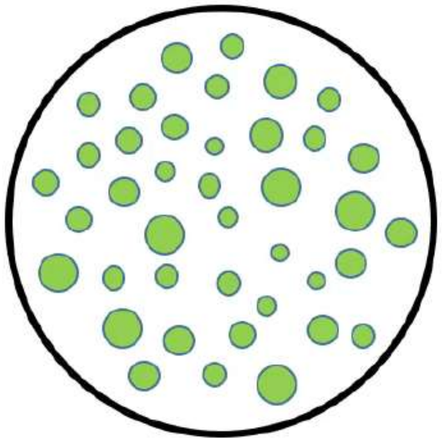
$\bar{x}_{combined}$

$\sigma_{combined}$

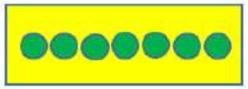
$n_{combined}$

Find the combined mean and standard deviation

Case 2:



μ, σ



$$\mu_{combined} = \mu \times 7 + \mu_{package}$$

$$\sigma_{combined} = \sqrt{\sigma^2 \times 7 + \sigma^2_{package}}$$

(2023) Question 3

(a) Find the sum to infinity for the series $27 + 9 + 3 + \cdots$

$$r = \frac{T_2}{T_1} = \frac{9}{27} = \frac{1}{3} \qquad r = \frac{T_3}{T_2} = \frac{3}{9} = \frac{1}{3}$$

constant ratio *r*, so it is a geometric progression

$$r = \frac{1}{3}, \qquad a = 27$$

$$S_\infty = \frac{a}{1-r} = \frac{27}{1-\frac{1}{3}} = \mathbf{40.5}$$

(b) Find the values of a, b, c, d and e such that

$$(3+2x)^a = b + 216x + cx^2 + 96x^d + ex^4$$

$\boldsymbol{a = 4}$

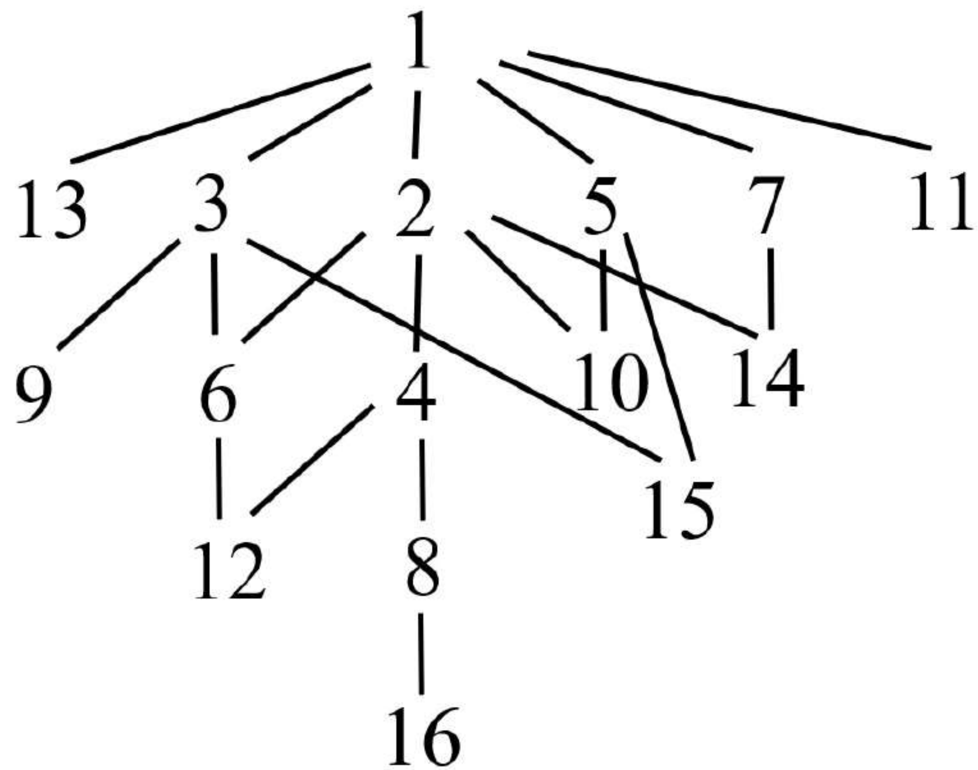
$1 + x$	1 1
$(1+x)^2$	1 2 1
$(1+x)^3$	1 3 3 1
$(1+x)^4$	1 4 6 4 1

$$(3+2x)^4 = 1(3^4) + 4(3^3)(2x) + 6(3^2)(2x)^2 + 4(3^1)(2x)^3 + 1(2x)^4$$

$$= 81 + 216x + 216x^2 + 96x^3 + 16x^4$$

$\boldsymbol{b = 81, \quad c = 216, \quad d = 3, \quad e = 16}$

(c) Draw a Hasse diagram to represent the relation "exactly divide" on the natural numbers from 1 to 16 inclusive.



(2023) Question 4

A series is defined by $S_n = \sum_{r=1}^{n} 2^r + r$

(a) Find the first 3 terms of the series.

$$T_r = 2^r + r$$

$$T_1 = 2^1 + 1 = \mathbf{3}$$

$$T_2 = 2^2 + 2 = \mathbf{6}$$

$$T_3 = 2^3 + 3 = \mathbf{11}$$

(b) Use the method of induction to prove that $S_n = 2(2^n - 1) + \frac{n}{2}(n+1)$

Proof: $T_r = 2^r + r,$ $S_n = 2(2^n - 1) + \frac{n}{2}(n+1)$

(1) Base case

$T_1 = 2^1 + 1 = 3,$ $S_1 = 2(2^1 - 1) + \frac{1}{2}(1+1) = 3$

Therefore, $T_1 = S_1$

(2) Assumption

There is at least one value of k $(1 \leq k < n)$ such that

$$S_k = 2\left(2^k - 1\right) + \frac{k}{2}(k+1)$$

(3) Statement

If the formula is true, then $S_{k+1} = 2\left(2^{k+1} - 1\right) + \frac{(k+1)}{2}(k+2)$

$$T_r = 2^r + r$$

$$S_n = 2(2^n - 1) + \frac{n}{2}(n+1)$$

(4) Induction

$$S_{k+1} = S_k + T_{k+1}$$

$$= 2(2^k - 1) + \frac{k}{2}(k+1) + 2^{k+1} + (k+1)$$

$$= 2\left(2^k - 1\right) + 2^{k+1} + \frac{k}{2}(k+1) + (k+1)$$

$$= 2^{k+1} - 2 + 2^{k+1} + \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= 2(2^{k+1}) - 2 + \frac{k(k+1) + 2(k+1)}{2}$$

$$= 2(2^{k+1} - 1) + \frac{(k+1)(k+2)}{2}$$