(2023) Question 1

$$A = \begin{pmatrix} 3 & 9 \\ -2 & 8 \end{pmatrix} \qquad B = \begin{pmatrix} a \\ -1 \end{pmatrix} \qquad C = \begin{pmatrix} 9 \\ b \end{pmatrix}$$

$AB = C$ $\qquad$ a and b are constants

(a) Find the values of a and b

$$AB = \begin{pmatrix} 3 & 9 \\ -2 & 8 \end{pmatrix}\begin{pmatrix} a \\ -1 \end{pmatrix} = \begin{pmatrix} 3a - 9 \\ -2a - 8 \end{pmatrix} = \begin{pmatrix} 9 \\ b \end{pmatrix}$$

$$3a - 9 = 9$$
$$3a = 18$$
$$\boldsymbol{a = 6}$$

$$-2a - 8 = b$$
$$-2(6) - 8 = b$$
$$\boldsymbol{b = -20}$$

$$A = \begin{pmatrix} 3 & 9 \\ -2 & 8 \end{pmatrix} \qquad B = \begin{pmatrix} a \\ -1 \end{pmatrix} \qquad C = \begin{pmatrix} 9 \\ b \end{pmatrix}$$

$AB = C$ $\qquad$ a and b are constants

(b) Find the inverse of A

$$A = \begin{pmatrix} 3 & 9 \\ -2 & 8 \end{pmatrix}$$

$$|A| = (3)(8) - (9)(-2) = 24 + 18 = 42$$

$$A^{-1} = \frac{1}{42}\begin{pmatrix} 8 & -9 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \mathbf{4/21} & \mathbf{-3/14} \\ \mathbf{1/21} & \mathbf{1/14} \end{pmatrix}$$

(c) $\begin{cases} 3x + 9y = 33 \\ -2x + 8y = 20 \end{cases}$

Using your answer to part (b), or otherwise, find the values of x and y

$$A = \begin{pmatrix} 3 & 9 \\ -2 & 8 \end{pmatrix} \qquad A^{-1} = \frac{1}{42}\begin{pmatrix} 8 & -9 \\ 2 & 3 \end{pmatrix} \qquad N = \begin{pmatrix} 33 \\ 20 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}N = \frac{1}{42}\begin{pmatrix} 8 & -9 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 33 \\ 20 \end{pmatrix} = \frac{1}{42}\begin{pmatrix} 264 - 180 \\ 66 + 60 \end{pmatrix}$$

$$= \frac{1}{42}\begin{pmatrix} 84 \\ 126 \end{pmatrix} = \begin{pmatrix} \mathbf{2} \\ \mathbf{3} \end{pmatrix}$$

(d) Explain why the matrix $D = \begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}$ has no inverse

$$|D| = (3)(4) - (2)(6) = 12 - 12 = 0$$

$|D| = 0$, so matrix D has no inverse.

(2023) Question 2

With respect to a fixed origin, O, the straight lines L_1 and L_2 are:

$$L_1: \quad \underline{r} = i + 2j + \lambda(i + j)$$

$$L_2: \quad \frac{x-6}{3} = 3 - y \quad \Longrightarrow \quad \frac{x-6}{3} = \frac{y-3}{-1}$$

(a) Find the point of intersection between L_1 and L_2.

$$L_1: \quad x = 1 + \lambda \qquad\qquad L_2: \quad x = 6 + 3\mu$$
$$\qquad y = 2 + \lambda \qquad\qquad\qquad y = 3 - \mu$$

(a) Find the point of intersection between L_1 and L_2.

(continue)

L_1: $x = 1 + \lambda$
$y = 2 + \lambda$

L_2: $x = 6 + 3\mu$
$y = 3 - \mu$

$$\begin{cases} 1 + \lambda = 6 + 3\mu \\ 2 + \lambda = 3 - \mu \end{cases} \quad (-)$$

$$-1 = 3 + 4\mu$$

$$-4\mu = 4$$

$$\mu = -1$$

$$x = 6 + 3(-1) = 3$$

$$y = 3 - (-1) = 4$$

the point of intersection is **(3, 4)**

(b) Line L_3 intersects L_1 at the point $2i + 3j$ and it intersects L_2 at the point $9i + 2j$. Find the Vector equation of the line L_3

L_3: $\quad \overrightarrow{OA} = 2i + 3j \quad$ point $A(2,3)$

$\quad \overrightarrow{OB} = 9i + 2j \quad$ point $B(9,2)$

$$\underline{r} = \overrightarrow{OA} + \beta(\overrightarrow{OB} - \overrightarrow{OA})$$

$$= 2i + 3j + \beta(9i + 2j - (2i + 3j))$$

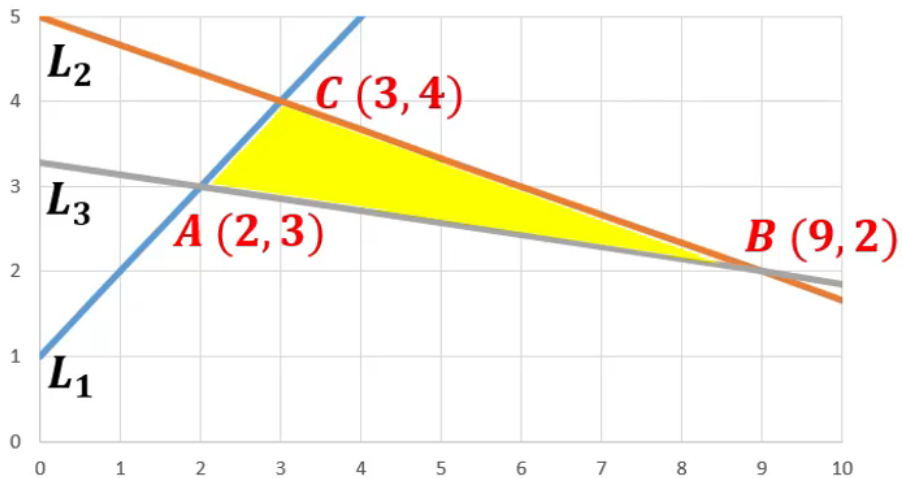
$$= \boldsymbol{2i + 3j + \beta(7i - j)}$$

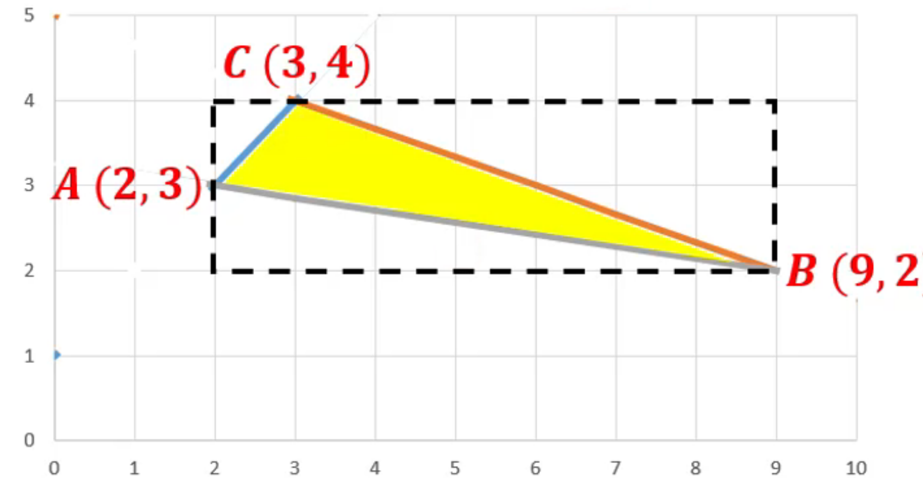
(c) Calculate the area of the triangle enclosed between the three lines

Line L_3 intersects L_1 at the point $A(2,3)$

Line L_3 intersects L_2 at the point $B(9,2)$

Line L_1 intersects L_2 at the point $C(3,4)$





$$area = 2 \times 7 - \frac{1 \times 7}{2} - \frac{1 \times 1}{2} - \frac{6 \times 2}{2} = \mathbf{4}$$

(2023) Question 3

(a) (i) The letters of the word DIVIDED are placed at random in a line. How many different orders of the letters are possible?

DIVIDED

7 letters (3 D, 2 I) $\qquad \dfrac{7!}{(3!)(2!)} = \mathbf{420}$

(ii) In how many of the possible orders are the three Ds next to each other?

DDDIVIE

5 letters (2 I) $\qquad \dfrac{5!}{2!} = \mathbf{60}$

(iii) Find the probability that the first two letters in the order contain at least one D (method #1)

DIVIDED

D V
V D (D, D, I, I, E) $\frac{5!}{(2!)(2!)} = 30$

D I
I D (D, D, I, V, E) $\frac{5!}{2!} = 60$

D E
E D (D, D, I, I, V) $\frac{5!}{(2!)(2!)} = 30$

D D (D, I, I, V, E) $\frac{5!}{2!} = 60$

$$30 \times 2 + 60 \times 2 + 30 \times 2 + 60 = 300$$

$$P(at\ least\ one\ D\ in\ the\ first\ two\ letters) = \frac{300}{420} = \mathbf{\frac{5}{7}} = \mathbf{0.714}$$

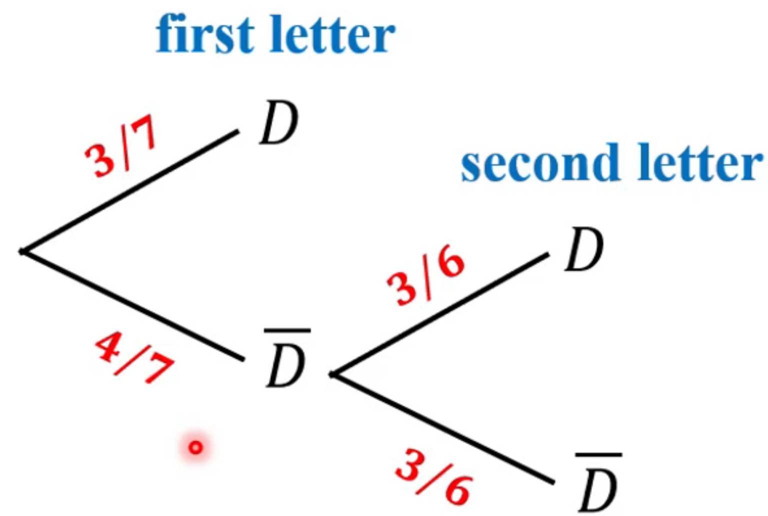
(iii) Find the probability that the first two letters in the order contain at least one D (method #2)

DIVIDED 7 letters (3D)

D

$P(first\ letter\ is\ D) = 3/7$

(I, E, V) D

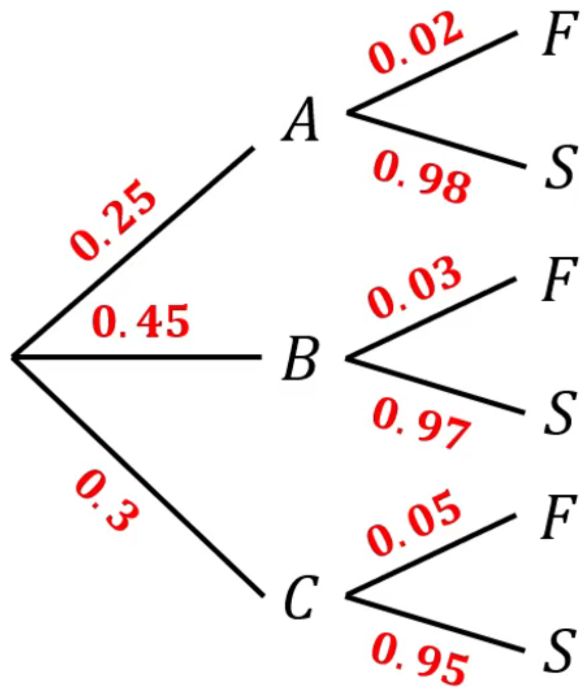


$P(second\ letter\ is\ D | first\ letter\ is\ \overline{D}) = 3/6$

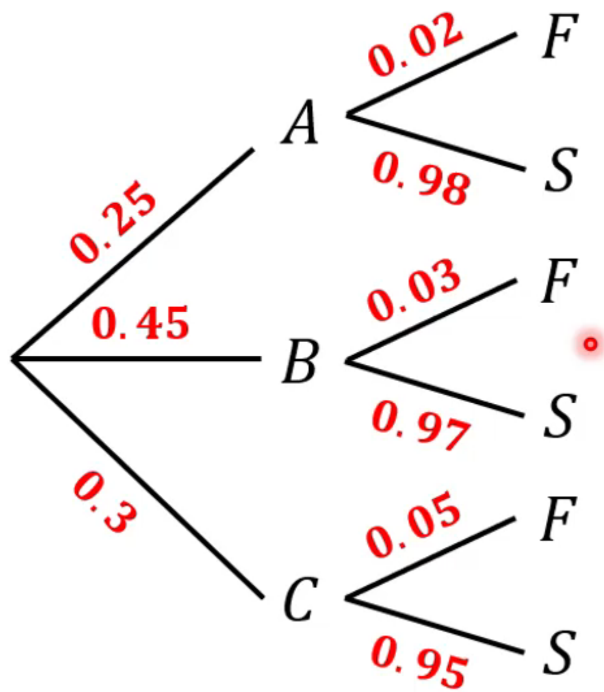
$$P(at\ least\ one\ D\ in\ the\ first\ two\ letters) = \frac{3}{7} + \frac{4}{7} \times \frac{3}{6} = \mathbf{\frac{5}{7}} = \mathbf{0.714}$$

(b) In a factory a particular component is manufactured on three separate machines. Machine A makes 25% of the components, machine B produces 45% and machine C makes the remainder. Of the components made by each machine 2% of those produced by machine A are found to be faulty with the corresponding figures for machines B and C being 3% and 5% respectively

(i) Draw a tree diagram to represent this information



(ii) The components produced by all three machines are merged. Calculate the probability that a component picked at random is faulty



The component is produced by machine A and it is faulty $(0.25)(0.02) = 0.005$

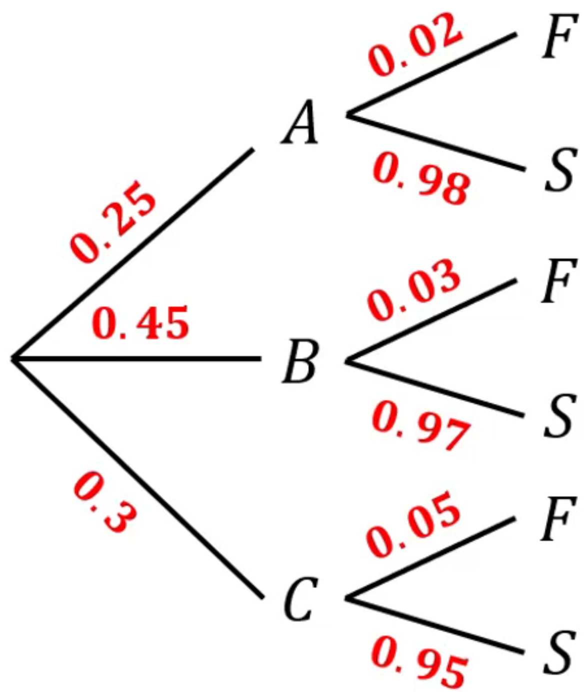
The component is produced by machine B and it is faulty $(0.45)(0.03) = 0.0135$

The component is produced by machine C and it is faulty $(0.3)(0.05) = 0.015$

The component is faulty

$0.005 + 0.0135 + 0.015 = \mathbf{0.0335}$

(iii) Given that the chosen component is faulty, calculate the chance that it was produced by machine B



The component is produced by machine B and it is faulty $(0.45)(0.03) = 0.0135$

The component is faulty 0.0335

$$P(machine\ B \mid faulty)$$

$$= \frac{P(machine\ B \cap faulty)}{P(faulty)}$$

$$= \frac{0.0135}{0.0335} = \mathbf{0.403}$$

(2023) Question 4

(a) Differentiate the following:

(i) $y = 4x^3$

$y' = (4x^3)' = \mathbf{12x^2}$

(ii) $y = (3x + 7)^5$

$$\frac{d}{dx}[x^n] = n \cdot x^{n-1}$$

(outer function) (inner function)

$f(x) = x^5$ $g(x) = 3x + 7$

$f'(x) = 5x^4$ $g'(x) = 3$

$y' = 5(\ 3x + 7\ )^4(3) = \mathbf{15(3x + 7)^4}$

$y = x^{\boldsymbol{n}}$

$y = (\underbrace{3x + 7}_{\boldsymbol{g(x)}})^{\mathbf{5}}$

(b) Evaluate the following:

(i) $\int_1^3 (3x^2 + 2x)\, dx$

$= 3\left(\frac{x^3}{3}\right) + 2\left(\frac{x^2}{2}\right)$

$= x^3 + x^2 \Big|_{\mathbf{1}}^{\mathbf{3}}$

$= (3^3 + 3^2) - (1^3 + 1^2)$

$= \mathbf{34}$

(ii) $\int_{y=2}^{4} \boxed{\int_{x=1}^{3} (2x + y)\, dx}dy$

$$\int_{x=1}^{3} (2x + y)\, dx = 2\left(\frac{x^2}{2}\right) + xy = x^2 + xy \Big|_{\boldsymbol{x = 1}}^{\mathbf{3}}$$

$$= (3^2 + 3y) - (1^2 + y) = \boxed{8 + 2y}$$

$$\int_{y=2}^{4} (8 + 2y)\, dy = 8y + 2\left(\frac{y^2}{2}\right) = 8y + y^2 \Big|_{\boldsymbol{y = 2}}^{\mathbf{4}}$$

$$= (8(4) + 4^2) - (8(2) + 2^2) = \mathbf{28}$$