

(2023) Question 1

(a) Perform the following Octal calculations without changing base. Show your working.

(i) $47_8 + 63_8$

$$47_8 + 63_8 = \mathbf{132_8}$$

$$\begin{array}{r} 47_8 \\ + 63_8 \\ \hline 132 \end{array}$$

$$7 + 3 = 10_{10} = \mathbf{12_8}$$

$$4 + 6 + \mathbf{1} = 11_{10} = \mathbf{13_8}$$

(a) Perform the following Octal calculations without changing base. Show your working.

(ii) $47_8 \times 63_8$

$$47_8 \times 63_8 = \mathbf{3705_8}$$

$$\begin{array}{r} 47_8 \\ \times 63_8 \\ \hline 165 \\ + 3520 \\ \hline 3705 \end{array}$$

$$3 \times 7 = 21_{10} = \mathbf{25_8}$$

$$3 \times 4 + \mathbf{2} = 14_{10} = \mathbf{16_8}$$

$$6 \times 7 = 42_{10} = \mathbf{52_8}$$

$$6 \times 4 + \mathbf{5} = 29_{10} = \mathbf{35_8}$$

$$5 + 0 = 5_{10} = 5_8$$

$$6 + 2 = 8_{10} = \mathbf{10_8}$$

$$1 + 5 + \mathbf{1} = 7_{10} = 7_8$$

(b) Solve the following

(i) $4x + 7 = 51$

$$4x + 7 = 51$$

$$4x = 51 - 7$$

$$4x = 44$$

$$**x = 11**$$

(b) Solve the following

$$(ii) \begin{cases} 5x - 7y = -3 & (1) \times 3 \\ 2x + 3y = 22 & (2) \times 7 \end{cases}$$

$$\begin{cases} 15x - 21y = -9 \\ 14x + 21y = 154 \end{cases} \quad (+)$$

$$29x = 145$$

$$x = 5 \quad (3)$$

Substitute (3) into (2)

$$2(5) + 3y = 22$$

$$10 + 3y = 22$$

$$3y = 12$$

$$y = 4$$

(b) Solve the following

(iii) $5x^2 + 3x - 6 = 0$

$$a = 5, \quad b = 3, \quad c = -6$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(5)(-6)}}{2(5)} = \mathbf{0.84, -1.44}$$

(2023) Question 2

(a) (i) Find the mean and standard deviation for the following set of data

x	10	20	30	40	50
f	2	3	3	1	1

Mean $\bar{x} = 26$

Standard deviation $\sigma = 12$

Number of items $n = 2 + 3 + 3 + 1 + 1 = 10$

(ii) A further 20 items with a mean of 25 and standard deviation of 10 are added to the initial set of data. Find the combined mean and standard deviation for all 30 items.

group 1: $\bar{x}_1 = 26, \sigma_1 = 12, n_1 = 10$

group 2: $\bar{x}_2 = 25, \sigma_2 = 10, n_2 = 20$

combined mean of 30 items

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} = \frac{(10)(26) + (20)(25)}{10 + 20} = \mathbf{25.33}$$

(ii) A further 20 items with a mean of 25 and standard deviation of 10 are added to the initial set of data. Find the combined mean and standard deviation for all 30 items.

group 1: $\bar{x}_1 = 26, \sigma_1 = 12, n_1 = 10$

group 2: $\bar{x}_2 = 25, \sigma_2 = 10, n_2 = 20$

combined standard deviation of 30 items

$$\sum x^2 = n(\sigma^2 + \bar{x}^2) \quad \leftarrow \quad \sigma = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2}$$

group 1: $\sum x_1^2 = 10(12^2 + 26^2) = 8200$

group 2: $\sum x_2^2 = 20(10^2 + 25^2) = 14500$

$$\begin{aligned} &= \sqrt{\frac{8200 + 14500}{10 + 20} - (25.33)^2} \\ &= \mathbf{10.73} \end{aligned}$$

(b) Widgets are packed in boxes ready for shipping, with 10 widgets per full box. The mean weight of an individual widget is 12g and the standard deviation is 3g. The corresponding figures for the boxes are $\mu = 15g$ and $\sigma = 4g$. Find the combined mean and standard deviation of the weight of a full box of widgets.

One widget: $\mu_{\text{widget}} = 12, \quad \sigma_{\text{widget}} = 3$

An empty box: $\mu_{\text{empty box}} = 15, \quad \sigma_{\text{empty box}} = 4$

A full box of widgets: $\mu_{\text{full box}} = 12 \times 10 + 15 = \mathbf{135 (g)}$

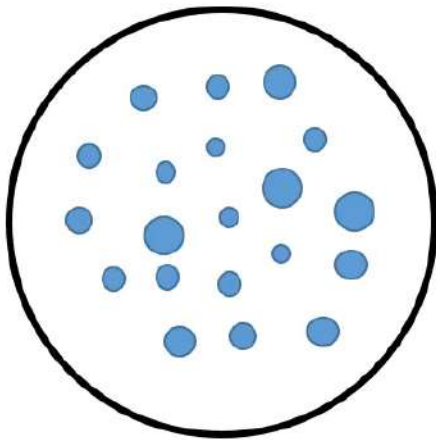
$$\sigma^2_{\text{full box}} = 3^2 \times 10 + 4^2 = 106$$

$$\sigma_{\text{full box}} = \sqrt{106} = \mathbf{10.30 (g)}$$

Find the combined mean and standard deviation

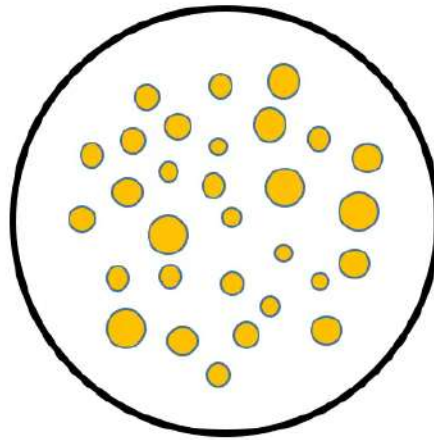
Case 1:

group 1 (20 items)



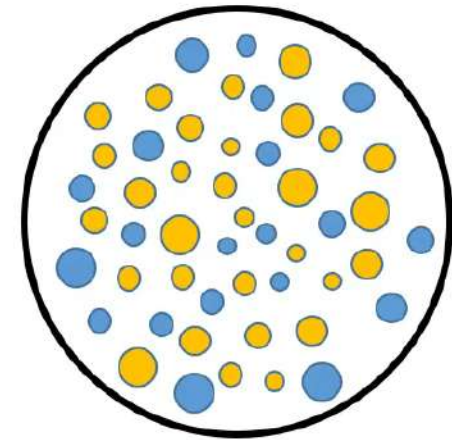
\bar{x}_1, σ_1, n_1

group 2 (30 items)



\bar{x}_2, σ_2, n_2

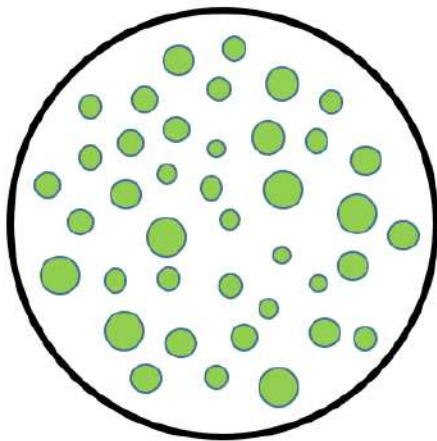
Mix group 1 and group 2
(50 items)



$\bar{x}_{combined}$
 $\sigma_{combined}$
 $n_{combined}$

Find the combined mean and standard deviation

Case 2:



μ, σ



$$\underline{\mu_{combined}} = \underline{\mu} \times \underline{7} + \underline{\mu_{package}}$$

$$\underline{\sigma_{combined}} = \sqrt{\underline{\sigma^2 \times 7 + \sigma_{package}^2}}$$

(2023) Question 3

(a) Find the sum to infinity for the series $27 + 9 + 3 + \dots$

$$r = \frac{T_2}{T_1} = \frac{9}{27} = \frac{1}{3}$$

$$r = \frac{T_3}{T_2} = \frac{3}{9} = \frac{1}{3}$$

constant ratio r , so it is a geometric progression

$$r = \frac{1}{3}, \quad a = 27$$

$$S_{\infty} = \frac{a}{1 - r} = \frac{27}{1 - \frac{1}{3}} = \mathbf{40.5}$$

(b) Find the values of a , b , c , d and e such that

$$(3 + 2x)^a = b + 216x + cx^2 + 96x^d + ex^4$$

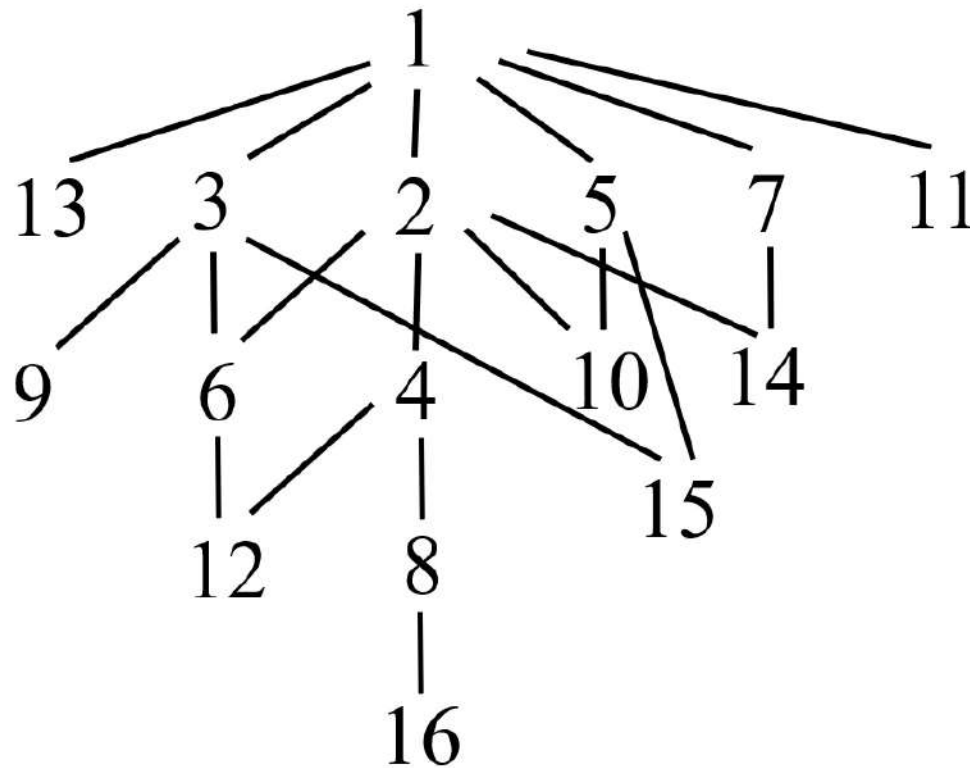
$$a = 4$$

$$\begin{array}{rcccccc} & & 1 & + & x & & & & & & 1 & & 1 \\ & & & & & & & & & & 1 & & 2 & & 1 \\ (1+x)^2 & & & & & & & & & & 1 & & 3 & & 3 & & 1 \\ (1+x)^3 & & & & & & & & & & 1 & & 4 & & 6 & & 4 & & 1 \\ (1+x)^4 & & & & & & & & & & & & & & & & & & \end{array}$$

$$\begin{aligned} (3 + 2x)^4 &= 1(3^4) + 4(3^3)(2x) + 6(3^2)(2x)^2 + 4(3^1)(2x)^3 + 1(2x)^4 \\ &= 81 + 216x + 216x^2 + 96x^3 + 16x^4 \end{aligned}$$

$$b = 81, \quad c = 216, \quad d = 3, \quad e = 16$$

(c) Draw a Hasse diagram to represent the relation “exactly divide” on the natural numbers from 1 to 16 inclusive.



(2023) Question 4

A series is defined by $S_n = \sum_{r=1}^n 2^r + r$

(a) Find the first 3 terms of the series.

$$T_r = 2^r + r$$

$$T_1 = 2^1 + 1 = 3$$

$$T_2 = 2^2 + 2 = 6$$

$$T_3 = 2^3 + 3 = 11$$

(b) Use the method of induction to prove that $S_n = 2(2^n - 1) + \frac{n}{2}(n + 1)$

Proof: $T_r = 2^r + r,$ $S_n = 2(2^n - 1) + \frac{n}{2}(n + 1)$

(1) Base case

$$T_1 = 2^1 + 1 = 3,$$

$$S_1 = 2(2^1 - 1) + \frac{1}{2}(1 + 1) = 3$$

$$\text{Therefore, } T_1 = S_1$$

(2) Assumption

There is at least one value of k ($1 \leq k < n$) such that

$$S_k = 2(2^k - 1) + \frac{k}{2}(k + 1)$$

(3) Statement

If the formula is true, then $S_{k+1} = 2(2^{k+1} - 1) + \frac{(k + 1)}{2}(k + 2)$

(4) Induction

$$S_{k+1} = S_k + T_{k+1}$$

$$= 2(2^k - 1) + \frac{k}{2}(k+1) + 2^{k+1} + (k+1)$$

$$= 2(2^k - 1) + 2^{k+1} + \frac{k}{2}(k+1) + (k+1)$$

$$= 2^{k+1} - 2 + 2^{k+1} + \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= 2(2^{k+1}) - 2 + \frac{k(k+1) + 2(k+1)}{2}$$

$$= 2(2^{k+1} - 1) + \frac{(k+1)(k+2)}{2}$$

$$T_r = 2^r + r$$

$$S_n = 2(2^n - 1) + \frac{n}{2}(n+1)$$