参考答案

1. (10分)设$D(x)=\begin{cases}0, & x\in\mathbb{Q}\bigcap[0,1]\\ 1, & x\in[0,1]\backslash\mathbb{Q}\end{cases}$, $R(x)=\begin{cases}\frac{1}{p}, & x=\frac{q}{p}(\text{既约})\in\mathbb{Q}\bigcap(0,1]\\ 0, & x\in\{0\}\bigcup\{[0,1]\backslash\mathbb{Q}\}\end{cases}$.

问是否$D(x)\in R[0,1]$, $R(x)\in R[0,1]$. 简要说明理由.

【解】: $D(x)\notin R[0,1]$, $R(x)\in R[0,1]$.

对$[0,1]$的任意分割Δ, $D(x)$的振幅和$\sum\limits_{i=1}^{n}\omega_i(D(x),\Delta)\equiv 1$;

$\forall\varepsilon>0,\forall\delta>0$, $\exists$分割Δ使得 $\sum\limits_{\omega_i(R(x),\Delta)>\delta}|\Delta x_i|<\varepsilon$.

或者, $D(x)$的间断点集为$[0,1]$, 测度为1; $R(x)$的间断点集为$\mathbb{Q}\bigcap[0,1]$, 测度为0.

2. (10分) 计算积分$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\cos^2 x}{x(\pi-2x)}\,\mathrm{d}x$.

【解】:

$$I\xlongequal[x=\frac{\pi}{2}-t]{t=\frac{\pi}{2}-x}\int_{\frac{\pi}{3}}^{\frac{\pi}{6}}\frac{\sin^2 t}{t(\pi-2t)}(-\,\mathrm{d}t)=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sin^2 x}{x(\pi-2x)}\,\mathrm{d}x;$$

$$I=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\cos^2 x+\sin^2 x}{x(\pi-2x)}\,\mathrm{d}x=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{x(\pi-2x)}\,\mathrm{d}x=\frac{1}{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\frac{1}{x}+\frac{2}{\pi-2x}\right)\mathrm{d}x$$

$$=\frac{1}{2\pi}\ln\frac{x}{\pi-2x}\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}=\frac{\ln 2}{\pi}.$$

3. (15分) 求极坐标曲线$r^2=\cos 2\theta$绕

(a)极轴旋转所成的旋转面的面积;

(b)过极点垂直于极轴的直线旋转所成的旋转面的面积.

【解】:

$$S_a=\int_{*}^{**}2\pi y\sqrt{(dx)^2+(dy)^2}=4\pi\int_0^{\frac{\pi}{4}}r\sin\theta\sqrt{r^2(\theta)+r'(\theta)^2}\,\mathrm{d}\theta;(\text{左右两半})$$

$$S_b=\int_{*}^{**}2\pi x\sqrt{(dx)^2+(dy)^2}=4\pi\int_0^{\frac{\pi}{4}}r\cos\theta\sqrt{r^2(\theta)+r'(\theta)^2}\,\mathrm{d}\theta;(\text{上下两半})$$

$$r=\sqrt{\cos 2\theta},\quad r'(\theta)=\frac{-\sin 2\theta}{\sqrt{\cos 2\theta}},\quad \sqrt{r^2(\theta)+r'(\theta)^2}=\frac{1}{\sqrt{\cos 2\theta}}=\frac{1}{r}.$$

$$S_a=4\pi\int_0^{\frac{\pi}{4}}\sin\theta\,\mathrm{d}\theta=4\pi\left(1-\frac{\sqrt{2}}{2}\right)=2\pi(2-\sqrt{2}).$$

$$S_b=4\pi\int_0^{\frac{\pi}{4}}\cos\theta\,\mathrm{d}\theta=4\pi\frac{\sqrt{2}}{2}=2\sqrt{2}\pi.$$

4. (10分) 讨论二元极限$\lim\limits_{(x,y)\to(0,0)}\frac{x^2y^2}{x^2y^2+(x-y)^2}$是否存在. 若存在, 计算其值; 若不存在, 说明理由.

【解】:

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx^2+x}}\frac{x^2y^2}{x^2y^2+(x-y)^2}=\lim_{x\to 0}\frac{x^2(x+kx^2)^2}{x^2(x+kx^2)^2+k^2x^4}=\lim_{x\to 0}\frac{(1+kx)^2}{(1+kx)^2+k^2}=\frac{1}{1+k^2},$$

路径极限依赖路径. 所以极限不存在.

5. (10分) 证明函数$f(x,y)=\sqrt{x^2+y^2}$在$\mathbb{R}^2$上一致连续.

【证明】:

对任意两(不同的点)点$(x_1,y_1),(x_2,y_2)\in\mathbb{R}^2$,

2

$$|f(x_1,y_1)-f(x_2,y_2)|=\left|\sqrt{x_1^2+y_1^2}-\sqrt{x_2^2+y_2^2}\right|=\frac{\left|(x_1^2+y_1^2)-(x_2^2+y_2^2)\right|}{\sqrt{x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}}$$

$$\leqslant\frac{|x_1^2-x_2^2|}{\sqrt{x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}}+\frac{|y_1^2-y_2^2|}{\sqrt{x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}}$$

8

$$=\frac{|x_1-x_2||x_1+x_2|}{\sqrt{x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}}+\frac{|y_1-y_2||y_1+y_2|}{\sqrt{x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}}$$

$$\leqslant\frac{|x_1-x_2||x_1+x_2|}{|x_1|+|x_2|}+\frac{|y_1-y_2||y_1+y_2|}{|y_1|+|y_2|}\leqslant|x_1-x_2|+|y_1-y_2|.$$

6. (15分)

4=2+2 (a)给出一个二元函数$f(x,y)$使得其在(x_0,y_0)点不连续但是$\dfrac{\partial f(x_0,y_0)}{\partial x}$、$\dfrac{\partial f(x_0,y_0)}{\partial y}$都存在;

4=2+2 (b)给出一个二元函数$f(x,y)$使得其在(x_0,y_0)点$\dfrac{\partial f(x_0,y_0)}{\partial x}$、$\dfrac{\partial f(x_0,y_0)}{\partial y}$都不存在, 但是对任何单位向量$\boldsymbol{\ell}=(\cos\alpha,\cos\beta)$, $\dfrac{\partial f(x_0,y_0)}{\partial \boldsymbol{\ell}}$ 都存在;

7=3+2+2 (c)给出一个二元函数$f(x,y)$使得其在(x_0,y_0)点$\dfrac{\partial f(x,y)}{\partial x}$、$\dfrac{\partial f(x,y)}{\partial y}$都不连续, 但是$f(x,y)$在$(x_0,y_0)$点可微.

【解】:

(a) $f(x,y)=\begin{cases}\frac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$ 于$(0,0)$点. 2

(b) $f(x,y)=\sqrt{x^2+y^2}$ 于$(0,0)$点. 2

(c) 偏导数连续并非可微的必要条件. 例如:

$$f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & x^2+y^2\neq0\\ 0, & x^2+y^2=0\end{cases}$$

在$(0,0)$点可微, 但其两个偏导数在$(0,0)$点不连续.

事实上, $f(0+\Delta x,0+\Delta y)-f(0,0)=[(\Delta x)^2+(\Delta y)^2]\sin\dfrac{1}{(\Delta x)^2+(\Delta y)^2}$,

$$\lim_{\Delta x\to0,\Delta y\to0}\frac{f(0+\Delta x,0+\Delta y)-f(0,0)}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$=\lim_{\Delta x\to0,\Delta y\to0}\sqrt{(\Delta x)^2+(\Delta y)^2}\sin\frac{1}{(\Delta x)^2+(\Delta y)^2}=0,$$

即$f(0+\Delta x,0+\Delta y)-f(0,0)=o\left(\sqrt{(\Delta x)^2+(\Delta y)^2}\right)$, $\sqrt{(\Delta x)^2+(\Delta y)^2}\to0$,

所以,

$\exists 0,0$ s.t. $f(0+\Delta x,0+\Delta y)-f(0,0)=0\cdot\Delta x+0\cdot\Delta y+o\left(\sqrt{(\Delta x)^2+(\Delta y)^2}\right)$,

所以$f(x,y)$于$(0,0)$点可微.

但是, $f'_x(x,y)=\begin{cases}2x\sin\frac{1}{x^2+y^2}-\frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2}, & x^2+y^2\neq0\\ 0, & x^2+y^2=0\end{cases}$

$$f'_y(x,y)=\begin{cases}2y\sin\frac{1}{x^2+y^2}-\frac{2y}{x^2+y^2}\cos\frac{1}{x^2+y^2}, & x^2+y^2\neq0\\ 0, & x^2+y^2=0\end{cases}$$

在$(0,0)$点都不连续.

7. (10分)设$u(x,y)$是无零点的二阶连续可微函数(二阶偏导数连续). 证明：

$u(x,y)=f(x)g(y)$的充分必要条件是 $u\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y}$, 其中$f,g$为可导函数.

【证明】:

$$u(x,y)=f(x)g(y) \Rightarrow \frac{\partial u}{\partial x}=f'(x)g(y),\ \frac{\partial u}{\partial y}=f(x)g'(y),\ \frac{\partial u}{\partial x\partial y}=f'(x)g'(y)$$

$$\Rightarrow u\frac{\partial^2 u}{\partial x\partial y}=f(x)g(y)f'(x)g'(y)=\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}.$$

另一方面,

$$u\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} \Rightarrow u\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}=0 \Rightarrow \frac{u\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}}{u^2}=0.(u\text{无零点})$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\frac{\partial u}{\partial x}}{u}\right)=0 \Rightarrow \frac{\frac{\partial u}{\partial x}}{u}=\varphi(x) \Rightarrow \frac{\partial}{\partial x}\ln u=\varphi(x)$$

$$\Rightarrow \ln u=\int\varphi(x)dx+\psi(y) \Rightarrow u=e^{\int\varphi(x)dx+\psi(y)}=f(x)g(y).$$

5 2 2 1

8. (10分) 设$\varphi_k(x)\in C[0,1]$, $k\in\mathbb{N}$; $\int_0^1\varphi_k^2(x)\,\mathrm{d}x=1$, $\forall k\in\mathbb{N}$. 证明：

$\forall n\in\mathbb{N},\exists\ c_1,\cdots,c_n\in\mathbb{R}\ \ s.t.\ \ \max\limits_{x\in[0,1]}\sum\limits_{k=1}^n c_k\varphi_k(x)\geqslant\sqrt{n}$, 且$\sum\limits_{k=1}^n c_k^2=1$.

【证明】:

$$\max_{x\in[0,1]}\sum_{k=1}^n c_k\varphi_k(x)\geqslant\sqrt{n} \Leftarrow \exists\xi\in[0,1]\ \text{s.t.}\ \sum_{k=1}^n c_k\varphi_k(\xi)=\sqrt{n}.$$

$$\int_0^1\varphi_k^2(x)\,\mathrm{d}x=1 \Rightarrow \int_0^1\sum_{k=1}^n\varphi_k^2(x)dx=n \Rightarrow \sum_{k=1}^n\varphi_k^2(\xi)=n.\text{(积分中值定理)}$$

令$c_k=\dfrac{\varphi_k(\xi)}{\sqrt{\sum_{k=1}^n\varphi_k^2(\xi)}}$, 则$\sum\limits_{k=1}^n c_k\varphi_k(\xi)=\sqrt{\sum\limits_{k=1}^n\varphi_k^2(\xi)}=\sqrt{n}$. 得证.

2 4 4

9. (10分) 设$F\subset\mathbb{R}^n$是紧集, $E\subset\mathbb{R}^n$是开集, 且$F\subset E$. 证明：

存在开集O使得$F\subset O\subset\overline{O}\subset E$.

【证明】:

$F\subset\mathbb{R}^n$紧 $\Rightarrow$ F有界闭.

E开, $F\subset E \Rightarrow \forall x\in F,\ \exists\delta_x>0$ s.t. $B(x,\delta_x)\subset E \Rightarrow F\subset\bigcup\limits_{x\in F}B(x,\delta_x)\subset E$.

进一步, $F\subset\bigcup\limits_{x\in F}B(x,\dfrac{\delta_x}{2})\subset\bigcup\limits_{x\in F}B(x,\delta_x)\subset E$. 4

由于F是紧集, 所以$\exists x_1,\cdots,x_m\in F$ s.t. $F\subset\bigcup\limits_{j=1}^m B(x_j,\dfrac{\delta_{x_j}}{2})\subset\bigcup\limits_{j=1}^m B(x_j,\delta_{x_j})\subset E$.

$\bigcup\limits_{j=1}^m B(x_j,\dfrac{\delta_{x_j}}{2})$是开集, 并且, $\overline{\bigcup\limits_{j=1}^m B(x_j,\dfrac{\delta_{x_j}}{2})}\subset\bigcup\limits_{j=1}^m\overline{B(x_j,\dfrac{\delta_{x_j}}{2})}\subset\bigcup\limits_{j=1}^m B(x_j,\delta_{x_j})\subset E$.

注意上式中关系 $\overline{\bigcup\limits_{j=1}^m B(x_j,\dfrac{\delta_{x_j}}{2})}\subset\bigcup\limits_{j=1}^m B(x_j,\delta_{x_j})$ 成立的条件是"有限并";

对无限并, 有可能包含关系相反.

这样, 令$O=\bigcup\limits_{j=1}^m B(x_j,\dfrac{\delta_{x_j}}{2})$, 则定理得证.

4 2