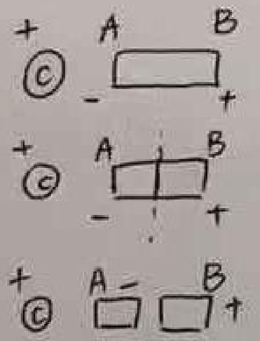
1.1 电荷的本质

物体吸引轻小物质的能力的衡量

玻璃+丝绸：正

橡胶+毛皮：负



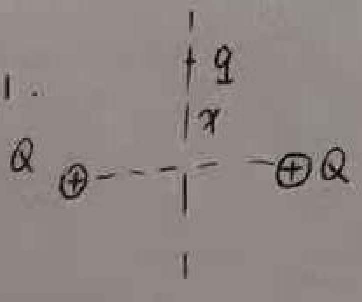
中性的导体 → 两块带异种电荷的金属块

当重新接触后

⇒ 电荷守恒

库仑定律是静电场的基本原理

eg1.



$$F = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{(r^2+x^2)^{\frac{3}{2}}}\cdot x$$

$$F' = 0 \Rightarrow \frac{(r^2+x^2)^{\frac{3}{2}} - \frac{3}{2}(r^2+x^2)^{\frac{1}{2}}\cdot 2x\cdot x}{(r^2+x^2)^3} = 0$$

$$x = \frac{\sqrt{2}}{2}r$$

1.2 接触力 & 非接触力

$\underbrace{\text{场、实物}}_{\text{物质}}$ 能量、动量、角动量

eg1: $\frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1}{x_0^2} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_2}{(l-x_0)^2} \Rightarrow l - x_0 = \sqrt{2}x_0,\ x_0 = \frac{l}{1+\sqrt{2}} = (\sqrt{2}-1)l$

? 平衡

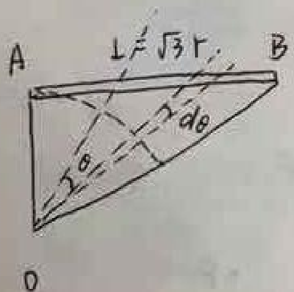
$$F = q\left(\frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1}{(x_0+r)^2} - \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_2}{(l-x_0-r)^2}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{q_1}{x_0^2}\left(1-\frac{2r}{x_0}\right) - q_2\cdot\frac{1}{(l-x_0)^2}\left(1+\frac{2r}{l-x_0}\right)\right)$$

$$= -\frac{q}{4\pi\varepsilon_0}\left(\frac{q_1}{x_0^3}+\frac{q_2}{(l-x_0)^3}\right)\cdot 2r \Rightarrow \text{稳定平衡}$$

eg2: 

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{4\pi\varepsilon_0}\cdot\frac{\lambda R d\theta}{R^2}\cdot\cos\theta = \frac{\lambda}{4\pi\varepsilon_0 R}\cdot\sqrt{2}$$

eg3: 

$$dE_{直} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\cdot\frac{r}{\cos^2\theta}\cdot d\theta}{\left(\frac{r}{\cos\theta}\right)^2} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda r d\theta}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda d\theta}{r}$$

$$dE_{圆} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda r d\theta}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda d\theta}{r}$$

$\Rightarrow$ 等效

$$E = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r}\cdot \overset{\cos\theta}{d\theta} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q/\sqrt{3}r}{r}$$

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{\sqrt{3}Q}{3r^2}$$

1.3 电通量 & 流体通量

有源、无旋

$\uparrow$ $\uparrow$

高斯 静电场环路

推导：

立体角 $d\Omega = \frac{dS}{r^2} = \frac{r\sin\theta d\varphi \cdot r d\theta}{r^2} = \sin\theta d\theta d\varphi$

全角 $\Omega = \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \sin\theta d\theta d\varphi = 2\cdot 2\pi = 4\pi$ （φ描述平面，θ描述高度 当φ旋转了2π时，θ只需从0到π 环带从顶到底）

高斯定理 点电荷 $\int_0^{4\pi} \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\cdot r^2 d\Omega = \frac{q}{\varepsilon_0}$

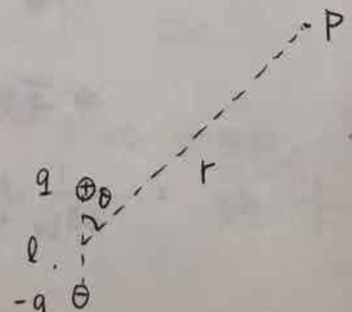
⇒ 电荷体系同样成立

eg1： 球冠立体角 $\int_0^{\alpha} \frac{2\pi r\sin\theta\cdot r d\theta}{r^2} = 2\pi(1-\cos\alpha)$

或 $\int_0^{\alpha}\int_{\varphi=0}^{2\pi}\sin\theta d\theta d\varphi = 2\pi(1-\cos\theta)$

⇒ 电通量相同 $q_1(1-\cos\alpha)\cdot\frac{2\pi}{4\pi} = q_2(1-\cos\beta)\frac{2\pi}{4\pi}$

$\beta = \cos^{-1}\left(1-\frac{q_1}{q_2}(1-\cos\alpha)\right)$



均匀球&球壳：对称+高斯定理

长直导线：与圆弧等效

平板：平移、镜像对称、+高斯定理

电偶极子：

$$E_r = \frac{1}{4\pi\varepsilon_0}\frac{q}{(r-\frac{l}{2}\cos\theta)^2} - \frac{1}{4\pi\varepsilon_0}\frac{q}{(r+\frac{l}{2}\cos\theta)^2} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r^2}(1+\frac{l}{r}\cos\theta) - \frac{1}{r^2}(1-\frac{l}{r}\cos\theta)\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\cdot\frac{2l}{r^3}\cos\theta$$

$$E_\theta = \frac{1}{4\pi\varepsilon_0}\frac{q\sin\theta}{(r-\frac{l}{2}\cos\theta)^2}\cdot\frac{l}{2r} + \frac{1}{4\pi\varepsilon_0}\frac{q\sin\theta}{(r+\frac{l}{2}\cos\theta)^2}\cdot\frac{l}{2r} = \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{r^2}\cdot\frac{l}{r}\sin\theta = \frac{q}{4\pi\varepsilon_0}\cdot\frac{l}{r^3}\sin\theta$$

eg2: $E = \frac{\iiint E\,dV}{V} = \frac{\iiint \frac{1}{4\pi\varepsilon_0}\frac{q}{r_i^2}\cdot\hat{r}_i\cdot dV}{V} = \iiint \frac{1}{4\pi\varepsilon_0}\cdot\frac{q/V}{r_i^2}\cdot dV\cdot\hat{r}_i$

等效于一个半径为R，电荷密度为 $\frac{q}{V}$ 的带电球体在P点产生的场强

$\therefore E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q/V\cdot\frac{4}{3}\pi r^3}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}$ 将两个物理模型用数学表达式连系

1.4

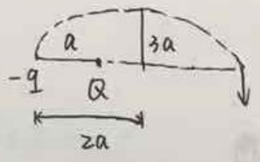
只有保守场才有势的概念

电势差是绝对的，电势是相对的

(能量与能量差)

电势就是电场对路径的积分

eg1. 角守+能守



$A = 2a$

解1: $v_1 a = v_2\cdot 3a$

$\frac{1}{2}mv_1^2 - \frac{KQq}{a} = \frac{1}{2}mv_2^2 - \frac{KQq}{3a}$

$\Rightarrow v_1 = \sqrt{\frac{3KQq}{2ma}}$

$v_2 = \sqrt{\frac{KQq}{6ma}}$

又 $V_3 = \sqrt{\frac{KQq}{3ma}} \Rightarrow \Delta E = \frac{1}{2}m\Delta v^2 = \frac{KQq}{12a}$

解2: 万有引力，记半长轴为A，r_1为近轴长，r_2为远轴长

$v_{近} = \sqrt{\frac{GM}{A}\cdot\frac{r_2}{r_1}} = \sqrt{\frac{KQq}{2ma}\cdot 3}$ $\quad v_{远} = \sqrt{\frac{GM}{A}\cdot\frac{r_1}{r_2}} = \sqrt{\frac{KQq}{6ma}}$

又圆运动 $v_{圆} = \sqrt{\frac{GM}{R}} = \sqrt{\frac{KQq}{3ma}} \Rightarrow \Delta E = \frac{KQq}{12a}$

解3: 几何方法 $\rho = \frac{(1+y'^2)^{\frac{3}{2}}}{|y''|}$ $\quad \rho_{最大} = \frac{a^2}{b}$, $\rho_{最小} = \frac{b^2}{a}$, $A = 2a$, $B = \sqrt{3}a$

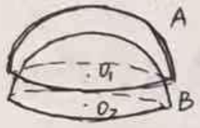
$\Rightarrow \cancel{\rho_{大} = \frac{r_2^2}{r_1} = \frac{4}{3}a}$, $\rho_{小} = \frac{B^2}{A} = \cancel{\frac{1}{3}a}\ \frac{3}{2}a$

$\Rightarrow m\frac{v_1^2}{\rho_{小}} = \frac{KQq}{a^2}$, $m\frac{v_2^2}{\rho_{小}} = \frac{KQq}{(3a)^2}$ $\Rightarrow v_1 = \sqrt{\frac{3KQq}{2ma}}$, $v_2 = \sqrt{\frac{KQq}{6ma}}$

又 $v_{圆} = \sqrt{\frac{KQq}{3ma}} \Rightarrow \Delta E = \frac{KQq}{12a}$

1.: 点乘满足分配律 + 场强叠加原理

eg2.



$$U_{A1} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2R}, \quad U_{B2} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2R}$$

$$U_1 = U_{A1} + U_{B1}, \quad U_2 = U_{A2} + U_{B2}$$

$$U_{A2} + U_{B1} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R}$$

$$\therefore U_2 = \frac{1}{4\pi\varepsilon_0}\frac{2Q}{R} - U_1$$

eg3. 四面等势，且内部无电荷无电场，即为等势体

$$\phi_0 = \frac{\phi_1 + \phi_2 + \phi_3 + \phi_4}{4}$$

1.5

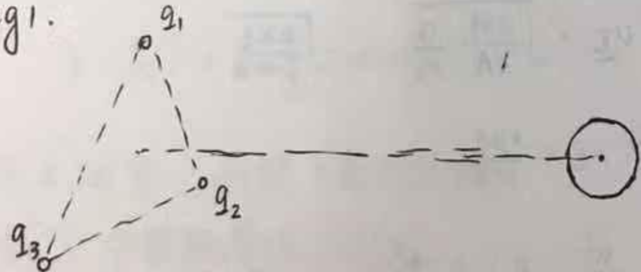
导体的特点：自由电荷

静电感应：在外场力作用下导体表面出现电荷分布

当电荷分布稳定后，不再有电荷运动与电场改变 → 静电平衡

⇒ 场强为0，反证，互相影响互相制约

eg1.



设大球带电Q，半径R，小球半径r，距离为l，三角形边为a

有连接后等势 $\frac{k(Q-q_1)}{R} + \frac{kq_1}{l} = \frac{kq_1}{r} + \frac{k(Q-q_1)}{l}$

$$\frac{k(Q-q_1-q_2)}{R} + \frac{kq_1}{l} + \frac{kq_2}{l} = \frac{kq_2}{r} + \frac{kq_1}{a} + \frac{k(Q-q_1-q_2)}{l}$$

$$k(Q-q_1-q_2-q_3)/R + \frac{k(q_1+q_2+q_3)}{l} = \frac{k(q_1+q_2)}{a} + \frac{kq_3}{r} + \frac{k(Q-q_1-q_2-q_3)}{l}$$

$$\frac{Q-q_1}{R}=\frac{q_1}{r},$$ ①

有 $l>>a$，即 $$\frac{Q-q_1-q_2}{R}=\frac{q_2}{r}+\frac{q_1}{a},$$ ②

$$\frac{Q-q_1-q_2-q_3}{R}=\frac{q_3}{r}+\frac{q_1}{a}+\frac{q_2}{a}$$ ③

$-③+②$: $q_3\left(\frac{1}{R}+\frac{1}{r}\right)=q_2\left(\frac{1}{r}-\frac{1}{a}\right)$

$①-②$: $q_2\left(\frac{1}{R}+\frac{1}{r}\right)=q_1\left(\frac{1}{r}-\frac{1}{a}\right)$ $\Rightarrow q_3=\frac{q_2^2}{q_1}$

由高斯定理，无场强则无电荷

导体表面场强 & 无限大导体平板场强

eg2. $\frac{k(Q+q)}{r^2}=\frac{kQ}{a^2} \Rightarrow q=Q\left(\frac{r^2}{a^2}-1\right)$

又 $dq=\rho\cdot 4\pi r^2 dr$

$Q\cdot\frac{2r}{a^2}=\rho\cdot 4\pi r^2 dr$

$\therefore \rho=\frac{Q}{4\pi a^2}\cdot\frac{2}{r}$

内表面的电荷分布等效于一个相反电荷的带电体，用高斯定理可求其电荷面密度

eg3. $Q_1, -Q_1, Q_1+Q_2$

$\frac{Q_1}{4\pi\varepsilon_0}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)+\frac{Q_1+Q_2}{4\pi\varepsilon_0}\cdot\frac{1}{R_3}$

1.6

静电能（静电势能）$\begin{cases} \text{自能：带电体的电荷从无穷远处移到一起外力作的功} \\ \text{互能：将一个带电体从电荷体系移到无穷远电场力作的功} \end{cases}$

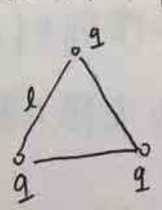
无穷远 → 电荷体系、外力作功

⇔ 电荷体系 → 无穷远 电场力作功

$E = E_1 = E_2 = \frac{1}{2}(E_1 + E_2)$ ⇒ 电势能是共同具有的

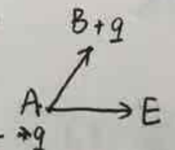
点电荷组 $E = \sum_{\substack{i,j=1 \\ i<j}}^{n} k\frac{q_i q_j}{r_{ij}} = \sum_{i,j=1}^{n} k\frac{q_i q_j}{2r_{ij}}$ ①数学 ③电势与电量成正比 线性求和取半即可

2 （也可以理解为任意两个点电荷间的能量算了两遍）②总体



$\frac{3kq^2}{l}$

扩展：电偶极子在外场中的静电能



$E_p = q \cdot d\varphi\big|_{AB} = -q \cdot \vec{E} \cdot d\vec{l}$

$= -\vec{p} \cdot \vec{E}$

$M = \vec{p} \times \vec{E}$，$F = -P\dot{E}$，$\dot{E}$ 指电场沿电偶极矩的变化率

eg1.

任取离子 A_0，则与第1对离子有互能 $E = \frac{-2ke^2}{a}$

与第2对离子有互能 $E = \frac{2ke^2}{2a}$

⋮

与第i对离子有互能 $E = \begin{cases} \frac{2ke^2}{ia} & i\text{为偶} \\ -\frac{2ke^2}{ia} & i\text{为奇} \end{cases}$

任一离子与电荷体系共同具有电势能

$$E_0 = \frac{2ke^2}{a}\left(-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots\right) = -\frac{2ke^2}{a}\ln 2$$

则总体系的能量 $\frac{2N \cdot E_0}{2} = -N \cdot \frac{2ke^2}{a} \cdot \ln 2$

注： $\frac{1}{1+x} = 1 + \frac{-x}{1!(1+x)^2} + \frac{2x^2}{2!(1+x)^3} + \frac{-3!x^3}{3!(1+x)^4} + \cdots$

$= 1 - x + x^2 - x^3 + \cdots$

两边积分 $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

取 $x=1$，有 $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

带电体与电荷体系没有区别

$W = \frac{1}{2}\iiint_V \rho\, dV \cdot U$

$= \frac{1}{2}\iint_S \sigma\, ds \cdot U$

$= \frac{1}{2}\int_l \eta\, dl \cdot U$

或者 λ

总结：自能恒为正

总静电能是自能与互能之和

孤立带电体

eg2.

$r = \frac{R}{\sqrt[3]{2}}$

$E_0 = \frac{1}{2}\cdot\frac{kQ_0^2}{R}$ $\quad E' = 2\cdot\frac{1}{2}\frac{k\left(\frac{Q_0}{2}\right)^2}{r} = \frac{1}{4}\cdot\frac{kQ_0^2}{r} = \frac{1}{4}\cdot\frac{kQ_0^2}{R}\cdot\sqrt[3]{2}$

$\Delta E = E_0 - E' = \frac{1}{2}\frac{kQ_0^2}{R}\left(1 - \frac{\sqrt[3]{2}}{2}\right) \Rightarrow E_k = \frac{\Delta E}{2} = \frac{1}{8}Q_0 U_0 (2 - \sqrt[3]{2})$

孤立带电球

$\Delta\varphi = \int_r^R \frac{1}{4\pi\varepsilon_0}\frac{\rho\cdot\frac{4}{3}\pi r^3}{r^2}\cdot dr = \frac{1}{4\pi\varepsilon_0}\cdot\rho\cdot\frac{4}{3}\pi\cdot\frac{R^2-r^2}{2}$

$\varphi_0 = \frac{1}{4\pi\varepsilon_0}\cdot\rho\cdot\frac{4}{3}\pi R^2$ $\qquad \frac{1}{2}\cdot\frac{\rho^2}{4\pi\varepsilon_0}\cdot\frac{8\pi^2}{3}\cdot\left(\frac{R^5}{3} - \frac{R^5}{5}\right)$

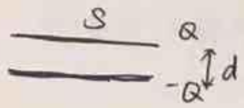
$\Rightarrow$

$\Rightarrow E_p \cancel{Q} = \frac{1}{2}\cdot\int_0^R \rho\cdot 4\pi r^2 dr\cdot\frac{1}{4\pi\varepsilon_0}\cdot\rho\,\frac{4}{3}\pi\,\frac{R^2-r^2}{2}$

$+ \frac{1}{2}\cdot\rho\,\frac{4}{3}\pi R^3\cdot\frac{1}{4\pi\varepsilon_0}\cdot\rho\cdot\frac{4}{3}\pi R^2 \Rightarrow \frac{1}{2}\cdot\frac{\rho^2}{4\pi\varepsilon_0}\cdot\frac{16}{9}\pi^2 R^5$

$= \frac{\rho^2}{4\pi\varepsilon_0}\cdot\frac{8}{45}\pi^2 R^5 + \frac{\rho^2}{4\pi\varepsilon_0}\cdot\frac{8}{9}\pi^2 R^5 = \frac{\rho^2}{4\pi\varepsilon_0}\pi^2 R^5\cdot\frac{48}{45} =$

平行板

1. 

中线电势为0

正极电势为 $\frac{Q/S}{\varepsilon_0}\cdot\frac{d}{2}$

负极电势为 $-\frac{Q/S}{\varepsilon_0}\cdot\frac{d}{2}$

$\Rightarrow E_p=\frac{Q^2/S}{\varepsilon_0}\cdot\frac{d}{2}$

导体球

$\varphi=\frac{1}{4\pi\varepsilon_0}\frac{Q}{R}$

$\Rightarrow E_p=\frac{Q\varphi}{2}$

$=\frac{Q^2}{8\pi\varepsilon_0 R}$

eg3. 如上

2. 电场的能量

导体球&平行板体系的能量都知道了

但是能量是电荷携带的吗？

在静电场中没有结论，电磁波说明了静电能是定域于静电场的

3. $w_e=\frac{1}{2}\varepsilon_0E^2$

在R处取微分

4. $\int_R^\infty w_e\cdot 4\pi r^2dr=\frac{Q^2}{8\pi\varepsilon_0}\frac{1}{R}\Rightarrow -w_e 4\pi r^2dr=\frac{-Q^2}{8\pi\varepsilon_0}\cdot\frac{dr}{r^2}$

$\therefore\Rightarrow w_e=\frac{Q^2}{32\pi^2\varepsilon_0R^4}$ 积分上限确定，变化积分下限

平板要简单一些，$w_e=\frac{\frac{Q^2/S}{\varepsilon_0}\cdot\frac{d}{2}}{Sd}=\frac{Q^2}{2\varepsilon_0d^2S^2}$

$\therefore w_e=\frac{1}{2}\varepsilon_0E^2$

eg4. $\int_R^\infty\frac{1}{2}\varepsilon_0\left(\frac{Q}{4\pi\varepsilon_0r^2}\right)^2\cdot4\pi r^2dr$

$=\int_R^\infty\frac{1}{2}\cdot\frac{Q^2}{4\pi\varepsilon_0}\cdot\frac{dr}{r^2}=\frac{Q^2}{8\pi\varepsilon_0}\cdot\frac{1}{R}$

孤立导体的电容 ⟶ 物理意义（与水容器的类比）

取决于导体的形状与线度

导体球 $C = 4\pi\varepsilon_0 R$

不可能只用一个导体的性质来描述

↑

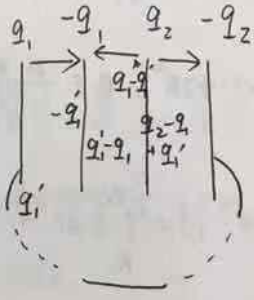
电容器：屏蔽其他导体的影响（位置，形状，感应电荷）

$C = \dfrac{Q}{U_+ - U_-}$　　记~~正极~~带正电的导体为正极，负电导体为负极

$\dfrac{\text{电量}}{\text{电势差}}$

⇒ 导体的形状、线度与相对位置有关

eg1.



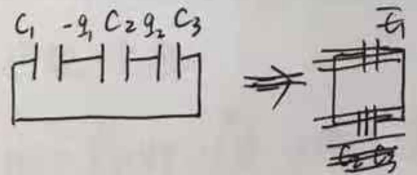
解1：设板1带电 q_1'

$$q_1' + (q_1' - q_1) + (q_2 - q_1 + q_1') = 0$$

$$\Rightarrow q_1' = -\frac{q_2 - 2q_1}{3}$$

$$\Rightarrow U = \frac{q_1' - q_1}{\varepsilon_0 S} \cdot d = -\frac{q_1 + q_2}{3\varepsilon_0 S} \cdot d$$　（设转移电荷量为 x 也可）

解2：



$C_1 = \dfrac{\varepsilon_0 S}{d}$

$C_2 = \dfrac{\varepsilon_0 S}{d} = C_3$

也可以直接设电容电量

⇒ $Q_1 + Q_2 + Q_3 = 0$，Q_i 为 C_i 的正极(左板)电量

$$\begin{cases} Q_1 - Q_3 = q_1 - q_2 \\ Q_2 - Q_1 = -q_1 \\ Q_3 - Q_2 = q_2 \end{cases} \Rightarrow \begin{aligned} Q_1 &= \frac{2q_1 - q_2}{3} \\ Q_2 &= -\frac{q_1 + q_2}{3} \\ Q_3 &= \frac{2q_2 - q_1}{3} \end{aligned} \Rightarrow U_2 = \frac{Q_2}{C_2} = -\frac{q_1 + q_2}{3\varepsilon_0 S} d$$

平行板：

$$E=\frac{\sigma}{\varepsilon_0}=\frac{q}{\varepsilon_0 S} \Rightarrow U=Ed=\frac{q}{\varepsilon_0 S}d$$

$$\Rightarrow C=\frac{Q}{U}=\frac{\varepsilon_0 S}{d}$$

同轴圆柱：

①可近似，$S=2\pi Rl$，$C=\dfrac{\varepsilon_0\cdot 2\pi Rl}{d}$

② R_1, R_2

$$E=\frac{\lambda l/\varepsilon_0}{2\pi r\cdot l}=\frac{\lambda}{2\pi\varepsilon_0}\cdot\frac{1}{r} \Rightarrow U=\int_{R_1}^{R_2}E\,dr=\frac{\lambda}{2\pi\varepsilon_0}\cdot\ln\frac{R_2}{R_1}$$

2 $$\Rightarrow C=\frac{\lambda l}{U}=\frac{2\pi\varepsilon_0 l}{\ln\frac{R_2}{R_1}}$$

同心球：

①可近似，$S=4\pi R^2$，$C=\dfrac{\varepsilon_0\cdot 4\pi R^2}{d}$

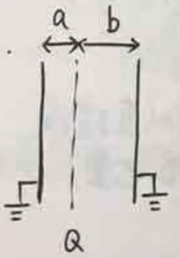
② R_1, R_2

3 $$E=\frac{Q}{4\pi\varepsilon_0 R^2} \Rightarrow U=\int_{R_1}^{R_2}E\,dr=\frac{Q}{4\pi\varepsilon_0}\cdot\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

$$\Rightarrow C=\frac{Q}{U}=\frac{4\pi\varepsilon_0}{\frac{1}{R_1}-\frac{1}{R_2}}$$

4

eg2.



$$\frac{Q_1}{\varepsilon_0 S}\cdot a=\frac{Q_2}{\varepsilon_0 S}\cdot b$$

5 $$\Rightarrow Q_1=\frac{-b}{a+b}Q$$

$$Q_2=-\frac{a}{a+b}Q$$

6 eg3.

解1：令A带电Q

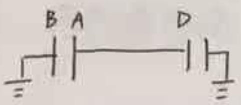
则 $U_A=\int_a^b\frac{Q}{4\pi\varepsilon_0}\cdot\frac{1}{r^2}dr=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$

则 $q_D=d\cdot\frac{4\pi\varepsilon_0 Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)=Q\left(\frac{1}{a}-\frac{1}{b}\right)d$

$$\therefore C=\frac{Q+q_D}{U_A}$$

$$=4\pi\varepsilon_0\frac{\frac{1}{a}-\frac{1}{b}+\frac{1}{d}}{\frac{1}{a}-\frac{1}{b}}\cdot d$$

$$=4\pi\varepsilon_0\left(d+\frac{ab}{b-a}\right)$$

解2：

⇒ 并联，两端连线相同

$$C = C_{AB} + C_D = 4\pi\varepsilon_0\left(\frac{ab}{b-a}\right) + 4\pi\varepsilon_0 d$$

串联：两极依次相连，电压无等势处

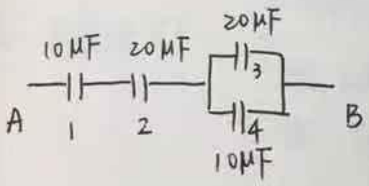
并联：两极等势

$$U_总 = U_1 + U_2 + \cdots + U_n = \frac{Q_1}{C_1} + \frac{Q_2}{C_2} + \cdots + \frac{Q_n}{C_n} = Q\left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}\right)$$

$$\Rightarrow C_总 = \frac{1}{C_1} + \cdots + \frac{1}{C_n}$$

$$U_总 = U,\quad Q_总 = C_1U + C_2U + \cdots + C_nU$$

$$\Rightarrow C_总 = C_1 + \cdots + C_n$$

eg4. 

有电压分压 $\frac{6}{11}U, \frac{3}{11}U, \frac{2}{11}U$

$$\Rightarrow U_{max} = \cancel{180V} \frac{11}{3}\cdot 60V$$
$$= 220V$$

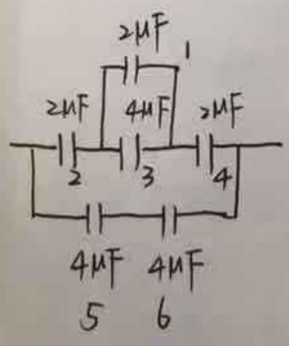
将一个充电的电容器两极用导线放电，(短路)可见火花，利用火花的热能甚至可以熔焊金属，即"电熔焊"

电能 → 热能 + 光能

$$W = \int_0^Q U\,dq = \int_0^Q \frac{1}{C}\cdot q\,dq = \frac{1}{2}\cdot\frac{1}{C}q^2\Big|_0^Q = \frac{1}{2}\frac{Q^2}{C}$$

带电体的能量公式 $\sum \frac{1}{2} q_i U_i$

①数学
②分摊能量 ⇒ 自洽、完整
③重复计算
④电容

eg5. 

$U_2 = U_4 = \frac{600}{7}V$，$U_1 = U_3 = \frac{200}{7}V$

$U_5 = U_6 = 100V$ ⇒ $Q_1 = C_1U_1$，$E_1 = \frac{1}{2}C_1U_1^2$

⋮

1.8 静电场习题课

eg1. 记空间内某点到圆心的距离为 r

1\. 当 $r \le R$ 时

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r^2}, \quad Q = \int_0^r 4\pi r^2 \cdot dr \cdot \rho_0 (1 - \frac{r}{R}) = 4\pi\rho_0 (\frac{r^3}{3} - \frac{r^4}{4R})$$

$$\therefore E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{r^2} (4\pi\rho_0 (\frac{r^3}{3} - \frac{r^4}{4R}))$$

$$= \frac{\rho_0}{\varepsilon_0} \cdot (\frac{r}{3} - \frac{r^2}{4R})$$

2\. 当 $r > R$ 时

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q_0}{r^2}, \quad Q_0 = 4\pi\rho_0 (\frac{R^3}{3} - \frac{R^3}{4})$$

$$= \frac{1}{3}\pi R^3 \rho_0$$

3\. $\therefore E = \dfrac{\rho_0 R^3}{12\varepsilon_0 r^2}$

4\. $\therefore E_{max} = \dfrac{\rho_0 R^3}{9\varepsilon_0 R^2}, \quad r = \dfrac{2}{3} R$

$$= \frac{\rho_0 R}{9\varepsilon_0}$$

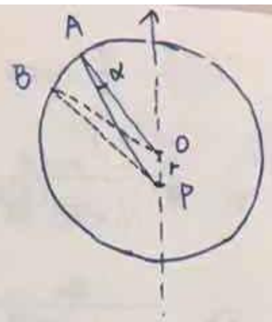
5\. eg2. 取小圆柱面高斯面，半径 r，高 x （$r, x \ll R$）

$$E_{侧} \cdot 2\pi r x = E_{底} \cdot \pi r^2$$

6 又 $E_{底} = \dfrac{1}{4\pi\varepsilon_0} \cdot \dfrac{Q}{(R^2+x^2)^{\frac{3}{2}}} \cdot x = \dfrac{1}{4\pi\varepsilon_0} \cdot \dfrac{Qx}{R^3}$

$$\therefore E_{侧} = \frac{Qr}{8\pi\varepsilon_0 R^3}$$

eg五. 解2. 用数学方法



记圆上某点到P点距离为 x

位矢与正极轴夹角为 θ

x与半径、OP构成三角形

$$x^2+r^2-2xr\cos\theta = R^2 \Rightarrow x=\frac{1}{2}\left(2r\cos\theta+\sqrt{(2r\cos\theta)^2-4(r^2-R^2)}\right)$$

$$=r\cos\theta+\sqrt{R^2-r^2\sin^2\theta}$$

又已知 $r\ll R$, $x=r\cos\theta+R\left(1-\frac{r^2}{2R^2}\sin^2\theta\right)$

令位矢变化小幅角 $d\theta$

有 $\widehat{AB}$ 长度为 dl

三种计算方法：① $\cancel{\frac{dl}{\cos\alpha}}\ dl\cdot\cos\alpha = x\,d\theta$

③ $(dl)^2=\sqrt{(dx)^2+(x\,d\theta)^2}^{\,2}$

② $x^2+(x')^2-2xx'\cos d\theta=(dl)^2$

①: $dl=\dfrac{x\,d\theta}{\cos\alpha}=\dfrac{x\,d\theta}{1-\frac{1}{2}\left(\frac{r\sin\theta}{R}\right)^2}$

②: $dl=\left(x^2+x^2+2x\,dx+(dx)^2-2x(x+\cancel{dx}\,dx)\cos d\theta\right)^{\frac{1}{2}}$

$$=\left(2x^2\cdot\frac{1}{2}(d\theta)^2+(dx)^2+2x\,dx\cdot\frac{1}{2}(d\theta)^2\right)^{\frac{1}{2}}$$

$$=\cancel{(x^2+x\,dx)^{\frac{1}{2}}d\theta=\left(1+\frac{dx}{x}\right)^{\frac{1}{2}}\cdot x\,d\theta}$$

$$\left(x^2(d\theta)^2+(dx)^2+x\,dx\,(d\theta)^2\right)^{\frac{1}{2}}$$

$$=x\,d\theta\cdot\left(1+\left(\frac{dx}{x\,d\theta}\right)^2\right)^{\frac{1}{2}}$$

$$=x\,d\theta\left(1+\left(-\frac{r\sin\theta+R\cdot\frac{r^2}{2R^2}2\sin\theta\cos\theta}{r\cos\theta+R\left(1-\frac{r^2}{2R^2}\sin^2\theta\right)}\right)^2\right)^{\frac{1}{2}}$$

$$=x\,d\theta\left(1+\left(\frac{r}{R}\cdot\frac{\sin\theta+\frac{r}{R}\sin\theta\cos\theta}{\frac{r}{R}\cos\theta+\left(1-\frac{r^2}{2R^2}\sin^2\theta\right)}\right)^2\right)^{\frac{1}{2}}=x\,d\theta\left(1+\left(\frac{r}{R}\cdot\sin\theta\right)^2\right)^{\frac{1}{2}}$$

$$dE = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\lambda\, dl}{x^2} \cdot \cancel{\frac{r}{x}} \cos\theta$$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{\lambda}{x^2} \cdot \cancel{x} \cdot \frac{x\, d\theta}{1 - \frac{1}{2}\left(\frac{r\sin\theta}{R}\right)^2} \cdot \cos\theta$$

$$= \frac{\lambda \cancel{x}}{4\pi\varepsilon_0} \cdot \frac{1}{x^{\cancel{2}}} \cdot \frac{d\theta}{1 - \frac{1}{2}\left(\frac{r\sin\theta}{R}\right)^2} \cdot \cos\theta$$

$$\Rightarrow E = \int_0^{2\pi} dE = \frac{\lambda}{4\pi\varepsilon_0} \cdot \frac{d\theta \cos\theta}{R\left(1 + \frac{r}{R}\cos\theta - \frac{r^2}{2R^2}\sin^2\theta\right)\left(1 - \frac{1}{2}\left(\frac{r\sin\theta}{R}\right)^2\right)}$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \cdot \frac{1}{R} \int_0^{2\pi} \frac{\cos\theta\, d\theta}{1 + \frac{r}{R}\cos\theta - \frac{r^2}{2R^2}\sin^2\theta - \frac{r^2}{2R^2}\sin^2\theta}$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \cdot \frac{1}{R} \cdot \int_0^{2\pi} \left(1 - \frac{r}{R}\cos\theta + \frac{r^2}{R^2}\sin^2\theta\right) \cos\theta\, d\theta$$

$$= \frac{\lambda}{4\pi\varepsilon_0 R} \cdot \left(0 - \frac{r}{R} \cdot \frac{-1}{2} \cdot 2\pi + \frac{r^2}{R^2} \cdot 0\right)$$

$$= \frac{1}{4\pi\varepsilon_0 R} \cdot \frac{Q}{2\pi R} \cdot \frac{r}{R}\pi = \frac{Qr}{8\pi\varepsilon_0 R^3}$$

例3

$$\lambda(\theta)=\frac{1}{2}\cdot R\sin\theta\cdot 2\pi\cdot Rd\theta\cdot\sigma\cdot\frac{1}{Rd\theta}=\sigma\pi R^2\sin\theta=\frac{Q}{4R}\sin\theta$$

⇒ 球壳压成圆环，电场为0（直径上），电势为0

⇒ 带电球体压成圆饼，电场线性分布

例4 (1) $q_{1左}=q_{n右}=\dfrac{\sum_{i=1}^{n}Q_i}{2}$

(2) $q_{1右}=Q_1-q_{1左}=\dfrac{Q_1-\sum_{i=2}^{n}Q_i}{2}$

$q_{2左}=-q_{1右}=\dfrac{\sum_{2}^{n}Q_i-Q_1}{2}$，$q_{2右}=Q_2-q_{2左}=\dfrac{Q_1+Q_2-\sum_{3}^{n}Q_i}{2}$

$q_{3左}=-q_{2右}=\dfrac{\sum_{3}^{n}Q_i-\sum_{1}^{2}Q_i}{2}$，$q_{3右}=Q_3-q_{3左}=\dfrac{\sum_{1}^{3}Q_i-\sum_{4}^{n}Q_i}{2}$

⋮

$q_{k左}=\dfrac{\sum_{k}^{n}Q_i-\sum_{1}^{k-1}Q_i}{2}$，$q_{k右}=\dfrac{\sum_{1}^{k}Q_i-\sum_{k+1}^{n}Q_i}{2}$

⋮

$q_{n左}=\dfrac{Q_n-\sum_{1}^{n-1}Q_i}{2}$，$q_{n右}=\dfrac{\sum_{1}^{n}Q_i}{2}$

例5

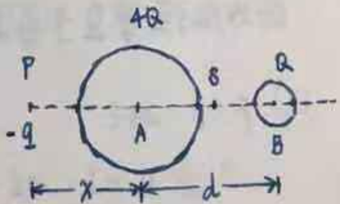
~~I. 进入球A前~~

$\cancel{E_0=-\frac{1}{4\pi\varepsilon_0}\left(\frac{4Qq}{x}+\frac{Qq}{x+d}\right)}$

~~IV. 到达B球心~~

$\cancel{E_p'''=-\frac{1}{4\pi\varepsilon_0}\frac{4Qq}{d}}$

$\cancel{-\frac{1}{4\pi\varepsilon_0}\cdot Qq}$



~~II. 球A内~~

$\cancel{E_p'=\frac{1}{4\pi\varepsilon_0}\left(\frac{4Qq}{R}+\frac{Qq}{d}\right)}$

$$\frac{1}{4\pi\varepsilon_0}\frac{4Q}{AS^2}=\frac{1}{4\pi\varepsilon_0}\frac{Q}{BS^2}\Rightarrow AS=\frac{2}{3}d,\quad BS=\frac{1}{3}d$$

~~III. 进入球B前~~

$\cancel{E_p''=\frac{1}{4\pi\varepsilon_0}\cdot\frac{4Qq}{d-r}-\frac{1}{4\pi\varepsilon_0}\cdot\frac{Qq}{r}}$

$$\frac{1}{4\pi\varepsilon_0}\cdot\frac{4Qq}{x}+\frac{1}{4\pi\varepsilon_0}\frac{Qq}{x+d}=\frac{1}{4\pi\varepsilon_0}\frac{4Qq}{\frac{2}{3}d}+\frac{1}{4\pi\varepsilon_0}\frac{Qq}{\frac{1}{3}d}$$

$$\Rightarrow x=\frac{2}{9}(\sqrt{10}-1)d$$

2.4

均匀磁场对载流线圈的力矩

$$\vec M = \oint \vec r\times(I\,d\vec l\times\vec B) = \frac12\oint\left(\vec r\times(I\,d\vec l\times\vec B) + I\,d\vec l\times(\vec r\times\vec B)\right)$$
$$\frac12\oint\left(\vec r\times(I\,d\vec l\times\vec B) - I\,d\vec l\times(\vec r\times\vec B)\right)$$

$(d\vec l = d\vec r)$

$$= \frac12\oint d\left(\vec r\times(I\,d\vec r\times\vec B)\right) + \frac12\oint(\vec r\times I\,d\vec r)\times\vec B$$

$$= \frac12\cdot 0 + \frac12\cdot I\cdot 2\vec S\times\vec B = I\vec S\times\vec B$$

(注: $A\times(B\times C) - B\times(A\times C) = (AC)B - (AB)C - (BC)A + (BA)C$

$$= (AC)B - (BC)A$$
$$= -C\times(A\times B)$$
$$= (A\times B)\times C\,)$$

eg1. 单位面积受力 $p = \frac12 B_0\cdot\sigma$, (1) $F = p\cdot\cancel{2\pi R l}\ 2R\cdot l$

应该~~乘载面~~截面积的2Rl $\quad = \frac12\cdot\frac{\mu_0 I}{2\pi R}\cdot\frac{I}{2\pi R}\cdot\cancel{2\pi R l}\ 2R\cdot l$

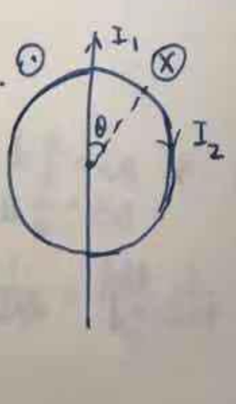
而非表面积 $\quad = \frac{\mu_0 I^2 \cancel{R} l}{4\pi^2 R}$

因为压强垂直于表面,所以合力对应表面的投影

(2) $F = p\cdot 2Rl$

$$= \frac12\mu_0 nI\cdot nI\cdot 2Rl = \mu_0 n^2 I^2 R l$$

eg2.



$$B_1(r) = \frac{\mu_0 I_1}{2\pi r}$$

$$F_2 = \int I_2\cdot R\,d\theta\cdot B_1(R\sin\theta)\cdot\sin\theta$$
$$= \int_0^{2\pi} I_2\cdot R\,d\theta\cdot\frac{\mu_0 I_1}{2\pi R\sin\theta}\cdot\sin\theta = \mu_0 I_1 I_2$$

eg3. (1) $dB = \frac{\mu_0}{4\pi} \cdot \frac{\lambda dr \cdot \frac{\omega}{2\pi} \cdot 2\pi r}{r^2} = \frac{\mu_0 \lambda}{2\pi} \frac{\omega}{2\pi} \cdot \frac{dr}{r}$

$\Rightarrow B = \frac{\mu_0 \omega \lambda}{4\pi} \cdot \ln \frac{a+b}{a}$

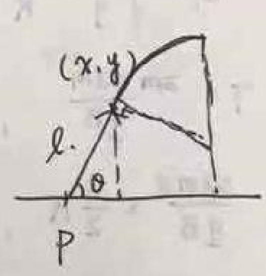
(2) $p_m = \int_a^{a+b} \lambda dr \cdot \frac{\omega}{2\pi} \cdot \pi r^2 = \frac{1}{2}\omega\lambda \cdot \frac{1}{3}\left((a+b)^3 - a^3\right)$

(3) $B_0 = \frac{\mu_0 \omega \lambda}{4\pi} \cdot \left(\frac{b}{a}\right)$, $p_m = \frac{1}{6}\omega\lambda \cdot 3 \cdot a^2 b$

$= \frac{1}{2}\omega\lambda a^2 b$

$\Rightarrow$ 点电荷的运动磁场与磁矩

7.6 习题



eg1. 由几何关系 $|x| + l\cos\theta = a$

$y = l\sin\theta$

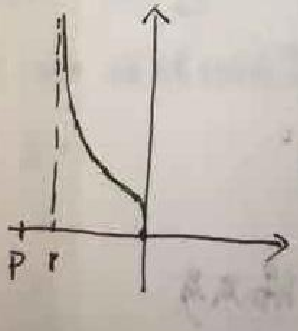
$|x| = R\sin\theta$

消去 l、θ，有 $|x| + \frac{y}{\frac{|x|}{R}} \cdot \left(1 - \left(\frac{|x|}{R}\right)^2\right)^{\frac{1}{2}} = a$

化简有 $y = \frac{a - |x|}{\left(1 - \left(\frac{x}{R}\right)^2\right)^{\frac{1}{2}}} \cdot \frac{|x|}{R}$，即 $y^2 = \frac{(a - |x|)^2}{R^2 - x^2} \cdot x^2$ ，$0 \le |x| < r$

是对y轴对称的边界

① $R < a$, $\theta \in [0, \frac{\pi}{2})$

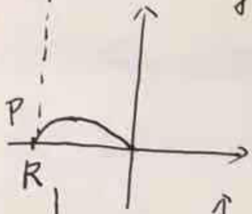


有 $y = \frac{a - |x|}{(R^2 - x^2)^{\frac{1}{2}}} \cdot |x|$

$y' = \frac{(a - 2x)(R^2 - x^2)^{\frac{1}{2}} - (a - |x|)|x| \cdot \frac{1}{2}(R^2 - x^2)^{-\frac{1}{2}} \cdot (-2x)}{(R^2 - x^2)} = 0$

$\Rightarrow$ $x \to 0$ 时，$y' \to +\infty$

② $R=a$

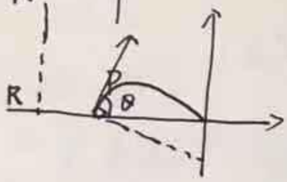


$y^2=\frac{(a-|x|)^2}{R^2-x^2}\cdot x^2$

$|x|=a$ 时，$y^2=\frac{(R-|x|)^2}{(R-|x|)(R+|x|)}=\frac{R-|x|}{R+|x|}\cdot x^2$

$=0$

③



$|x|=a$ 时，仍有 $y=0$

有 θ_{max} 满足 $R\sin\theta=a$

有 $\theta_{max}=\sin^{-1}\frac{a}{R}=\sin^{-1}\left(\frac{qBa}{mv}\right)$

eg2. (1) $R=\frac{mv}{qB_1}=\frac{(2mE)^{\frac{1}{2}}}{qB_1}=\frac{(2mqU_0)^{\frac{1}{2}}}{qB_1}=\sqrt{\frac{2mU_0}{q}}\cdot\frac{1}{B_1}$

$\therefore B_1=\sqrt{\frac{2mU_0}{q}}\cdot\frac{1}{R}$

(2) $T=\frac{\cancel{2m}\,2\pi m}{qB}$，$vT=\frac{\pi}{2}R$

$\Rightarrow \frac{2\pi mv}{qB}=\frac{\pi}{2}R$，$B=\frac{4}{R}\sqrt{\frac{2mU_0}{q}}$

注：若无 B_1，电子会沿垂直于环平面的方向漂移

因角动量守恒，可得 v_ϕ（法向速度），由动力方程可求漂移速度 v_z

$mv_\phi r=mv_0R \Rightarrow v_\phi=\frac{R}{r}v_0$

$\cancel{qv_\phi dB=m\frac{dv_z}{dt}\Rightarrow q\frac{R}{r}v_0}$

$q\vec{v}\times\vec{B}=qv_r\cdot B=m\frac{dv_z}{dt}$，$qBv_r dt=m\,dv_z$

$\therefore v_z=\frac{qB}{m}(R-r)$，又 $r>R$，即 v_z 向反方面漂移

由 r_{max} 时，$v_r=0$

$v_z^2+v_\phi^2=const=v_0^2 \Rightarrow \left[\frac{qB}{m}(R-r)\right]^2+\left(\frac{R}{r}v_0\right)^2=v_0^2$

即 $\left(\frac{qBR}{mv_0}\right)^2\cdot\left(1-\frac{r}{R}\right)^2+\left(\frac{R}{r}\right)^2=1 \Rightarrow r>R$，且不会偏太多

电介质

真空下 $\oint \vec{E}\cdot d\vec{s} = \frac{1}{\varepsilon_0}\sum q$

有介质时 $\oint \vec{E}\, d\vec{s} = \frac{1}{\varepsilon_0}\sum q_{自} + \frac{1}{\varepsilon_0}\sum q_{极}$

又 $\oint \vec{P}\, d\vec{s} = -\sum q_{极}$

$\therefore \oint (\vec{E} + \frac{1}{\varepsilon_0}\vec{P})\cdot d\vec{s} = \frac{1}{\varepsilon_0}\sum q_{自}$

记 $\vec{D} = \vec{E} + \frac{1}{\varepsilon_0}\vec{P}$

磁介质

真空下 $\oint \vec{B}\cdot d\vec{l} = \mu_0 \sum I$

有介质时 $\oint \vec{B}\cdot d\vec{l} = \mu_0 \sum I_{自} + \mu_0 \sum I_{磁}$

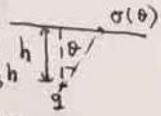
又 $\oint \vec{M}\cdot d\vec{l} = \cancel{\mu_0}\sum I_{磁}$

$\therefore \oint (\vec{B} - \mu_0 \vec{M})\cdot d\vec{l} = \mu_0 \sum I_{自}$

记 $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$

eg2 ① 点电荷 q 周围的极化电荷使得 $q \to \frac{q}{\varepsilon_r}$

② 设 $\sigma(\theta)$



$E_{上n} = \frac{\sigma(\theta)}{2\varepsilon_0} + \frac{1}{4\pi\varepsilon_0\varepsilon_r}\cdot\frac{q}{r^3}h$

$E_{下n} = -\frac{\sigma(\theta)}{2\varepsilon_0} + \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{q}{r^3}h$

$D_{上n} = D_{下n}$

$\Rightarrow \sigma(\theta) = \frac{q\cos^3\theta}{2\pi h^2}\ \frac{\varepsilon_r - 1}{\varepsilon_r(\varepsilon_r+1)}$

$\therefore q' = \int_0^{\frac{\pi}{2}} \sigma(\theta)\cdot 2\pi \frac{h\sin\theta}{\cos\theta}\cdot\frac{h}{\cos^2\theta}\, d\theta$

$= \frac{\varepsilon_r - 1}{\varepsilon_r(\varepsilon_r+1)}\ q$

$F = \frac{1}{4\pi\varepsilon_0}\cdot\frac{qq'}{(2h)^2} = \cdots$

麦克斯韦方程组

$$\begin{cases} \oint \vec{E}\cdot d\vec{l} = 0 \\ \oint \vec{D}\, d\vec{s} = \sum \frac{q_{自}}{\varepsilon_0} \\ \oint \vec{B}\cdot d\vec{s} = 0 \\ \oint \vec{H}\cdot d\vec{l} = \sum I_{自} \end{cases}$$

eg3. $B_0 = \frac{\mu_0}{4\pi}\cdot\frac{I\cdot 2\pi r}{r^2} = \frac{\mu_0 I}{2r}$

$\Rightarrow$ ~~M~~ 设磁化强度矢量为 M, $dB = \frac{\mu_0}{4\pi}\ i\cdot 2\pi r\sin\theta$

$\Rightarrow B' =$

eg1. 设介质球内电场为 E，原电场为 E_0，极化强度矢量为 P

有 $\vec{P} = \varepsilon_0(\varepsilon_r - 1)\vec{E}$

且 $\vec{E}_P = \frac{1}{4\pi\varepsilon_0}\frac{\rho\cdot\frac{4}{3}\pi r^3}{r^2} = \frac{\rho}{3\varepsilon_0}\cdot\vec{r}$

附加电场/退极化场 $\vec{E'} = \frac{\rho d}{3\varepsilon_0} = \frac{\sigma}{3\varepsilon_0} = \frac{\vec{P}}{3\varepsilon_0}$

$\therefore \vec{E_0} - \vec{E'} = \vec{E}$

$\therefore (1 + \frac{\varepsilon_r - 1}{3})\vec{E} = \vec{E_0} \Rightarrow \vec{E} = \frac{3}{2+\varepsilon_r}\vec{E_0}$

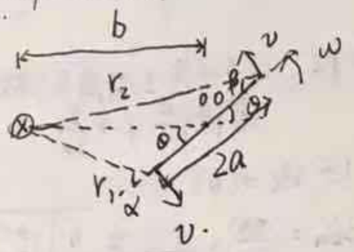
$\therefore \vec{P} = \frac{3(\varepsilon_r - 1)}{2+\varepsilon_r}\vec{E_0}\varepsilon_0$, $\vec{p_m} = \vec{P}\cdot\frac{4}{3}\pi a^3$

$\therefore \overline{P} = n\cdot\cancel{\ }\cdot\vec{p_m} = (\varepsilon_r' - 1)\overline{\overline{E}} = \frac{\varepsilon_r' - 1}{\varepsilon_r'}$

$\therefore \varepsilon_r' = (1 - \frac{3(\varepsilon_r - 1)}{2+\varepsilon_r}\cdot n\cdot\frac{4}{3}\pi a^3)^{-1}$

电磁感应习题课

eg1.



$$r_1^2 = b^2 + a^2 - 2ab\cos\theta$$

$$r_2^2 = b^2 + a^2 + 2ab\cos\theta$$

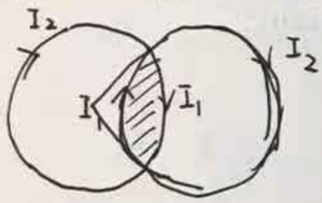
$$\varepsilon_1 = \frac{\mu_0 I}{2\pi r_1} \cdot 2a \cdot \omega a \cdot \sin\alpha$$

$$\varepsilon_2 = \frac{\mu_0 I}{2\pi r_2} \cdot 2a \cdot \omega a \cdot \sin\beta$$

$$\left\{ \begin{aligned} \frac{\sin\alpha}{b} &= \frac{\sin\theta}{r_1} \\ \frac{\sin\beta}{b} &= \frac{\sin\theta}{r_2} \end{aligned} \right.$$

$$\Rightarrow I = \frac{\varepsilon_1 - \varepsilon_2}{R} = \frac{2\mu_0 I \omega a^2 b \sin\omega t}{2\pi R}\left(\frac{1}{a^2+b^2-2ab\cos\omega t} - \frac{1}{a^2+b^2+2ab\cos\omega t} \right)$$

eg2. $\otimes B$.



$$I_2 \cdot \frac{5}{6}R + I_1 \cdot \frac{1}{6}R = \pi r^2 \frac{dB}{dt}$$

$$S_{阴} = 2\left(\frac{\pi R^2}{6} - \frac{\sqrt{3}}{4}R^2\right)$$

$$I_1 \cdot \frac{1}{3}R = \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) r^2 \cdot \frac{dB}{dt}$$

$$\Rightarrow I_1 = \left(\pi - \frac{3}{2}\sqrt{3}\right) r^2 \frac{dB}{dt} \cdot \frac{1}{R}$$

$$I_2 = \left(\frac{5}{6}\pi + \frac{\sqrt{3}}{4}\right) r^2 \cdot \frac{dB}{dt} \cdot \frac{1}{\frac{5}{6}R} = \left(\pi + \frac{3}{10}\sqrt{3}\right) r^2 \frac{1}{R} \cdot \frac{dB}{dt}$$

$$\Rightarrow F = B I_1 r - B I_2 r$$

$$= B \cdot r \cdot \frac{9}{5}\sqrt{3} r^2 \cdot \frac{dB}{dt} \cdot \frac{1}{R}$$

$$\Rightarrow \int F\,dt = \frac{9}{5}\sqrt{3} r^3 \cdot \frac{1}{R} \cdot \frac{1}{2} B_0^2 = \frac{9\sqrt{3}}{10} \frac{B_0^2 r^3}{R}$$

eg6.

(1) $Bl v_t = \frac{Q}{C}$, 又 $Bl \cdot Q = -m(v_t - v_0)$ $\therefore v_t = \frac{m v_0}{m + B^2 l^2 C}$

有 $Bl v - \frac{Q}{C} = IR$

$BIl = -m\frac{dv}{dt}$

代入: $-BIl = m \cdot \frac{1}{Bl}\left(\frac{dI}{dt}R + \frac{I}{C}\right)$

即 $\frac{m}{B^2l^2}R \cdot \frac{dI}{dt} + \left(\frac{m}{B^2l^2}\frac{1}{C}+1\right)I = 0$

$\Rightarrow I\left(1+\frac{m}{B^2l^2C}\right) = -\frac{mR}{B^2l^2} \cdot \frac{dI}{dt}$

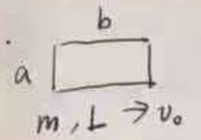
$I = \frac{Blv_0}{R} \cdot e^{-\frac{1}{1+\frac{m}{B^2l^2C}} \cdot \frac{mR}{B^2l^2}t}$

$= \frac{Blv_0}{R} \cdot e^{-\frac{mRC}{B^2l^2C+m}t}$

$\Rightarrow \int dv = -\frac{Bl}{m}\int I dt$

$v - v_0 = -\frac{(Bl)^2 v_0}{mR} \cdot \frac{B^2l^2C+m}{mRC}\left(1-e^{-\frac{mRC}{B^2l^2C+m}t}\right)$

eg3.



$\otimes B_0$

$\Rightarrow L\cdot\frac{dI}{dt} = B_0 a v$　回路欧姆定律

$m\frac{dv}{dt} = -B_0 I a$　受力方程

代入有　$B_0 a v = L\cdot\left(-\frac{m}{B_0 a}\cdot\frac{d^2 v}{dt^2}\right) = -\frac{mL}{B_0 a}\frac{d^2 v}{dt^2}$

$\therefore \frac{d^2 v}{dt^2} + \frac{(B_0 a)^2}{mL} v = 0 \Rightarrow v = v_0\cos\left(\frac{B_0 a}{\sqrt{mL}}t\right)$

$x = \frac{\sqrt{mL}}{B_0 a} v_0 \sin\left(\frac{B_0 a}{\sqrt{mL}}t\right)$

eg4. 设线框中电流为I，由 $-L\frac{dI}{dt} = S\cdot\frac{dB_z}{dt}$

$-LI = \Delta B_z\cdot a^2$

$\Rightarrow$ ~~初始~~ $I = \frac{\Delta B_z a^2}{L} = \frac{\alpha z\cdot a^2}{L} = \frac{\alpha a^2 z}{L}$

$\Rightarrow F = -mg - (B_y\cdot\frac{a}{2}\cdot I\cdot 2) = -mg - \alpha a\cdot aI = -mg - \frac{\alpha^2 a^4}{L}\cdot z$

即 $k = \frac{\alpha^2 a^4}{L}$，$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{\alpha^2 a^4}{mL}} = \frac{\alpha a^2}{\sqrt{mL}}$

$\Rightarrow x = \frac{mgL}{\alpha^2 a^4}\left(\cos\left(\frac{\alpha a^2}{\sqrt{mL}}t\right) - 1\right)$

eg5　$B = \frac{\mu_0}{4\pi}\frac{I\cdot 2\pi r}{r^2} = \frac{\mu_0 I}{2r} \Rightarrow \Phi = B\cdot\pi r^2 = \frac{1}{2}\mu_0\pi I r$

$\Rightarrow \varepsilon = \frac{d\Phi}{dt} = \frac{1}{2}\mu_0\pi r\cdot\frac{dI}{dt} = I\cdot R$

即 $\frac{1}{2}\mu_0\pi r\cdot\Delta I = IR\Delta t$

$\Rightarrow R = \frac{\mu_0\pi r_0 I}{2I\Delta t} = \rho\frac{l}{S} = \rho\frac{2\pi r}{\pi(\frac{d}{2})^2}$

$\therefore \rho = \frac{R\pi(\frac{d}{2})^2}{2\pi r} = \frac{Rd^2}{8r}$

eg7.

$\cancel{\varphi = B\pi R^2 \cos\sin\theta}$

$\cancel{\Rightarrow \varepsilon = \frac{d\varphi}{dt} = B\pi R^2 \cos\theta\cdot\omega}$ $\qquad$ $r = \cancel{\rho}\,\frac{2\pi R}{\sigma\cdot\pi\frac{d^2}{4}}$

$\cancel{\Rightarrow I = \frac{\varepsilon}{r} = \frac{B\pi R^2}{r}\cdot\omega\cos\theta}$

$\cancel{\Rightarrow P = I^2 r = \frac{(B\pi R^2)^2}{r}\,\omega^2\cos^2\theta}$

$\cancel{E_p = mgR = \rho\cdot 2\pi R\cdot\pi\frac{d^2}{4}\,g\cdot R}$

$\cancel{\vec{m} = IS}$

$\cancel{\Rightarrow \vec{M} = \vec{m}\times\vec{B} = mB\cos\theta}$

$\cancel{mgR\sin\theta = I\cdot\pi R^2 B\cos\theta =}$

(1) 设倒下用时 T，并简化计算模型，设其 $\omega = \frac{\frac{\pi}{2}}{T} = \frac{\pi}{2T}$

有 $E_K = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}mr^2 + mr^2\right)\omega^2 = \frac{1}{2}\cdot\frac{3}{2}mr^2\cdot\frac{\pi^2}{4T^2} = \frac{3\pi^2 mr^2}{16T^2}$

平行轴定理

并设其动生电动势 $\varepsilon = \frac{\Delta\phi}{T}$

有 $E_Q = I^2 R\cdot T = \left(\frac{\varepsilon}{R}\right)^2 RT = \frac{\varepsilon^2}{R}T = \frac{\Delta\phi^2}{RT}$

$= \frac{(B\cdot\pi r^2)^2}{RT} = \frac{\pi^2B^2r^4}{RT}$

$\therefore \frac{E_K}{E_Q} = \frac{3mR}{16B^2r^2T}$，又 $m = \rho\cdot 2\pi r\cdot\pi\left(\frac{d}{2}\right)^2$，$R = \frac{1}{\sigma}\cdot\frac{2\pi r}{\pi\left(\frac{d}{2}\right)^2}$

无磁场，因为楞次定律

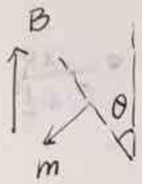
$\therefore \frac{E_K}{E_Q} = \frac{3\pi^2 \cancel{mR}\,\rho}{4B^2T\sigma}$，有 $T \geq T_0 \approx \frac{\frac{\pi}{2}}{\sqrt{\frac{mgr}{\frac{3}{2}mr^2}}} = \frac{\pi}{2}\sqrt{\frac{3r}{2g}}$

$\therefore \frac{E_K}{E_Q} < \frac{3\pi\rho}{2B^2T\sigma}\cdot\sqrt{\frac{2g}{3r}} = 5.5\times10^{-2}$，即相比之下，动能几乎没有

电流的定义

稳恒电流 ⇒ 电流场

$\vec{j} = nq\vec{v}$



(2) 近似假设：忽略动能

$\Rightarrow M_g = M_m$，即 $mgr\sin\theta = \vec{m}\times\vec{B} = \bar{m}B\cos\theta$

又 $\varepsilon = \frac{d\varphi}{dt} = \frac{d}{dt}(BS\sin\theta) = B\cdot\pi r^2\cos\theta\cdot\omega$

$\therefore i = \frac{\varepsilon}{R} = \frac{\omega B\pi r^2}{R}\cos\theta \Rightarrow \bar{m} = i\cdot S = \frac{\omega B\pi^2 r^4}{R}\cos\theta$

$\therefore mgr\sin\theta = \frac{\omega(B\pi r^2)^2}{R}\cos^2\theta$

$\Rightarrow \frac{d\theta}{dt} = \frac{mgrR}{(B\pi r^2)^2}\cdot\frac{\sin\theta}{\cos^2\theta}$，即 $\frac{d\theta}{1}\cdot\frac{\cos^2\theta}{\sin\theta} = \frac{\rho\cdot 2\pi r\cdot\pi(\frac{d}{2})^2\cdot\frac{1}{\sigma}\cdot\frac{2\pi r}{\pi(\frac{d}{2})^2}\cdot gr}{(B\pi r^2)^2}dt$

即右式 $= \frac{4\frac{\rho}{\sigma}\cdot gr}{B^2 r^2}\cdot\Delta t = \Delta t\cdot\left(\frac{B^2 r\sigma}{4\rho g}\right)^{-1}$

左式 $= \frac{\cos^2\theta}{\sin\theta}d\theta = \frac{1-\sin^2\theta}{\sin\theta}d\theta = \frac{\sin\theta d\theta}{\sin^2\theta} - \sin\theta d\theta$

$= \frac{d\cos\theta}{\cos^2\theta-1} + d\cos\theta = \left(\frac{1}{\cos\theta-1} - \frac{1}{\cos\theta+1}\right)\cdot\frac{d\cos\theta}{2} + d\cos\theta$

$= \frac{1}{2}d\left(\ln|\cos\theta-1| - \ln|\cos\theta+1|\right) + d\cos\theta$

$= \left(\frac{1}{2}\cdot\ln\left|\frac{\cos\theta-1}{\cos\theta+1}\right| + \cos\theta\right)\Big|_{\theta=0.1}^{\frac{\pi}{2}} = \cancel{\frac{1}{2}\ln\frac{\cos0.1-1}{\cos0.1+1}} + (-\cos0.1)$

$\ln\tan\frac{\theta}{2}$　$\frac{1}{\tan0.05}$

$= \ln\frac{1+\cos0.1}{-1+\cos0.1} - \cos0.1 \doteq \cancel{2.99}\ -2$

$\therefore t = \dfrac{B^2 r\sigma}{4\rho g}\cdot\left(\ln\tan\frac{\theta}{2}+\cos\theta\right)\Big|_{0.1}^{\frac{\pi}{2}} = 1.125$

例18. 首先磁棒与磁偶极子等效

即 $B = \dfrac{\mu_0}{4\pi}\cdot\dfrac{p\cdot 2}{r^3} \sim \dfrac{1}{z^3}$，即 $B_z = \dfrac{B_0}{z^3}$

由磁场高斯定理： $B_r\cdot 2\pi a\cdot dz = dB_z\cdot \pi a^2$

$\therefore B_r = \dfrac{1}{2}\cdot\dfrac{B_0}{z^4}\cdot 3a = \dfrac{3a}{2}\dfrac{B_0}{z^4}$

(1) 平衡时，有受力

$F_安 = mg \Leftrightarrow B_r\cdot i\cdot 2\pi a = mg$

又 $Li = B_z\cdot\pi a^2 \Rightarrow i = \dfrac{\pi a^2}{L}\cdot\dfrac{B_0}{z^3}$

$\Big\}\ \dfrac{3a}{2}\cdot\dfrac{B_0}{z^3}\cdot\dfrac{\pi a^2 B_0}{zL}\cdot 2\pi a = mg$

$\therefore \dfrac{1}{z} = \sqrt[7]{\dfrac{mgL}{3\pi^2 a^4 B_0^2}} = \sqrt[7]{\dfrac{3\pi^2 a^4 B_0^2}{mgL}}$

(2) 受小扰动后，$z = z_0 + \Delta z$

有 $F_安 = \dfrac{3a}{2}\cdot\dfrac{\pi a^2 B_0^2}{L}\cdot 2\pi a\cdot\dfrac{1}{z^4} = \dfrac{3\pi^2 a^4 B_0^2}{L}\cdot\dfrac{1}{(z_0+\Delta z)^7}$

$= \dfrac{3\pi^2 a^4 B_0^2}{L}\cdot\dfrac{1}{z_0^7}\left(1 - 7\cdot\dfrac{\Delta z}{z_0}\right) \sim$ 简振

$\therefore T = 2\pi\sqrt{\dfrac{m}{\dfrac{\pi^2 a^4 B_0^2}{z_0^8 L}\cdot 3}} = 2\pi\sqrt{\dfrac{m z_0^7 L}{7\pi^2 a^4 B_0^2\cdot 3}\cdot z_0} = 2\pi\sqrt{\dfrac{z_0}{7g}}$

恒定电流

eg1.

(1) $R=\rho\cdot\frac{l}{\pi(r_2^2-r_1^2)}$

(2) $R=\int_{r_1}^{r_2}\rho\cdot\frac{dr}{2\pi rl}=\frac{\rho}{2\pi l}\cdot\ln\frac{r_2}{r_1}$

(3) $\frac{1}{R}=\int\frac{1}{\rho\cdot\frac{2\pi r}{l\,dr}}=\frac{l}{\rho\cdot 2\pi}\cdot\ln\frac{r_2}{r_1}$

$\therefore R=\frac{2\pi\rho}{l}\cdot\frac{1}{\ln\frac{r_2}{r_1}}$

eg2 求电荷量 → 求电流 $\left\{\begin{array}{l}\text{宏观}\\ \text{微观} \Leftarrow \text{我们要求的}\end{array}\right.$

目标是求等效阻力f

对相同的线圈有

$\varepsilon=E\cdot 2\pi r=\frac{\varepsilon}{p}IR$

又 $I=neSv$，$R=\rho\cdot\frac{2\pi r}{S}$

即 $E=nev\cdot\rho$

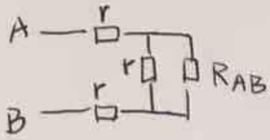
$\Rightarrow f=eE=ne^2\rho\cdot v$，即阻力与速度成正比

$f=ma\Rightarrow ne^2\rho\cdot v=-m\frac{dv}{dt}$ 又 $nesv\,dt=dq$

即 $\cancel{ne^2\rho\cdot dl=-mdv}$，$\cancel{l=\frac{mv_0}{ne^2\rho}}$

$nesv\cdot dt\cdot e\rho=-mdv\cdot S$，$Q=\frac{mv_0}{e\rho}\cdot S$

eg3. (1)

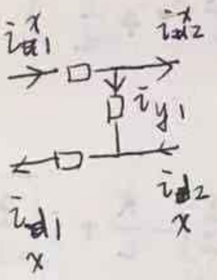


有 $R_{AB} = 2r + \frac{r R_{AB}}{r + R_{AB}}$

即 $R_{AB} = (1+\sqrt{3})\, r$.

(2)

对第1个网格



有 $i_{x1} = \frac{E}{(2+\sqrt{3})\, r} = (2-\sqrt{3})\frac{E}{r}$.

且 $\begin{cases} i_{y1} \cdot r = i_{x2} \cdot R_{AB} \\ i_{y1} + i_{x2} = i_{x1} \end{cases}$

即 $i_{x2} = (2-\sqrt{3})\, i_{x1}$

$i_{y1} = (\sqrt{3}-1)\, i_{x1}$

即对第k个网格，有 $i_{xk} = (2-\sqrt{3})\, i_{xk-1}$

$i_{yk} = (\sqrt{3}-1)\, i_{xk}$

即 $i_{xk} = (2-\sqrt{3})^{k-1}\, i_{x1} = (2-\sqrt{3})^{k}\frac{E}{r}$

$i_{yk} = (\sqrt{3}-1)\cdot(2-\sqrt{3})^{k-1}\, i_{x1} = (\sqrt{3}-1)\cdot(2-\sqrt{3})^{k}\frac{E}{r}$.

RLC：
$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=0$$

即 $\ddot{q}+\frac{R}{L}\cdot\dot{q}+\frac{1}{CL}q=0$

设 $q=Ae^{\alpha x}$

$\Rightarrow \alpha^2+\frac{R}{L}\alpha+\frac{1}{CL}=0 \Rightarrow \alpha=\frac{1}{2}\left(-\frac{R}{L}\pm\sqrt{\left(\frac{R}{L}\right)^2-4\frac{1}{CL}}\right)$

① 阻尼振动

$\left(\frac{R}{L}\right)^2-\frac{4}{CL}<0$，即 $R^2<4\frac{L}{C}$，$\alpha_1=\frac{1}{2}\left(-\frac{R}{L}+i\sqrt{\frac{4}{CL}-\left(\frac{R}{L}\right)^2}\right)$

$\alpha_2=\frac{1}{2}\left(-\frac{R}{L}-i\sqrt{\frac{4}{CL}-\left(\frac{R}{L}\right)^2}\right)$

② 临界阻尼

$\left(\frac{R}{L}\right)^2=\frac{4}{CL}$，即 $R^2=4\frac{L}{C}$

$\alpha=-\frac{1}{2}\cdot\frac{R}{L}$

即 $q=(A+Bt)e^{-\alpha t}$

③ 过阻尼.

$\alpha_1=\frac{1}{2}\left(-\frac{R}{L}+\sqrt{\left(\frac{R}{L}\right)^2-\frac{4}{CL}}\right)$，$\alpha_2=\frac{1}{2}\left(-\frac{R}{L}-\sqrt{\left(\frac{R}{L}\right)^2-\frac{4}{CL}}\right)$