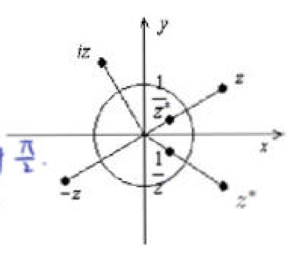
《数学物理方法》第一章《复数的基本概念》习题

1. 已知一复数，在复平面上画出$\mathrm{i}z, -z, z^*, \frac{1}{z^*}, \frac{1}{z}$的位置，并指出它们之间的几何关系。

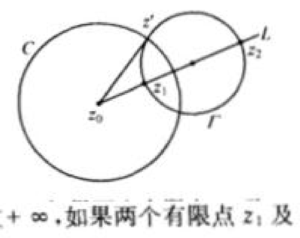
[Solution].

Geometric relations:



- iz is the point obtained by rotating z counterclockwise by $\frac{\pi}{2}$.
- $-z$ is the symmetric point of z about the origin.
- z^* is the symmetric point of z about the real axis.
- $\frac{1}{z^*}$ is the symmetric point of z about the unit circle (i.e. the unit circle is an Apollonius circle of z & $\frac{1}{z^*}$). → See reference below.
- $\frac{1}{z}$ is the symmetric point of $\frac{1}{z^*}$ about the x-axis.

[Reference]. 严镇军. 复变函数[M]. 合肥: 中国科学技术大学出版社, 2001. (第8章第3节).



定义 3 设已给圆周 $C: |z-z_0| = R, 0 < R < +\infty$，如果两个有限点 z_1 及 z_2 在自 z_0 点出发的同一条射线上，且

$$|z_1 - z_0| \cdot |z_2 - z_0| = R^2.$$

则称 z_1 及 z_2 关于圆周 C 对称。

由定义可见，圆周上的点和自身对称。我们还规定圆心 z_0 及 ∞ 关于 C 对称。

2. 若$|z|=1$，试证明

$$\left|\frac{az+b}{b^*z+a^*}\right| = 1,$$

a, b为任意复数。

[Solution].

Since $zz^* = |z|^2 = 1$ and $|z^*| = |z| = 1$, we have

$$|az+b| = |z^*|\cdot|az+b| = |azz^* + bz^*| = |a + bz^*|$$
$$= |(a+bz^*)^*| = |b^*z + a^*|.$$

$$\Rightarrow \left|\frac{az+b}{b^*z+a^*}\right| = \frac{|az+b|}{|b^*z+a^*|} = 1.$$

3. 计算下列数值：

(1) $\sqrt{1+\mathrm{i}\sqrt{3}}$

(2) $(1-\mathrm{i})/(1+i)$

(3) i^{i}

(4) $\sqrt[\mathrm{i}]{i}$

(5) $\sin\theta + \sin 2\theta + \sin 3\theta + \cdots + \sin n\theta$

(6) $\cos\theta + \cos 2\theta + \cos 3\theta + \cdots + \cos n\theta$

[Solution]

(1). $\sqrt{1+i\sqrt{3}} = (2e^{\frac{\pi}{3}i + 2\pi i N})^{\frac{1}{2}} = \sqrt{2}\, e^{\frac{\pi}{6}i + \pi i N}, N \in \mathbb{Z}$

$$\Rightarrow \sqrt{1+i\sqrt{3}} = \begin{cases} \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i & N \text{ is even} \\ -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i & N \text{ is odd} \end{cases}$$

$$\Rightarrow \sqrt{1+i\sqrt{3}} = \pm\left(\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i\right).$$

[Note]. In complex analysis, the square root function is multi-valued.

(2). $\frac{1-i}{1+i} = \frac{(1-i)^2}{(1+i)(1-i)} = \frac{-2i}{2} = -i.$

(3). $i^i = \exp(i \cdot \mathrm{Ln}\, i) = \exp(i \cdot \mathrm{Ln}\, e^{(2N+\frac{1}{2})\pi i})$

$= \exp(i \cdot (2N+\frac{1}{2})\pi i) = e^{-(2N+\frac{1}{2})\pi}, N \in \mathbb{Z}.$

[Note]. Power functions are multi-valued when it comes to complex analysis. So do logrithmic functions.

(4). $\sqrt[i]{i} = \exp(\frac{1}{i} \cdot \mathrm{Ln}\, i) = \exp(-i \cdot \mathrm{Ln}\, e^{(2N+\frac{1}{2})\pi i})$

$= \exp(-i \cdot (2N+\frac{1}{2})\pi i) = e^{(2N+\frac{1}{2})\pi}, N \in \mathbb{Z}.$

(5). When $\sin\frac{\theta}{2} = 0$, $\sum_{k=1}^{n} \sin k\theta = 0$. Otherwise,

$$\sum_{k=1}^{n} \sin k\theta = \sum_{k=1}^{n} \frac{1}{2i}(e^{ik\theta} - e^{-ik\theta})$$
$$= \frac{1}{2i}\left(\sum_{k=1}^{n}(e^{i\theta})^k - \sum_{k=1}^{n}(e^{-i\theta})^k\right)$$
$$= \frac{1}{2i}\left(\frac{e^{i\theta}(1-e^{in\theta})}{1-e^{i\theta}} - \frac{e^{-i\theta}(1-e^{-in\theta})}{1-e^{-i\theta}}\right)$$
$$= \frac{1}{2i}\left(\frac{e^{\frac{i}{2}\theta} - e^{i(n+\frac{1}{2})\theta}}{e^{-\frac{i}{2}\theta} - e^{\frac{i}{2}\theta}} - \frac{e^{-\frac{i}{2}\theta} - e^{-i(n+\frac{1}{2})\theta}}{e^{\frac{i}{2}\theta} - e^{-\frac{i}{2}\theta}}\right)$$
$$= -\frac{1}{2i}\,\frac{(e^{i(n+\frac{1}{2})\theta} + e^{-i(n+\frac{1}{2})\theta}) - (e^{\frac{i}{2}\theta} + e^{-\frac{i}{2}\theta})}{e^{\frac{i}{2}\theta} - e^{-\frac{i}{2}\theta}}$$
$$= -\frac{1}{2i}\,\frac{2\cos(n+\frac{1}{2})\theta - 2\cos\frac{1}{2}\theta}{2i\sin\frac{1}{2}\theta}$$
$$= \frac{\cos\frac{\theta}{2} - \cos(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}.$$

Hence
$$\sum_{k=1}^{n} \sin k\theta = \begin{cases} 0 & \theta = 2N\pi, N \in \mathbb{Z} \\ \frac{\cos\frac{\theta}{2} - \cos(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} & \text{otherwise} \end{cases}$$

(6). When $\sin\frac{\theta}{2} = 0$, $\sum_{k=1}^{n} \cos k\theta = n$. Otherwise,

$$\sum_{k=1}^{n} \cos k\theta = \sum_{k=1}^{n} \frac{1}{2}(e^{ik\theta} + e^{-ik\theta})$$
$$= \frac{1}{2}\left(\sum_{k=1}^{n}(e^{i\theta})^k + \sum_{k=1}^{n}(e^{-i\theta})^k\right)$$
$$= \frac{1}{2}\left(\frac{e^{i\theta}(1-e^{in\theta})}{1-e^{i\theta}} + \frac{e^{-i\theta}(1-e^{-in\theta})}{1-e^{-i\theta}}\right)$$
$$= \frac{1}{2}\left(\frac{e^{\frac{i}{2}\theta} - e^{i(n+\frac{1}{2})\theta}}{e^{-\frac{i}{2}\theta} - e^{\frac{i}{2}\theta}} + \frac{e^{-\frac{i}{2}\theta} - e^{-i(n+\frac{1}{2})\theta}}{e^{\frac{i}{2}\theta} - e^{-\frac{i}{2}\theta}}\right)$$
$$= \frac{1}{2}\cdot\frac{-(e^{\frac{i}{2}\theta} - e^{-\frac{i}{2}\theta}) + (e^{i(n+\frac{1}{2})\theta} - e^{-i(n+\frac{1}{2})\theta})}{e^{\frac{i}{2}\theta} - e^{-\frac{i}{2}\theta}}$$
$$= \frac{1}{2}\cdot\frac{-2i\sin\frac{1}{2}\theta + 2i\sin(n+\frac{1}{2})\theta}{2i\sin\frac{\theta}{2}}$$
$$= \frac{\sin(n+\frac{1}{2})\theta - \sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}}.$$

Thus
$$\sum_{k=1}^{n} \cos k\theta = \begin{cases} n & \theta = 2N\pi, N \in \mathbb{Z} \\ \frac{\sin(n+\frac{1}{2})\theta - \sin\frac{\theta}{2}}{\sin\frac{\theta}{2}} & \text{otherwise} \end{cases}$$

[Remark].

Some people may consider $\sum e^{ik\theta} = \sum \cos k\theta + i\sum \sin k\theta$, then compare real & imaginary parts on both sides, but this method is not convincing when θ is a complex number.

4. 试写出经过点a且与复数b所代表的矢量平行的直线方程。

[Solution].

Let $a = (a_x, a_y)$, and write $b = \mathrm{Re}\,b + i\,\mathrm{Im}\,b$.

The answer is $\mathrm{Im}\,b \cdot x - \mathrm{Re}\,b \cdot y + \mathrm{Re}\,b \cdot a_y - \mathrm{Im}\,b \cdot a_x = 0$.

Or we take the complex plane and denote $A = a_x + i a_y$, then the line could be represented as $b^*(z-A) = b(z-A)^*$, that is,

$(ib)^* z + ibz^* - ((ib)^* a + iba^*) = 0.$

[Note]. A line in the complex plane has a standard equation $B^*z + Bz^* + C = 0$, where $B \in \mathbb{C}, |B| > 0, C \in \mathbb{R}$.

5. 证明圆上四点z_1, z_2, z_3 and z_4满足条件：$\mathrm{Im}\left(\frac{z_1-z_3}{z_1-z_4}\Big/\frac{z_2-z_3}{z_2-z_4}\right) = 0$。

[Proof].

On the complex plane, we can write the circle as $z = z_0 + Re^{i\theta}$, where $z_0 \in \mathbb{C}$ and $R \in \mathbb{R}$ are constants, $\theta \in [0, 2\pi)$ is a variable.

Write $z_1 = z_0 + Re^{i\theta_1}$, $z_2 = z_0 + Re^{i\theta_2}$,
$z_3 = z_0 + Re^{i\theta_3}$, $z_4 = z_0 + Re^{i\theta_4}$,

$$w = \frac{z_1 - z_3}{z_1 - z_4}\Big/\frac{z_2 - z_3}{z_2 - z_4} = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)},$$

then
$$w = \frac{R(e^{i\theta_1} - e^{i\theta_3}) \cdot R(e^{i\theta_2} - e^{i\theta_4})}{R(e^{i\theta_1} - e^{i\theta_4}) \cdot R(e^{i\theta_2} - e^{i\theta_3})}$$
$$= \frac{e^{i(\theta_1+\theta_2)} + e^{i(\theta_3+\theta_4)} - e^{i(\theta_2+\theta_3)} - e^{i(\theta_1+\theta_4)}}{e^{i(\theta_1+\theta_2)} + e^{i(\theta_3+\theta_4)} - e^{i(\theta_2+\theta_4)} - e^{i(\theta_1+\theta_3)}}.$$

Now we calculate w^*, and multiply $e^{i(\theta_1+\theta_2+\theta_3+\theta_4)}$ on both the numerator & the denominator.

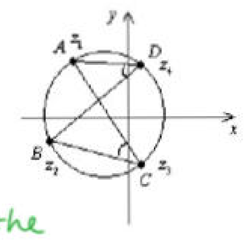
$$\Rightarrow w^* = \frac{e^{-i(\theta_1+\theta_2)} + e^{-i(\theta_3+\theta_4)} - e^{-i(\theta_2+\theta_3)} - e^{-i(\theta_1+\theta_4)}}{e^{-i(\theta_1+\theta_2)} + e^{-i(\theta_3+\theta_4)} - e^{-i(\theta_2+\theta_4)} - e^{-i(\theta_1+\theta_3)}}$$
$$= \frac{e^{i(\theta_3+\theta_4)} + e^{i(\theta_1+\theta_2)} - e^{i(\theta_1+\theta_4)} - e^{i(\theta_2+\theta_3)}}{e^{i(\theta_3+\theta_4)} + e^{i(\theta_1+\theta_2)} - e^{i(\theta_1+\theta_3)} - e^{i(\theta_2+\theta_4)}}$$
$$= w.$$

Therefore, w is real and $\mathrm{Im}\, w = 0$. □

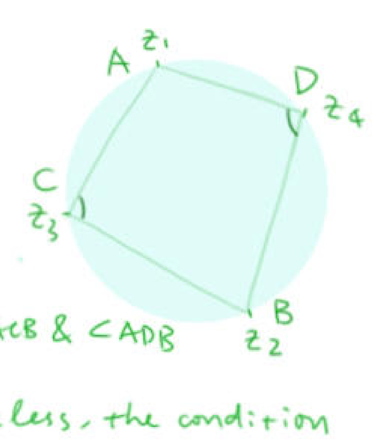
[Remark].

This problem could be explained geometrically as below.

如图若四点共圆，则有$\angle ACB = \angle ADB$（同弧所对圆周角相等）。反之也成立。写成复数形式即为$\frac{z_1-z_3}{z_2-z_3}\Big/\frac{z_1-z_4}{z_2-z_4} =$实数。



But it may not be an absolutely rigorous "proof" due to the huge uncertainty when translating the original question into a geometric one. For instance, consider the diagram below.



Suddenly, the relation between the arguments of $\frac{z_1-z_3}{z_2-z_3}$ & $\frac{z_1-z_4}{z_2-z_4}$ and $\angle ACB$ & $\angle ADB$ becomes unclear. Nevertheless, the condition $\angle ACB = \angle ADB$ fails.

《数学物理方法》第二章《复变函数》习题

1. 试证明极坐标下的柯西－黎曼条件：

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

[Solution].

It suffices to show such equations are equivalent to the C-R ones when u & v are real differentiable on $\mathbb{C}\setminus\{0\}$.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{1}{r}\frac{\partial x}{\partial\theta} & \frac{1}{r}\frac{\partial y}{\partial\theta} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}.$$

On the one hand, starting from $\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$,

$$\Rightarrow \begin{cases} \frac{\partial u}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y} = \cos\theta\frac{\partial v}{\partial y} - \sin\theta\frac{\partial v}{\partial x} = \frac{1}{r}\frac{\partial v}{\partial\theta} \\ \frac{\partial v}{\partial r} = \cos\theta\frac{\partial v}{\partial x} + \sin\theta\frac{\partial v}{\partial y} = -\cos\theta\frac{\partial u}{\partial y} + \sin\theta\frac{\partial u}{\partial x} = -\frac{1}{r}\frac{\partial u}{\partial\theta}. \end{cases}$$

On the other hand, starting from $\begin{cases} \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta} \\ \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial\theta} \end{cases}$,

since
$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\theta} \end{bmatrix},$$

$$\Rightarrow \begin{cases} \frac{\partial u}{\partial x} = \cos\theta\frac{\partial u}{\partial r} + \sin\theta\frac{1}{r}\frac{\partial u}{\partial\theta} = \cos\theta\frac{1}{r}\frac{\partial v}{\partial\theta} - \sin\theta\frac{\partial v}{\partial r} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{1}{r}\frac{\partial u}{\partial\theta} = -\sin\theta\frac{1}{r}\frac{\partial v}{\partial\theta} - \cos\theta\frac{\partial v}{\partial r} = -\frac{\partial v}{\partial x}. \end{cases}$$

As a consequence, $\begin{cases} \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta} \\ \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial\theta} \end{cases}$ counts.

[Remark]

By using similar methods, one can use $\begin{cases} z = x+iy \\ z^* = x-iy \end{cases}$ and $\begin{cases} \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}) \\ \frac{\partial}{\partial z^*} = \frac{1}{2}(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}) \end{cases}$ to prove the following lemma.

[Lemma]

For real differentiable functions $u(x,y)$ and $v(x,y)$, $f(x,y) = u(x,y) + iv(x,y)$ is complex differentiable if and only if $\left(\frac{\partial f}{\partial z^*}\right)_z = 0$. Furthurmore, $\frac{df}{dz} = \left(\frac{\partial f}{\partial z}\right)_{z^*}$ holds.

2. 设$\rho = \rho(r,\theta)$及$\varphi = \varphi(r,\theta)$是实变量$x$, y的实函数。若$f(z) = \rho(\cos\varphi + \mathrm{i}\sin\varphi)$是$z = x + \mathrm{i}y$的解析函数，试证：

$$\frac{\partial\rho}{\partial x} = \rho\frac{\partial\varphi}{\partial y}, \quad \frac{\partial\rho}{\partial y} = -\rho\frac{\partial\varphi}{\partial x}. \tag{2}$$

[Solution].

$f = u + iv$ with $u = \rho\cos\varphi$ and $v = \rho\sin\varphi$.

C-R equations $\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases}$ obtain

$$\begin{cases} \frac{\partial\rho}{\partial x}\cos\varphi - \rho\sin\varphi\frac{\partial\varphi}{\partial x} = \frac{\partial\rho}{\partial y}\sin\varphi + \rho\cos\varphi\frac{\partial\varphi}{\partial y} & ① \\ \frac{\partial\rho}{\partial y}\cos\varphi - \rho\sin\varphi\frac{\partial\varphi}{\partial y} = -\frac{\partial\rho}{\partial x}\sin\varphi - \rho\cos\varphi\frac{\partial\varphi}{\partial x} & ② \end{cases}$$

Hence ①$\cos\varphi$ + ②$\sin\varphi$ gives $\frac{\partial\rho}{\partial x} = \rho\frac{\partial\varphi}{\partial y}$, ③

and ②$\cos\varphi$ − ①$\sin\varphi$ gives $\frac{\partial\rho}{\partial y} = -\rho\frac{\partial\varphi}{\partial x}$. ④

(Conversely, ③$\cos\varphi$ − ④$\sin\varphi$ gives ① and ③$\sin\varphi$ + ④$\cos\varphi$ gives ②.)

[Remark].

Some people may use another appoach, that is from $\begin{cases} u = \rho\cos\varphi \\ v = \rho\sin\varphi \end{cases}$ to derive

$$\begin{bmatrix} \frac{\partial\rho}{\partial u} & \frac{\partial\rho}{\partial v} \\ \frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial\rho} & \frac{\partial u}{\partial\varphi} \\ \frac{\partial v}{\partial\rho} & \frac{\partial v}{\partial\varphi} \end{bmatrix}^{-1} = \begin{bmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{bmatrix}^{-1} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\frac{1}{\rho}\sin\varphi & \frac{1}{\rho}\cos\varphi \end{bmatrix},$$

and then write

$$\begin{cases} \frac{\partial\rho}{\partial x} = \frac{\partial\rho}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\rho}{\partial v}\frac{\partial v}{\partial x} = \rho\frac{\partial\varphi}{\partial v}\frac{\partial v}{\partial y} + (-\rho\frac{\partial\varphi}{\partial u})(-\frac{\partial u}{\partial x}) = \rho\frac{\partial\varphi}{\partial y} \\ \frac{\partial\rho}{\partial y} = \frac{\partial\rho}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial\rho}{\partial v}\frac{\partial v}{\partial y} = \rho\frac{\partial\varphi}{\partial v}(-\frac{\partial v}{\partial x}) + (-\rho\frac{\partial\varphi}{\partial u})\frac{\partial u}{\partial x} = -\rho\frac{\partial\varphi}{\partial x}, \end{cases}$$

which is what we need. But such a method has one disadvantage. It is hard to deal with the case where $\rho = 0$ and singularities occur.

3. 若函数$f(z) = u + \mathrm{i}v$在G内解析，且$f(z) \neq$常数，试讨论下列函数是否也是G内的解析函数：

(1) $u - \mathrm{i}v$; (2) $-u - \mathrm{i}v$;

(3) $-v + \mathrm{i}u$; (4) $v + \mathrm{i}u$.

[Solution].

(1) Assume $g(z) = u - iv$ is analytic, then

$u = \frac{1}{2}(f+g)$ is analytic $\Rightarrow u =$ constant.
$v = \frac{1}{2i}(f-g)$ is analytic $\Rightarrow v =$ constant.
→ see remark below

$\Rightarrow f = u + iv =$ constant. Contradiction!

$\Rightarrow g = u - iv$ must not be analytic.

(2) $-u - iv = -f$ must be analytic.

(3) $-v + iu = if$ must be analytic.

(4) Assume $v + iu$ is analytic, then $-i(v+iu) = u - iv = g(z)$ is analytic.

By (1), it's impossible.

$\Rightarrow v + iu$ must not be analytic.

[Remark].

If $f(z)$ is analytic on G and $f(z) \in \mathbb{R}$ for any $z \in G$, then $f(z)$ must be a constant.

It's because if we write $f = u + iv$, then $v = 0$, the C-R equations $\Rightarrow \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0 \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0 \end{cases} \Rightarrow u =$ constant

$\Rightarrow f = u + iv = u$ is constant.

4. 若$f(z) = u(x,y) + \mathrm{i}v(x,y)$解析，且

$$u - v = (x-y)(x^2 + 4xy + y^2),$$

试求$f(z)$。

[Solution].

f is analytic $\Rightarrow (1+i)f$ is analytic

$\Rightarrow (u-v) + i(u+v)$ is analytic.

C-R equations $\Rightarrow \begin{cases} \frac{\partial(u+v)}{\partial x} = -\frac{\partial(u-v)}{\partial y} = -3x^2 + 6xy + 3y^2 \\ \frac{\partial(u+v)}{\partial y} = \frac{\partial(u-v)}{\partial x} = 3x^2 + 6xy - 3y^2. \end{cases}$

$\Rightarrow u + v = \int (-3x^2 + 6xy + 3y^2)dx + (3x^2 + 6xy - 3y^2)dy$

$= -x^3 + 3x^2y + 3xy^2 - y^3 + C', \quad C' \in \mathbb{R}.$

$\Rightarrow f = u + iv = \frac{1+i}{2}(u+v) + \frac{1-i}{2}(u-v)$

$= 3x^2y - y^3 + i(3xy^2 - x^3) + \frac{1+i}{2}C'$

Let $C = \frac{1}{2}C'$, and use $x = \frac{z+z^*}{2}$ & $y = \frac{z-z^*}{2i}$.

$\Rightarrow f = -iz^3 + (1+i)C, \quad C \in \mathbb{R}.$

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