《数学物理方法》第三章《初等函数》习题

1. 试根据下列条件确定解析函数$f(z)=u+\mathrm{i}\,v$：

(1) $u=x+y$; (2) $u=\sin x\cosh y$;

(3) $v=\frac{x}{x^2+y^2}$; (4) $v=\mathrm{tg}^{-1}\frac{y}{x}$.

[Solution].

(1). $\nabla^2 u=0$ on $\mathbb{R}^2$.

The C-R equations obtain

$$\begin{cases}\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=-1\\ \frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}=1\end{cases}\quad\Rightarrow v=\int -dx+dy=-x+y+C$$

$$\Rightarrow f(z)=u+iv=x+y+i(y-x)+C=(1-i)z+iC,\ C\in\mathbb{R}.$$

(2). Similarly, $\nabla^2 u=0$ on $\mathbb{R}^2$ with $\begin{cases}\frac{\partial v}{\partial x}=-\sin x\sinh y\\ \frac{\partial v}{\partial y}=\cos x\cosh y\end{cases}$.

$$\Rightarrow v=\int -\sin x\sinh y\,dx+\cos x\cosh y\,dy=\cos x\cosh y+C$$

$$\Rightarrow f(z)=u+iv=\sin x\cosh y+i\cos x\sinh y+iC=\sin z+iC,\ C\in\mathbb{R}.$$

(3). $\nabla^2 v=0$ on $\mathbb{R}^2\setminus\{(0,0)\}$, on which it could be derived that $\begin{cases}\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=-\frac{2xy}{(x^2+y^2)^2}\\ \frac{\partial u}{\partial y}=-\frac{\partial u}{\partial x}=\frac{x^2-y^2}{(x^2+y^2)^2}\end{cases}$.

$$\Rightarrow u=\int -\frac{2xy}{(x^2+y^2)^2}dx+\frac{x^2-y^2}{(x^2+y^2)^2}dy=\frac{y}{x^2+y^2}+C.$$

$$\Rightarrow f(z)=u+iv=\frac{y}{x^2+y^2}+i\frac{x}{x^2+y^2}+C=\frac{i}{z}+C,\ C\in\mathbb{R},\ z\neq 0.$$

(4). $\nabla^2 v=0$ on $\{(x,y)\mid x>0\}$ & $\{(x,y)\mid x<0\}$, on each of which, evidently we have $\begin{cases}\frac{\partial u}{\partial x}=\frac{x}{x^2+y^2}\\ \frac{\partial u}{\partial y}=\frac{y}{x^2+y^2}\end{cases}$, implying that $u=\int\frac{x}{x^2+y^2}dx+\frac{y}{x^2+y^2}dy=\frac12\ln|x^2+y^2|+C$.

$$\Rightarrow f(z)=\frac12\ln|x^2+y^2|+i\,\mathrm{atg}\frac{y}{x}+C$$

$$=\begin{cases}\ln|z|+i\arg(z)+C,\ C\in\mathbb{R} & \mathrm{Arg}(z)\in(-\frac{\pi}{2}+2k\pi,\frac{\pi}{2}+2k\pi),\ k\in\mathbb{Z}\\ \ln|z|+i(\arg(z)\pm\pi)+C,\ C\in\mathbb{R}, & \mathrm{Arg}(z)\in(\frac{\pi}{2}+2k\pi,\frac{3\pi}{2}+2k\pi),\ k\in\mathbb{Z}.\\ \text{where "+" for } \arg(z)\in(-\frac{\pi}{2},-\pi) & \\ \text{and "−" for } \arg(z)\in(\frac{\pi}{2},\pi]. & \end{cases}$$

Here we denote $\mathrm{Arg}(z)$ as the arguments of z and $\arg(z)$ as the principal one, satisfying $\arg(z)\in(-\pi,\pi]$.

[Note] $f(z)=\frac12\ln|x^2+y^2|+i\,\mathrm{atg}\frac{y}{x}$ represents only two of the analytic branches on half-planes of $f(z)=\ln z$, hence they are not the same.

2. 求出下列函数值：

(1) $e^{2+\mathrm{i}}$; (2) $\sin\mathrm{i}$;

(3) $\cos(5-\mathrm{i})$; (4) $\ln(-1)$.

[Solution].

(1) $e^{2+i}=e^2\cdot e^i=e^2(\cos 1+i\sin 1)=e^2\cos 1+ie^2\sin 1$.

(2) $\sin i=i\sinh 1=\frac{i}{2}(e-\frac1e)$.

(3) $\cos(5-i)=\frac12(e^{5i-1}+e^{-5i+1})=\frac12\left((e+\frac1e)\cos 5+i(e-\frac1e)\sin 5\right)$.

(4). $\ln(-1)=\ln(\exp(i\pi+2N\pi))=\ln 1+i\pi+2iN\pi=(2N+1)\pi i,\ N\in\mathbb{Z}$.

3. 证明：$w=-\mathrm{i}\,\frac{z-1}{z+1}=-\mathrm{i}+\mathrm{i}\,\frac{2}{z+1}$将直线$y=ax$变为圆。

[Notice]. Here $a\neq 0$, otherwise it will be a line.

[Proof].

Since $z=x+iy$, $y=ax\Leftrightarrow\frac{a+i}{2}z+\frac{a-i}{2}z^*=0$.

Let $B=\frac{a-i}{2}\neq 0$, then $B^*z+Bz^*=0$ for the line.

$w=-i+i\frac{2}{z+1}$ implies $z=\frac{2i}{w+i}-1$.

$$\Rightarrow B^*(\frac{2i}{w+i}-1)+B(\frac{2i}{w+i}-1)^*=0$$

$$B^*\frac{i}{w+i}+B(\frac{i}{w+i})^*=\frac{B+B^*}{2}$$

$$B^*(-iw+1)^*+B(-iw+1)=\frac12(B+B^*)(-iw+1)^*(-iw+1)$$

Let $A=-\frac{2iB^*}{B+B^*}\Rightarrow ww^*+A^*w+Aw^*-1=0$.

On the complex plane of w, write $w=x+iy$.

$$\Rightarrow x^2+y^2+2\mathrm{Re}A\cdot x+2\mathrm{Im}A\cdot y=1$$

$$(x+\mathrm{Re}A)^2+(y+\mathrm{Im}A)^2=1+|A|^2$$

This is obviously a circle. □

[Remark].

In fact, every rational function transforms a line or a circle into a line or a circle on the complex plane. Here is the reference.

严镇军.复变函数[M]. 合肥：中国科学技术大学出版社，2001. (第8章第3节).

由函数

$$w=\frac{az+b}{cz+d}\tag{8.1}$$

所确定的变换称为分式线性变换 M，这里，a,b,c,d 是复常数，且 $ad-bc\neq 0$. 这后一要求是必要的，否则 $w\equiv$ 常数. 当 $c=0$ 时，M 成为变换 $L: w=\alpha z+\beta$，$\alpha=a/d\neq 0$，$\beta=b/d$，它称为整线性变换.

下面几个变换是函数(8.1)的特殊情形：

1) 平移变换

$$T: w=z+b.$$

这是整个平面的一个平移，每个点移动同一个向量 b.

2) 旋转变换

$$R: w=e^{i\theta}z,\quad \theta\text{ 为实数}.$$

这是以原点为中心的一个旋转，转动角为 θ.

3) 相似变换

$$S: w=rz,\quad r>0.$$

这是一个以原点为相似中心，而伸张系数为 r 的相似变换.

4) 倒数变换

$$I: w=\frac{1}{z}.$$

这个变换把单位圆周 $|z|=1$ 变成单位圆周 $|w|=1$. 把单位圆内(或外)部变成单位圆外(或内)部.

任何一个分式线性变换(8.1)都可以表成上述四类变换的乘积. 事实上，当 $c=0$ 时，则

$$w=\frac{az+b}{d}=\frac{a}{d}\left(z+\frac{b}{a}\right)=\left|\frac{a}{d}\right|e^{i\theta}\left(z+\frac{b}{a}\right),$$

式中，$\theta=\arg\frac{a}{d}$. 由此可见它是以下三个变换的乘积：

$$z_1=z+\frac{b}{a},\quad z_2=e^{i\theta}z_1,\quad w=\left|\frac{a}{d}\right|z_2.$$

当 $c\neq 0$ 时，则

$$w=\frac{az+b}{cz+d}=\frac{a}{c}+\frac{bc-ad}{c^2\left(z+\frac{d}{c}\right)}.$$

读者自己不难把它分解成上述四类变换的乘积.

我们知道保形变换把一个很小很小的圆周变成一个和圆周差不多的东西. 分式线性变换的最大特点，就是它把任意圆照样变成圆周. 不过这里的所谓圆周，也包括直线在内. 直线认为是通过无限远点的圆周.

定理 3（保圆性） 分式线性变换把圆周变为圆周.

证 由于分式线性变换可以表成 T,R,S,I 的乘积，而 T,R,S 显然把圆周变为圆周，故只要证变换 $w=1/z$ 有此性质就行了.

圆周或直线方程可表示为

$$Az\bar{z}+\bar{B}z+B\bar{z}+C=0,$$

Here, $\bar{z}$ means z^*.

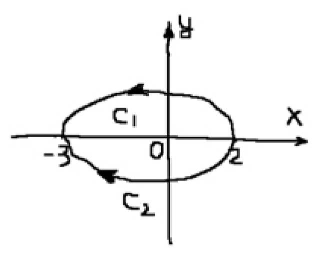
式中，A 及 C 为实数，且 $|B|^2>AC$（当 $A=0$ 时是直线），经变换 $w=1/z$ 后，上述圆周或直线成为

$$Cw\bar{w}+Bw+\bar{B}\bar{w}+A=0.$$

它是 w 平面上的圆周或直线（视 C 是否为零而定）方程. 定理得证.

一个普通(有限的)圆周经过分式线性变换后，究竟是变成直线还是普通圆周，只要看它上面有没有点变成无穷远点即可确定.

4. 函数$w=z+\sqrt{z-1}$，规定$w(2)=1$，试分别求出当z沿着图中的C_1和C_2连续变化时$w(-3)$的值。



[Solution].

$$w=z+\sqrt{z-1}=z+\sqrt{|z-1|}\exp\left(\frac{i}{2}\mathrm{Arg}(z-1)\right).$$

$$w(2)=1\Rightarrow\exp\left(\frac{i}{2}\mathrm{Arg}(z-1)\right)\Big|_{z=2}=-1.$$

Through C_1, $\mathrm{Arg}(z-1)\big|_{z=-3}=\mathrm{Arg}(z-1)\big|_{z=2}+\pi$

$$\Rightarrow\exp\left(\frac{i}{2}\mathrm{Arg}(z-1)\right)\Big|_{z=-3}=-i$$

$$w(-3)=-3+2\exp\left(\frac{i}{2}\mathrm{Arg}(z-1)\right)\Big|_{z=-3}=-3-2i.$$

Through C_2, $\mathrm{Arg}(z-1)\big|_{z=-3}=\mathrm{Arg}(z-1)\big|_{z=2}-\pi$

$$\Rightarrow\exp\left(\frac{i}{2}\mathrm{Arg}(z-1)\right)\Big|_{z=-3}=i$$

$$w(-3)=-3+2\exp\left(\frac{i}{2}\mathrm{Arg}(z-1)\right)\Big|_{z=-3}=-3+2i.$$

5. 函数$w=\sqrt{(z-a)(z-b)}$的割线有多少种可能的作法？试在两种不同作法下讨论单值分枝的规定。设a,b为实数，且$a\neq b$。

[Solution].

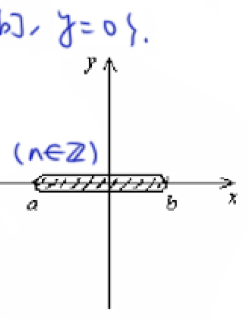
There are definitely indefinite valid choices to construct branch cuts, because the branch cut could be a line segment, a curve segment, two rays, two bent rays, etc.

Now we consider 2 of the simpliest cases.

- Case 1.

Branch cut is $\{z=x+iy\mid x\in[a,b],\ y=0\}$.

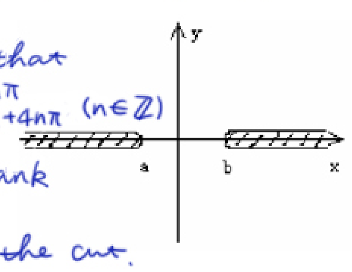
There are 2 branches that satisfy $\mathrm{Arg}(z-a)+\mathrm{Arg}(z-b)=\pm\pi+4n\pi\ (n\in\mathbb{Z})$ respectively on the upper bank of the cut.



- Case 2.

Branch cut is $\{z=x+iy\mid x\notin(a,b),\ y=0\}$.

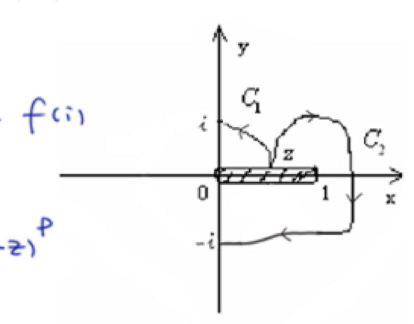
There are 2 branches that satisfy $\mathrm{Arg}(z-a)+\mathrm{Arg}(z-b)=\begin{cases}4n\pi\\ 2\pi+4n\pi\end{cases}(n\in\mathbb{Z})$ respectively on the upper bank of the right-half part of the cut.



6. 设$f(z)=\frac{z^{1-p}(1-z)^p}{2z}$，$-1<p<2$。在实轴上沿0到1作割线，规定沿割线上岸$\arg z=\arg(1-z)=0$，试计算$f(\pm\mathrm{i})$。

[Solution].

We may use C_1 to touch $f(i)$ and C_2 to reach $f(-i)$.



$$f(z)=\frac{z^{1-p}(1-z)^p}{2z}=\frac12 z^{-p}(1-z)^p$$

$$=\frac12|z|^{-p}|1-z|^p\exp\left(-ip\mathrm{Arg}(z)+ip\mathrm{Arg}(1-z)\right)$$

Through C_1, $\mathrm{Arg}(z)=\frac{\pi}{2}$, $\mathrm{Arg}(1-z)=-\frac{\pi}{4}$

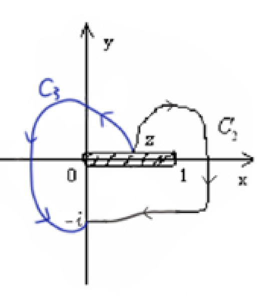
$$\Rightarrow f(i)=2^{\frac{p}{2}-1}\exp\left(-\frac34 p\pi i\right).$$

Through C_2, $\mathrm{Arg}(z)=-\frac{\pi}{2}$, $\mathrm{Arg}(1-z)=-\frac74\pi$

$$\Rightarrow f(-i)=2^{\frac{p}{2}-1}\exp\left(-\frac54 p\pi i\right).$$

[Remark].

Some people may doubt that the arguments of z and $1-z$ at $z=i$ could change if we go through C_3 instead of C_2.



But it doesn't matter because their combination $-p\mathrm{Arg}(z)+p\mathrm{Arg}(1-z)$ doesn't essentially change ($\mathrm{Arg}(z)=\frac32\pi$, $\mathrm{Arg}(1-z)=\frac{\pi}{4}$, $-p\mathrm{Arg}(z)+p\mathrm{Arg}(1-z)$ is still $-\frac54 p\pi$), making $f(-i)$ the same. In fact, $f(-i)$ is the same whatever the reaching curve is, since winding the branch cut each time only causes $\Delta(-p\mathrm{Arg}(z)+p\mathrm{Arg}(1-z))=-p\times 2\pi+p\times 2\pi=0$.

On the other hand, if $f(-i)$ is still multi-valued, such a branch cut is not valid at all due to the failure of avoiding multi-valuedness. Hence as long as the branch cut is valid, we have no need to worrying about how to choose a reaching curve from the nonhomotopic list.

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