《数学物理方法（下）》第七章《正交曲面坐标系》习题

1. 选择写出下列任意一个正交曲面坐标系中的拉普拉斯算子：

   (a) 椭圆坐标系(ξ,η,z),

   $$x=a\xi\eta,\quad y=a\sqrt{(\xi^2-1)(1-\eta^2)},\quad z=z;$$

   (b) 抛物线柱坐标系(λ,μ,z),

   $$x=\frac{1}{2}(\lambda-\mu),\quad y=\sqrt{\lambda\mu},\quad z=z;$$

   (c) 锥面坐标系(r,λ,μ),

   $$x=\frac{r}{\alpha}\sqrt{(\alpha^2-\lambda)(\alpha^2+\mu)}, \tag{1}$$
   $$y=\frac{r}{\beta}\sqrt{(\beta^2+\lambda)(\beta^2-\mu)}, \tag{2}$$
   $$z=\frac{r\sqrt{\lambda\mu}}{\alpha\beta};\quad (\alpha^2+\beta^2=1) \tag{3}$$

   (d) 椭球坐标系(λ,μ,ν),

   $$x^2=\frac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)}, \tag{4}$$
   $$y^2=\frac{(b^2+\lambda)(b^2+\mu)(b^2+\nu)}{(b^2-c^2)(b^2-a^2)}, \tag{5}$$
   $$z^2=\frac{(c^2+\lambda)(c^2+\mu)(c^2+\nu)}{(c^2-a^2)(c^2-b^2)}. \tag{6}$$

2. 在圆域$0\le r\le a$上求解，任选一题：

   (a)

   $$\begin{cases}\nabla^2u=-4,\\ u|_{r=a}=0;\end{cases}$$

(b)

$$\begin{cases} \nabla^2 u = -4r\sin\varphi, \\ u\big|_{r=a} = 0; \end{cases}$$

(c)

$$\begin{cases} \nabla^2 u = -4r^2\sin 2\varphi, \\ u\big|_{r=a} = 0. \end{cases}$$

3. 求解球内的定解问题：

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\kappa}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = 0, \\ u\big|_{r=0} = \text{有界}, \quad u\big|_{r=1} = A\mathrm{e}^{-(p\pi)^2\kappa t}, \\ u\big|_{t=0} = 0. \end{cases}$$

提示：

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru).$$