Quantum Mechanics: Fall 2017
Solution to Midterm Exam

NOTE: Problems start on page 2. Bold symbols are 3-component vectors.

Some useful facts:

- 1D harmonic oscillator: $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2$.
  Here $\hat{x}$ is position operator, $\hat{p}$ is momentum operator, $[\hat{x}, \hat{p}] = \mathbb{i}\hbar$, and in position representation $\hat{p} = -\mathbb{i}\hbar\frac{\partial}{\partial x}$. Define $\hat{a}_{\mp} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} \pm \mathbb{i}\frac{1}{m\omega}\hat{p}) = \sqrt{\frac{m\omega}{2\hbar}}(x \pm \frac{\hbar}{m\omega}\frac{\partial}{\partial x})$. Then $[\hat{a}_-, \hat{a}_+] = 1$ and $\hat{H} = \hbar\omega\,(\hat{a}_+\hat{a}_- + \frac{1}{2})$. It has a unique ground state $|\psi_0\rangle$ with $\hat{a}_-|\psi_0\rangle = 0$, and excited states $|\psi_n\rangle \equiv \frac{1}{\sqrt{n!}}(\hat{a}_+)^n|\psi_0\rangle$ with energy $E_n = (n + \frac{1}{2})\hbar\omega$. The ground state wavefunction is $\psi_0(x) = (\frac{m\omega}{\pi\hbar})^{1/4}\exp(-\frac{m\omega}{2\hbar}x^2)$.

- $\int_0^{+\infty} x^n e^{-x}\mathrm{d}x = \Gamma(n+1) = n!$, for non-negative integer n.

- Central potential problem: $\hat{H} = \frac{1}{2m}\hat{\boldsymbol{p}}^2 + V(r)$.
  Here $\hat{\boldsymbol{p}}$ is the 3D momentum $-\mathbb{i}\hbar\frac{\partial}{\partial \boldsymbol{r}}$, and $r = |\boldsymbol{r}|$ is the radius. Under polar coordinates (r, θ, ϕ), the eigenfunctions are generally $\psi_{E,\ell,m} = \frac{u(r)}{r} \cdot Y_\ell^m(\theta, \phi)$, where $Y_\ell^m(\theta, \phi)$ is the spherical harmonics, and $u(r)$ satisfies $-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right] \cdot u = E \cdot u$. Here $\ell = 0, 1, \ldots$ is the angular momentum quantum number; $m = -\ell, -\ell+1, \ldots, \ell$ is the azimuthal angular momentum quantum number; E is the energy eigenvalue.

  - The spherical harmonics are orthonormal, and are eigenfunctions of $\hat{\boldsymbol{L}}^2$ and $\hat{L}_z$.
    $Y_0^0 = \frac{1}{\sqrt{4\pi}}$, $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$, $Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm\mathbb{i}\phi}$, $\ldots$.

- Generic angular momentum: $[\hat{J}_x, \hat{J}_y] = \mathbb{i}\hbar\hat{J}_z$, $[\hat{J}_y, \hat{J}_z] = \mathbb{i}\hbar\hat{J}_x$, $[\hat{J}_z, \hat{J}_x] = \mathbb{i}\hbar\hat{J}_y$.
  For eigenstate $|j, m\rangle$ of $\hat{\boldsymbol{J}}^2$ and $\hat{J}_z$, $\hat{\boldsymbol{J}}^2|j, m\rangle = j(j+1)\hbar^2|j, m\rangle$, $\hat{J}_z|j, m\rangle = m\hbar|j, m\rangle$, and $(\hat{J}_x \pm \mathbb{i}\hat{J}_y)|j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\hbar|j, m \pm 1\rangle$.
  Here $2j$ is non-negative integer, $m = -j, -j+1, \ldots, j$.

  - Orbital angular momentum: $\hat{\boldsymbol{L}} \equiv \hat{\boldsymbol{r}} \times \hat{\boldsymbol{p}}$.
  - Spin-1/2: basis states $|\uparrow\rangle$ and $|\downarrow\rangle$.
    Under this basis, $\hat{S}_a = \frac{\hbar}{2}\sigma_a$ where $\sigma_{x,y,z}$ are Pauli matrices.
    Generic wavefunction under this basis is $\begin{pmatrix}\psi_\uparrow \\ \psi_\downarrow\end{pmatrix}$, which means $\psi_\uparrow|\uparrow\rangle + \psi_\downarrow|\downarrow\rangle$.

Problem 1. (45 points) Consider the 1D harmonic oscillator $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2$. Set the initial(time $t = 0$) wavefunction to be $\psi(x, t = 0) = A \cdot x^2 \cdot \exp(-\frac{m\omega}{2\hbar}x^2)$.

(a) (5pts) Find the normalization constant A.

(b) (10pts) Solve the coefficients c_n in the expansion $\psi(x, t = 0) = \sum_n c_n\psi_n(x)$ in terms of $\hat{H}$ eigenstates. [Hint: this expansion contains only a few terms.]

(c) (5pts) Write down the wavefunction $\psi(x, t)$ at time t.

(d) (5pts) Measure energy in $\psi(x, t)$, what values can you get, with what probabilities?

(e) (20pts) Compute the expectation values $\langle\hat{x}\rangle$, $\langle\hat{p}\rangle$, $\langle\hat{x}^2\rangle$, $\langle\hat{p}^2\rangle$ in the state $\psi(x, t)$. Check that the uncertainty principle for $\hat{x}, \hat{p}$ is satisfied.

Solution:

(a) A should satisfy that $\int_{x=-\infty}^{\infty} |\psi(x, t = 0)|^2 \,\mathrm{d}x = |A|^2 \int_{x=-\infty}^{\infty} x^4 \exp(-\frac{m\omega}{\hbar}x^2) \,\mathrm{d}x = 1$.

From the basic Gaussian integral formula, $\int_{-\infty}^{\infty} e^{-ax^2}\mathrm{d}x = \sqrt{\pi}a^{-1/2}$, take second derivative with respect to a, $\int_{-\infty}^{\infty} x^4 e^{-ax^2}\mathrm{d}x = \sqrt{\pi}a^{-5/2} \cdot \frac{1}{2} \cdot \frac{3}{2}$.

Then $|A|^2\sqrt{\pi}(\frac{m\omega}{\hbar})^{-5/2} \cdot \frac{3}{4} = 1$. We can choose $A = \pi^{-1/4}(\frac{m\omega}{\hbar})^{5/4}\sqrt{\frac{4}{3}}$.

(b) Method #1: direct decomposition,

for notation simplicity, define length $a = \sqrt{\frac{\hbar}{m\omega}}$,

$\psi(x, t = 0) = (\frac{1}{\pi a^2})^{1/4}\sqrt{\frac{4}{3}}(\frac{x}{a})^2 \exp(-\frac{1}{2}(\frac{x}{a})^2)$;

$\hat{a}_+ = \frac{1}{\sqrt{2}}(\frac{x}{a} - a\frac{\partial}{\partial x})$.

$\psi_0(x) = (\frac{1}{\pi a^2})^{1/4} \exp(-\frac{1}{2}(\frac{x}{a})^2)$;

$\psi_1(x) = \hat{a}_+\psi_0(x) = (\frac{1}{\pi a^2})^{1/4}\sqrt{2}(\frac{x}{a}) \exp(-\frac{1}{2}(\frac{x}{a})^2)$;

$\psi_1(x) = \frac{1}{\sqrt{2!}}(\hat{a}_+)^2\psi_0(x) = \frac{1}{\sqrt{2}}\hat{a}_+\psi_1(x) = (\frac{1}{\pi a^2})^{1/4}\frac{1}{\sqrt{2}}[2(\frac{x}{a})^2 - 1] \exp(-\frac{1}{2}(\frac{x}{a})^2)$.

Combining these results, $\psi(x, t = 0) = \sqrt{\frac{1}{3}}\psi_0(x) + \sqrt{\frac{2}{3}}\psi_2(x)$.

As a consistency check, the expansion coefficient is normalized, $\sum_n |c_n|^2 = 1$.

Method #2: use ladder operators

$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_- + \hat{a}_+)$.

$\psi(x, t = 0) = \sqrt{\frac{4}{3}\frac{m\omega}{\hbar}}\hat{x}^2\psi_0(x) = \sqrt{\frac{4}{3}}\frac{1}{2}(\hat{a}_- + \hat{a}_+)^2\psi_0(x)$

By the commutation relation $[\hat{a}_-, \hat{a}_+] = 1$, $(\hat{a}_- + \hat{a}_+)^2 = \hat{a}_-^2 + \hat{a}_+^2 + 2\hat{a}_+\hat{a}_- + 1$.

Use $\hat{a}_-\psi_0(x) = 0$, and $\psi_n(x) = \frac{1}{\sqrt{n!}}\hat{a}_+^n\psi_0(x)$,

$\psi(x,t=0)=\sqrt{\frac{1}{3}}\cdot[0+\sqrt{2}\psi_2(x)+2\cdot 0+\psi_0(x)]=\sqrt{\frac{1}{3}}\psi_0(x)+\sqrt{\frac{2}{3}}\psi_2(x).$

(c) Use the result of (b),
$\psi(x,t)=\sum_n c_n e^{-\mathrm{i}E_n t/\hbar}\psi_n(x)=\sqrt{\frac{1}{3}}e^{-\mathrm{i}\omega t/2}\psi_0(x)+\sqrt{\frac{2}{3}}e^{-5\mathrm{i}\omega t/2}\psi_2(x).$

(d) From the result of (c), possible energy measurement results are $E_0=\frac{1}{2}\hbar\omega$ with possibility $\frac{1}{3}$; and $E_2=\frac{5}{2}\hbar\omega$ with possibility $\frac{2}{3}$.

(e)
$|\psi(t)\rangle=\sqrt{\frac{1}{3}}e^{-\mathrm{i}E_0t/\hbar}|\psi_0\rangle+\sqrt{\frac{2}{3}}e^{-\mathrm{i}E_2t/\hbar}|\psi_2\rangle.$
Rewrite $\hat{x}$ and $\hat{p}$ by the ladder operators, $\hat{x}=\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_-+\hat{a}_+)$, $\hat{p}=-\mathrm{i}\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_--\hat{a}_+)$.
$\hat{x}^2=\frac{\hbar}{2m\omega}(\hat{a}_-+\hat{a}_+)^2=\frac{\hbar}{2m\omega}(\hat{a}_-^2+\hat{a}_+^2+2\hat{a}_+\hat{a}_-+1).$
$\hat{p}^2=-\frac{\hbar m\omega}{2}(\hat{a}_--\hat{a}_+)^2=\frac{\hbar m\omega}{2}(-\hat{a}_-^2-\hat{a}_+^2+2\hat{a}_+\hat{a}_-+1).$
$\langle\psi(t)|\hat{a}_+|\psi(t)\rangle=0$, because $\hat{a}_+|\psi(t)\rangle$ contains only $|\psi_1\rangle$ and $|\psi_3\rangle$.
$\langle\psi(t)|\hat{a}_-|\psi(t)\rangle=\langle\psi(t)|\hat{a}_+|\psi(t)\rangle^*=0.$
$\langle\psi(t)|\hat{a}_+^2|\psi(t)\rangle=\sqrt{\frac{2}{3}}e^{\mathrm{i}E_2t\hbar}\cdot\sqrt{2}\cdot\sqrt{\frac{1}{3}}e^{-\mathrm{i}E_0t\hbar}=\frac{2}{3}e^{2\mathrm{i}\omega t}.$
$\langle\psi(t)|\hat{a}_-^2|\psi(t)\rangle=\langle\psi(t)|\hat{a}_+^2|\psi(t)\rangle^*=\frac{2}{3}e^{-2\mathrm{i}\omega t}.$
$\langle\psi(t)|\hat{a}_+\hat{a}_-|\psi(t)\rangle=\sqrt{\frac{2}{3}}e^{\mathrm{i}E_2t\hbar}\cdot 2\cdot\sqrt{\frac{2}{3}}e^{-\mathrm{i}E_2t\hbar}=\frac{4}{3}.$
Finally,
$\langle\hat{x}\rangle=0$, $\langle\hat{p}\rangle=0$, $\langle\hat{x}^2\rangle=\frac{\hbar}{2m\omega}(\frac{4}{3}\cos(2\omega t)+\frac{8}{3}+1))$, $\langle\hat{p}^2\rangle=\frac{\hbar m\omega}{2}(-\frac{4}{3}\cos(2\omega t)+\frac{8}{3}+1))$.
$\sigma_x^2\cdot\sigma_p^2=\frac{\hbar^2}{4}[(\frac{11}{3})^2-(\frac{4}{3})^2\cos^2(2\omega t)]>\frac{\hbar^2}{4}.$

Problem 2. (20 points) Consider a 3D central potential problem $\hat{H}=\frac{1}{2m}\hat{\boldsymbol{p}}^2-\alpha\cdot\delta(r-R)$. Here α, R are positive constants, δ is Dirac-δ function. Consider only the s-wave bound state with $\ell=m=0$, the eigenstate is $\psi_{E,0,0}=\frac{1}{r}u(r)$ with energy eigenvalue $E<0$.

(a) (5pts) Draw qualitatively the function $u(r)$.

(b) (15pts) Express $u(r)$ in terms of elementary functions, and write down the equation for the energy eigenvalue E. [Hint: Use $\kappa\equiv\frac{\sqrt{-2mE}}{\hbar}$. You will not be able to solve E.]

Solution:

$u(r)$ satisfies $\frac{-\hbar^2}{2m}\frac{\mathrm{d}^2u}{\mathrm{d}r^2}-\alpha\cdot\delta(r-R)\cdot u=E\cdot u.$

Define $\kappa = \sqrt{-2mE}/\hbar$.

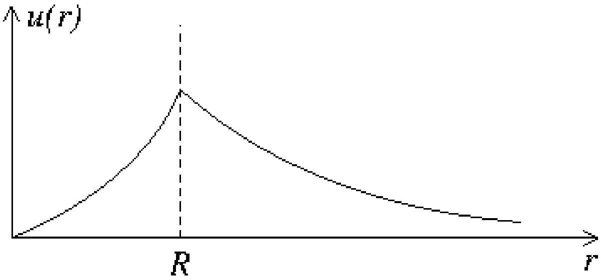
(a) At $r = R$, $u(r)$ is continuous, but its derivative is discontinuous.

For $r \to 0$, $u(r) \sim r^{\ell+1}$, so $u(r)$ should vanish linearly at $r = 0$.

For $r \to \infty$, $u(r)$ $e^{-\kappa r}$ decays exponentially.

For $r \neq R$, $\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} = \kappa^2 u$. Then if $u(r) > 0$, $u(r)$ is a convex function.

A schematic picture is the following,



(b) $u(r)$ should be linear combinations of $e^{-\kappa r}$ and $e^{\kappa r}$ in each region where the potential is zero.

For $r > R$, it must be proportional to $e^{-\kappa r}$, in order to be a bound state; but for $r < R$ it should contain both $e^{-\kappa r}$ and $e^{\kappa r}$.

$$u(r) = \begin{cases} Ae^{-\kappa r}, & r > R; \\ Be^{-\kappa r} + Ce^{\kappa r}, & 0 < r < R. \end{cases}$$

Because $u(r)$ is continuous at $r = R$, and $u(r = 0) = 0$, the coefficients are related by $Ae^{-\kappa R} = Be^{-\kappa R} + Ce^{\kappa R}$ and $B + C = 0$.

Finally, $u(r) = A \cdot \begin{cases} e^{-\kappa r}, & r > R; \\ e^{-\kappa R} \cdot \frac{e^{-\kappa r} - e^{\kappa r}}{e^{-\kappa R} - e^{\kappa R}}, & 0 < r < R. \end{cases}$

The equation at $r = R$ produces the relation, $-\frac{\hbar^2}{2m}\left(\frac{\mathrm{d}u}{\mathrm{d}r}|_{r=R+} - \frac{\mathrm{d}u}{\mathrm{d}r}|_{r=R-}\right) - \alpha \cdot u(r = R) = 0$.

This simplifies to $\frac{\hbar^2}{2m}\kappa \cdot \left(\frac{2}{1 - e^{-2\kappa R}}\right) = \alpha$, or equivalently $e^{-2\kappa R} = 1 - \frac{\hbar^2 \kappa}{m\alpha}$.

Note: when $R \to +\infty$, this simplifies to the condition of 1D bound state on a δ-potential.

Note: this equation may not have a solution, if $R < \frac{\hbar^2}{m\alpha}$.

Problem 3. (15 points) Consider spin-1/2 particle on the unit sphere($r = 1$). The spinor wavefunction is generically $\begin{pmatrix} \psi_\uparrow(\theta, \phi) \\ \psi_\downarrow(\theta, \phi) \end{pmatrix}$. The total angular momentum is $\hat{\boldsymbol{J}} = \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}}$.

(a) (5pts) If the particle has orbital angular momentum ℓ [*i.e.* $\hat{\boldsymbol{L}}^2$ has eigenvalue $\ell(\ell +$

1)$\hbar^2$]. What is the possible total angular momentum j, or the possible eigenvalue of $\hat{\boldsymbol{J}}^2$?

(b) (10pts) For the largest possible j in (a), write down the normalized spinor wavefunction for eigenstate of eigenvalues $\hat{\boldsymbol{J}}^2 = j(j+1)\hbar^2$ and $\hat{\boldsymbol{L}}^2 = \ell(\ell+1)\hbar^2$ and $\hat{J}_z = m\hbar$, in terms of spherical harmonics. Here $m = -j, -j+1, \dots, j$.

Solution:

(a) j can be $(\ell - \frac{1}{2})$ or $(\ell + \frac{1}{2})$, for $\ell > 0$; if $\ell = 0$, j can only be $\frac{1}{2}$.

(b) For $j = \ell + \frac{1}{2}$ and highest possible $m = j$, $|j, m = j\rangle = |\ell, m_\ell = \ell\rangle|\uparrow\rangle = \begin{pmatrix} Y_l^l \\ 0 \end{pmatrix}$.

Apply the lowering ladder operator repeatedly, we can generate all $|j, m\rangle$ states.

Generically $|j, m\rangle = c_{m,\uparrow}|\ell, m - \frac{1}{2}\rangle|\uparrow\rangle + c_{m,\downarrow}|\ell, m + \frac{1}{2}\rangle|\downarrow\rangle$.

Then $|j, m-1\rangle = \frac{1}{\sqrt{(j+m)(j-m+1)}\hbar}\hat{J}_-|j, m\rangle = \frac{1}{\sqrt{(j+m)(j-m+1)}\hbar}(\hat{L}_- + \hat{S}_-)|j, m\rangle$

$= \frac{\sqrt{(j+m-1)(j-m+1)}}{\sqrt{(j+m)(j-m+1)}}c_{m,\uparrow}|\ell, m - \frac{3}{2}\rangle|\uparrow\rangle + \frac{\sqrt{(j+m)(j-m)}}{\sqrt{(j+m)(j-m+1)}}c_{m,\downarrow}|\ell, m - \frac{1}{2}\rangle|\downarrow\rangle$
$+ \frac{1}{\sqrt{(j+m)(j-m+1)}}c_{m,\uparrow}|\ell, m - \frac{1}{2}\rangle|\downarrow\rangle$.

This produces recursion relation, $c_{m-1,\uparrow} = c_{m,\uparrow}\frac{\sqrt{j+m-1}}{\sqrt{j+m}}$, and
$c_{m-1,\downarrow} = c_{m,\downarrow}\frac{\sqrt{j-m}}{\sqrt{j-m+1}} + \frac{1}{\sqrt{(j+m)(j-m+1)}}c_{m,\uparrow}$.

From $c_{j,\uparrow} = 1$, we have $c_{m,\uparrow} = \frac{\sqrt{j+m}}{\sqrt{2j}}$.

By normalization condition, $c_{m,\downarrow} = \frac{\sqrt{j-m}}{\sqrt{2j}}$, which satisfies the above recursion relation.

$|j = \ell + \frac{1}{2}, m\rangle = \begin{pmatrix} \frac{\sqrt{j+m}}{\sqrt{2j}} Y_\ell^{m-\frac{1}{2}} \\ \frac{\sqrt{j-m}}{\sqrt{2j}} Y_\ell^{m+\frac{1}{2}} \end{pmatrix}$.

Problem 4. (15 points) Consider $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 + \alpha \cdot \delta(x)$. Here α is a positive constant, δ is Dirac-δ function.

(a) (10pts) Draw qualitatively the wavefunctions for the ground state, first excited state, and second excited state. What can you say about the properties of these wavefunctions?

(b) (5pts) What can you say about the ground state energy(*e.g.* compared to harmonic oscillator energy levels)?

Solution:

Denote the eigenstates of this Hamiltonian as ψ'_n, for $n = 0, 1, 2, \dots$, in ascending order of their energies E'_n.

Denote the eigenstates of original harmonic oscillator as ψ_n with energies $E_n = (n+\frac{1}{2})\hbar\omega$. Denote the original harmonic oscillator Hamiltonian by $\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2$.

Note that this Hamiltonian $\hat{H}$ has spatial inversion symmetry (invariant under $x \to -x$).

Note that $\langle\psi_0|\hat{H}|\psi_0\rangle \geq \langle\psi_0'|\hat{H}|\psi_0'\rangle$, because we can expand ψ_0 in terms of ψ_n', $\psi_0 = \sum_n c_n'\psi_n'$, with $\sum_n |c_n'|^2 = 1$, then $\langle\psi_0|\hat{H}|\psi_0\rangle = \sum_n |c_n'|^2 E_n' \geq \sum_n |c_n'|^2 E_0' = E_0'$.

Similarly $\langle\psi_0'|\hat{H}_0|\psi_0'\rangle > E_0$.

(a)

(2pts) Ground and second excited states are parity even, $\psi_0'(x) = \psi_0(-x)$, $\psi_2'(x) = \psi_2'(-x)$;

(1pts) first excited state should be parity odd, $\psi_1'(x) = -\psi_1'(-x)$.

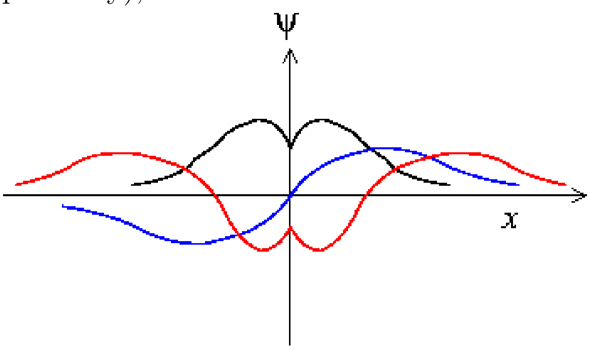
(3pts) ψ_0' and ψ_2' has a "dip" toward horizontal axis at $x = 0$; ψ_1' is smooth at $x = 0$.

(2pts) $\psi_n'(x)$ has n nodes.

(1pts) All ψ_n' decay as $\exp(-\frac{m\omega}{2\hbar}x^2)$ for $|x| \to \infty$, like the harmonic oscillator eigenstates.

(1pts) In fact ψ_1' is the same as the first excited state of original harmonic oscillator.

Schematic picture of the three states is the following (black, blue, red lines are for $\psi_{0,1,2}$ respectively),



(b).

(3pts) $E_0' > E_0$. Because $E_0' = \langle\psi_0'|\frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2} + \alpha\delta(x)|\psi_0'\rangle > \langle\psi_0'|\frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}|\psi_0'\rangle \geq E_0$.

(1pts) $E_0' < E_1' = E_1$.

(1pts) $E_0' < E_0 + \langle\psi_0|\alpha\delta(x)|\psi_0\rangle = E_0 + \alpha\sqrt{\frac{m\omega}{\pi\hbar}}$. Because $E_0' < \langle\psi_0|\frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2} + \alpha\delta(x)|\psi_0\rangle$.

Problem 5 (5 points) Can the probability current density $\boldsymbol{J} = \frac{i\hbar}{2m}(\psi\partial_{\boldsymbol{r}}\psi^* - \psi^*\partial_{\boldsymbol{r}}\psi)$ be *nonzero* for a stationary bound state? If yes, try to give an example; if not, why?

Solution:

Yes, in 2-dimension and higher spatial dimensions.

For stationary state, the probability density $\rho(\boldsymbol{r},t) \equiv |\psi(\boldsymbol{r},t)|^2$ does not change over time. By the continuity equation, $\boldsymbol{J}$ has to satisfy $\mathrm{div}\boldsymbol{J} = 0$.

In one-dimension, this divergence-free condition requires that $\boldsymbol{J}$ is a uniform constant in 1D space (does not vanish at spatial infinity). This is not compatible with bound states.

In 2-dimension or higher spatial dimensions, $\boldsymbol{J}$ can be divergence-free circulating currents, whose magnitude vanishes at spatial infinity. This usually comes from complex combinations of degenerate energy levels.

Example: hydrogen atom eigenstate $\psi_{2,1,1}$ whose angular part is $\sin\theta\, e^{\mathbb{i}\phi}$.

Example: for 3D harmonic oscillator, the state $\frac{1}{\sqrt{2}}(|n_x = 1, n_y = 0, n_z = 0\rangle + \mathbb{i}|n_x = 0, n_y = 1, n_z = 0\rangle)$ with wavefunction proportional to $(x + \mathbb{i}y)\exp[-\frac{m\omega}{2\hbar}(x^2 + y^2 + z^2)]$.