Quantum Mechanics: Fall 2023
Final Exam: Brief Solutions

Possibly useful facts:

- 1D harmonic oscillator: $\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2$.
  $[\hat{x}, \hat{p}_x] = \mathbb{i}\hbar$, and in position representation $\hat{p}_x = -\mathbb{i}\hbar\frac{\partial}{\partial x}$. Define $\hat{a}_\mp = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} \pm \mathbb{i}\frac{1}{m\omega}\hat{p}_x) = \sqrt{\frac{m\omega}{2\hbar}}(x \pm \frac{\hbar}{m\omega}\frac{\partial}{\partial x})$. Then $[\hat{a}_-, \hat{a}_+] = 1$ and $\hat{H} = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)$. It has a unique ground state $|\psi_0\rangle$ with $\hat{a}_-|\psi_0\rangle = 0$, and excited states $|\psi_n\rangle \equiv \frac{1}{\sqrt{n!}}(\hat{a}_+)^n|\psi_0\rangle$ with energy $E_n = (n+\frac{1}{2})\hbar\omega$. The ground state wavefunction is $\psi_0(x) = (\frac{m\omega}{\pi\hbar})^{1/4}\exp(-\frac{m\omega}{2\hbar}x^2)$.

- $\int_{-\infty}^{\infty} x^{2n}e^{-x^2/(2a)}\mathrm{d}x = (2n-1)!! \cdot a^n \cdot \sqrt{2\pi a}$, for $a > 0$ and non-negative integer n.

- Generic angular momentum: $[\hat{J}_x, \hat{J}_y] = \mathbb{i}\hbar\hat{J}_z$, $[\hat{J}_y, \hat{J}_z] = \mathbb{i}\hbar\hat{J}_x$, $[\hat{J}_z, \hat{J}_x] = \mathbb{i}\hbar\hat{J}_y$.
  For eigenstate $|j,m\rangle$ of $\hat{\boldsymbol{J}}^2$ and $\hat{J}_z$, $\hat{\boldsymbol{J}}^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle$, $\hat{J}_z|j,m\rangle = m\hbar|j,m\rangle$, and $(\hat{J}_x \pm \mathbb{i}\hat{J}_y)|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\hbar|j,m\pm 1\rangle$.
  Here $2j$ is non-negative integer, $m = -j, -j+1, \ldots, j$.

  - Spin-1/2: basis states $|\uparrow\rangle$ and $|\downarrow\rangle$, namely $|S_z = +\frac{1}{2}\hbar\rangle$ and $|S_z = -\frac{1}{2}\hbar\rangle$.
    Under this basis, $\hat{S}_a = \frac{\hbar}{2}\sigma_a$ where $\sigma_{x,y,z}$ are Pauli matrices.
    $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -\mathbb{i} \\ \mathbb{i} & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

- (Degenerate) Time-independent perturbation theory: $\hat{H} = \hat{H}^{(0)} + \hat{H}^{(1)}$. Denote the (degenerate) orthonormal eigenstates of $\hat{H}^{(0)}$ by $|\psi_{n\alpha}^{(0)}\rangle$, $\hat{H}^{(0)}|\psi_{n\alpha}^{(0)}\rangle = E_n^{(0)}|\psi_{n\alpha}\rangle$.
  Suppose $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$, E_n is close to $E_n^{(0)}$, then $(E_n - E_n^{(0)})$ is an eigenvalue of "secular equation" matrix, $\langle\psi_{n\beta}^{(0)}|\hat{H}^{(1)}|\psi_{n\alpha}^{(0)}\rangle + \sum_{m,m\neq n}\frac{1}{E_n^{(0)}-E_m^{(0)}}\langle\psi_{n\beta}^{(0)}|\hat{H}^{(1)}|\psi_m^{(0)}\rangle\langle\psi_m^{(0)}|\hat{H}^{(1)}|\psi_{n\alpha}^{(0)}\rangle$ up to second order. Here β & α are row/column index, the sum is over all eigenstates of $\hat{H}^{(0)}$ with energy different from $E_n^{(0)}$. In non-degenerate case, this is a 1×1 matrix.

- Dyson series: Solution to $\frac{\partial}{\partial t}c_n(t) = \sum_m V_{n,m}(t)c_m(t)$ is formally,
  $$c_n(t) = c_n(t=0) + \sum_m \int_0^t V_{n,m}(t')\mathrm{d}t' \cdot c_m(t=0) + \sum_m\sum_{m'}\int_0^t V_{n,m}(t')\left[\int_0^{t'} V_{m,m'}(t'')\mathrm{d}t''\right]\mathrm{d}t' \cdot c_{m'}(t=0) + \ldots$$

Problem 1. (30 points) Consider a non-relativistic particle of mass m moving on a ring of circumference L. This can be viewed as a 1D problem defined on x-axis with periodic boundary condition for the wavefunction, $\psi(x+L)=\psi(x)$, and normalization condition $\int_{-\frac{L}{2}}^{\frac{L}{2}} |\psi(x)|^2 \,\mathrm{d}x = 1$.

(a) (5pts) For free particle, $\hat{H}_0 = \frac{\hat{p}^2}{2m}$, with this periodic boundary condition, *write down all the energy eigenvalues* $E_n^{(0)}$ *and normalized eigenstate wavefunctions* $\psi_n^{(0)}(x)$.

(b) (15pts) Add a time-independent perturbation $V(x) = V_0 \cos(2\pi x/L)$ to $\hat{H}_0$, here V_0 is a "small" real parameter. *Compute the 1st and 2nd order (in terms of* V_0*) corrections to the energy eigenvalues.* [Note: degenerate perturbation theory may be avoided.]

(c) (10pts) Consider a variational wavefunction $\psi(x) = A + B\cos(2\pi x/L)$, where A, B are real parameters. *Compute the energy expectation value* $E(A,B)$ *(the expectation value of* $\hat{H} = \hat{H}_0 + V(x)$*) under* $\psi(x)$*. Minimize* $E(A,B)$ *with respect to parameters* A, B.

Solution

(a) This is one of homework problems.

plane wave basis: simultaneous basis of $\hat{H}_0$ and $\hat{p}$,

$E_n = \frac{\hbar^2}{2m}(\frac{2\pi n}{L})^2 = \frac{2\pi^2\hbar^2}{mL^2}n^2$,

$\psi_n(x) = \frac{1}{\sqrt{L}} e^{\mathrm{i}\frac{2\pi n}{L}x}$, here n is an integer,

standing wave basis: even/odd (superscript $^{(\mathrm{e})}$ and $^{(\mathrm{o})}$) functions of x,

$E_0 = 0$, $E_n = \frac{\hbar^2}{2m}(\frac{2\pi n}{L})^2$,

$\psi_0^{(\mathrm{e})}(x) = \frac{1}{\sqrt{L}}$, $\psi_n^{(\mathrm{e})}(x) = \sqrt{\frac{2}{L}}\cos(\frac{2\pi n}{L}x)$, $\psi_n^{(\mathrm{o})}(x) = \sqrt{\frac{2}{L}}\sin(\frac{2\pi n}{L}x)$, here $n = 1, 2, \ldots$ is a positive integer.

(b) the perturbation is an even function of x, therefore the eigenstates of $\hat{H}_0 + V(x)$ are either even or odd functions of x, we can solve this problem in the subspace of even/odd functions separately. In the even or odd subspace, the eigenvalues of $\hat{H}_0$ are nondegenerate,

even functions subspace:

$E_n^{(0)} = \frac{\hbar^2}{2m}(\frac{2\pi n}{L})^2,$

$\psi_n^{(0)}(x) = \begin{cases} \frac{1}{\sqrt{L}}, & n=0; \\ \sqrt{\frac{2}{L}}\cos(\frac{2\pi n}{L}x), & n>0 \end{cases}$

for $n = 0, 1, \dots,$

$\langle\psi_n^{(0)}|V(x)|\psi_m^{(0)}\rangle = \begin{cases} \frac{V_0}{\sqrt{2}}, & m-1=n=0, \text{ or } n-1=m=0; \\ \frac{V_0}{2}, & m-1=n>0, \text{ or } n-1=m>0; \\ 0, & \text{otherwise} \end{cases}$

$E_0^{(e)}(V_0) \approx 0 + 0 + \frac{|V_0/\sqrt{2}|^2}{0-\frac{2\pi^2\hbar^2}{mL^2}} = -\frac{V_0^2L^2m}{4\pi^2\hbar^2}$

$E_1^{(e)}(V_0) \approx \frac{2\pi^2\hbar^2}{mL^2} + 0 + \frac{|V_0/2|^2}{E_1^{(0)}-E_2^{(0)}} + \frac{|V_0/\sqrt{2}|^2}{E_1^{(0)}-E_0^{(0)}} = \frac{2\pi^2\hbar^2}{mL^2} + \frac{V_0^2L^2m}{4\pi^2\hbar^2}\cdot\frac{5}{6},$

$E_n^{(e)}(V_0) \approx \frac{2\pi^2\hbar^2}{mL^2}n^2 + 0 + \frac{|V_0/2|^2}{E_n^{(0)}-E_{n+1}^{(0)}} + \frac{|V_0/2|^2}{E_n^{(0)}-E_{n-1}^{(0)}} = \frac{2\pi^2\hbar^2}{mL^2}n^2 + \frac{V_0^2L^2m}{4\pi^2\hbar^2}\cdot\frac{1}{4n^2-1}$, for $n = 2, 3, \dots$.

odd functions subspace:

$E_n^{(0)} = \frac{\hbar^2}{2m}(\frac{2\pi n}{L})^2,$

$\psi_n^{(0)}(x) = \sqrt{\frac{2}{L}}\sin(\frac{2\pi n}{L}x),$

for $n = 1, 2, \dots,$

$\langle\psi_n^{(0)}|V(x)|\psi_m^{(0)}\rangle = \begin{cases} \frac{V_0}{2}, & m-1=n, \text{ or } n-1=m; \\ 0, & \text{otherwise} \end{cases}$

$E_1^{(o)}(V_0) \approx \frac{2\pi^2\hbar^2}{mL^2} + 0 + \frac{|V_0/2|^2}{E_1^{(0)}-E_2^{(0)}} = \frac{2\pi^2\hbar^2}{mL^2} - \frac{V_0^2L^2m}{4\pi^2\hbar^2}\cdot\frac{1}{6}$

$E_n^{(o)}(V_0) \approx \frac{2\pi^2\hbar^2}{mL^2}n^2 + 0 + \frac{|V_0/2|^2}{E_n^{(0)}-E_{n+1}^{(0)}} + \frac{|V_0/2|^2}{E_n^{(0)}-E_{n-1}^{(0)}} = \frac{2\pi^2\hbar^2}{mL^2}n^2 + \frac{V_0^2L^2m}{4\pi^2\hbar^2}\cdot\frac{1}{4n^2-1}$, for $n = 2, 3, \dots$.

NOTE: as a check, in either even or odd subspace, the trace (sum of eigenvalues) of $\hat{H}_0 + V(x)$ should be independent of V_0, because $\mathrm{Tr}[V(x)] = \frac{1}{L}\int_{-L/2}^{L/2} V(x)\,\mathrm{d}x = 0$.

(c) use $\psi(x) = \sqrt{L}\cdot[A\psi_0^{(e)}(x) + \frac{B}{\sqrt{2}}\psi_1^{(e)}(x)]$, and the matrix elements of $V(x)$ in even subspace in (b),

$E(A,B) = \frac{E_0^{(0)}A^2 + E_1^{(0)}B^2/2 + V_0AB}{A^2+B^2/2}$

let $A = \cos(\theta)$, $B/\sqrt{2} = \sin(\theta)$,

$E(A,B) = \frac{E_0^{(0)}+E_1^{(0)}}{2} + \frac{E_0^{(0)}-E_1^{(0)}}{2}\cos(2\theta) + \frac{V_0}{\sqrt{2}}\sin(2\theta)$

$= \frac{E_0^{(0)}+E_1^{(0)}}{2} + \sqrt{\frac{(E_0^{(0)}-E_1^{(0)})^2}{4} + \frac{V_0^2}{2}}\cos(2\theta+\phi)$

here $\cos(\phi) = \frac{(E_0^{(0)}-E_1^{(0)})/2}{\sqrt{\frac{(E_0^{(0)}-E_1^{(0)})^2}{4}+\frac{V_0^2}{2}}}$, $\sin(\phi) = \frac{V_0/\sqrt{2}}{\sqrt{\frac{(E_0^{(0)}-E_1^{(0)})^2}{4}+\frac{V_0^2}{2}}}$,

therefore $\min E(A,B) = \frac{\pi^2\hbar^2}{mL^2} - \sqrt{(\frac{\pi^2\hbar^2}{mL^2})^2 + \frac{V_0^2}{2}}$

Note: this variational ground state energy is the same as the perturbation theory result $E_0^{(e)}(V_0)$ in (b) up to $O(V_0^2)$ order.

Problem 2. (25 points) Consider a two-dimensional harmonic oscillator, $\hat{H}_0 = (\frac{1}{2m}\hat{p}_x^2 + \frac{m\omega^2}{2}x^2) + (\frac{1}{2m}\hat{p}_y^2 + \frac{m\omega^2}{2}y^2)$. Here $\hat{p}_x = -\mathbb{i}\hbar\partial_x$, $\hat{p}_y = -\mathbb{i}\hbar\partial_y$. Its eigenvalues and eigenstates can be labeled by two non-negative integers n_x, n_y as $E_{n_x,n_y} = \hbar\omega\cdot(n_x+n_y+1)$, $\psi_{n_x,n_y}(x,y) = \psi_{n_x}(x)\cdot\psi_{n_y}(y)$, here $\psi_n(x)$ is the nth excited state of 1D harmonic oscillator (see page 1.)

(a) (5pts) Consider a time-dependent perturbation $\hat{V}(t) = -f\cdot[x\sin(\Omega t) + y\cos(\Omega t)]$. Here f, Ω are positive parameters. Suppose the solution to Schrödinger equation $\mathbb{i}\hbar\frac{\partial}{\partial t}\psi(x,y,t) = [\hat{H}_0 + \hat{V}(t)]\psi(x,y,t)$ is $\psi(x,y,t) = \sum_{n_x,n_y} c_{n_x,n_y}(t)e^{-\mathbb{i}E_{n_x,n_y}t/\hbar}\psi_{n_x,n_y}(x,y)$. *Derive the differential equations for the coefficients $c_{n_x,n_y}(t)$ in the form of $\frac{\mathrm{d}}{\mathrm{d}t}c_{n_x,n_y}(t) = \dots$.* [NOTE: the right-hand-side of these equations should be expressed in terms of known quantities.]

(b) (10pts) Use the result of (a), *compute the transition probability from ground state to excited states over time t, namely $P_{(0,0)\to(1,0)}(t)$ and $P_{(0,0)\to(0,1)}(t)$, to lowest nontrivial order of f.*

(c) (5pts*) *Compute the transition probability $P_{(0,0)\to(1,1)}$ to lowest nontrivial order of f.* [NOTE: may need to go beyond 1st order approximation of $c_{n_x,n_y}(t)$]

(d) (5pts***) Define rotating coordinates $\tilde{x}(t) = x\cos(\Omega t) - y\sin(\Omega t)$, $\tilde{y}(t) = x\sin(\Omega t) + y\cos(\Omega t)$, and $\tilde{\psi}(\tilde{x}(t),\tilde{y}(t),t) = \psi(x,y,t)$. *Derive the Schrödinger equation for $\tilde{\psi}$, namely $\mathbb{i}\hbar\frac{\partial}{\partial t}\tilde{\psi}(\tilde{x},\tilde{y},t) = \tilde{H}\tilde{\psi}$. Check that $\tilde{H}$ does not explicitly depend on t. Solve the exact eigenvalues and eigenstates of $\tilde{H}$.* [Hint: define ladder operators for $\tilde{x}$ and $\tilde{y}$, make linear combinations of these ladder operators so that $\tilde{H}$ becomes two decoupled harmonic oscillators under constant force.]

Solution:

(a) Define $\hat{a}_{x,\pm} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} \mp \frac{\mathbb{i}}{m\omega}\hat{p}_x)$, $\hat{a}_{y,\pm} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{y} \mp \frac{\mathbb{i}}{m\omega}\hat{p}_y)$, then $[\hat{a}_{i,-}, \hat{a}_{j,+}] = \delta_{i,j}$, $[\hat{a}_{i,-}, \hat{a}_{j,-}] = 0$, for $i, j = x, y$.

$\hat{H}_0 = \hbar\omega(\hat{a}_{x,+}\hat{a}_{x,-} + \hat{a}_{y,+}\hat{a}_{y,-} + 1)$,

the eigenstates of $\hat{H}_0$ are

$\psi_{n_x,n_y}(x,y) = \psi_{n_x}(x)\cdot\psi_{n_y}(y) = \frac{1}{\sqrt{n_x! n_y!}}(\hat{a}_{x,+})^{n_x}(\hat{a}_{y,+})^{n_y}\psi_{0,0}(x,y)$, with $E_{n_x,n_y} = E_{n_x} + E_{n_y} = \hbar\omega(n_x + n_y + 1)$, where $n_x, n_y = 0, 1, \ldots$.

$\hat{V}(t) = -f\sqrt{\frac{\hbar}{2m\omega}}[(\hat{a}_{x,+} + \hat{a}_{x,-})\sin(\Omega t) + (\hat{a}_{y,+} + \hat{a}_{y,-})\cos(\Omega t)]$

assume $|\psi(t)\rangle = \sum_{n_x,n_y} c_{n_x,n_y}(t)e^{-\mathbb{i}E_{n_x,n_y}t/\hbar}|\psi_{n_x,n_y}\rangle$, then

$\frac{\mathrm{d}}{\mathrm{d}t}c_{n_x,n_y}(t) = \frac{1}{\mathbb{i}\hbar}\sum_{n'_x,n'_y} e^{\mathbb{i}(n_x+n_y-n'_x-n'_y)\omega t}\langle\psi_{n_x,n_y}|\hat{V}(t)|\psi_{n'_x,n'_y}\rangle \cdot c_{n'_x,n'_y}(t)$,

where $\langle\psi_{n_x,n_y}|\hat{V}(t)|\psi_{n'_x,n'_y}\rangle = -f\sqrt{\frac{\hbar}{2m\omega}}[(\sqrt{n_x}\delta_{n_x,n'_x+1}\delta_{n_y,n'_y} + \sqrt{n'_x}\delta_{n_x+1,n'_x}\delta_{n_y,n'_y})\sin(\Omega t) + (\sqrt{n_y}\delta_{n_x,n'_x}\delta_{n_y,n'_y+1} + \sqrt{n'_y}\delta_{n_x,n'_x}\delta_{n_y+1,n'_y})\cos(\Omega t)]$.

(b) for computing $P_{(0,0)\to(n_x,n_y)}(t)$, the initial condition of $c_{n'_x,n'_y}(t)$ is $c_{0,0}(t=0)=1$, and all other $c_{n'_x,n'_y}(t=0)=0$,

under 1st order approximation,

$c_{1,0}(t) \approx \frac{1}{\mathbb{i}\hbar}\int_0^t \mathrm{d}t\, e^{\mathbb{i}\omega t}\langle\psi_{1,0}|\hat{V}(t)|\psi_{0,0}\rangle = \frac{1}{\mathbb{i}\hbar}\int_0^t \mathrm{d}t\, e^{\mathbb{i}\omega t}(-f)\sqrt{\frac{\hbar}{2m\omega}}\sin(\Omega t)$
$= \mathbb{i}\frac{f}{\sqrt{2\hbar m\omega}}\int_0^t \mathrm{d}t\, e^{\mathbb{i}\omega t}\cdot\frac{e^{\mathbb{i}\Omega t}-e^{-\mathbb{i}\Omega t}}{2\mathbb{i}} = -\frac{\mathbb{i}}{2}\frac{f}{\sqrt{2\hbar m\omega}}(\frac{e^{\mathbb{i}(\omega+\Omega)t}-1}{\omega+\Omega} - \frac{e^{\mathbb{i}(\omega-\Omega)t}-1}{\omega-\Omega})$

then $P_{(0,0)\to(1,0)}(t) = |c_{1,0}(t)|^2 = \frac{1}{4}\frac{f^2}{2\hbar m\omega}|\frac{e^{\mathbb{i}(\omega+\Omega)t}-1}{\omega+\Omega} - \frac{e^{\mathbb{i}(\omega-\Omega)t}-1}{\omega-\Omega}|^2$

similarly, under 1st order approximation,

$c_{0,1}(t) \approx \frac{1}{\mathbb{i}\hbar}\int_0^t \mathrm{d}t\, e^{\mathbb{i}\omega t}\langle\psi_{0,1}|\hat{V}(t)|\psi_{0,0}\rangle = \frac{1}{\mathbb{i}\hbar}\int_0^t \mathrm{d}t\, e^{\mathbb{i}\omega t}(-f)\sqrt{\frac{\hbar}{2m\omega}}\cos(\Omega t)$
$= \mathbb{i}\frac{f}{\sqrt{2\hbar m\omega}}\int_0^t \mathrm{d}t\, e^{\mathbb{i}\omega t}\cdot\frac{e^{\mathbb{i}\Omega t}+e^{-\mathbb{i}\Omega t}}{2} = \frac{1}{2}\frac{f}{\sqrt{2\hbar m\omega}}(\frac{e^{\mathbb{i}(\omega+\Omega)t}-1}{\omega+\Omega} + \frac{e^{\mathbb{i}(\omega-\Omega)t}-1}{\omega-\Omega})$

then $P_{(0,0)\to(0,1)}(t) = |c_{0,1}(t)|^2 = \frac{1}{4}\cdot\frac{f^2}{2\hbar m\omega}|\frac{e^{\mathbb{i}(\omega+\Omega)t}-1}{\omega+\Omega} + \frac{e^{\mathbb{i}(\omega-\Omega)t}-1}{\omega-\Omega}|^2$

(c) Method #1:

nontrivial contribution to $c_{1,1}(t)$ is from 2nd order term in Dyson series, $c_{1,1}(t) = \sum_{n'_x,n'_y}\frac{1}{\mathbb{i}\hbar}\int_0^t \mathrm{d}t_1\, e^{\mathbb{i}(2-n'_x-n'_y)\omega t_1}\langle\psi_{1,1}|\hat{V}(t_1)|\psi_{n'_x,n'_y}\rangle\cdot\frac{1}{\mathbb{i}\hbar}\int_0^{t_1}\mathrm{d}t_2\, e^{\mathbb{i}(n'_x+n'_y)\omega t_2}\langle\psi_{n'_x,n'_y}|\hat{V}(t_2)|\psi_{0,0}\rangle$.

only $(n'_x, n'_y) = (1,0)$ and $(0,1)$ terms contribute,

note that $\langle\psi_{1,1}|\hat{V}(t)|\psi_{0,1}\rangle = \langle\psi_{1,0}|\hat{V}(t)|\psi_{0,0}\rangle = -f\sqrt{\frac{\hbar}{2m\omega}}\sin(\Omega t)$, and $\langle\psi_{1,1}|\hat{V}(t)|\psi_{1,0}\rangle = \langle\psi_{0,1}|\hat{V}(t)|\psi_{0,0}\rangle = -f\sqrt{\frac{\hbar}{2m\omega}}\cos(\Omega t)$,

use the identity,

$\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\, f(t_1)g(t_2) + \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\, g(t_1)f(t_2) = [\int_0^t \mathrm{d}t_1\, f(t_1)] \cdot [\int_0^t \mathrm{d}t_2\, g(t_2)]$,

$c_{1,1}(t) = [\mathbb{i}\frac{f}{\sqrt{2\hbar m\omega}} \int_0^t \mathrm{d}t\, e^{\mathbb{i}\omega t} \sin(\Omega t)] \cdot [\mathbb{i}\frac{f}{\sqrt{2\hbar m\omega}} \int_0^t \mathrm{d}t\, e^{\mathbb{i}\omega t} \cos(\Omega t)]$

$= -\frac{\mathbb{i}}{4} \cdot \frac{f^2}{2\hbar m\omega}[(\frac{e^{\mathbb{i}(\omega+\Omega)t}-1}{\omega+\Omega})^2 - (\frac{e^{\mathbb{i}(\omega-\Omega)t}-1}{\omega-\Omega})^2]$

$P_{(0,0)\to(1,1)}(t) = |c_{1,1}(t)|^2 = \frac{1}{16} \cdot \frac{f^4}{(2\hbar m\omega)^2}|(\frac{e^{\mathbb{i}(\omega+\Omega)t}-1}{\omega+\Omega})^2 - (\frac{e^{\mathbb{i}(\omega-\Omega)t}-1}{\omega-\Omega})^2|^2$

Method #2:

view this as two independent 1D systems(x and y), and compute the transition probabilities separately, $P_{(0,0)\to(1,1)}(t) = P_{x,0\to 1}(t) \cdot P_{y,0\to 1}(t)$,

and $P_{x,0\to 1}(t) \approx P_{(0,0)\to(1,0)}(t)$, $P_{y,0\to 1}(t) \approx P_{(0,0)\to(0,1)}(t)$ under 1st order approximation.

(d)

note that

$\hat{p}_{\tilde{x}} \equiv -\mathbb{i}\hbar(\frac{\partial}{\partial \tilde{x}})_{\tilde{y}} = \hat{p}_x \cos(\Omega t) - \hat{p}_y \sin(\Omega t)$, $\hat{p}_{\tilde{y}} \equiv -\mathbb{i}\hbar(\frac{\partial}{\partial \tilde{y}})_{\tilde{x}} = \hat{p}_x \sin(\Omega t) + \hat{p}_y \cos(\Omega t)$, and

$\tilde{x}^2 + \tilde{y}^2 = x^2 + y^2$, $\hat{p}_{\tilde{x}}^2 + \hat{p}_{\tilde{y}}^2 = \hat{p}_x^2 + \hat{p}_y^2$,

then $\hat{H}_0 = \frac{\hat{p}_{\tilde{x}}^2 + \hat{p}_{\tilde{y}}^2}{2m} + \frac{m\omega^2}{2}(\tilde{x}^2 + \tilde{y}^2)$,

use the chain rule, $[\frac{\partial}{\partial t}\psi(x,y,t)]_{x,y} = [\frac{\partial}{\partial t}\tilde{\psi}(\tilde{x},\tilde{y},t)]_{\tilde{x},\tilde{y}} + (\frac{\partial \tilde{x}}{\partial t})_{x,y}(\frac{\partial \tilde{\psi}}{\partial \tilde{x}})_{\tilde{y},t} + (\frac{\partial \tilde{y}}{\partial t})_{x,y}(\frac{\partial \tilde{\psi}}{\partial \tilde{y}})_{\tilde{x},t}$

note that $(\frac{\partial \tilde{x}}{\partial t})_{x,y} = -\Omega\tilde{y}$, and $(\frac{\partial \tilde{y}}{\partial t})_{x,y} = \Omega\tilde{x}$.

then $[\frac{\partial}{\partial t}\psi(x,y,t)]_{x,y} = [\frac{\partial}{\partial t}\tilde{\psi}(\tilde{x},\tilde{y},t)]_{\tilde{x},\tilde{y}} - \Omega\tilde{y}(\frac{\partial \tilde{\psi}}{\partial \tilde{x}})_{\tilde{y},t} + \Omega\tilde{x}(\frac{\partial \tilde{\psi}}{\partial \tilde{y}})_{\tilde{x},t}$

$= (\frac{\partial \tilde{\psi}}{\partial t})_{\tilde{x},\tilde{y}} + \frac{\mathbb{i}}{\hbar}\Omega \cdot (\tilde{x}\hat{p}_{\tilde{y}} - \tilde{y}\hat{p}_{\tilde{x}})\tilde{\psi} = (\frac{\partial \tilde{\psi}}{\partial t})_{\tilde{x},\tilde{y}} + \frac{\mathbb{i}}{\hbar}\Omega \cdot \hat{\tilde{L}}_z\tilde{\psi}$

finally from $\mathbb{i}\hbar[\frac{\partial}{\partial t}\psi(x,y,t)]_{x,y} = [\hat{H}_0 + \hat{V}(t)]\psi$, we have

$\mathbb{i}\hbar[\frac{\partial}{\partial t}\tilde{\psi}(\tilde{x},\tilde{y},t)]_{\tilde{x},\tilde{y}} = [\frac{\hat{p}_{\tilde{x}}^2 + \hat{p}_{\tilde{y}}^2}{2m} + \frac{m\omega^2}{2}(\tilde{x}^2 + \tilde{y}^2) - f\tilde{y} + \Omega\hat{\tilde{L}}_z]\tilde{\psi}$

namely,

$\hat{\tilde{H}} = \frac{\hat{p}_{\tilde{x}}^2 + \hat{p}_{\tilde{y}}^2}{2m} + \frac{m\omega^2}{2}(\tilde{x}^2 + \tilde{y}^2) - f\tilde{y} + \Omega\hat{\tilde{L}}_z$, which obviously does not explicitly depend on time t.

define ladder operators for $\tilde{x}$ and $\tilde{y}$,

$\hat{a}_{\tilde{x},\pm} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{\tilde{x}} \mp \frac{\mathbb{i}}{m\omega}\hat{p}_{\tilde{x}})$, $\hat{a}_{\tilde{y},\pm} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{\tilde{y}} \mp \frac{\mathbb{i}}{m\omega}\hat{p}_{\tilde{y}})$, then

$[\hat{a}_{i,-}, \hat{a}_{j,+}] = \delta_{i,j}$, $[\hat{a}_{i,-}, \hat{a}_{j,-}] = 0$, for $i,j = \tilde{x},\tilde{y}$.

$\hat{\tilde{H}} = \hbar\omega \cdot (\hat{a}_{\tilde{x},+}\hat{a}_{\tilde{x},-} + \hat{a}_{\tilde{y},+}\hat{a}_{\tilde{y},-} + 1) - f\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_{\tilde{y},+} + \hat{a}_{\tilde{y},-}) + \hbar\Omega \cdot (-\mathbb{i}\hat{a}_{\tilde{x},+}\hat{a}_{\tilde{y},-} + \mathbb{i}\hat{a}_{\tilde{y},+}\hat{a}_{\tilde{y},-})$

the bilinear terms can be “diagonalized” by defining another set of ladder operators,

$\hat{a}_{\pm,+} = \frac{1}{\sqrt{2}}(\mp\mathbb{i}\hat{a}_{\tilde{x},+} + \hat{a}_{\tilde{y},+})$, $\hat{a}_{\pm,-} = \frac{1}{\sqrt{2}}(\pm\mathbb{i}\hat{a}_{\tilde{x},-} + \hat{a}_{\tilde{y},-})$, then

$[\hat{a}_{i,-}, \hat{a}_{j,+}] = \delta_{i,j}$, $[\hat{a}_{i,-}, \hat{a}_{j,-}] = 0$, for $i,j = +,-$.

$\hat{a}_{\tilde{x},+}\hat{a}_{\tilde{x},-} + \hat{a}_{\tilde{y},+}\hat{a}_{\tilde{y},-} = \hat{a}_{+,+}\hat{a}_{+,-} + \hat{a}_{-,+}\hat{a}_{-,-}$, and
$(-\mathbb{i}\hat{a}_{\tilde{x},+}\hat{a}_{\tilde{y},-} + \mathbb{i}\hat{a}_{\tilde{y},+}\hat{a}_{\tilde{y},-}) = \hat{a}_{+,+}\hat{a}_{+,-} - \hat{a}_{-,+}\hat{a}_{-,-}$,

the hamiltonian becomes $\hat{\tilde{H}} =$
$\hbar(\omega+\Omega)\hat{a}_{+,+}\hat{a}_{+,-} - \frac{f}{2}\sqrt{\frac{\hbar}{m\omega}}(\hat{a}_{+,+} + \hat{a}_{+,-}) + \hbar(\omega-\Omega)\hat{a}_{-,+}\hat{a}_{-,-} - \frac{f}{2}\sqrt{\frac{\hbar}{m\omega}}(\hat{a}_{-,+} + \hat{a}_{-,-}) + \hbar\omega$,
which looks like two decoupled harmonic oscillators under constant forces (ground states are "coherent states")

define shifted ladder operators,
$\hat{a}'_{+,\pm} = \hat{a}_{+,\pm} - \frac{(f/2)\sqrt{\hbar/m\omega}}{\hbar(\omega+\Omega)}$, $\hat{a}'_{-,\pm} = \hat{a}_{-,\pm} - \frac{(f/2)\sqrt{\hbar/m\omega}}{\hbar(\omega-\Omega)}$, then
$[\hat{a}'_{i,-}, \hat{a}'_{j,+}] = \delta_{i,j}$, $[\hat{a}'_{i,-}, \hat{a}'_{j,-}] = 0$, for $i, j = +, -$,

$\tilde{H} = \hbar(\omega+\Omega)(\hat{a}'_{+,+}\hat{a}'_{+,-}) + \hbar(\omega-\Omega)(\hat{a}'_{-,+}\hat{a}'_{-,-}) + \hbar\omega - \frac{f^2}{2m(\omega^2-\Omega^2)}$

therefore the energy eigenvalues are
$E_{n_+,n_-} = \hbar(\omega+\Omega)\cdot n_+ + \hbar(\omega-\Omega)\cdot n_- + \hbar\omega - \frac{f^2}{2m(\omega^2-\Omega^2)}$,

eigenstates are $\tilde{\psi}_{n_+,n_-} = \frac{1}{\sqrt{n_+!n_-!}}(\hat{a}'_{+,+})^{n_+}(\hat{a}'_{-,+})^{n_-}\tilde{\psi}_{0,0}$, and
the unique ground state $\tilde{\psi}_{0,0}$ satisfies $\hat{a}'_{+,-}\tilde{\psi}_{0,0} = \hat{a}'_{-,-}\tilde{\psi}_{0,0} = 0$.

(not required) $\tilde{\psi}_{0,0}(\tilde{x},\tilde{y})$ is a 2D coherent state and can be solved explicitly, the condition $\hat{a}'_{+,-}\tilde{\psi}_{0,0} = \hat{a}'_{-,-}\tilde{\psi}_{0,0} = 0$ is equivalent to
$\hat{a}_{\tilde{x},-}\tilde{\psi}_{0,0} = \mathbb{i}f\sqrt{\frac{\hbar}{2m\omega}}\cdot\frac{\Omega}{\hbar(\omega^2-\Omega^2)}\cdot\tilde{\psi}_{0,0}$, and
$\hat{a}_{\tilde{y},-}\tilde{\psi}_{0,0} = f\sqrt{\frac{\hbar}{2m\omega}}\cdot\frac{\omega}{\hbar(\omega^2-\Omega^2)}\cdot\tilde{\psi}_{0,0}$, therefore
$\tilde{\psi}_{0,0}(\tilde{x},\tilde{y}) = \exp[\mathbb{i}\frac{f\Omega}{\hbar(\omega^2-\Omega^2)}\tilde{x}]\psi_0(\tilde{x})\cdot\psi_0(\tilde{y} - \frac{f}{m(\omega^2-\Omega^2)})$

Problem 3. (45 points) Consider two identical particles in the 1D ring defined in Problem 1, with Hamiltonian $\hat{H}_0 = \frac{1}{2m}(\hat{p}_1^2 + \hat{p}_2^2)$. Subscripts $_1$ and $_2$ label the two particles. $\hat{p}_1 = -\mathbb{i}\hbar\frac{\partial}{\partial x_1}$, $\hat{p}_2 = -\mathbb{i}\hbar\frac{\partial}{\partial x_2}$. The two-body wavefunction is periodic in both x_1 and x_2, $\psi(x_1,x_2) = \psi(x_1+L,x_2) = \psi(x_1,x_2+L)$, and has normalization $\int_{-L/2}^{L/2}\mathrm{d}x_1\int_{-L/2}^{L/2}\mathrm{d}x_2\,|\psi(x_1,x_2)|^2 = 1$.

(a) (5pts) For two identical spinless BOSONS under $\hat{H}_0$, *write down the normalized ground state(s) and first excited state(s), and second excited state(s) wavefunctions, and corresponding energies.*

(b) (5pts) For two identical spinless FERMIONS under $\hat{H}_0$, *write down the normalized ground state(s) and first excited state(s), and second excited state(s) wavefunctions, and corresponding energies.*

(c) (10pts) For spin-1/2 BOSONS and FERMIONS under $\hat{H}_0$, *write down the normalized ground state(s) and first excited state(s), and corresponding energies, for the boson/fermion cases respectively.* [Note: see page 1 for spin-1/2 properties]

(d) (10pts) *In the first excited state(s) of (c), for spin-1/2 BOSONS and FERMION cases respectively, compute the expectation values of* $\cos[\frac{2\pi}{L}(x_1 - x_2)]$.

(e) (5pts) In the ground states of (c), for spin-1/2 BOSONS and FERMION cases respectively, measure $\hat{S}_{1,x} + \hat{S}_{2,x}$, here $\hat{S}_{i,x}$ is the x-component of spin operators for particle $i = 1, 2$. *What are the possible measurement results and corresponding probabilities?*

(f) (5pts**) Add a time-independent perturbation $\hat{H}_1 = \lambda \cdot \delta(x_1 - x_2)$. Here λ is a "small" real parameter. *Compute the corrections to all the BOSON case energies in (c) to lowest nontrivial order of* λ. [NOTE: results may contain sums of infinite series]

(g) (5pts**) *Derive the exact equation for ground state energy of* $\hat{H}_0 + \hat{H}_1$ *for two spin-1/2 BOSONS.* [Hint: change variables from x_1, x_2 to center-of-mass coordinate $X = \frac{x_1+x_2}{2}$ and relative coordinate $x = x_2 - x_1$, be careful about the periodic condition and permutation symmetry of ψ]

Solution

use superscript $^{(A)}$ to label antisymmetric (spinless fermion) orbital wave functions, and superscript $^{(S)}$ to label symmetric (spinless boson) orbital wave functions,

(a) for two spinless bosons,

ground state is $\psi^{(S)}_{0,0}(x_1, x_2) = \frac{1}{L}$, $E_{0,0} = 0$.

if using plane wave basis for single particle states, first excited states are

$\psi^{(S)}_{0,1}(x_1,x_2) = \frac{1}{L\sqrt{2}}(e^{\mathrm{i}\frac{2\pi}{L}x_1} + e^{\mathrm{i}\frac{2\pi}{L}x_2})$, $\psi^{(S)}_{0,-1}(x_1,x_2) = \frac{1}{L\sqrt{2}}(e^{-\mathrm{i}\frac{2\pi}{L}x_1} + e^{-\mathrm{i}\frac{2\pi}{L}x_2})$,

energy $E_{0,1} = E_{0,-1} = \frac{2\pi^2\hbar^2}{mL^2}$,

second excited states are

$\psi^{(S)}_{1,1}(x_1,x_2) = \frac{1}{L}e^{\mathrm{i}\frac{2\pi}{L}(x_1+x_2)}$, $\psi^{(S)}_{-1,-1}(x_1,x_2) = \frac{1}{L}e^{-\mathrm{i}\frac{2\pi}{L}(x_1+x_2)}$,

$\psi^{(S)}_{1,-1}(x_1,x_2) = \frac{1}{L\sqrt{2}}(e^{\mathrm{i}\frac{2\pi}{L}(x_1-x_2)} + e^{\mathrm{i}\frac{2\pi}{L}(x_2-x_1)}) = \frac{\sqrt{2}}{L}\cos[\frac{2\pi}{L}(x_1-x_2)]$,

energy $E_{1,1} = E_{-1,-1} = E_{1,-1} = \frac{2\pi^2\hbar^2}{mL^2}\cdot 2$,

if using standing wave basis for single particle states,

first excited states are

$\psi^{(S)}_{0,1\text{e}}(x_1,x_2) = \frac{1}{L}(\cos(\frac{2\pi}{L}x_1) + \cos(\frac{2\pi}{L}x_2))$, $\psi^{(S)}_{0,1\text{o}}(x_1,x_2) = \frac{1}{L}(\sin(\frac{2\pi}{L}x_1) + \sin(\frac{2\pi}{L}x_2))$,

energy $E_{0,1\text{e}} = E_{0,1\text{o}} = \frac{2\pi^2\hbar^2}{mL^2}$,

second excited states are

$\psi^{(S)}_{1\text{e},1\text{e}}(x_1,x_2) = \frac{2}{L}\cos(\frac{2\pi}{L}x_1)\cos(\frac{2\pi}{L}x_2)$, $\psi^{(S)}_{1\text{o},1\text{o}}(x_1,x_2) = \frac{2}{L}\sin(\frac{2\pi}{L}x_1)\sin(\frac{2\pi}{L}x_2)$,

$\psi^{(S)}_{1\text{e},1\text{o}}(x_1,x_2) = \frac{2}{L\sqrt{2}}(\cos(\frac{2\pi}{L}x_1)\sin(\frac{2\pi}{L}x_2) + \sin(\frac{2\pi}{L}x_1)\cos(\frac{2\pi}{L}x_2)) = \frac{\sqrt{2}}{L}\sin[\frac{2\pi}{L}(x_1+x_2)]$,

energy $E_{1\text{e},1\text{e}} = E_{1\text{o},1\text{o}} = E_{1\text{e},1\text{o}} = \frac{2\pi^2\hbar^2}{mL^2}\cdot 2$,

(b) for two spinless fermions,

if using plane wave basis for single particle states,

ground states are

$\psi^{(A)}_{0,1}(x_1,x_2) = \frac{1}{L\sqrt{2}}(e^{\mathrm{i}\frac{2\pi}{L}x_1} - e^{\mathrm{i}\frac{2\pi}{L}x_2})$, $\psi^{(A)}_{0,-1}(x_1,x_2) = \frac{1}{L\sqrt{2}}(e^{-\mathrm{i}\frac{2\pi}{L}x_1} - e^{-\mathrm{i}\frac{2\pi}{L}x_2})$,

energy $E_{0,1} = E_{0,-1} = \frac{2\pi^2\hbar^2}{mL^2}$,

first excited states are

$\psi^{(A)}_{1,-1}(x_1,x_2) = \frac{1}{L\sqrt{2}}(e^{\mathrm{i}\frac{2\pi}{L}(x_1-x_2)} - e^{\mathrm{i}\frac{2\pi}{L}(x_2-x_1)}) = \mathrm{i}\frac{\sqrt{2}}{L}\sin[\frac{2\pi}{L}(x_1-x_2)]$,

energy $E_{1,-1} = \frac{2\pi^2\hbar^2}{mL^2}\cdot 2$,

second excited states are

$\psi^{(A)}_{0,2}(x_1,x_2) = \frac{1}{L\sqrt{2}}(e^{\mathrm{i}\frac{\pi}{L}x_1} - e^{\mathrm{i}\frac{\pi}{L}x_2})$, $\psi^{(A)}_{0,-2}(x_1,x_2) = \frac{1}{L\sqrt{2}}(e^{-\mathrm{i}\frac{\pi}{L}x_1} - e^{-\mathrm{i}\frac{\pi}{L}x_2})$,

energy $E_{0,2} = E_{0,-2} = \frac{2\pi^2\hbar^2}{mL^2}\cdot 4$,

if using standing wave basis for single particle states,

ground states are

$\psi^{(A)}_{0,1\text{e}}(x_1,x_2) = \frac{1}{L}(\cos(\frac{2\pi}{L}x_1) - \cos(\frac{2\pi}{L}x_2))$, $\psi^{(A)}_{0,1\text{o}}(x_1,x_2) = \frac{1}{L}(\sin(\frac{2\pi}{L}x_1) - \sin(\frac{2\pi}{L}x_2))$, energy $E_{0,1\mathrm{e}} = E_{0,1\mathrm{o}} = \frac{2\pi^2\hbar^2}{mL^2}$,

first excited states are

$\psi^{(\mathrm{A})}_{1\mathrm{e},1\mathrm{o}}(x_1, x_2) = \frac{2}{L\sqrt{2}}(\cos(\frac{2\pi}{L}x_1)\sin(\frac{2\pi}{L}x_2) - \sin(\frac{2\pi}{L}x_1)\cos(\frac{2\pi}{L}x_2)) = \frac{\sqrt{2}}{L}\sin[\frac{2\pi}{L}(x_2 - x_1)]$,

energy $E_{1\mathrm{e},1\mathrm{o}} = \frac{2\pi^2\hbar^2}{mL^2} \cdot 2$,

second excited states are

$\psi^{(\mathrm{A})}_{0,2\mathrm{e}}(x_1, x_2) = \frac{1}{L}(\cos(\frac{\pi}{L}x_1) - \cos(\frac{\pi}{L}x_2))$, $\psi^{(\mathrm{A})}_{0,2\mathrm{o}}(x_1, x_2) = \frac{1}{L}(\sin(\frac{\pi}{L}x_1) - \sin(\frac{\pi}{L}x_2))$,

energy $E_{0,2\mathrm{e}} = E_{0,2\mathrm{o}} = \frac{2\pi^2\hbar^2}{mL^2} \cdot 4$,

(c) Factorize the eigenbasis into orbital and spin wavefunctions, the spin part of wavefunctions can be spin single $|S = 0, S_z = 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$, or spin triplet states $|S = 1, S_z = 1\rangle = |\uparrow\rangle|\uparrow\rangle$, $|S = 1, S_z = -1\rangle = |\downarrow\rangle|\downarrow\rangle$, and $|S = 1, S_z = 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle)$,

Use standing wave single particle basis hereafter.

For two spin-1/2 bosons,

the ground states are $|\psi_{0,0}(x_1, x_2)\rangle|S = 1, S_z\rangle$, where $S_z = +1, 0, -1$, with energy $E_{0,0} = 0$

,

the first excited states are $|\psi^{(\mathrm{S})}_{0,1\mathrm{e}}(x_1, x_2)\rangle|S = 1, S_z\rangle$, and $|\psi^{(\mathrm{S})}_{0,1\mathrm{o}}(x_1, x_2)\rangle|S = 1, S_z\rangle$, where $S_z = +1, 0, -1$; and $|\psi^{(\mathrm{A})}_{0,1\mathrm{e}}(x_1, x_2)\rangle|S = 0, S_z = 0\rangle$, and $|\psi^{(\mathrm{A})}_{0,1\mathrm{o}}(x_1, x_2)\rangle|S = 0, S_z = 0\rangle$, with energy $E_{0,1} = \frac{2\pi^2\hbar^2}{mL^2}$.

For two spin-1/2 fermions,

the ground state is $|\psi_{0,0}(x_1, x_2)\rangle|S = 0, S_z = 0\rangle$, with energy $E_{0,0} = 0$,

the first excited states are $|\psi^{(\mathrm{A})}_{0,1\mathrm{e}}(x_1, x_2)\rangle|S = 1, S_z\rangle$, and $|\psi^{(\mathrm{A})}_{0,1\mathrm{o}}(x_1, x_2)\rangle|S = 1, S_z\rangle$, where $S_z = +1, 0, -1$; and $|\psi^{(\mathrm{S})}_{0,1\mathrm{e}}(x_1, x_2)\rangle|S = 0, S_z = 0\rangle$, and $|\psi^{(\mathrm{S})}_{0,1\mathrm{o}}(x_1, x_2)\rangle|S = 0, S_z = 0\rangle$, with energy $E_{0,1} = \frac{2\pi^2\hbar^2}{mL^2}$.

(d) the observable does not depend on spin, so we just need to evaluate the following expectation values for spinless bosons or fermions,

$\langle\psi^{(\mathrm{S})}_{0,1\mathrm{e}}|\cos(\frac{2\pi}{L}(x_1 - x_2))|\psi^{(\mathrm{S})}_{0,1\mathrm{e}}\rangle$

$= \frac{1}{L^2}\int_{-L/2}^{L/2}\mathrm{d}x_1\int_{-L/2}^{L/2}\mathrm{d}x_2\,[\cos(\frac{2\pi}{L}x_1) + \cos(\frac{2\pi}{L}x_2)]^2\cos(\frac{2\pi}{L}(x_1 - x_2))$

$= \frac{1}{2}$

$\langle\psi^{(S)}_{0,1o}|\cos(\frac{2\pi}{L}(x_1 - x_2))|\psi^{(S)}_{0,1o}\rangle$

$= \frac{1}{L^2}\int_{-L/2}^{L/2}\mathrm{d}x_1\int_{-L/2}^{L/2}\mathrm{d}x_2\,[\sin(\frac{2\pi}{L}x_1) + \sin(\frac{2\pi}{L}x_2)]^2\cos(\frac{2\pi}{L}(x_1 - x_2))$

$= \frac{1}{2}$

$\langle\psi^{(A)}_{0,1e}|\cos(\frac{2\pi}{L}(x_1 - x_2))|\psi^{(S)}_{0,1e}\rangle$

$= \frac{1}{L^2}\int_{-L/2}^{L/2}\mathrm{d}x_1\int_{-L/2}^{L/2}\mathrm{d}x_2\,[\cos(\frac{2\pi}{L}x_1) - \cos(\frac{2\pi}{L}x_2)]^2\cos(\frac{2\pi}{L}(x_1 - x_2))$

$= -\frac{1}{2}$

$\langle\psi^{(A)}_{0,1o}|\cos(\frac{2\pi}{L}(x_1 - x_2))|\psi^{(S)}_{0,1o}\rangle$

$= \frac{1}{L^2}\int_{-L/2}^{L/2}\mathrm{d}x_1\int_{-L/2}^{L/2}\mathrm{d}x_2\,[\sin(\frac{2\pi}{L}x_1) - \sin(\frac{2\pi}{L}x_2)]^2\cos(\frac{2\pi}{L}(x_1 - x_2))$

$= -\frac{1}{2}$

Finally for the observable $\cos(\frac{2\pi}{L}(x_1 - x_2))$, under the first excited states of two spin-1/2 bosons or fermions, if the orbital wavefunction is symmetric, the expectation value is $+\frac{1}{2}$; if the orbital wavefunction is antisymmetric, the expectation value is $-\frac{1}{2}$.

NOTE: using the planewave single particle basis will reach the same conclusion.

(e)

This observable depends only on the spin wavefunction, it has
eigenvalue $+\hbar$, with eigenstate $\frac{1}{2}(|\uparrow\rangle + |\downarrow\rangle)\otimes(|\uparrow\rangle + |\downarrow\rangle)$;
eigenvalue $-\hbar$, with eigenstate $\frac{1}{2}(|\uparrow\rangle - |\downarrow\rangle)\otimes(|\uparrow\rangle - |\downarrow\rangle)$;
eigenvalue 0, with eigenstates $\frac{1}{\sqrt{2}}(|\uparrow\rangle|\uparrow\rangle - |\downarrow\rangle|\downarrow\rangle)$, and $\frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$.

The probability of getting each result under each of the spin wavefunction basis is listed in the following table,

measurement result	$\|S_0, S_z = 0\rangle$	$\|S = 1, S_z = 1\rangle$	$\|S = 1, S_z = 0\rangle$	$\|S = 1, S_z = -1\rangle$
$+\hbar$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$-\hbar$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
0	1	$\frac{1}{2}$	0	$\frac{1}{2}$

NOTE: you can also use the $|S = 1, S_x = m\hbar\rangle$ basis for spin triplet states.

(f) The hamiltonian with perturbation does not depend on spin, therefore preserves S and S_z quantum numbers,

The hamiltonian also has the following discrete symmetries:

inversion (even or odd), $x_1 \to -x_1$ and $x_2 \to -x_2$;

particle exchange (symmetric or antisymmetric), $x_1 \leftrightarrow x_2$

In a subspace with certain parity and exchange symmetry and S and S_z, the two energy levels in (c) for two spin-1/2 bosons are non-degenerate, therefore we can use non-degenerate perturbation theory.

For first order corrections, we need the following expectation values,

$\langle \psi_{0,0} | \lambda\delta(x_1 - x_2) | \psi_{0,0} \rangle = \frac{1}{L^2} \int_{-L/2}^{L/2} \mathrm{d}x_1 \int_{-L/2}^{L/2} \mathrm{d}x_2 \, \lambda\delta(x_1 - x_2) = \frac{\lambda}{L}$

The lowest order nontrivial energy correction to ground state energy is $\frac{\lambda}{L}$

$\langle \psi_{0,1e}^{(S)} | \lambda\delta(x_1 - x_2) | \psi_{0,1e}^{(S)} \rangle = \frac{2\lambda}{L}$

$\langle \psi_{0,1o}^{(S)} | \lambda\delta(x_1 - x_2) | \psi_{0,1o}^{(S)} \rangle = \frac{2\lambda}{L}$

The lowest order nontrivial energy correction to first excited states with symmetric orbital wavefunctions is $\frac{2\lambda}{L}$

It is easy to see that

$\langle \psi | \lambda\delta(x_1 - x_2) | \psi_{0,1e}^{(A)} \rangle = 0$, and $\langle \psi | \lambda\delta(x_1 - x_2) | \psi_{0,1o}^{(A)} \rangle = 0$, for any $\psi(x_1, x_2)$, therefore there is no energy correction to first excited states with antisymmetric orbital wavefunctions.

(g)

let $\tilde{\psi}(X, x) = \psi(x_1, x_2)$, the normalization is $\int_0^L \mathrm{d}X \int_{-L/2}^{L/2} \mathrm{d}x \, |\tilde{\psi}|^2 = 1$,

then $\psi(x_1 + L, x_2) = \psi(x_1, x_2 + L) = \psi(x_1, x_2)$ becomes

$\tilde{\psi}(X + L/2, x - L) = \tilde{\psi}(X + L/2, x + L) = \tilde{\psi}(X, x)$, then $\tilde{\psi}$ also satisfies $\tilde{\psi}(X + L, x) = \tilde{\psi}(X, x)$

for bosons, $\psi(x_2, x_1) = \psi(x_1, x_2)$, then $\tilde{\psi}(X, x) = \tilde{\psi}(X, -x)$,

define $\hat{p}_X = -\mathrm{i}\hbar(\frac{\partial}{\partial X})_x = \hat{p}_1 + \hat{p}_2$, $\hat{p}_x = -\mathrm{i}\hbar(\frac{\partial}{\partial x})_X = \frac{1}{2}(\hat{p}_2 - \hat{p}_1)$,

then $\hat{H}_0 + \hat{H}_1 = \frac{1}{4m}\hat{p}_X^2 + \frac{1}{m}\hat{p}_x^2 + \lambda\delta(x)$, which contains two independent hamiltonians for X and x respectively, the eigenstates can be chosen as a tensor product, $\tilde{\psi}(X, x) = \phi(X)\varphi(x)$,

for the X system, the ground state of $\frac{1}{4m}\hat{p}_X^2$ is $\phi(X) = \frac{1}{\sqrt{L}}$, with $E_X = 0$,

then for the x system with hamiltonian $\frac{1}{m}\hat{p}_x^2 + \lambda\delta(x)$, the eigenstate $\varphi(x)$ should satisfy $\varphi(x + L) = \varphi(x) = \varphi(-x)$, therefore $\varphi(x) = \varphi(L - x)$, namely $\varphi(x)$ is symmetric about $x = L/2$, and is smooth at $x = L/2$ because the hamiltonian there is not singular,

suppose the eigenvalue of x system is $E_x = \frac{\hbar^2 k^2}{m}$, then

$$\varphi(x) = \begin{cases} A \cdot \cos[k(x - L/2)], & 0 \le x \le L/2; \\ A \cdot \cos[k(x + L/2)], & -L/2 \le x \le 0 \end{cases}.$$

the boundary condition for $\frac{\partial \varphi}{\partial x}$ at $x = 0$ is

$-\frac{\hbar^2}{m}\frac{\partial \varphi}{\partial x}\Big|_{x=-0}^{+0} + \lambda\varphi(x=0) = 0$,

therefore $-\frac{\hbar^2}{m} \cdot 2Ak\sin(kL/2) + \lambda A\cos(kL/2) = 0$,

the ground state energy of the entire system is $E = E_X + E_x = E_x$,

the equation for $k = \sqrt{2mE}/\hbar$ is $\frac{2\hbar^2}{m\lambda}k = \cot(\frac{kL}{2})$

NOTE: if $\lambda < 0$, then $E < 0$, k is pure imaginary.

(not required) for small λ, the ground state k is also small, the equation for k is approximately

$-\frac{\hbar^2}{m} \cdot 2k \cdot (kL/2) + \lambda(1 - \frac{(kL/2)^2}{2}) \approx 0$,

then $k \approx \frac{2}{L}\sqrt{\frac{\lambda}{\lambda/2 + (4\hbar^2/mL)}} \approx \frac{\sqrt{m\lambda}}{\hbar\sqrt{L}}$,

$E \approx \frac{\lambda}{L}$, consistent with the perturbation theory result in (f).