Quantum Mechanics: Fall 2019
Midterm Exam: Brief Solution

NOTE: Problems start on page 2. Bold symbols are 3-component vectors.

Some useful facts: You can use them directly.

- 1D harmonic oscillator: $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2$.
  Here $\hat{x}$ is position operator, $\hat{p}$ is momentum operator, $[\hat{x}, \hat{p}] = \mathbb{i}\hbar$, and in position representation $\hat{p} = -\mathbb{i}\hbar\frac{\partial}{\partial x}$. Define $\hat{a}_\mp = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} \pm \mathbb{i}\frac{1}{m\omega}\hat{p}) = \sqrt{\frac{m\omega}{2\hbar}}(x \pm \frac{\hbar}{m\omega}\frac{\partial}{\partial x})$. Then $[\hat{a}_-, \hat{a}_+] = 1$ and $\hat{H} = \hbar\omega\,(\hat{a}_+\hat{a}_- + \frac{1}{2})$. It has a unique ground state $|\psi_0\rangle$ with $\hat{a}_-|\psi_0\rangle = 0$, and excited states $|\psi_n\rangle \equiv \frac{1}{\sqrt{n!}}(\hat{a}_+)^n|\psi_0\rangle$ with energy $E_n = (n+\frac{1}{2})\hbar\omega$. The ground state wavefunction is $\psi_0(x) = (\frac{m\omega}{\pi\hbar})^{1/4}\exp(-\frac{m\omega}{2\hbar}x^2)$.

- $\int_{-\infty}^{+\infty} e^{-x^2/a}\mathrm{d}x = \sqrt{\pi a}$, for $a > 0$. Applying $\frac{\partial}{\partial a}$ can produce $\int_{-\infty}^{+\infty} x^{2n}e^{-x^2/a}\mathrm{d}x$.

- Central potential problem: $\hat{H} = \frac{1}{2m}\hat{\boldsymbol{p}}^2 + V(r)$.
  Here $\hat{\boldsymbol{p}}$ is the 3D momentum $-\mathbb{i}\hbar\frac{\partial}{\partial \boldsymbol{r}}$, and $r = |\boldsymbol{r}|$ is the radius. Under polar coordinates (r, θ, ϕ), the eigenfunctions are generally $\psi_{E,\ell,m} = \frac{u(r)}{r}\cdot Y_\ell^m(\theta, \phi)$, where $Y_\ell^m(\theta, \phi)$ is the spherical harmonics, and $u(r)$ satisfies $-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]\cdot u = E\cdot u$. Here $\ell = 0, 1, \ldots$ is the angular momentum quantum number; $m = -\ell, -\ell+1, \ldots, \ell$ is the azimuthal angular momentum quantum number; E is the energy eigenvalue.
  - The spherical harmonics are orthonormal, and are eigenfunctions of $\hat{\boldsymbol{L}}^2$ and $\hat{L}_z$.
    $Y_0^0 = \frac{1}{\sqrt{4\pi}}$, $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$, $Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm\mathbb{i}\phi}$, ...

- Generic angular momentum: $[\hat{J}_x, \hat{J}_y] = \mathbb{i}\hbar\hat{J}_z$, $[\hat{J}_y, \hat{J}_z] = \mathbb{i}\hbar\hat{J}_x$, $[\hat{J}_z, \hat{J}_x] = \mathbb{i}\hbar\hat{J}_y$.
  For eigenstate $|j, m\rangle$ of $\hat{\boldsymbol{J}}^2$ and $\hat{J}_z$, $\hat{\boldsymbol{J}}^2|j, m\rangle = j(j+1)\hbar^2|j, m\rangle$, $\hat{J}_z|j, m\rangle = m\hbar|j, m\rangle$, and $(\hat{J}_x \pm \mathbb{i}\hat{J}_y)|j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\hbar|j, m \pm 1\rangle$.
  Here $2j$ is non-negative integer, $m = -j, -j+1, \ldots, j$.
  - Orbital angular momentum: $\hat{\boldsymbol{L}} \equiv \hat{\boldsymbol{r}} \times \hat{\boldsymbol{p}}$.
  - Spin-1/2: basis states $|\uparrow\rangle$ and $|\downarrow\rangle$.
    Under this basis, $\hat{S}_a = \frac{\hbar}{2}\sigma_a$ where $\sigma_{x,y,z}$ are Pauli matrices.
    Generic wavefunction under this basis is $\begin{pmatrix}\psi_\uparrow \\ \psi_\downarrow\end{pmatrix}$, which means $\psi_\uparrow|\uparrow\rangle + \psi_\downarrow|\downarrow\rangle$.

Problem 1. (35 points) Consider a 1D harmonic oscillator $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}$. (see page 1)

(a) (5pts) The initial wavefunction is $\varphi(x, t = 0) = (A + Bx^2) \cdot \exp(-\frac{m\omega}{2\hbar}x^2)$, where A, B are complex numbers. *Solve the condition on A, B so that $\varphi(x, t = 0)$ is normalized.*

(b) (5pts) Measure energy(namely $\hat{H}$) under $\varphi(x, t = 0)$. *What are the possible measurement results, and their corresponding probabilities?*

(c) (5pts) Evolve the state by the harmonic oscillator Hamiltonian. *Solve the wavefunction $\varphi(x, t)$ at time t.*

(d) (20pts) *Compute the expectation values $\langle\hat{x}\rangle$, $\langle\hat{p}\rangle$, $\langle\hat{x}^2\rangle$, $\langle\hat{p}^2\rangle$ in the state $\varphi(x, t)$. Check that the uncertainty relation for $\hat{x}, \hat{p}$ is satisfied.*

Solution:

(a) Method #1: computing integrals

Use $\int_{-\infty}^{\infty} |\varphi(x, t = 0)|^2 \mathrm{d}x = 1$,

$\int_{-\infty}^{\infty} e^{-x^2/a}\mathrm{d}x = \sqrt{\pi a}$, $\int_{-\infty}^{\infty} x^2 e^{-x^2/a}\mathrm{d}x = \sqrt{\pi a} \cdot \frac{a}{2}$, $\int_{-\infty}^{\infty} x^4 e^{-x^2/a}\mathrm{d}x = \sqrt{\pi a} \cdot \frac{a}{2} \cdot \frac{3a}{2}$.

$\sqrt{\frac{\pi\hbar}{m\omega}} \cdot (|A|^2 + (A^*B + B^*A)\frac{\hbar}{2m\omega} + |B|^2\frac{3\hbar^2}{4m^2\omega^2}) = 1.$

Method #2: expanding into orthonormal eigenbasis

If $\varphi(x, t = 0) = \sum_{n=0}^{\infty} c_n\psi_n(x)$, where $\psi_n(x) = \frac{1}{\sqrt{n!}}(\hat{a}_+)^n\psi_0(x)$ are orthonormal eigenbasis for $\hat{H}$, then $\sum_{n=0}^{\infty} |c_n|^2 = 1$.

For notation simplicity, define $a_0 = \sqrt{\frac{\hbar}{m\omega}}$. then $\hat{a}_+ = \frac{1}{\sqrt{2}}(\frac{\hat{x}}{a_0} - a_0\partial_x)$.

$\psi_0(x) = (\frac{1}{\pi a_0^2})^{1/4} \exp(-\frac{x^2}{2a_0^2})$,

$\psi_1(x) = \hat{a}_+\psi_0(x) = (\frac{1}{\pi a_0^2})^{1/4}\sqrt{2}(\frac{x}{a_0}) \exp(-\frac{x^2}{2a_0^2})$,

$\psi_2(x) = \frac{1}{\sqrt{2}}(\hat{a}_+)^2\psi_0(x) = (\frac{1}{\pi a_0^2})^{1/4}\frac{1}{\sqrt{2}}[2(\frac{x}{a_0})^2 - 1] \exp(-\frac{x^2}{2a_0^2})$, ...

So $\varphi(x, t = 0) = (\pi a_0^2)^{1/4}[\frac{Ba_0^2}{\sqrt{2}}\psi_2 + (A + \frac{Ba_0^2}{2})\psi_0]$.

Namely, $c_0 = (\pi a_0^2)^{1/4}(A + \frac{Ba_0^2}{2})$, $c_2 = (\pi a_0^2)^{1/4}\frac{Ba_0^2}{\sqrt{2}}$, and other $c_n = 0$.

This expansion can also be obtained by using $\hat{x}^2 = \frac{a_0^2}{2}(\hat{a}_+\hat{a}_+ + \hat{a}_-\hat{a}_- + 2\hat{a}_+\hat{a}_- + 1)$ in the $A + Bx^2$ factor.

Then $(\pi a_0^2)^{1/2}\left[|B|^2\frac{a_0^4}{2}+|A+B\frac{a_0^2}{2}|^2\right]=1.$

(b) According to the Method #2 of (a), energy measurement results can be
$E_0=\frac{1}{2}\hbar\omega$, with probability $(\pi a_0^2)^{1/2}|A+B\frac{a_0^2}{2}|^2$;
$E_2=\frac{5}{2}\hbar\omega$, with probability $(\pi a_0^2)^{1/2}|B|^2\frac{a_0^4}{2}$.

(c) According to the Method #2 of (a), and the general solution to Schrödinger equation for time-independent Hamiltonian,
$\varphi(x,t)=\sum_{n=0}^{\infty}c_n e^{-\mathrm{i}E_n t/\hbar}\psi_n=(\pi a_0^2)^{1/4}[\frac{Ba_0^2}{\sqrt{2}}e^{-\mathrm{i}E_2 t/\hbar}\psi_2+(A+\frac{Ba_0^2}{2})e^{-\mathrm{i}E_0 t/\hbar}\psi_0]$
$=[Bx^2e^{-\mathrm{i}5\omega t/2}+Ae^{-\mathrm{i}\omega t/2}+\frac{B\hbar}{2m\omega}(e^{-\mathrm{i}\omega t/2}-e^{-\mathrm{i}5\omega t/2})]\exp(-\frac{m\omega}{2\hbar}x^2)$.

(d) $\hat{x}=\frac{a_0}{\sqrt{2}}(\hat{a}_-+\hat{a}_+)$, $\hat{p}=-\mathrm{i}\frac{\hbar}{\sqrt{2}a_0}(\hat{a}_--\hat{a}_+)$.
Under $\varphi(x,t)$, $\langle\hat{x}\rangle=0$, $\langle\hat{p}\rangle=0$.
This can also be seen from the fact that $\varphi(x,t)$ is even with respect to x.
Use $\hat{x}^2=\frac{a_0^2}{2}(\hat{a}_+\hat{a}_++\hat{a}_-\hat{a}_-+2\hat{a}_+\hat{a}_-+1)$, $\hat{p}^2=\frac{\hbar^2}{2a_0^2}(-\hat{a}_+\hat{a}_+-\hat{a}_-\hat{a}_-+2\hat{a}_+\hat{a}_-+1)$.
$\langle\hat{x}^2\rangle=\frac{a_0^2}{2}[\sqrt{2}(c_2^*c_0e^{\mathrm{i}(E_2-E_0)t/\hbar}+c_0^*c_2e^{-\mathrm{i}(E_2-E_0)t/\hbar})+5|c_2|^2+|c_0|^2]$;
$\langle\hat{p}^2\rangle=\frac{\hbar^2}{2a_0^2}[-\sqrt{2}(c_2^*c_0e^{\mathrm{i}(E_2-E_0)t/\hbar}+c_0^*c_2e^{-\mathrm{i}(E_2-E_0)t/\hbar})+5|c_2|^2+|c_0|^2]$.
Here c_0,c_2 are given above in Method #2 of (a). $E_2-E_0=2\hbar\omega$.

(4pts) The uncertainty relation can be checked as follows,
$(\langle\hat{x}^2\rangle-\langle\hat{x}\rangle^2)(\langle\hat{p}^2\rangle-\langle\hat{p}\rangle^2)=\frac{\hbar^2}{4}\left[(5|c_2|^2+|c_0|^2)^2-2(c_2^*c_0e^{\mathrm{i}2\omega t}+c_0^*c_2e^{-\mathrm{i}2\omega t})^2\right]$
$\geq\frac{\hbar^2}{4}\left[(5|c_2|^2+|c_0|^2)^2-8|c_2|^2|c_0|^2\right]=\frac{\hbar^2}{4}\left[25|c_2|^4+2|c_2|^2|c_0|^2+|c_0|^4\right]$
$\geq\frac{\hbar^2}{4}\left[|c_2|^4+2|c_2|^2|c_0|^2+|c_0|^4\right]=\frac{\hbar^2}{4}(|c_2|^2+|c_0|^2)^2=\frac{\hbar^2}{4}$.

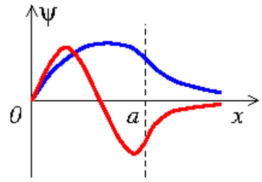
Problem 2. (15 points) Consider a 1D non-relativistic particle, with $\hat{H}=\frac{\hat{p}^2}{2m}+V(x)$, in the "half-infinite" square potential well, $V(x)=\begin{cases}+\infty, & x<0;\\ -V_0, & 0<x<a;\\ 0, & a<x.\end{cases}$. Here a,V_0 are positive constants.

(a) (5pts) Assume that the ground state and first excited state are bound states. *Draw qualitatively the wavefunctions for these two bound states.*

(b) (10pts) *Derive the equations for energy eigenvalues of bound states. Determine the condition on V_0 so that there are at least two bound states.* [Note: you will not be able to solve the energy eigenvalues for arbitrary V_0 and a.]

Solution: This is roughly the same as problem 2.40 in textbook (a homework problem).

(a) Schematic picture of the ground state and first excited state.



Important features:

both wavefunctions vanishes for $x \leq 0$;

both wavefunctions are smooth (with continuous derivative) at $x = a$;

both are exponentially decaying for $x > a$;

the ground state has no node; the first excited state has one node.

(b) The bound state energy E should satisfy $-V_0 < E < 0$.

Define $K = \sqrt{-2mE}/\hbar$, $k = \sqrt{2m(E + V_0)}/\hbar$.

The eigenstate should be $\psi(x) = \begin{cases} 0, & x < 0; \\ A\sin(kx), & 0 < x < a; \\ B\exp(-Kx), & a < x. \end{cases}$.

$\psi(x)$ and $\partial_x \psi(x)$ should both be continuous at $x = a$.

$A\sin(ka) = B\exp(-Ka)$, $kA\cos(ka) = -KB\exp(-Ka)$. So

$Ka = -(ka)\cot(ka)$, $(Ka)^2 + (ka)^2 = \frac{2mV_0a^2}{\hbar^2}$.

This is exactly the equation for odd parity solutions of finite square potential well.

To have at least two solutions, $\frac{2mV_0a^2}{\hbar^2} \geq (\frac{3\pi}{2})^2$, or $V_0 \geq (\frac{3\pi}{2})^2(\frac{\hbar^2}{2ma^2})$.

Problem 3. (35 points) Consider a particle in 3D space confined on the sphere of radius $r \equiv \sqrt{x^2 + y^2 + z^2} = R$. We label its position just by polar and azimuthal angles θ, ϕ. The wavefunction $\psi(\theta, \phi)$ satisfy normalization $\int_0^\pi \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \, |\psi(\theta, \phi)|^2 = 1$. The Hamiltonian is just its kinetic energy $\hat{H}_0 = \frac{\hat{\boldsymbol{L}}^2}{2mR^2}$, where $\hat{\boldsymbol{L}}$ is the angular momentum operator (page 1).

(a) (5pts) *Write down all the energy eigenvalues of $\hat{H}_0$. Write down the normalized ground state and first excited state wavefunctions for $\hat{H}_0$.*

(b) (5pts) If the particle has spin-1/2 spin angular momentum, so its wavefunction should be a spinor wavefunction $\begin{pmatrix} \psi_\uparrow(\theta, \phi) \\ \psi_\downarrow(\theta, \phi) \end{pmatrix}$. And the Hamiltonian is $\hat{H} = \hat{H}_0 + \lambda \hat{\boldsymbol{L}} \cdot \hat{\boldsymbol{S}}$. Here $\hat{\boldsymbol{S}}$ is spin angular momentum operator, $\hat{\boldsymbol{L}} \cdot \hat{\boldsymbol{S}} \equiv \hat{L}_x \hat{S}_x + \hat{L}_y \hat{S}_y + \hat{L}_z \hat{S}_z$ is the "spin-orbit coupling" term, λ is a positive constant. Define the total angular momentum operator $\hat{\boldsymbol{J}} = \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}}$. *Show that the following commutators vanish,* $[\hat{H}_0, \lambda \hat{\boldsymbol{L}} \cdot \hat{\boldsymbol{S}}] = 0, [\hat{\boldsymbol{J}}^2, \hat{H}] = 0, [\hat{J}_z, \hat{H}] = 0$.

(c) (5pts) According to (b), we can find simultaneous eigenstates of $\hat{H}$, $\hat{\boldsymbol{J}}^2$, $\hat{J}_z$, $\hat{\boldsymbol{L}}^2$. *Write down the possible combinations of eigenvalues of the above four operators.* [Hint: it may be convenient to rewrite the "spin-orbit coupling" term in terms of $\hat{\boldsymbol{J}}$]

(d) (5pts) *What is the degeneracy of the ground states of $\hat{H}$?* [Hint: be careful that the ground states of $\hat{H}$ may not be simply related to the ground state of $\hat{H}_0$]

(e) (10pts) (*) We can remove the ground state degeneracy in (d) by adding a "Zeeman field" term to the Hamiltonian, so it becomes $\hat{H} - B_z \hat{J}_z$. Here B_z is a small positive constant($B_z \ll \lambda\hbar, \frac{\hbar}{mR^2}$). *Solve the unique normalized ground state spinor wavefunction in this case.* [Hint: use the ladder operators, you can represent the results by spherical harmonics]

(f) (5pts) In the ground state in (e), measure $\hat{S}_x$. *What are the possible measurement results and their probabilities?*

Solution:

(a) Ground state: $E_0 = 0$, eigenstate wavefunction $Y_0^0 = \sqrt{\frac{1}{4\pi}}$.

First excited states: $E_1 = \frac{2\hbar^2}{2mR^2}$, 3-fold degenerate,

eigenstate wavefunctions $Y_1^{-1} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{-\mathrm{i}\phi}$, $Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta$, $Y_1^1 = \sqrt{\frac{3}{8\pi}} \sin\theta e^{\mathrm{i}\phi}$.

(b) Rewrite the spin-orbit coupling term in terms of $\hat{\boldsymbol{J}}$.

$\lambda \hat{\boldsymbol{L}} \cdot \hat{\boldsymbol{S}} = \frac{\lambda}{2}[(\hat{\boldsymbol{L}} + \hat{\boldsymbol{S}})^2 - \hat{\boldsymbol{L}}^2 - \hat{\boldsymbol{S}}^2] = \frac{\lambda}{2}(\hat{\boldsymbol{J}}^2 - \hat{\boldsymbol{L}}^2 - \hat{\boldsymbol{S}}^2)$.

$\hat{H} = \frac{\hat{\boldsymbol{L}}^2}{2mR^2} + \frac{\lambda}{2}(\hat{\boldsymbol{J}}^2 - \hat{\boldsymbol{L}}^2 - \hat{\boldsymbol{S}}^2)$.

$\hat{\boldsymbol{S}}^2 = \frac{3\hbar^2}{4}$ is a constant for spin-1/2 particles.

So we just need to prove that $[\hat{\boldsymbol{L}}^2, \hat{\boldsymbol{J}}^2] = 0$, $[\hat{\boldsymbol{L}}^2, \hat{J}_z] = 0$.

Use $[\hat{J}_a, \hat{L}_b] = \mathrm{i}\hbar\epsilon_{abc}\hat{L}_c$, and $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$, then $[\hat{J}_a, \hat{\boldsymbol{L}}^2] = [\hat{J}_a, \hat{L}_b\hat{L}_b] = \mathrm{i}\hbar(\epsilon_{abc}\hat{L}_c\hat{L}_b + \hat{L}_b\epsilon_{abc}\hat{L}_c) = \mathrm{i}\hbar(\epsilon_{abc}\hat{L}_c\hat{L}_b + \hat{L}_c\epsilon_{acb}\hat{L}_b) = \mathrm{i}\hbar\hat{L}_c\hat{L}_b(\epsilon_{abc} + \epsilon_{acb}) = 0$. Here we have used the Einstein convention of implicit summation over repeated indices.

Then $[\hat{\boldsymbol{J}}^2, \hat{\boldsymbol{L}}^2] = \hat{J}_a[\hat{J}_a, \hat{\boldsymbol{L}}^2] + [\hat{J}_a, \hat{\boldsymbol{L}}^2]\hat{J}_0 = 0$.

(c) Use the form of $\hat{H}$ in (b).

$\hat{\boldsymbol{L}}^2$ has eigenvalues $\hbar^2\ell(\ell+1)$, $\ell = 0, 1, \ldots$.

$\hat{\boldsymbol{J}}^2$ has eigenvalues $\hbar^2 j(j+1)$, j can be $\ell + \frac{1}{2}$, and $\ell - \frac{1}{2}$ if $\ell > 0$.

$\hat{J}_z$ has eigenvalues $m\hbar$ with $m = -j, -j+1, \ldots, j$.

The possible combinations of these eigenvalues are summarized in the table below,

	$j = \ell + \frac{1}{2}$ case	$j = \ell - \frac{1}{2}$ case
ℓ values	$\ell = 0, 1, \ldots$	$\ell = 1, 2, \ldots$
$\hat{H}$ eigenvalue	$\frac{\hbar^2}{2mR^2}\ell(\ell+1) + \frac{\lambda}{2}\hbar^2\ell$	$\frac{\hbar^2}{2mR^2}\ell(\ell+1) - \frac{\lambda}{2}\hbar^2(\ell+1)$
$\hat{\boldsymbol{J}}^2$ eigenvalue	$\hbar^2(\ell+\frac{1}{2})(\ell+\frac{3}{2})$	$\hbar^2(\ell-\frac{1}{2})(\ell+\frac{1}{2})$
$\hat{J}_z$ eigenvalue	$j_z\hbar$, $j_z = -j, -j+1, \ldots, j$	
$\hat{\boldsymbol{L}}^2$ eigenvalue	$\hbar^2\ell(\ell+1)$	

(d). Degeneracy should be $2j+1$. Note that $\lambda > 0$. Then according to the result of (c),
if $\lambda < \frac{1}{mR^2}$, the ground state has $j = \frac{1}{2}$, $\ell = 0$, 2-fold degeneracy;
if $\frac{1}{mR^2} < \lambda < \frac{4}{mR^2}$, the ground state has $j = \frac{1}{2}$, $\ell = 1$, 2-fold degeneracy;
if $\frac{2\ell}{mR^2} < \lambda < \frac{2\ell+2}{mR^2}$, for $\ell = 2, 3, \ldots$, the ground state has $j = \ell - \frac{1}{2}$, 2ℓ-fold degeneracy.

(not required) The "borderline values" for λ in the above inequalities will have higher accidental degeneracy = sum of degeneracy on both sides,
if $\lambda = \frac{1}{mR^2}$, then $j = \frac{1}{2}, \ell = 0$, or $j = \frac{1}{2}, \ell = 1$, 4-fold degeneracy;
if $\lambda = \frac{2\ell}{mR^2}$ for some $\ell = 2, 3, \ldots$, then $j = \ell - \frac{1}{2}$ or $j = (\ell-1) - \frac{1}{2}$, total degeneracy is $4\ell - 2$.

(e). The unique ground state should have the highest possible J_z quantum number $j_z = j$ among the degenerate ground states in(d).

(3pts) If $\lambda < \frac{1}{mR^2}$, the ground state is $|j = \frac{1}{2}, j_z = \frac{1}{2}; \ell = 0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

If $\frac{1}{mR^2} < \lambda$, the ground state is $|j = \ell - \frac{1}{2}, j_z = \ell - \frac{1}{2}\rangle$. The ℓ value is given in (d). We need to represent this state by linear combinations of $|\ell, m\rangle|s = \frac{1}{2}, s_z\rangle$. Here $|\ell, m\rangle$ is the spherical harmonic Y_ℓ^m. (see Problem 4.51 in textbook).

Assume that $|j = \ell - \frac{1}{2}, j_z = \ell - \frac{1}{2}\rangle = A|\ell, m = \ell - 1\rangle|\uparrow\rangle + B|\ell, m = \ell\rangle|\downarrow\rangle$.

$\hat{J}_+|j = \ell - \frac{1}{2}, j_z = \ell - \frac{1}{2}\rangle = 0 = (\hat{L}_+ + \hat{S}_+)(A|\ell, m = \ell - 1\rangle|\uparrow\rangle + B|\ell, m = \ell\rangle|\downarrow\rangle)$

$= \hbar A\sqrt{2\ell}|\ell, m = \ell\rangle|\uparrow\rangle + \hbar B|\ell, m = \ell\rangle|\uparrow\rangle$.

Therefore $A = \sqrt{\frac{1}{2\ell+1}}$, $B = -\sqrt{\frac{2\ell}{2\ell+1}}$, up to overall complex phase factors.

(7pts) The ground state for $\frac{1}{mR^2} < \lambda$ cases is $\begin{pmatrix} \sqrt{\frac{1}{2\ell+1}} Y_\ell^{\ell-1} \\ -\sqrt{\frac{2\ell}{2\ell+1}} Y_\ell^\ell \end{pmatrix}$.

(not required) For the "borderline value" cases, we should choose the side with higher j. However $\lambda = \frac{1}{mR^2}$ case is special, because both sides have $j = \frac{1}{2}$, so the degeneracy cannot be completely removed.

(f). $\hat{S}_x$ has eigenvalue $+\frac{\hbar}{2}$ with normalized eigenstate $|\hat{S}_x = +\frac{\hbar}{2}\rangle \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$;

and eigenvalue $-\frac{\hbar}{2}$ with normalized eigenstate $|\hat{S}_x = -\frac{\hbar}{2}\rangle \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$.

Under a spinor wavefunction $|\psi\rangle \equiv \begin{pmatrix} \psi_\uparrow(\theta,\phi) \\ \psi_\downarrow(\theta,\phi) \end{pmatrix}$, the probability to obtain the $\pm\frac{\hbar}{2}$ eigenvalue is $|\langle \hat{S}_x = \pm\frac{\hbar}{2}|\psi\rangle|^2 \equiv \int_0^\pi \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\, \frac{1}{2}|\psi_\uparrow(\theta,\phi) \pm \psi_\downarrow(\theta,\phi)|^2$.

For all the cases of ground state in (e), $\int_0^\pi \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\, \psi_\uparrow^*(\theta,\phi)\psi_\downarrow(\theta,\phi) = 0$, then the probabilities are both $\int_0^\pi \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\, \frac{1}{2}[|\psi_\uparrow(\theta,\phi)|^2 + |\psi_\downarrow(\theta,\phi)|^2] = \frac{1}{2}$. Measurement results can be $\pm\frac{\hbar}{2}$ with probability $\frac{1}{2}$ for each case.

4. (15pts) Consider a 1D non-relativistic particle, with $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$. The potential is a periodic array of attractive δ-functions (a "Dirac comb"), $V(x) = -\alpha \cdot \sum_{n=-\infty}^{\infty} \delta(x - nL)$, where α, L are positive constants.

(a) (10pts) (*) Assume the positive energy eigenstate is $A_n e^{\mathbb{i}k\cdot(x-nL)} + B_n e^{-\mathbb{i}k\cdot(x-nL)}$ for $nL < x < (n+1)L$. Here k is the wavevector related to energy by $E = \frac{\hbar^2 k^2}{2m}$. *Solve A_{n+1}, B_{n+1} in terms of A_n, B_n.*

(b) (5pts) (*) *Are there negative energy normalizable bound states? Prove your answer.*

Solution: This is related to Problem 2.53 of textbook (a homework problem).

(a) This is almost the same as Problem 2.53(c).

From $\psi((n+1)L+0) = \psi((n+1)L-0)$ and $\partial_x\psi|_{x=(n+1)L-0}^{x=(n+1)L+0} = -\frac{2m\alpha}{\hbar^2}\psi((n+1)L)$,

$A_{n+1} + B_{n+1} = A_n e^{\mathbb{i}kL} + B_n e^{-\mathbb{i}kL}$,

$\mathbb{i}k \cdot (A_{n+1} - B_{n+1}) = \mathbb{i}k \cdot (A_n e^{\mathbb{i}kL} - B_n e^{-\mathbb{i}kL}) - \frac{2m\alpha}{\hbar^2}(A_n e^{\mathbb{i}kL} + B_n e^{-\mathbb{i}kL})$

For notation simplicity, define $\beta = \frac{m\alpha}{\hbar^2 k}$, this is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} e^{\mathbb{i}kL}, & e^{-\mathbb{i}kL} \\ e^{\mathbb{i}kL}(1+2\mathbb{i}\beta), & -e^{-\mathbb{i}kL}(1-2\mathbb{i}\beta) \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix}$$

Then $\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} e^{\mathbb{i}kL}(1+\mathbb{i}\beta) & \mathbb{i}\beta e^{-\mathbb{i}kL} \\ -\mathbb{i}\beta e^{\mathbb{i}kL} & e^{-\mathbb{i}kL}(1-\mathbb{i}\beta) \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix}$.

(b) There are NO negative energy normalizable bound states.

Assume the negative energy eigenstate is

$\psi(x) = A_n e^{-K\cdot(x-nL)} + B_n e^{K\cdot(x-nL)}$, for $nL < x < (n+1)L$, here $K = \sqrt{-2mE}/\hbar$.

The result of (a) can still be used, with replacement $k \to \mathbb{i}K$, $\beta \to -\mathbb{i}\tilde{\beta}$ with $\tilde{\beta} = \frac{m\alpha}{\hbar^2 K}$.

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} e^{-KL}(1+\tilde{\beta}) & \tilde{\beta} e^{KL} \\ -\tilde{\beta} e^{-KL} & e^{KL}(1-\tilde{\beta}) \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix}.$$

It would be more convenient to use $\tilde{A}_n = e^{-KL/2} A_n$ and $\tilde{B}_n = e^{KL/2} B_n$.

Then $\begin{pmatrix} \tilde{A}_{n+1} \\ \tilde{B}_{n+1} \end{pmatrix} = \begin{pmatrix} (1+\tilde{\beta})e^{-KL} & \tilde{\beta} \\ -\tilde{\beta} & (1-\tilde{\beta})e^{KL} \end{pmatrix} \begin{pmatrix} \tilde{A}_n \\ \tilde{B}_n \end{pmatrix}$.

The normalization of $\psi(x)$ would be given by

$\int_{-\infty}^{\infty} |\psi(x)|^2 \,\mathrm{d}x = \sum_{n=-\infty}^{\infty} [\frac{\sinh(KL)}{KL}(|\tilde{A}_n|^2 + |\tilde{B}_n|^2) + \tilde{A}_n^* \tilde{B}_n + \tilde{B}_n^* \tilde{A}_n] \cdot L$

The summand is bounded on both sides, $0 \le (\frac{\sinh(KL)}{KL} - 1)(|\tilde{A}_n|^2 + |\tilde{B}_n|^2)$ $\le \frac{\sinh(KL)}{KL}(|\tilde{A}_n|^2 + |\tilde{B}_n|^2) + \tilde{A}_n^* \tilde{B}_n + \tilde{B}_n^* \tilde{A}_n \le (\frac{\sinh(KL)}{KL} + 1)(|\tilde{A}_n|^2 + |\tilde{B}_n|^2)$.

So $\psi(x)$ is normalizable if and only if $\sum_{n=-\infty}^{\infty}(|\tilde{A}_n|^2 + |\tilde{B}_n|^2)$ is finite, which is also equivalent to the condition that both $\sum_{n=-\infty}^{\infty} |\tilde{A}_n|^2$ and $\sum_{n=-\infty}^{\infty} |\tilde{B}_n|^2$ are finite.

The "transfer matrix" $M \equiv \begin{pmatrix} (1+\tilde{\beta})e^{-KL} & \tilde{\beta} \\ -\tilde{\beta} & (1-\tilde{\beta})e^{KL} \end{pmatrix}$ is non-singular, and $\det(M) = 1$, $\mathrm{Tr}(M) = 2[\cosh(KL) - \tilde{\beta}\sinh(KL)]$.

In most cases, M has two distinct eigenvalues with linearly independent right-eigenvectors, $M\vec{v}_i = \lambda_i \vec{v}_i$, $i = 1, 2$. Then $\begin{pmatrix} \tilde{A}_n \\ \tilde{B}_n \end{pmatrix} = c_1 \lambda_1^n \vec{v}_1 + c_2 \lambda_2^n \vec{v}_2$, where $c_{1,2}$ are complex numbers. No matter whether $|\lambda_{1,2}|$ are larger or smaller than unity, this cannot satisfy the condition that $\sum_{n=-\infty}^{\infty}(|\tilde{A}_n|^2 + |\tilde{B}_n|^2)$ is finite, because $\begin{pmatrix} \tilde{A}_n \\ \tilde{B}_n \end{pmatrix}$ will not tend to zero at either $n \to +\infty$ or $n \to -\infty$.

(not required) A special case: $\mathrm{Tr}(M) = \pm 2$. Then M seems to have two-fold degenerate eigenvalue $\lambda = +1$ for $\mathrm{Tr}(M) = +2$ [$\lambda = -1$ for $\mathrm{Tr}(M) = -2$]. However there is only one nonzero right-eigenvector $\vec{v}_1$, and M is related to the Jordan canonical form $\begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$ by a similarity transformation, namely $M \cdot \begin{pmatrix} \vec{v}_1, & \vec{v}_2 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} \cdot \begin{pmatrix} \vec{v}_1, & \vec{v}_2 \end{pmatrix}$. The eigenstate solution must be $\begin{pmatrix} \tilde{A}_n \\ \tilde{B}_n \end{pmatrix} = c\lambda^n \vec{v}_1$, which cannot satisfy $\sum_{n=-\infty}^{\infty}(|\tilde{A}_n|^2 + |\tilde{B}_n|^2) < \infty$.