Quantum Mechanics: Fall 2022
Final Exam (B): Brief Solutions

Possibly useful facts:

- 1D harmonic oscillator: $\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2$.
  $[\hat{x}, \hat{p}_x] = \mathbb{i}\hbar$, and in position representation $\hat{p}_x = -\mathbb{i}\hbar\frac{\partial}{\partial x}$. Define $\hat{a}_\mp = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} \pm \mathbb{i}\frac{1}{m\omega}\hat{p}_x) = \sqrt{\frac{m\omega}{2\hbar}}(x \pm \frac{\hbar}{m\omega}\frac{\partial}{\partial x})$. Then $[\hat{a}_-, \hat{a}_+] = 1$ and $\hat{H} = \hbar\omega\,(\hat{a}_+\hat{a}_- + \frac{1}{2})$. It has a unique ground state $|\psi_0\rangle$ with $\hat{a}_-|\psi_0\rangle = 0$, and excited states $|\psi_n\rangle \equiv \frac{1}{\sqrt{n!}}(\hat{a}_+)^n|\psi_0\rangle$ with energy $E_n = (n + \frac{1}{2})\hbar\omega$. The ground state wavefunction is $\psi_0(x) = (\frac{m\omega}{\pi\hbar})^{1/4}\exp(-\frac{m\omega}{2\hbar}x^2)$.

- $\int_{-\infty}^{\infty} x^{2n} e^{-x^2/(2a)}\mathrm{d}x = (2n-1)!! \cdot a^n \cdot \sqrt{2\pi a}$, for $a > 0$ and non-negative integer n.

- Generic angular momentum: $[\hat{J}_x, \hat{J}_y] = \mathbb{i}\hbar\hat{J}_z$, $[\hat{J}_y, \hat{J}_z] = \mathbb{i}\hbar\hat{J}_x$, $[\hat{J}_z, \hat{J}_x] = \mathbb{i}\hbar\hat{J}_y$.
  For eigenstate $|j,m\rangle$ of $\hat{\boldsymbol{J}}^2$ and $\hat{J}_z$, $\hat{\boldsymbol{J}}^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle$, $\hat{J}_z|j,m\rangle = m\hbar|j,m\rangle$, and $(\hat{J}_x \pm \mathbb{i}\hat{J}_y)|j,m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\hbar|j, m \pm 1\rangle$.
  Here $2j$ is non-negative integer, $m = -j, -j+1, \ldots, j$.
  - Spin-1/2: basis states $|\uparrow\rangle$ and $|\downarrow\rangle$, namely $|S_z = +\frac{1}{2}\hbar\rangle$ and $|S_z = -\frac{1}{2}\hbar\rangle$.
    Under this basis, $\hat{S}_a = \frac{\hbar}{2}\sigma_a$ where $\sigma_{x,y,z}$ are Pauli matrices.
    $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -\mathbb{i} \\ \mathbb{i} & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

- (Degenerate) Time-independent perturbation theory: $\hat{H} = \hat{H}^{(0)} + \hat{H}^{(1)}$. Denote the (degenerate) orthonormal eigenstates of $\hat{H}^{(0)}$ by $|\psi_{n\alpha}^{(0)}\rangle$, $\hat{H}^{(0)}|\psi_{n\alpha}^{(0)}\rangle = E_n^{(0)}|\psi_{n\alpha}\rangle$.
  Suppose $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$, E_n is close to $E_n^{(0)}$, then $(E_n - E_n^{(0)})$ is an eigenvalue of "secular equation" matrix, $\langle\psi_{n\beta}^{(0)}|\hat{H}^{(1)}|\psi_{n\alpha}^{(0)}\rangle + \sum_{m, m\neq n}\frac{1}{E_n^{(0)} - E_m^{(0)}}\langle\psi_{n\beta}^{(0)}|\hat{H}^{(1)}|\psi_m^{(0)}\rangle\langle\psi_m^{(0)}|\hat{H}^{(1)}|\psi_{n\alpha}^{(0)}\rangle$ up to second order. Here β & α are row/column index, the sum is over all eigenstates of $\hat{H}^{(0)}$ with energy different from $E_n^{(0)}$. In non-degenerate case, this is a 1×1 matrix.

- Dyson series: Solution to $\frac{\partial}{\partial t}c_n(t) = \sum_m V_{n,m}(t)c_m(t)$ is formally,
  $c_n(t) = c_n(t=0) + \sum_m \int_0^t V_{n,m}(t')\mathrm{d}t' \cdot c_m(t=0)$
  $\quad + \sum_m\sum_{m'}\int_0^t V_{n,m}(t')\left[\int_0^{t'} V_{m,m'}(t'')\mathrm{d}t''\right]\mathrm{d}t' \cdot c_{m'}(t=0) + \ldots$

- Series inversion: from series $x = z + a_1 z^2 + a_2 z^3 + a_3 z^4 + O(z^5)$ for $|z| \ll 1$,
  solve z, then $z = x + (-a_1)x^2 + (2a_1^2 - a_2)x^3 + (-5a_1^3 + 5a_1 a_2 - a_3)x^4 + O(x^5)$.

Problem 1. (35 points) Consider a non-relativistic particle of mass m moving on a ring of circumference L. This can be viewed as a 1D problem defined on x-axis with periodic boundary condition for the wavefunction, $\psi(x+L) = \psi(x)$, inner product $\langle\phi(x)|\psi(x)\rangle$ is defined as $\int_{x=-\frac{L}{2}}^{\frac{L}{2}} \phi^*(x)\psi(x)\,\mathrm{d}x = 1$.

(a) (5pts) For free particle, $\hat{H}_0 = \frac{\hat{p}^2}{2m}$, with this periodic boundary condition, *write down all the energy eigenvalues $E_n^{(0)}$ and normalized eigenstate wavefunctions $\psi_n^{(0)}(x)$.*

(b) (10pts) Add a perturbation $V(x) = V_0 \cos(\frac{2\pi x}{L})$ to $\hat{H}_0$, here V_0 is a "small" real parameters. *Compute the correction to the ground state energy in (a) to second order of V_0.*

(c) (10pts*) *Compute the correction to* all *excited states energies by $V(x)$ to second order of V_0.* [Note: degenerate perturbation theory can be avoided]

(d) (10pts*) Consider a variational state $\psi_A(x) = 1 + A\cos(\frac{2\pi x}{L})$ with real variational parameter A, *compute the expectation value of $\hat{H} = \hat{H}_0 + V(x)$ under $\psi_A(x)$, $E(A) = \frac{\langle\psi_A(x)|\hat{H}|\psi_A(x)\rangle}{\langle\psi_A(x)|\psi_A(x)\rangle}$; then minimize $E(A)$ with respect to A.*

Solution:

(a) this is the same as a midterm problem,

$E_n^{(0)} = \frac{\hbar^2}{2m}(\frac{2\pi n}{L})^2$, $\psi_n^{(0)}(x) = \frac{1}{\sqrt{L}} e^{\mathrm{i}\frac{2\pi n}{L}x}$, n is an integer.

For later convenience, we can use another set of eigenbasis,

$\psi_{0,\text{even}}^{(0)} = \frac{1}{\sqrt{L}}$, $\psi_{n,\text{even}}^{(0)}(x) = \sqrt{\frac{2}{L}}\cos(\frac{2\pi n}{L}x)$ and $\psi_{n,\text{odd}}^{(0)}(x) = \sqrt{\frac{2}{L}}\sin(\frac{2\pi n}{L}x)$ for $n = 1, 2, \ldots$, with $E_n^{(0)} = \frac{\hbar^2}{2m}(\frac{2\pi n}{L})^2$.

(b)

The ground state in (a) is $\psi_0^{(0)}(x) = \psi_{0,\text{even}}^{(0)}(x)$,

$V(x)|\psi_0^{(0)}\rangle = V_0\cos(\frac{2\pi x}{L}) \cdot \frac{1}{\sqrt{L}} = \frac{V_0}{\sqrt{2}}|\psi_{1,\text{even}}^{(0)}\rangle$.

The 1st order correction $\langle\psi_0^{(0)}|V(x)|\psi_0^{(0)}\rangle = 0$.

The correction to ground state energy to 2nd order is $0 + \frac{|\frac{V_0}{\sqrt{2}}|^2}{E_0^{(0)} - E_1^{(0)}} = -\frac{V_0^2 m L^2}{4\pi^2\hbar^2}$

(c) $\hat{H}_0 + V(x)$ is invariant under $x \to -x$ (inversion symmetry), by considering the subspace of even/odd functions separately, we can avoid degenerate perturbation theory.

(even, $n = 1$) case:
$V(x)|\psi^{(0)}_{1,\text{even}}\rangle = V_0 \cos(\frac{2\pi x}{L}) \cdot \sqrt{\frac{2}{L}} \cos(\frac{2\pi x}{L}) = V_0\sqrt{\frac{2}{L}} \cdot \frac{1}{2}[1 + \cos(\frac{4\pi x}{L})]$
$= \frac{V_0}{\sqrt{2}}|\psi^{(0)}_0\rangle + \frac{V_0}{2}|\psi^{(0)}_{2,\text{even}}\rangle$,
then $E_{1,\text{even}} \approx E^{(0)}_1 + 0 + \frac{|V_0/\sqrt{2}|^2}{E^{(0)}_1 - E^{(1)}_0} + \frac{|V_0/2|^2}{E^{(0)}_1 - E^{(1)}_2} = E^{(0)}_1 + \frac{V_0^2 mL^2}{4\pi^2\hbar^2} \cdot \frac{5}{6}$

(even, $n > 1$) case:
$V(x)|\psi^{(0)}_{n,\text{even}}\rangle = V_0 \cos(\frac{2\pi x}{L}) \cdot \sqrt{\frac{2}{L}} \cos(\frac{2\pi nx}{L}) = V_0\sqrt{\frac{2}{L}} \cdot \frac{1}{2}[\cos(\frac{2\pi(n-1)x}{L}) + \cos(\frac{2\pi(n+1)x}{L})]$
$= \frac{V_0}{2}|\psi^{(0)}_{n-1,\text{even}}\rangle + \frac{V_0}{2}|\psi^{(0)}_{n+1,\text{even}}\rangle$,
then $E_{n,\text{even}} \approx E^{(0)}_n + 0 + \frac{|V_0/2|^2}{E^{(0)}_n - E^{(1)}_{n-1}} + \frac{|V_0/2|^2}{E^{(0)}_n - E^{(1)}_{n+1}} = E^{(0)}_1 + \frac{V_0^2 mL^2}{4\pi^2\hbar^2} \cdot \frac{1}{4n^2-1}$

(odd, $n = 1$) case:
$V(x)|\psi^{(0)}_{1,\text{odd}}\rangle = V_0 \cos(\frac{2\pi x}{L}) \cdot \sqrt{\frac{2}{L}} \sin(\frac{2\pi x}{L}) = V_0\sqrt{\frac{2}{L}} \cdot \frac{1}{2} \sin(\frac{4\pi x}{L})$
$= \frac{V_0}{2}|\psi^{(0)}_{2,\text{odd}}\rangle$,
then $E_{1,\text{odd}} \approx E^{(0)}_1 + 0 + \frac{|V_0/2|^2}{E^{(0)}_1 - E^{(1)}_2} = E^{(0)}_1 - \frac{V_0^2 mL^2}{4\pi^2\hbar^2} \cdot \frac{1}{6}$

(odd, $n > 1$) case:
$V(x)|\psi^{(0)}_{n,\text{odd}}\rangle = V_0 \cos(\frac{2\pi x}{L}) \cdot \sqrt{\frac{2}{L}} \sin(\frac{2\pi nx}{L}) = V_0\sqrt{\frac{2}{L}} \cdot \frac{1}{2}[\sin(\frac{2\pi(n-1)x}{L}) + \sin(\frac{2\pi(n+1)x}{L})]$
$= \frac{V_0}{2}|\psi^{(0)}_{n-1,\text{odd}}\rangle + \frac{V_0}{2}|\psi^{(0)}_{n+1,\text{odd}}\rangle$,
then $E_{n,\text{odd}} \approx E^{(0)}_n + 0 + \frac{|V_0/2|^2}{E^{(0)}_n - E^{(1)}_{n-1}} + \frac{|V_0/2|^2}{E^{(0)}_n - E^{(1)}_{n+1}} = E^{(0)}_1 + \frac{V_0^2 mL^2}{4\pi^2\hbar^2} \cdot \frac{1}{4n^2-1}$

Note: as a consistency check, $\sum_{n=0}^{\infty} E_{n,\text{even}} = \sum_{n=0}^{\infty} E^{(0)}_n$, and $\sum_{n=1}^{\infty} E_{n,\text{odd}} = \sum_{n=1}^{\infty} E^{(0)}_n$, because $V(x)$ has no diagonal matrix elements, and should not change the trace of Hamiltonian in each subspace.

(d)
$\psi_A = \sqrt{L}(\psi^{(0)}_0 + \frac{A}{\sqrt{2}}\psi^{(0)}_{1,\text{even}})$,
therefore $\frac{\langle\psi_A|\hat{H}_0|\psi_A\rangle}{\langle\psi_A|\psi_A\rangle} = \frac{E^{(0)}_0 + E^{(0)}_1 \cdot |A/\sqrt{2}|^2}{1+|A/\sqrt{2}|^2} = \frac{2\pi^2\hbar^2}{mL^2} \cdot \frac{A^2}{2+A^2}$

Use the result in (b)(c), $V(x)|\psi_A\rangle = V_0\sqrt{L}(\frac{1}{\sqrt{2}}|\psi^{(0)}_{1,\text{even}}\rangle + \frac{A}{2}|\psi^{(0)}_{0,\text{even}}\rangle + \frac{A}{2\sqrt{2}}|\psi^{(0)}_{2,\text{even}}\rangle)$
$\frac{\langle\psi_A|V(x)|\psi_A\rangle}{\langle\psi_A|\psi_A\rangle} = V_0\frac{A}{1+|A/\sqrt{2}|^2} = \frac{2V_0A}{2+A^2}$

The variational energy is $E(A) = \frac{E_1^{(0)}A^2+2V_0A}{2+A^2} = E_1^{(0)} + 2\frac{V_0A-E_1^{(0)}}{2+A^2}$

From $0 = \frac{\mathrm{d}}{\mathrm{d}A}E(A) = 2\cdot\frac{(V_0)(2+A^2)-(V_0A-E_1^{(0)})(2A)}{(2+A^2)^2} = 2\cdot\frac{-V_0A^2+2E_1^{(1)}A+2V_0}{(2+A^2)^2}$,

note that when $V_0 \to 0$, we should have $A \to 0$,

$A = \frac{E_1^{(1)}}{V_0} - \sqrt{(\frac{E_1^{(1)}}{V_0})^2+2} \approx -\frac{V_0}{E_1^{(1)}} + O(V_0^3)$

to simplify the calculation, use $A^2 = \frac{2}{V_0^2}\left[(E_1^{(0)})^2 + V_0^2 - E_1^{(0)}\sqrt{(E_1^{(0)})^2+2V_0^2}\right]$,

Minimal variational energy is $\min E(A) = E_1^{(0)} - \frac{\sqrt{(E_1^{(0)})^2+2V_0^2}}{(1/V_0^2)[(E_1^{(0)})^2+2V_0^2-E_1^{(0)}\sqrt{(E_1^{(0)})^2+2V_0^2}]}$

$= \frac{1}{2}\left[E_1^{(0)} - \sqrt{(E_1^{(0)})^2+2V_0^2}\right] \approx -\frac{1}{2}\frac{V_0^2}{E_1^{(0)}}$, consistent with 2nd order perturbation.

Method #2 for minimization: let $A = \sqrt{2}\tan(k)$, then

$E(A) = E_1^{(0)}\sin^2(k) + \sqrt{2}V_0\sin(k)\cos(k) = \frac{1}{2}[E_1^{(0)} - E_1^{(0)}\cos(2k) + \sqrt{2}V_0\sin(2k)]$

$= \frac{1}{2}[E_1^{(0)} - \sqrt{(E_1^{(0)})^2+2V_0^2}\cos(2k+\phi)]$, where

$\cos(\phi) = \frac{E_1^{(0)}}{\sqrt{(E_1^{(0)})^2+2V_0^2}}$, $\sin(\phi) = \frac{\sqrt{2}V_0}{\sqrt{(E_1^{(0)})^2+2V_0^2}}$,

then $\min E(A) = \frac{1}{2}[E_1^{(0)} - \sqrt{(E_1^{(0)})^2+2V_0^2}]$, when $2k = \phi$.

Method #3 for minimization:

the result of $E(A)$ can be rewritten as $[E(A) - E_1^{(0)}]A^2 + 2V_0A + 2E(A) = 0$,

for A to have real solution, $(2V_0)^2 - 4\cdot[E(A) - E_1^{(0)}]\cdot[2E(A)] \geq 0$,

namely, $V_0^2 + 2E_1^{(0)}E(A) - 2[E(A)]^2 \geq 0$,

then $E(A) \geq \frac{1}{2}E_1^{(0)} - \sqrt{[\frac{1}{2}E_1^{(0)}]^2 + \frac{V_0^2}{2}}$,

Problem 2. (25 points) Consider 1D harmonic oscillator $\hat{H}_0 = \frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}x^2$ (see page 1). Add a time-dependent perturbation ("non-linear coupling to external oscillating force") $V(t) = -f_1\cdot\sin(\Omega t)\cdot x - f_2\cdot\sin(2\Omega t)\cdot x^2$, here Ω is a positive parameter, f_1, f_2 are "small" real parameters.

(a) (5pts) Assume the solution to $\mathrm{i}\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = [\hat{H}_0 + V(t)]|\psi(t)\rangle$ is $|\psi(t)\rangle = \sum_n c_n(t)e^{-\mathrm{i}E_nt/\hbar}|\psi_n\rangle$, here E_n, ψ_n are eigenvalues/states of $\hat{H}_0$ (see page 1). *Derive* *the differential equations for coefficients $c_n(t)$, the results should not contain unknown matrix elements.*

(b) (10pts) With initial condition, $c_0(t=0)=1$ and other $c_n(t=0)=0$, *solve $c_n(t)$ to first order of f_1, f_2.*

(c) (5pts*) Use the approximate solutions to $c_n(t)$ in (b), *compute the approximate expectation value of position x under $\psi(t)$, $\langle\psi(t)|\,x\,|\psi(t)\rangle$*

(d) (5pts) Suppose $\Omega \ll \omega$, by the adiabatic theorem, $\psi(t)$ will be approximately the instantaneous ground state of $\hat{H}_0 + V(t)$ (up to overall phase factor), *under this approximation, compute $\langle\psi(t)|\,x\,|\psi(t)\rangle$.* [Hint: $\hat{H}_0 + V(t)$ is still a harmonic oscillator]

Solution:

(a) $\mathbb{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}c_n(t) = \sum_m e^{\mathbb{i}\omega_{nm}t}\langle\psi_n|V(t)|\psi_m\rangle c_m(t),$ where $\omega_{nm} = (E_n - E_m)/\hbar = (n-m)\omega$,

use $x = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)$, $x^2 = \frac{\hbar}{2m\omega}(\hat{a}_+ + \hat{a}_-)^2 = \frac{\hbar}{2m\omega}(\hat{a}_+^2 + \hat{a}_-^2 + 2\hat{a}_+\hat{a}_- + 1)$,

The involved matrix element is $\langle\psi_n|V(t)|\psi_m\rangle$

$= -\sqrt{\frac{\hbar}{2m\omega}}f_1\sin(\Omega t)(\sqrt{n}\delta_{n,m+1} + \sqrt{n+1}\delta_{n,m-1})$

$-\frac{\hbar}{2m\omega}f_2\sin(2\Omega t)(\sqrt{n(n-1)}\delta_{n,m+2} + \sqrt{(n+1)(n+2)}\delta_{n,m-2} + (2n+1)\delta_{n,m})$

(b) With given initial condition, to first order of f_1, f_2, non-trivial $c_n(t)$ are

$c_0(t) \approx 1 + \int_0^t \frac{1}{\mathbb{i}\hbar}[-\frac{\hbar}{2m\omega}f_2\sin(2\Omega t)]\cdot 1\,\mathrm{d}t = 1 + \mathbb{i}\frac{\hbar}{2m\omega}f_2\cdot\frac{1}{2\hbar\Omega}[1-\cos(2\Omega t)]$

$= 1 + \mathbb{i}\frac{\hbar}{2m\omega}f_2\cdot\frac{1}{\hbar\Omega}\sin^2(\Omega t)$

$c_1(t) \approx 0 + \int_0^t \frac{1}{\mathbb{i}\hbar}e^{\mathbb{i}\omega t}[-\sqrt{\frac{\hbar}{2m\omega}}f_1\sin(\Omega t)]\cdot 1\,\mathrm{d}t = \sqrt{\frac{\hbar}{2m\omega}}\frac{f_1}{2\hbar}\int_0^t[e^{\mathbb{i}(\omega+\Omega)t} - e^{\mathbb{i}(\omega-\Omega)t}]\,\mathrm{d}t$

$= \sqrt{\frac{\hbar}{2m\omega}}\frac{f_1}{2\hbar}[\frac{e^{\mathbb{i}(\omega+\Omega)t}-1}{\mathbb{i}(\omega+\Omega)} - \frac{e^{\mathbb{i}(\omega-\Omega)t}-1}{\mathbb{i}(\omega-\Omega)}] = \sqrt{\frac{\hbar}{2m\omega}}\frac{f_1}{\hbar}[e^{\mathbb{i}(\omega+\Omega)t/2}(\frac{\sin((\omega+\Omega)t/2)}{(\omega+\Omega)}) - e^{\mathbb{i}(\omega-\Omega)t/2}(\frac{\sin((\omega-\Omega)t/2)}{(\omega-\Omega)})]$

$c_2(t) \approx 0 + \int_0^t \frac{1}{\mathbb{i}\hbar}e^{\mathbb{i}2\omega t}[-\frac{\hbar}{2m\omega}f_2\sin(2\Omega t)\cdot\sqrt{2}]\cdot 1\,\mathrm{d}t = \frac{\sqrt{2}\hbar}{2m\omega}\frac{f_2}{2\hbar}\int_0^t[e^{\mathbb{i}2(\omega+\Omega)t} - e^{\mathbb{i}2(\omega-\Omega)t}]\,\mathrm{d}t$

$= \frac{\sqrt{2}\hbar}{2m\omega}\frac{f_2}{2\hbar}[\frac{e^{\mathbb{i}2(\omega+\Omega)t}-1}{\mathbb{i}2(\omega+\Omega)} - \frac{e^{\mathbb{i}2(\omega-\Omega)t}-1}{\mathbb{i}2(\omega-\Omega)}] = \frac{\sqrt{2}\hbar}{2m\omega}\frac{f_2}{\hbar}[e^{\mathbb{i}(\omega+\Omega)t}(\frac{\sin((\omega+\Omega)t)}{2(\omega+\Omega)}) - e^{\mathbb{i}(\omega-\Omega)t}(\frac{\sin((\omega-\Omega)t)}{2(\omega-\Omega)})]$

Other $c_n(t)$ (for $n>2$) vanish at first order of f_1, f_2.

(c) Generically $\langle\psi(t)|x|\psi(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\sum_n[c^*_{n+1}(t)c_n(t)e^{\mathbb{i}\omega t} + c^*_n(t)c_{n+1}(t)e^{-\mathbb{i}\omega t}]$

The result for plugging in results of (b) is omitted here.

To lowest order of f_1, f_2, $\langle\psi(t)|x|\psi(t)\rangle \approx \sqrt{\frac{\hbar}{2m\omega}}[c^*_1(t)c_0(t)e^{\mathbb{i}\omega t} + c^*_0(t)c_1(t)e^{-\mathbb{i}\omega t}]$
$\approx \frac{f_1}{2m\omega}\cdot 2\mathrm{Re}[e^{\mathbb{i}(\omega-\Omega)t/2}(\frac{\sin((\omega+\Omega)t/2)}{(\omega+\Omega)}) - e^{\mathbb{i}(\omega+\Omega)t/2}(\frac{\sin((\omega-\Omega)t/2)}{(\omega-\Omega)})]$
$= \frac{f_1}{m\omega}\cdot[-\sin(\omega t)\frac{\Omega}{\omega^2-\Omega^2} + \sin(\Omega t)\frac{\omega}{\omega^2-\Omega^2}]$

(d) At time t, $\hat{H}(t) = \frac{\hat{p}^2}{2m} + [\frac{m\omega^2}{2} - f_2\sin(2\Omega t)]x^2 - f_1\sin(\Omega t)x$
$= \frac{\hat{p}^2}{2m} + \frac{m(\omega')^2}{2}x^2 - f_1\sin(\Omega t)x = \frac{\hat{p}^2}{2m} + \frac{m(\omega')^2}{2}(x-a)^2 - \frac{m(\omega')^2}{2}a^2$
here $\omega' = \sqrt{\omega^2 - 2\frac{f_2}{m}\sin(2\Omega t)}$, $a = \frac{f_1\sin(\Omega t)}{m(\omega')^2}$,
therefore under the instantaneous ground state of $\hat{H}(t)$, $\langle x\rangle = a = \frac{f_1\sin(\Omega t)}{m\omega^2 - 2f_2\sin(2\Omega t)}$
this is consistent with the result of (c), if we keep only the first order term of f_1, f_2 and assuming $\Omega \ll \omega$.

Problem 3. (25 points) Consider two identical particles moving on the ring of length L defined in Problem 1, with Hamiltonian $\hat{H}_0 = \frac{1}{2m}(\hat{p}_1^2 + \hat{p}_2^2)$. Subscripts $_1$ and $_2$ label the two particles. $\hat{p}_1 = -\mathbb{i}\hbar\frac{\partial}{\partial x_1}$, $\hat{p}_2 = -\mathbb{i}\hbar\frac{\partial}{\partial x_2}$.

(a) (5pts) For two identical spinless BOSONS under $\hat{H}_0$, *write down the normalized ground state(s) and first excited state(s), and second excited state(s) wavefunctions, and corresponding energies.* You can use the single particle basis in Problem 1(a) to represent these states.

(b) (5pts) For two identical spinless FERMIONS under $\hat{H}_0$, *write down the normalized ground state(s) and first excited state(s), and second excited state(s) wavefunctions, and corresponding energies.*

(c) (10pts) For spin-1/2 BOSONS and FERMIONS under $\hat{H}_0$, *write down the normalized ground state(s) and first excited state(s), and corresponding energies, for the boson/fermion cases respectively.*

(d) (5pts**) Add a perturbation $\hat{H}_1 = \lambda \cdot [1 - \cos(2\pi \cdot \frac{x_1 - x_2}{L})] \cdot (\hat{S}_{1z} \cdot \hat{S}_{2z})$, here λ is a "small" real parameter. *Compute the corrections to all the FERMION case energies in (c) to lowest nontrivial order of* λ

Solution:

In this problem we omit the superscript $^{(0)}$ for the single-particle basis in Problem 1, and will use the even/odd basis.

(a) for two spinless bosons,

ground state $\psi^{(B,\text{spinless})}_{0,0}(x_1, x_2) = \psi_0(x_1)\psi_0(x_2)$, with energy $E_{0,0} = E_0 + E_0 = 0$.

first excited states energy is $E_{0,1} = E_0 + E_1 = \frac{2\pi^2\hbar^2}{mL^2}$, states are

$\psi^{(B,\text{spinless})}_{0,(1,\text{even})}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_{1,\text{even}}(x_2) + \psi_{1,\text{even}}(x_1)\psi_0(x_2)]$,

$\psi^{(B,\text{spinless})}_{0,(1,\text{odd})}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_{1,\text{odd}}(x_2) + \psi_{1,\text{odd}}(x_1)\psi_0(x_2)]$,

NOTE: if using planewave basis for single particle, the two boson basis are

$\psi^{(B,\text{spinless})}_{0,\pm 1}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_{\pm 1}(x_2) + \psi_{\pm 1}(x_1)\psi_0(x_2)]$

second excited states energy is $E_{1,1} = E_1 + E_1 = \frac{4\pi^2\hbar^2}{mL^2}$, states are

$\psi^{(B,\text{spinless})}_{(1,\text{even}),(1,\text{even})}(x_1, x_2) = \psi_{(1,\text{even})}(x_1)\psi_{(1,\text{even})}(x_2)$,

$\psi^{(B,\text{spinless})}_{(1,\text{odd}),(1,\text{odd})}(x_1, x_2) = \psi_{(1,\text{odd})}(x_1)\psi_{(1,\text{odd})}(x_2)$,

$\psi^{(B,\text{spinless})}_{(1,\text{even}),(1,\text{odd})}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_{(1,\text{even})}(x_1)\psi_{1,\text{odd}}(x_2) + \psi_{1,\text{odd}}(x_1)\psi_{(1,\text{even})}(x_2)]$

NOTE: if using planewave basis for single particle, the two boson basis are

$\psi^{(B,\text{spinless})}_{1,1}(x_1, x_2) = \psi_1(x_1)\psi_1(x_2)$,

$\psi^{(B,\text{spinless})}_{-1,-1}(x_1, x_2) = \psi_{-1}(x_1)\psi_{-1}(x_2)$,

$\psi^{(B,\text{spinless})}_{1,-1}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_1(x_1)\psi_{-1}(x_2) + \psi_{-1}(x_1)\psi_1(x_2)]$

(b) for two spinless fermions,

ground states energy is $E_{0,1} = E_0 + E_1 = \frac{2\pi^2\hbar^2}{mL^2}$, states are

$\psi^{(F,\text{spinless})}_{0,(1,\text{even})}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_{1,\text{even}}(x_2) - \psi_{1,\text{even}}(x_1)\psi_0(x_2)]$,

$\psi^{(F,\text{spinless})}_{0,(1,\text{odd})}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_{1,\text{odd}}(x_2) - \psi_{1,\text{odd}}(x_1)\psi_0(x_2)]$,

NOTE: if using planewave basis for single particle, the two fermion basis are

$\psi^{(F,\text{spinless})}_{0,\pm 1}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_{\pm 1}(x_2) - \psi_{\pm 1}(x_1)\psi_0(x_2)]$

first excited state energy is $E_{1,1} = E_1 + E_1 = \frac{4\pi^2\hbar^2}{mL^2}$, state is
$\psi^{(F,\text{spinless})}_{(1,\text{even}),(1,\text{odd})}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_{(1,\text{even})}(x_1)\psi_{1,\text{odd}}(x_2) - \psi_{1,\text{odd}}(x_1)\psi_{(1,\text{even})}(x_2)]$
NOTE: if using planewave basis for single particle, the two boson basis is
$\psi^{(F,\text{spinless})}_{1,-1}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_1(x_1)\psi_{-1}(x_2) - \psi_{-1}(x_1)\psi_1(x_2)]$

second excited states energy is $E_{0,2} = E_0 + E_2 = \frac{8\pi^2\hbar^2}{mL^2}$, states are
$\psi^{(F,\text{spinless})}_{0,(2,\text{even})}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_{2,\text{even}}(x_2) - \psi_{2,\text{even}}(x_1)\psi_0(x_2)]$,
$\psi^{(F,\text{spinless})}_{0,(2,\text{odd})}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_{2,\text{odd}}(x_2) - \psi_{2,\text{odd}}(x_1)\psi_0(x_2)]$,
NOTE: if using planewave basis for single particle, the two boson basis are
$\psi^{(F,\text{spinless})}_{0,\pm 2}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_{\pm 2}(x_2) - \psi_{\pm 2}(x_1)\psi_0(x_2)]$

(c) Factorize the eigenbasis into orbital and spin wavefunctions, the spin part of wavefunctions can be spin single $|S = 0, S_z = 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$, or spin triplet states $|S = 1, S_z = 1\rangle = |\uparrow\rangle|\uparrow\rangle$, $|S = 1, S_z = -1\rangle = |\downarrow\rangle|\downarrow\rangle$, and $|S = 1, S_z = 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle)$,

For two spin-1/2 bosons,

- ground states are $\psi^{(B,\text{spinless})}_{0,0}(x_1, x_2)|S = 1, S_z\rangle$, for $S_z = -1, 0, 1$, with energy $E_{0,0} = 0$

- first excited states are
  $\psi^{(B,\text{spinless})}_{0,(1,\text{even})}(x_1, x_2)|S = 1, S_z\rangle$ and $\psi^{(B,\text{spinless})}_{0,(1,\text{odd})}(x_1, x_2)|S = 1, S_z\rangle$ for $S_z = -1, 0, 1$, and
  $\psi^{(F,\text{spinless})}_{0,(1,\text{even})}(x_1, x_2)|S = 0, S_z = 0\rangle$, and $\psi^{(F,\text{spinless})}_{0,(1,\text{odd})}(x_1, x_2)|S = 0, S_z = 0\rangle$,
  with energy $E_{0,1}$

For two spin-1/2 fermions,

- ground states are $\psi^{(B,\text{spinless})}_{0,0}(x_1, x_2)|S = 0, S_z = 0\rangle$, with energy $E_{0,0} = 0$

- first excited states are
  $\psi^{(F,\text{spinless})}_{0,(1,\text{even})}(x_1, x_2)|S = 1, S_z\rangle$ and $\psi^{(F,\text{spinless})}_{0,(1,\text{odd})}(x_1, x_2)|S = 1, S_z\rangle$ for $S_z = -1, 0, 1$, and
  $\psi^{(B,\text{spinless})}_{0,(1,\text{even})}(x_1, x_2)|S = 0, S_z = 0\rangle$, and $\psi^{(B,\text{spinless})}_{0,(1,\text{odd})}(x_1, x_2)|S = 0, S_z = 0\rangle$,
  with energy $E_{0,1}$

(d) The perturbation preserves $\hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z}$, note that $\hat{S}_{1z} \cdot \hat{S}_{2z} = \frac{1}{2}(\hat{S}_{1z} + \hat{S}_{2z})^2 - \frac{1}{2}(\hat{S}_{1z}^2 + \hat{S}_{2z}^2) = \frac{1}{2}(\hat{S}_{1z} + \hat{S}_{2z})^2 - \frac{\hbar^2}{4}$

The perturbation is invariant under inversion, $x_{1,2} \to -x_{1,2}$, so we can divide the Hilbert space into even/odd functions under this inversion, note that $\psi^{(B/F,\text{spinless})}_{(n,\text{even}),(m,\text{even})}$ is even function, $\psi^{(B/F,\text{spinless})}_{(n,\text{even}),(m,\text{odd})}$ is odd function, $\psi^{(B/F,\text{spinless})}_{(n,\text{odd}),(m,\text{odd})}$ is even function.

By considering the exchange of only the coordinates, $x_1 \leftrightarrow x_2$, we have $\langle\psi^{(B,\text{spinless})}_{?,?}(x_1,x_2)|[1-\cos(2\pi\frac{x_1-x_2}{L})]|\psi^{(F,\text{spinless})}_{?,?}(x_1,x_2)\rangle = 0$, therefore if we can always factorize the wavefunction into orbital and spin parts, we can divide the Hilbert space into subspaces which is even/odd under $x_1 \leftrightarrow x_2$ (like spinless boson or spinless fermion).

Finally we can divide the two spin-1/2 fermion Hilbert space by S_z quantum number$(1,0,-1)$, inversion symmetry(even or odd), exchange symmetry of orbital part of wavefunction(spinless-boson-like, or spinless-fermion-like).

Then for the ground and first excited two spin-1/2 fermion states in (c), we can completely avoid degenerate perturbation theory, and all the first order energy correction $\langle\psi|\hat{H}_1|\psi\rangle$ is non-zero (see below).

Use the following results, $\langle S, S_z|\hat{S}_{1z}\cdot\hat{S}_{2z}|S,S_z\rangle = \frac{\hbar^2}{2}(S_z^2 - \frac{1}{2})$,

$$\langle\psi^{(B,\text{spinless})}_{0,0}(x_1,x_2)|[1-\cos(2\pi\frac{x_1-x_2}{L})]|\psi^{(B,\text{spinless})}_{0,0}(x_1,x_2)\rangle$$
$$= \int_0^L \mathrm{d}x_1 \int_0^L \mathrm{d}x_2\, (\tfrac{1}{L})^*[1-\cos(2\pi\tfrac{x_1-x_2}{L})](\tfrac{1}{L}) = 1,$$

$$\langle\psi^{(F,\text{spinless})}_{0,(1,\text{even})}(x_1,x_2)|[1-\cos(2\pi\frac{x_1-x_2}{L})]|\psi^{(F,\text{spinless})}_{0,(1,\text{even})}(x_1,x_2)\rangle$$
$$= \int_0^L \mathrm{d}x_1 \int_0^L \mathrm{d}x_2\, [\tfrac{1}{L}(\cos\tfrac{2\pi x_1}{L} - \cos\tfrac{2\pi x_2}{L})]^*[1-\cos(2\pi\tfrac{x_1-x_2}{L})][\tfrac{1}{L}(\cos\tfrac{2\pi x_1}{L} - \cos\tfrac{2\pi x_2}{L})] = \tfrac{3}{2},$$

$$\langle\psi^{(F,\text{spinless})}_{0,(1,\text{odd})}(x_1,x_2)|[1-\cos(2\pi\frac{x_1-x_2}{L})]|\psi^{(F,\text{spinless})}_{0,(1,\text{odd})}(x_1,x_2)\rangle$$
$$= \int_0^L \mathrm{d}x_1 \int_0^L \mathrm{d}x_2\, [\tfrac{1}{L}(\sin\tfrac{2\pi x_1}{L} - \sin\tfrac{2\pi x_2}{L})]^*[1-\cos(2\pi\tfrac{x_1-x_2}{L})][\tfrac{1}{L}(\sin\tfrac{2\pi x_1}{L} - \sin\tfrac{2\pi x_2}{L})] = \tfrac{3}{2},$$

$$\langle\psi^{(B,\text{spinless})}_{0,(1,\text{even})}(x_1,x_2)|[1-\cos(2\pi\frac{x_1-x_2}{L})]|\psi^{(B,\text{spinless})}_{0,(1,\text{even})}(x_1,x_2)\rangle$$
$$= \int_0^L \mathrm{d}x_1 \int_0^L \mathrm{d}x_2\, [\tfrac{1}{L}(\cos\tfrac{2\pi x_1}{L} + \cos\tfrac{2\pi x_2}{L})]^*[1-\cos(2\pi\tfrac{x_1-x_2}{L})][\tfrac{1}{L}(\cos\tfrac{2\pi x_1}{L} + \cos\tfrac{2\pi x_2}{L})] = \tfrac{1}{2},$$

$$\langle\psi^{(B,\text{spinless})}_{0,(1,\text{odd})}(x_1,x_2)|[1-\cos(2\pi\frac{x_1-x_2}{L})]|\psi^{(B,\text{spinless})}_{0,(1,\text{odd})}(x_1,x_2)\rangle$$
$$= \int_0^L \mathrm{d}x_1 \int_0^L \mathrm{d}x_2\, [\tfrac{1}{L}(\sin\tfrac{2\pi x_1}{L} + \sin\tfrac{2\pi x_2}{L})]^*[1-\cos(2\pi\tfrac{x_1-x_2}{L})][\tfrac{1}{L}(\sin\tfrac{2\pi x_1}{L} + \sin\tfrac{2\pi x_2}{L})] = \tfrac{1}{2},$$

The states and their symmetry and first order correction to their energies are listed in the table below

S_z	$x_{1,2} \to -x_{1,2}$	$x_1 \leftrightarrow x_2$	state	1st order correction to energy
0	even	even	$\psi^{(B,\text{spinless})}_{0,0}(x_1,x_2)\|S=0,S_z=0\rangle$	$\lambda\cdot(-\frac{\hbar^2}{4})\cdot 1$
1	even	odd	$\psi^{(F,\text{spinless})}_{0,(1,\text{even})}(x_1,x_2)\|S=1,S_z=1\rangle$	$\lambda\cdot(\frac{\hbar^2}{4})\cdot\frac{3}{2}$
0	even	odd	$\psi^{(F,\text{spinless})}_{0,(1,\text{even})}(x_1,x_2)\|S=1,S_z=0\rangle$	$\lambda\cdot(-\frac{\hbar^2}{4})\cdot\frac{3}{2}$
-1	even	odd	$\psi^{(F,\text{spinless})}_{0,(1,\text{even})}(x_1,x_2)\|S=1,S_z=-1\rangle$	$\lambda\cdot(\frac{\hbar^2}{4})\cdot\frac{3}{2}$
1	odd	odd	$\psi^{(F,\text{spinless})}_{0,(1,\text{odd})}(x_1,x_2)\|S=1,S_z=1\rangle$	$\lambda\cdot(\frac{\hbar^2}{4})\cdot\frac{3}{2}$
0	odd	odd	$\psi^{(F,\text{spinless})}_{0,(1,\text{odd})}(x_1,x_2)\|S=1,S_z=0\rangle$	$\lambda\cdot(-\frac{\hbar^2}{4})\cdot\frac{3}{2}$
-1	odd	odd	$\psi^{(F,\text{spinless})}_{0,(1,\text{odd})}(x_1,x_2)\|S=1,S_z=-1\rangle$	$\lambda\cdot(\frac{\hbar^2}{4})\cdot\frac{3}{2}$
0	even	even	$\psi^{(B,\text{spinless})}_{0,(1,\text{even})}(x_1,x_2)\|S=0,S_z=0\rangle$	$\lambda\cdot(-\frac{\hbar^2}{4})\cdot\frac{1}{2}$
0	odd	even	$\psi^{(B,\text{spinless})}_{0,(1,\text{odd})}(x_1,x_2)\|S=0,S_z=0\rangle$	$\lambda\cdot(-\frac{\hbar^2}{4})\cdot\frac{1}{2}$

Problem 4. (15 points*) Consider a two-dimensional harmonic oscillator, $\hat{H}_0 = (\frac{1}{2m}\hat{p}_x^2 + \frac{m\omega^2}{2}x^2) + (\frac{1}{2m}\hat{p}_y^2 + \frac{m\omega^2}{2}y^2)$. Here $\hat{p}_x = -\mathbb{i}\hbar\partial_x$, $\hat{p}_y = -\mathbb{i}\hbar\partial_y$.

(a) (5pts) *Write down the ground state, first excited states, and second excited states of* $\hat{H}_0$. You can use the 1D harmonic oscillator eigenstates to represent them, and do not need to write down the explicit formula. [Hint: it would be convenient to define ladder operators for the x- and y- parts of $\hat{H}_0$].

(b) (10pts*) Add a time-dependent perturbation $V(t) = -f\cdot[x\cos(\Omega t) + y\sin(\Omega t)]$, here f,Ω are real parameters. *By time-dependent perturbation theory, compute the transition probability from the ground state in (a) to the second excited states, over time t, to lowest nontrivial order of f.*

Solution:

(a) Define $\hat{a}_{x,\pm} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} \mp \frac{\mathbb{i}}{m\omega}\hat{p}_x)$, $\hat{a}_{y,\pm} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{y} \mp \frac{\mathbb{i}}{m\omega}\hat{p}_y)$, then $[\hat{a}_{i,-},\hat{a}_{j,+}] = \delta_{i,j}$, $[\hat{a}_{i,-},\hat{a}_{j,-}] = 0$,

$\hat{H}_0 = \hbar\omega(\hat{a}_{x,+}\hat{a}_{x,-} + \hat{a}_{y,+}\hat{a}_{y,-} + 1)$ is the sum of two independent 1D harmonic oscillators.

The ground state of $\hat{H}_0$ satisfies $\hat{a}_{x,-}\psi_0(x,y) = \hat{a}_{y,-}\psi_0(x,y) = 0$, and is $\psi_0(x,y) = \psi_0(x)\cdot\psi_0(y)$, with $E_{0,0} = E_0 + E_0 = \hbar\omega$.

the first excited states of $\hat{H}_0$ are
$\psi_{1,0}(x,y) = \psi_1(x)\cdot\psi_0(y) = \hat{a}_{x,+}\psi_0(x)\cdot\psi_0(y)$, and
$\psi_{0,1}(x,y) = \psi_0(x)\cdot\psi_1(y) = \psi_0(x)\cdot\hat{a}_{y,+}\psi_0(y)$, with $E_{1,0} = E_{0,1} = E_0 + E_1 = 2\hbar\omega$.

the second excited states of $\hat{H}_0$ are
$\psi_{2,0}(x,y) = \psi_2(x)\cdot\psi_0(y) = \frac{1}{\sqrt{2}}\hat{a}_{x,+}^2\psi_0(x)\cdot\psi_0(y)$, and
$\psi_{0,2}(x,y) = \psi_0(x)\cdot\psi_2(y) = \psi_0(x)\cdot\frac{1}{\sqrt{2}}\hat{a}_{y,+}^2\psi_0(y)$, and
$\psi_{1,1}(x,y) = \psi_1(x)\cdot\psi_1(y) = \hat{a}_{x,+}\psi_0(x)\cdot\hat{a}_{y,+}\psi_0(y)$, with
$E_{2,0} = E_{0,2} = E_{1,1} = 3\hbar\omega$.

Generically the eigenstates of $\hat{H}_0$ are
$\psi_{n_x,n_y}(x,y) = \psi_{n_x}(x)\cdot\psi_{n_y}(y) = \frac{1}{\sqrt{n_x!n_y!}}(\hat{a}_{x,+})^{n_x}(\hat{a}_{y,+})^{n_y}\psi_{0,0}(x,y)$, with
$E_{n_x,n_y} = E_{n_x} + E_{n_y} = \hbar\omega(n_x + n_y + 1)$, where $n_x, n_y = 0, 1, \ldots$.

(b)
$\hat{V}(t) = -f\sqrt{\frac{\hbar}{2m\omega}}[(\hat{a}_{x,+} + \hat{a}_{x,-})\cos(\Omega t) + (\hat{a}_{y,+} + \hat{a}_{y,-})\sin(\Omega t)]$

We can assume $|\psi(t)\rangle = \sum_{n_x,n_y} c_{n_x,n_y}(t)e^{-\mathrm{i}E_{n_x,n_y}t/\hbar}|\psi_{n_x,n_y}\rangle$, then
$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}c_{n_x,n_y}(t) = \sum_{n'_x,n'_y} e^{\mathrm{i}(n_x+n_y-n'_x-n'_y)\omega t}\langle\psi_{n_x,n_y}|\hat{V}(t)|\psi_{n'_x,n'_y}\rangle\cdot c_{n'_x,n'_y}(t)$,

However to compute the transition amplitude from ground state to second excited states, we will need 2nd order term of the Dyson series, for the processes $(0,0)\to(1,0)\to(2,0)$, $(0,0)\to\left\{\begin{array}{l}(1,0)\\(0,1)\end{array}\right.\to(1,1)$, $(0,0)\to(0,1)\to(0,2)$,

the following identities will be useful,
$\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\, f(t_1)f(t_2) = \frac{1}{2}[\int_0^t \mathrm{d}t_1\, f(t_1)]^2$,
$\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\, f(t_1)g(t_2) + \int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\, g(t_1)f(t_2) = [\int_0^t \mathrm{d}t_1\, f(t_1)]\cdot[\int_0^t \mathrm{d}t_2\, g(t_2)]$,

For $(0,0)\to(1,0)\to(2,0)$, the transition amplitude to lowest order is
$(\frac{1}{\mathrm{i}\hbar})^2\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\, e^{\mathrm{i}\omega t_1}(-f\sqrt{\frac{\hbar}{2m\omega}})\cos(\Omega t_1)\sqrt{2}\cdot e^{\mathrm{i}\omega t_2}(-f\sqrt{\frac{\hbar}{2m\omega}})\cos(\Omega t_2)$
$= -\frac{1}{\sqrt{2}}\cdot\frac{f^2}{2m\omega\hbar}[\int_0^t \mathrm{d}t_1\, e^{\mathrm{i}\omega t_1}\cos(\Omega t_1)]^2 = -\frac{1}{\sqrt{2}}\cdot\frac{f^2}{2m\omega\hbar}\cdot\{\frac{1}{2}[\frac{e^{\mathrm{i}(\omega+\Omega)t}-1}{\mathrm{i}(\omega+\Omega)} + \frac{e^{\mathrm{i}(\omega-\Omega)t}-1}{\mathrm{i}(\omega-\Omega)}]\}^2$
So $P_{(0,0)\to(2,0)}(t) \approx \frac{1}{128}\frac{f^4}{m^2\hbar^2\omega^2}|\frac{e^{\mathrm{i}(\omega+\Omega)t}-1}{(\omega+\Omega)} + \frac{e^{\mathrm{i}(\omega-\Omega)t}-1}{(\omega-\Omega)}|^4$

For $(0,0) \to (0,1) \to (0,2)$, the transition amplitude to lowest order is

$(\frac{1}{\mathrm{i}\hbar})^2 \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\, e^{\mathrm{i}\omega t_1}(-f\sqrt{\frac{\hbar}{2m\omega}}) \sin(\Omega t_1)\sqrt{2} \cdot e^{\mathrm{i}\omega t_2}(-f\sqrt{\frac{\hbar}{2m\omega}}) \sin(\Omega t_2)$

$= -\frac{1}{\sqrt{2}} \cdot \frac{f^2}{2m\omega\hbar}[\int_0^t \mathrm{d}t_1\, e^{\mathrm{i}\omega t_1} \sin(\Omega t_1)]^2 = -\frac{1}{\sqrt{2}} \cdot \frac{f^2}{2m\omega\hbar} \cdot \{\frac{1}{2\mathrm{i}}[\frac{e^{\mathrm{i}(\omega+\Omega)t}-1}{\mathrm{i}(\omega+\Omega)} - \frac{e^{\mathrm{i}(\omega-\Omega)t}-1}{\mathrm{i}(\omega-\Omega)}]\}^2$

So $P_{(0,0)\to(0,2)}(t) \approx \frac{1}{128}\frac{f^4}{m^2\hbar^2\omega^2}|\frac{e^{\mathrm{i}(\omega+\Omega)t}-1}{(\omega+\Omega)} - \frac{e^{\mathrm{i}(\omega-\Omega)t}-1}{(\omega-\Omega)}|^4$

For $(0,0) \to \begin{cases} (1,0) \\ (0,1) \end{cases} \to (1,1)$ processes, to total transition amplitude to lowest order is

$(\frac{1}{\mathrm{i}\hbar})^2 \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\, e^{\mathrm{i}\omega t_1}(-f\sqrt{\frac{\hbar}{2m\omega}}) \sin(\Omega t_1) \cdot e^{\mathrm{i}\omega t_2}(-f\sqrt{\frac{\hbar}{2m\omega}}) \cos(\Omega t_2)$

$+(\frac{1}{\mathrm{i}\hbar})^2 \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\, e^{\mathrm{i}\omega t_1}(-f\sqrt{\frac{\hbar}{2m\omega}}) \cos(\Omega t_1) \cdot e^{\mathrm{i}\omega t_2}(-f\sqrt{\frac{\hbar}{2m\omega}}) \sin(\Omega t_2)$

$= -\frac{f^2}{2m\omega\hbar}[\int_0^t \mathrm{d}t_1\, e^{\mathrm{i}\omega t_1} \sin(\Omega t_1)][\int_0^t \mathrm{d}t_2\, e^{\mathrm{i}\omega t_2} \cos(\Omega t_2)]$

$= -\frac{f^2}{2m\omega\hbar} \cdot \frac{1}{4\mathrm{i}}[\frac{e^{\mathrm{i}(\omega+\Omega)t}-1}{\mathrm{i}(\omega+\Omega)} - \frac{e^{\mathrm{i}(\omega-\Omega)t}-1}{\mathrm{i}(\omega-\Omega)}][\frac{e^{\mathrm{i}(\omega+\Omega)t}-1}{\mathrm{i}(\omega+\Omega)} + \frac{e^{\mathrm{i}(\omega-\Omega)t}-1}{\mathrm{i}(\omega-\Omega)}]$

So $P_{(0,0)\to(1,1)}(t) \approx \frac{1}{64}\frac{f^4}{m^2\hbar^2\omega^2}|\frac{e^{\mathrm{i}(\omega+\Omega)t}-1}{(\omega+\Omega)} - \frac{e^{\mathrm{i}(\omega-\Omega)t}-1}{(\omega-\Omega)}|^2|\frac{e^{\mathrm{i}(\omega+\Omega)t}-1}{(\omega+\Omega)} + \frac{e^{\mathrm{i}(\omega-\Omega)t}-1}{(\omega-\Omega)}|^2$

The overall transition probability to 2nd excited states are

$P_{(0,0)\to(2,0)}(t) + P_{(0,0)\to(0,2)}(t) + P_{(0,0)\to(1,1)}(t) \approx \frac{1}{32}\frac{f^4}{m^2\hbar^2\omega^2}[|\frac{e^{\mathrm{i}(\omega+\Omega)t}-1}{(\omega+\Omega)}|^2 + |\frac{e^{\mathrm{i}(\omega-\Omega)t}-1}{(\omega-\Omega)}|^2]^2$

We can also view this as two independent 1D systems(x and y), and compute the transition probabilities separately,

$P_{(0,0)\to(2,0)}(t) = P_{x,0\to2}(t) \cdot P_{y,0\to0}(t)$,

$P_{(0,0)\to(0,2)}(t) = P_{x,0\to0}(t) \cdot P_{y,0\to2}(t)$,

$P_{(0,0)\to(1,1)}(t) = P_{x,0\to1}(t) \cdot P_{y,0\to1}(t)$.