Quantum Mechanics: Fall 2020
Final Exam: Brief Solutions

NOTE: Sentences in italic fonts are questions to be answered.

Possibly useful facts:

- 1D harmonic oscillator: $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2$.
  $[\hat{x}, \hat{p}] = \mathrm{i}\hbar$, and in position representation $\hat{p} = -\mathrm{i}\hbar\frac{\partial}{\partial x}$. Define $\hat{a}_\mp = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} \pm \mathrm{i}\frac{1}{m\omega}\hat{p}) = \sqrt{\frac{m\omega}{2\hbar}}(x \pm \frac{\hbar}{m\omega}\frac{\partial}{\partial x})$. Then $[\hat{a}_-, \hat{a}_+] = 1$ and $\hat{H} = \hbar\omega\,(\hat{a}_+\hat{a}_- + \frac{1}{2})$. It has a unique ground state $|\psi_0\rangle$ with $\hat{a}_-|\psi_0\rangle = 0$, and excited states $|\psi_n\rangle \equiv \frac{1}{\sqrt{n!}}(\hat{a}_+)^n|\psi_0\rangle$ with energy $E_n = (n+\frac{1}{2})\hbar\omega$. The ground state wavefunction is $\psi_0(x) = (\frac{m\omega}{\pi\hbar})^{1/4}\exp(-\frac{m\omega}{2\hbar}x^2)$.
- $\int_{-\infty}^{\infty} x^{2n}e^{-x^2/(2a)}\mathrm{d}x = (2n-1)!! \cdot a^n \cdot \sqrt{2\pi a}$, for $a > 0$ and non-negative integer n.
- Generic angular momentum: $[\hat{J}_x, \hat{J}_y] = \mathrm{i}\hbar\hat{J}_z$, $[\hat{J}_y, \hat{J}_z] = \mathrm{i}\hbar\hat{J}_x$, $[\hat{J}_z, \hat{J}_x] = \mathrm{i}\hbar\hat{J}_y$.
  For eigenstate $|j,m\rangle$ of $\hat{\boldsymbol{J}}^2$ and $\hat{J}_z$, $\hat{\boldsymbol{J}}^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle$, $\hat{J}_z|j,m\rangle = m\hbar|j,m\rangle$, and $(\hat{J}_x \pm \mathrm{i}\hat{J}_y)|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\hbar|j,m\pm 1\rangle$.
  Here $2j$ is non-negative integer, $m = -j, -j+1, \ldots, j$.
  - Spin-1/2: basis states $|\uparrow\rangle$ and $|\downarrow\rangle$, namely $|S_z = +\frac{1}{2}\hbar\rangle$ and $|S_z = -\frac{1}{2}\hbar\rangle$.
    Under this basis, $\hat{S}_a = \frac{\hbar}{2}\sigma_a$ where $\sigma_{x,y,z}$ are Pauli matrices.
    $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
  - Spherical harmonics Y_ℓ^m are orthonormal, and are eigenfunctions of $\hat{\boldsymbol{L}}^2$ and $\hat{L}_z$.
    $Y_0^0 = \frac{1}{\sqrt{4\pi}}$, $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$, $Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm\mathrm{i}\phi}$, ...
- (Degenerate) Time-independent perturbation theory: $\hat{H} = \hat{H}^{(0)} + \hat{H}^{(1)}$. Denote the (degenerate) orthonormal eigenstates of $\hat{H}^{(0)}$ by $|\psi_{n\alpha}^{(0)}\rangle$, $\hat{H}^{(0)}|\psi_{n\alpha}^{(0)}\rangle = E_n^{(0)}|\psi_{n\alpha}\rangle$.
  Suppose $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$, with E_n close to $E_n^{(0)}$, then $(E_n - E_n^{(0)})$ is the eigenvalue of "secular equation", $\langle\psi_{n\beta}^{(0)}|\hat{H}^{(1)}|\psi_{n\alpha}^{(0)}\rangle + \sum_{m,m\neq n}\frac{1}{E_n^{(0)} - E_m^{(0)}}\langle\psi_{n\beta}^{(0)}|\hat{H}^{(1)}|\psi_m^{(0)}\rangle\langle\psi_m^{(0)}|\hat{H}^{(1)}|\psi_{n\alpha}^{(0)}\rangle$ up to second order. Here β & α are row/column index, the sum is over all eigenstates of $\hat{H}^{(0)}$ with energy different from $E_n^{(0)}$. In non-degenerate case, this is a 1×1 matrix.
- Some Taylor expansions: $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \ldots$; $\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} + \ldots$; $\frac{x}{\sin(x)} = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \ldots$; $\frac{1}{\cos(x)} = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \ldots$.

Problem 1. (20 points) Consider the 1D harmonic oscillator, $\hat{H}^{(0)} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}$. Add a time-independent perturbation $\hat{H}^{(1)} = \lambda \cdot (\hat{a}_-\hat{a}_- + \hat{a}_+\hat{a}_+)$, where λ is a small real parameter, $\hat{a}_\pm$ are ladder operators (see page 1). The full Hamiltonian is $\hat{H} = \hat{H}^{(0)} + \hat{H}^{(1)}$.

(a) (10pts) *Compute the approximate eigenvalues* E_n *of* $\hat{H}$ *up to 2nd order of* λ*, which corresponds to the unperturbed nth energy level* $E_n^{(0)} = \hbar\omega \cdot (n + \frac{1}{2})$ *of* $\hat{H}^{(0)}$ *when* $\lambda = 0$.

(b) (5pts*) Use the variational method to compute the approximate ground state energy of $\hat{H}$. Consider the unnormalized variational wave function $\psi_A(x) = (1 + A \cdot \frac{m\omega}{\hbar} \cdot x^2) \cdot \psi_0(x)$. Here A is a complex variational parameter, ψ_0 is the ground state wave function of unperturbed harmonic oscillator $\hat{H}^{(0)}$ (see page 1). *Compute energy expectation value* $E(A) = \frac{\langle\psi_A|\hat{H}|\psi_A\rangle}{\langle\psi_A|\psi_A\rangle}$. *Solve the minimal value of* $E(A)$ *with respect to* A.

(c) (5pts) *Solve the eigenvalues* E_n *of* $\hat{H}$ *exactly. Expand the results as power series of* λ *up to 2nd order and compare with the results of (a).* [Hint: rewrite $\hat{H}$ in terms of $\hat{x}$ and $\hat{p}$]

Solution:

(a) directly use the formula for non-degenerate perturbation theory (page 1). Denote the eigenstates of unperturbed harmonic oscillator by $\psi_n^{(0)} = \frac{1}{\sqrt{n!}}(\hat{a}_+)^n\psi_0^{(0)}$. Then $\hat{a}_-\psi_n^{(0)} = \sqrt{n}\psi_{n-1}^{(0)}$, $\hat{a}_+\psi_n^{(0)} = \sqrt{n+1}\psi_{n+1}^{(0)}$.

The matrix elements involved are $\langle\psi_m|\hat{a}_-\hat{a}_-|\psi_n\rangle = \sqrt{n(n-1)}\delta_{m,n-2}$ and $\langle\psi_m|\hat{a}_+\hat{a}_+|\psi_n\rangle = (\langle\psi_n|\hat{a}_-\hat{a}_-|\psi_m\rangle)^* = \sqrt{(n+1)(n+2)}\delta_{m,n+2}$. (2pts for these two results)

Therefore $E_n \approx E_n^{(0)} + \langle\psi_n^{(0)}|\hat{H}^{(1)}|\psi_n^{(0)}\rangle + \sum_{m,m\neq n} \frac{|\langle\psi_m^{(0)}|\hat{H}^{(1)}|\psi_n^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$
$= \hbar\omega \cdot (n + \frac{1}{2}) + 0 + \frac{|\lambda\sqrt{n(n-1)}|^2}{2\hbar\omega} + \frac{|\lambda\sqrt{(n+1)(n+2)}|^2}{-2\hbar\omega} = (\hbar\omega - 2\frac{\lambda^2}{\hbar\omega}) \cdot (n + \frac{1}{2})$

(b) $\psi_1(x) = \hat{a}_+\psi_0(x) = \sqrt{2}\sqrt{\frac{m\omega}{\hbar}}x \cdot \psi_0(x)$, $\psi_2(x) = \frac{1}{\sqrt{2}}\hat{a}_+\psi_1(x) = \frac{1}{\sqrt{2}}(2(\frac{m\omega}{\hbar})x^2 - 1) \cdot \psi_0(x)$, therefore $\psi_A = (1 + \frac{A}{2})\psi_0 + \frac{A}{\sqrt{2}}\psi_2$. (1pt for this result, if later parts are missing)

$\langle\psi_A|\psi_A\rangle = |1 + \frac{A}{2}|^2 + |\frac{A}{\sqrt{2}}|^2 = 1 + \frac{1}{2}A^* + \frac{1}{2}A + \frac{3}{4}A^*A$.

$\langle\psi_A|\hat{H}|\psi_A\rangle = |1 + \frac{A}{2}|^2 \cdot \frac{1}{2}\hbar\omega + |\frac{A}{\sqrt{2}}|^2 \cdot \frac{5}{2}\hbar\omega + \lambda\sqrt{2}[(1 + \frac{A}{2})^*(\frac{A}{\sqrt{2}}) + (\frac{A}{\sqrt{2}})^*(1 + \frac{A}{2})]$
$= \frac{\hbar\omega}{2} + (\frac{\hbar\omega}{4} + \lambda)A^* + (\frac{\hbar\omega}{4} + \lambda)A + (\frac{11\hbar\omega}{8} + \lambda)A^*A$

$E(A) = \frac{\frac{\hbar\omega}{2} + (\frac{\hbar\omega}{4} + \lambda)A^* + (\frac{\hbar\omega}{4} + \lambda)A + (\frac{11\hbar\omega}{8} + \lambda)A^*A}{1 + \frac{1}{2}A^* + \frac{1}{2}A + \frac{3}{4}A^*A} = \frac{\hbar\omega}{2} + \frac{(A^* + A + A^*A)\lambda + \hbar\omega A^*A}{1 + \frac{1}{2}A^* + \frac{1}{2}A + \frac{3}{4}A^*A}$. (3pts up to here)

Method #1 to minimize $E(A)$ (not rigorous, but will be accepted):

This can be rewritten as $[\frac{\hbar\omega}{2} + (\frac{\hbar\omega}{4} + \lambda)A^* + (\frac{\hbar\omega}{4} + \lambda)A + (\frac{11\hbar\omega}{8} + \lambda)A^*A]$ $-E(A) \cdot [1 + \frac{1}{2}A^* + \frac{1}{2}A + \frac{3}{4}A^*A] = 0$, or

$$\begin{pmatrix} 1 & A^* \end{pmatrix} \begin{pmatrix} \frac{\hbar\omega}{2} - E(A), & (\frac{\hbar\omega}{4} + \lambda) - \frac{1}{2}E(A) \\ (\frac{\hbar\omega}{4} + \lambda) - \frac{1}{2}E(A), & (\frac{11\hbar\omega}{8} + \lambda) - \frac{3}{4}E(A) \end{pmatrix} \begin{pmatrix} 1 \\ A \end{pmatrix} = 0.$$

Therefore the 2×2 hermitian matrix in the middle must have two eigenvalues of opposite sign (necessary but not sufficient), namely that its determinant must be negative,

$[\frac{\hbar\omega}{2} - E(A)] \cdot [(\frac{11\hbar\omega}{8} + \lambda) - \frac{3}{4}E(A)] - [(\frac{\hbar\omega}{4} + \lambda) - \frac{1}{2}E(A)]^2$

$= \frac{1}{2}[E(A)]^2 - (\frac{3}{2}\hbar\omega) \cdot E(A) + (\frac{5}{8}\hbar^2\omega^2 - \lambda^2) < 0,$

$(\frac{3}{2}\hbar\omega) - \sqrt{\hbar^2\omega^2 + 2\lambda^2} < E(A) < (\frac{3}{2}\hbar\omega) + \sqrt{\hbar^2\omega^2 + 2\lambda^2}.$

So minimal $\min E(A) = (\frac{3}{2}\hbar\omega) - \sqrt{\hbar^2\omega^2 + 2\lambda^2} \approx \frac{1}{2}\hbar\omega - \frac{\lambda^2}{\hbar\omega}$.

(Not required) When $E(A)$ is minimal, the 2×2 matrix is singular, $\begin{pmatrix} 1 \\ A \end{pmatrix}$ is its null vector, $A = -\frac{\hbar\omega/2 - \min E(A)}{\hbar\omega/4 + \lambda - (1/2)\min E(A)} = -2\left(\frac{\sqrt{\hbar^2\omega^2+2\lambda^2} - \hbar\omega}{\sqrt{\hbar^2\omega^2+2\lambda^2} - \hbar\omega + 2\lambda}\right)$. As a consistency check, $A \to 0$ as $\lambda \to 0$.

Method #2 to minimize $E(A)$: normalize ψ_A first,

Reparametrize ψ_A as $\psi_A = \sqrt{|1 + A/2|^2 + |A|^2/2} \cdot e^{\mathrm{i}\mathrm{Arg}(A)} \cdot [\cos\theta e^{\mathrm{i}\phi} \cdot \psi_0 + \sin\theta \cdot \psi_2]$.

Here θ, ϕ are real, $(\cos\theta, \sin\theta) = \frac{1}{\sqrt{|1+A/2|^2+|A|^2/2}} \cdot (|1 + A/2|, |A|/\sqrt{2})$, and $\phi = \mathrm{Arg}(\frac{1+A/2}{A/\sqrt{2}})$.

(Not required) To be rigorous, one needs to show that any (θ, ϕ) corresponds to some A.

Then $E(A) = \cos^2\theta \cdot \frac{1}{2}\hbar\omega + \sin^2\theta \cdot \frac{5}{2}\hbar\omega + 2\cos\theta\cos\phi\sin\theta \cdot \sqrt{2}\lambda$

$= \frac{1}{2}\hbar\omega + \hbar\omega \cdot [1 - \cos(2\theta)] + \sqrt{2}\lambda \cdot \cos\phi\sin(2\theta)$

$= \frac{3}{2}\hbar\omega - \sqrt{\hbar^2\omega^2 + 2\lambda^2\cos^2\phi} \cdot \cos(2\theta + \varphi).$

Here φ satisfies $\cos\varphi = \frac{\hbar\omega}{\sqrt{\hbar^2\omega^2+2\lambda^2\cos^2\phi}}$ and $\sin\varphi = \frac{\sqrt{2}\lambda\cdot\cos\phi}{\sqrt{\hbar^2\omega^2+2\lambda^2\cos^2\phi}}$.

Therefore the minimum of $E(A)$ is $\frac{3}{2}\hbar\omega - \sqrt{\hbar^2\omega^2 + 2\lambda^2}$, when $\phi = 0 \mod \pi$ and $2\theta + \varphi = 0 \mod 2\pi$.

(3) $\hat{H} = (\frac{1}{2m} - \lambda\frac{1}{\hbar m\omega})\hat{p}^2 + (\frac{m\omega^2}{2} + \lambda\frac{m\omega}{\hbar})\hat{x}^2$.

Define $m^* = \frac{m}{1 - 2\frac{\lambda}{\hbar\omega}}$, $\omega^* = \omega\sqrt{1 - 4(\frac{\lambda}{\hbar\omega})^2}$, then $\hat{H} = \frac{1}{2m^*}\hat{p}^2 + \frac{m^*(\omega^*)^2}{2}\hat{x}^2$.

So $E_n = \hbar\omega^* \cdot (n + \frac{1}{2}) = \hbar\omega \cdot \sqrt{1 - 4(\frac{\lambda}{\hbar\omega})^2} \cdot (n + \frac{1}{2}) \approx (\hbar\omega - 2\frac{\lambda^2}{\hbar\omega}) \cdot (n + \frac{1}{2})$.

Problem 2. (20 points) Consider a spin-1/2 moment (see page 1) under rotating magnetic field $\boldsymbol{B}(t) = (B_\perp \cos(\omega t), B_\perp \sin(\omega t), B_z)$. The Hamiltonian is $\hat{H}(t) = -\gamma \boldsymbol{B}(t) \cdot \hat{\boldsymbol{S}}$. Here $\gamma, B_\perp, B_z$ are real constants. Treat the $B_\perp$ part $\hat{H}' = -\gamma B_\perp \cdot [\hat{S}_x \cos(\omega t) + \hat{S}_y \sin(\omega t)]$ as perturbation. Unperturbed $\hat{H}^{(0)} = -\gamma B_z \hat{S}_z$ obviously has eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ for eigenvalues $\mp \gamma B_z \frac{\hbar}{2}$ respectively. The state $|\psi(t)\rangle$ satisfies $\mathbb{i}\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$.

(a) (5pts) Assume $|\psi(t)\rangle = c_\uparrow(t) \cdot e^{\mathbb{i}\gamma B_z t/2}|\uparrow\rangle + c_\downarrow(t) \cdot e^{-\mathbb{i}\gamma B_z t/2}|\downarrow\rangle$, *derive the differential equations for $c_\uparrow$ and $c_\downarrow$.* [Note: the matrix elements of $\hat{H}'$ should be explicitly evaluated]

(b) (10pts) The initial state is $|\psi(t=0)\rangle = |\uparrow\rangle$. *Use the result of (a) to compute $|c_\uparrow(t)|^2$ and $|c_\downarrow(t)|^2$ up to lowest non-trivial order of $B_\perp$.*

(c) (5pts**) $|\psi(t)\rangle$ can be solved exactly (Rabi oscillation). Assume that $|\psi(t)\rangle = c'_\uparrow(t) \cdot e^{-\mathbb{i}\omega t/2}|\uparrow\rangle + c'_\downarrow(t) \cdot e^{\mathbb{i}\omega t/2}|\downarrow\rangle$. *Derive and solve the differential equations for $c'_\uparrow$ and $c'_\downarrow$ under the initial condition in (b).* [Hint: as a consistency check, $|c'_\uparrow(t)|^2$ and $|c'_\downarrow(t)|^2$ should reduce to the result of (b) in the limit of small $B_\perp$]

Solution: this is basically the same as Homework Problem 9.7.

$$\hat{H}(t) = \begin{pmatrix} -\gamma B_z \hbar/2 & -\gamma B_\perp e^{-\mathbb{i}\omega t}\hbar/2 \\ -\gamma B_\perp e^{\mathbb{i}\omega t}\hbar/2 & \gamma B_z \hbar/2 \end{pmatrix}$$

(a) $\mathbb{i}\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = [(\mathbb{i}\hbar\frac{\partial}{\partial t}c_\uparrow(t)) + c_\uparrow(t)\cdot(-\gamma B_z\hbar/2)]\cdot e^{\mathbb{i}\gamma B_z t/2}|\uparrow\rangle$ $+[(\mathbb{i}\hbar\frac{\partial}{\partial t}c_\downarrow(t)) + c_\downarrow(t)\cdot(\gamma B_z\hbar/2)]\cdot e^{-\mathbb{i}\gamma B_z t/2}|\downarrow\rangle$

$\hat{H}(t)|\psi(t)\rangle = [c_\uparrow(t)\cdot(-\gamma B_z\hbar/2)\cdot e^{\mathbb{i}\gamma B_z t/2} + c_\downarrow(t)\cdot(-\gamma B_\perp e^{-\mathbb{i}\omega t}\hbar/2)\cdot e^{-\mathbb{i}\gamma B_z t/2}]|\uparrow\rangle$ $+[c_\downarrow(t)\cdot(\gamma B_z\hbar/2)\cdot e^{-\mathbb{i}\gamma B_z t/2} + c_\uparrow(t)\cdot(-\gamma B_\perp e^{\mathbb{i}\omega t}\hbar/2)\cdot e^{\mathbb{i}\gamma B_z t/2}]|\downarrow\rangle$

Therefore $\mathbb{i}\hbar\frac{\partial}{\partial t}c_\uparrow(t) = c_\downarrow(t)\cdot(-\gamma B_\perp e^{-\mathbb{i}\omega t}\hbar/2)\cdot e^{-\mathbb{i}\gamma B_z t}$, $\mathbb{i}\hbar\frac{\partial}{\partial t}c_\downarrow(t) = c_\uparrow(t)\cdot(-\gamma B_\perp e^{\mathbb{i}\omega t}\hbar/2)\cdot e^{\mathbb{i}\gamma B_z t}$.

(b) $c_\uparrow(t=0) = 1$, $c_\downarrow(t=0) = 0$.

Then to lowest nontrivial order of $B_\perp$,

$c_\downarrow(t) = \frac{1}{\mathbb{i}\hbar}\int_0^t \mathrm{d}t\, c_\uparrow(t)(-\gamma B_\perp e^{\mathbb{i}\omega t}\hbar/2)\cdot e^{\mathbb{i}\gamma B_z t} \approx \frac{1}{\mathbb{i}\hbar}\int_0^t \mathrm{d}t\, c_\uparrow(t=0)(-\gamma B_\perp e^{\mathbb{i}\omega t}\hbar/2)\cdot e^{\mathbb{i}\gamma B_z t}$ $= \frac{1}{\mathbb{i}\hbar}\int_0^t \mathrm{d}t\,(-\gamma B_\perp e^{\mathbb{i}\omega t}\hbar/2)\cdot e^{\mathbb{i}\gamma B_z t} = \frac{1}{2}\frac{\gamma B_\perp}{\gamma B_z+\omega}[e^{\mathbb{i}(\gamma B_z+\omega)t} - 1]$.

Then $|c_\downarrow(t)|^2 \approx (\frac{\gamma B_\perp}{\gamma B_z+\omega})^2 \sin^2(\frac{\gamma B_z+\omega}{2}t)$, $|c_\uparrow(t)|^2 = 1 - |c_\downarrow(t)|^2 \approx 1 - (\frac{\gamma B_\perp}{\gamma B_z+\omega})^2 \sin^2(\frac{\gamma B_z+\omega}{2}t)$.

You can also directly compute $c_\uparrow(t) = 1 + \frac{1}{\mathbb{i}\hbar}\int_0^t \mathrm{d}t\, c_\downarrow(t)(-\gamma B_\perp e^{-\mathbb{i}\omega t}\hbar/2)\cdot e^{-\mathbb{i}\gamma B_z t}$. However you need to use the 1st-order result for $c_\downarrow(t)$ in the right-hand-side.

$c_\uparrow(t) \approx 1 + \frac{1}{\mathbb{i}\hbar}\int_0^t \mathrm{d}t\, \frac{1}{2}\frac{\gamma B_\perp}{\gamma B_z+\omega}[e^{\mathbb{i}(\gamma B_z+\omega)t} - 1]\cdot(-\gamma B_\perp e^{-\mathbb{i}\omega t}\hbar/2)\cdot e^{-\mathbb{i}\gamma B_z t}$

$= 1 + \frac{1}{4}\frac{(\gamma B_\perp)^2}{\gamma B_z+\omega}\cdot[\mathbb{i}t - \frac{1}{\gamma B_z+\omega}\cdot(e^{-\mathbb{i}(\gamma B_z+\omega)t} - 1)]$

$= 1 - \frac{1}{2}\frac{(\gamma B_\perp)^2}{(\gamma B_z+\omega)^2}\sin^2(\frac{\gamma B_z+\omega}{2}t) + \mathbb{i}\frac{1}{4}\frac{(\gamma B_\perp)^2}{\gamma B_z+\omega}[t + \frac{1}{\gamma B_z+\omega}\sin((\gamma B_z+\omega)t)]$.

Then $|c_\uparrow(t)|^2 \approx 1 - 2\cdot\frac{1}{2}\frac{(\gamma B_\perp)^2}{(\gamma B_z+\omega)^2}\sin^2(\frac{\gamma B_z+\omega}{2}t) + O(B_\perp^4)$.

(c) Similar to (a), we have $\mathbb{i}\hbar\frac{\partial}{\partial t}\begin{pmatrix} c'_\uparrow(t) \\ c'_\downarrow(t)\end{pmatrix} = \begin{pmatrix} -\gamma B_z\hbar/2 - \omega\hbar/2 & -\gamma B_\perp\hbar/2 \\ -\gamma B_\perp\hbar/2 & \gamma B_z\hbar/2 + \omega\hbar/2 \end{pmatrix}\begin{pmatrix} c'_\uparrow(t) \\ c'_\downarrow(t)\end{pmatrix}$

$= -\frac{\hbar}{2}((\gamma B_z+\omega)\sigma_z + \gamma B_\perp\sigma_x)\begin{pmatrix} c'_\uparrow(t) \\ c'_\downarrow(t)\end{pmatrix}$.

This can be solved as $\begin{pmatrix} c'_\uparrow(t) \\ c'_\downarrow(t)\end{pmatrix} = \exp[\frac{\mathbb{i}}{2}((\gamma B_z+\omega)\sigma_z + \gamma B_\perp\sigma_x)t]\begin{pmatrix} c'_\uparrow(t=0) \\ c'_\downarrow(t=0)\end{pmatrix}$.

Use the result of Homework Problem 4.56(e),

$\exp[\frac{\mathbb{i}}{2}((\gamma B_z+\omega)\sigma_z + \gamma B_\perp\sigma_x)t] = \cos(\Omega\cdot t/2)\sigma_0 + \mathbb{i}\sin(\Omega\cdot t/2)(\frac{(\gamma B_z+\omega)}{\Omega}\sigma_z + \frac{(\gamma B_\perp)}{\Omega}\sigma_x)$,

where $\Omega = \sqrt{(\gamma B_z+\omega)^2 + (\gamma B_\perp)^2}$, then

$\begin{pmatrix} c'_\uparrow(t) \\ c'_\downarrow(t)\end{pmatrix} = \begin{pmatrix} \cos(\Omega t/2) + \mathbb{i}\sin(\Omega t/2)\frac{(\gamma B_z+\omega)}{\Omega} \\ \mathbb{i}\sin(\Omega t/2)\frac{(\gamma B_\perp)}{\Omega}\end{pmatrix}$

Note that $|c'_\downarrow(t)|^2 = \frac{(\gamma B_\perp)^2}{\Omega^2}\sin^2(\Omega t/2)$, and to lowest order of $B_\perp$ we can approximate Ω by $(\gamma B_z+\omega)$ here, which reproduce the result of (a).

Another way to solve this differential equation is to find the "eigenmodes".

Define $a = (\gamma B_z+\omega)$, $b = \gamma B_\perp$, from Textbook Problem 4.30, $(a\sigma_z + b\sigma_x)$ has eigenvector $\chi_+ = \frac{1}{\sqrt{2\sqrt{a^2+b^2}(\sqrt{a^2+b^2}-a)}}\begin{pmatrix} b \\ \sqrt{a^2+b^2}-a\end{pmatrix}$ for eigenvalue $\sqrt{a^2+b^2}$; and eigenvector $\chi_- = \frac{1}{\sqrt{2\sqrt{a^2+b^2}(\sqrt{a^2+b^2}+a)}}\begin{pmatrix} b \\ -\sqrt{a^2+b^2}-a\end{pmatrix}$ for eigenvalue $-\sqrt{a^2+b^2}$.

Initial state $\begin{pmatrix}1\\0\end{pmatrix} = \frac{b}{\sqrt{2\sqrt{a^2+b^2}(\sqrt{a^2+b^2}-a)}}\chi_+ + \frac{b}{\sqrt{2\sqrt{a^2+b^2}(\sqrt{a^2+b^2}+a)}}\chi_-$, then

$\begin{pmatrix} c'_\uparrow(t) \\ c'_\downarrow(t)\end{pmatrix} = \exp(\mathbb{i}\sqrt{a^2+b^2}\cdot t/2)\frac{b}{\sqrt{2\sqrt{a^2+b^2}(\sqrt{a^2+b^2}-a)}}\chi_+ + \exp(-\mathbb{i}\sqrt{a^2+b^2}\cdot t/2)\frac{b}{\sqrt{2\sqrt{a^2+b^2}(\sqrt{a^2+b^2}+a)}}\chi_-$

$= \cos(\sqrt{a^2+b^2}\cdot t/2)\begin{pmatrix}1\\0\end{pmatrix} + \mathbb{i}\sin(\sqrt{a^2+b^2}\cdot t/2)\frac{b}{2\sqrt{a^2+b^2}\cdot b^2}\begin{pmatrix} (\sqrt{a^2+b^2}+a)\cdot b - (\sqrt{a^2+b^2}-a)\cdot b \\ (\sqrt{a^2+b^2}+a)(\sqrt{a^2+b^2}-a) + (\sqrt{a^2+b^2}-a)(\sqrt{a^2+b^2}+a)\end{pmatrix}$

$= \begin{pmatrix} \cos(\sqrt{a^2+b^2}\cdot t/2) + \mathbb{i}\sin(\sqrt{a^2+b^2}\cdot t/2)\frac{a}{\sqrt{a^2+b^2}} \\ \mathbb{i}\sin(\sqrt{a^2+b^2}\cdot t/2)\frac{b}{\sqrt{a^2+b^2}}\end{pmatrix}$

Problem 3. (40 points) Consider non-relativistic particle(s) in 3D harmonic potential. The Hamiltonian is $\hat{H}_{\text{1-body}} = H_{\text{1-body}}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{p}}) = \frac{1}{2m}\hat{\boldsymbol{p}}^2 + \frac{m\omega^2}{2}\hat{\boldsymbol{r}}^2$. Here m, ω are positive constants. It can be views as three copies of independent 1D harmonic oscillators, $\hat{H}_{\text{1-body}} = (\frac{\hat{p}_x^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}) + (\frac{\hat{p}_y^2}{2m} + \frac{m\omega^2\hat{y}^2}{2}) + (\frac{\hat{p}_z^2}{2m} + \frac{m\omega^2\hat{z}^2}{2})$. Define ladder operators, $\hat{a}_{i,\pm} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{r}_i \mp \mathbb{i}\frac{\hat{p}_i}{m\omega})$, for $i = x, y, z$ respectively. Then $[\hat{a}_{i,-}, \hat{a}_{j,+}] = \delta_{i,j}$. $\hat{H}_{\text{1-body}}$ has a unique normalized ground state $\varphi_0(\boldsymbol{r}) = \psi_0(x)\psi_0(y)\psi_0(z)$, satisfying $\hat{a}_{i,-}\varphi_0 = 0$. The first excited states are 3-fold degenerate, their orthonormal wavefunctions are $\varphi_{1,x}(\boldsymbol{r}) = \hat{a}_{x,+}\varphi_0 = \psi_1(x)\psi_0(y)\psi_0(z)$, $\varphi_{1,y}(\boldsymbol{r}) = \hat{a}_{y,+}\varphi_0 = \psi_0(x)\psi_1(y)\psi_0(z)$, $\varphi_{1,z}(\boldsymbol{r}) = \hat{a}_{z,+}\varphi_0 = \psi_0(x)\psi_0(y)\psi_1(z)$. Here ψ_0 and ψ_1 are eigenstates of 1D harmonic oscillator (see page 1). In this problem, we restrict single-particle wave functions to be linear combinations of φ_0, $\varphi_{1,x}$, $\varphi_{1,y}$, $\varphi_{1,z}$, namely restrict the single-particle Hilbert space to be the 4-dimensional space spanned by these four basis states.

(a) (5pts) The single-particle orbital angular momentum is $\hat{\boldsymbol{L}}_{\text{1-body}} = \boldsymbol{L}_{\text{1-body}}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{p}}) = \hat{\boldsymbol{r}} \times \hat{\boldsymbol{p}}$. *Find the eigenvalues and normalized eigenstates $\varphi_{\ell,m}$ of $\hat{\boldsymbol{L}}^2_{\text{1-body}}$ and $\hat{L}_{\text{1-body},z}$ in terms of the basis φ_0, $\varphi_{1,x}$, $\varphi_{1,y}$, $\varphi_{1,z}$.* [Hint: directly compare to spherical harmonics (see page 1), or compute and diagonalize the matrices of these operators]

(b) (10pts) Consider **two identical bosons**, *write down a complete orthonormal basis for 2-boson wave functions $\psi_{\text{2-boson}}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ in terms of single-particle eigenbasis $\varphi_{\ell,m}$ in (a).*

(c) (10pts) Consider **two identical fermions**, *write down a complete orthonormal basis for 2-fermion wave functions $\psi_{\text{2-fermion}}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ in terms of eigenbasis $\varphi_{\ell,m}$ in (a).*

(d) (10pts*) Consider the 2-body total angular momentum $\hat{\boldsymbol{L}}_{\text{2-body}} = \boldsymbol{L}_{\text{1-body}}(\hat{\boldsymbol{r}}_1, \hat{\boldsymbol{p}}_1) + \boldsymbol{L}_{\text{1-body}}(\hat{\boldsymbol{r}}_2, \hat{\boldsymbol{p}}_2) = \hat{\boldsymbol{r}}_1 \times \hat{\boldsymbol{p}}_1 + \hat{\boldsymbol{r}}_2 \times \hat{\boldsymbol{p}}_2$. Here $\hat{\boldsymbol{p}}_i = -\mathbb{i}\hbar\frac{\partial}{\partial \boldsymbol{r}_i}$. *Find the eigenvalues and normalized eigenstates of $\hat{\boldsymbol{L}}^2_{\text{2-body}}$ and $\hat{L}_{\text{2-body},z}$, for the 2-boson case and 2-fermion case respectively.* [Hint: these are special cases of "addition of angular momentum", in fact most basis in (b) and (c) are already the eigenstates of $\hat{\boldsymbol{L}}^2_{\text{2-body}}$ and $\hat{L}_{\text{2-body},z}$]

(e) (5pts***) The unperturbed 2-body Hamiltonian is $\hat{H}_{\text{2-body}} = H_{\text{1-body}}(\hat{\boldsymbol{r}}_1, \hat{\boldsymbol{p}}_1) + H_{\text{1-body}}(\hat{\boldsymbol{r}}_2, \hat{\boldsymbol{p}}_2) = \frac{\hat{\boldsymbol{p}}_1^2 + \hat{\boldsymbol{p}}_2^2}{2m} + \frac{m\omega^2(\hat{\boldsymbol{r}}_1^2 + \hat{\boldsymbol{r}}_2^2)}{2}$. The full Hamilotian is $\hat{H} = \hat{H}_{\text{2-body}} + \lambda \cdot (\hat{\boldsymbol{r}}_1 \cdot \hat{\boldsymbol{r}}_2)$

where λ is a small real parameter. *Compute the approximate eigenvalues of $\hat{H}$ **for two identical fermions** up to λ^1 order.* [Hint: use ladder operators or Gaussian integrals to evaluate matrix elements, degenerate perturbation theory may be avoided.]

Solution

(a) $Y_0^0 = \sqrt{\frac{1}{4\pi}}$, $Y_0^1 = \sqrt{\frac{3}{4\pi}}\frac{z}{r}$, $Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\frac{x \pm \mathbb{i} y}{r}$.

$\varphi_0 = (\frac{m\omega}{\pi\hbar})^{3/4}\exp(-\frac{m\omega}{2\hbar}r^2) = Y_0^0 \cdot \sqrt{4\pi}(\frac{m\omega}{\pi\hbar})^{3/4}\exp(-\frac{m\omega}{2\hbar}r^2)$,

$\varphi_{1,x} = (\frac{m\omega}{\pi\hbar})^{3/4}\sqrt{2}\sqrt{\frac{m\omega}{\hbar}}x\exp(-\frac{m\omega}{2\hbar}r^2) = \frac{-Y_1^1 + Y_1^{-1}}{\sqrt{2}} \cdot \sqrt{\frac{4\pi}{3}}(\frac{m\omega}{\pi\hbar})^{3/4}\sqrt{2}\sqrt{\frac{m\omega}{\hbar}}r\exp(-\frac{m\omega}{2\hbar}r^2)$,

$\varphi_{1,y} = (\frac{m\omega}{\pi\hbar})^{3/4}\sqrt{2}\sqrt{\frac{m\omega}{\hbar}}y\exp(-\frac{m\omega}{2\hbar}r^2) = \frac{-Y_1^1 - Y_1^{-1}}{\sqrt{2}\mathbb{i}} \cdot \sqrt{\frac{4\pi}{3}}(\frac{m\omega}{\pi\hbar})^{3/4}\sqrt{2}\sqrt{\frac{m\omega}{\hbar}}r\exp(-\frac{m\omega}{2\hbar}r^2)$,

$\varphi_{1,z} = (\frac{m\omega}{\pi\hbar})^{3/4}\sqrt{2}\sqrt{\frac{m\omega}{\hbar}}z\exp(-\frac{m\omega}{2\hbar}r^2) = Y_1^0 \cdot \sqrt{\frac{4\pi}{3}}(\frac{m\omega}{\pi\hbar})^{3/4}\sqrt{2}\sqrt{\frac{m\omega}{\hbar}}r\exp(-\frac{m\omega}{2\hbar}r^2)$.

Therefore $\hat{\boldsymbol{L}}^2_{\text{1-body}}$ has eigenvalue $\hbar^2\ell(\ell+1)$, and $\hat{L}_{\text{1-body},z}$ has eigenvalue $\hbar m$, for $\ell = 0; m = 0$ and $\ell = 1; m = -1, 0, 1$.

Eigenstates are $\varphi_{\ell=0,m=0} = \varphi_0$,

$\varphi_{\ell=1,m=1} = -\frac{1}{\sqrt{2}}\varphi_{1,x} - \frac{\mathbb{i}}{\sqrt{2}}\varphi_{1,y} = -\sqrt{\frac{m\omega}{\hbar}}(x + \mathbb{i}y)\varphi_0(\boldsymbol{r})$,

$\varphi_{\ell=1,m=0} = \varphi_{1,z} = \sqrt{\frac{m\omega}{\hbar}}\sqrt{2}z \cdot \varphi_0(\boldsymbol{r})$,

$\varphi_{\ell=1,m=-1} = \frac{1}{\sqrt{2}}\varphi_{1,x} - \frac{\mathbb{i}}{\sqrt{2}}\varphi_{1,y} = \sqrt{\frac{m\omega}{\hbar}}(x - \mathbb{i}y)\varphi_0(\boldsymbol{r})$.

Or you can use $\hat{L}_{\text{1-body},x} = -\mathbb{i}\hbar(y\partial_z - z\partial_y) = \hbar(-\mathbb{i}\hat{a}_{y,+}\hat{a}_{z,-} + \mathbb{i}\hat{a}_{z,+}\hat{a}_{y,-})$,

$\hat{L}_{\text{1-body},y} = -\mathbb{i}\hbar(z\partial_x - x\partial_z) = \hbar(-\mathbb{i}\hat{a}_{z,+}\hat{a}_{x,-} + \mathbb{i}\hat{a}_{x,+}\hat{a}_{z,-})$,

$\hat{L}_{\text{1-body},z} = -\mathbb{i}\hbar(x\partial_y - y\partial_x) = \hbar(-\mathbb{i}\hat{a}_{x,+}\hat{a}_{y,-} + \mathbb{i}\hat{a}_{y,+}\hat{a}_{x,-})$,

to evaluate the matrix elements of these operators under the $\varphi_0, \varphi_{1,x}, \varphi_{1,y}, \varphi_{1,z}$ basis, then $\hat{L}_{\text{1-body},x} = \hbar\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbb{i} \\ 0 & 0 & \mathbb{i} & 0 \end{pmatrix}$, $\hat{L}_{\text{1-body},x} = \hbar\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{i} \\ 0 & 0 & 0 & 0 \\ 0 & -\mathbb{i} & 0 & 0 \end{pmatrix}$, $\hat{L}_{\text{1-body},z} = \hbar\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathbb{i} & 0 \\ 0 & \mathbb{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and $\hat{\boldsymbol{L}}^2_{\text{1-body}} = \hbar^2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$. So you only need to diagonalize $\hat{L}_{\text{1-body},z}$.

(b)

$\varphi_{0,0}(\boldsymbol{r}_1)\varphi_{0,0}(\boldsymbol{r}_2) \equiv |0,0;0,0\rangle$,

$\varphi_{1,1}(\boldsymbol{r}_1)\varphi_{1,1}(\boldsymbol{r}_2) \equiv |1,1;1,1\rangle$,

$\varphi_{1,0}(\boldsymbol{r}_1)\varphi_{1,0}(\boldsymbol{r}_2) \equiv |1,0;1,0\rangle$,

$\varphi_{1,-1}(\boldsymbol{r}_1)\varphi_{1,-1}(\boldsymbol{r}_2) \equiv |1,-1;1,-1\rangle$,

$$\frac{1}{\sqrt{2}}[\varphi_{0,0}(\boldsymbol{r}_1)\varphi_{1,1}(\boldsymbol{r}_2)+\varphi_{1,1}(\boldsymbol{r}_1)\varphi_{0,0}(\boldsymbol{r}_2)] \equiv \frac{1}{\sqrt{2}}(|0,0;1,1\rangle+|1,1;0,0\rangle),$$
$$\frac{1}{\sqrt{2}}[\varphi_{0,0}(\boldsymbol{r}_1)\varphi_{1,0}(\boldsymbol{r}_2)+\varphi_{1,0}(\boldsymbol{r}_1)\varphi_{0,0}(\boldsymbol{r}_2)] \equiv \frac{1}{\sqrt{2}}(|0,0;1,0\rangle+|1,0;0,0\rangle),$$
$$\frac{1}{\sqrt{2}}[\varphi_{0,0}(\boldsymbol{r}_1)\varphi_{1,-1}(\boldsymbol{r}_2)+\varphi_{1,-1}(\boldsymbol{r}_1)\varphi_{0,0}(\boldsymbol{r}_2)] \equiv \frac{1}{\sqrt{2}}(|0,0;1,-1\rangle+|1,-1;0,0\rangle),$$
$$\frac{1}{\sqrt{2}}[\varphi_{1,1}(\boldsymbol{r}_1)\varphi_{1,0}(\boldsymbol{r}_2)+\varphi_{1,0}(\boldsymbol{r}_1)\varphi_{1,1}(\boldsymbol{r}_2)] \equiv \frac{1}{\sqrt{2}}(|1,1;1,0\rangle+|1,0;1,1\rangle),$$
$$\frac{1}{\sqrt{2}}[\varphi_{1,1}(\boldsymbol{r}_1)\varphi_{1,-1}(\boldsymbol{r}_2)+\varphi_{1,-1}(\boldsymbol{r}_1)\varphi_{1,1}(\boldsymbol{r}_2)] \equiv \frac{1}{\sqrt{2}}(|1,1;1,-1\rangle+|1,-1;1,1\rangle),$$
$$\frac{1}{\sqrt{2}}[\varphi_{1,0}(\boldsymbol{r}_1)\varphi_{1,-1}(\boldsymbol{r}_2)+\varphi_{1,-1}(\boldsymbol{r}_1)\varphi_{1,0}(\boldsymbol{r}_2)] \equiv \frac{1}{\sqrt{2}}(|1,0;1,-1\rangle+|1,-1;1,0\rangle).$$

(c)

$$\frac{1}{\sqrt{2}}[\varphi_{0,0}(\boldsymbol{r}_1)\varphi_{1,1}(\boldsymbol{r}_2)-\varphi_{1,1}(\boldsymbol{r}_1)\varphi_{0,0}(\boldsymbol{r}_2)] \equiv \frac{1}{\sqrt{2}}(|0,0;1,1\rangle-|1,1;0,0\rangle),$$
$$\frac{1}{\sqrt{2}}[\varphi_{0,0}(\boldsymbol{r}_1)\varphi_{1,0}(\boldsymbol{r}_2)-\varphi_{1,0}(\boldsymbol{r}_1)\varphi_{0,0}(\boldsymbol{r}_2)] \equiv \frac{1}{\sqrt{2}}(|0,0;1,0\rangle-|1,0;0,0\rangle),$$
$$\frac{1}{\sqrt{2}}[\varphi_{0,0}(\boldsymbol{r}_1)\varphi_{1,-1}(\boldsymbol{r}_2)-\varphi_{1,-1}(\boldsymbol{r}_1)\varphi_{0,0}(\boldsymbol{r}_2)] \equiv \frac{1}{\sqrt{2}}(|0,0;1,-1\rangle-|1,-1;0,0\rangle),$$
$$\frac{1}{\sqrt{2}}[\varphi_{1,1}(\boldsymbol{r}_1)\varphi_{1,0}(\boldsymbol{r}_2)-\varphi_{1,0}(\boldsymbol{r}_1)\varphi_{1,1}(\boldsymbol{r}_2)] \equiv \frac{1}{\sqrt{2}}(|1,1;1,0\rangle-|1,0;1,1\rangle),$$
$$\frac{1}{\sqrt{2}}[\varphi_{1,1}(\boldsymbol{r}_1)\varphi_{1,-1}(\boldsymbol{r}_2)-\varphi_{1,-1}(\boldsymbol{r}_1)\varphi_{1,1}(\boldsymbol{r}_2)] \equiv \frac{1}{\sqrt{2}}(|1,1;1,-1\rangle-|1,-1;1,1\rangle),$$
$$\frac{1}{\sqrt{2}}[\varphi_{1,0}(\boldsymbol{r}_1)\varphi_{1,-1}(\boldsymbol{r}_2)-\varphi_{1,-1}(\boldsymbol{r}_1)\varphi_{1,0}(\boldsymbol{r}_2)] \equiv \frac{1}{\sqrt{2}}(|1,0;1,-1\rangle-|1,-1;1,0\rangle).$$

(d) Treat this as "addition of angular momentum" for distinguishable particles first.

$\hat{\boldsymbol{L}}^2_{\text{2-body}}$ has eigenvalue $\hbar^2\ell_{\text{2-body}}(\ell_{\text{2-body}}+1)$, and $\hat{L}_{\text{2-body},z}$ has eigenvalue $\hbar m_{\text{2-body}}$ for $m_{\text{2-body}} = -\ell_{\text{2-body}}, \dots, \ell_{\text{2-body}}$.

If both particles have $\ell = 0$, then $\ell_{\text{2-body}} = 0$, $m_{\text{2-body}} = 0$, this is 2-boson state $|\ell_{\text{2-body}} = 0, m_{\text{2-body}} = 0; \ell_{1,2} = (0,0); \text{boson}\rangle = |0,0;0,0\rangle$.

If one particle has $\ell = 0$, the other has $\ell = 1$, then $\ell_{\text{2-body}} = 1$, $m_{\text{2-body}} = 1, 0, -1$, this can be 2-boson state(symmetric) or 2-fermion state(anti-symmetric),

$|\ell_{\text{2-body}} = 1, m_{\text{2-body}}; \ell_{1,2} = (0,1); \text{boson}\rangle = \frac{1}{\sqrt{2}}(|0,0;1,m_{\text{2-body}}\rangle + |1,m_{\text{2-body}};0,0\rangle)$,

$|\ell_{\text{2-body}} = 1, m_{\text{2-body}}; \ell_{1,2} = (0,1); \text{fermion}\rangle = \frac{1}{\sqrt{2}}(|0,0;1,m_{\text{2-body}}\rangle - |1,m_{\text{2-body}};0,0\rangle)$.

If both particles have $\ell = 1$, then $\ell_{\text{2-body}}$ can be 2 or 1 or 0.

$\ell_{\text{2-body}} = 2$ states are symmetric with respect to exchange, are 2-boson states.

$|\ell_{\text{2-body}} = 2, m_{\text{2-body}} = 2; \ell_{1,2} = (1,1); \text{boson}\rangle = |1,1;1,1\rangle$;

$|\ell_{\text{2-body}} = 2, m_{\text{2-body}} = 1; \ell_{1,2} = (1,1); \text{boson}\rangle = \frac{1}{\sqrt{2}}(|1,0;1,1\rangle + |1,1;1,0\rangle)$;

$|\ell_{\text{2-body}} = 2, m_{\text{2-body}} = 0; \ell_{1,2} = (1,1); \text{boson}\rangle = \frac{1}{\sqrt{6}}(|1,-1;1,1\rangle + 2|1,0;1,0\rangle + |1,1;1,-1\rangle)$;

$|\ell_{\text{2-body}} = 2, m_{\text{2-body}} = -1; \ell_{1,2} = (1,1); \text{boson}\rangle = \frac{1}{\sqrt{2}}(|1,0;1,1\rangle + |1,1;1,0\rangle)$;

$|\ell_{\text{2-body}} = 2, m_{\text{2-body}} = -2; \ell_{1,2} = (1,1); \text{boson}\rangle = |1,-1;1,-1\rangle$.

$\ell_{\text{2-body}} = 1$ states are anti-symmetric with respect to exchange, are 2-fermion states.

$|\ell_{\text{2-body}} = 1, m_{\text{2-body}} = 1; \ell_{1,2} = (1,1); \text{fermion}\rangle = \frac{1}{\sqrt{2}}(|1,0;1,1\rangle - |1,1;1,0\rangle)$;

$|\ell_{\text{2-body}} = 1, m_{\text{2-body}} = 0; \ell_{1,2} = (1,1); \text{fermion}\rangle = \frac{1}{\sqrt{2}}(|1,-1;1,1\rangle - |1,1;1,-1\rangle)$;

$|\ell_{\text{2-body}} = 1, m_{\text{2-body}} = -1; \ell_{1,2} = (1,1); \text{fermion}\rangle = \frac{1}{\sqrt{2}}(|1,-1;1,0\rangle - |1,0;1,-1\rangle)$.

$\ell_{\text{2-body}} = 0$ state is symmetric with respect to exchange, is 2-boson state.

$|\ell_{\text{2-body}} = 0, m_{\text{2-body}} = 0; \ell_{1,2} = (1,1); \text{boson}\rangle = \frac{1}{\sqrt{3}}(|1,-1;1,1\rangle - |1,0;1,0\rangle + |1,1;1,-1\rangle)$.

(e) The perturbation $\lambda \cdot (\boldsymbol{r}_1 \cdot \boldsymbol{r}_2) = \frac{\lambda}{2}[(\boldsymbol{r}_1 + \boldsymbol{r}_2)^2 - (\boldsymbol{r}_1 - \boldsymbol{r}_2)^2]$ is obviously invariant under spatial rotation, so it commutes with each component of total angular momentum $\hat{\boldsymbol{L}}_{\text{2-body}} = \hat{\boldsymbol{L}}_1 + \hat{\boldsymbol{L}}_2$, where $\hat{\boldsymbol{L}}_i = \hat{\boldsymbol{r}}_i \times \hat{\boldsymbol{p}}_i$ for $i = 1,2$. The proof is similar to Homework Problem 4.19. $[\hat{L}_{i,a}, \hat{r}_{i,b}] = \mathbb{i}\hbar\epsilon_{abc}\hat{r}_{i,c}$, then $[\hat{L}_{\text{2-body},a}, (\hat{\boldsymbol{r}}_1 \pm \hat{\boldsymbol{r}}_2)_b] = \mathbb{i}\hbar\epsilon_{abc}(\hat{\boldsymbol{r}}_1 \pm \hat{\boldsymbol{r}}_2)_c$, $[\hat{L}_{\text{2-body},a}, (\hat{\boldsymbol{r}}_1 \pm \hat{\boldsymbol{r}}_2)^2] = 0$.

Therefore the perturbation term does not change eigenvalues of $\hat{\boldsymbol{L}}^2_{\text{2-body}}$ and $\hat{L}_{\text{2-body},z}$.

According to (d), for two identical fermions, $\ell_{\text{2-body}} = 1$, $m_{\text{2-body}} = 1, 0, -1$. Given $(\ell_{\text{2-body}}, m_{\text{2-body}})$ there are two 2-fermion states, and in this 2-dimensional subspace there is no degeneracy for the unperturbed $\hat{H}_{\text{2-body}}$. This is summarized in the following table.

state	$\ell_{\text{2-body}}$	$m_{\text{2-body}}$	$\hat{H}_{\text{2-body}}$ eigenvalue
$\frac{1}{\sqrt{2}}(\|0,0;1,1\rangle - \|1,1;0,0\rangle) = \frac{1}{\sqrt{2}}\sqrt{\frac{m\omega}{\hbar}}[(x_1 - x_2) + \mathbb{i}(y_1 - y_2)] \cdot \varphi_0(\boldsymbol{r}_1)\varphi_0(\boldsymbol{r}_2)$	1	1	$\hbar\omega \cdot (\frac{3}{2} + \frac{5}{2})$
$\frac{1}{\sqrt{2}}(\|1,0;1,1\rangle - \|1,1;1,0\rangle) = \frac{m\omega}{\hbar}[(x_1 + \mathbb{i}y_1)z_2 - (x_2 + \mathbb{i}y_2)z_1] \cdot \varphi_0(\boldsymbol{r}_1)\varphi_0(\boldsymbol{r}_2)$	1	1	$\hbar\omega \cdot (\frac{5}{2} + \frac{5}{2})$
$\frac{1}{\sqrt{2}}(\|0,0;1,0\rangle - \|1,0;0,0\rangle) = \sqrt{\frac{m\omega}{\hbar}}(z_2 - z_1) \cdot \varphi_0(\boldsymbol{r}_1)\varphi_0(\boldsymbol{r}_2)$	1	0	$\hbar\omega \cdot (\frac{3}{2} + \frac{5}{2})$
$\frac{1}{\sqrt{2}}(\|1,-1;1,1\rangle - \|1,1;1,-1\rangle) = \frac{m\omega}{\hbar}[\sqrt{2}\mathbb{i}(y_1 x_2 - x_1 y_2)] \cdot \varphi_0(\boldsymbol{r}_1)\varphi_0(\boldsymbol{r}_2)$	1	0	$\hbar\omega \cdot (\frac{5}{2} + \frac{5}{2})$
$\frac{1}{\sqrt{2}}(\|0,0;1,-1\rangle - \|1,-1;0,0\rangle) = -\frac{1}{\sqrt{2}}\sqrt{\frac{m\omega}{\hbar}}[(x_1 - x_2) - \mathbb{i}(y_1 - y_2)] \cdot \varphi_0(\boldsymbol{r}_1)\varphi_0(\boldsymbol{r}_2)$	1	-1	$\hbar\omega \cdot (\frac{3}{2} + \frac{5}{2})$
$\frac{1}{\sqrt{2}}(\|1,-1;1,0\rangle - \|1,0;1,-1\rangle) = \frac{m\omega}{\hbar}[(x_1 - \mathbb{i}y_1)z_2 - (x_2 - \mathbb{i}y_2)z_1] \cdot \varphi_0(\boldsymbol{r}_1)\varphi_0(\boldsymbol{r}_2)$	1	-1	$\hbar\omega \cdot (\frac{5}{2} + \frac{5}{2})$

So we just need to evaluate the expectation value of $\lambda\cdot(\hat{\boldsymbol{r}}_1\cdot\hat{\boldsymbol{r}}_2)$ under each of the above six 2-fermion states. There are several simplifications due to symmetry.

Note that because of the single-particle expectation value $\langle\varphi_{\ell=1,m}|\hat{\boldsymbol{r}}|\varphi_{\ell=1,m'}\rangle = 0$ (see "selection rule" in Section 9.3.3, or just use the fact that $\varphi_{\ell=1,m}$ are odd functions of $\boldsymbol{r}$), if both fermions are in $\ell=1$ states, the expectation value of $\lambda\cdot(\hat{\boldsymbol{r}}_1\cdot\hat{\boldsymbol{r}}_2)$ must vanish.

If $\hat{O}$ commutes with $\hat{L}_{x,y,z}$, then $\langle\ell,m|\hat{O}|\ell,m\rangle = \langle\ell,m|\hat{O}\cdot\frac{\hat{L}_-}{\hbar\sqrt{(\ell+m+1)(\ell-m)}}|\ell,m+1\rangle$ $=\langle\ell,m|\frac{\hat{L}_-}{\hbar\sqrt{(\ell+m+1)(\ell-m)}}\cdot\hat{O}|\ell,m+1\rangle = \langle\ell,m+1|\hat{O}|\ell,m+1\rangle$ is independent of m.

Therefore the expectation values of $\lambda\cdot(\hat{\boldsymbol{r}}_1\cdot\hat{\boldsymbol{r}}_2)$ under $\frac{1}{\sqrt{2}}(|0,0;1,m\rangle - |1,m;0,0\rangle)$ states are independent of m. Choose $m=0$ state, we just need to evaluate a Gaussian integral, or expectation value under ground state $\varphi_0(\boldsymbol{r}_1)\varphi_0(\boldsymbol{r}_2)$, $\langle\lambda(x_1x_2+y_1y_2+z_1z_2)\cdot\frac{m\omega}{\hbar}(z_2-z_1)^2\rangle_0$ $=\lambda\frac{m\omega}{\hbar}\langle z_1z_2\cdot(-2z_1z_2)\rangle_0 = -2\lambda\frac{m\omega}{\hbar}\cdot(\frac{\hbar}{2m\omega})^2 = -\frac{\lambda\hbar}{2m\omega}$.

Finally, up to λ^1 order,
for $|\ell_{\text{2-body}}=1, m_{\text{2-body}}; \ell_{1,2}=(0,1); \text{fermion}\rangle$ states, energy$\approx 4\hbar\omega - \frac{\lambda\hbar}{2m\omega}$;
for $|\ell_{\text{2-body}}=1, m_{\text{2-body}}; \ell_{1,2}=(1,1); \text{fermion}\rangle$ states, energy$\approx 5\hbar\omega$.

Another method is to solve $\hat{H}$ exactly. Define $\boldsymbol{r}_\pm = \frac{\boldsymbol{r}_1\pm\boldsymbol{r}_2}{\sqrt{2}}$. Then $\hat{H} = \frac{-\hbar^2}{2m}(\partial^2_{\boldsymbol{r}_+}+\partial^2_{\boldsymbol{r}_-}) + (\frac{m\omega^2}{2}+\frac{\lambda}{2})\boldsymbol{r}_+^2 + (\frac{m\omega^2}{2}-\frac{\lambda}{2})\boldsymbol{r}_-^2$. We have two 3D harmonic oscillators with frequencies $\omega_\pm = \sqrt{\omega^2\pm\frac{\lambda}{m}} \approx \omega\pm\frac{\lambda}{2m\omega}$. Denote their eigenstates by $\varphi_0^{(\pm)}$ and $\varphi_{1,a}^{(\pm)}$ with $a=x,y,z$, respectively.

When $\lambda=0$,
$|\ell_{\text{2-body}}=1, m_{\text{2-body}}=0; \ell_{1,2}=(0,1); \text{fermion}\rangle = -\varphi_0^{(+)}(\boldsymbol{r}_+)\varphi_{1,z}^{(-)}(\boldsymbol{r}_-)$;
$|\ell_{\text{2-body}}=1, m_{\text{2-body}}=1; \ell_{1,2}=(0,1); \text{fermion}\rangle = \frac{1}{\sqrt{2}}\varphi_0^{(+)}(\boldsymbol{r}_+)[\varphi_{1,x}^{(-)}(\boldsymbol{r}_-)+\mathbb{i}\varphi_{1,y}^{(-)}(\boldsymbol{r}_-)]$;
$|\ell_{\text{2-body}}=1, m_{\text{2-body}}=-1; \ell_{1,2}=(0,1); \text{fermion}\rangle = -\frac{1}{\sqrt{2}}\varphi_0^{(+)}(\boldsymbol{r}_+)[\varphi_{1,x}^{(-)}(\boldsymbol{r}_-)-\mathbb{i}\varphi_{1,y}^{(-)}(\boldsymbol{r}_-)]$;
$|\ell_{\text{2-body}}=1, m_{\text{2-body}}=0; \ell_{1,2}=(1,1); \text{fermion}\rangle = \frac{\mathbb{i}}{\sqrt{2}}[\varphi_{1,x}^{(+)}(\boldsymbol{r}_+)\varphi_{1,y}^{(-)}(\boldsymbol{r}_-)-\varphi_{1,y}^{(+)}(\boldsymbol{r}_+)\varphi_{1,x}^{(-)}(\boldsymbol{r}_-)]$;
$|\ell_{\text{2-body}}=1, m_{\text{2-body}}=1; \ell_{1,2}=(1,1); \text{fermion}\rangle$
$=-\frac{1}{2}[(\varphi_{1,x}^{(+)}(\boldsymbol{r}_+)+\mathbb{i}\varphi_{1,y}^{(+)}(\boldsymbol{r}_+))\varphi_{1,z}^{(-)}(\boldsymbol{r}_-)-\varphi_{1,z}^{(+)}(\boldsymbol{r}_+)(\varphi_{1,x}^{(-)}(\boldsymbol{r}_-)+\mathbb{i}\varphi_{1,y}^{(-)}(\boldsymbol{r}_-))]$;
$|\ell_{\text{2-body}}=1, m_{\text{2-body}}=-1; \ell_{1,2}=(1,1); \text{fermion}\rangle$
$=-\frac{1}{2}[(\varphi_{1,x}^{(+)}(\boldsymbol{r}_+)-\mathbb{i}\varphi_{1,y}^{(+)}(\boldsymbol{r}_+))\varphi_{1,z}^{(-)}(\boldsymbol{r}_-)-\varphi_{1,z}^{(+)}(\boldsymbol{r}_+)(\varphi_{1,x}^{(-)}(\boldsymbol{r}_-)-\mathbb{i}\varphi_{1,y}^{(-)}(\boldsymbol{r}_-))]$.

When $\lambda \neq 0$,

energy of $|\ell_{\text{2-body}} = 1, m_{\text{2-body}}; \ell_{1,2} = (0,1); \text{fermion}\rangle$ states are $\frac{3}{2}\hbar\omega_+ + \frac{5}{2}\hbar\omega_- \approx 4\hbar\omega - \frac{\lambda\hbar}{2m\omega}$;

energy of $|\ell_{\text{2-body}} = 1, m_{\text{2-body}}; \ell_{1,2} = (1,1); \text{fermion}\rangle$ states are $\frac{5}{2}\hbar\omega_+ + \frac{5}{2}\hbar\omega_- \approx 5\hbar\omega$.

Problem 4 (10 points) Consider **two identical bosons** in 1D free space with an attractive δ-potential interaction. The Hamiltonian is $\hat{H} = -\frac{\hbar^2}{2m}(\partial_{x_1}^2 + \partial_{x_2}^2) - \alpha \cdot \delta(x_1 - x_2)$. Here m, α are positive constants.

(a) (6pts) Assume the eigenstate is $\psi(x_1, x_2) = Ae^{\mathbb{i}k_1x_1}e^{\mathbb{i}k_2x_2} + Be^{\mathbb{i}k_2x_1}e^{\mathbb{i}k_1x_2}$ for $x_1 > x_2$. *Derive the equation(s) for the constants* A, B, k_1, k_2. [Note: the eigenvalue is obviously $E = \frac{\hbar^2}{2m}(k_1^2 + k_2^2)$.]

(b) (4pts*) The two bosons can form bound state such that $|\psi(x_1, x_2)| \to 0$ as $|x_1 - x_2| \to +\infty$. *Solve these bound states wave functions and their energy eigenvalues.* [Hint: use the result of (a) and assume that k_1, k_2 are complex; or use the center-of-mass coordinate $X = \frac{x_1+x_2}{2}$ and relative coordinate $x = x_1 - x_2$. The result will contain free parameter(s).]

Solution:

(a). $\psi(x_1, x_2) = \begin{cases} Ae^{\mathbb{i}k_1x_1}e^{\mathbb{i}k_2x_2} + Be^{\mathbb{i}k_2x_1}e^{\mathbb{i}k_1x_2}, & x_1 > x_2; \\ Be^{\mathbb{i}k_1x_1}e^{\mathbb{i}k_2x_2} + Ae^{\mathbb{i}k_2x_1}e^{\mathbb{i}k_1x_2}, & x_1 < x_2. \end{cases}$

It obviously satisfy $\psi(x_1 = x_2 + 0, x_2) = \psi(x_1 = x_2 - 0, x_2) = (A + B)\exp(\mathbb{i}(k_1 + k_2)x_2)$.

$\partial_{x_1}\psi(x_1, x_2) = \begin{cases} \mathbb{i}k_1Ae^{\mathbb{i}k_1x_1}e^{\mathbb{i}k_2x_2} + \mathbb{i}k_2Be^{\mathbb{i}k_2x_1}e^{\mathbb{i}k_1x_2}, & x_1 > x_2; \\ \mathbb{i}k_1Be^{\mathbb{i}k_1x_1}e^{\mathbb{i}k_2x_2} + \mathbb{i}k_2Ae^{\mathbb{i}k_2x_1}e^{\mathbb{i}k_1x_2}, & x_1 < x_2. \end{cases}$

$\partial_{x_2}\psi(x_1, x_2) = \begin{cases} \mathbb{i}k_2Ae^{\mathbb{i}k_1x_1}e^{\mathbb{i}k_2x_2} + \mathbb{i}k_1Be^{\mathbb{i}k_2x_1}e^{\mathbb{i}k_1x_2}, & x_1 > x_2; \\ \mathbb{i}k_2Be^{\mathbb{i}k_1x_1}e^{\mathbb{i}k_2x_2} + \mathbb{i}k_1Ae^{\mathbb{i}k_2x_1}e^{\mathbb{i}k_1x_2}, & x_1 < x_2. \end{cases}$

Therefore $-\frac{\hbar^2}{2m}(\partial_{x_1}^2 + \partial_{x_2}^2)\psi(x_1, x_2) = E \cdot \psi(x_1, x_2)$

$+(-\frac{\hbar^2}{2m})\delta(x_1 - x_2) \cdot 2[(\mathbb{i}k_1A + \mathbb{i}k_2B) - (\mathbb{i}k_1B + \mathbb{i}k_2A)]\exp(\mathbb{i}(k_1 + k_2)x_2)$.

Both $\partial_{x_1}^2$ and $\partial_{x_2}^2$ contribute to the δ-function term. Finally,

$\frac{\hbar^2}{m} \cdot \mathbb{i}(k_1 - k_2)(A - B) + \alpha \cdot (A + B) = 0.$

Another method to derive this is to use the center-of-mass coordinate $X = \frac{x_1+x_2}{2}$ and $x = x_1 - x_2$, then $\hat{H} = \frac{-\hbar^2}{2 \cdot 2m}\partial_X^2 + [\frac{-\hbar^2}{2 \cdot m/2}\partial_x^2 - \alpha\delta(x)]$,

$$\psi = \begin{cases} (Ae^{\mathbb{i}(k_1-k_2)x/2} + Be^{-\mathbb{i}(k_1-k_2)x/2})\exp(\mathbb{i}(k_1+k_2)X), & x_1 > x_2; \\ (Be^{\mathbb{i}(k_1-k_2)x/2} + Ae^{-\mathbb{i}(k_1-k_2)x/2})\exp(\mathbb{i}(k_1+k_2)X), & x_1 < x_2. \end{cases}$$

We only need to deal with the x-dependent parts of $\hat{H}$ and the x-dependent factor of ψ, which is a 1D δ-potential problem for single particle.

(b) From $\psi = \begin{cases} (Ae^{\mathbb{i}(k_1-k_2)x/2} + Be^{-\mathbb{i}(k_1-k_2)x/2})\exp(\mathbb{i}(k_1+k_2)X), & x_1 > x_2; \\ (Be^{\mathbb{i}(k_1-k_2)x/2} + Ae^{-\mathbb{i}(k_1-k_2)x/2})\exp(\mathbb{i}(k_1+k_2)X), & x_1 < x_2. \end{cases}$, bound states must have $k_1 - k_2 = 2\mathbb{i}\kappa$ where $\kappa > 0$ and $B = 0$ (or equivalently $k_1 - k_2 = -2\mathbb{i}\kappa$ and $A = 0$).

Then from the result of (a), $\kappa = \frac{m\alpha}{2\hbar^2}$. (1pt for this result)

$E = \frac{\hbar^2}{2m}(k_1^2 + k_2^2) = \frac{\hbar^2}{2\cdot 2m}(k_1+k_2)^2 + \frac{\hbar^2}{2\cdot m/2}(\frac{k_1-k_2}{2})^2 = \frac{\hbar^2}{2\cdot 2m}K^2 - \frac{m\alpha^2}{4\hbar^2}$, where $K = k_1 + k_2$.
$\psi = Ae^{-\kappa|x_1-x_2|}e^{\mathbb{i}K(x_1+x_2)/2}$.

(Not required) K must be real, otherwise $|\psi(x_1, x_2)|$ will not uniformly tend to zero when $|x_1 - x_2| \to \infty$. Together with the fact that E must be real, we have that $k_1 - k_2$ must be either real or pure imaginary.

Problem 5 (10 points) Consider a spin-1/2 moment with Hamiltonian $\hat{H} = -\gamma B_z \hat{S}_z$ and initial state $|\psi(t=0)\rangle = |\uparrow\rangle$. Do a sequence of measurements at time $t_n = \frac{T}{N} \cdot n$ for observable $\hat{O}_n = \hat{S}_z \cos(\frac{\pi}{N} \cdot n) + \hat{S}_x \sin(\frac{\pi}{N} \cdot n)$, here N is a positive integer, T is a positive constant, $n = 1, 2, \ldots, N$.

(a) (6pts) *What are the possible measurement results for $\hat{O}_n$? And what are the corresponding collapsed states $|\psi(t = t_n + 0)\rangle$ immediately after measuring $\hat{O}_n$.* [Note: results of previous problems might help; the overall phase factor of $|\psi(t = t_n + 0)\rangle$ is unimportant.]

(b) (4pts**) *Compute the final probability of $|\downarrow\rangle$ at time $T+0$ after the last measurement.* [Note: derive a recursion relation between probability distributions of two consecutive measurements; be careful about the time evolution due to $\hat{H}$ between two measurements; you may not be able to simplify a product $\prod_n(\ldots)$. When $N \to \infty$ this probability should become unity, similar to the "quantum Zeno effect".]

Solution:

(a) $\hat{O}_n = \frac{\hbar}{2}(\cos(\theta_n)\sigma_z + \sin(\theta_n)\sigma_x)$, where $\theta_n = \frac{\pi}{N} \cdot n$.

Its eigenvalues are $\pm\frac{\hbar}{2}$, normalized eigenvectors are

$$\chi_+ = \begin{pmatrix} \cos(\theta_n/2) \\ \sin(\theta_n/2) \end{pmatrix} \text{ and } \chi_- = \begin{pmatrix} \sin(\theta_n/2) \\ -\cos(\theta_n/2) \end{pmatrix} \text{ [see Problem 2(c)].}$$

nth measurement results are

$+\frac{\hbar}{2}$ with collapsed state $\begin{pmatrix} \cos(\theta_n/2) \\ \sin(\theta_n/2) \end{pmatrix}$; and $-\frac{\hbar}{2}$ with collapsed state $\begin{pmatrix} \sin(\theta_n/2) \\ -\cos(\theta_n/2) \end{pmatrix}$.

(b) Define the probability of $\pm\frac{\hbar}{2}$ for nth measurement as $P_\pm(n)$, then $P_+(n) + P_-(n) = 0$.

Suppose the nth measurement got $+\frac{\hbar}{2}$ result, $\psi(t = t_n + 0) = \begin{pmatrix} \cos(\theta_n/2) \\ \sin(\theta_n/2) \end{pmatrix}$,

immediately before $(n+1)$th measurement, $\psi(t = t_{n_1} - 0) = \begin{pmatrix} \cos(\theta_n/2)e^{\mathrm{i}\gamma B_z \frac{T}{N}/2} \\ \sin(\theta_n/2)e^{-\mathrm{i}\gamma B_z \frac{T}{N}/2} \end{pmatrix}$,

then the conditional probability of $+\frac{\hbar}{2}$ for $(n+1)$th measurement would be

$P(O_{n+1} = +\frac{\hbar}{2} | O_n = +\frac{\hbar}{2}) = \left| (\cos(\theta_{n+1}/2), \sin(\theta_{n+1}/2)) \begin{pmatrix} \cos(\theta_n/2)e^{\mathrm{i}\gamma B_z \frac{T}{N}/2} \\ \sin(\theta_n/2)e^{-\mathrm{i}\gamma B_z \frac{T}{N}/2} \end{pmatrix} \right|^2$

$= |\cos(\gamma B_z \frac{T}{N}/2)\cos(\frac{\theta_{n+1} - \theta_n}{2}) + \mathrm{i}\sin(\gamma B_z \frac{T}{N}/2)\cos(\frac{\theta_{n+1} + \theta_n}{2})|^2$

$= \cos^2(\gamma B_z \frac{T}{N}/2)\cos^2(\frac{\pi}{2N}) + \sin^2(\gamma B_z \frac{T}{N}/2)\cos^2(\frac{(2n+1)\pi}{2N})$

$= 1 - \sin^2(\frac{\pi}{2N}) - \sin^2(\gamma B_z \frac{T}{N}/2)\sin(\frac{n\pi}{N})\sin(\frac{(n+1)\pi}{N})$.

Then $P(O_{n+1} = -\frac{\hbar}{2} | O_n = +\frac{\hbar}{2}) = 1 - P(O_{n+1} = +\frac{\hbar}{2} | O_n = +\frac{\hbar}{2})$

$= \sin^2(\frac{\pi}{2N}) + \sin^2(\gamma B_z \frac{T}{N}/2)\sin(\frac{n\pi}{N})\sin(\frac{(n+1)\pi}{N})$.

Similarly we can find

$P(O_{n+1} = +\frac{\hbar}{2} | O_n = -\frac{\hbar}{2}) = \sin^2(\frac{\pi}{2N}) + \sin^2(\gamma B_z \frac{T}{N}/2)\sin(\frac{n\pi}{N})\sin(\frac{(n+1)\pi}{N})$,

and $P(O_{n+1} = -\frac{\hbar}{2} | O_n = -\frac{\hbar}{2}) = 1 - \sin^2(\frac{\pi}{2N}) - \sin^2(\gamma B_z \frac{T}{N}/2)\sin(\frac{n\pi}{N})\sin(\frac{(n+1)\pi}{N})$.

Therefore $\begin{pmatrix} P_+(n+1) \\ P_-(n+1) \end{pmatrix} = \begin{pmatrix} 1 - w_n & w_n \\ w_n & 1 - w_n \end{pmatrix} \begin{pmatrix} P_+(n) \\ P_-(n) \end{pmatrix}$,

where $w_n = \sin^2(\frac{\pi}{2N}) + \sin^2(\gamma B_z \frac{T}{N}/2)\sin(\frac{n\pi}{N})\sin(\frac{(n+1)\pi}{N})$.

$P_+(n+1) - P_-(n+1) = (1 - 2w_n)(P_+(n) - P_-(n))$, and $P_+(0) = 1$, $P_-(0) = 0$.

Therefore $P_+(N) - P_-(N) = \prod_{n=1}^{N}(1 - 2w_n)$.

For the last measurement, χ_+ is just $|\downarrow\rangle$. So the final probability of $|\downarrow\rangle$ is just $P_+(N) = \frac{1}{2}[1 + \prod_{n=1}^{N}(1 - 2w_n)]$, where $w_n = \sin^2(\frac{\pi}{2N}) + \sin^2(\gamma B_z \frac{T}{N}/2)\sin(\frac{n\pi}{N})\sin(\frac{(n+1)\pi}{N})$.

(Not required) When $N \to \infty$, $w_n \sim O(\frac{1}{N^2})$, $\prod_{n=1}^{N}(1 - 2w_n) \to 1$, so $P_+(N) \to 1$.