Quantum Mechanics: Fall 2017
Final Exam: Brief Solutions

NOTE: Problems start on page 2. Bold symbols are 3-component vectors.

Some useful facts:

- 1D harmonic oscillator: $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2$.
  Here $\hat{x}$ is position operator, $\hat{p}$ is momentum operator, $[\hat{x},\hat{p}] = \mathbb{i}\hbar$, and in position representation $\hat{p} = -\mathbb{i}\hbar\frac{\partial}{\partial x}$. Define $\hat{a}_\mp = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} \pm \mathbb{i}\frac{1}{m\omega}\hat{p}) = \sqrt{\frac{m\omega}{2\hbar}}(x \pm \frac{\hbar}{m\omega}\frac{\partial}{\partial x})$. Then $[\hat{a}_-, \hat{a}_+] = 1$ and $\hat{H} = \hbar\omega\,(\hat{a}_+\hat{a}_- + \frac{1}{2})$. It has a unique ground state $|\psi_0\rangle$ with $\hat{a}_-|\psi_0\rangle = 0$, and excited states $|\psi_n\rangle \equiv \frac{1}{\sqrt{n!}}(\hat{a}_+)^n|\psi_0\rangle$ with energy $E_n = (n+\frac{1}{2})\hbar\omega$. The ground state wavefunction is $\psi_0(x) = (\frac{m\omega}{\pi\hbar})^{1/4}\exp(-\frac{m\omega}{2\hbar}x^2)$.
- $\int_{-\infty}^{\infty} x^{2n}e^{-x^2/(2a)}\mathrm{d}x = (2n-1)!!\cdot a^n \cdot \sqrt{2\pi a}$, for $a > 0$ and non-negative integer n.
- Generic angular momentum: $[\hat{J}_x, \hat{J}_y] = \mathbb{i}\hbar\hat{J}_z$, $[\hat{J}_y, \hat{J}_z] = \mathbb{i}\hbar\hat{J}_x$, $[\hat{J}_z, \hat{J}_x] = \mathbb{i}\hbar\hat{J}_y$.
  For eigenstate $|j,m\rangle$ of $\hat{\boldsymbol{J}}^2$ and $\hat{J}_z$, $\hat{\boldsymbol{J}}^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle$, $\hat{J}_z|j,m\rangle = m\hbar|j,m\rangle$, and $(\hat{J}_x \pm \mathbb{i}\hat{J}_y)|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\hbar|j,m\pm 1\rangle$.
  Here $2j$ is non-negative integer, $m = -j, -j+1, \ldots, j$.
  - Spin-1/2: basis states $|\uparrow\rangle$ and $|\downarrow\rangle$, namely $|S_z = +\frac{1}{2}\hbar\rangle$ and $|S_z = -\frac{1}{2}\hbar\rangle$.
    Under this basis, $\hat{S}_a = \frac{\hbar}{2}\sigma_a$ where $\sigma_{x,y,z}$ are Pauli matrices.
- Eigenvalues of $a_0\sigma_0 + a_1\sigma_x + a_2\sigma_y + a_3\sigma_z$ are $a_0 \pm \sqrt{a_1^2 + a_2^2 + a_3^2}$, for real $a_{0,1,2,3}$.
- (Degenerate) Time-independent perturbation theory: $\hat{H} = \hat{H}_0 + \hat{V}$.
  Denote the (degenerate) orthonormal eigenstates of $\hat{H}_0$ by $|\psi^{(0)}_{n\alpha}\rangle$, $\hat{H}_0|\psi^{(0)}_{n\alpha}\rangle = E^{(0)}_n|\psi_{n\alpha}\rangle$. Suppose $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$, with E_n close to $E^{(0)}_n$, then $(E_n - E^{(0)}_n)$ is the eigenvalue of the secular equation matrix, $\langle\psi^{(0)}_{n\beta}|\hat{V}|\psi^{(0)}_{n\alpha}\rangle + \sum_{m,m\neq n}\frac{1}{E^{(0)}_n - E^{(0)}_m}\langle\psi^{(0)}_{n\beta}|\hat{V}|\psi^{(0)}_m\rangle\langle\psi^{(0)}_m|\hat{V}|\psi^{(0)}_{n\alpha}\rangle$ up to second order. Here β & α are column/row index, the sum is over all eigenstates of $\hat{H}_0$ with energy different from $E^{(0)}_n$. In non-degenerate case, this is a 1×1 matrix.
- Some Taylor expansions: $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \ldots$; $\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} + \ldots$.
- Change of variables: if $x'_i = \sum_j A_{ij}x_j$, where A is a non-singular constant matrix. Then $\frac{\partial}{\partial x'_i} = \sum_j (A^{-1})_{ji}\frac{\partial}{\partial x_j}$, where A^{-1} is the inverse matrix of A.

Problem 1. (30 points) Consider two spin-1/2 moments, labeled by subscripts $_1$ and $_2$ respectively. One set of complete orthonormal basis of the entire Hilbert space is the tensor products of S_z-eigenbasis $|S_{1,z}\rangle|S_{2,z}\rangle$, namely $|\uparrow\rangle|\uparrow\rangle$, $|\uparrow\rangle|\downarrow\rangle$, $|\downarrow\rangle|\uparrow\rangle$, and $|\downarrow\rangle|\downarrow\rangle$.

(a) (8pts) *Compute the eigenvalues and normalized eigenstates of* $\hat{H}_0 = -J\hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2 = -J\cdot(\hat{S}_{1,x}\hat{S}_{2,x}+\hat{S}_{1,y}\hat{S}_{2,y}+\hat{S}_{1,z}\hat{S}_{2,z})$. J is a positive constant. [Hint: $\hat{H}_0$ is related to $(\hat{\boldsymbol{S}}_1+\hat{\boldsymbol{S}}_2)^2$.]

(b) (8pts) Let the initial state $|\psi(t=0)\rangle = |\uparrow\rangle|\downarrow\rangle$. Evolve it under $\hat{H}_0$, namely $\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = \hat{H}_0|\psi(t)\rangle$. *Solve* $|\psi(t)\rangle$ *in terms of the* S_z*-eigenbasis. Evaluate the expectation values* $\langle\psi(t)|\hat{S}_{1,x}|\psi(t)\rangle$, $\langle\psi(t)|\hat{S}_{1,y}|\psi(t)\rangle$, $\langle\psi(t)|\hat{S}_{1,z}|\psi(t)\rangle$.

(c) (8pts) Consider $\hat{H} = \hat{H}_0 + B\cdot(\hat{S}_{1,z} - \hat{S}_{2,z})$. where B is a "small" real parameter. *Solve up to second order of* B *the ground state energies by perturbation theory.* [Hint: the ground states of $\hat{H}_0$ are degenerate, but degenerate perturbation theory can be avoided]

(d) (6pts) *Solve the exact energy eigenvalues of* $\hat{H}$ *in (c). Expand the results to second order of* B *and compare with the result of (c).*

Solution.

(a). Define total spin $\hat{\boldsymbol{S}} = \hat{\boldsymbol{S}}_1 + \hat{\boldsymbol{S}}_2$.

According to the "addition of angular momentum", the total spin quantum number can be 1 or 0, namely $(\hat{\boldsymbol{S}}_1+\hat{\boldsymbol{S}}_2)^2$ has eigenvalues $\hbar^2\cdot S\cdot(S+1)$ with $S=1$ or 0. The eigenstates of $\hat{\boldsymbol{S}}^2$ and $|\hat{S}_z\rangle$, $|S,S_z\rangle$, can be written as linear combinations of $|S_{1,z}\rangle|S_{2,z}\rangle$ with $S_z = S_{1,z}+S_{2,z}$.

Then $|S=1,S_z=1\rangle$ must be $|\uparrow\rangle|\uparrow\rangle$ up to overall phase factor.

Similarly $|S=1,S_z=-1\rangle$ must be $|\downarrow\rangle|\downarrow\rangle$.

$|S=1,S_z=0\rangle = \frac{1}{\sqrt{2}}\hat{S}_-|S=1,S_z=1\rangle = \frac{1}{\sqrt{2}}(\hat{S}_{1,-}+\hat{S}_{2,-})|\uparrow\rangle|\uparrow\rangle = \frac{1}{\sqrt{2}}(|\downarrow\rangle|\uparrow\rangle + |\uparrow\rangle|\downarrow\rangle)$.

$|S=0,S_z=0\rangle$ is a linear combination of $|\downarrow\rangle|\uparrow\rangle$ and $|\uparrow\rangle|\downarrow\rangle$, and must be orthogonal to $|S=1,S_z=0\rangle$. so must be $\frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$ up to overall phase factor.

Because $\hat{H}_0 = -\frac{J}{2}[(\hat{\boldsymbol{S}}_1+\hat{\boldsymbol{S}}_2)^2 - \hat{\boldsymbol{S}}_1^2 - \hat{\boldsymbol{S}}_2^2]$, and $\hat{\boldsymbol{S}}_1^2$ and $\hat{\boldsymbol{S}}_2^2$ are both constant $\hbar^2\frac{1}{2}\cdot(\frac{1}{2}+1) = \frac{3\hbar^2}{4}$, the $|S,S_z\rangle$ states are eigenstates of $\hat{H}_0$ with eigenvalues $-\frac{J\hbar^2}{2}(S\cdot(S+1)-\frac{3}{2})$.

The final results are

S	S_z	$\lvert S, S_z\rangle$	H_0 eigenvalue
1	1	$\lvert\uparrow\rangle\lvert\uparrow\rangle$	$-\frac{J\hbar^2}{4}$
1	0	$\frac{1}{\sqrt{2}}(\lvert\downarrow\rangle\lvert\uparrow\rangle+\lvert\uparrow\rangle\lvert\downarrow\rangle)$	$-\frac{J\hbar^2}{4}$
1	-1	$\lvert\downarrow\rangle\lvert\downarrow\rangle$	$-\frac{J\hbar^2}{4}$
0	0	$\frac{1}{\sqrt{2}}(\lvert\uparrow\rangle\lvert\downarrow\rangle-\lvert\downarrow\rangle\lvert\uparrow\rangle)$	$\frac{3J\hbar^2}{4}$

These can also be obtained by directly diagonalize $\hat{H}_0$. Note that $\hat{\boldsymbol{S}}_1\cdot\hat{\boldsymbol{S}}_2 = \hat{S}_{1,z}\hat{S}_{2,z}+\frac{1}{2}(\hat{S}_{1,+}\hat{S}_{2,-}+\hat{S}_{1,-}\hat{S}_{2,+})$, under the basis given in main text of this problem, $\hat{H}_0 = -J\hbar^2\cdot\begin{pmatrix}\frac{1}{4} & 0 & 0 & 0\\ 0 & -\frac{1}{4} & \frac{1}{2} & 0\\ 0 & \frac{1}{2} & -\frac{1}{4} & 0\\ 0 & 0 & 0 & \frac{1}{4}\end{pmatrix}$. You just need to diagonalize the central 2×2 block.

(b) $|\psi(t=0)\rangle = |\uparrow\rangle|\downarrow\rangle = \frac{1}{\sqrt{2}}(|S=1,S_z=0\rangle+|S=0,S_z=0\rangle)$.

Therefore $|\psi(t)\rangle = \frac{1}{\sqrt{2}}(e^{-\mathrm{i}(-J\hbar^2/4)\cdot t/\hbar}|S=1,S_z=0\rangle + e^{-\mathrm{i}(3J\hbar^2/4)\cdot t/\hbar}|S=0,S_z=0\rangle)$ $= e^{-\mathrm{i}(-J\hbar^2/4)\cdot t/\hbar}\cdot\left(\frac{1+e^{-\mathrm{i}J\hbar t}}{2}|\uparrow\rangle|\downarrow\rangle+\frac{1-e^{-\mathrm{i}J\hbar t}}{2}|\downarrow\rangle|\uparrow\rangle\right)$.

The matrix elements of $\hat{S}_{1,a}$ $(a=x,y,z)$ are

$\hat{S}_{1,z}|\uparrow\rangle|?\rangle = \frac{\hbar}{2}|\uparrow\rangle|?\rangle$, $\hat{S}_{1,z}|\downarrow\rangle|?\rangle = -\frac{\hbar}{2}|\downarrow\rangle|?\rangle$;

$\hat{S}_{1,x}|\uparrow\rangle|?\rangle = \frac{\hbar}{2}|\downarrow\rangle|?\rangle$, $\hat{S}_{1,x}|\downarrow\rangle|?\rangle = -\frac{\hbar}{2}|\uparrow\rangle|?\rangle$;

$\hat{S}_{1,y}|\uparrow\rangle|?\rangle = \mathrm{i}\frac{\hbar}{2}|\downarrow\rangle|?\rangle$, $\hat{S}_{1,y}|\downarrow\rangle|?\rangle = -\mathrm{i}\frac{\hbar}{2}|\uparrow\rangle|?\rangle$.

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$\langle\hat{S}_{1,z}\rangle = \frac{\hbar}{2}\cdot(|\frac{1+e^{-\mathrm{i}J\hbar t}}{2}|^2-|\frac{1-e^{-\mathrm{i}J\hbar t}}{2}|^2) = \frac{\hbar}{2}\cos(J\hbar t)$, $\langle\hat{S}_{1,x}\rangle = \langle\hat{S}_{1,y}\rangle = 0$.

(c) Use the result of (a) as the basis, $\hat{H}_0$ is the diagonal matrix $\frac{J\hbar^2}{4}\begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 3\end{pmatrix}$.

The ground states are three-fold degenerate $|S=1,S_z\rangle$ states.

The perturbation $B\cdot(\hat{S}_{1,z}-\hat{S}_{2,z})$ under this basis is $\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & B\hbar\\ 0 & 0 & 0 & 0\\ 0 & B\hbar & 0 & 0\end{pmatrix}$.

The 2nd order secular equation matrix for the three-fold degenerate ground states is

$$\begin{pmatrix} -\frac{J\hbar^2}{4} & 0 & 0 \\ 0 & -\frac{J\hbar^2}{4} + \frac{B\hbar \cdot B\hbar}{-\frac{J\hbar^2}{4} - \frac{3J\hbar^2}{4}} & 0 \\ 0 & 0 & -\frac{J\hbar^2}{4} \end{pmatrix}.$$

So the perturbed ground state energies up to 2nd order are $-\frac{J\hbar^2}{4} - \frac{B^2}{J}, -\frac{J\hbar^2}{4}, -\frac{J\hbar^2}{4}$.

(d) Note that the perturbation term commutes with $\hat{S}_z \equiv \hat{S}_{1,z} + \hat{S}_{2,z}$, therefore it will not mix states with different S_z quantum number.

For the subspace with $S_z = 1$, there is only one state $|S = 1, S_z = 1\rangle$ with is the eigenstate of $\hat{H}$ with exact eigenvalue $-\frac{J\hbar^2}{4}$.

For the subspace with $S_z = -1$, there is only one state $|S = 1, S_z = -1\rangle$ with is the eigenstate of $\hat{H}$ with exact eigenvalue $-\frac{J\hbar^2}{4}$.

For the subspace with $S_z = 0$, there are two basis states $|S = 1, S_z = 0\rangle$ and $|S = 0, S_z = 0\rangle$, $\hat{H}$ in this subspace is a 2×2 matrix $\begin{pmatrix} -\frac{J\hbar^2}{4} & B\hbar \\ B\hbar & \frac{3J\hbar^2}{4} \end{pmatrix} = \frac{J\hbar^2}{4}\sigma_0 - \frac{J\hbar^2}{2}\sigma_y + B\hbar\sigma_x$, use the facts on page 1, the exact eigenvalues are $\frac{J\hbar^2}{4} \pm \sqrt{(\frac{J\hbar^2}{2})^2 + (B\hbar)^2}$ $= \frac{J\hbar^2}{4} \pm \frac{J\hbar^2}{2} \cdot \sqrt{1 + \frac{4B^2}{J^2\hbar^2}} \approx \frac{J\hbar^2}{4} \pm \frac{J\hbar^2}{2} \cdot (1 + \frac{2B^2}{J^2\hbar^2}) = \begin{cases} -\frac{J\hbar^2}{4} - \frac{B^2}{J}; \\ \frac{3J\hbar^2}{4} + \frac{B^2}{J}. \end{cases}$.

Problem 2. (20 points) Consider a 1D harmonic oscillator $\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2$, with a time-independent perturbation, $\hat{V} = -f \cdot \hat{x}$. Here f is a real constant. The full Hamiltonian is $\hat{H} = \hat{H}_0 + \hat{V}$.

(a) (8pts) Suppose $\psi(x,t) = \sum_n c_n(t) e^{-\mathrm{i}E_n t/\hbar}\psi_n(x)$ satisfy the Schrödinger equation $\mathrm{i}\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi$. Here $E_n = \hbar\omega(n + \frac{1}{2})$ and $\psi_n(x)$ are eigenvalues and normalized eigenfunctions of $\hat{H}_0$ (see page 1). *Derive the differential equations for the coefficients $c_n(t)$ in terms of known quantities.* [Hint: use the ladder operators to compute the matrix elements.]

(b) (8pts) Suppose the initial state is $\psi(x, t = 0) = \psi_0(x)$. *Compute $c_1(t)$ for the lowest non-trivial order.*

(c) (4pts) With the same conditions of (b), *compute $c_2(t)$ for the lowest non-trivial order.* [Hint: result of (b) will be useful]

Solution.

(a). $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_- + \hat{a}_+)$.

$\hat{a}_+|\psi_n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}_+)^{n+1}|\psi_0\rangle = \sqrt{n+1}|\psi_{n+1}\rangle$, namely $\langle\psi_m|\hat{a}_+|\psi_n\rangle = \sqrt{n+1}\cdot\delta_{m,n+1}$. Take hermitian conjugate, we see that $\langle\psi_n|\hat{a}_-|\psi_m\rangle = \sqrt{n+1}\cdot\delta_{m,n+1}$, then $\hat{a}_-|\psi_{n+1}\rangle = \sqrt{n+1}|\psi_n\rangle$.

Apply $\hat{H} = \hat{H}_0 + \hat{V}$ on $|\psi(t)\rangle = \sum_n c_n(t)e^{-\mathrm{i}E_n t/\hbar}|\psi_n\rangle$,

$\hat{H}|\psi(t)\rangle = \sum_n c_n(t)e^{-\mathrm{i}E_n t/\hbar}\left[E_n|\psi_n\rangle + (-f\sqrt{\frac{\hbar}{2m\omega}})\cdot(\sqrt{n}|\psi_{n-1}\rangle + \sqrt{n+1}|\psi_{n+1}\rangle)\right]$,

here for $n=0$ define $|\psi_{n-1}\rangle = 0$.

$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = \sum_n\left[(\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}c_n(t))e^{-\mathrm{i}E_n\cdot t/\hbar} + c_n(t)\cdot E_n\cdot e^{-\mathrm{i}E_n\cdot t/\hbar}\right]|\psi_n\rangle$.

Compare the coefficient of $|\psi_n\rangle$, we have

$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}c_n(t) = (-f\sqrt{\frac{\hbar}{2m\omega}})\cdot(\sqrt{n+1}c_{n+1}(t)e^{-\mathrm{i}\omega t} + \sqrt{n}c_{n-1}(t)e^{\mathrm{i}\omega t})$.

(b). The initial condition is $c_0(t=0) = 1$ and $c_{n>0}(t=0) = 0$.

Use $\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}c_1(t) = (-f\sqrt{\frac{\hbar}{2m\omega}})\cdot(\sqrt{2}c_2(t)e^{-\mathrm{i}\omega t} + \sqrt{1}c_0(t)e^{\mathrm{i}\omega t})$, approximate $c_2(t)\sim 0$ and $c_0(t)\sim 1$ on the right-hand-side.

$c_1(t) \approx \int_0^t \mathrm{d}t\,\frac{1}{\mathrm{i}\hbar}(-f\sqrt{\frac{\hbar}{2m\omega}})e^{\mathrm{i}\omega t} = \frac{f}{\hbar\omega}\sqrt{\frac{\hbar}{2m\omega}}(e^{\mathrm{i}\omega t} - 1)$. .

(c) Use $\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}c_2(t) = (-f\sqrt{\frac{\hbar}{2m\omega}})\cdot(\sqrt{3}c_3(t)e^{-\mathrm{i}\omega t} + \sqrt{2}c_1(t)e^{\mathrm{i}\omega t})$, approximate $c_3(t)\sim 0$ and use the result of (b) for $c_1(t)$,

$c_2(t) \approx \int_0^t \mathrm{d}t\,\frac{1}{\mathrm{i}\hbar}(-f\sqrt{\frac{\hbar}{2m\omega}})\cdot\sqrt{2}\cdot\frac{f}{\hbar\omega}\sqrt{\frac{\hbar}{2m\omega}}(e^{\mathrm{i}\omega t} - 1)\cdot e^{\mathrm{i}\omega t}$

$= \frac{f^2}{\hbar\omega^2}\frac{\hbar}{2m\omega}\cdot\sqrt{2}\cdot[\frac{1}{2}(e^{2\mathrm{i}\omega t} - 1) - (e^{\mathrm{i}\omega t} - 1)] = \frac{f^2}{\hbar\omega^2}\frac{\hbar}{2m\omega}\cdot\frac{1}{\sqrt{2}}\cdot(e^{\mathrm{i}\omega t} - 1)^2$.

Problem 3. (40 points) Consider two identical particles in 1D harmonic potential, the Hamiltonian is $\hat{H}_0 = -\frac{\hbar^2}{2m}\left[(\frac{\partial}{\partial x_1})^2 + (\frac{\partial}{\partial x_2})^2\right] + \frac{m\omega^2}{2}(x_1^2 + x_2^2)$, where x_1, x_2 are coordinates of the particles. The generic wavefunction is $\psi(x_1, x_2)$ with certain symmetry properties.

(a) (12pts) *If these particles are bosons, write down the three lowest energy eigenvalues of $\hat{H}_0$, and write down the corresponding normalized eigenstate wavefunctions in terms of the normalized single particle eigenstate ψ_n defined on page 1.* [Note: be careful about degeneracy and normalization]

(b) (10pts) *If these particles are fermions, redo the questions in (a) for the four lowest* *energy eigenvalues.*

(c) (8pts) If these particles have interaction $\hat{V} = V \cdot (x_1 - x_2)^2$, where V is a small parameter. *Compute the ground state energy of $\hat{H}_0 + \hat{V}$ to first order perturbation, for the case of bosons and fermions respectively.* [Hint: it may be more convenient to use ladder operators]

(d) (6pts) *Redo the calculation in (c) to second order perturbation, for the case of bosons and fermions respectively.* [Hint: the result of (a)(b) will be helpful.]

(e) (4pts) *Solve the ground state energy of $\hat{H}_0 + \hat{V}$ exactly, for the case of bosons and fermions respectively. Compare the results to those of (d) and (e).* [Hint: use the center-of-mass coordinate $x_{\text{com}} \equiv \frac{x_1+x_2}{2}$ and the relative coordinate $X \equiv x_2 - x_1$, be careful about the symmetry properties of the wavefunction.]

Solution.

$\hat{H}_0$ can be viewed as two decoupled harmonic oscillators.

Define two sets of ladder operators, $\hat{a}_{i,\pm} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x}_i \mp \frac{\mathrm{i}}{m\omega}\hat{p}_i)$, for $i = 1, 2$.

Then $(\hat{a}_{i,+})^\dagger = \hat{a}_{i,-}$, $[\hat{a}_{i,-}, \hat{a}_{j,+}] = \delta_{i,j}$. And $\hat{H}_0 = \hbar\omega \cdot (\hat{a}_{1,+}\hat{a}_{1,-} + \hat{a}_{2,+}\hat{a}_{2,-} + 1)$.

If we ignore the (anti-)symmetry of the wavefunction for identical particles, a basis of normalized eigenstates of $\hat{H}_0$ are $\psi_n(x_1)\psi_m(x_2)$ with energy eigenvalue $\hbar\omega \cdot (n + m + 1)$. Here $\psi_{n,m}$ are normalized eigenstate wavefunction of one harmonic oscillator given on page 1. This state will be denoted by $|\psi_n\rangle|\psi_m\rangle$ hereafter.

(a). For bosons, the wavefunction is symmetric with respect to the exchange of x_1,x_2. The symmetrized basis states are $\psi_n(x_1)\psi_n(x_2)$, or $\frac{1}{\sqrt{2}}[\psi_n(x_1)\psi_m(x_2) + \psi_m(x_1)\psi_n(x_2)]$ for $n < m$.

The lowest three energy levels and eigenstates are

$E_{0,0} = \hbar\omega$, for $\psi_{0,0}(x_1, x_2) = \psi_0(x_1)\psi_0(x_2)$;

$E_{0,1} = 2\hbar\omega$, for $\psi_{0,1}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2)]$;

$E_{0,2} = E_{1,1} = 3\hbar\omega$, for $\psi_{0,2}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_2(x_2) + \psi_2(x_1)\psi_0(x_2)]$, and
$\psi_{1,1}(x_1, x_2) = \psi_1(x_1)\psi_1(x_2)$.

(b) For fermions, the wavefunction is anti-symmetric with respect to the exchange of x_1,x_2. The anti-symmetrized basis states are $\frac{1}{\sqrt{2}}[\psi_n(x_1)\psi_m(x_2) - \psi_m(x_1)\psi_n(x_2)]$ for $n < m$.

The lowest four energy levels and eigenstates are

$E_{0,1} = 2\hbar\omega$, for $\psi_{0,1}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_1(x_2) - \psi_1(x_1)\psi_0(x_2)]$;

$E_{0,2} = 3\hbar\omega$, for $\psi_{0,2}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_0(x_2)]$;

$E_{0,3} = E_{1,2} = 4\hbar\omega$, for $\psi_{0,3}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_3(x_2) - \psi_3(x_1)\psi_0(x_2)]$, and $\psi_{1,2}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_1(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2)]$;

$E_{0,4} = E_{1,3} = 5\hbar\omega$, for $\psi_{0,4}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_4(x_2) - \psi_4(x_1)\psi_0(x_2)]$, and $\psi_{1,3}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_1(x_1)\psi_3(x_2) - \psi_3(x_1)\psi_1(x_2)]$.

(c) $\hat{V} = V \cdot \frac{\hbar}{2m\omega}(\hat{a}_{1,-} + \hat{a}_{1,+} - \hat{a}_{2,-} - \hat{a}_{2,+})^2$

$$= \frac{V\hbar}{2m\omega} \cdot \big[(\hat{a}_{1,-}^2 + \hat{a}_{2,-}^2 - 2\hat{a}_{1,-}\hat{a}_{2,-}) + (\hat{a}_{1,+}^2 + \hat{a}_{2,+}^2 - 2\hat{a}_{1,+}\hat{a}_{2,+})$$
$$+ (2 + 2\hat{a}_{1,+}\hat{a}_{1,-} + 2\hat{a}_{2,+}\hat{a}_{2,-} - 2\hat{a}_{1,+}\hat{a}_{2,-} - 2\hat{a}_{2,+}\hat{a}_{1,-})\big].$$

The ground state for both boson case and fermion case is unique. The first order correction to ground state energy is $\langle \text{ground state}|\hat{V}|\text{ground state}\rangle$.

For bosons, $\hat{V}|\psi_{0,0}\rangle = \hat{V}|\psi_0\rangle|\psi_0\rangle$

$$= \frac{V\hbar}{2m\omega} \cdot [(0) + (\sqrt{2}|\psi_2\rangle|\psi_0\rangle + \sqrt{2}|\psi_0\rangle|\psi_2\rangle - 2|\psi_1\rangle|\psi_1\rangle) + (2)|\psi_0\rangle|\psi_0\rangle]$$
$$= \frac{V\hbar}{2m\omega} \cdot [2|\psi_{0,2}\rangle - 2|\psi_{1,1}\rangle + 2|\psi_{0,0}\rangle].$$

Therefore the ground state energy for bosons to 1st order is $\hbar\omega + \frac{V\hbar}{2m\omega} \cdot 2 = \hbar\omega + \frac{V\hbar}{m\omega}$.

For fermions, $\hat{V}|\psi_{0,1}\rangle = \hat{V}\frac{1}{\sqrt{2}}(|\psi_0\rangle|\psi_1\rangle - |\psi_1\rangle|\psi_0\rangle)$

$$= \frac{V\hbar}{2m\omega} \cdot \frac{1}{\sqrt{2}}[(0) + (\sqrt{2}|\psi_2\rangle|\psi_1\rangle - \sqrt{6}|\psi_3\rangle|\psi_0\rangle) + (\sqrt{6}|\psi_0\rangle|\psi_3\rangle - \sqrt{2}|\psi_1\rangle|\psi_2\rangle)$$
$$- 2(\sqrt{2}|\psi_1\rangle|\psi_2\rangle - \sqrt{2}|\psi_2\rangle|\psi_1\rangle) + (2)(|\psi_0\rangle|\psi_1\rangle - |\psi_1\rangle|\psi_0\rangle) - 2|\psi_1\rangle|\psi_0\rangle + 2|\psi_0\rangle|\psi_1\rangle$$
$$- 2|\psi_1\rangle|\psi_0\rangle + 2|\psi_0\rangle|\psi_1\rangle]$$
$$= \frac{V\hbar}{2m\omega} \cdot [-3\sqrt{2}|\psi_{1,2}\rangle + \sqrt{6}|\psi_{0,3}\rangle + 6|\psi_{0,1}\rangle].$$

Therefore the ground state energy for fermions to 1st order is $2\hbar\omega + \frac{V\hbar}{2m\omega} \cdot 6 = 2\hbar\omega + \frac{3V\hbar}{m\omega}$.

(d) We have already computed the matrx elements $\langle\psi_{n,m}|\hat{V}|\text{ground state}\rangle$ in (c). Then we can directly use the second order perturbation theory result given on page 1, for the non-degenerate ground state.

For bosons, the second order correction is $|\frac{V\hbar}{2m\omega}|^2 \cdot \frac{2^2+2^2}{-2\hbar\omega} = -\frac{V^2\hbar}{m\omega^3}$.

Therefore the ground state energy for bosons to 2nd order is $\hbar\omega + \frac{V\hbar}{m\omega} - \frac{V^2\hbar}{m\omega^3}$.

For fermions, the second order correction is $|\frac{V\hbar}{2m\omega}|^2 \cdot \frac{(-3\sqrt{2})^2+(\sqrt{6})^2}{-2\hbar\omega} = -\frac{3V^2\hbar}{m\omega^3}$.

Therefore the ground state energy for fermions to 2nd order is $2\hbar\omega + \frac{3V\hbar}{m\omega} - \frac{3V^2\hbar}{m\omega^3}$.

(e). Define $\hat{x}_{\text{com}} = \frac{\hat{x}_1+\hat{x}_2}{2}$, $\hat{p}_{\text{com}} = -\mathrm{i}\hbar\frac{\partial}{\partial x_{\text{com}}} = \hat{p}_1 + \hat{p}_2$; and $\hat{X} = \hat{x}_2 - \hat{x}_1$, $\hat{P} = -\mathrm{i}\hbar\frac{\partial}{\partial X} = \frac{1}{2}(\hat{p}_2 - \hat{p}_1)$.

Then $[\hat{x}_{\text{com}}, \hat{p}_{\text{com}}] = [\hat{X}, \hat{P}] = \mathrm{i}\hbar$, and other commutators are zero.

$$\hat{H} = \frac{\hat{p}_{\text{com}}^2}{2(2m)} + \frac{\hat{P}^2}{2(m/2)} + \frac{m\omega^2}{2}(2\hat{x}_{\text{com}}^2 + \frac{\hat{X}^2}{2}) + V\cdot\hat{X}^2$$
$$= \left[\frac{\hat{p}_{\text{com}}^2}{2(2m)} + \frac{(2m)\omega^2}{2}\hat{x}_{\text{com}}^2\right] + \left[\frac{\hat{P}^2}{2(m/2)} + \frac{(m/2)(\omega^2+\frac{4V}{m})}{2}\hat{X}^2\right].$$

This looks like two decoupled harmonic oscillators. The "center-of-mass" harmonic oscillator has mass $2m$ and frequency ω as single particle case. The "relative position" harmonic oscillator has mass $m/2$ and frequency $\sqrt{\omega^2 + 4V/m}$.

the basis of eigenstates are $\psi_{\text{com},n}(x_{\text{com}})\psi_{\text{rel},n'}(X)$, where $\psi_{\text{com},n}$ is the eigenstate for the "center-of-mass" harmonic oscillator, $\psi_{\text{rel},n'}$ is the eigenstate for the "relative position" harmonic oscillator. The energy is $\hbar\omega\cdot(n+\frac{1}{2}) + \hbar\sqrt{\omega^2+4V/m}\cdot(n'+\frac{1}{2})$.

For bosons, the wavefunction must be an even function of X, so n' must be even. The ground state is $\psi_{\text{com},0}(x_{\text{com}})\psi_{\text{rel},0}(X)$, with energy $\hbar\omega\cdot(\frac{1}{2}) + \hbar\sqrt{\omega^2+4V/m}\cdot(\frac{1}{2})$
$= \hbar\omega\cdot\frac{1}{2}\cdot(1+\sqrt{1+\frac{4V}{m\omega^2}})$
$\approx \hbar\omega\cdot\frac{1}{2}\cdot[1+1+\frac{1}{2}\frac{4V}{m\omega^2} - \frac{1}{8}(\frac{4V}{m\omega^2})^2] = \hbar\omega + \frac{V\hbar}{m\omega} - \frac{V^2}{m\omega^3}$.

For fermions, the wavefunction must be an odd function of X, so n' must be odd. The ground state is $\psi_{\text{com},0}(x_{\text{com}})\psi_{\text{rel},1}(X)$, with energy $\hbar\omega\cdot(\frac{1}{2}) + \hbar\sqrt{\omega^2+4V/m}\cdot(1+\frac{1}{2})$
$= \hbar\omega\cdot\frac{1}{2}\cdot(1+3\sqrt{1+\frac{4V}{m\omega^2}})$
$\approx \hbar\omega\cdot\frac{1}{2}\cdot[1+3+3\frac{1}{2}\frac{4V}{m\omega^2} - 3\frac{1}{8}(\frac{4V}{m\omega^2})^2] = 2\hbar\omega + \frac{3V\hbar}{m\omega} - \frac{3V^2}{m\omega^3}$.

Problem 4 (10 points) Consider two identical bosons confined on a ring of length L. The wavefunction $\psi(x_1,x_2)$ is periodic, $\psi(x_1,x_2) = \psi(x_1+L,x_2) = \psi(x_1,x_2+L)$, and symmetric

$\psi(x_1, x_2) = \psi(x_2, x_1)$. The normalization condition is $\int_0^L \mathrm{d}x_1 \int_0^L \mathrm{d}x_2\, |\psi(x_1, x_2)|^2 = 1$.

The two particles have δ-function interaction. The full Hamiltonian is $\hat{H} = -\frac{\hbar^2}{2m}\left[(\frac{\partial}{\partial x_1})^2 + (\frac{\partial}{\partial x_2})^2\right] + \frac{\hbar^2}{ma}\sum_n \delta(x_1 - x_2 - nL)$, where the sum is over all integer n, and a is a real parameter of the potential strength. Our goal is to solve the eigenstate wavefunction satisfying $\hat{H}\psi = E\psi$.

(a) (6pts) Assume the following "Bethe ansatz" form of the eigenfunction:
for $0 \le x_1 \le x_2 \le L$, $\psi(x_1, x_2) = A \cdot \left(e^{\mathbb{i}k_1 x_1} e^{\mathbb{i}k_2 x_2} + e^{\mathbb{i}\theta_{k_1,k_2}} e^{\mathbb{i}k_2 x_1} e^{\mathbb{i}k_1 x_2}\right)$;
for $0 \le x_2 \le x_1 \le L$, $\psi(x_1, x_2) = \psi(x_2, x_1) = A \cdot \left(e^{\mathbb{i}k_1 x_2} e^{\mathbb{i}k_2 x_1} + e^{\mathbb{i}\theta_{k_1,k_2}} e^{\mathbb{i}k_2 x_2} e^{\mathbb{i}k_1 x_1}\right)$;
and other cases can be obtained by the periodicity with respect to $x_{1,2}$.
Here A is the unimportant normalization constant, θ_{k_1,k_2} is a real number depending on the wavevectors k_1, k_2 only. *Derive the equations for θ_{k_1,k_2}, k_1, k_2, and energy eigenvalue E.* You will not be able to solve the final transcendental equation. [Hint: consider the periodicity, namely the boundary condition at $x_{1,2} = 0$ or L; apply $\hat{H}$ on ψ and consider the boundary condition at $x_1 = x_2$.]

(b) (4pts) For attractive interaction $a < 0$. The solution of (a) contains the "bound state" solutions, with complex k_1, k_2 and $\mathrm{Im}(k_1) < 0$ and $\mathrm{Im}(k_2) > 0$. *Derive the equation for the exact ground state energy in this case.* [Hint: you can use the result of (a), then θ_{k_1,k_2} is not real; or use the center-of-mass and relative coordinates given in the hint for Problem 3(e), be careful about the periodicity and symmetry properties]

Solution

(a) Consider $\psi(x_1, x_2)$
$= \left[\Theta(x_2 - x_1)(e^{\mathbb{i}k_1 x_1} e^{\mathbb{i}k_2 x_2} + e^{\mathbb{i}\theta_{k_1,k_2}} e^{\mathbb{i}k_2 x_1} e^{\mathbb{i}k_1 x_2}) + \Theta(x_1 - x_2)(e^{\mathbb{i}k_1 x_2} e^{\mathbb{i}k_2 x_1} + e^{\mathbb{i}\theta_{k_1,k_2}} e^{\mathbb{i}k_2 x_2} e^{\mathbb{i}k_1 x_1})\right]$,

for $0 \le x_1, x_2 \le L$. Here $\Theta(x) = \begin{cases} 1, & x > 0; \\ \frac{1}{2}, & x = 0; \\ 0, & x < 0. \end{cases}$ is the step function. $\frac{\mathrm{d}}{\mathrm{d}x}\Theta(x) = \delta(x)$.

$\psi(x_1, x_2)$ is continuous with respect to x_1 or x_2, $\psi(x_1, x_2 = x_1) = e^{\mathbb{i}(k_1+k_2)x_1} \cdot (1 + e^{\mathbb{i}\theta_{k_1,k_2}})$.

Apply $\hat{H}$ on $\psi(x_1, x_2)$. Note that
$\frac{\partial}{\partial x_1}\psi(x_1, x_2) = \Theta(x_2 - x_1)(\mathbb{i}k_1 e^{\mathbb{i}k_1 x_1} e^{\mathbb{i}k_2 x_2} + \mathbb{i}k_2 e^{\mathbb{i}\theta_{k_1,k_2}} e^{\mathbb{i}k_2 x_1} e^{\mathbb{i}k_1 x_2})$
$+\Theta(x_1 - x_2)(\mathbb{i}k_2 e^{\mathbb{i}k_1 x_2} e^{\mathbb{i}k_2 x_1} + \mathbb{i}k_1 e^{\mathbb{i}\theta_{k_1,k_2}} e^{\mathbb{i}k_2 x_2} e^{\mathbb{i}k_1 x_1})$, and

$\frac{\partial}{\partial x_2}\psi(x_1,x_2) = \Theta(x_2-x_1)(\mathbb{i}k_2 e^{\mathbb{i}k_1x_1}e^{\mathbb{i}k_2x_2} + \mathbb{i}k_1 e^{\mathbb{i}\theta_{k_1,k_2}}e^{\mathbb{i}k_2x_1}e^{\mathbb{i}k_1x_2})$
$+\Theta(x_1-x_2)(\mathbb{i}k_1 e^{\mathbb{i}k_1x_2}e^{\mathbb{i}k_2x_1} + \mathbb{i}k_2 e^{\mathbb{i}\theta_{k_1,k_2}}e^{\mathbb{i}k_2x_2}e^{\mathbb{i}k_1x_1})$.

Then $\frac{\partial^2}{\partial x_1^2}\psi(x_1,x_2) = \Theta(x_2-x_1)(-k_1^2 e^{\mathbb{i}k_1x_1}e^{\mathbb{i}k_2x_2} - k_2^2 e^{\mathbb{i}\theta_{k_1,k_2}}e^{\mathbb{i}k_2x_1}e^{\mathbb{i}k_1x_2})$
$+\Theta(x_1-x_2)(-k_2^2 e^{\mathbb{i}k_1x_2}e^{\mathbb{i}k_2x_1} - k_1^2 e^{\mathbb{i}\theta_{k_1,k_2}}e^{\mathbb{i}k_2x_2}e^{\mathbb{i}k_1x_1})$
$+\delta(x_2-x_1)\cdot\mathbb{i}(k_2-k_1)\cdot(e^{\mathbb{i}(k_1+k_2)x_1} - e^{\mathbb{i}\theta_{k_1,k_2}}e^{\mathbb{i}(k_1+k_2)x_1})$, and
$\frac{\partial^2}{\partial x_2^2}\psi(x_1,x_2) = \Theta(x_2-x_1)(-k_2^2 e^{\mathbb{i}k_1x_1}e^{\mathbb{i}k_2x_2} - k_1^2 e^{\mathbb{i}\theta_{k_1,k_2}}e^{\mathbb{i}k_2x_1}e^{\mathbb{i}k_1x_2})$
$+\Theta(x_1-x_2)(-k_1^2 e^{\mathbb{i}k_1x_2}e^{\mathbb{i}k_2x_1} - k_2^2 e^{\mathbb{i}\theta_{k_1,k_2}}e^{\mathbb{i}k_2x_2}e^{\mathbb{i}k_1x_1})$
$+\delta(x_2-x_1)\cdot\mathbb{i}(k_2-k_1)\cdot e^{\mathbb{i}(k_1+k_2)x_1}\cdot(1-e^{\mathbb{i}\theta_{k_1,k_2}})$.

Finally we get $\hat{H}\psi(x_1,x_2)$
$= \frac{\hbar^2(k_1^2+k_2^2)}{2m}\psi(x_1,x_2) + \delta(x_2-x_1)\cdot e^{\mathbb{i}(k_1+k_2)x_1}\cdot[\frac{\hbar^2}{ma}\cdot(1+e^{\mathbb{i}\theta_{k_1,k_2}}) - \frac{\hbar^2}{m}\mathbb{i}(k_2-k_1)\cdot(1-e^{\mathbb{i}\theta_{k_1,k_2}})]$.

Therefore $E = \frac{\hbar^2(k_1^2+k_2^2)}{2m}$, and $\frac{1+e^{\mathbb{i}\theta_{k_1,k_2}}}{1-e^{\mathbb{i}\theta_{k_1,k_2}}} = \mathbb{i}\cot(\theta_{k_1,k_2}/2) = \mathbb{i}(k_2-k_1)a$.

Consider the periodicity of ψ,
$\psi(0,x_2) = e^{\mathbb{i}k_2x_2} + e^{\mathbb{i}\theta_{k_1,k_2}}e^{\mathbb{i}k_1x_2} = \psi(L,x_2) = \psi(x_2,L) = e^{\mathbb{i}k_1x_2}e^{\mathbb{i}k_2L} + e^{\mathbb{i}\theta_{k_1,k_2}}e^{\mathbb{i}k_2x_2}e^{\mathbb{i}k_1L}$.
Namely $e^{\mathbb{i}k_1x_2}(e^{\mathbb{i}k_2L} - e^{\mathbb{i}\theta_{k_1,k_2}}) - e^{\mathbb{i}k_2x_2}(1-e^{\mathbb{i}\theta_{k_1,k_2}}e^{\mathbb{i}k_1L}) = 0$.
For this to be true for any x_2, we must have $e^{\mathbb{i}\theta_{k_1,k_2}} = e^{\mathbb{i}k_2L} = e^{-\mathbb{i}k_1L}$.

Finally,
$E = \frac{\hbar^2}{2m}[k_1^2+k_2^2]$.
$k_1 = \frac{2\pi}{L}n - k_2$, where n is an integer.
$\theta_{k_1,k_2} = -k_1L \mod (2\pi) = k_2L \mod (2\pi)$.
$\cot(\frac{k_2L}{2}) = (2k_2 - \frac{\pi}{L}n)\cdot a$.

(b) Method #1: Use the result of (a).

Define $k_+ = \frac{k_1+k_2}{2} = \frac{\pi}{L}n$, and $k_- = \frac{k_2-k_1}{2}$. Then
$E = \frac{\hbar^2}{m}(k_+^2+k_-^2)$, $\theta_{k_1,k_2} = k_-L + n\pi$, $\cot(\frac{k_-L}{2}+\frac{n\pi}{2}) = k_-\cdot a$.

$$\psi(x_1,x_2) = \begin{cases} e^{\mathbb{i}k_+(x_1+x_2)}\cdot(e^{\mathbb{i}k_-(x_2-x_1)} + e^{\mathbb{i}\theta_{k_1,k_2}}e^{\mathbb{i}k_-(x_1-x_2)}), & 0\le x_1 < x_2 \le L; \\ e^{\mathbb{i}k_+(x_1+x_2)}\cdot(e^{\mathbb{i}k_-(x_1-x_2)} + e^{\mathbb{i}\theta_{k_1,k_2}}e^{\mathbb{i}k_-(x_2-x_1)}), & 0\le x_2 < x_1 \le L. \end{cases}$$
$= e^{\mathbb{i}k_+(x_1+x_2)}\cdot(e^{\mathbb{i}k_-\cdot|x_2-x_1|} + e^{\mathbb{i}\theta_{k_1,k_2}}e^{-\mathbb{i}k_-\cdot|x_1-x_2|})$, for $0\le x_1,x_2\le L$.

For $a<0$, the bound state will have $k_+ = 0$ ($n=0$), and pure imaginary $k_- = \mathbb{i}\kappa$.
The ground state energy is $-\frac{\hbar^2\kappa^2}{2m}$. And $\coth(\frac{\kappa L}{2}) = -\kappa\cdot a$.
There is always one solution to this equation.

Method #2: use the "center-of-mass" and "relative" coordinates defined in problem 3(e).

$\hat{H} = \frac{\hat{p}_{\text{com}}^2}{2(2m)} + \frac{\hat{P}^2}{2(m/2)} + \frac{\hbar^2}{ma}\sum_n \delta(X - nL)$.

The eigenfunction is $e^{\mathbb{i}p_{\text{com}}x_{\text{com}}/\hbar} \cdot \phi(X)$, where $\phi(X)$ is the eigenfunction of $\frac{\hat{P}^2}{2(m/2)} + \frac{\hbar^2}{ma}\sum_n \delta(X - nL)$.

From $\psi(x_1 + L, x_2 + L) = \psi(x_1, x_2)$, we have $p_{\text{com}}L/\hbar = 2\pi \cdot n$ where n is an integer.

From $\psi(x_1, x_2 + L) = \psi(x_1, x_2)$, we have $\phi(X + L) \cdot e^{\mathbb{i}\pi n} = \phi(X)$.

From $\psi(x_2, x_1) = \psi(x_1, x_2)$, we have $\phi(-X) = \phi(X)$.

The ground state will have zero center-of-mass momentum ($n = 0$). So we just need to solve a periodic even eigenfunction $\phi(X) = \phi(-X) = \phi(X + L)$.

Suppose the energy eigenvalue is $E = -\frac{\hbar^2}{2m}\kappa^2$.

Then for $0 < X < L$, we can choose $\phi(X) = e^{-\kappa X} + e^{-\kappa(L-X)} = 2e^{-\kappa L/2}\cosh(\kappa(X - \frac{L}{2}))$.

By periodicity, for $-L < X < 0$, $\phi(X) = 2e^{-\kappa L/2}\cosh(\kappa(X + \frac{L}{2}))$.

The boundary condition at $X = 0$ is $-\frac{\hbar^2}{2m}(\frac{\partial\phi}{\partial X}|_{X=0+} - \frac{\partial\phi}{\partial X}|_{X=0-}) + \frac{\hbar^2}{ma}\phi|_{X=0} = 0$, this is $-\frac{\hbar^2}{2m} \cdot (-4e^{-\kappa L/2}\kappa\sinh(\kappa\frac{L}{2})) + \frac{\hbar^2}{ma} \cdot 2e^{-\kappa L/2}\cosh(\kappa\frac{L}{2})$, or $\coth(\frac{\kappa L}{2}) = -\kappa a$.