刘川 理论力学 A 2023 春 期中考试参考答案（民间版）

1.

坐标：

$$\begin{cases} x_1 = l\sin\phi_1\,, \qquad y_1 = l\cos\phi_1 \\ x_2 = l(\sin\phi_1 + \sin\phi_2), \qquad y_2 = l(\cos\phi_1 + \cos\phi_2) \end{cases}$$

速度：

$$\begin{cases} v_{x1} = l\cos\phi_1\,\dot{\phi}_1, \qquad v_{y1} = -l\sin\dot{\phi}_1 \\ v_{x2} = l\big(\cos\phi_1\,\dot{\phi}_1 + \cos\phi_2\,\dot{\phi}_2\big), \qquad v_{y1} = -l\big(\sin\phi_1\,\dot{\phi}_1 + \sin\phi_2\,\dot{\phi}_2\big) \end{cases}$$

拉氏量（直接写成化简后的）：

$$L = T - V = \frac{1}{2}ml^2\left(2\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2\right) - \frac{1}{2}mgl(2\phi_1^2 + \phi_2^2) + Const.$$

同动能的展开方式，耗散函数可以写作：

$$\mathcal{F} = \frac{1}{2}kl^2\left(2\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2\right)$$

运动方程：

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}_1}\right) = ml^2\big(2\ddot{\phi}_1 + \ddot{\phi}_2\big), \qquad \frac{\partial L}{\partial\phi_1} = -2mgl\phi_1, \qquad \frac{\partial\mathcal{F}}{\partial\dot{\phi}_1} = kl^2\big(2\dot{\phi}_1 + \dot{\phi}_2\big)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}_2}\right) = ml^2\big(\ddot{\phi}_1 + \ddot{\phi}_2\big), \qquad \frac{\partial L}{\partial\phi_2} = -mgl\phi_2, \qquad \frac{\partial\mathcal{F}}{\partial\dot{\phi}_2} = k\big(\dot{\phi}_1 + \dot{\phi}_2\big)$$

从而运动方程为：

$$\begin{cases} ml^2\big(2\ddot{\phi}_1 + \ddot{\phi}_2\big) + 2mgl\phi_1 + \ kl^2\big(2\dot{\phi}_1 + \dot{\phi}_2\big) = 0 \\ ml^2\big(\ddot{\phi}_1 + \ddot{\phi}_2\big) + mgl\phi_2 + \ kl^2\big(\dot{\phi}_1 + \dot{\phi}_2\big) = 0 \end{cases}$$

2.

(a)

长度微元：

$$ds^2 = R^2 d\theta^2 + R^2\sin^2\theta\, d\phi^2$$

写成矩阵形式：

$$ds^2 = (d\theta \quad d\phi)\begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}\begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

度规的协变-协变形式：

$$g_{ij} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}$$

逆变-逆变形式：

$$g^{ij} = \begin{pmatrix} 1/R^2 & 0 \\ 0 & 1/(R^2\sin^2\theta) \end{pmatrix}$$

(b)

仅有$g_{22,1}$分量不为零：

$$g_{22,1} = 2R^2 \sin\theta \cos\theta$$

(c)

总长度：

$$S = \int_0^{s_0} \sqrt{g_{ij}\dot{q}^i\dot{q}^j}\, ds$$

$$\delta S = \int_0^{s_0} \frac{1}{2} \frac{g_{ij,k}\delta q^k \dot{q}^i \dot{q}^j + g_{ij}\frac{d}{ds}(\delta q^i)\dot{q}^j + g_{ij}\dot{q}^i \frac{d}{ds}(\delta q^j)}{\sqrt{g_{ij}\dot{q}^i\dot{q}^j}} ds$$

合并同类项，并注意到分母为 1：

$$\delta S = \int_0^{s_0} \frac{1}{2}\left(g_{ij,k}\delta q^k \dot{q}^i \dot{q}^j + 2g_{ij}\frac{d}{ds}(\delta q^i)\dot{q}^j\right) ds$$

式中

$$\int_0^{s_0} g_{ij}\frac{d}{ds}(\delta q^i)\dot{q}^j ds$$

$$= \int_0^{s_0} g_{ij}\dot{q}^j\, d(\delta q^i)$$

$$= g_{ij}\dot{q}^j \delta q^i\Big|_0^{s_0} - \int_0^{s_0} g_{ij}\frac{d^2 q^j}{ds^2}\delta q_i\, ds - \int_0^{s_0} g_{ij,k}\dot{q}^j \dot{q}^k\, \delta q_i\, ds$$

交换傀标可得

$$\int_0^{s_0} g_{ij,k}\delta q^k \dot{q}^i \dot{q}^j\, ds = \int_0^{s_0} g_{jk,i}\delta q^i \dot{q}^j \dot{q}^k\, ds$$

故

$$\delta S = \int_0^{s_0} \left(\frac{1}{2}g_{jk,i}\dot{q}^j\dot{q}^k - g_{ij}\frac{d^2 q^j}{ds^2} - g_{ij,k}\dot{q}^k\dot{q}^j\right)\delta q_i\, ds$$

由δq_i的任意性得

$$g_{ij}\frac{d^2 q^j}{ds^2} = \frac{1}{2}g_{jk,i}\dot{q}^j\dot{q}^k - g_{ij,k}\dot{q}^j\dot{q}^k, \qquad i = 1,2,3,\dots,n$$

这就是所求的方程。

(d)

注意到仅当$(i,j,k) = (2,2,1)$时有$g_{ij,k} \neq 0$，易得球面上的方程为

$$\begin{cases} \ddot{\theta} = \sin\theta\cos\theta\,\dot{\phi}^2 \\ \ddot{\phi} = -2\cot\theta\,\dot{\theta}\dot{\phi} \end{cases}$$

3.

(a)

显然

$$\Omega = \sqrt{\frac{G(M_1 + M_2)}{R^3}}$$

(b)

显然

$$\begin{cases} \overrightarrow{x_1}(t) = \dfrac{M_2 R}{M_1 + M_2} \cdot (\cos \Omega t \quad \sin \Omega t)^T \\ \overrightarrow{x_2}(t) = -\dfrac{M_1 R}{M_1 + M_2} \cdot (\cos \Omega t \quad \sin \Omega t)^T \\ \vec{X}(t) = R \cdot (\cos \Omega t \quad \sin \Omega t)^T \end{cases}$$

(c)

记

$$r_1 = \frac{M_2 R}{M_1 + M_2}, \qquad r_2 = \frac{M_1 R}{M_1 + M_2}$$

在方位角θ处的势能：

$$V(r,\theta) = -\frac{GM_1 m}{\sqrt{r^2 + r_1^2 - 2rr_1 \cos\theta}} - \frac{GM_2 m}{\sqrt{r^2 + r_2^2 + 2rr_2 \cos\theta}}$$

使用展开式

$$(1 - 2x\cos\theta + x^2)^{-1/2} = 1 + \cos\theta \cdot x + \frac{3\cos^2\theta - 1}{2} \cdot x^2 + o(x^2)$$

可得

$$V(r,\theta) = -\frac{Gm}{r}\left((M_1 + M_2) + \cos\theta \cdot \frac{M_1 r_1 - M_2 r_2}{r} + \frac{3\cos^2\theta - 1}{2} \cdot \frac{M_1 r_1^2 + M_2 r_2^2}{r^2}\right) + o\left(\frac{R^2}{r^2}\right)$$

求平均：

$$\bar{V}(r) = \frac{1}{2\pi}\int_0^{2\pi} V(r,\theta)\, d\theta$$

注意到

$$\int_0^{2\pi} \cos\theta \; d\theta = 0$$

且

$$\frac{M_1 r_1^2 + M_2 r_2^2}{r^2} = \frac{M_1 M_2}{M_1 + M_2} = \mu$$

积分得

$$\bar{V}(r) = -\frac{G(M_1+M_2)m}{r} - \frac{G\mu m R^2}{4r^3}$$

接下来计算角度：

$$\delta\phi = \frac{\partial}{\partial J}\left(\frac{2m}{J}\int_0^{\pi} r^2 \delta V(r)\, d\phi\right)$$

代入$\delta V(r)$的具体形式：

$$\delta\phi = \frac{\partial}{\partial J}\left(\frac{2m}{J}\int_0^{\pi} -\frac{G\mu m R^2}{4r}\, d\phi\right)$$

再代入

$$\frac{l_0}{r} = 1 + e\cos\phi$$

可得

$$\begin{aligned}\delta\phi &= \frac{\partial}{\partial J}\left(\frac{2m}{J}\int_0^{\pi} -\frac{G\mu m R^2}{4l_0}(1+e\cos\phi)\, d\phi\right)\\ &= \frac{\partial}{\partial J}\left(\frac{-\pi G\mu m^2 R^2}{2Jl_0}\right)\end{aligned}$$

代入

$$l_0 = \frac{J^2}{G(M_1+M_2)m^2}$$

整理计算得

$$\delta\phi = \frac{3\pi}{2}\left(\frac{R}{l_0}\right)^2 \frac{M_1 M_2}{(M_1+M_2)^2}$$

讨论：当$l_0 \gg R$时，近似较好。

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