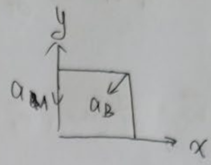
一、

$$a_A = a_0 + \varepsilon \times r_A + \omega \times (\omega \times r_A)$$

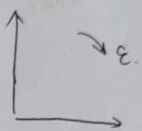
$$a_B = a_0 + \varepsilon \times r_B + \omega \times (\omega \times r_B)$$



故 $\vec{a}_B - \vec{a}_A = \varepsilon \times \vec{r}_{AB} + \omega \times (\omega \times \vec{r}_{AB})$

又因为 $\vec{a}_B - \vec{a}_A = (-4, -2)\ cm/s^2$

故可知: $\begin{cases} \varepsilon = 1\ \cancel{cm}\ s^{-2} \\ \omega = \sqrt{2}\ \ s^{-1}. \end{cases}$ 方向:



对于C点,

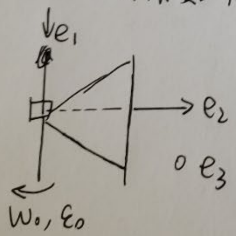
$$\vec{a}_C - \vec{a}_B = \varepsilon \times \vec{r}_{BC} + \omega \times (\omega \times \vec{r}_{BC}) = (-2, 4)\ cm/s^2$$

故 $\vec{a}_C = (-6, 0)\ cm/s^2$, $a_C = 6\ cm/s^2$.

二、在图中, $\angle BOA = 2\alpha$

设 $OA = r$, 则 $AC = r\tan 2\alpha$

设动坐标系如下, 则有:



$$\vec{\omega} = \omega_0 e_1 + \omega_1 e_2$$

其中, $\omega_1 \cdot r\tan 2\alpha - \omega_0 \cdot r = 0$

$$\Rightarrow \omega_1 = \frac{\omega_0}{\tan 2\alpha}$$

故 $\vec{\omega} = \omega_0 \vec{e}_1 + \dfrac{\omega_0}{\tan 2\alpha} \vec{e}_2$ 为角速度.

$$|\vec{\omega}| = \omega_0 \frac{1}{\sin 2\alpha}$$

对 $\vec{\omega} = \omega_0 \vec{e_1} + \frac{\omega_0}{\tan 2\alpha} \vec{e_2}$ 两边求导，

并注意：$\frac{d}{dt}\vec{e_1} = 0$，$\frac{d}{dt}\vec{e_2} = \omega_0 \vec{e_3}$，$\frac{d}{dt}\omega_0 = \varepsilon_0$

故

$$\vec{\varepsilon} = \frac{d}{dt}\vec{\omega} = \frac{d}{dt}\left[\omega_0\left(\vec{e_1} + \frac{\vec{e_2}}{\tan 2\alpha}\right)\right] = \varepsilon_0\left(\vec{e_1} + \frac{\vec{e_2}}{\tan 2\alpha}\right) + \omega_0 \cdot \frac{\omega_0}{\tan 2\alpha}\vec{e_3}$$

$$= \varepsilon_0 \vec{e_1} + \frac{\varepsilon_0}{\tan 2\alpha}\vec{e_2} + \frac{\omega_0^2}{\tan 2\alpha}\vec{e_3}$$

$$|\vec{\varepsilon}| = \sqrt{\varepsilon_0^2 \frac{1}{\sin^2 2\alpha} + \frac{\omega_0^4}{\tan^2 2\alpha}}$$ 为角加速度.

三、

由角速度公式，

$$w = \dot{\psi}\cdot\vec{z} + \dot{\theta}\cdot\vec{m} + \dot{\varphi}\cdot\vec{\zeta}$$

其中 $[x \quad y \quad z]\begin{bmatrix}\cos\psi & -\sin\psi & \\ \sin\psi & \cos\psi & \\ & & 1\end{bmatrix} = [m \quad n \quad z]$

$$[m \quad n \quad z]\begin{bmatrix}1 & & \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta\end{bmatrix} = [m \quad s \quad t]$$

$$[m \quad s \quad t]\begin{bmatrix}\cos\varphi & -\sin\varphi & \\ \sin\varphi & \cos\varphi & \\ & & 1\end{bmatrix} = [\xi \quad \eta \quad \zeta]$$

$\therefore \vec{\zeta} = -\sin\theta\cdot n + \cos\theta\cdot z$

$= -\sin\theta\cdot(-x\sin\psi + y\cos\psi) + z\cdot\cos\theta$

$\vec{m} = x\cos\psi + y\sin\psi$

故 $w = \dot{\psi}\vec{z} + \dot{\theta}(\cos\psi\,\vec{x} + \sin\psi\,\vec{y}) + \dot{\varphi}(\sin\theta\sin\psi\cdot\vec{x} - \sin\theta\cos\psi\cdot\vec{y} + \cos\theta\cdot\vec{z})$

$$= [x \quad y \quad z]\begin{bmatrix}\sin\theta\sin\psi\,\dot{\varphi} + \cos\psi\,\dot{\theta} \\ -\sin\theta\cos\psi\,\dot{\varphi} + \sin\psi\,\dot{\theta} \\ \dot{\psi} + \dot{\varphi}\cos\theta\end{bmatrix}.$$

在空间坐标系下，$w = \begin{bmatrix}\frac{\sqrt{3}}{2}\sin(\frac{\pi}{2}+ant)\cdot n \\ -\frac{\sqrt{3}}{2}\cos(\frac{\pi}{2}+ant)\cdot n \\ \frac{n}{2} + an\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{3}}{2}\cos(ant)\cdot n \\ \frac{\sqrt{3}}{2}\sin(ant)\cdot n \\ \frac{n}{2} + an\end{bmatrix}$

角加速度 $\varepsilon = \dot{w} = \begin{bmatrix}-\frac{\sqrt{3}}{2}\sin(ant)\cdot an^2 \\ \frac{\sqrt{3}}{2}\cos(ant)\cdot an^2 \\ 0\end{bmatrix}$

四. 设 OC 长度为 x, 则有 $\begin{cases} \dot{x}(0) = -u \\ \ddot{x}(0) = -a_0 \\ x(0) = 2r \end{cases}$

则 φ 满足:

$$\sin\varphi = \frac{r}{x}.$$

则有: 一阶导 $\cos\varphi \cdot \dot{\varphi} = -\frac{r}{x^2}\dot{x}$

$\varphi(0) = 30^\circ$, $x(0) = 2r$

则 $\frac{\sqrt{3}}{2}\dot{\varphi} = -\frac{1}{4r}\cdot(-u)$

$$\dot{\varphi} = \frac{u}{2\sqrt{3}r}$$

二阶导 $-\sin\varphi\cdot(\dot{\varphi})^2 + \cos\varphi\cdot\ddot{\varphi} = \frac{2r}{x^3}\dot{x}^2 - \frac{r}{x^2}\ddot{x}$

代入, 得 $-\frac{1}{2}\dot{\varphi}^2 + \frac{\sqrt{3}}{2}\ddot{\varphi} = \frac{u^2}{4r^2} + \frac{a_0}{4r}$

$$\frac{\sqrt{3}}{2}\ddot{\varphi} = \frac{u^2}{r^2}\left(\frac{1}{4} + \frac{1}{24}\right) + \frac{a_0}{4r} = \frac{7}{24}\frac{u^2}{r^2} + \frac{a_0}{4r}$$

$$\ddot{\varphi} = \frac{1}{\sqrt{3}}\left(\frac{7}{12}\frac{u^2}{r^2} + \frac{a_0}{2r}\right)$$

故角速度 $\dot{\varphi} = \frac{u}{2\sqrt{3}r}$

角加速度 $\ddot{\varphi} = \frac{1}{\sqrt{3}}\left(\frac{7}{12}\frac{u^2}{r^2} + \frac{a_0}{2r}\right)$

五：(1) 以单摆顶点为重力势能零点，

则前段，人 $E = E_{p,初始} = -\frac{\sqrt{3}}{2} l \cdot mg$

总机械能

(2) 在最低处时，能量守恒：

$$\frac{1}{2}mv^2 = mgl(1-\frac{\sqrt{3}}{2})$$

$$v = \sqrt{gl(2-\sqrt{3})} = \frac{\sqrt{6}-\sqrt{2}}{2}\sqrt{gl}$$

则动量矩 $L = mv \cdot L = m\sqrt{gl(2-\sqrt{3})} \cdot \frac{1}{2}l$

$$= \frac{1}{2}ml\sqrt{gl(2-\sqrt{3})} = \frac{\sqrt{6}-\sqrt{2}}{4} ml\sqrt{gl}$$

(3) 由于动能、势能均不变，故总机械能 $E_0 = -\frac{\sqrt{3}}{2} l \cdot mg$

六.

(1) 代入 Binet 方程，

$\frac{d^2}{d\varphi^2}u + u = -\frac{m}{m^2h^2}\frac{1}{u^2}F(u)$，其中 $u = \frac{1}{r}$，$h = r \cdot v = ka \cdot a = ka^2$

$$\therefore \frac{d^2}{d\varphi^2}u + u = \frac{m}{m^2 \cdot k^2a^4} u^{-2} \cdot mk^2a^4 \cdot u^3 = u$$

$$\Rightarrow \frac{d^2}{d\varphi^2}u = 0,\quad u = A\varphi + B. \Rightarrow r = \frac{1}{A\varphi+B}$$

而 $\dot{r} = -\frac{A\dot{\varphi}}{(A\varphi+B)^2}$，$\dot{r}(0) = \frac{-ka}{\pi}$，$\dot{\varphi}(0) = k$

代入，得 $\begin{cases} A \cdot \pi + B = \frac{1}{a} \\ A = \frac{1}{\pi a} \end{cases} \Rightarrow \begin{cases} A = \frac{1}{\pi a} \\ B = 0 \end{cases}$，轨迹为 $r = \frac{\pi a}{\varphi}$

(2) 由能量守恒，势能 $V(r) = -\frac{mk^2a^4}{2r^2}$，初始机械能 $E_0 = \frac{mk^2a^2}{2\pi^2}$

故有：

$$-\frac{mk^2a^4}{2r^2} + \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) = \frac{mk^2a^2}{2\pi^2}，且\ \dot{r} = -\frac{\pi a}{\varphi^2}\dot{\varphi}$$

$$\Rightarrow -\frac{mk^2a^4}{2r^2} + \frac{m}{2}(\dot{r}^2 + \dot{r}^2\frac{\pi^2a^2}{r^2}) = \frac{mk^2a^2}{2\pi^2}$$

两边求导得：

$$\frac{k^2a^4}{r^3}+\ddot{r}\left(1+\frac{\pi^2a^2}{r^2}\right)-\dot{r}^2\frac{\pi^2a^2}{r^3}=0$$

~~可解得 $r(t)=$~~

且 ~~$\varphi(t)$ 与 $r(t)$ 满足~~

$$\cancel{\varphi(t)=\frac{\pi a}{r(t)}}$$

且由角动量守衡，

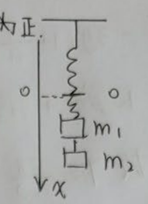
$$r^2\dot{\varphi}=ka^2,\quad r=\frac{\pi a}{\varphi}$$

$$\Rightarrow \dot{\varphi}=\frac{k}{\pi^2}\varphi^2 \Rightarrow \frac{1}{\pi}-\frac{1}{\varphi}=\frac{k}{\pi^2}t$$

故 $\varphi(t)=\dfrac{1}{\frac{1}{\pi}-\frac{k}{\pi^2}t}$，$r(t)=\pi a\left(\frac{1}{\pi}-\frac{k}{\pi^2}t\right)$

$$=a-\frac{ak}{\pi}t.$$

七、以k悬挂m_1时的平衡位置为O点，向下为正。



则运动方程为：

$$m_1\ddot{x}+kx=0$$

且 $\begin{cases} x(0)=\dfrac{m_2 g}{k} \\ \dot{x}(0)=0 \end{cases}$

则 $x=A\sin(\sqrt{\tfrac{k}{m_1}}\,t)+B\cos(\sqrt{\tfrac{k}{m_1}}\,t)$

$\quad = A\sin(2t)+B\cos(2t)$

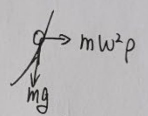
且 $\begin{cases} x(0)=0.4\,m \\ \dot{x}(0)=0 \end{cases} \Rightarrow x=0.4\cos(2t)$ （单位：m）

频率：$f=\dfrac{\omega}{2\pi}=\dfrac{1}{\pi}\ s^{-1}$

圆频率：$\omega=2\ s^{-1}$

周期：$T=\dfrac{2\pi}{\omega}=\pi\ s$

八、(1) 匀速圆周运动时，有：



$$\frac{m\omega^2\rho}{mg}=\tan\theta$$

$$=\frac{dz}{d\rho}=\frac{1}{2a^3}(2a^2\rho+4\rho^3)=\frac{\rho}{a}+2\frac{\rho^3}{a^3}$$

故 $\omega=\sqrt{\dfrac{3g}{a}}$，　$v=\omega\cdot a=\sqrt{3ga}$

(2) 其角动量守恒，有：$\omega\rho^2=\sqrt{3ga^3}$

并由能量守恒，

$$\frac{m}{2}(\dot{\rho}^2+\dot{z}^2+\omega^2\rho^2)+z\cdot mg=E_0，且\ \dot{z}=\left(\frac{\rho}{a}+2\frac{\rho^3}{a^3}\right)\dot{\rho}$$

故为：$\dfrac{m}{2}\left(\dot{\rho}^2+\dot{\rho}^2\left(\dfrac{\rho}{a}+2\dfrac{\rho^3}{a^3}\right)^2+\dfrac{3ga^3}{\rho^2}\right)+mg\dfrac{1}{2a^3}(a^2\rho^2+\rho^4)=E_0$

两边求导，得：（认为$\dot{\rho}$、$\ddot{\rho}$与$\rho-a$是一阶小量，并只保留一阶）

$$\ddot{\rho}\left(1+\left(\frac{\rho}{a}+2\frac{\rho^3}{a^3}\right)^2\right)-\frac{3ga^3}{\rho^3}+g\left(\frac{\rho}{a}+2\frac{\rho^3}{a^3}\right)=0$$

之后，令 $\rho-a=x$，并作一阶近似：

$$10\ddot{x} - 3g\left(1+\frac{x}{a}\right)^{-3} + g\left(1+\frac{x}{a} + 2\left(1+\frac{x}{a}\right)^{3}\right) = 0$$

∴有：$10\ddot{x} + 16\frac{g}{a}x = 0$，$\omega = \sqrt{\frac{16g}{10a}} = \frac{4\sqrt{g}}{\sqrt{10a}} = \frac{2}{5}\sqrt{10}\sqrt{\frac{g}{a}}$

故 $T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{16g}{10a}}} = \frac{\sqrt{10}}{2}\pi \cdot \sqrt{\frac{a}{g}}$