# Strategyproofness-Exposing Descriptions of Matching Mechanisms*

Yannai A. Gonczarowski† Ori Heffetz‡ Clayton Thomas§

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**Abstract**

A *menu description* exposes strategyproofness by presenting a mechanism to player $i$ in two steps. Step (1) uses others' reports to describe $i$'s *menu* of potential outcomes. Step (2) uses $i$'s report to select $i$'s favorite outcome from her menu. How can canonical matching mechanisms be explained using menu descriptions? We provide simple menu descriptions of Deferred Acceptance (DA) and Top Trading Cycles (TTC). For TTC, our description additionally exposes a simple argument for the strategyproofness of the traditional description. For DA, we prove that this is impossible, and that simple descriptions cannot convey both strategyproofness and the full matching.

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†Department of Economics and Department of Computer Science, Harvard University — *E-mail*: yannai@gonch.name.

‡Johnson Graduate School of Management, Cornell University, Bogen Department of Economics and Federmann Center for Rationality, The Hebrew University of Jerusalem, and NBER — *E-mail*: oh33@cornell.edu.

§Microsoft Research — *E-mail*: thomas.clay95@gmail.com.

# 1 Introduction

Strategyproof mechanisms are often considered desirable. Under standard economic assumptions, these mechanisms eliminate the need for players to strategize, since straightforward play is a dominant strategy.[1] In practice, however, real participants in strategyproof mechanisms often play theoretically dominated strategies, raising the possibility that they do not perceive the mechanisms as strategyproof.[2]

In this paper, we posit that the way mechanisms are *described* can influence the extent to which participants perceive strategyproofness. In contrast to other recent works, which have sought to implement a given choice rule via different interactive mechanisms in order to discourage non-straightforward play arising from behavioral factors,[3] we propose only changing the ex ante description of the static, direct-revelation mechanism. We propose a general outline—called *menu descriptions*—for describing a mechanism to one player at a time in a way that exposes strategyproofness. Namely, strategyproofness of menu descriptions holds via an elementary, one-sentence argument.

Our focus is on matching, and particularly on two canonical mechanisms: the stable Deferred Acceptance (henceforth DA) and the Pareto efficient Top Trading Cycles (henceforth TTC). While these mechanisms are quite successful, we posit that their traditional *outcome descriptions*—detailed and explicit algorithms for calculating the matching outcome—may not expose strategyproofness, since demonstrating this property from these descriptions is conventionally done using delicate and technical mathematical arguments.

Our paper has four main results. The first two present novel menu descriptions of DA and TTC, which expose their strategyproofness while maintaining complexity comparable to that of the traditional descriptions. For TTC, our description is sufficiently similar to the traditional description that it also yields a simple proof that the traditional description is strategyproof. Our third and fourth main results are

[1] We use the term "straightforward" to describe the strategy an agent would play under classic economic assumptions. While often referred to as the "truthtelling" strategy, we avoid this morally laden term, since deviations from this strategy should not be thought of as dishonesty.

[2] Evidence of such non-straightforward behavior comes both from the lab and the field (where participants are typically explicitly informed of the mechanism's strategyproofness). For surveys of the literature, see Hakimov and Kübler (2021); Rees-Jones and Shorrer (2023).

[3] These factors include contingent-reasoning failures (Li, 2017; Pycia and Troyan, 2023) and expectations-based loss aversion (Dreyfuss et al., 2022; Meisner and von Wangenheim, 2023).

negative findings for DA. They formally show that (a) in contrast to TTC, *no* menu description of DA yields a simple proof for the strategyproofness of its traditional description, and (b) that simple descriptions of DA face a trade-off between conveying the outcome matching and conveying strategyproofness. This illustrates the strategic complexity of DA compared to TTC, and further motivates our menu description.

Before proceeding, consider a foundational, illustrative example: the Serial Dictatorship (henceforth, SD) mechanism. When matching applicants to institutions,[4] the traditional description of SD is as follows: in some order, say $i = 1, \dots, n$, applicant $i$ is matched to her highest-ranked not-yet-matched institution. Strategyproofness is quite evident from this description: Applicant $i$ cannot influence the set of not-yet-matched institutions, and straightforward reporting guarantees $i$ her favorite not-yet-matched institution. Our paper asks: To what extent can we find descriptions of other mechanisms that make strategyproofness as evident as in SD?

Our paper begins in Section 2 with preliminaries. We study descriptions in terms of the classic notion of a *menu* (Hammond, 1979)—the set of all institutions an applicant might match to, given others' reports. In particular, a *menu description* for applicant $i$ has the following two-step outline:

**Step (1)** uses only the reports of other applicants to describe $i$'s menu.

**Step (2)** describes $i$'s match as her highest-ranked institution from her menu.

In contrast with some traditional descriptions, strategyproofness can be readily seen from any menu description: Applicant $i$ cannot effect her menu in Step (1), and straightforward reporting guarantees $i$ her favorite possible institution in Step (2).

In Section 3, we present our first main result: A novel menu description of DA, summarized in Table 1. It describes applicant $i$'s menu as all institutions that prefer $i$ to their outcome in "flipped-side-proposing" deferred acceptance, excluding $i$.

In many ways, our menu description in Table 1 is comparable in complexity to the traditional description of DA, an important consideration for use with real-world participants.[5] rior to our work, it was not clear how to construct DA's menu, except via a trivial solution involving running the traditional description many times to

[4] In all mechanisms we consider, the only strategic players are the applicants. The institutions are not strategic, and thus their preferences over the applicants are by convention called *priorities*. We consider one-to-one matching for concreteness, but our results generalize substantially; see Section 3.

[5] We also note that Table 1 gives a useful theoretical tool for reasoning about DA. For example, it immediately implies that when one applicant's priority improves at a set of institutions, her match cannot become worse (Balinski and Sönmez, 1999).

Table 1: Two equivalent descriptions of DA (the applicant-optimal stable match)

| Traditional Descr.: | Menu Description: |
|---|---|
| The applicants and institutions will be matched using the *applicant-proposing deferred acceptance* algorithm. | We will run *institution-proposing deferred acceptance* with all applicants *except you*, to obtain a hypothetical matching. Your menu consists of every institution that ranks you higher than its hypothetically matched applicant.<br>You will be matched to the institution that you *ranked highest* out of your menu. |

**Note:** In the menu description, others' hypothetical matches need not be their matches in DA.

separately check if different institutions are on $i$'s menu—an approach to describing the mechanism that might be viewed as complicated, implausible, or confusing.

In Section 4, we present our second main result: A novel menu description of TTC, which is in fact contained within a slight tweak of the traditional description. In particular, it features an extra third step after the menu description for applicant $i$:

**Step (1)** uses only the reports of other applicants to describe $i$'s menu.

**Step (2)** describes $i$'s match as her highest-ranked institution from her menu.

**Step (3)** describes the rest of the matching (for all other applicants).

We call a description with this three-step outline—i.e., an outcome description that contains a menu description—an *individualized dictatorship*.

Our new description of TTC gives a simple, intuitive proof that TTC is strategyproof. The traditional description of TTC works in terms of "eliminating trading cycles," and it is not hard to argue that these cycles can be eliminated in any order. Our new description differs from the traditional one only by changing the order in which cycles are eliminated (eliminating the cycle involving applicant $i$ as late as possible). Thus, the match of applicant $i$ equals her match in a menu description, making TTC's strategyproofness clear.[6]

We find our menu description of DA (Section 3) nearly as appealing as our individualized dictatorship for TTC (Section 4). However, one may wonder: Like TTC,

[6] Following the appearance of our paper, the survey article Morrill and Roth (2024) used our individualized dictatorship to give a self-contained proof that TTC is strategyproof. Regarding the potential real-world adoption of TTC for public school choice, Morrill and Roth write:

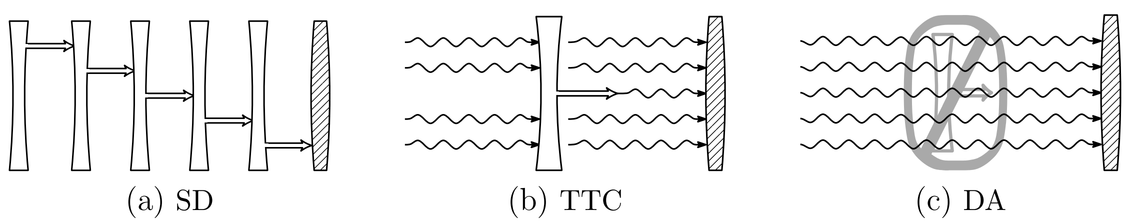
> "Our experience [. . . ] taught us that when we worked with school districts, we should help design not just a mechanism, but also the communication package that explained that mechanism [. . . ]. Perhaps if we had already known of the proof of [Gonczarowski, Heffetz, and Thomas] we could have explained [TTC's strategyproofness] more clearly."

does DA have an individualized dictatorship that is a slight tweak of, and results in an intuitive proof of the strategyproofness of, its traditional description?

In Section 5, we present our third main result: that the answer to the above question is a robust *no*. Consider a permissive view of which descriptions are a "slight tweak of the traditional description of DA." First, they are "applicant-proposing"—i.e., ones that look at applicants' preferences in favorite-to-least-favorite order. Second, they construct the outcome by keeping track of (and iteratively modifying) a single tentative matching, without performing more complicated bookkeeping—we capture (a flexible, necessary condition for) this property by considering descriptions that keep track of an amount of memory roughly linear in the number of applicants. We prove that, in contrast to (SD and) TTC, no individualized dictatorship for DA satisfies these two properties; hence, no individualized dictatorship for DA is a slight tweak of its traditional description. In fact, we prove that any applicant-proposing individualized dictatorships for DA must use *quadratic* memory, roughly equivalent to keeping track of a separate matching for each applicant.

These second and third main results complete a striking trichotomy among SD, TTC, and DA. In SD, strategyproofness is exposed "automatically." For TTC, this is not the case, but a slight tweak of the traditional description suffices to expose strategyproofness for one applicant. For DA, a comparable result is impossible, in the precise and strong sense discussed above. See Figure 1 for an illustration.

Figure 1: Trichotomy for (tweaks of) traditional descriptions of SD, TTC, and DA




**Notes:** Each figure depicts an applicant-proposing and linear-memory description. Each description progresses from left to right and calculates the outcome matching, which is depicted as a shaded rectangle. Light rectangles and doubled arrows depict Step (2) of a menu description. Wavy arrows depict other calculations. Panel (c) depicts our main impossibility theorem, which proves that DA cannot be described in finer detail as in the other two panels.

Our findings for DA presented thus far include positive (Section 3) and negative (Section 5) results. However, in pursuit of an ideal description of DA, one may wonder: Can we find an additional simple description of DA—beyond the traditional one and our new menu description—of a class not yet considered?

In Section 6, we present our fourth main result: that—within a broad classification of matching mechanism descriptions—the answer to the above question is *no*. We consider applicant-proposing descriptions (like the traditional outcome description of DA), and institution-proposing descriptions (like our new menu description of DA). Furthermore, we consider our three description outlines: Menu descriptions, outcome descriptions, and individualized dictatorships. We examine for DA all the resulting six classes of such descriptions (proposing side / outline combinations). We construct a description in each class not precluded by our main impossibility result (Section 5). Unfortunately, each new such description is an exceedingly unintuitive and convoluted algorithms that is impractical for use as a real-world description.

This fourth main result indicates that simple descriptions of DA face a trade-off between conveying strategyproofness and conveying the full matching outcome. The traditional (outcome) description of DA, used in countless real-life matching markets, is at one corner of this tradeoff. Our new menu description is at the other corner, exposing perhaps the most relevant property for applicants: Strategyproofness.

Table 2 summarizes our four main results, which construct and classify strategy-proofness-exposing descriptions of DA and TTC. In Section 7, we review related work, including our empirical companion paper investigating our menu description of DA. We conclude in Section 8, where we also discuss potential practical concerns.

# 2 Preliminaries

## 2.1 Mechanisms

This paper studies (static, direct-revelation) matching mechanisms. This environment consists of $n$ applicants $\{1, \ldots, n\}$ who are matched to institutions.[7] Applicant $i$ has a strict ordinal preference $\succ_i$ over institutions, also called her *report* or her *type*.[8] We let $\mathcal{T}_i$ denote the set of possible types of applicant $i$, and let $A$ denote the

[7]We denote applicants $i$ whenever the mechanism (or a concept such as strategyproofness, or a menu) is described specifically to applicant $i$; we otherwise denote applicants by $d$ (mnemonic: doctor). We denote institutions by $h$ (mnemonic: hospital), and matchings by $\mu$.

[8]Applicants may go unmatched, and their preference lists may be partial (indicating that they prefers to remain unmatched over institutions not on her preference list). We also let $h_1 \succ_d h_2$ indicate that applicant $d$ prefers $h_1$ to $h_2$; $h_1 \succeq_d h_2$ indicate $h_1 \succ_d h_2$ or $h_1 = h_2$; $\emptyset \succ_d h$ indicate that $d$ does not rank $h$; $\mu(d)$ denote the match of $d$ in $\mu$; and $\mu(d) = \emptyset$ denote $d$ going unmatched.

Table 2: Summary of main results

| Section | Description Type(s) | Result and Interpretation |
|---|---|---|
| Sec. 3 | Institution-proposing menu description for **DA** | **Positive Result**: Description 1 is an alternative description of an applicant's match in DA that exposes strategyproofness. |
| Sec. 4 | Applicant-proposing individualized dictatorship for **TTC** | **Positive Result**: Description 2 gives the above (↑) benefit, and in addition, exposes the strategyproofness of TTC's traditional description. |
| Sec. 5 | Applicant-proposing individualized dictatorship for **DA** | **Robust Negative Result:** There are no such linear-memory descriptions. Hence, menu descriptions cannot convey the strategyproofness of DA's traditional description. |
| Sec. 6 | One-side-proposing, menu, outcome, or individualized-dictatorship for **DA** | **Additional Negative Finding:** Except the traditional description and Description 1, all new descriptions we find are convoluted and impractical. Hence, with these descriptions we can convey either strategyproofness or the outcome matching, but not both. |

set of matchings.[9] Following much of the matching literature, we treat (only) the applicants as strategic agents who submit preferences, and focus on mechanisms that are strategyproof for the applicants.

**Definition 2.1.** A *(matching) mechanism* is any mapping $f : \mathcal{T}_1 \times \ldots \times \mathcal{T}_n \to A$ from the types of all applicants to a matching.[10] A mechanism is *strategyproof* if, for every $\succ_i, \succ_i' \in \mathcal{T}_i$ and $\succ_{-i} \in \mathcal{T}_{-i}$, we have $f_i(\succ_i, \succ_{-i}) \succeq_i f_i(\succ_i', \succ_{-i})$.

We study the canonical strategyproof mechanisms SD, TTC, and DA. These mechanisms are defined with respect to *priorities* of the institutions over the applicants. We treat the priorities as a non-strategic part of the mechanism (although, our descriptions of mechanisms also treat the priorities as input; see below). SD uses a priority order $\succ$ listing all applicants; TTC and SD use a profile of priority orders $\{\succ_h\}_h$, one for each institution $h$.

**Definition 2.2** (SD)**.** For a given priority order $\succ$, Serial Dictatorship (SD) is defined as follows. Applicants are considered in order of highest-to-lowest priority, and each

[9] As is standard, $\mathcal{T}_{-i}$ denotes the set $\mathcal{T}_1 \times \ldots \times \mathcal{T}_{i-1} \times \mathcal{T}_{i+1} \ldots \mathcal{T}_n$, and for $\succ_i \in \mathcal{T}_i$ and $\succ_{-i} \in \mathcal{T}_{-i}$, we write $(\succ_i, \succ_{-i})$ for the naturally corresponding element of $\mathcal{T}_1 \times \ldots \times \mathcal{T}_n$.

[10] For an applicant $i$, we let $f_i(\succ_1, \ldots, \succ_n)$ denote $i$'s match $\mu(i)$ in $\mu = f(\succ_1, \ldots, \succ_n)$.

applicant is permanently matched to her favorite not-yet-matched institution.

**Definition 2.3** (TTC)**.** For a given profile of institutions' priorities $\{\succ_h\}_h$, Top Trading Cycles (TTC) is defined as follows. Repeat the following until everyone is matched (or has exhausted their preference lists): Every remaining (i.e., currently unmatched) applicant "points" to her favorite remaining institution, and every remaining institution points to its highest priority remaining applicant. There must be some cycle in this directed graph (since there is only a finite number of vertices). Choose any such cycle and "eliminate" that cycle by permanently matching each applicant in the cycle to the institution she is pointing to (and remove all matched agents from consideration for later cycles).

**Definition 2.4** (DA)**.** For a given profile of institutions' priorities $\{\succ_h\}_h$, Deferred Acceptance (DA) is defined as follows. Repeat the following until every applicant is matched (or has exhausted her preference list): A currently unmatched applicant is chosen to "propose" to her favorite institution which has not yet "rejected" her. The institution then rejects every proposal except for the *top priority applicant* who has proposed to it thus far. Rejected applicants become (currently) unmatched, while the top priority applicant is tentatively matched to the institution. At the end, the tentative allocation becomes final.

As is well-known, TTC and DA are the two canonical matching mechanisms which are priority-based and strategyproof. TTC is Pareto-efficient for the applicants, and DA is stable. Note that DA refers to the (direct-revelation) mechanism defined by applicant-proposing DA (i.e., outputting the applicant-optimal stable matching); when confusion might arise, we use APDA (and for institution-proposing, we use IPDA). Note also that the outcomes of TTC and of DA are independent of the order in which steps of the above algorithms are chosen (Corollary E.4 and Proposition E.9).

## 2.2 Descriptions

This paper studies ex ante *descriptions* of matching mechanisms, i.e., descriptions that are given before any concrete inputs are known. When matching markets are described in detail to participants, this is typically done by specifying a set of explicit, precise, step-by-step instructions for calculating the result, i.e., by specifying an algorithm.[11] Thus, we formally define a description to be any algorithm that uses as input the preferences of the applicants and the priorities of the institutions, and calculates some result (e.g., an outcome matching).[12]

For SD, TTC, and DA, we refer to the description in the above Definitions 2.2 through 2.4 as the *traditional description* of the corresponding mechanism. Formally, for mechanism $f$, these are algorithm which, using input $\succ_1, \ldots, \succ_n$ (and the priorities of institutions), output $f(\succ_1, \ldots, \succ_n)$. We refer more generally to such algorithms as *outcome descriptions* for $f$.

Beyond outcome descriptions, we study two other description outlines: menu descriptions and individualized dictatorship descriptions; these are defined in Sections 2.3 and 2.4, respectively.

## 2.3 Menus and Menu Descriptions

The starting point of our framework for changing mechanism descriptions is the following characterization of strategyproofness in terms of applicants' *menus*.[13]

**Definition 2.5** (Menu)**.** For any matching mechanism $f$, the *menu* $\mathcal{M}_{t_{-i}}$ of applicant $i$ with respect to types $\succ_{-i} \in \mathcal{T}_{-i}$ of other applicants is the set of all institutions $h$ for which there exists some $\succ_i \in \mathcal{T}_i$ such that $f_i(\succ_i, \succ_{-i}) = h$. That is,

$$\mathcal{M}_{\succ_{-i}} = \{\ f_i(\succ_i, \succ_{-i}) \mid \succ_i \in \mathcal{T}_i\ \}\,.$$

**Theorem 2.6** (Hammond, 1979)**.** *A matching mechanism $f$ is strategyproof if and only if each applicant $i$ always receives her favorite institution from her menu. That is, for every $\succ_{-i} \in \mathcal{T}_{-i}$ and $\succ_i \in \mathcal{T}_i$, it holds that $f_i(\succ_i, \succ_{-i}) \succeq_i h$ for all $h \in \mathcal{M}_{\succ_{-i}}$.*

[11] Of course, the way a description/algorithm is actually conveyed to participants can vary. One common real-world approach to relaying matching algorithms is an illustrative video using an example (see, e.g., Figure 3 in Section 5.1 for such a video for DA). In our paper, we abstract over exactly how the algorithm is relayed.

[12] Algorithms can be defined in full mathematical detail in various ways; any such definition will suffice for our purposes. For completeness, in Appendix A we give a definition from first-principles that captures all results in this paper.

[13] Definition 2.5 has been considered under many different names in many different contexts (e.g., *sets that decentralize the mechanism* in Hammond (1979); *option sets* in Barberà et al. (1991); *proper budget sets* in Leshno and Lo (2021); *feasible sets* in Katuščák and Kittsteiner (2025); and likely others). We follow the "economics and computation" literature (Hart and Nisan, 2017; Dobzinski, 2016; and follow-ups) in calling these sets "menus." This notion is distinct from many other definitions of menus (e.g., those of Mackenzie and Zhou, 2022; Bó and Hakimov, 2023, and many others).

*Proof.* Suppose $f$ is strategyproof and fix $\succ_{-i} \in \mathcal{T}_{-i}$. For every $\succ_i \in \mathcal{T}_i$, it holds by definition that for any $h = f_i(\succ'_i, \succ_{-i}) \in \mathcal{M}_{\succ_{-i}}$, we have $f_i(\succ'_i, \succ_{-i}) \succeq_i h$. On the other hand, if applicant $i$ always receives her favorite institution from her menu, then she always prefers reporting $\succ_i$ at least as much as any $\succ'_i$, so $f$ is strategyproof. □

We use menus to describe mechanisms while exposing their strategyproofness:

**Definition 2.7** (Menu Description)**.** A *menu description* of mechanism $f$ for applicant $i$ is a description with the following outline:

**Step (1)** uses only $\succ_{-i} \in \mathcal{T}_{-i}$ to calculate the menu $\mathcal{M}_{\succ_{-i}}$ of applicant $i$.

**Step (2)** uses $\succ_i \in \mathcal{T}_i$ to match applicant $i$ to her favorite institution in $\mathcal{M}_{\succ_{-i}}$.

Formally, a menu description for $i$ is thus an algorithm that initially receives only $\succ_{-i}$ as input and calculates $\mathcal{M}_{\succ_{-i}}$ as an intermediate result, then additionally receives $\succ_i$ as input and uses it to calculate $i$'s favorite choice from $\mathcal{M}_{\succ_{-i}}$ as the final result.

The central premise of our paper is that menu descriptions are one way to expose strategyproofness. This is because the strategyproofness of any menu description can immediately be seen via a simple, one-sentence argument: First, applicant $i$'s report cannot affect her menu, and second, straightforward reporting ("truthtelling") gets applicant $i$ her favorite institution from the menu.[14]

## 2.4 Individualized Dictatorships

Beyond menu descriptions, outcome descriptions that contain menu descriptions also play a key role in our results. We term such descriptions *individualized dictatorships*.

**Definition 2.8** (Individualized Dictatorship)**.** An *individualized dictatorship (description)* of mechanism $f$ for applicant $i$ is a description with the following outline:

**Step (1)** uses only $\succ_{-i} \in \mathcal{T}_{-i}$ to calculate the menu $\mathcal{M}_{\succ_{-i}}$ of applicant $i$.

[14] There is also a precise sense in which menu descriptions are the *only* ones for which the above argument for strategyproofness goes through. In particular, suppose a description calculates the match of applicant $i$ in some mechanism $f$, and has the following outline:

**Step (1)** uses $\succ_{-i}$ to calculate a set $S$ of institutions.

**Step (2)** uses $\succ_i$ to match $i$ to her top-ranked institution in $S$.

Then, it is not hard to show that the set $S$ must be $i$'s menu.

**Step (2)** uses $\succ_i \in \mathcal{T}_i$ to match applicant $i$ to her favorite institution from $\mathcal{M}_{\succ_{-i}}$.

**Step (3)** uses both $\succ_i$ and $\succ_{-i}$ to calculate the full outcome matching $f(\succ_i, \succ_{-i})$.

Formally, an individualized dictatorship for $i$ is thus an algorithm that initially receives $\succ_{-i}$ as input and calculates $\mathcal{M}_{\succ_{-i}}$, then additionally receives $\succ_i$ as input and calculates $i$'s favorite choice from $\mathcal{M}_{\succ_{-i}}$, and finally proceeds to calculate the entire outcome matching $f(\succ_i, \succ_{-i})$ as the final result.

For example, consider SD (Definition 2.2). This mechanism is easily seen to be strategyproof, directly from its traditional description (and even for many students encountering it for the first time). This is reflected by the fact that applicants are matched in SD via menu descriptions. In particular, when applicants are prioritized $1 \succ 2 \succ \ldots \succ n$, the traditional description of SD can be divided into three steps:

(1) Each applicant $j < i$ is matched, in order, to her top-ranked remaining institution.

(2) Applicant $i$ is matched to her top-ranked remaining institution.

(3) Each applicant $j > i$ is matched, in order, to her top-ranked remaining institution.

Steps (1) and (2) form menu description, but this menu description is contained within the (traditional) outcome description, and thus Steps (1) through (3) form an individualized dictatorship.

## 2.5 Uses of Menu Descriptions

Throughout this paper, we argue that menu descriptions are useful both as an alternative way to describe a mechanism to participants, and as a way to gain insights into the traditional description of the mechanism. Importantly, not all menu descriptions achieve these goals. For example:[15]

**Example 2.9** (A "brute force" menu description)**.** Consider any strategyproof matching mechanism $f$ with a traditional description $D$. For each institution $h$, let $\{h\}$ denote the preference list that ranks only $h$ as acceptable. Then, consider the following description for applicant $i$:

(1) Start with $M = \emptyset$. For each institution $h$ separately, evaluate $D$ on $(\{h\}, \succ_{-i})$; if $i$ matches to $h$, then add $h$ to $M$.

[15] This menu description was also identified by Katuščák and Kittsteiner (2025).

(2) Match $i$ to her highest-ranked institution in $M$.

By strategyproofness, $h$ will be included in $M$ in Step (1) if and only if $h$ is on the menu. Thus, the above provides a menu description of $f$.

While all strategyproof mechanisms have menu descriptions as in Example 2.9, such descriptions do not seem useful. First, we suspect that real participants would find the above description complicated and confusing, precluding its real-world use as an alternative description. Second, without already knowing that the traditional description is strategyproof, there is no clear relation between the outcomes of the above description and those of the traditional description, precluding Example 2.9 as a tool to aid in conveying the traditional description's strategyproofness.

Given the above, we look for simple or appealing new menu descriptions (e.g., our description of DA in Section 3). We also look for menu descriptions that help to illustrate the strategyproofness of traditional description (e.g., our description of TTC in Section 4, which is additionally an individualized dictatorship).

# 3 A Menu Description of DA

In this section, we present our first main result: A novel menu description of DA. This is Description 1 (rephrased from Table 1 in the introduction).

**Description 1** A menu description of (*applicant*-proposing) DA for applicant $i$

(1) Run *institution*-proposing DA with applicant $i$ removed from the market, to get a matching $\mu_{-i}$. Let $M$ be the set of institutions $h$ such that $i \succ_h \mu_{-i}(h)$.

(2) Match $i$ to $i$'s highest-ranked institution in $M$.

Description 1 seems comparable in complexity to the traditional description of DA (from Definition 2.4). In fact, it only adds an easy-to-state "menu calculation and matching" step on top of a modified traditional description of DA. We speculate that many real market participants would find such a description understandable. (In fact, our companion paper Gonczarowski et al. (2024) gives empirical evidence that many lab participants can learn this description—see Section 7.)

Crucially, Description 1 uses the *institution*-proposing DA algorithm to describe DA (traditionally described via *applicant*-proposing DA); to give intuition for why the proposing side is flipped, we show via an example that using applicant-proposing DA in this description would not suffice.

**Example 3.1.** Consider a market with three applicants $i, d_1, d_2$ and two institutions $h_1, h_2$. Applicants have preferences $d_1 : h_1 \succ h_2$ and $d_2 : h_2 \succ h_1$, and institutions have priorities $h_1 : d_2 \succ i \succ d_1$ and $h_2 : d_1 \succ i \succ d_2$. Running applicant-proposing DA on these preferences without $i$ gives matching $\{(d_1, h_1), (d_2, h_2)\}$, and both $h_1$ and $h_2$ prefer $i$ to their match. However, neither $h_1$ nor $h_2$ are on $i$'s menu, since having $i$ propose to any $h_i \in \{h_1, h_2\}$ (after running applicant-proposing DA without $i$) causes a "rejection cycle" that results in $h_i$ rejecting $i$. Intuitively, institution-proposing DA fixes this issue by outputting a matching that has no potential "applicant-proposing rejection cycles."[16]

Formally, the following theorem establishes the correctness of Description 1:

**Theorem 3.2.** *Description 1 is a menu description of DA.*

*Proof.* Fix institutions' priorities, an applicant $i$, and preferences $\succ_{-i}$ of applicants other than $i$. Let $\{h\}$ denote the preference list of $i$ that reports only institution $h$ as acceptable, and let $\emptyset$ denote the preference list of $i$ that reports *no* institution as acceptable. For clarity, denote applicant-proposing DA by $\mathit{APDA}(\cdot) = \mathit{DA}(\cdot)$ and denote institution-proposing DA by $\mathit{IPDA}(\cdot)$.

Now, for any institution $h$, we observe the following chain of equivalences:

$$
\begin{aligned}
&h \text{ is in the menu of } i \text{ in } \mathit{APDA} \text{ (with respect to } \succ_{-i})\\
&\qquad \iff \big(\text{By strategyproofness of } \mathit{APDA}\text{; Theorem E.8}\big)\\
&i \text{ is matched to } h \text{ by } \mathit{APDA}(\{h\}, \succ_{-i})\\
&\qquad \iff \big(\text{By the Lone Wolf / Rural Hospitals Theorem; Theorem E.6}\big)\\
&i \text{ is matched to } h \text{ by } \mathit{IPDA}(\{h\}, \succ_{-i})\\
&\qquad \iff \big(\mathit{IPDA}(\{h\}, \succ_{-i}) \text{ and } \mathit{IPDA}(\emptyset, \succ_{-i}) \text{ coincide until } h \text{ proposes to } i\big)\\
&h \text{ proposes to } i \text{ in } \mathit{IPDA}(\emptyset, \succ_{-i})\\
&\qquad \iff \big(\mathit{IPDA}(\emptyset, \succ_{-i}) \text{ and } \mathit{IPDA}(\succ_{-i}) \text{ produce the same matching;}\\
&\qquad\qquad\quad \text{in } \mathit{IPDA}, h \text{ proposes in favorite-to-least-favorite order}\big)\\
&h \text{ prefers } i \text{ to its match in } \mathit{IPDA}(\succ_{-i}) \text{ (in the market without } i).
\end{aligned}
$$
□

[16]This intuition regarding "applicant-proposing rejection cycles" is related to the concept of an institution-improving rotation as in Gusfield and Irving (1989).

In addition to giving a perhaps-appealing alternative description of DA, Theorem 3.2 provides a characterization of the menu in DA which is useful for reasoning about DA's properties. We briefly highlight two applications. First, one can immediately see from Description 1 that, if one applicant's priorities increase at some set of institutions, then (all other things being equal) the match of that applicant in DA can only improve (Balinski and Sönmez, 1999). Second, a short argument using Description 1, which we provide in Remark B.3, shows that in a market with $n+1$ applicants, $n$ institutions, and uniformly random full length preference lists, applicants receive in DA roughly their $n/\log(n)$th choice in expectation—rather lower than in the case with $n$ applicants, where they receive their $\log(n)$th choice—re-proving results from Ashlagi et al. (2017); Cai and Thomas (2022).

Description 1 generalizes to a broader class of stable matching markets. In fact, in Remark B.2, we observe that the same arguments as in the above proof show that a natural generalization of Description 1 characterizes the menu of DA in many-to-one markets, and even in a general class of markets with contracts, namely, those considered by Hatfield and Milgrom (2005).

Finally, we remark that Description 1 can facilitate a proof from first-principles of the strategyproofness of (traditionally described) DA (without relying on this fact, as in the proof above). We give such a proof in Appendix B. While we view this proof as theoretically appealing, and perhaps useful for classroom instruction, we believe this approach remains far too mathematically involved to convey the strategyproofness of DA's traditional description to real-world participants. In contrast, if a clearinghouse directly adopts Description 1 as a way to describe participants' matches in explicit detail, then strategyproofness follows via a simple and clear proof.

# 4 An Individualized Dictatorship for TTC

In this section, we present our second main result: A novel individualized dictatorship for TTC that additionally yields a simple, intuitive proof that the traditional description of TTC is strategyproof. This is Description 2.

Steps (1) and (2) of Description 2 give a menu description of TTC that seems very similar in complexity to the traditional description (from Definition 2.3). Moreover, Description 2 as a whole modifies the traditional one *only* by delaying matching

**Description 2** An individualized dictatorship for TTC for applicant $i$

(1) Using $\succ_{-i}$, iteratively eliminate as many cycles not involving applicant $i$ as possible. Let $M$ denote the set of remaining institutions.

(2) Using $\succ_i$, match $i$ to her highest-ranked institution in $M$. Call this institution $h$.

(3) Using $(\succ_i, \succ_{-i})$, eliminate the cycle created when $i$ points to $h$, then continue to eliminate cycles until all applicants match (or exhaust their preference lists).

---

applicant $i$ as long as possible.[17] This accurately describes the full outcome matching since, as is well known, TTC is independent of the order in which cycles are chosen to be eliminated and matched. Formally:

**Theorem 4.1.** *Description 2 is an individualized dictatorship description of TTC.*

*Proof.* Fix an applicant $i$. By construction, Step (1) of Description 2 does not use $\succ_i$. Thus, to prove the theorem, it suffices to show that the set $M$ in Step (1) is $i$'s menu, and that the matching output in Step (3) is the outcome of TTC. We use the fact that TTC is independent of the order in which cycles are eliminated (Proposition E.9).

To see that $M$ is $i$'s menu, first observe that (by Proposition E.9) any institution matched during Step (1) is not on $i$'s menu. Second, observe that after the conclusion of Step (1), once $i$ points to *any* remaining institution, this must complete a cycle. Since this cycle necessarily involves applicant $i$, she gets the institution she pointed to, and thus this institution is on $i$'s menu. See Figure 2 for an illustration.

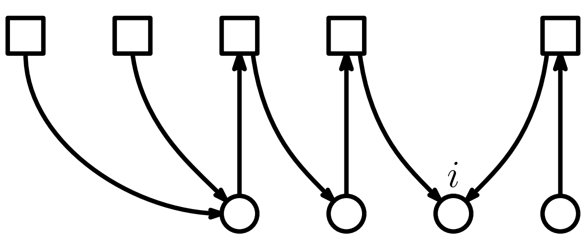



Figure 2: Illustration of the menu calculation in Description 2

**Notes:** Circles are applicants; squares are institutions; each applicant except $i$ (resp. institution) points to her favorite remaining institution (resp. applicant).

To see that Step (3) calculates the outcome of TTC, observe that given the above (and given Proposition E.9), the entire run of Description 2 constitutes a valid or-

[17]This can also be thought of as running TTC, with a twist: During the first stage, applicant $i$ does not point to any institutions. This stage lasts until no cycles exist, after which $i$ points as normal (and immediately gets matched to what she points to).

dering of cycle elimination in the traditional description of TTC. This prove the theorem. □

In addition to constructing a new menu description of TTC, Theorem 4.1 yields a simple proof that the traditional description of TTC is strategyproof. In particular, Theorem 4.1 demonstrates—given (only) the fact that TTC is independent of the order in which cycles are eliminated—that in the traditional description of TTC, any applicant $i$ is matched according to a menu description. Hence, TTC is strategyproof.

The above simple proof is enabled by two crucial facts regarding Description 2. First, it contains a menu description. Second, it only slightly tweaks TTC's traditional (outcome) description. Crucially, a description cannot achieve these two tasks without being an outcome description that contains a menu description, i.e., an individualized dictatorship.

All told, our description of TTC—and the simple argument for the strategyproofness it provides—give new, promising ways to explain TTC's strategyproofness, both in the classroom, and for real-world market participants.[18]

# 5 Main Impossibility Result for DA

In this section, we present our third main result, which is our main impossibility theorem for DA. We show a formal sense in which—in contrast to what our individualized dictatorship for TTC achieved in Section 4—no menu description of DA yields a simple proof of the strategyproofness of DA's traditional description.

Concretely, we show that no slight tweak of the traditional description of DA is an individualized dictatorship. First, we formalize our notion of "no slight tweak." Second, we state and discuss our impossibility theorem. Third, we present its proof.

## 5.1 Applicant-Proposing and Linear-Memory Descriptions

We now identify two simplicity desiderata satisfied by the traditional descriptions of SD, TTC, and DA. First, these descriptions only consider the preferences of each applicant once, in a specific, natural order—from favorite to least favorite. We call this property *applicant-proposing*. Second, they require a small amount of bookkeeping

[18]See footnote 6.

as they run—little more than the bookkeeping required to remember a single matching. We formalize this property through *linear-memory*, which stipulates that the bookkeeping used by the description is not much more than that of a single match.

Before formally defining these simplicity conditions, we illustrate how they are used to relay the traditional description of DA in one of its most celebrated practical applications: matching medical doctors to residencies in the US National Residency Matching Program (NRMP). Figure 3 shows a screenshot of a video that describes DA in this market by applying it to a small example. The explanation in the video is aided by two visual elements: crossing off institutions from applicants' lists as the description progresses, and keeping track of a "current tentative matching" illustrated by the yellow-highlighted names. These two simple visual elements are enabled precisely by our two simplicity desiderata. First, the fact that the description is applicant-proposing is necessary for the video to cross off institutions from applicants' lists as the description progresses. Second, the linear memory requirement—which stipulates that the description use only a small amount of bookkeeping per applicant—is necessary for the yellow highlighting in the video to capture the entire required bookkeeping.

Figure 3: An illustration of the traditional description of DA through an example


**Note:** Screenshot taken from https://youtu.be/kVTwXNawpbk (NMS, 2020), a video produced by National Matching Services (the company providing matching software to the NRMP).

Without taking a specific stance on what a "slight tweak of DA" is, we take the stance that all slight tweaks of the traditional description of DA (and of SD and TTC) are—as leveraged in Figure 3—applicant-proposing and linear-memory. In particular, slight tweaks of traditional descriptions should read applicants' preferences in a similar way (without, e.g., directly checking if applicant $d$ prefers $h_1$ or $h_2$, and without reading applicants' preference lists multiple times), and should not require dramatically more bookkeeping (than the flexible linear-memory requirement allows).

We now define applicant-proposing and linear-memory descriptions.[19]

**Definition 5.1** (Applicant-proposing and Linear-memory)**.**

- In a matching environment, an description is *applicant-proposing* if it satisfies the following. The description uses applicants' preferences only by querying a single applicant at a time, such that the $j^{\text{th}}$ query to applicant $d$ depends only on the $j^{\text{th}}$ institution on $d$'s preference list. (The priorities of the institutions, on the other hand, can be used by the description in any way.)

  Formally, an applicant-proposing description is thus an algorithm which maintains some *state* which is iteratively updated by querying applicants' preference lists (one applicant at a time), with the following property. For any possible inputs and for any applicant $d$, suppose the algorithm queries $d$'s preference in states $s_1, s_2, \ldots, s_k$ as it runs, and for each $j = 1, \ldots, k$, let $s'_j$ denote the updated state which the algorithm reaches immediately after querying $d$'s preferences in $s_j$. Then, $s'_j$ depends only on ($s_j$ and) the $j^{\text{th}}$ institution on $d$'s preference list (which is considered to be the "empty institution" if $d$'s lists fewer than $j$ institutions).[20]

- The *memory requirement* of a description is the number of bits required to represent the state of the description. (Intuitively, this is the amount of extra bookkeeping or "scratch paper" required by the description.)

  Formally, the memory requirement of a description, or any algorithm, is the logarithm in base 2 of the number of possible states of the algorithm.

[19]As discussed in Section 2.2, we formally define descriptions to be algorithms. For a self-contained mathematical model of algorithms, see Appendix A.

[20]While we call this property "applicant-*proposing*," it also applies to descriptions of TTC that one might call "applicant-pointing".

In a matching environment with $n$ applicants and $n$ institutions, we say a description is *linear-memory* if its memory requirement is at most $\widetilde{\mathcal{O}}(n)$.[21]

Linear memory is the minimum possible memory requirement for outcome descriptions (as well as for menu descriptions) of matching mechanisms. Indeed, $\widetilde{\mathcal{O}}(n)$ is exactly (up to the precise logarithmic factors) the number of bits of memory required to describe a single matching (or a single applicant's menu).[22]

As discussed above, the traditional (outcome) descriptions of each of SD, TTC, and DA are applicant-proposing and linear-memory. Formally:

**Observation 5.2.** *Each of SD, TTC, and DA has an applicant-proposing and linear-memory outcome description.*

More generally, we take the stance that "all slight tweaks" of these traditional descriptions must be applicant-proposing linear-memory outcome descriptions.[23] Like the corresponding traditional description, such a slight tweak must satisfy the two mild simplicity conditions discussed above, and must reach the same final result (namely, the outcome matching).

If such a "slight tweak" description additionally exposes strategyproofness using a menu, then it must additionally be an individualized dictatorship (i.e., an outcome description containing a menu description). For example, this is true of SD's traditional description and our description of TTC from Section 4. Formally:

**Corollary 5.3.** *SD and TTC each have an applicant-proposing and linear-memory individualized-dictatorship description.*

## 5.2 Main Impossibility Theorem

We now present our main impossibility result. Using the simplicity desiderata of Section 5.1, we prove that no slight tweak of the traditional description of DA is an individualized dictatorship. Formally:

[21] The standard computer-science notation $\widetilde{\mathcal{O}}(n)$ means $\mathcal{O}(n \log^{\alpha} n)$ for some constant $\alpha$. That is, for large enough $n$, memory is upper-bounded by $cn \log^{\alpha} n$ for some constants $c, \alpha$ that do not depend on $n$. Using $\widetilde{\mathcal{O}}(n)$ memory means using only nearly constant bookkeeping per applicant.

[22] To see this formally, note that there are $n! = 2^{O(n \log n)}$ distinct matchings involving $n$ applicants and $n$ institutions (and exactly $2^n$ possible menus). Intuitively, this means that the number of letters it takes to write down a single matching with $n$ applicants and $n$ institutions (or, a subset of the $n$ institutions) is roughly proportional to $n$.

[23] Note that we do not view *every* description satisfying these properties as a slight tweak of a traditional one. Instead, we take (only) the stance that all slight tweaks must satisfy these properties.

**Theorem 5.4.** *DA has no applicant-proposing, linear-memory individualized dictatorship. In fact, with $n$ applicants and institutions, any applicant-proposing individualized dictatorship for DA requires $\Omega(n^2)$ memory.*[24]

Theorem 5.4 shows a precise sense in which slight tweaks of DA's traditional description cannot expose its strategyproofness, in a sharp contrast to (SD and) TTC (Corollary 5.3). We prove the theorem below in Section 5.3.

Additionally, Theorem 5.4 is a strong and robust impossibility result. Namely, we show that applicant-proposing individualized dictatorships for DA require *quadratic* memory—$\Omega(n^2)$ bits. This nearly matches the memory requirement of simply memorizing all applicants' preferences—$\widetilde{\mathcal{O}}(n^2)$ bits.[25] If an applicant-proposing description memorizes all applicants' preferences, then it can calculate *any* desired result (formally, by querying each applicant's entire preference list in order, with a separate state of the algorithm's memory for each possible preference profile, and returning a separate desired result for each such state). This shows that quadratic memory is the highest possible amount of memory that an algorithm might require. Thus, where applicant-proposing individualized dictatorships of (SD and) TTC use memory as *low* as possible (linear, see Section 5.1), for DA the memory requirement is as *high* as possible (quadratic). See Figure 4 for an illustration of the qualitative gap between these two memory requirements.

Figure 4: Linear versus quadratic memory

$$\underbrace{\boxed{\{\ (h_1, d_3),\ (h_2, d_1),\ (h_3, \emptyset),\ (h_4, d_4),\ (h_5, d_2)\ \}}}_{\text{Linear memory}}$$

$$\underbrace{\boxed{\begin{array}{l} d_1:\ h_1 \succ h_2 \succ h_3 \succ h_4 \succ \emptyset \\ d_2:\ h_1 \succ h_4 \succ h_2 \succ h_3 \succ h_5 \succ \emptyset \\ d_3:\ h_2 \succ h_1 \succ h_4 \succ h_3 \succ \emptyset \\ d_4:\ h_1 \succ h_2 \succ h_4 \succ h_3 \succ h_5 \succ \emptyset \\ d_4:\ h_1 \succ h_5 \succ h_3 \succ \emptyset \end{array}}}_{\text{Quadratic memory}}$$

[24]The standard computer-science notation $\Omega(n^2)$ means that, for large enough $n$, memory is lower-bounded by $cn^2$ for some constant $c$ that does not depend on $n$.

[25]To see this formally, observe that there are $(n!)^n = 2^{O(n^2 \log(n))}$ possible preference profiles for all applicants. Intuitively, this means that the number of letters it takes to write down $n$ applicants' preferences over all $n$ institutions is roughly proportional to $n^2$.

Theorem 5.4 is also tight in the following sense. The theorem shows that descriptions of DA cannot simultaneously satisfy four criteria: being an outcome description, containing a menu description, being applicant-proposing, and using linear-memory. The impossibility only holds when all four of these criteria are assumed. We establish this as follows. First, DA's traditional description is an applicant-proposing, linear-memory outcome description. Second, DA has an applicant-proposing *quadratic memory* individualized dictatorship (i.e., an outcome description containing a menu description), since (as discussed above) quadratic-memory is as high as possible. Third and fourth, we show below in Section 6 and Appendix C that DA has an applicant-proposing linear-memory menu description, and a linear-memory individualized dictatorship which is not applicant-proposing.[26] Hence, Theorem 5.4 captures the complexity of DA in our framework very precisely.

All told, there is a stark trichotomy in our framework between SD, TTC, and DA. The traditional description of SD is already a menu description, simultaneously for all applicants, exposing its strategyproofness easily.[27] The traditional description of TTC must be slightly tweaked and specialized to each individual applicant in order to expose strategyproofness.[28] However, once this is done, strategyproofness is easy to see. For DA, in contrast with both other mechanisms, no small tweak of the traditional description suffices to expose strategyproofness using a menu, in the robust and strong sense provided by Theorem 5.4. See Figure 1 in the introduction for an illustration.

## 5.3 Proof of Main Impossibility Theorem

Theorem 5.4 states that applicant-proposing individualized dictatorships for DA require high memory. Recall that such descriptions must—while querying applicants' preferences only once in favorite-to-least-favorite order—calculate $i$'s menu using $\succ_{-i}$, and then proceed to calculate the full matching using $(\succ_i, \succ_{-i})$.

To prove Theorem 5.4, we construct a set of applicant preferences that, intuitively

[26] We construct an applicant-proposing linear-memory menu description of DA in Section C.2. We construct a linear-memory individualized dictatorship for DA in Section C.3; this description is *institution*-proposing.

[27] SD has an (S)OSP implementation (Li, 2017; Pycia and Troyan, 2023) for a similar reason.

[28] One can show that if a mechanism is not OSP-implementable—as is the case for TTC (Li, 2017)—then any description of the mechanism *must* be specialized to a given applicant $i$ in order to contain a menu description for $i$. In Remark B.4 we give a short direct proof that TTC's order requires such specialization.

speaking, has two properties: (A) to calculate $i$'s menu given preferences in this set, essentially the full preference list of every applicant other than $i$ must be read in its entirety, and (B) to calculate the final matching, essentially all this information must be remembered in full. Property (B) then establishes the $\Omega(n^2)$ memory requirement.

*Proof of Theorem 5.4.* Fix an applicant $i$ and let $D$ be any applicant-proposing individualized dictatorship description of DA for $i$.

We now describe a set $\mathcal{S} \subseteq \mathcal{T}_{-i}$ of possible inputs to DA, illustrated in Figure 5, which allows us to establish property (B) discussed above (intuitively, by allowing $i$'s possible reports to affect the outcome matching in a different way for each different $\succ_{-i} \in \mathcal{S}$). For simplicity, let the number of applicants and institutions $n$ be a multiple of 4. Other than $i$, there are applicants and institutions $d_j, d'_j, h_j, h'_j$ for each $j \in \{1, \ldots, n/2\}$. There are $n/2$ total "cycles" containing two applicants and two institutions each. Cycle $j$ has applicants $d_j$ and $d'_j$ and institutions $h_j$ and $h'_j$. The cycles are divided into two classes, "top" cycles (for $j \in \{1, \ldots, n/4\}$) and "bottom" cycles (for $j \in \{n/4 + 1, \ldots, n/2\}$).

The institutions' priorities are fixed, and defined as follows:

For top cycles ($j \leq n/4$):

$$h_j: \quad d'_j \succ i \succ d_j$$
$$h'_j: \quad d_j \succ d'_j$$

For bottom 2-cycles ($j > n/4$):

$$h_j: \quad d'_j \succ d_1 \succ d_2 \succ \ldots \succ d_{n/4} \succ d_j$$
$$h'_j: \quad d_j \succ d'_j.$$

For the top cycle applicants ($d_j$ with $j \leq n/4$), the preferences vary (in a way we will specify momentarily). Other applicants' preference are fixed, as follows:

For bottom cycles ($j > n/4$): $\quad d_j: \quad h_j \succ h'_j$

For all cycles ($j \in \{1, \ldots, n/2\}$): $\quad d'_j: \quad h'_j \succ h_j.$

Let $\mathcal{S}$ denote the set of preference profiles where we additionally have:

For top cycles ($j \leq n/4$): $\quad d_j: \quad h_j \succ B_j \succ h'_j,$

where $B_j$ is an arbitrary subset of $\{h_k \mid k > n/4\}$, ranked in any fixed order (say, increasing order of $j$). Any such collection of $\left(T_j\right)_{j=1}^{n/4}$defines a distinct preference profile in $\mathcal{S}$. Note that $|\mathcal{S}| = 2^{(n/4)^2}$. See Figure 5 for an illustration.

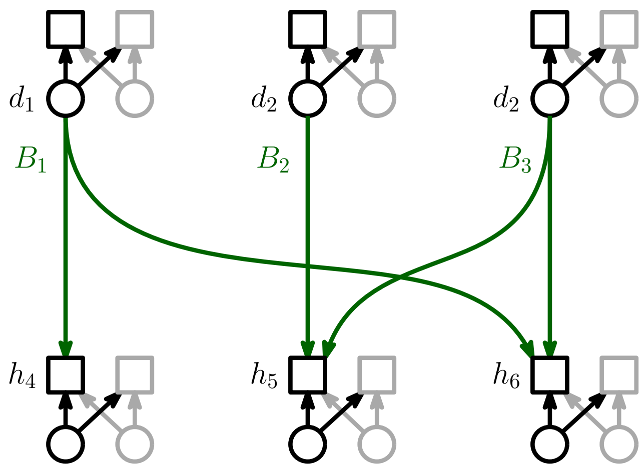



Figure 5: Illustration of the construction used to prove Theorem 5.4

**Notes:** Dark nodes represent $d_j$ or $h_j$ for some $j$, and grey nodes represent $d'_j$ or $h'_j$. The green arrows directed outwards from a top cycle $d_j$ represent the sets $B_j$.

We additionally define a set of inputs $\mathcal{S}' \supseteq \mathcal{S}$, which allow us to establish property (A) discussed above (intuitively, by making $i$'s menu depend on the final institution ranked above $\emptyset$ on other applicants' lists). Specifically, let $\mathcal{S}'$ denote the set containing every element of $\mathcal{S}$, and additionally any top cycle applicant $d_j$ ($j \le n/4$) may or may not truncate the final institution $h'_i$ off her list, marking it as unacceptable. In other words, in addition to the sets $\big(B_j\big)_{j=1}^{n/4}$, an element of $\mathcal{S}'$ is defined by bits $(c_j)_{j=1}^{n/4}$, such that, for each top cycle $j$ ($j \le n/4$):[29]

$$\begin{aligned} &\text{When } c_j = 0: && d_j : \quad h_j \succ B_j \succ h'_j \\ &\text{When } c_j = 1: && d_j : \quad h_j \succ B_j. \end{aligned}$$

We now proceed to prove the two crucial properties of DA, and the description $D$, when run on this family of preference profiles. The following lemmas formalize, respectively, properties (A) and (B) discussed above.

**Lemma 5.5.** *Consider a preference profile in $\mathcal{S}'$. For each top cycle $j$ (with $j \le n/4$), we have that $h_j$ is in applicant $i$'s menu if and only if $d_j$ does not rank $h'_j$ (i.e., $c_j = 1$). Hence, to correctly calculate $i$'s menu, description $D$ must read the entire preference list of each such $d_j$ (up to the position of $h'_j$).*

To prove this lemma, consider the execution of APDA when $i$ submits a list containing only $h_j$. First, $d_j$ is rejected, then she proposes to every institution $h_k \in B_j$. This "rotates" the bottom cycle containing $h_k$; in more detail, $h_k$ will accept the

[29] This collection of preferences can also be constructed with full preference lists by adding some unmatched institution $h_\emptyset$ to represent truncating $d_i$'s list.

proposal from $d_j$, then $d_k$ will propose to $h'_k$, then $d'_k$ with propose to $h_k$, and $d_j$ will be rejected from $h_k$. This will occur for every $h_k \in B_j$, so $d_j$ will not match to any $h_k$ with $k \in \{n/4+1, \ldots, n/2\}$.

Finally, after getting rejected from each institution in $B_j$, applicant $d_j$ may or may not propose to $h'_j$, depending on the bit $c_j$. If she does not, then $d_*$ remains matched to $h_j$ and in this case $h_j$ is on $i$'s menu. If she does, then $h'_j$ will reject $d'_j$, who will propose to $h_j$, which will reject $i$. So $i$ will go unmatched, and thus in this case $h_j$ is not on $i$'s menu.

The final sentence of the lemma then follows from the fact that $D$ is an applicant-proposing and must calculate $i$'s menu. This proves Lemma 5.5.

**Lemma 5.6.** *Each distinct preference profile $\succ_{-i} \in \mathcal{S}$ induces a distinct function $APDA(\cdot, \succ_{-i}) : \mathcal{T}_{-i} \to A$ from applicant $i$'s report to outcome matchings. Hence, to correctly calculate the outcome matching, the description $D$ must—across all states where it finishes calculating $i$'s menu—have at least one state for each element of $\mathcal{S}$.*

To prove this lemma, consider two distinct preference profiles in $\mathcal{S}$, one profile $\succ_{-i}$ corresponding to $\big(B_j\big)_{j=1}^{n/4}$, and the other profile $\succ'_{-i}$ corresponding to $\big(B'_j\big)_{j=1}^{n/4}$. Without loss of generality, there is some $j$ and $k$ such that $h_k \in B_j \setminus B'_j$. Suppose now that $i$'s report $\succ_i$ lists only $h_j$. Then, consider execution of APDA under $(\succ_i, \succ_{-i})$ and under $(\succ_i, \succ'_{-i})$. Under $\succ_{-i}$, the bottom tier cycle containing $h_k$ will be "rotated," i.e. since $h_k \in B_j$, the sequence of rejections will cause $h_k$ to match to $d'_k$. However, this is not the case under $\succ'_{-i}$, since $h_k \notin B_j$. Thus, $APDA(\cdot, \succ_{-i}) \neq APDA(\cdot, \succ'_{-i})$.

We now prove the final sentence of the lemma. As argued in Lemma 5.5, $D$ must have read all top cycle applicants' preferences in order to calculate $i$'s menu. Moreover, since $D$ is an individualized dictatorship, it must do so before learning $\succ_i$. Hence, to calculate the outcome matching correctly at the end, $D$ must remember the entirety of $\big(B_j\big)_{j=1}^{n/4}$. This proves Lemma 5.6.

We now prove Theorem 5.4. Together, Lemma 5.5 and Lemma 5.6 show that when $D$ has just calculated the menu of applicant $i$, the description must be in a distinct state for each distinct $\succ_{-i} \in \mathcal{S}$. There is one such $\succ_{-i}$ for each possible way of assigning the sets $B_j \subseteq \{h_k \mid k > n/4\}$ for all $j \in \{1, \ldots, n/4\}$. There are $2^{(n/4)^2} = 2^{\Omega(n^2)}$ possible ways to set this collection $\big(B_j\big)_{j=1}^{n/4}$. Thus, the description requires at least this many states, and thus requires memory $\Omega(n^2)$. This finishes the proof. $\square$

# 6 On Additional Descriptions of DA

In this section, we present our fourth main result, which is an additional negative finding for DA. We examine a broad classification of mechanism descriptions. While we uncover additional descriptions of DA, we find that all such uncovered descriptions (beyond the traditional one and Description 1) are unintuitive and convoluted algorithms that are impractical for real-world use.

To motivate our search for additional descriptions of DA, consider the outline of individualized dictatorships, which provided our highly-useful Description 2 for TTC. Our main impossibility theorem (Theorem 5.4) shows that applicant-proposing individualized dictatorships for DA must be complex; formally, they cannot be linear-memory. However, this theorem does not give any impossibility result for individualized dictatorships for DA which—like our menu description of DA, Description 1—are *institution*-proposing.[30] Given this, one might still hope for a useful institution-proposing individualized dictatorship for DA, which might yield an alternative outcome description of DA together with a simple proof of its strategyproofness.

Perhaps surprisingly, in Appendix C we construct a new institution-proposing individualized dictatorship of DA which is, in fact, linear memory; unfortunately, this description is *exceedingly* unintuitive and convoluted. Indeed, as one can see from the details in Appendix C, this description is a highly technical algorithm that requires careful bookkeeping to maintain its linear-memory. Thus, in contrast to DA's traditional description and our Description 1, this algorithm is impractical for describing DA to real-world participants.

In Appendix D, we additionally use an established formal simplicity condition to demonstrate a sense in which institution-proposing individualized dictatorships of DA *must be* convoluted and impractical. Our linear-memory condition used in Theorem 5.4 does not suffice for this purpose (since our convoluted descriptions in Appendix C satisfy the flexible linear-memory condition); instead, we use the *pick-an-object* simplicity condition of Bó and Hakimov (2023). We prove that institution-proposing individualized dictatorships for DA cannot be pick-an-object.[31] Briefly and informally, this means that all such descriptions must learn the match of some ap-

[30] We use the term institution-proposing to mean the definition perfectly analogous to applicant-proposing (Definition 5.1), in which sides of the market are interchanged.

[31] In Appendix D, we demonstrate more generally that for DA, institution-proposing outcome descriptions—and thus individualized dictatorships as a special case—cannot be pick-an-object.

plicant $d$ when making queries which seem unrelated to $d$, showing a precise sense in which such descriptions cannot be simple. Combined with our robust main impossibility result (Section 5), this shows that one-side-proposing individualized dictatorships for DA cannot (in appropriate senses) be simple.

More broadly, in pursuit of potentially useful descriptions of DA, we consider a broad classification of matching mechanism descriptions. We consider applicant-proposing descriptions (like traditional ones), and institution-proposing descriptions (like Description 1). We consider our three description outlines: menu descriptions, outcome descriptions, and individualized dictatorships. Altogether, this gives six classes of one-side-proposing descriptions with one of these outlines. In Appendix C, we construct linear-memory descriptions of DA of *every* class that is not ruled out by our main impossibility result Theorem 5.4. Unfortunately, all of the additional descriptions in Appendix C are (like our institution-proposing individualized dictatorship) exceedingly unintuitive and convoluted algorithms. See Table 3 for an overview of all our descriptions and results for DA.

Table 3: Classification of descriptions of DA

| | Menu Description | Outcome Description | Individualized Dictatorship |
|---|---|---|---|
| Applicant proposing | Unintuitive, convoluted algorithm in Section C.2. | Traditional DA algorithm. | **Impossible** (without quadratic memory) by Theorem 5.4. |
| Institution proposing | Description 1 in Section 3. | Unintuitive, convoluted algorithm in Section C.1. | Unintuitive, convoluted (e.g., not pick-an-object) algorithm in Section C.3. |

**Notes:** We consider descriptions which either read preferences in an applicant-proposing manner or read priorities in an institution-proposing manner. We consider three description outlines: menu descriptions (conveying strategyproofness), outcome descriptions (conveying the fully matching), or individualized dictatorship (conveying both).

All told, our results exhaustively consider all classes of descriptions of DA that are one-side-proposing and fit one of our three description outlines. Within this classification, we find two simple and practical descriptions of DA: the traditional one, and our menu description. This indicates that within our framework, simple descriptions of DA face a trade-off between conveying strategyproofness and conveying the full outcome matching.

# 7 Related work

Our paper is most directly inspired by the contemporary "strategic simplicity" program in mechanism design theory, which largely considers different dynamic implementations of mechanisms. A cornerstone of this literature is Li (2017), which introduces obviously strategyproof (OSP) mechanisms as a way to expose strategyproofness. Unfortunately, TTC (Li, 2017) and DA (Ashlagi and Gonczarowski, 2018) do not have OSP mechanisms (except in rare special cases of institutions' priorities; see Troyan, 2019; Mandal and Roy, 2021; Thomas, 2021).[32]

In contrast to the above literature, we consider different ex ante descriptions of (static, direct-revelation) mechanisms. Breitmoser and Schweighofer-Kodritsch (2022) provide empirical evidence that framing a static auction as an OSP (ascending-clock) auction can be effective towards conveying strategyproofness. Since DA and TTC do not have OSP implementations, they cannot be framed in this way. Nonetheless, by relaying the match of only a single applicant at a time, menu descriptions frame the mechanism in a way that is OSP for that applicant (and in fact *strongly* OSP; Pycia and Troyan 2023).

The experimental paper of Katuščák and Kittsteiner (2025) also suggests describing matching mechanisms to participants via menu descriptions, but does not investigate any menu description beyond that of Example 2.9, which essentially calculates the menu by iterating over all possible reports and running the traditional mechanism description each time.

We are not aware of any prior characterizations of the menu in DA. Our characterization builds on a large literature developing techniques for reasoning about stable matchings.[33] The menu in DA is different than other commonly considered

[32] A different line of work also considers notions of strategic simplicity that are weaker than strategyproofness (Börgers and Li, 2019; Fernandez, 2020; Troyan and Morrill, 2020; Chen and Möller, 2024; Mennle and Seuken, 2021).

[33] In particular, our proof of Theorem 3.2 in Appendix B analyzes DA by incrementally modifying preference lists. Similar techniques appear in Gale and Sotomayor (1985); Teo et al. (2001); Immorlica and Mahdian (2005); Hatfield and Milgrom (2005); Gonczarowski (2014); Ashlagi et al. (2017); Cai and Thomas (2022), for example. Our proof of Theorem 3.2 in Section 3 uses the strategyproofness of DA; to our knowledge, this is a fairly novel technique.

Certain other properties of DA (e.g., in Blum et al., 1997; Adachi, 2000) and of unit-demand auctions (e.g., in Gul and Stacchetti, 2000; Alaei et al., 2016), despite not being studied with definitions in the theory of stable matching, such as applicant $i$'s set of stable partners (Gale and Shapley, 1962) or her budget set of institutions $h$ where she is above the $h$'s cutoff (Segal, 2007; Azevedo and Leshno, 2016; Luflade, 2017; Azevedo and Budish, 2019; Immorlica et al., 2020). In particular, in finite matching markets, these other commonly-considered sets depend on applicant $i$'s report, and hence do not equal $i$'s menu. We provide explicit examples and more discussion in Remark B.5 and Remark B.6.

Proposition 2 in Leshno and Lo (2021) characterizes the menu in TTC in a different way from our Description 2. Their characterization does not give an individualized dictatorship for TTC, and hence cannot be used in the same way as Description 2 to derive a simple proof of the strategyproofness of TTC's traditional description.

Our paper is also loosely inspired by the literature within computer science studying menus. These works largely focus on single-player selling mechanisms (e.g., Hart and Nisan, 2019; Daskalakis et al., 2017; Babaioff et al., 2022; Saxena et al., 2018; Gonczarowski, 2018).[34] Papers considering menus in multi-player mechanisms include Dobzinski (2016) and Dobzinski et al. (2022), who use menus as a tool for bounding communication complexity. We do not know of any prior algorithmic work on menus of matching mechanisms, nor of any prior work that analyzes different ways to describe multi-player mechanisms in terms of menus.

The present paper is part of our broader research agenda. In the working paper version (Gonczarowski, Heffetz, and Thomas, 2023), we consider more general environments, study a basic extension of our theory for auctions, and conduct an experiment for a second-price auction and median voting. The theoretical computer science paper Gonczarowski and Thomas (2024) investigates a number of complexity questions related to our main theorems (particularly, to Theorems 3.2, 4.1, and 5.4).

Most relevantly, the empirical companion paper Gonczarowski, Heffetz, Ishai, and Thomas (2024) investigates participants' responses to two different descriptions of DA—namely, the traditional one, and Description 1 (our menu description). We find evidence that, while Description 1 is significantly more complex for participants than the traditional one, many participants can understand Description 1 and calculate its outcomes. Interestingly, while levels of strategyproofness-understanding are

relation to menus, bear some technical similarity to the menu calculation in Description 1. However, the proofs seem unrelated.

[34] Brânzei and Procaccia (2015); Golowich and Li (2022) study the computational complexity of checking whether a mechanism, given its extensive- or normal-form representation, is strategyproof.

similar under both descriptions of DA, we see very high levels of strategyproofness-understanding under a less-complex, stripped-down menu description which omits the details of how the menu is calculated. This stripped-down menu description—which relays *only* strategyproofness—yields even higher strategyproofness-understanding levels than a description of strategyproofness inspired by textbook definitions, highlighting its potential as a practical way to convey strategyproofness. This stripped-down menu description could naturally complement real-world usage of Description 1 by offering an accessible summary focusing on the aspects of Description 1 that are important for strategyproofness.

# 8 Discussion

Strategyproofness has long been proposed as a way to make mechanisms fair by leveling the playing field for players who do not strategize well (Pathak and Sönmez, 2008). We warmly embrace this agenda. However, we observe that if participants do not all *understand* strategyproofness, then disparities may remain. Menu descriptions aim to improve this understanding. From a menu description, participants can more directly see why the mechanism is strategyproof, offering an alternative to status-quo tactics such as appeals-to-authority asserting that the mechanism is strategyproof.[35]

While menu descriptions expose strategyproofness, they may obscure other properties of the mechanism. For example, since Description 1—our menu description of DA—relays each applicant's match separately, it is unclear why this description always produces a feasible matching, a fact that is clear in the traditional description.[36] Description 2—our individualized dictatorship for TTC—might be used to simultaneously convey strategyproofness and feasibility. However, for DA, our negative results suggest a sense in which, for a broad class of simple descriptions (see

[35] One common prior approach taken by clearinghouses is to encourage straightforward reporting without explaining strategyproofness. For example, Dreyfuss et al. (2022) notes that an informative video by the National Resident Matching Program (NRMP) was formerly introduced with the text:

> Research on the algorithm was the basis for awarding the 2012 Nobel Prize in Economic Sciences. To make the matching algorithm work best for you, create your rank order list in order of your true preferences, not how you think you will match.

[36] While traditional mechanism descriptions require participants to trust the description (as noted in, e.g., Akbarpour and Li, 2020), the fact that menu descriptions obscure feasibility may influence some participants' levels of trust. While our work focuses on understanding, trust may be an interesting direction for future theoretical or empirical work.

Section 6), the tradeoff between conveying strategyproofness and conveying feasibility is unavoidable. Future empirical work may present TTC to lab participants using our Description 2—or (as suggested by Morrill and Roth, 2024) use this description to explain the strategyproofness of TTC's traditional one—and measure participants' understanding of both strategyproofness and feasibility.

In this paper and its experimental companion (Gonczarowski, Heffetz, Ishai, and Thomas, 2024), we suggest that some principled alternative framings of mechanisms (namely, menu descriptions) might better convey their properties (namely, strategyproofness), and we analyze such framings theoretically and empirically. We view the general premise behind our work—of formally reasoning about the simplicity of, and properties exposed by, different descriptions of the same mechanism—as being of potential broader use. Future theoretical work might consider other properties one may wish to expose (e.g., fairness or optimality) and study opportunities and tradeoffs for exposing these properties in a variety of different mechanisms and settings.

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# Appendix

## A Detailed Mathematical Model of Algorithms

In this appendix, we define from first-principles a mathematical model of descriptions of mechanisms which can express all our results.

We introduce the notion of an *extensive-form description*. For generality, we state this definition in terms of a general mechanism design environment with players $1, \ldots, n$, type spaces $\mathcal{T}_1, \ldots, \mathcal{T}_n$, and outcome space $A$. At a technical level, an extensive-form description is similar to an extensive-form mechanism, except that different branches may "merge," i.e., the underlying game tree is actually a directed acyclic graph (DAG).[1] Note, however, that the interpretation is different from that of an extensive-form mechanism: Rather than modeling an interactive process where the players may act multiple times, an extensive-form description spells out the steps used to calculate some result by iteratively querying the directly-reported types of the players.

We are interested in three types of extensive-form descriptions, corresponding to our three description outlines: outcome descriptions, menu descriptions, and individualized dictatorships.

**Definition A.1** (Extensive-Form Descriptions)**.**

- An *extensive-form description* in some environment is defined by a directed graph on some set of vertices $V$.[2] There is a (single) root vertex $s \in V$, and the vertices of $V$ are organized into *layers* $j = 1, \ldots, L$ such that each edge goes between layer $j$ and $j + 1$ for some $j$. For a vertex $v$, let $S(v)$ denote the edges outgoing from $v$. Each vertex $v$ with out-degree at least 2 is associated with some player $i$, whom the vertex is said to *query*, and some *transition function* $\ell_v : \mathcal{T}_i \to S(v)$ from types of player $i$ to edges outgoing from $v$. (It will be convenient to also allow vertices with out-degree 1, which are not associated with any player.) For each type profile $(t_1, \ldots, t_n)$, the *evaluation path* on

[1] Alternatively, extensive-form descriptions can be viewed as finite automata where state transitions are given by querying the types of players.

[2] Formally, a directed graph $G$ on vertices $V$ is some set of ordered pairs $G \subseteq V \times V$. An element $(v, w) \in G$ is called an *edge* from $v$ to $w$. A *source* (resp., *sink*) vertex is any $v$ where there exists no vertex $w$ with an edge from $w$ to $v$ (resp., from $v$ to $w$).

$(t_1, \ldots, t_n) \in \mathcal{T}_1 \times \ldots \times \mathcal{T}_n$ is defined as follows: Start in the root vertex $s$, and whenever reaching any non-terminal vertex $v$ that queries a player $i$ and has transition function $\ell_v$, follow the edge $\ell_v(t_i)$.

- An *extensive-form outcome description* of a mechanism $f$ is an extensive-form description in which each terminal vertex is labeled by an outcome, such that for each type profile $(t_1, \ldots, t_n) \in \mathcal{T}_1 \times \mathcal{T}_n$, the terminal vertex reached by following the evaluation path on $t \in T$ is labeled by the outcome $f(t_1, \ldots, t_n)$.

- An *extensive-form menu description* of a social choice function $f$ for player $i$ is an extensive-form description with $k + 1$ layers, such that (a) each vertex preceding layer $k$ queries some player other than $i$, (b) each vertex $v$ in layer $k$ queries player $i$ and is labeled by some set $M(v) \subseteq A_i$, such that if $v$ is on the evaluation path on a type profile $(t_1, \ldots, t_n) \in \mathcal{T}_1 \times \mathcal{T}_n$, then $M(v) = \mathcal{M}_{t_{-i}}$ is the menu of player $i$ with respect to $t_{-i}$ in $f$, and (c) each (terminal) vertex $v$ in the final layer $k + 1$ is labeled by an outcome for player $i$,[3] such that if $v$ is reached by following the evaluation path on a type profile $(t_1, \ldots, t_n)$, then $v$ is labeled by $i$'s outcome in $f(t_1, \ldots, t_n)$.

- An *extensive-form individualized dictatorship (description)* of $f$ for player $i$ is an extensive-form outcome description such that, for some $k$, the first $k + 1$ layers are an extensive-form menu description.

For a concrete example of an extensive-form description, we consider a menu description of a second price auction.[4] In this mechanism, a bidder's menu consists of two options: winning the item and playing the highest bid placed by any other bidder, or winning nothing and paying nothing. Thus, a menu description can be given as follows:

(1) Your "price to win" the item will be set to the highest bid placed by any other player.

[3]Formally, in a general mechanism design environment, an *outcome of player* $i$ (or, an $i$*-outcome*) is a maximal set $E$ of outcomes such that all possible types of player $i$ in $\mathcal{T}_i$ view each outcome in $E$ as equally desirable.

[4]While we have not formally defined menus or menu descriptions in non-matching environments, they naturally generalize by considering the menu of $i$ induced by reports $t_{-i}$ to be the set of $i$'s outcomes consistent with $t_{-i}$.

(2) If your bid is higher than this "price to win," then you will win the item and pay this price. Otherwise, you will win nothing and pay nothing.

An extensive-form description can formalize this menu description by querying the other bidders one-by-one, while keeping track of only the highest bid placed by any of them. Figure A.1 provides an illustration.

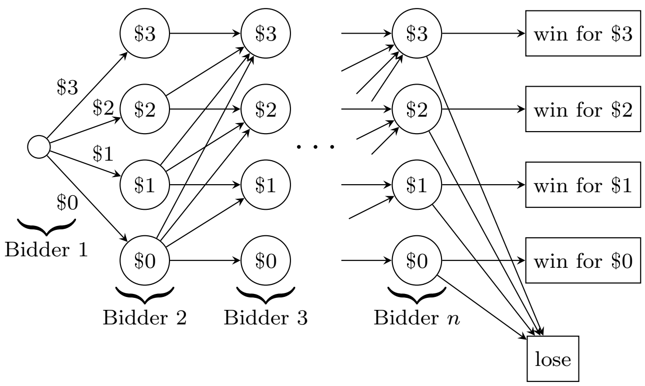


Figure A.1: An extensive-form menu description for bidder $n$ in a second-price auction

**Note:** The second-to-last layer is labeled with bidder $n$'s menu, abbreviated in the figure by the price she must pay to win the item.

More broadly, any precise algorithm taking players types as inputs induces an extensive-form description in a natural way: the vertices in layer $j$ are the possible states of the algorithm after querying the types of different players altogether $j$ times. In particular, our positive results (Description 1 and Description 2) correspond to extensive-form descriptions. The definitions of all our simplicity desiderata (Definition 5.1 and Definition D.1 below) also extend naturally to extensive-form descriptions. Moreover, the proofs of our impossibility theorems (Theorem 5.4 and Proposition D.2 below) hold, mutatis mutandis, for the relevant class of extensive form descriptions.

In addition to providing a self-contained mathematical language for expressing our results, the definition of an extensive-form description allows us to clarify some ways in which our impossibility results are strong. Namely, while algorithms are often required to work for any number of players, our impossibility results hold even if one can use a separate extensive-form description for each number of players $n$, and regardless of the computational complexity of such a description. Relatedly, our impossibility results follow from direct combinatorial arguments and do not depend on any complexity-theoretic conjectures such as $P \neq NP$.

# B Omitted Proofs and Remarks

In this appendix, we provide proofs and remarks omitted from the main text.

We start by proving Theorem 3.2, which shows that Description 1 is a menu description of DA,without assuming the strategyproofness of DA. This providing an alternative, instructive approach for proving DA's strategyproofness. For additional completeness, known results used in the proof are stated with full proofs in Appendix E.

*Direct Proof of Theorem 3.2.* For this proof, we denote applicant $i$—to whom the menu is described—by $d_*$. We also let $P$ be the preference profile, let $P_{d_*}$ denote $d_*$'s preference in $P$. Let $h_* = APDA_{d_*}(P)$ denote the match of $d_*$ according to applicant-proposing DA. We wish to show that $h_*$ is the $P_{d_*}$-favorite institution in the set containing (1) the "outside option" of going unmatched, and (2) all institutions $h$ such that $h$ prefers $d_*$ to $IPDA_h(P_{-d_*})$ (the match of $h$ according to institution-proposing DA in the market without $d_*$).

Let $P|_{d_*:\emptyset}$ denote the preference profile obtained by altering $P$ so that $d_*$ reports an empty preference list (i.e., marking all institutions as unacceptable). Note that $IPDA(P_{-d_*})$ and $IPDA(P|_{d_*:\emptyset})$ produce the same matching (ignoring $d_*$), and furthermore, the institutions $h$ that prefer $d_*$ to $IPDA_h(P_{-d_*})$ are exactly those that propose to $d_*$ during (the calculation of) $IPDA(P|_{d_*:\emptyset})$. We therefore wish to prove:

1. If $h_* \neq \emptyset$, then then $h_*$ proposes to $d_*$ during $IPDA(P|_{d_*:\emptyset})$.

2. $d_*$ gets no proposal in $IPDA(P|_{d_*:\emptyset})$ that is $P_{d_*}$-preferred to $h_*$.

We start with the first claim. Assume that $h_* \neq \emptyset$. Let $P|_{d_*:\{h_*\}}$ denote the preference profile obtained by altering $P$ so that $d_*$ reports a preference list consisting only of $h_*$ (i.e., marking all other institutions as unacceptable). Observe that $APDA(P)$, the applicant-proposing DA outcome for preferences $P$, is stable under preferences $P|_{d_*:\{h_*\}}$. Thus, by the Lone Wolf / Rural Hospitals Theorem (Roth, 1986, see Theorem E.6), since $d_*$ is matched in $APDA(P)$, she must be matched in $IPDA(P|_{d_*:\{h_*\}})$ as well. Thus, $IPDA(P|_{d_*:\{h_*\}}) = h_*$. Since regardless of the order in which we choose to make proposals in DA, the same proposals are made and the same outcome is reached (Dubins and Freedman, 1981, see Corollary E.3), the following is a valid run of $IPDA(P|_{d_*:\emptyset})$: first run $IPDA(P|_{d_*:\{h_*\}})$, then have $d_*$ reject $h_*$, then continue running (according to $P|_{d_*:\emptyset}$) until IPDA concludes. Thus, $h_*$ proposes to $d_*$ during $IPDA(P|_{d_*:\emptyset})$, proving the first claim.

We move on to the second claim. Let $T$ denote $d_*$'s preference list, truncated just above $h^*$ (i.e., obtained by altering $d_*$'s list in $P$ by removing any institution she does not strictly prefer to $h_*$). Let $P|_{d_*:T}$ denote the preference profile replacing $d_*$'s preference list in $P$ by the truncated list $T$. To prove the second claim, it suffices to prove that $d_*$ is not matched in $IPDA(P|_{d_*:T})$. To see why this suffices, note that if this is the case, then $d_*$ rejects all proposals made to it during $IPDA(P|_{d_*:T})$, and hence this run also constitutes also a valid run of $IPDA(P|_{d_*:\emptyset})$, and since $d_*$ gets no proposals $P_{d_*}$-preferred to $h_*$ in the former, neither does it receive such proposals in the latter, proving the second claim. It therefore remains to prove that $d_*$ is not matched in $IPDA(P|_{d_*:T})$.

Suppose for contradiction that $d_*$ is matched in $\mu' = IPDA(P|_{d_*:T})$. Since DA always results in a stable matching under the reported preferences (Gale and Shapley, 1962, see Lemma E.1), $\mu'$ is stable for $P|_{d_*:T}$. But by the fact that APDA results in the applicant-optimal stable matching (Gale and Shapley, 1962, see Corollary E.3), and since $d_*$ prefers her match in $\mu'$ to her match in $APDA(P)$, $\mu'$ is not stable for $P$. Therefore, there is a blocking pair for $\mu'$ under $P$. Since such a pair must not block under $P|_{d_*:T}$ (since $\mu'$ is stable under these preferences), it must involve applicant $d_*$, as her preference order is the only one that differs between $P$ and $P|_{d_*:T}$. Let $(d_*, h)$ be this blocking pair. Therefore, $h \succ^P_{d_*} \mu'(d)$. But $\mu'(d)$ is still on $d_*$'s truncated list $T$ (used in $P|_{d_*:T}$), and thus $h$ is on this list as well. Thus, this pair blocks for $\mu'$ under $P|_{d_*:T}$ as well, and so $\mu'$ is unstable for $P|_{d_*:T}$, a contradiction. □

**Remark B.1.** As noted in Section 3, Theorem 3.2 extends to many-to-one markets with substitutable priorities. To quickly see why this extension holds in the special case in which institutions have responsive preferences (i.e., the special case in which each institution has a master preference order and a capacity), fix a many-to-one market, and following a standard approach, consider a one-to-one market where each institution from the original market is split into "independent copies." That is, the number of copies of each institution equals the capacity of the institution, each "copied" institution has the same preference list as the original institution, and each applicant ranks all the copies of the institution (in any order) in the same way she ranked the original institution. Ignoring the artificial difference between copies of the same institution, the run of applicant-proposing DA is equivalent under these two markets. Thus, an applicant's menu is equivalent under both markets, and so by Theorem 3.2, a menu description for the many-to-one market can be given through institution-proposing DA under the corresponding one-to-one market, which in turn is equivalent to institution-proposing DA under the original market (where at each step, each institution proposes to a number of applicants up to its capacity). The only change in Description 1 in this case would be replacing the condition $d \succ_h \mu_{-d}(h)$ with $\exists d' \in \mu_{-d}(h) : d \succ_h d'$.

**Remark B.2.** As additionally noted in Section 3, Theorem 3.2 also extends to many-to-one markets with contracts in which the institutions have substitutable preferences that satisfy the law of aggregate demand (the conditions under which Hatfield and Milgrom (2005) prove that the strategyproofness of applicant-proposing DA and the rural hospitals theorem hold). Description A.1 gives a menu description of DA in this environment, which generalizes Description 1 as follows: (1) Description A.1 uses the generalized Gale–Shapley algorithm of Hatfield and Milgrom (2005) starting from $(\emptyset, X)$ (where $X$ is the set of all possible contracts) to calculate the institution-optimal stable outcome without $d_*$ to get a matching $\mu_{-d_*}$. (2) A given contract $c = (d_*, h, c)$ (i.e., an $(\text{applicant}, \text{institution}, \text{term})$ tuple) is on $d_*$'s menu if and only if $h$ would choose $(d_*, h, c)$ if given a choice from the set containing $(d_*, c)$ and its matches in $\mu_{-d_*}$ (in the notation of Hatfield and Milgrom (2005), $c \in C_h(\mu_{-d_*}(h) \cup \{c\})$). Under this modification, each step of the proof of Theorem 3.2 in Section 3 holds by a completely analogous argument for this market.

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**Description A.1** A menu description of the applicant-optimal stable matching in a many-to-one market with contracts

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(1) Calculate the institution-optimal stable matching with applicant $d$ removed from the market using the generalized Gale–Shapley algorithm of Hatfield and Milgrom (2005). Call the resulting matching $\mu_{-d}$. Let $M$ be the set of contracts $c = (d, h, t)$ involving applicant $d$ such that $c \in C_h(\mu_{-d}(h) \cup \{c\})$.

(2) Match $d$ to $d$'s highest-ranked contract in $M$.

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**Remark B.3.** In this remark, we show how Theorem 3.2, which characterizes the menu in DA in terms of Description 1, can be used to prove results from Ashlagi et al. (2017) via arguments similar to Cai and Thomas (2022). Consider a randomized market with $n+1$ applicants and $n$ institutions, where such that each applicant/institution draws a full-length preference list uniformly at random, and let $\mu$ be the result of (applicant-optimal) DA with these preferences. We prove that the expected rank each applicant receives on their preference list (formally, the expectation of $|\{h : h \succeq_d \mu(d)\}|$ for any $d$) is at least $(1-\epsilon)n/\log(n)$ for any $\epsilon > 0$ and large enough $n$.

Fix an applicant $d_*$, and consider calculating $d_*$'s menu using Description 1 in this market. This is equivalent to considering IPDA in a market where $d_*$ rejects all proposals, and setting $d_*$'s menu to consist of all proposals she receives. By the principle of deferred decisions, this run of IPDA can be constructed by letting each institution $h$ proposes to a uniformly random applicant (among those $h$ has not yet proposed to) each time she proposes. Observe that this run of IPDA will terminate as soon as each of the $n$ applicants other than $d_*$ receives a proposal. Thus (much like the standard case of $n$ applicants and $n$ institutions in APDA Wilson (1972)), the total number of proposals made in this run of IPDA is stochastically dominated by a coupon collector random variable. Thus, intuitively, the total number of proposals will be $n\log(n)$, and $\log(n)$ of these will go to $d_*$ in expectation, and $d_*$'s top choice out of these $\log(n)$ proposals will be their $n/\log(n)$th ranked choice overall.

Formally, let $Y$ denote the number of proposals $d_*$ receives, and let $\overline{Y}$ denote the same quantity in a market where each institution makes each proposal completely uniformly at random (without regard to prior proposals); it follows that $Y$ is stochastically dominated by $\overline{Y}$. Let $\overline{Z_i}$ denote the total number of proposals between the $(i-1)$th and $i$th distinct applicant in $\mathcal{D} \setminus \{d_*\}$ receiving a proposal (in the market with repeated proposals). The expected value of $Z_i$ is exactly $(n+1)/(n+1-i)$, and each of these $Z_i$ proposals (except for the final one) has a $1/i$ probability of going to $d_*$. Thus, we have

$$\mathbb{E}[Y] \le \mathbb{E}\left[\overline{Y}\right] = \sum_{i=1}^{n} \frac{1}{i}\left(\frac{n+1}{n+1-i} - 1\right) = \sum_{i=1}^{n} \frac{1}{i}\left(\frac{i}{n+1-i}\right) = H_n \le \log(n) + 1.$$

Now, let $R = |\{h : h \succeq_d h_*\}|$, where $h_*$ is $d_*$'s top-ranked proposal received (i.e., $d_*$'s match in APDA). One can show that, conditioned on $Y = y$, we have the expected value of $R$ exactly equal to $(n+1)/(y+1)$ (see for example (Cai and Thomas, 2022, Claim A.1)). Thus, by Jensen's inequality, we have

$$\mathbb{E}[R] = \mathop{\mathbb{E}}_{y \sim Y}\left[\frac{n+1}{y+1}\right] \geq \frac{n+1}{\mathbb{E}[Y]+1} \geq \frac{n+1}{\log(n)+2} \geq (1-\epsilon)\frac{n}{\log(n)}$$

for any $\epsilon > 0$ and large enough $n$, as desired.

**Remark B.4.** We now formally show that, unlike SD, a description of TTC *must* be specialized to individual applicants in order to contain a menu description for them.

To do this, it suffices to construct an instance containing two applicants $d_1$ and $d_2$ such that each of their menus depends on the other. For example, consider an instance where $h_i : d_i \succ d_{3-i}$ and $d_i : h_{3-i} \succ h_i$ for $i \in \{1, 2\}$. Under this instance, for each $i \in \{1, 2\}$, institution $h_{3-i}$ is on $d_i$'s menu, but if applicant $d_{3-i}$ changed her preference list, this would no longer be true. Hence, a description cannot calculate either applicant's menu before the description queries the other applicant's type.

**Remark B.5.** We now show that in (finite-market) DA, budget sets and menus are different sets, and moreover, neither set includes the other. For a fixed profile of preferences and priorities, denote an applicant $i$'s budget set $B(i) = \{h | i \succeq_h \mu(i)\}$, where $\mu$ is the outcome of DA. Let $M(i) = \mathcal{M}_{\succ_{-i}}$ denote $i$'s menu.

Now, consider the market with institutions $h_1, h_2, h_3$, and $h_4$, and applicants $d_1, d_2, d_3$, and $d_4$. Let the preferences and priorities be as follows:

$$\begin{array}{ll} h_1 : d_1 \succ d_2 \succ \ldots & \qquad d_1 : h_1 \succ \ldots \\ h_2 : d_4 \succ d_3 \succ d_2 \succ d_1 \succ \ldots & \qquad d_2 : h_1 \succ h_2 \succ h_4 \succ \ldots \\ h_3 : d_3 \succ \ldots & \qquad d_3 : h_3 \succ \ldots \\ h_4 : d_2 \succ d_4 \succ \ldots & \qquad d_4 : h_4 \succ h_2 \succ \ldots \end{array}$$

Then, one can check that DA pairs $h_i$ to $d_i$ for each $i = 1, \ldots, 4$, and that $h_2 \in B(d_3) \setminus M(d_3)$, and also $h_2 \in M(d_1) \setminus B(d_1)$. Thus, neither the menu nor the budget set contain the other. Moreover, the relationship between the two sets does not seem to be restricted in a straightforward way based on priorities and the outcome of DA: despite the fact that $d_3 \succ_{h_2} d_2$, we have $h_2 \notin M(d_3)$; despite $d_1 \prec_{h_2} d_2$, we have $h_2 \in M(d_1)$.

**Remark B.6.** We now show that in DA, an applicant's set of stable partners is a (possibly strict) subset of her menu. For a given profile of preferences and priorities, let $S(i)$ denote the set of stable partners of applicant $i$, and let $M(i)$ denote her menu. We begin by showing that $M(i) \neq S(i)$. Consider any instance with two institutions $h_1, h_2$ which both rank $i$ above all other applicants. Both $h_1$ and $h_2$ must be in $i$'s menu. However, $i$ ranks $h_1$ above all other institutions, then $h_1$ is $i$'s unique stable partner; thus $h_2 \in M(i) \setminus S(i)$.

We now show that $S(i) \subseteq M(i)$. Suppose the profile of preferences and priorities is $P$. Consider any $h \in S(i)$, and let $\mu$ be a stable matching with $\mu(i) = h$. Then, let $\widetilde{P}$ denote modifying $P$ by having $i$ submit a list which ranks only $h$. Then, observe that $\mu$ is also stable under $\widetilde{P}$. Thus, by the Rural Hospital Theorem (Theorem E.6), $i$ and $h$ must be matched in every stable matching under $\widetilde{P}$, in particular, in $DA(\widetilde{P})$. Thus, $h \in M(i)$, and $S(i) \subseteq M(i)$.

# C Unintuitive and Convoluted Descriptions of DA

In this appendix, we present additional descriptions of DA. While these are interesting on a technical level, we believe these descriptions are vastly more complicated than traditional descriptions of DA, and quite impractical. For notational convenience, in this appendix, we refer to the priorities of institutions as "preferences." We also denote the set of applicants by $\mathcal{D}$, the set of institutions by $\mathcal{H}$, and (when relevant) we describe the menu to applicant $d_*$.

## C.1 Institution-proposing outcome description of DA

First, we construct an *institution*-proposing linear-memory outcome description of DA. Interestingly, essentially this same algorithm was used as a lemma by Ashlagi et al. (2017) (henceforth, AKL).[5]

**Theorem C.1** (Adapted from Ashlagi et al., 2017)**.** *Description A.2 computes the applicant-optimal stable outcome. Moreover, Description A.2 is an institution-proposing and $\widetilde{\mathcal{O}}(n)$-memory description.*

[5]For context, Ashlagi et al. (2017) needs such an algorithm to analyze (for a random matching market) the expected "gap" between the applicant and institution optimal stable matching. Their algorithm builds on the work of Immorlica and Mahdian (2005), and is also conceptually similar to algorithms for constructing the "rotation poset" in a stable matching instance Gusfield and Irving (1989a) (see also Cai and Thomas (2019)).

*Proof.* AKL refer to the sides of the market as "men" and "women", and define "Algorithm 2 (MOSM to WOSM)", a men-proposing algorithm for the women-optimal stable matching. Description A.2 follows the exact same order of proposals as this algorithm from AKL. The only difference apart from rewriting the algorithm in a more "pseudocode" fashion is that Description A.2 performs bookkeeping in a slightly different way—Algorithm 2 from AKL maintains *two* matchings, and their list $V$ keeps track of only women along a rejection chain; our list $V$ keeps track of both applicants and institutions along the rejection chain (and can thus keep track of the "difference between" the two matchings which AKL tracks).

Moreover, the algorithm is institution-proposing, by construction. Furthermore, as it runs it stores only a single matching $\mu$, a set $\mathcal{D}_{\text{term}} \subseteq \mathcal{D}$, and the "rejection chain" $V$ (which can contain each applicant $d \in \mathcal{D}$ *at most once*). Thus, it uses memory $\widetilde{O}(n)$. □

## C.2 Applicant-proposing menu description of DA

In this section, we construct an applicant-proposing linear-memory menu description of DA. On an intuitive level, the algorithm works as per Example 2.9, but avoiding the need to "restart many times" by using the various properties of DA and by careful bookkeeping (to intuitively "simulate all of the separate runs of the brute-force description on top of each other").

On a formal level, we describe the algorithm as a variant of Description A.2. The proof constructing this algorithm uses a bijection between one applicant's menu in DA under some preferences, and some data concerning the *institution*-optimal stable matching under a related set of preferences. Our applicant-proposing menu description is then phrased as a variation of Description A.2, which (reversing the roles of applicants and institutions from the presentation in Description A.2) is able to compute the institution-optimal matching using an applicant-proposing algorithm.

Fix an applicant $d_*$ and set $P$ that contains (1) the preferences of all applicants $\mathcal{D} \setminus \{d_*\}$ *other than* $d_*$ over $\mathcal{H}$ and (2) the preferences of all institutions $\mathcal{H}$ over all applicants $\mathcal{D}$ (including $d_*$). We now define the "related set of preferences" mentioned above. Define the *augmented preference list* $P'$ as follows: For each $h_i \in \mathcal{H}$, we create two additional applicants $d_i^{\text{try}}, d_i^{\text{fail}}$ and two additional institutions $h_i^{\text{try}}, h_i^{\text{fail}}$. The

**Description A.2** An institution-proposing outcome description of DA

**Input:** Preferences of all applicants $\mathcal{D}$ and institutions $\mathcal{H}$
**Output:** The result of applicant-proposing deferred acceptance

▷ *We start from the institution-optimal outcome, and slowly "improve the match for the applicants"* ◁
Let $\mu$ be the result of institution-proposing DA
Let $\mathcal{D}_{\text{term}}$ be all applicants unmatched in $\mu$ ▷ *$\mathcal{D}_{\text{term}}$ is all applicants at their optimal stable partner*
**while** $\mathcal{D}_{\text{term}} \neq \mathcal{D}$ **do**
  Pick any $\widehat{d} \in \mathcal{D} \setminus \mathcal{D}_{\text{term}}$, and set $d = \widehat{d}$
  Let $h = \mu(d)$ and set $V = [(d, h)]$
  **while** $V \neq []$ **do**
    Let $d \leftarrow$ NextAcceptingApplicant$(\mu, h)$
    **if** $d = \emptyset$ or $d \in \mathcal{D}_{\text{term}}$ **then**
      ▷ *In this case, all the applicants in $V$ have reached their optimal stable partner.* ◁
      Add every applicant which currently appears in $V$ to $\mathcal{D}_{\text{term}}$
      Set $V = []$
    **else if** $d \neq \emptyset$ and $d$ does not already appear in $V$ **then** ▷ *Record this in the rejection chain*
      Add $(d, \mu(d))$ to the end of $V$
      Set $h \leftarrow \mu(d)$ ▷ *The next proposing institution will be the "old match" of $d$.*
    **else if** $d \neq \emptyset$ and $d$ appears in $V$ **then**
      ▷ *A new "rejection rotation" should be written to $\mu$* ◁
      WriteRotation$(\mu, V, d, h)$ ▷ *Updates the value of $\mu$, $V$, and (possibly) $h$*
**Return** $\mu$

**function** NextAcceptingApplicant$(\mu, h)$
  **repeat**
    **Query** $h$'s preference list to get their next choice $d$
  **until** $d = \emptyset$ or $h \succ_d \mu(d)$
  **Return** d

**procedure** WriteRotation$(\mu, V, d, h)$
  Let $T = (d_1, h_1), \ldots, (d_k, h_k)$ be the suffix of $V$ starting with the first occurrence of $d = d_1$
  Update $\mu$ such that $\mu(h_i) = d_{i+1}$ (for each $i = 1, \ldots, k$, with indices taken mod $k$)
  ▷ *Now we fix $V$ and $h$ to reflect the new $\mu$* ◁
  Update $V$ by removing $T$ from the end of $V$
  **if** $V \neq \emptyset$ **then**
    Let $(d_0, h_0)$ denote the final entry remaining in $V$
    ▷ *The next proposing institution will either $h_k$ or $h_0$, depending on which $d_1$ prefers* ◁
    **if** $h_k \succ_{d_1} h_0$ **then**
      Set $h \leftarrow h_0$
    **else if** $h_0 \succ_{d_1} h_k$ **then**
      Add $(d_1, h_k)$ to the end of $V$
      Set $h \leftarrow h_k$

entire preference lists of these additional agents in $P'$ are as follows: for each $h_i \in \mathcal{H}$:

$$d_i^{\text{try}} : h_i^{\text{try}} \succ h_i \succ h_i^{\text{fail}} \qquad\qquad d_i^{\text{fail}} : h_i^{\text{fail}} \succ h_i^{\text{try}}$$
$$h_i^{\text{try}} : d_i^{\text{fail}} \succ d_i^{\text{try}} \qquad\qquad h_i^{\text{fail}} : d_i^{\text{try}} \succ d_i^{\text{fail}}$$

We need to modify the preference lists of the pre-existing institutions as well. But this modification is simple: for each $h_i \in \mathcal{H}$, replace $d_*$ with $d_i^{\text{try}}$. The institution-optimal matching for this augmented set of preferences $P'$ will encode the menu, as we need.[6]

**Proposition C.2.** *An institution $h_i \in \mathcal{H}$ is on $d_*$'s menu in APDA with preferences $P$ if and only if in the institution-optimal stable matching with the augmented preferences $P'$, we have $h_i^{\text{try}}$ matched to $d_i^{\text{try}}$.*

*Proof.* For both directions of this proof, we use the following lemma, which is a special case of the main technical lemma in Cai and Thomas (2022):

**Lemma C.3.** *In $P'$, each $h_i^{\text{try}}$ has a unique stable partner if and only if, when $h_i^{\text{try}}$ rejects $d_i^{\text{try}}$ (i.e. if $h_i^{\text{try}}$ submitted a list containing only $d_i^{\text{fail}}$, and all other preferences remained the same), $h_i^{\text{try}}$ goes unmatched (say, in the applicant-optimal matching).*

Note that each $h_i^{\text{try}}$ is matched to $d_i^{\text{try}}$ in the applicant-optimal matching with preferences $P'$ (and the matching among all original applicants and institutions is the same as $\mu_{\text{app}}$).

($\Leftarrow$) By the lemma, if $h_i^{\text{try}}$ is matched to $d_i^{\text{try}}$ in the institution-optimal matching under $P'$, then $h_i^{\text{try}}$ must go unmatched when $h_i^{\text{try}}$ rejects $d_i^{\text{try}}$. But, after $h_i^{\text{try}}$, we know $d_i^{\text{try}}$ will propose to $h_i$, and some rejection chain may be started. Because $d_i^{\text{try}}$'s very next choice is $h_i^{\text{fail}}$ (and proposing there would lead directly to $h_i^{\text{try}}$ receiving a proposal from $d_i^{\text{fail}}$), the *only* way for $h_i^{\text{try}}$ to remain unmatched is if $d_i^{\text{try}}$ remains matched to $h_i$. But because (relative to all the original applicants) $d_i^{\text{try}}$ is in the same

[6] For the reader familiar with the rotation poset of stable matchings (Gusfield and Irving, 1989b), the intuition for this construction is the following: having $h_i^{\text{try}}$ reject applicant $d_i^{\text{try}}$ corresponds to $d_*$ "trying" to get $h_i \in \mathcal{H}$, i.e., "trying to see if $h_i$ is on their menu." If $d_*$ would be rejected by $h_i$ after proposing, either immediately or after some "rejection rotation," then so will $d_i^{\text{try}}$ (because they serve the same role as $d_*$ at $h_i$). So if a rotation swapping $h_i^{\text{try}}$ and $h_i^{\text{fail}}$ exists (e.g., in the institution optimal matching) then $h_i$ is *not* on $d_*$'s menu. On the other hand, if $d_*$ could actually permanently match to $h_i$, then $d_i^{\text{try}}$ proposing to $h_i$ will result in a rejection chain that ends at some other applicant (either exhausting their preference list or proposing to an institution in $\mathcal{H}_{\text{term}}$), which does not result in finding a rotation (or writing a new set of matches as we "work towards the institution-optimal match"). Thus, if $h_i^{\text{try}}$ and $h_i^{\text{fail}}$ do not swap their matches in the institution-optimal stable outcome, then $h_i$ *is* on $d_*$'s menu.

place as $d_*$ on $h_i$'s preference list, the resulting set of rejections in $P'$ will be precisely the same as those resulting from $d_*$ submitting a preference list in $P$ which contains only $h_i$. In particular, $d_*$ would remain matched at $h_i$ in $P$ if they submitted such a list. Thus, $h_i$ is on $d_*$'s menu.

($\Rightarrow$) Suppose $h_i^{\text{try}}$ is matched to $d_i^{\text{fail}}$ in the institution optimal matching under $P'$. Again, $h_i^{\text{try}}$ must receive a proposal from $d_i^{\text{fail}}$ when $h_i^{\text{try}}$ rejects $d_i^{\text{try}}$. But this can only happen if $d_i^{\text{try}}$ is rejected by $h_i$ (then proposes to $h_i^{\text{fail}}$). But because the preferences of the original applicants in $P'$ exactly corresponds to those in $P$, we know that $d_*$ would get rejected by $h_i$ if they proposed to them in $\mu_{\text{app}}$ under $P$. But then $h_i$ cannot be on $d_*$'s menu. $\square$

With this lemma in hand, we can now show that there is an applicant-proposing linear-memory menu description of (applicant-optimal) DA. This description is given in Description A.3.

**Description A.3** An applicant-proposing menu description of DA

**Input:** An applicant $d_*$ and preferences of all applicants $\mathcal{D} \setminus \{d_*\}$ and institutions $\mathcal{H}$
**Output:** The menu of $d_*$ in applicant-optimal DA given these preferences

1: Simulate the flipped-side version of Description A.2 (such that applicants propose) on preferences $P'$ to get a matching $\mu$
2: **Return** the set of all institutions $h_i$ such that $h_i^{\text{try}}$ is matched to $d_i^{\text{try}}$ in $\mu$

**Theorem C.4.** *There is an applicant-proposing, $\widetilde{O}(n)$ memory menu description of (applicant-optimal) DA.*

*Proof.* The algorithm proceeds by simulating a run of Description A.2 on preferences $P'$ (interchanging the role of applicants and institutions, so that applicants are proposing). This is easy to do while still maintaining the applicant-proposing and $\widetilde{O}(n)$ memory. In particular, $P'$ adds only $O(n)$ applicants and institutions, with each $d_i^{\text{try}}$ and $d_i^{\text{fail}}$ making a predictable set of proposals. Moreover, the modification made to the preferences lists of the institutions $h \in \mathcal{H}$ is immaterial—when such institutions receive a proposal from $d_i^{\text{try}}$, the algorithm can just query their lists for $d_*$. $\square$

## C.3 Institution-proposing individualized dictatorship description of DA

In this section, we construct an institution-linear individualized dictatorship description of (applicant-optimal) DA.[7] Throughout this section, let $P|_{d_i:L}$ denote altering preferences $P$ by having $d_i$ submit list $L$.

Unlike our applicant-proposing menu description of DA from Section C.2, our institution-proposing individualized-dictatorship description cannot be "reduced to" another algorithm such as Description A.2. However, the algorithm is indeed a modified version of Description A.2 that "embeds" our simple institution-proposing menu algorithm Description 1 (i.e., IPDA where an applicant $d_*$ submits an empty preference list) as the "first phase." The key difficulty the algorithm must overcome is being able to "undo one of the rejections" made in the embedded run of Description 1. Namely, the algorithm must match $d_*$ to her top choice from her menu, and "undo" all the rejections caused by $d_*$ rejecting her choice.[8] To facilitate this, the description has $d_*$ reject institutions that propose to $d_*$ "as slowly as possible," and maintains a delicate $\widetilde{O}(n)$-bit data structure that allows it to undo one of $d_*$'s rejections.[9] The

[7] For some technical intuition on why such a description might exist, consider the construction used in Theorem 5.4, and consider an individualized dictatorship for applicant $i$ executed on these preferences. To find the menu in this construction with an applicant-proposing algorithm, all of the "top tier rotations" must be "rotated", but to find the correct final matching after learning $t_i$, some arbitrary subset of the rotations must be "unrolled" (leaving only the subset of rotations which $t_i$ actually proposes to). Theorem 5.4 shows that all of this information must thus be remembered in full. Now consider a run of Description A.2 on these preferences (or on a modified form of these preferences where institutions' preference lists determine which top tier rotations propose to bottom tier rotations). Some subset of top-tier institutions will propose to applicant $i$. To continue on with a run of Description A.2, it suffices to undo *exactly one* of these proposals. So, if two or more top-tier rotations trigger a bottom-tier rotation, then we can be certain that the bottom-tier rotation will be rotated, and we only have to remember which bottom-tier rotations are triggered by exactly one top-tier rotation (which takes $\widetilde{O}(n)$ bits).

[8] Description A.2 is independent of the order in which proposals are made. Moreover, one can even show that $d_*$ receives proposals from all $h$ on her menu in Description A.2. However, this does not suffice to construct our individualized dictatorship simply by changing the order of Description A.2. The main reason is this: in Description A.2, the preferences of $d_*$ are already known, so $d_*$ can reject low-ranked proposals without remembering the effect that accepting their proposal might have on the matching. While the "unrolling" approach of Description A.4 is inspired by the way Description A.2 effectively "unrolls rejection chains" (by storing rejections in a list $V$ and only writing these rejections to $\mu$ when it is sure they will not be "unrolled"), the bookkeeping of Description A.4 is far more complicated (in particular, the description maintains a DAG $\Delta$ instead of a list $V$).

[9] Interestingly, this "rolled back state" is *not* the result of institution-proposing DA on preferences $(P, d_i : \{h_j\})$, where $h_j$ is $d_i$'s favorite institution on her menu. Instead, it is a "partial state" of Description A.2 (when run on these preferences), which (informally) may perform ad-

way this data structure works is involved, but one simple feature that illustrates how and why it works is the following: *exactly one* rejection from $d_*$ will be undone, so if some event is caused by *more than one* (independent) rejection from $d_*$, then this event will be caused regardless of what $d_*$ picks from the menu.

We present our algorithm in Description A.4. For notational convenience, we define a related set of preferences $P_{\text{hold}}$ as follows: For each $h_i \in \mathcal{H}$, add a "copy of $d_*$" called $d_i^{\text{hold}}$ to $P_{\text{hold}}$. The only acceptable institution for $d_i^{\text{hold}}$ is $h_i$, and if $d_*$ is on $h_i$'s list, replace $d_*$ with $d_i^{\text{hold}}$ on $h_i$'s list. Given what we know from Section 3, the proof that this algorithm calculates the menu is actually fairly simple:

**Lemma C.5.** *The set $\mathcal{H}_{\text{menu}}$ output by Description A.4 is the menu of $d_*$ in (applicant-proposing) DA.*

*Proof.* Ignoring all bookkeeping, Phase 1 of this algorithm corresponds to a run of $IPDA(P|_{d_*:\emptyset})$. The only thing changed is the order in which $d_*$ performs rejections, but DA is invariant under the order in which rejections are performed. Moreover, $\mathcal{H}_{\text{menu}}$ consists of exactly all institutions who propose to $d$ during this process, i.e. $d_*$'s menu (according to Section 3). □

The correctness of the matching, on the other hand, requires an involved proof. The main difficulty surrounds the "unroll DAG" $\Delta$, which must be able to "undo some of the rejections" caused by $d_*$ rejecting different $h$. We start by giving some invariants of the state maintained by the algorithm (namely, the values of $\Delta$, $\mu$, $P$, and $h$):

**Lemma C.6.** *At any point outside of the execution of* AdjustUnrollDAG*:*

*(1) $P$ contains all nodes in $\Delta$ of the form $(d, h)$ (where $h$ is the "currently proposing" $h \in \mathcal{H}$).*

*(2) All of the nodes in $P$ have out-degree $0$.*

*(3) The out-degree of every node in $\Delta$ is at most $1$.*

*(4) Every source node in $\Delta$ is of the form $(d_*, h_i)$ for some $h_i \in \mathcal{H}_{\text{menu}}$.*

*(5) For every edge $(d_0, h_0)$ to $(d_1, h_1)$ in $\Delta$, we have $\mu(d_1) = h_0$.*

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ditional "applicant-improving rotations" on top of the result, and thus we can continue running Description A.2 until we find the applicant-optimal outcome.

**Description A.4** An institution-proposing individualized dictatorship description of DA

**Phase 1 input:** An applicant $d_*$ and preferences of applicants $\mathcal{D} \setminus \{d_*\}$ and institutions $\mathcal{H}$
**Phase 1 output:** The menu $\mathcal{H}_{\text{menu}}$ presented to $d_*$ in (applicant-proposing) DA
**Phase 2 input:** The preference list of applicant $d_*$
**Phase 2 output:** The result of (applicant-proposing) DA

▷ ***Phase 1:*** ◁
Simulate a run of $IPDA(P_{\text{hold}})$ and call the result $\mu'$
Let $\mathcal{H}_*$ be all those institutions $h_i \in \mathcal{H}$ matched to $d_i^{\text{hold}}$ in $\mu'$ ▷ *These institutions "currently sit at $d_*$"*
Let $\mu$ be $\mu'$, ignoring all matches of the form $(d_i^{\text{hold}}, h)$
Let $\mathcal{H}_{\text{menu}}$ be a copy of $\mathcal{H}_*$ ▷ *We will grow $\mathcal{H}_{\text{menu}}$*
Let $\Delta$ be an empty graph ▷ *The "unroll DAG". After Phase 1, we'll "unroll a chain of rejections"*
**while** $\mathcal{H}_* \neq \emptyset$ **do**
    Pick some $h \in \mathcal{H}_*$ and remove $h$ from $\mathcal{H}_*$
    Add $(d_*, h)$ to $\Delta$ as a source node
    Set $P = \{(d_*, h)\}$ ▷ *This set stores the "predecessors of the next rejection"*
    **while** $h \neq \emptyset$ **do**
        Let $d \leftarrow$ NextInterestedApplicant$(\mu, \Delta, h)$
        AdjustUnrollDag$(\mu, \Delta, P, d, h)$ ▷ *Updates each of these values*
**Return** $\mathcal{H}_{\text{menu}}$
▷ ***Phase 2:*** *We now additionally have access to $d_*$'s preferences* ◁
Permanently match $d_*$ to their top pick $h_{\text{pick}}$ from $\mathcal{H}_{\text{menu}}$
$(\mu, \mathcal{D}_{\text{term}}) \leftarrow$ UnrollOneChain$(\mu, \Delta, h_{\text{pick}})$
**Continue** running the Description A.2 until its end, using this $\mu$ and $\mathcal{D}_{\text{term}}$, starting from Description 4
**Return** the matching resulting from Description A.2

**function** NextInterestedApplicant$(\mu, \Delta, h)$
    **repeat**
        **Query** $h$'s preference list to get their next choice $d$
    **until** $d \in \{\emptyset, d_*\}$ OR ($d$ is in $\Delta$, paired with $h'$ in $\Delta$, and $h \succ_d h'$) OR ($d$ is not in $\Delta$ and $h \succ_d \mu(d)$)
    **Return** d

**procedure** UnrollOneChain$(\mu, \Delta, h_{\text{pick}})$
    Let $(d_0, h_0), (d_1, h_1), \ldots, (d_k, h_k)$ be the (unique) longest chain in $\Delta$ starting from $(d_0, h_0) = (d^*, h_{\text{pick}})$
    Set $\mu(d_i) = h_i$ for $i = 0, \ldots, k$
    Set $\mathcal{D}_{\text{term}} = \{d_*, d_1, \ldots, d_k\}$
    **return** $(\mu, \mathcal{D}_{\text{term}})$

**procedure** AdjustUnrollDag($\mu$, $\Delta$, P, $d$, $h$)

  **if** $d = \emptyset$ **then**

    Set $h = \emptyset$ ▷ *Continue and pick a new h*

  **else if** $d = d^*$ **then** ▷ *$h$ proposes to $d_*$, so we've found a new $h$ in the menu*

    Add $h$ to $\mathcal{H}_{\text{menu}}$

    Add $(d_*, h)$ to $\Delta$

    Add $(d_*, h)$ to the set $P$ ▷ *$h$ still proposes; the next rejection will have multiple predecessors*

  **else if** $d$ does not already appear in $\Delta$ **then** ▷ *Here $h \succ_d \mu(d)$*

    Add $(d, \mu(d))$ to $\Delta$ ▷ *Record this in the rejection DAG*

    Add an edge from each $p \in P$ to $(d, \mu(d))$ in $\Delta$, and set $P = \{(d, \mu(d))\}$

    Set $h' \leftarrow \mu(d)$, then $\mu(d) \leftarrow h$, then $h \leftarrow h'$

    ▷ *The next proposing institution will be the "old match" of $d$.* ◁

  **else if** $d$ appears in $\Delta$ **then**

    AdjustUnrollDagCollision($\mu$, $\Delta$, $P$, $d$, $h$) ▷ *Updates each of these values*

**procedure** AdjustUnrollDagCollision($\mu$, $\Delta$, P, $d$, $h$)

  Let $p_1 = (d_1, h_1)$ be the pair where $d = d_1$ appears in $\Delta$ ▷ *We know $h \succ_{d_1} h_1$*

  Let $P_1$ be the set of all predecessors of $p_1$ in $\Delta$

  ▷ *First, we drop all rejections from $\Delta$ which we are now sure we won't have to unroll* ◁

  Let $(d_1, h_1), \ldots, (d_k, h_k)$ be the (unique) longest possible chain in $\Delta$ starting from $(d_1, h_1)$ such that *each node $(d_j, h_j)$ for $j > 1$ has exactly one predecessor*

  Remove each $(d_i, h_i)$ from $\Delta$, for $i = 1, \ldots, k$, and remove all edges pointing to these nodes

  ▷ *Now, we adjust the nodes to correctly handle $d_1$ (which might have to "unroll to $h_{min}$")* ◁

  Let $h_{\min}$ be the institution among $\{\mu(d_1), h\}$ which $d_1$ prefers least

  Let $p_{\text{new}} = (d_1, h_{\min})$; add $p_{\text{new}}$ to $\Delta$

  **if** $h_{\min} = h$ **then** ▷ *We replace $p_1$ with $p_{new}$*

    Add an edge from each $p \in P_1$ to $p_{\text{new}}$

    Add $p_{\text{new}}$ to $P$ ▷ *$h$ is still going to propose next*

  **else** ▷ *Here $h_{min} = \mu(d_1)$; we add $p_{new}$ below the predecessors $P$*

    Add an edge from every $p \in P$ to $p_{\text{new}}$

    Set $P = P_1 \cup \{p_{\text{new}}\}$

    Set $h' \leftarrow \mu(d_1)$, then $\mu(d) \leftarrow h$, then $h \leftarrow h'$ ▷ *$d_1$'s old match will propose next*

*(6) For each $d \in \mathcal{D} \setminus \{d_*\}$, there is at most one node in $\Delta$ of the form $(d, h_i)$ for some $h_i$.*

Each of these properties holds trivially at the beginning of the algorithm, and it is straightforward to verify that each structural property is maintained each time AdjustUnrollDag runs.

We now begin to model the properties that $\Delta$ needs to maintain as the algorithm runs.

**Definition C.7.** At some point during the run of any institution-proposing algorithm with preferences $Q$, define the *truncated revealed preferences* $\overline{Q}$ as exactly those institution preferences which have been queried so far, and assuming that all further queries to all institutions will return $\emptyset$ (that is, assume that all institution preference lists end right after those preferences learned so far).

For some set of preferences $Q$ we say the revealed truncated preferences $\overline{Q}$ and the pair $(\mu', \mathcal{D}'_{\text{term}})$ is a *partial AKL state* for preferences $Q$ if there exists some execution order of Description A.2 and a point along that execution path such that the truncated revealed preferences are $\overline{Q}$, and $\mu$ and $\mathcal{D}_{\text{term}}$ in Description A.2 take the values $\mu'$ and $\mathcal{D}'_{\text{term}}$

Let $Q$ be a set of preferences which does not include preference of $d_*$, and let $\overline{Q}$ a truncated revealed preferences of $Q$. Call a pair $(\mu, \Delta)$ *unroll-correct for $Q$ at $\overline{Q}$* if 1) $\mu$ is the result of $IPDA(\overline{Q})$, and moreover, for every $h \in \mathcal{H}_{\text{menu}}$, the revealed preferences $\overline{Q}$ and pair UnrollOneChain$(\mu, \Delta, h)$ is a valid partial AKL state of preferences $(\overline{Q}, d_* : \{h\})$.

The following is the main technical lemma we need, which inducts on the total number of proposals made in the algorithm, and shows that $(\mu, \Delta)$ remain correct every time the algorithm changes their value:

**Lemma C.8.** *Consider any moment where we query some institution's preferences list withing* NextInterestedApplicant *in Description A.4. Let $h$ be the just-queried institution, let $d$ be the returned applicant, and suppose that the truncated revealed preferences before that query are $\overline{Q}$, and fix the current values of $\mu$ and $\Delta$. Suppose that $(\mu, \Delta)$ are unroll-correct for $Q$ at $\overline{Q}$.*

*Now let $\overline{Q}'$ be the revealed preferences after adding $d$ to $h$'s list, and let $\mu'$ and $\Delta'$ be the updated version of these values after Description A.4 processes this proposal* *(formally, if* NextInterestedApplicant *returns $d$, fix $\mu'$ and $\Delta'$ to the values of $\mu$ and $\Delta$ after the algorithm finishes running* AdjustUnrollDag*; if* NextInterestedApplicant *does not return $d$, set $\mu' = \mu$ and $\Delta' = \Delta$). Then $(\mu', \Delta')$ are unroll-correct for $Q$ at $\overline{Q}'$.*

*Proof.* First, observe that if $h$'s next choice is $\emptyset$, then the claim is trivially true, because $\overline{Q} = Q$ (and AdjustUnrollDAG does not change $\mu$ or $\Delta$). Now suppose $h$'s next choice is $d \neq \emptyset$, but is not returned by NextInterestedApplicant. This means that: 1) $d \neq d_*$, 2) $\mu(d) \succ_d h$, and 3) either $d$ does not appear in $\Delta$, or $d$ does appear in $\Delta$, in which case $d$ matched to some $h'$ such that $h' \succ_d h$. Because $(\mu, \Delta)$ are unroll-correct for $Q$ at $\overline{Q}$, and because Lemma C.6 says that $d$ can appear at most once in $\Delta$, the only possible match which $d$ could be unrolled to at truncated revealed preferences $\overline{Q}$ is $h'$ (formally, if the true complete preferences were $\overline{Q}$, then for all $h_* \in \mathcal{H}_{\text{menu}}$, the partial AKL state under preferences $(\overline{Q}, d : \{h_*\})$ to which we we would unroll would match $d$ to either $\mu(d)$ or $h'$). But $d$ would not reject $\mu(d)$ in favor of $h$, nor would she reject $h'$ in favor of $h$. Thus, (for all choices of $h_* \in \mathcal{H}_{\text{menu}}$) we know $h$ will always be rejected by $d$, and $(\mu, \Delta)$ are already unroll-correct for $Q$ at $\overline{Q}'$.

Now, consider a case where $h$'s next proposal $d \neq \emptyset$ is returned by NextInterestedApplicant. There are a number of ways in which AdjustUnrollDAG may change $\Delta$. We go through these cases.

First, suppose $d = d_*$. In this case, the menu of $d_*$ in $\overline{Q}'$ contains exactly one more institution than the menu in $\overline{Q}$, namely, institution $h$. Moreover, for any $h_* \in \mathcal{H}_{\text{menu}} \setminus \{h\}$, the same partial AKL state is valid under both preferences $(\overline{Q}, d : \{h_*\})$ and $(\overline{Q}', d : \{h_*\})$ (the only difference in $(\overline{Q}', d : \{h_*\})$ is a single additional proposal from $h$ to $d_*$, which is rejected; the correct value of $\mathcal{D}_{\text{term}}$ is unchanged). For $h_* = h$, the current matching $\mu$, modified to match $h$ to $d_*$, is a valid partial AKL state for $(\overline{Q}', d : \{h\})$, and this is exactly the result of UnrollOneChain (with $\mathcal{D}_{\text{term}} = \{d_*\}$, which is correct for preferences $(\overline{Q}', d : \{h\})$). Thus, (using also the fact from Lemma C.6 that $P$ contains all nodes in $\Delta$ involving $h$), each possible result of UnrollOneChain is a correct partial AKL state for each $(\overline{Q}', d : \{h_*\})$, so $(\mu', \Delta')$ is unroll-correct for $Q$ at $\overline{Q}'$.

Now suppose $d \notin \{\emptyset, d_*\}$ is returned from AdjustUnrollDAG, and $d$ does not already appear in $\Delta$. In this case, $h \succ_d \mu(d)$, and for every $h_* \in \mathcal{H}_{\text{menu}}$, the unrolled state when preferences $(\overline{Q}, d : \{h_*\})$ will pair $d$ to $\mu(d)$. Under preferences $(\overline{Q}', d : \emptyset)$, a single additional proposal will be made on top of the proposals of $(\overline{Q}, d : \emptyset)$, namely, $h$ will propose to $d$ and $d$ will reject $\mu(d)$. However, *if $h_*$ is such that $h$ is "unrolled"* (formally, if $h_*$ is such that $\textsc{UnrollOneChain}(\mu, \Delta, h_*)$ changes the partner of $h$) then $h$ cannot propose to $d$ in $(\overline{Q}, d : \emptyset)$ (because all pairs in $\Delta$ can only "unroll" $h$ to partners before $\mu(h)$ on $h$'s list), nor in $(\overline{Q}', d : \emptyset)$ (because $\overline{Q}'$ only adds a partner to $h$'s list after $\mu(h)$). Thus, for all $h_*$ such that $h$ is unrolled, the pair $(d, \mu(d))$ should be unrolled as well. On the other hand, for all $h_*$ such that $h$ is not unrolled, $h$ will propose to $d$ (matched to $d'$), so $d$ will match to $h$ in the unrolled-to state. This is exactly how $\mu'$ and $\Delta'$ specify unrolling should go, as needed.

**(Hardest case: $\textsc{AdjustUnrollDagCollision}$.)** We now proceed to the hardest case, where $d \notin \{\emptyset, d_*\}$ is returned from $\textsc{AdjustUnrollDag}$, and $d$ already appears in $\Delta$. In this case, $\textsc{AdjustUnrollDagCollision}$ modifies $\Delta$. Define $p_1$, $P_1$, and $h_{\min}$, following the notation of $\textsc{AdjustUnrollDagCollision}$. Now consider any $h_* \in \mathcal{H}_{\text{menu}}$ under preferences $\overline{Q}$. There are several cases of how $h_*$ may interact with the nodes changed $\textsc{AdjustUnrollDagCollision}$, so we look at these cases and prove correctness. There are two important considerations which we must prove correct: first, we consider the way that $\textsc{AdjustUnrollDagCollision}$ removes nodes from $\Delta$ (starting on Description 19), and second, we consider the way that it creates a new node to handle $d$ (starting on Description 22).

**(First part of $\textsc{AdjustUnrollDagCollision}$.)** We first consider the way $\textsc{AdjustUnrollDagCollision}$ removes nodes from $\Delta$. There are several subcases based on $h_*$. First, suppose $\textsc{UnrollOneChain}(\mu, \Delta, h_*)$ does not contain $p_1$. Then, because $\textsc{AdjustUnrollDagCollision}$ only drops $p_1$ and nodes only descended through $p_1$, the chain unrolled by $\textsc{UnrollOneChain}(\mu', \Delta', h_*)$ is unchanged until $h$. (We will prove below that the behavior when this chain reaches $h$ is correct.) Thus, the initial part of this unrolled chain remains correct for $Q$ at $\overline{Q}'$.

On the other hand, suppose that $\textsc{UnrollOneChain}(\mu, \Delta, h_*)$ contains $p_1$. There are two sub-cases based on $\Delta$. First, suppose that there exists a pair $p \in P$ in $\Delta$ such that $p$ is a descendent of $p_1$ (i.e. there exists a $p = (d_x, h) \in P$ and a path from $p_1$ to $p$ in $\Delta$). In this case, under preferences $\overline{Q}$, $\textsc{UnrollOneChain}(\mu, \Delta, h_*)$ would unroll to each pair in the path starting at $h_*$, which includes $p_1$ and all nodes on the path from $p_1$ to $p$. Under $\Delta'$, however, *none of the nodes from $p_1$ to $p$* will be unrolled in this case. The reason is this: in Description A.2, the path from $p_1$ to $p$, including the proposal of $h$ to $d_1$, form an "improvement rotation" when the true preferences are

$\overline{Q}'$. Formally, under preferences $(\overline{Q}', d_* : \{h_*\})$, if $d_1$ rejected $h_1$, the rejections would follow exactly as in the path in $\Delta$ between $p_1$ and $p$, and finally $h$ would propose to $d_1$. Description A.2 would then call WriteRotation, and the value of $\mu$ would be updated for each $d$ on this path. So deleting these nodes is correct in this subcase.[10]

For the second subcase, suppose that there is *no* path between $p_1$ and any $p \in P$ in $\Delta$. In this case, there must be some source $(d_*, \overline{h})$ in $\Delta$ which is an ancestor of some $p \in P$, and such that the path from $(d_*, \overline{h})$ to $p$ does not contain any descendent of $p_1$. (This follows because each $p \in P$ must have at least one source as an ancestor, and no ancestor of any $p \in P$ can be descendent of $p_1$.) To complete the proof in this subcase, it suffices to show that at preferences $(\overline{Q}', d_* : \{h_*\})$, we "do not need to unroll" the path in $\Delta$ starting at $h_*$ after $p_1$ (formally, we want to show that if you unroll from $\mu'$ the path in $\Delta$ from $h_*$ to just before $p_1$ (including the new node added by the lines starting on Description 22), then this is a partial AKL state of $Q$ at $\overline{Q}'$). The key observation is this: in contrast to preferences $(\overline{Q}, d_* : \{h_*\})$, where pair $p_1$ is "unrolled", under preferences $(\overline{Q}', d_* : \{h_*\})$, we know *$h$ will propose to $d_1$ anyway*, because $d_*$ will certainly reject $\overline{h}$ (and trigger a rejection chain leading from $(d_*, \overline{h})$ to $h$ proposing to $d_1$).

**(Second part of AdjustUnrollDagCollision.)** We now consider the second major task of AdjustUnrollDagCollision, namely, creating a new node to handle $d$. The analysis will follow in the same way regardless of how the first part of AdjustUnrollDagCollision executed (i.e., regardless of whether there exists a path between $p_1$ and $P$). The analysis has several cases. First, suppose $(d_*, h_*)$ is not an ancestor of any node in $P_1 \cup P$ in $\Delta$. This will hold in $\Delta'$ as well, so neither UnrollOneChain$(\mu, \Delta, h_*)$ nor will UnrollOneChain$(\mu', \Delta', h_*)$ will not change the match of $d$. Instead, the match of $d$ under UnrollOneChain$(\mu', \Delta', h_*)$ will be $\mu'(d)$, which is a correct partial AKL state under $(\overline{Q}', d_* : \{h_*\})$, as desired.

Second, suppose $h_*$ is such that $(d_*, h_*)$ is an ancestor of some node in $P_1$ in $\Delta$. There are two subcases. If $h_{\min} = h$, then we have $\mu(d_1) = \mu'(d_1)$, but when UnrollOneChain$(\mu', \Delta', h_*)$ is run, we unroll $d_1$ to $h$. Correspondingly, in IPDA with preferences $(\overline{Q}', d_* : \{h_*\})$, we know $d_1$ will not receive a proposal from $\mu(d_1)$ (as this match is unrolled in $\overline{Q}$) but $d_1$ will receive a proposal from $h$ (as this additional proposal happens in $\overline{Q}'$ but not in $\overline{Q}$, regardless of whether this happens due to a "re-

[10]This is the core reason why Description A.4 cannot "unroll" to $IPDA(Q, d : \{h_i\})$—instead, it unrolls to a "partial state of AKL".

jection rotation" of AKL, or simply due to two rejection chains causing this proposal, as discussed above), which $d_1$ prefers to the unrolled-to match under preferences $\overline{Q}$. Thus, under preferences $(\overline{Q}', d_* : \{h_*\})$, we know $d_1$ will match to $h_{\min} = h$ in a valid partial AKL-state. So $(\mu', \Delta')$ is correct for $\overline{Q}'$ in this subcase. If, on the other hand, $h_{\min} = \mu(d_1)$, then in $\Delta'$, $\textsc{UnrollOneChain}(\mu', \Delta', h_*)$ will not contain the new node $p_{\text{new}}$. However, $\mu'(d_1) = h$, and we know $d$ would receive a proposal from $h$ $(\overline{Q}', d_* : \{h_*\})$, and would accept this proposal. So $(\mu', \Delta')$ is correct for $\overline{Q}'$ in this subcase.

Third and finally, suppose $h_*$ is such that $(d_*, h_*)$ is an ancestor of some node in $P$ in $\Delta$. The logic is similar to the previous paragraph, simply reversed. Specifically, there are two subcases. If $h_{\min} = h$, then when preferences are $(\overline{Q}', d_* : \{h_*\})$, then $d_1$ will no longer receive a proposal from $h$, but will still receive a proposal from $\mu(d_1)$. So $d_1$ should remain matched to $\mu(d_1)$ during $\textsc{UnrollOneChain}(\mu', \Delta', h_*)$, and $(\mu', \Delta')$ is correct for $\overline{Q}'$ in this subcase. If $h_{\min} = \mu(d_1)$, then $\mu'(d_1) = h$, and in $\Delta'$, $\textsc{UnrollOneChain}(\mu', \Delta', h_*)$ will contain the new node $p_{\text{new}}$, which unrolls $d_1$ to their old match $\mu(d_1)$. This is correct, because in $\overline{Q}$, according to $\Delta$, we know $h$ will be unrolled to some previous match, and correspondingly, in preferences $(\overline{Q}', d_* : \{h_*\})$, we know $d_1$ will never receive a proposal from $h$. So $(\mu', \Delta')$ is correct for $\overline{Q}'$ in this subcase.

Thus, for all cases, $(\mu', \Delta')$ are unroll-correct for $Q$ at $\overline{Q}'$, as required. □

To begin to wrap up, we bound the computational resources of the algorithm:

**Lemma C.9.** *Description A.4 is institution-proposing and uses memory $\widetilde{O}(n)$.*

*Proof.* The institution-proposing property holds by construction. To bound the memory, the only thing that we need to consider on top of AKL is the "unroll DAG" $\Delta$. This memory requirement is small, because there are at most $O(n)$ nodes of the form $(d_*, h)$ for different $h \in \mathcal{H}$, and by Lemma C.6, a given applicant $d \in \mathcal{D} \setminus \{d_*\}$ can appear *at most once* in $\Delta$. So the memory requirement is $\widetilde{O}(n)$. □

We can now prove our main result:

**Theorem C.10.** *Description A.4 is an institution-proposing, $\widetilde{O}(n)$ memory individualized dictatorship for DA.*

*Proof.* We know Description A.4 correctly computes the menu, and that it is institution-proposing and $\widetilde{O}(n)$ memory. So we just need to show that it correctly computes the final matching. To do this, it suffices to show that at the end of Phase 1 of Description A.4, $(\mu, \Delta)$ is unroll-correct for $Q$ at the truncated revealed preferences $\overline{Q}$ (for then, by definition, running Description A.2 after UnrollOneChain will correctly compute the final matching).

To see this, first note that an empty graph is unroll-correct for the truncated revealed preference after running $IPDA(P_{\text{hold}})$, as no further proposals beyond $d_*$ can be made in these truncated preferences. Second, each time we pick an $h \in \mathcal{H}_*$ on Description 8, a single $(d_*, h)$ added to $\Delta$ (with no edges) is unroll-correct for $Q$ at $\overline{Q}'$, by construction. Finally, by Lemma C.8, every other query to any institution's preference list keeps $(\mu, \Delta)$ unroll-correct after the new query. So by induction, $(\mu, \Delta)$ is unroll-correct at the end of Phase 1, as desired. □

# D Supplemental Impossibility Result for DA

In this appendix, we give a supplemental impossibility result for descriptions of DA. We prove that institution-proposing outcome descriptions—and hence institution-proposing individualized dictatorships as a special case—cannot satisfy the *pick-an-object* simplicity condition of Bó and Hakimov (2023).

Bó and Hakimov (2023) introduce the pick-an-object condition in the context of interactive mechanisms, where (informally speaking) agents are iteratively asked to pick their favorite objects from some set, and whenever the mechanism terminates, every agent is matched to their most recently picked object. For example, in a dynamic mechanism implementing DA, applicants can be iteratively asked to pick their favorite institution from the set of all institutions they have not yet proposed to. We consider the pick-an-object condition within the context of one-side-proposing outcome descriptions. In this context, the condition requires that when the description terminates, every agent on the proposing side must be matched to whichever agent they proposed to most recently.

Like the linear-memory condition we use in Section 5, the pick-an-object condition captures one feature of matching mechanism descriptions used to explain these mechanisms in practice. Indeed, the description in Figure 3 on page 16 that is used by the NRMP is pick-an-object, since the yellow highlighting in that figure tracks the most recent proposal of each applicant and, at the end of the description, relays the outcome matching. However, where linear-memory is a fairly permissive desideratum concerning the amount of bookkeeping used, pick-an-object is a more restrictive desideratum concerning the manner in which the bookkeeping is updated and used. Thus, we do not interpret our pick-an-object impossibility result as strongly as our linear-memory impossibility result, e.g., we do not argue that all small tweaks of the traditional description of DA should be pick-an-object. Nevertheless, our pick-an-object impossibility result is quite useful: It shows a potentially-desirable class of descriptions cannot satisfy an established and intuitive simplicity condition, and gives a specific barrier that hypothetical more-practical alternatives to our unintuitive and convoluted description in Appendix C would have to circumvent.

We now formally define pick-an-object, adapting the definition from Bó and Hakimov (2023) to focus on institution-proposing outcome descriptions.

**Definition D.1** (Pick-an-Object)**.** An institution-proposing outcome description is *pick-an-object* if, whenever the description terminates and calculates some outcome matching $\mu$, it satisfies the following. For every institution $h$, let $d_h$ be the most recently queried applicant from $h$'s preference list, i.e., if the description made $j$ queries to $h$, then $d$ is the $j^{\text{th}}$ applicant on $h$'s preference list. Then, $\mu(h) = d_h$ for every institution $h$.

Observe that the traditional descriptions of SD, TTC, and DA are applicant-proposing outcome descriptions that are pick-an-object (according to a definition perfectly analogous to Definition D.1, but interchanging the roles of the applicants and institutions). DA (the applicant-optimal stable matching mechanism) has a nontrivial institution-proposing outcome description as well (Section C.1). However, as we now show, such a description cannot be pick-an-object, giving a sense in which they cannot be simple. Formally:

**Proposition D.2.** *No institution-proposing outcome description of DA is pick-an-object.*

*Proof.* Assume for contradiction that $D$ is an institution-proposing outcome description of DA which is pick-an-object. Consider a market with institutions $h_1, h_2$ and applicants $d_1, d_2, d_3$. We first define preferences of three applicants as follows:

$$d_1 : h_2 \succ h_1$$
$$d_2 : h_1 \succ h_2$$
$$d_3 : \text{(any complete preference list)}$$

Next, we consider two possible preference lists for each of $h_1, h_2$:

$$\succ_1 : d_1 \succ d_2 \succ d_3 \qquad\qquad \succ_2 : d_2 \succ d_1 \succ d_3$$
$$\succ'_1 : d_1 \succ d_3 \succ d_2 \qquad\qquad \succ'_2 : d_2 \succ d_3 \succ d_1$$

One can check that DA (the applicant-optimal stable matching) produces outcome matching $\mu_1$ that assigns $d_1$ to $h_2$ and $d_2$ to $h_1$ when the priorities are $(\succ_1, \succ_2)$; on any other profile of priorities among those defined above, DA has as outcome the matching $\mu_2$ that assigns $d_1$ to $h_1$ and $d_2$ to $h_2$. Thus, our description $D$ can know the outcome on these inputs only when it has read the second-highest-priority spot of *both* $h_1$ and $h_2$. However, intuitively, this means that our institution-proposing description $D$ of DA cannot be pick-an-object, because the highest-priority applicant for both $h_1$ and $h_2$ must be read before we can know whether these institutions are assigned to these applicants.

Formally, consider the execution of $D$ when institutions have priorities $(\succ_1, \succ_2)$. Consider the final time during this execution when $D$ learns the difference between $\succ_j$ and $\succ'_j$ for some $j \in \{1, 2\}$; i.e., the latest possible state $s$ during the execution of the description with priority profile $Q = (\succ_1, \succ_2)$ where the execution diverges from that of some priority profile in $\{(\succ'_1, \succ_2), (\succ_1, \succ'_2)\}$. (Note that the description must learn this difference in order to calculate DA.) By the symmetry in the defined preferences, it is without loss of generality to suppose that in state $s$, the description queries the preferences of applicant 1, and thus has one successor state consistent with $Q$ and another consistent with $Q' = (\succ'_1, \succ_2)$. However, since $D$ is institution-proposing, this means that in state $s$, the description has already read $d_1$ off the priority list of $h_1$ (and is proceeding to read either $d_2$ or $d_3$ next). Since $D$ is pick-an-object, this means that $h_1$ cannot match to $d_1$ in any the final outcome matching of any execution of $D$ consistent with $s$. But this is a contradiction, since $h_1$ must match to $d_1$ in DA when the priority profile is $(\succ'_1, \succ_2)$. This finishes the proof. □

Proposition D.2 directly implies that DA has no institution-proposing individualized dictatorship satisfying the pick-an-object condition (since such a description is, in particular, an outcome description). Combined with our robust main impossibility result (Section 5), this establishes precise impossibilities for simple one-side-proposing individualized dictatorship descriptions of DA: Such applicant-proposing descriptions cannot be linear-memory, and such institution-proposing descriptions cannot be pick-an-object.

# E Proofs of Known Results

This appendix is dedicated to reproducing complete, from scratch proofs of all lemmas we need to reason about DA and TTC. The results in Section E.1 for DA, along with our direct proof of the correctness of our menu description of DA in Appendix B, demonstrate the strategyproofness of (the traditional description of) DA from first-principles. If desired, one can then compare this proof with one classical, direct proof presented in Section E.2. In Section E.3, we present a classical lemma for TTC: that the result is independent of the order in which cycles are eliminated (in the traditional description).

## E.1 Direct Proof of Stable Matching Lemmas

Here, we supply all lemmas needed for the direct proof of Theorem 3.2 given in Appendix B. Let $D$ denote the set of applicants, and $H$ the set of institutions. Recall that a matching $\mu$ is *stable* if $\mu(a) \succ_a \emptyset$ for all $a \in D \cup H$, and moreover there is no pair $d \in D, h \in H$ such that $h \succ_d \mu(h)$ and $d \succ_h \mu(d)$.

**Lemma E.1** (Gale and Shapley, 1962)**.** *The output of DA is a stable matching.*

*Proof.* Consider running the traditional description of DA on some profile of preferences (and priorities), and let the output matching be $\mu$. Consider a pair $d \in D$, $h \in H$ which is unmatched in $\mu$. Suppose for contradiction $h \succ_d \mu(d)$ and $d \succ_h \mu(h)$. In the DA algorithm, $d$ would propose to $h$ before $\mu(d)$. However, it's easy to observe from the traditional description of DA that once an institution is proposed to, they remain matched and can only increase their priority for their match. This contradicts the fact that $h$ was eventually matched to $\mu(h)$. □

Note that Lemma E.1 also proves that at least one stable matching always exists. Next, we show that DA (i.e., the matching output by the APDA algorithm) is (simultaniously) the best stable matching for all applicants.

**Lemma E.2** (Gale and Shapley, 1962)**.** *If an applicant $d \in D$ is ever rejected by an institution $h \in H$ during some run of the APDA algorithm, then no stable matching can pair $d$ to $h$.*

*Proof.* Let $\mu$ be any matching, not necessarily stable. We will show that if $h$ rejects $\mu(h)$ at any step of DA, then $\mu$ is not stable.

Consider the first time during in the run of APDA where such a rejection occurred. In particular, let $h$ reject $d \stackrel{\text{def}}{=} \mu(h)$ in favor of $\widetilde{d} \neq d$ (either because $\widetilde{d}$ proposed to $h$, or because $\widetilde{d}$ was already matched to $h$ and $d$ proposed). We have $\widetilde{d} \succ_h d$. We have $\mu(\widetilde{d}) \neq h$, simply because $\mu$ is a matching. Because this is the *first* time an applicant has been rejected by her match in $\mu$, $\widetilde{d}$ has not yet proposed to $\mu(\widetilde{d})$. This means $h \succ_{\widetilde{d}} \mu(\widetilde{d})$, and $\mu$ is not stable.

Thus, no institution can ever reject a stable partner in APDA. $\square$

The following corollaries are immediate:

**Corollary E.3** (Gale and Shapley, 1962)**.** *In DA, every applicant is matched to her favorite stable partner.*

**Corollary E.4** (Dubins and Freedman, 1981)**.** *The matching output by the traditional DA algorithm is independent of the order in which applicants are selected to propose.*

A phenomenon dual to Corollary E.3 occurs for the institutions:

**Lemma E.5** (McVitie and Wilson, 1971)**.** *In the match returned by APDA, every $h \in H$ is paired to her least-favorite stable partner.*

*Proof.* Let $d \in D$ and $h \in H$ be paired by applicant-proposing deferred acceptance. Let $\mu$ be any stable matching which does not pair $d$ and $h$. We must have $h \succ_d \mu(d)$, because $h$ is the $d$'s favorite stable partner. If $d \succ_h \mu(h)$, then $\mu$ is not stable. Thus, we must in fact have $\mu(h) \succ_h d$. $\square$

Finally, we show that the set of matched agents must be the same in each stable matching.

**Theorem E.6** (Lone Wolf / Rural Hospitals Theorem, Roth, 1986)**.** *The set of unmatched agents is the same in every stable matching.*

*Proof.* Consider any stable matching $\mu$ in which applicants $D^\mu$ and institutions $H^\mu$ are matched, and let $D^0$ and $H^0$ be matched in DA. By Corollary E.3, we know that for all $d \in D^\mu$, the match of $d$ can only improve in DA; in particular, $d$ is still matched in DA, and thus $D^\mu \subseteq D^0$. Similarly, Lemma E.5 implies that each agent in $H^0$ is matched in every stable outcome, so $H^0 \subseteq H^\mu$. But then, since the matching is one-to-one, we have $|D^0| = |H^0|$ as well as $|D^0| \geq |D^\mu| = |H^\mu| \geq |H^0|$, so the same number of agents (on each side) are matched in $\mu$ and in DA. Thus, $D^0 = D^\mu$ and $H^0 = H^\mu$. $\square$

## E.2 Direct Proof of the Strategyproofness of DA from its Traditional Description

To contrast between Section E.1 and Theorem 3.2, we also include a direct proof of the strategyproofness of DA, adapted from Gale and Sotomayor (1985). Note, however, that the following proof also shows that DA is *weakly group* strategyproof, whereas Theorem 3.2 does not.

**Lemma E.7** (Attributed to J.S. Hwang by Gale and Sotomayor, 1985)**.** *Let $\mu = APDA(P)$ and $\mu'$ be any other matching. Let $T$ denote the set of all applicants who* strictly *prefer their match in $\mu'$ to their match in $\mu$, and suppose $T \neq \emptyset$. Then there exists a blocking pair $(d, h)$ in $\mu'$* with *$d \notin T$.*

*Proof.* We consider two cases. Let $\mu(T)$ denote the set of matches of agents in $T$ under $\mu$ (and similarly define $\mu'(T)$).

Case 1: $\mu(T) \neq \mu'(T)$. Every applicant in $T$ is matched in $\mu'$, so $|\mu'(T)| \geq |\mu(T)|$. Thus, there exists some $h \in \mu'(T)$ but $h \notin \mu(T)$, that is, $h = \mu'(d')$ with $d' \in T$ but $h = \mu(d)$ with $d \notin T$. By the definition of $T$, we have $h = \mu(d) \succ_d \mu'(d)$. Because $h \succ_{d'} \mu(d')$, we know $d'$ would propose to $h$ in $APDA(P)$. So $d = \mu(h) \succ_h d'$. Thus, $(d, h)$ is a blocking pair in $\mu'$ (with $d \notin T$).

Case 2: $\mu(T) = \mu'(T)$. This case is a bit harder. Consider the run of $APDA(P)$. First, for any $d \in T$, note that $d$ must have proposed to $\mu'(d)$ before proposing to $\mu(D)$, so each institution in $\mu(T)$ receives at least two proposals from applicants in $T$.

Now, consider the *final* time in a run of $APDA(P)$ when an applicant $d_f \in T$ proposes to an institution $h$ in $\mu(T)$. As $h$ receives at least two proposals from applicants in $T$, we know $h$ must be tentatively matched, say to $d$, and $h$ must reject $d$ for $d_f$. However, $d$ cannot herself be in $T$, as then $d$ would need to make another proposal to $\mu(d) \in \mu(T)$ (and we assumed this is the final proposal from an applicant in $T$ to an institution in $\mu(T)$).

We claim that $(d, h)$ is a blocking pair in $\mu'$. Proof: As $d \notin T$ and $d$ proposes to $h$ during $APDA(P)$, we have $h \succ_d \mu(d) \succeq_d \mu'(d)$. Now, again consider when $h$

rejects $d$ in $APDA(P)$. At this point in time, $h$ has already rejected every agent in $T$, other than $d_f = \mu(h)$, who proposes to $h$ during $APDA(P)$. In particular, $h$ has already rejected $\mu'(h) \in T$ (who also proposes to $h$ in $APDA(P)$, as noted above), so $d \succ_h \mu'(h)$.

Thus, in either case there exists a blocking pair $(d, h)$ in $\mu'$ with $d \notin T$. $\square$

**Theorem E.8** (Roth, 1982; Dubins and Freedman, 1981)**.** *APDA is (weakly group-)strategyproof for the applicants.*

*Proof.* Suppose a set $L$ of applicants change their preferences, and each of them improve their match. In particular, if $\mu' = APDA(P')$, where $P'$ is the altered list of preferences, then $L \subseteq T$ as in Lemma E.7. Thus, there exists a blocking pair $(d, h)$ for $\mu'$ under preferences $P$, where we additionally have $d \notin T$. In particular, $d \notin L$. Thus, $d$ and $h$ each keep their preferences the same in $P'$ as in $P$. So, $(d, h)$ is also a blocking pair under preferences $P'$, so $\mu'$ cannot possibly be stable under $P'$. This is a contradiction. $\square$

## E.3 Direct Proof that TTC is Independent of the Order Of Cycle Elimination

We now prove that TTC is independent of the order in which the steps are chosen in the traditional description (analagous to Corollary E.4 for DA). Intuitively, this follows mostly from the observation that cycles in the traditional description of TTC must always be disjoint (since the pointing graph has out-degree 1), so choosing different orderings to eliminate cycles is actually immaterial; we formalize this idea below. See also Carroll (2014); Morrill and Roth (2024) for similar contemporary proofs.

**Proposition E.9** (Follows from Shapley and Scarf, 1974; Roth and Postlewaite, 1977)**.** *The TTC algorithm is independent of the order in which cycles are chosen and eliminated.*

*Proof.* Fix a profile of priorities and preferences. Define the *elimination graph* $G$ as follows. The vertices of $G$ are the set of all partial matchings between applicants and institutions. There is an edge $\mu_1 \to \mu_2$ in $G$ whenever $\mu_2$ differs from $\mu_1$ by the elimination of exactly one cycle, as defined in Definition 2.3, under the given preferences and priorities. Formally, this is defined as follows. Fix $\mu_1$, and consider the *pointing graph* $B = B_{\mu_1}$ given $\mu_1$ to be the bipartite graph formed by applicants and institutions who are unmatched in $\mu_1$, where each agent points to her top-ranked agent on the other side who is unmatched in $\mu_1$ (if any such agents on the other side remain). Then, we have an edge $\mu_1 \to \mu_2$ whenever there exists a cycle in $B$ such that, if $\mu_1$ is modified such that every applicant in the cycle is matched to the institution she points to, then the resulting matching is $\mu_2$. When $\mu_1 \to \mu_2$ in $G$, and the cycle $C$ in $B_{\mu_1}$ represents the difference between $\mu_2$ and $\mu_1$, we say that $C$ is *available* in $\mu_1$.

Now, define a *elimination sequence* $T$ to be any sequence $T = \mu_1 \to \mu_2 \to \ldots \to \mu_k$ of adjacent edges in $G$, such that $\mu_1$ is the empty matching which pairs no agents, and $T$ is of maximal possible length. Observe that the outcome of TTC is defined to be the final matching $\mu_k$ of an elimination sequence.

We make the following observations regarding any elimination sequence $T = \mu_1 \to \ldots \to \mu_k$:

- For any fixed pointing graph $B_{\mu_i}$, all of the cycles $C$ in $B_{\mu_i}$ are disjoint. This follows because the pointing graph has out-degree 1.

- If $C$ is available in some $\mu_x$, then there exists a $z > x$ such that $C$ is available in every subsequent $\mu_y$ for $x \le y < z$. This follows from the previous observation, since for each $\mu_y \to \mu_{y+1}$ with $x \le y < z$ with $y$ increasing inductively, the vertices in the cycle $C$ are not changed as we switch from $\mu_y$ to $\mu_{y+1}$, unless the cycle $C$ itself is eliminated. Thus, in particular, $\mu_z$ differs from $\mu_{z-1}$ by the elimination of $C$.

- Suppose that in $T$, cycle $C_1$ is available in some $\mu_x$, but $C_2 \neq C_1$ eliminated in $\mu_x$ to get $\mu_{x+1}$. Then, there exists another elimination sequence $T' = \mu_1 \to \mu_x \to \mu'_{x+1} \to \ldots \to \mu'_k$ which agrees with $T$ up until $\mu_x$, but $C_1$ is eliminated at $\mu_x$ to get $\mu'_{x+1}$, and which ends in the same final matching $\mu'_k = \mu_k$. To show this, we construct $T'$ as follows. After eliminating $C_1$ at $\mu_x$ to get $\mu'_{x+1}$, follow the same order of eliminating cycles as in $T$ until cycle $C_1$ is eliminated in $T$—i.e., go from $\mu'_{y+1}$ to $\mu'_{y+2}$ via the same cycle used to go from $\mu_y$ to $\mu_{y+1}$, for each $y \ge x$ such that $C_1$ is not eliminated in $\mu_y \to \mu_{y+1}$ in $T$. (All such cycles must be available as needed in $T'$, since before $C_1$ was eliminated in $T$, none of these cycles could have involved agents in $C_1$ in any way.) At some point, $C_1$ must be eliminated in $T$, say in $\mu_z \to \mu_{z+1}$. After this point, the elimination sequence $T'$ will from that point onward agree with $T$, i.e., $\mu_w = \mu'_w$ for $w \geq z+1$.

Now, suppose for contradiction that there are two elimination orderings $T_1$ and $T_2$ which produce different final matchings, and additionally suppose among all such pairs, the index $j > 1$ where $T_1$ and $T_2$ first disagree is *as large as possible*. Then, at index $j$, two cycles $C_1$ and $C_2$ are eliminated in $T_1$ and $T_2$, respectively. Then, by final observation listed above, we can consider the elimination sequence $T'_2$ that disagrees with $T_1$ at least one step later than $j$ (by eliminated $C_1$), but has the same final matching as $T_2$. This contradicts the assumption that $j$ was as large as possible.

This proves that all elimination sequences must produce the same final matching, which is the outcome of TTC. This proves the result. □

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