Competition in Matching Markets: The Short-Side Advantage

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1 Introduction

Stable matching mechanisms are ubiquitous in theory and in practice, especially in the "bipartite case" where agents lie in two disjoint groups and matches are made between members of different groups. The most commonly used stable matching mechanism is "one-side proposing differed acceptance", which has the nice properties of being simple to implement, fast to execute, and strategyproof for the proposing side [DF81] (moreover, it is the unique mechanism satisfying strategyproofness for one of the sides [GS85]).

Many structural properties are known about the set of stable matches for a given instance (known as the *core* of the instance). For instance, the core forms a distributive lattice with a compact (and algorithmically friendly) representation, and every distributive lattice is the core of some "not too large" stable marriage instance [IL86]. This latter fact means that, a priori, one cannot say much more about the core than one can say about arbitrary distributive lattices[1]. However, in addition to the structure that the core must necessarily have, it is very interesting to know what qualities the core *probably* has.

The "first" randomized model of stable matching is to take preferences completely uniform for each agent. For a while, it's been known that in a randomized balanced market, i.e. one with the same number of agents on each side, there's probably a large core and a big imbalance between the different sides – the proposing side gets matched which they favor significantly more [Pit89]. However, in a recent breakthrough paper [AKL17], it was found that this distinction *almost vanishes* in large markets when there is an imbalance of just *one* more agents on one side than the other. In this note we survey this result.

Organization. In section 2, we review the basic properties of differed acceptance and stable matchings, which will motivate some of the techniques to come. In section 3, we review the case of a balanced market, and discuss which parts of the analysis fail for unbalance markets. In section 4, we prove some results on unbalanced markets, taking a new proof

[1] One recent result which runs counter to this statement is given in [KGW18], where the authors use the size of the stable marriage instance to bound the number of elements in the distributive lattice of the core. There, the exact meaning of the words "not too large" becomes important.

approach based around differed acceptance which is closer to the analysis for the balanced market. In section 5, we discuss the proof technique given in [AKL17].

2 Review: Differed Acceptance and Stable Matching

The basic setup of stable matching has many variations. The central focus of this note is the setting with an unequal number of agents on each side. Some other import variations include adding "unacceptable matches" (agents do not rank every agent on the other side), "indifference" (agents sometimes don't distinguish their preference between some matches), "many-to-one matchings" (where several agents from one side are matched to one on the other), and "couples" (where certain agents on one side express their preferences/acceptable matches jointly). We use the "unacceptable matches" concept as a technical tool, but ignore the subtleties of the other variations.

Definition 2.1. *A matching market is a collection $\mathcal{M}$ of "men" and $\mathcal{W}$ of "women", where each man $m \in \mathcal{M}$ has a ranking (strict total order) denoted $\succ_m$ over women $w \in \mathcal{W}$, and vice-versa. If an agent i "deems a match to j unacceptable", i.e. they would rather go unmatched than be matched to j, then we denote this by $\emptyset \succ_i j$.*

A matching is a one-to-one assignment of men to women, which we denote by $\mu : \mathcal{M} \cup \mathcal{W} \to \mathcal{M} \cup \mathcal{W} \cup \{\emptyset\}$, such that all the reasonable conditions are met to make it a matching. We write $\mu(i) = \emptyset$ if agent i is unmatched.

A matching μ is stable *for a set of preferences $P = \{\succ_w\}_{w\in\mathcal{W}} \cup \{\succ_m\}_{m\in\mathcal{M}}$ if no unmatched man/woman pair $(m, w) \in \mathcal{M} \times \mathcal{W}$ is blocking under P, where (m, w) is called blocking if we simultaneously have $m \succ_w \mu(w)$ and $w \succ_m \mu(w)$. A* pair *(m, w) is called stable under P if $\mu(m) = w$ in* some *stable matching, and m is called a stable partner of w (and vice-versa).*

The man-proposing differed acceptance algorithm is given by Algorithm 1. We provide simple proof of the basic properties of this algorithm.

Algorithm 1 MPDA: Men-proposing differed acceptance algorithm

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Let $U = \mathcal{M}$ be the set of unmatched men
Let μ be an all empty matching
while $U \neq \emptyset$ and some $m \in U$ has not proposed to every woman **do**
  Pick such a m (in any order)
  m "proposes" to their highest-ranked woman w which they have not yet proposed to
  if $m \succ_w \mu(w)$ **then**
    If $\mu(w) \neq \emptyset$, add $\mu(w)$ to U
    Set $\mu(w) = m$, remove m from U

Claim 2.2. *The MPDA algorithm terminates.*

Proof. Every man will propose to every woman at most once. □

Intuitively, this algorithm starts with the men doing whatever they prefer the most, then doing the minimal amount of work to make the matching stable. Indeed, men propose in their order of preference. If a woman w ever rejected a man m they prefer over their current match, then *remained* with their current match, then (m, w) would clearly create an instability in the final matching.

Claim 2.3. *The output of MPDA is a stable matching.*

Proof. Consider a pair $m \in \mathcal{M}$, $w \in \mathcal{W}$ which is unmatched in the output matching μ. Suppose for contradiction $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$. Well, in the MPDA algorithm, m would propose to w before $\mu(m)$. However, it's easy to observe that once a woman is proposed to, they remains matched and can only increase their preference for their match. This contradicts the fact that w was eventually matched to $\mu(w)$. □

Note that this algorithm gives us a very interesting existence result: it was not at all clear that stable matching existed before we had this algorithm.

This next claim allow us to easily prove several corollaries. The proof follows this strategy: although it's not immediately easy to show an event can't happen, you can show it *can't happen for the first time.*

Claim 2.4. *If a man $m \in \mathcal{M}$ is ever rejected by a woman $w \in \mathcal{W}$ during some run of MPDA (that is, m proposes to w and w does not accept) then no stable matching can pair m to w.*

Proof. Let μ be any matching, not necessarily stable. We will show that if w rejects $\mu(w)$ at any step of MPDA, then μ is not stable.

Suppose that some pair, matched in μ, is rejected during MPDA. Consider the first time during in the run of MPDA where such a rejection occurs, i.e. a woman w rejects $\mu(w)$ but no other woman w' has rejected $\mu(w')$ so far. In particular, let w reject $m = \mu(w)$ in favor of $m' \neq m$ (either because m' proposed to w, or because m' was already matched to w and m proposed). We have $m' \succ_w m$. We have $\mu(m') \neq w$, simply because μ is a matching. Because this is the first time any man has been rejected by a match from μ, m' has not yet proposed to $\mu(m')$. Because men propose in their preference order, we have $w \succ_{m'} \mu(m')$. However, this means μ is not stable.

Thus, no woman can ever reject a stable partner in MPDA. □

We can now formalize our intuition that DA moves the men down their preference lists the minimal amount required to enforce stability.

Corollary 2.5. *Let* $\text{best}(m)$ *denote the most preferred match m can achieve in any stable matching, i.e. the maximum according to $\succ_m$ of the set $\{w \in \mathcal{W} : \exists \mu : \mu \text{ is stable and } \mu(m) = w\}$ (or write* $\text{best}(m) = \emptyset$ *if the above set is empty).*

In the match returned by MPDA, every $m \in \mathcal{M}$ is paired to $\text{best}(m)$.

A completely dual phenomenon occurs for the women's preferences:

Claim 2.6. *In the match returned by MPDA, every $w \in \mathcal{W}$ is paired to their worst stable match in $\mathcal{M}$.*

Proof. Let $m \in \mathcal{M}$ and $w \in \mathcal{W}$ be paired by MPDA. Let μ be any stable matching which does not pair m and w. We must have $w \succ_m \mu(m)$, because $w = \text{best}(m)$. If $m \succ_w \mu(w)$, then μ is not stable. Thus, w cannot be stably matched to any man she prefers less than m. $\square$

We also get this reassuring fact:

Corollary 2.7. *The matching output by the MPDA algorithm is independent of the order in which men are selected to propose.*

Using our results so far, we can prove the following weaker version of the rural hospital theorem[2]:

Claim 2.8 (Rural Hospital Theorem)**.** *Suppose preference lists are full length, that is, every agent ranks every other agent and prefers any match to being unmatched. Then the set of unmatched agents will be the same in every stable outcome.*

Proof. Suppose without loss of generality that $|\mathcal{M}| > |\mathcal{W}|$. Let $\overline{\mathcal{M}}$ be unmatched in MPDA, and $\overline{\mathcal{M}}'$ unmatched in some other stable matching μ. The men in $\overline{\mathcal{M}}$ are exactly those that are rejected by every woman in MPDA, so $\overline{\mathcal{M}} \subseteq \overline{\mathcal{M}}'$ by claim 2.4. Because full length preference lists are used, every stable matching must be maximal (because any pair of unmatched agents will be blocking). Thus $\mathcal{M}$ and $\mathcal{M}'$ must be the same size, and thus they are equal. $\square$

The collection of stable matchings satisfies very strong "lattice properties" giving a lot of useful structure. Here we will only use a few of the basic consequences of this.

First off, in order for one side to benefit, the other side must be worst off:

Claim 2.9. *Let μ, μ' be stable matchings, and say $\mu(m) = w$, but $\mu'(m) \neq w$. Then $\mu'(m) \succ_m w$ if and only if $\mu'(w) \prec_w m$.*

Proof. ($\Leftarrow$) "If w downgrades, then m upgrades". Suppose $\mu'(w) \prec_w m$. Because μ' is stable, yet m and w are not matched in μ', we must have $\mu'(m) \succ_m w$, or else (m, w) would form a blocking pair. (A rephrasing: this direction is easy because the definition of stability immediately makes it impossible for m and w to both downgrade).

($\Rightarrow$) "If w upgrades, then m downgrades". Let $m' = \mu'(w) \neq m$ and $w' = \mu'(m) \neq w$. Suppose that $m' \succ_w m$, and for contradiction suppose that $w' \succ_m w$. Because μ' is stable, (m', w') is not a blocking pair, so either $w \succ_{m'} w'$ or $m \succ_{w'} m'$. In the first case, (m', w) form a blocking pair in μ, and in the second case, (m, w') form a blocking pair in μ. Thus, in either case μ is not stable. $\square$

[2] The full rural hospital theorem [Rot86] is stronger in two ways. First, the assumption of full-length preference lists is dropped. Second, many-to-one matching markets are considered (i.e. several doctors can be matched to one hospital), and the conclusion is that if a hospital does not fill *all* its openings in *some* stable outcome, then it will always recieve the same doctors (and same number of openings slots) in *every* stable outcome.

3 Balanced Random Markets

For a man m and woman w in some preference set P, define $R_m(w) := |\{w' : w' \succeq_m w\}|$, i.e. the number of women preferred at least as much as w, i.e. the index of w on m's ordered preference list. We will say that m ranks w better than w' when $R_m(w) < R_m(w')$. Given a matching μ stable under preferences P, define

$$R_{men}(\mu) := \frac{1}{|\mathcal{M} \setminus \overline{\mathcal{M}}|} \sum_{m \in \mathcal{M} \setminus \overline{\mathcal{M}}} R_m(\mu(m))$$

That is, $R_{men}(\mu)$ is the average over matched men of the men's rank of their wives. Define $R_w(m)$ and $R_{women}(\mu)$ analogously.

A random matching market is defined by a set of randomly draw preferences $P = P_{n,m}$ with n men and m women, where each of the n men with uniformly random rankings over the m women, and m women with uniformly random rankings over the n men. Given a set of preferences P, let $MOSM(P)$ denote the man-optimal stable outcome, i.e. the result of running MPDA (men-proposing differed acceptance). Likewise, let $WOSM(P)$ denote the woman-optimal stable outcome, i.e. the result of running women-proposing differed acceptance.

The crucial observations to dealing with MPDA are the following:

- The sum of the ranks that the men have for their wives in the MOSM is exactly the number of proposals made during MPDA.
- When $n \leq m$, MPDA terminates as soon as n distinct women are proposed to. Thus, the sum of ranks of the men is essentially a *coupon collector* random variable, with expectation $O(\log n)$.

Proposition 3.1. *Let P be a random matching market with n men and $n+k$ women for $k \geq 0$. Then $\mathbb{E}\left[R_{men}(MOSM(P))\right] = O(\left(\frac{n+k}{n}\right) \log\left(\frac{n+k}{k}\right))$.*

Proof. Consider running men-proposing differed acceptance on M. At the steps requiring some unmatched man to be selected, select one uniformly at random. Let Y denote the random variable giving the total number of proposals made by men to women. By the so-called "principle of differed decisions", running this algorithm on a random matching market is equivalent to running the following randomized algorithm: women's preferences are taken as input, but men's are not, and whenever a man is ready to propose to a woman, simply select a woman he hasn't yet proposed to uniformly at random.

Let Z denote a "coupon collector" random variable defined by the following random process: at each step, a uniformly random number in $[n+k]$ is drawn, and Z returns the number of steps needed for n distinct numbers to be draw. Note that Z statistically dominates Y, because every time a man proposes to a woman, he is at least as likely to propose to an unmatched woman as the coupon collect step is to select an unselected outcome.

Now, we have $Z = \sum_{i=1}^{n} Z_i$, where Z_i is the number of steps needed before the ith distinct number is draw. Observe that Z_i is a geometric random variable with success probability

$\frac{n+k-i}{n+k}$. Thus:

$$\begin{aligned}
\mathbb{E}[Y] &\leq \mathbb{E}[Z] \\
&= \sum_{i=1}^{n} \mathbb{E}[Z_i] \\
&= \sum_{i=1}^{n} \frac{n+k}{n+k-i} \\
&= (n+k) \sum_{i=k}^{n+k} \frac{1}{i} \\
&= (n+k)[H_{n+k} - H_k] \\
&= \Theta\left((n+k)[\log(n+k) - \log(k)]\right)
\end{aligned}$$

Which gives exactly the desired bound on $\mathbb{E}[R_{men}(MOSM(P))] = \mathbb{E}[Y/n]$. $\square$

We can use the above claim to get an informal handle on the expectation of $R_{women}(MOSM(P))$ as well. In expectation, $O\left((n+k)\log\left(\frac{n+k}{k}\right)\right)$ proposals are made, so any fixed woman w receives an average of $O\left(\log\left(\frac{n+k}{k}\right)\right)$ proposals. The rank of the men proposing to w is essentially uniformly distributed on $[n]$. The minimum of h uniformly distributed random variables with expectation x has expectation x/h. Thus, the best-ranked man proposing to w has rank $\Omega\left(n/\log\left(\frac{n+k}{k}\right)\right)$ on average.

The previous paragraph can be made formal. A sequence of classical works (in order, [Wil72], [Knu97], [Pit89], [Pit92]) got quite a good handle on the asymptotic behavior of balanced matching markets. Performing a full analysis can get quite complicated, especially when counting the *number* of distinct stable matchings[3].

4 Unbalanced Random Markets: One Exposition

With a little effort, you can extend coupon-collector type arguments to get results in the unbalanced case. This next claim is adapted from [IM05]:

Claim 4.1. *Consider some matching market P (i.e. a set of preferences for men and women) and fix a woman w^* who is matched under any some stable matching for P. Construct preference sets $P_n, P_{n-1}, \ldots, P_1$ where in P_i we have w^* "truncate her list" after place i, that is, w^* reports her top i choices and deems all other matches unacceptable. Let m_i denote the match of w^* under $MPDA(P_i)$ (or write $m_i = \emptyset$ if w^* is unmatched) and let $L = (m_n, m_{n-1}, \ldots, m_1)$.*

We have the following:

[3] This is likely one reason that [AKL17] uses different statistics to measure the size of the core, namely, the fraction of agents with multiple distinct partners. For many applications, such as the strategic implication of the size of the core, this is the more important statistic anyway.

1. *For each i, μ_i is a stable matching for P_i with $\mu_i(w^*) \neq \emptyset$ if and only if μ_i is a stable matching for P.*

2. *L is a list containing all the stable matches of w, possibly with repetition and possibly terminating in a string of $\emptyset$s (which do not represent stable matches).*

3. *L is ranked from worst to best according to $\prec_w$.*

Proof. (1, $\Rightarrow$) If $\mu_i(w^*) = \emptyset$, then we immediately know that μ_i is not stable for P, by the rural hospital theorem (claim 2.8 suffices). Also, if μ_i is not stable for P_i, then any blocking pair in P_i will also be a blocking pair in μ_i, so μ_i will not be stable for P.

(1, $\Leftarrow$) Suppose w^* is matched in μ_i, and suppose for contradiction that (m, w) is a blocking pair for μ_i under P. All agents have identical preferences in P and P_i except for w^*, so we must have $w = w^*$. But because w^* is matched under P_i, we know $m \succ_{w^*} \mu_i(w^*)$ if and only if $m \succ^{(i)}_{w^*} \mu_i(w^*)$ (where $\succ^{(i)}_{w^*}$ denotes w^*'s preferences in P_i). Thus, (m, w) is also blocking for μ_i under P_i, a contradiction.

(2, 3) As in the case of full-length preference lists, $MPDA(P_i)$ will return a stable matching for P_i, and it will assign w^* to the worst stable partner w^* has under preferences P_i, provided one exists. By part 1, the stable partners of w^* under P_i are exactly the stable partners of w^* under P which are ranked better than place i.

Now, induct on j, with the inductive hypothesis being that $(m_n, m_{n-1}, \ldots, m_j)$ contains all stable partners ranked above j by woman w^* (in order according to $\prec_{w^*}$). The base case is simply the fact that m_n is the worst stable partner of w^* (claim 2.6). Now, for the inductive step, we have two cases. If m_j is still on the preference list of w^* in P_{j+1}, then μ_j will still be a stable match under preferences P_{j+1}. Again by claim 2.6, woman w^* will still be matched to m_j under preferences P_{j+1}. On the other hand, if m_j is no longer acceptable to w^* in P_{j+1}, then m_{j+1} will be the stable partner of w^* which is worst-ranked according to $\prec_{w^*}$ (or $\emptyset$ if w^* has already reached her best stable match) by the same logic. This finishes the proof of the inductive step and of claims 2 and 3.

□

We get this immediate corollary:

Corollary 4.2. *Suppose that when a woman w^* truncates her preference list after place i, she is unmatched by MPDA. Then w^* has no stable parters ranked better than i.*

We are now ready to prove the main result of this section. We focus on the case of n men and $n + 1$ women, for simplicity. I'm still working on making this bit fully formal.

Claim 4.3. *Consider a random matching market with n men and $n + 1$ women. Fix some woman w^*. With probability approaching 1 as $n \to \infty$, w^* has no stable partners ranked better than $\Omega(n/\log n)$.*

(Proof Sketch). Suppose w^* truncates her preference list at place $cn/\log n$ for some constant c. Denote this preference list by P^*. By corollary 4.2, w^* has a stable partner ranked better than $cn/\log n$ if and only if w^* is matched under $MPDA(P^*)$. This occurs if and only if w^* receives a proposal from a man ranked better than $cn/\log n$. But when a man proposes

to w^*, his rank by w^* is uniformly distributed (over those ranks w^* has not yet seen in the algorithm). To proceed, we upper bound the number of proposals that w^* likely receives, and thus the probability that she gets a proposal ranked higher than $cn/\log n$.

Let Y denote the random variable giving the number of proposals w^* sees during the run of $MPDA(P^*)$. First, note that $MPDA(P^*)$ will certainly terminate if all women other than w^* have been proposed to. Thus, Y is statistically dominated by the following variation on a coupon collector random variable: define a process which at each step selects a number uniformly at random from $[n+1]$. The process terminates when each number in $[n]$ has been selected, and Z outputs the number of times $n+1$ was selected. We have Y statistically dominated by Z.

Now, because the standard coupon collector random variable has expectation $O(n \log n)$, this modified coupon collector should have expectation $O(\log n)$. Thus, in expectation, the best ranked man who proposes to w^* has rank $\Omega(n/\log n)$.

□

5 [AKL17] and MOSM to WOSM Conversion

For this section, we always assume $|\mathcal{M}| = |\mathcal{W}| + k$ for $k > 0$. The main algorithm used in the analysis of [AKL17] is given here in Algorithm 2. To understand this algorithm fully, I've actually implemented it at `https://github.com/ClathomasPrime/CompetitiveStableMatching`. It is presented here in a more "imperative programming" style than that given in [AKL17]

The algorithm starts from the man-optimal stable outcome (MOSM) by running differed acceptance. Then it triggers a series of "divorces", i.e. it breaks a marriage $(m, \hat{w})$ and "continues running differed acceptance". This involves m again proposing down his list until accepted by some woman who prefers m to her current best known stable match. This "rejection chain" can involve cycles, where a woman accepts a proposal for the second time. Ignoring cycles for now, the chain can end in one of two ways: a "terminal phase" in which an element of $\overline{\mathcal{W}}$ (the set of women unmatched in the MOSM) receives a match, or an "improvement phase" in which $\hat{w}$ receives a better stable match. A terminal phase clearly cannot constitute a stable match, by the rural hospital theorem (claim 2.8). On the other hand, [AKL17] shows that any improvement phase yields a new stable outcome.

Some notes:

- The most recent stable match $\tilde{\mu}$ is stored, while the "partial match" μ may not correspond to any stable match.

- When $w \in V$, we compare m against w's match in $\tilde{\mu}$, not in μ. This is because we want to make the smallest changes possible that result in new stable outcomes, and w will still make an improvement from the old stable outcome by accepting this m.

- If a cycle occurs during a rejection chain, that cycle is immediately "recorded" in $\tilde{\mu}$. This is because this is equivalent to picking $\hat{w}$ to be any woman in the simple cycle, and (similar to differed acceptance) the execution of this algorithm is independent of the order in which women are picked.

Algorithm 2 MOSM to WOSM Conversion Algorithm

Let μ be the man-optimal stable matching
For each man m, let $R(m)$ be the set of women who rejected m during the run of $MPDA$.
Let S be the set of unmatched women $\overline{\mathcal{W}}$
while $S \neq \mathcal{W}$ **do**
  Store $\tilde{\mu} \leftarrow \mu$ ▷ $\tilde{\mu}$ stores the most recent *stable* match encountered
  Pick any $\hat{w} \in \mathcal{W} \setminus S$
  Let $m = \mu(\hat{w})$; let $V = (\hat{w})$
  Set $\mu(\hat{w}) = \emptyset$ and add $\hat{w}$ to $R(m)$ ▷ $\hat{w}$ rejects m
  while $V \neq ()$ **do**
    Let w be m's most preferred woman not in $R(m)$
    while $\tilde{\mu}(w) >_w m$ **do** ▷ while w has received a better stable match
      Add w to $R(m)$ ▷ w rejects m
      Update w to m's top woman not in $R(m)$
      ▷ If $|\mathcal{M}| < |\mathcal{W}|$, then m has had his proposal accepted by *some* woman w
    if $w \in S$ and $m >_w \tilde{\mu}(w)$ **then** ▷ End of a terminal phase
      Restore $\mu \leftarrow \tilde{\mu}$
      Add all women in V to S; Set $V = ()$
    else if $w \in V$ and $m >_w \tilde{\mu}(w)$ **then** ▷ New stable matching found
      if $w \neq \hat{w} = v_1$ **then**
        Set $m \leftarrow \mu(w)$
      Say $w = v_\ell$ and $V = v_1, \ldots, v_J$
      Update $\mu(w) = m$ and $\tilde{\mu}(v_i) = \mu(v_i)$ for $i = \ell, \ell + 1, \ldots, J$
      Remove $v_\ell, v_{\ell+1}, \ldots, v_J$ from V
    else if $w \notin V$, $w \notin S$, and $m >_w \tilde{\mu}(w)$ **then** ▷ w wants to upgrade to m
      Swap $\mu(w) \leftrightarrow m$; append (w) to the end of V ▷ w rejects $\mu(w)$

- We actually don't need to wait until the rejection chain hits $\overline{\mathcal{W}}$. Instead, we let S denote the set of women for whom we know that a phase starting with $\hat{w} \in S$ will be terminal. If at any point a woman in S then accepts the proposal, we know that the phase we are currently in will be terminal as well.

Formally, [AKL17] proves that the following all hold during the run of algorithm 2:

- $\tilde{\mu}$ is always a stable matching.
- When a phase is terminal, $\hat{w}$ is currently matched in $\tilde{\mu}$ to her optimal stable partner. Because we eliminate cycles, any other woman in V would also cause a terminal phase if she was chosen as $\hat{w}$. Thus, S always consists of women who have reached their optimal stable match.
- Every woman is at some point in time matched to every stable partner she has, in order from her least preferred to her most preferred.
- Algorithm 2 terminates with $\tilde{\mu}$ being the woman-optimal stable outcome.

Finally, [AKL17] shows that the core of an unbalanced market is probably small by showing that algorithm 2 probably terminates quickly. Thus, there cannot be a big difference between the man-optimal and the woman optimal stable matches: most agents have unique stable partners, and the average ranks for agent's partners is about the same in all matchings. This is a long, involved probabilistic argument, but is terribly difficult to carry out. The result is:

Theorem 5.1 ([AKL17], Appendix B)**.** *Consider any sequence of random matching markets with n men and $n + k(n)$ women (for any $k(n) \geq 1$). For any $\epsilon > 0$, with probability that approaches 1 as $n \to \infty$, each of the following events hold:*

1. *Define $s_k(n) := \left(\frac{n+k}{n}\right) \log\left(\frac{n+k}{k}\right)$. Then for every stable matching μ,*

$$R_{men}(\mu) \leq (1+\epsilon)s_k(n) \qquad\qquad R_{women}(\mu) \geq \frac{n}{1+(1+\epsilon)s_k(n)}$$

2. *Less than a $1/\sqrt{\log n}$ fraction of the men and women have multiple stable partners. In particular, this fraction approaches 0 as $n \to \infty$. Moreover, the total number of stable partners is at most $n + n/\sqrt{\log n}$.*

3. *There is little difference in ranking between the MOSM and the WOSM:*

$$\frac{R_{men}(WOSM)}{R_{men}(MOSM))} \leq 1+\epsilon \qquad\qquad \frac{R_{women}(WOSM)}{R_{women}(MOSM))} \geq 1-\epsilon$$

Comparing this to the results from section 3, we see that when imbalance is present, men *almost always* do as well in *any* stable matching as they do in the man-optimal stable matching. Two special cases are worth pointing out:

Corollary 5.2 (2.2 in [AKL17])**.** *Suppse $k(n) = 1$ for each n. Then for every $\epsilon > 0$, with probability approaching 1 as $n \to \infty$, we have*

$$R_{men}(\mu) \leq (1+\epsilon)\log n \qquad\qquad R_{women}(\mu) \geq (1-\epsilon)\frac{n}{\log n}$$

for every stable matching μ.

Corollary 5.3 (2.3 in [AKL17])**.** *Suppse there exists a $\lambda > 0$ with $k(n) = \lambda n$ for each n. Let $\kappa = (1+\lambda)\log(1+1/\lambda)$. Then for every $\epsilon > 0$, with probability approaching 1 as $n \to \infty$, we have*

$$R_{men}(\mu) \leq (1+\epsilon)\kappa \qquad\qquad R_{women}(\mu) \geq (1-\epsilon)\frac{n}{1+\kappa}$$

for every stable matching μ.

In a follow-up paper [**?**], many of these results are shown to be sharp, and the analysis is extended to count the average number of stable matchings in an unbalanced market. Interestingly, [**?**] is not able to improve the rate at which the fraction of agents with multiple stable partners tends to zero as $n \to \infty$, even though the arguments in section 4 seem to suggest that this fraction might be about $1/\log n$.

It's worth noting that algorithm 2 is almost identical to the algorithm *minimal-differences* described in [GI89] (a book which fully fleshes out ideas initiated in [IL86]). The purpose of algorithm *minimal-differences* is to traverse the matching lattice in a maximal chain from the MOSM to the WOSM, and along the way identify "the rotation poset" (i.e. the poset for which the matching lattice is the collection of downwards-closed sets of the poset, which is guaranteed to exist by Birkhoff's representation theorem). The only difference is that algorithm 2 does not explicitly identify the "rotations" (i.e. simple improvement cycles). [AKL17]'s careful analysis of this algorithm for unbalanced random markets essentially says that the rotation poset, and consequentially the stable matching lattice, is typically small (in a few very precise ways).

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