Verifying Robustness of Programs Under Structural Perturbations

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Motivation

- An attempt to synthesize the max function using PBE:
  - $(13, 15) \mapsto 15$
  - $(-23, 19) \mapsto 19$
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- Synthesized program: `P(a,b):=return b`
- Neither synthesized program, nor synthesizer are *robust*

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  would synthesize very different program
- Synthesize a robust program or develop robust synthesizer

Robustness Properties

- Continuity: small change to input $\Rightarrow$ small change to output

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Sort([1,4,3,6])=[1,3,4,6]
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- Permutation: permuting input permutes the output

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$$\sigma = (0 \mapsto 2, 1 \mapsto 3, 2 \mapsto 1, 3 \mapsto 0)$$

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Find([6,3,1,4], 4)=3
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- Permutation: permuting input leaves output invariant

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Verifying Continuity

- Consider

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if x ≥ 0 then
    r := y
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- Proof rule:

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- Only applicable to numerical perturbations

If-Elsif-Elsif-Else Cost

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  - Reason about each branch (4 branches)

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- Given `if-elsif-elsif-else`,
  - Reason about each branch (4 branches)
  - Check equivalence (3 comparisons)

Cartesian Hoare Logic

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  - Determinism:

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  - Determinism:

$$\|\vec{x_1} = \vec{x_2}\| f(\vec{x}) \| ret_1 = ret_2\|$$

  - Symmetry:

$$\|x_1 = y_2 \wedge x_2 = y_1\| f(x, y) \| ret_1 = ret_2\|$$

If-Elsif-Elsif-Else Cost

- Given `if-elsif-elsif-else`, must reason about product of each pair of branches (16 pairs):

$$
\begin{gathered}
P_1 \circledast P_1 \\
\vdots \\
P_1 \circledast P_4 \\
P_2 \circledast P_1 \\
\vdots \\
P_4 \circledast P_4
\end{gathered}
$$

Motivation

- "Sanity Checks"

- Functions invariant under order of list
  - max, sum, sort

- Data structures with representing something else
- Invariance under value it represents
  - binary search trees, heaps, hash sets

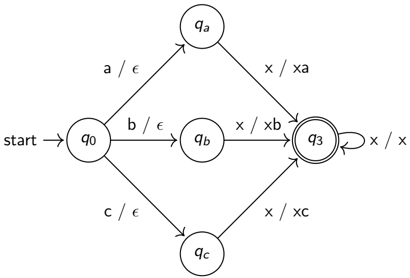
Lists – Invariance under order

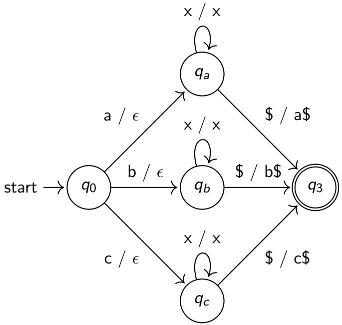
Given an array a

- Let a_{swap} be a with its first and second entry swapped
  - $[a[1], a[0], a[2], a[3], \ldots, a[n]]$
- Let a_{rot} be a rotated by 1
  - $[a[1], a[2], a[3], \ldots a[n], a[0]]$

Lemma: If for any a, $P(a) = P(a_{swap}) = P(a_{rot})$, then for any permutation a' of a, we have $P(a) = P(a')$.
Proof: Math

Automata – Invariance under order





Automata – Invariance under order

Theorem: Given a deterministic automata M, we can check if M is invariant under the order of its input in time $O(|\Sigma|^2|M|\log(|\Sigma||M|))$.

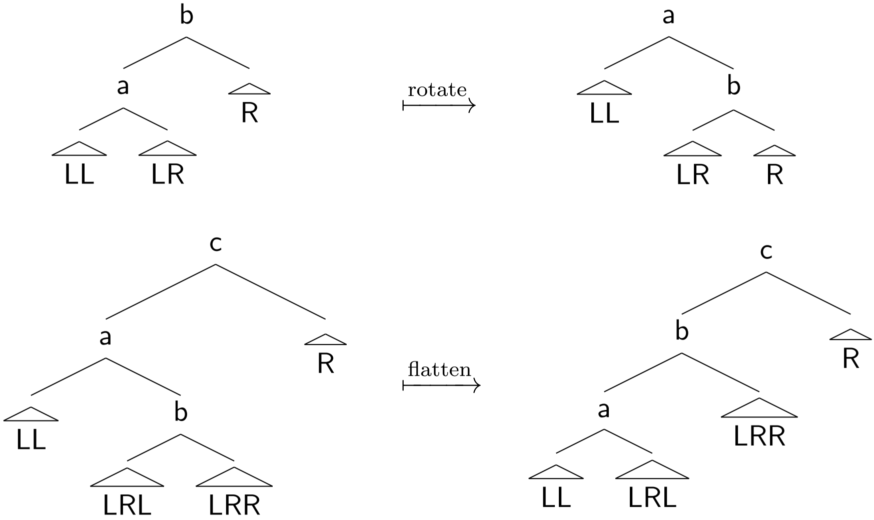
Proof:

- Construct the machines M_{swap} and M_{rot} by composing M with those machines
  - Requires $O(|\Sigma||M|)$ states
- Check if $L(M_{swap}) = L(M) = L(M_{rot})$
  - E.g. by state minimization in time $O(n|\Sigma|\log n)$

Binary Search Trees

- For lists, two simple permutations generated all permutations
- Goal: similar permutations for BSTs

Binary Search Trees



Binary Search Trees

It suffices to show

- Every tree can be transformed into a “normal form” (i.e. list)
- Every operation is reversable

Binary Search Trees – Proof by example

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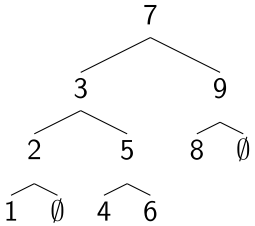

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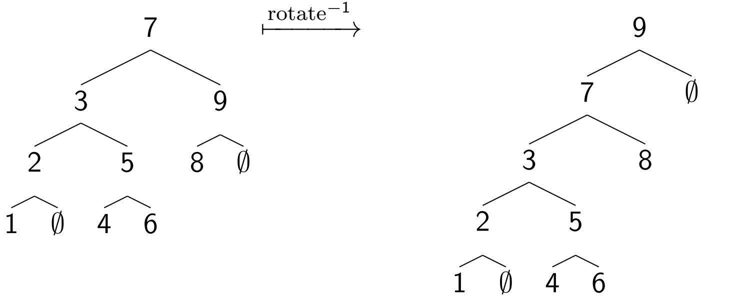

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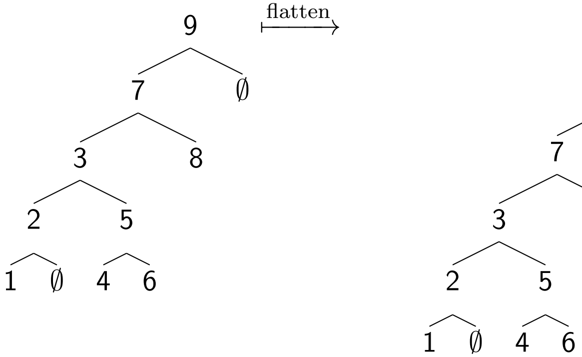

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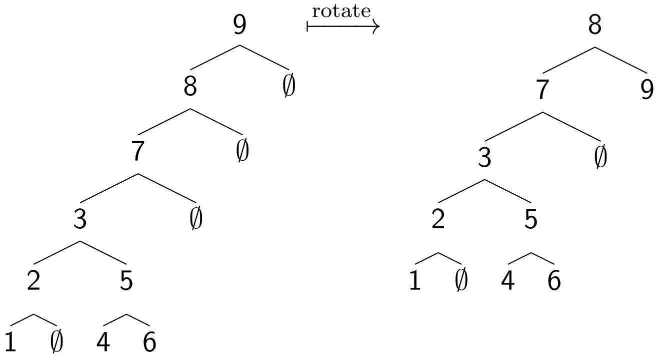

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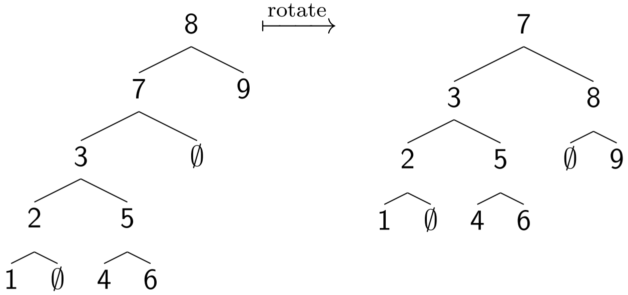

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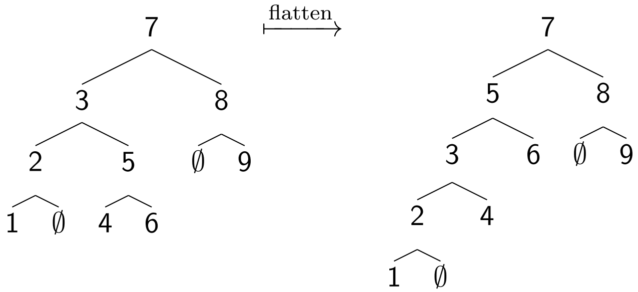

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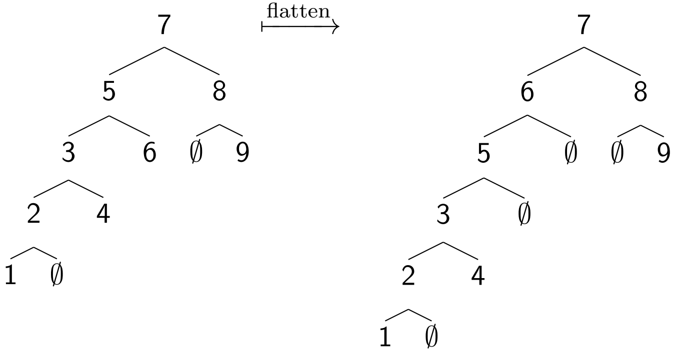

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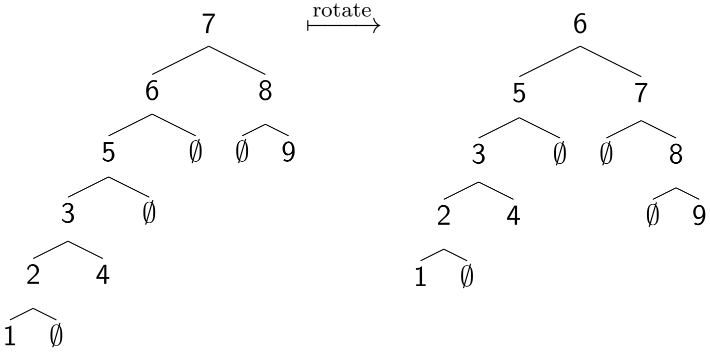

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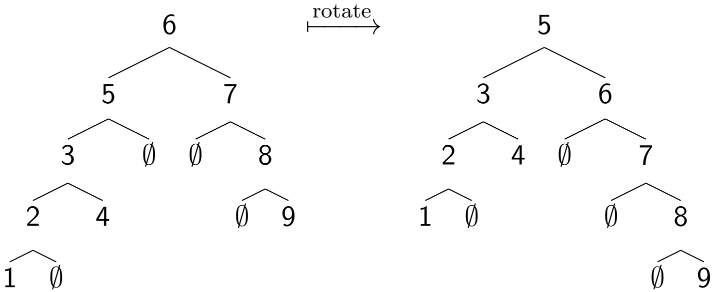

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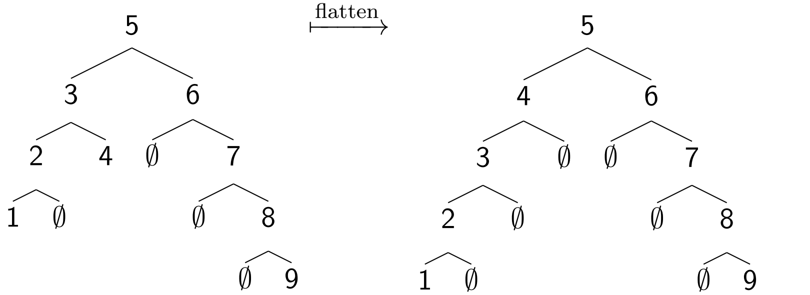

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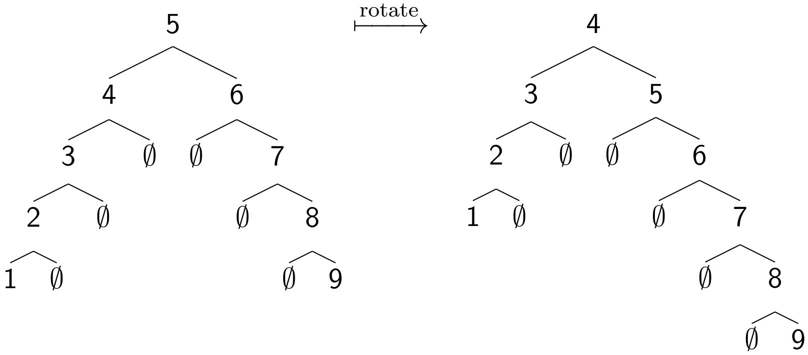

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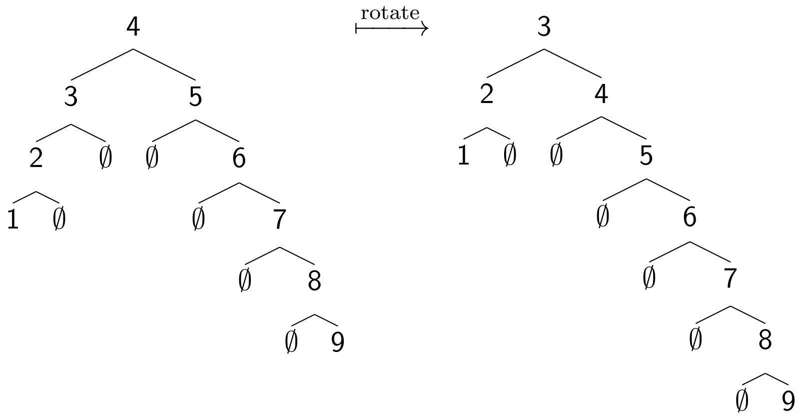

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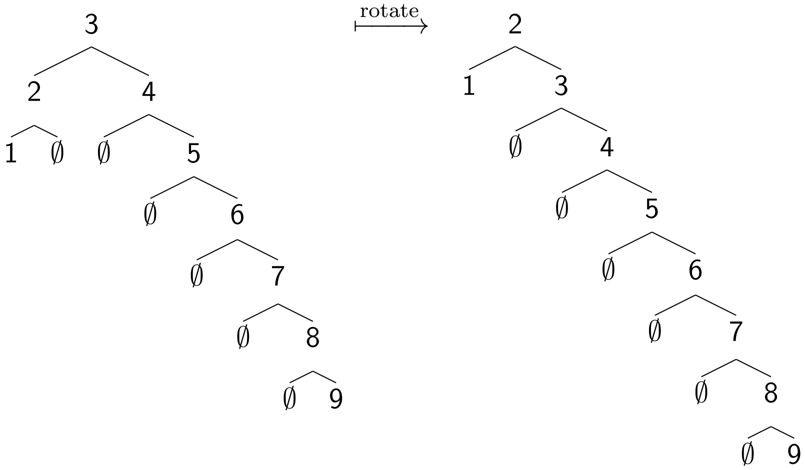

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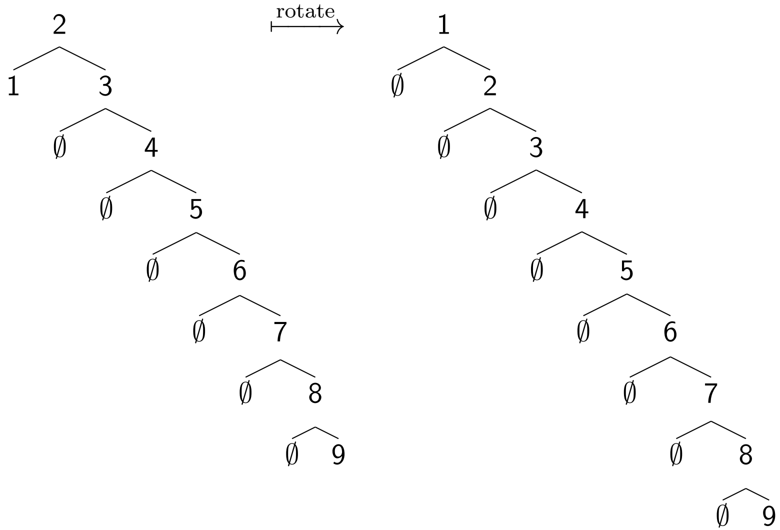

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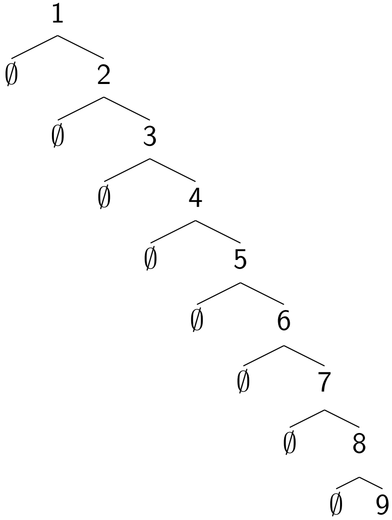
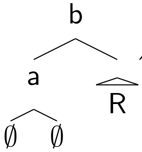
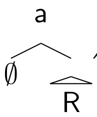


Checking BST code

- Goal: write a secondSmallest function on BSTs

secondSmallest (

) = b

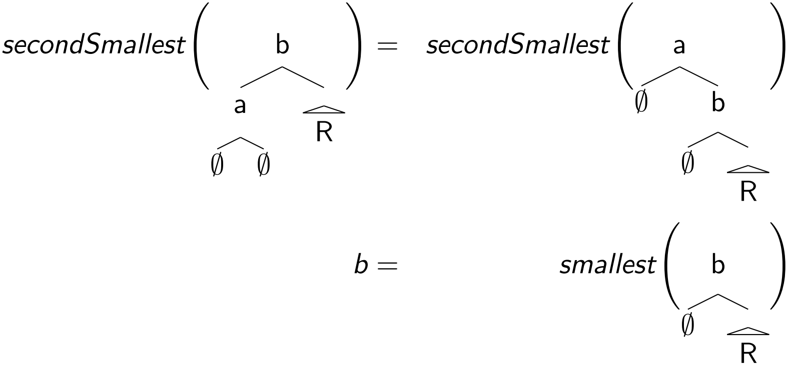
secondSmallest (

) = smallest(R)

(more cases)

Checking BST code

- One case of checking invariance under rotation:



More General Procedure

Invariance of a program $P : T \to Z$ relative to a function $f : T \to T'$

- E.g. $f : BST \to List$

Observation: The following are equivalent:

- $f(x) = f(y) \implies P(x) = P(y)$
- There exists a program $\widetilde{P} : T' \to Z$ such that $P = \widetilde{P} \circ f$

$$\begin{tikzcd}
T \arrow[rr, "P"] \arrow[dr, "f"'] & & Z \\
& T' \arrow[ur, dashed, "\widetilde{P}"'] &
\end{tikzcd}$$

More General Procedure

- Idea: Synthesize a witness to the invariance
    - A function $\widetilde{P} : T' \to Z$
- P and f provide a *full specification* of $\widetilde{P}$
- Counterexample guided inductive synthesis

$$\begin{tikzcd} T \arrow[rr, "P"] \arrow[dr, "f"'] & & Z \\ & T' \arrow[ur, dashed, "\widetilde{P}"'] & \end{tikzcd}$$

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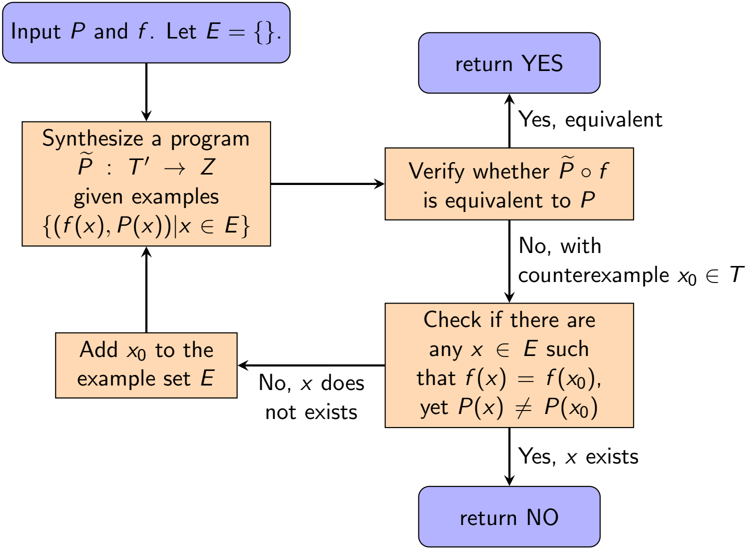

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asdf

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