Verifying Robustness of Programs Under Structural Perturbations

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Motivation

- An attempt to synthesize the max function using PBE:
  - $(13, 15) \mapsto 15$
  - $(-23, 19) \mapsto 19$
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- Synthesized program: `P(a,b):=return b`
- Neither synthesized program, nor synthesizer are *robust*

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  would synthesize very different program
- Synthesize a robust program or develop robust synthesizer

Robustness Properties

- **Continuity**: small change to input $\Rightarrow$ small change to output

```
Sort([1,4,3,6])=[1,3,4,6]
Sort([2,3,3,5])=[2,3,3,5]
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  `Sort([1,4,3,6])=[1,3,4,6]`
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- **Simultaneous Permutation**: permuting all inputs leaves output invariant (`Grade(responses,answers)`)

```
Grade([sqrt(x^2), 1/e, 6.5], [abs(x), e^-1, 13/2])=1
```

rearrange problem parts

```
Grade([1/e, 6.5, sqrt(x^2)], [e^-1, 13/2, abs(x)])=1
```

Verifying Continuity [1]

- Consider

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if x ≥ 0 then
    r := y
else
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- Proof rule:

$$\frac{\begin{array}{cc} c \vdash \mathrm{Cont}(P_1, \mathrm{In}, \mathrm{Out}) & c \vdash \mathrm{Cont}(P_2, \mathrm{In}, \mathrm{Out}) \\ c' \vdash \mathrm{Cont}(b, \mathrm{Var}(b)) & (c \wedge \neg c') \vdash \mathrm{Out}_{P_1} = \mathrm{Out}_{P_2} \end{array}}{c \vdash \mathrm{Cont}(\textbf{if } b \textbf{ then } P_1 \textbf{ else } P_2, \mathrm{In}, \mathrm{Out})}$$

Verifying Continuity [1]

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  1: **if** $x \geq 0$ **then**
  2: $\quad r := y$
  3: **else**
  4: $\quad r := z$

- If $y \neq z$, discontinuous at $x = 0$
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- Only applicable to numerical perturbations

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- Requires specifying property in first-order logic
- Not optimized for 2-safety properties

Our Contributions

Goals:

- Reason about invariance under discrete perturbations
- Want to optimize for our specific problem

Results:

- Small sets of perturbations that "generate" all perturbations
  - Lists, binary search trees
- Formulate "invariance with respect to a function"
  - General, sound procedure
- Sanity checks and bug finding

Lists – Invariance under order

Given an array a

- Let a_{swap} be a with its first and second entry swapped
  - $[a[1], a[0], a[2], a[3], \ldots, a[n]]$
- Let a_{rot} be a rotated by 1
  - $[a[1], a[2], a[3], \ldots a[n], a[0]]$

Lemma: If for any a, $P(a) = P(a_{swap}) = P(a_{rot})$, then for any permutation a' of a, we have $P(a) = P(a')$.
Proof: Math [3]

Programs – Invariance under order

- maxList([x]) = x
- maxList([x, ...xs...]) = max(x, maxList(xs))

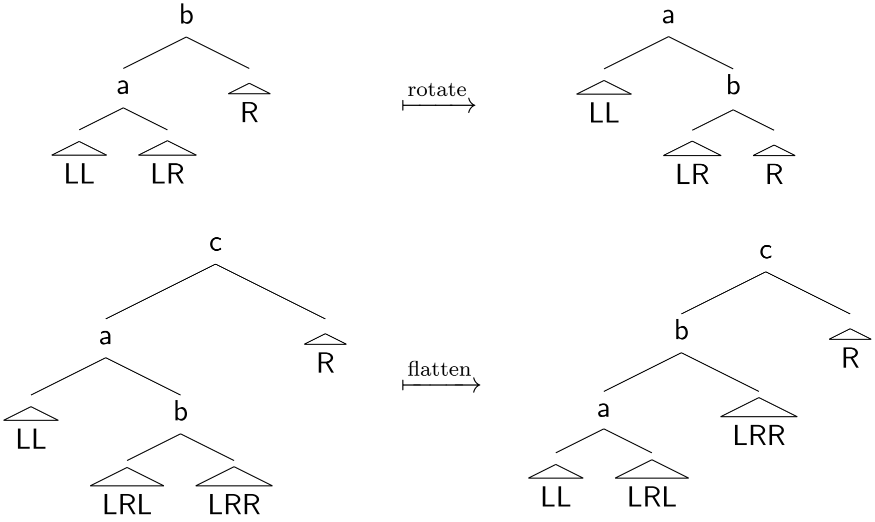
- Verifying maxList(a) = maxList(a_{swap}) has one case:

$$\begin{array}{rcl} maxList([x, y, ...xs...]) & \stackrel{?}{=} & maxList([y, x, ...xs...]) \\ \| & & \| \\ max(x, maxList([y, ...xs...])) & & max(y, maxList([x, ...xs...])) \\ \| & & \| \\ max(x, max(y, maxList(xs))) & & max(y, max(x, maxList(xs))) \\ \| & & \| \\ max(x, max(y, z)) & & max(y, max(x, z)) \end{array}$$

Binary Search Trees

- For lists, two simple permutations generated all permutations
- Goal: similar permutations for BSTs

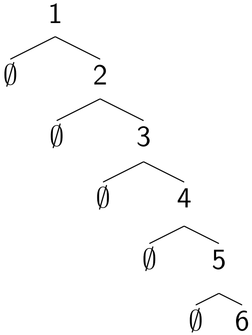
Binary Search Trees



Binary Search Trees

It suffices to show

- Every tree can be transformed into a "normal form" (i.e. list)
    - "flatten" straightens out the tree
    - "rotate" lets you straighten all the parts
- Every operation is reversable



Lists and Binary Search Trees

- Can check robustness under ALL permutations by checking just TWO permutations

More General Procedure

- Sets of permutations are case-by-case
- Goal: formulation of invariance
  - Useful
  - Easy to code/express
  - Checkable

More General Procedure

Invariance of a program $P : T \to Z$ relative to a function $f : T \to T'$

- $f(t)$ gives a "canonical representative" of t
- For concreteness, $f = list : BST \to List$

Observation: The following are equivalent:

- $list(x) = list(y) \implies P(x) = P(y)$
- There exists a program $\widetilde{P} : Lists \to Z$ such that $P(t) = \widetilde{P}(list(t))$

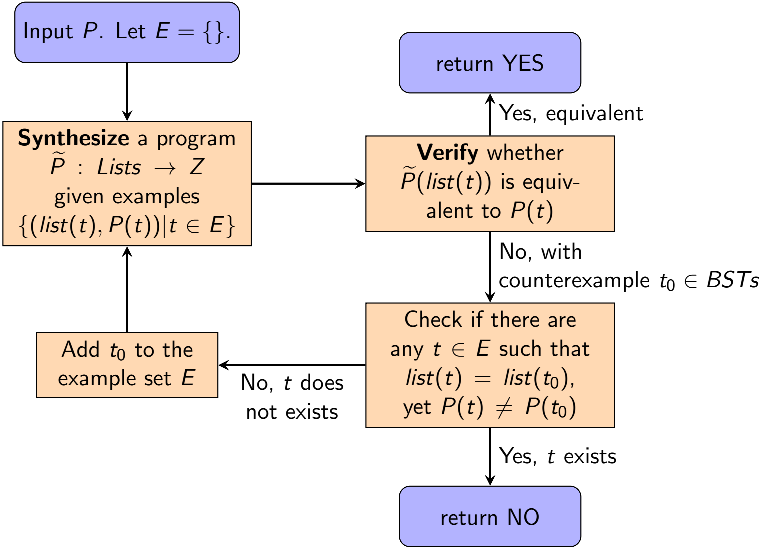
$$\begin{tikzcd}
BSTs \arrow[rr, "P"] \arrow[dr, "list"'] & & Z \\
& Lists \arrow[ur, dashed, "\widetilde{P}"'] &
\end{tikzcd}$$

More General Procedure

- Idea: Synthesize a witness to the invariance
  - A function $\widetilde{P} : \mathit{Lists} \to Z$
- P and *list* provide a *full specification* of $\widetilde{P}$
- Counterexample guided inductive synthesis [4]

$$
\begin{tikzcd}
\mathit{BSTs} \arrow[rr, "P"] \arrow[dr, "\mathit{list}"] & & Z \\
& \mathit{Lists} \arrow[ur, dashed, "\widetilde{P}"] &
\end{tikzcd}
$$

More General Procedure



Future Directions

- Develop proof rules for discrete perturbations
- Improved handling of branching programs by Cartesian Hoare Logic
- Working implementation of Cartesian Hoare Logic
- Find more data structures with small perturbation sets
- Speed up our general procedure
- Synthesis for verification?
- Implement!

References

S. Chaudhuri, S. Gulwani, and R. Lublinerman.
Continuity analysis of programs.
POPL '10, pages 57–70, New York, NY, USA, 2010. ACM.

S. Chaudhuri, S. Gulwani, and R. Lublinerman.
Continuity and robustness of programs.
Commun. ACM, 55(8):107–115, Aug. 2012.

D. S. Dummit and R. M. Foote.
Abstract Algebra.
John Wiley & Sons, 3rd edition, 2004.

A. Solar-Lezama, L. Tancau, R. Bodik, V. Saraswat, and S. Seshia.
Combinatorial sketching for finite programs.
In *ASPLOS-12*, pages 404–415, Oct 2016.

M. Sousa and I. Dillig.
Cartesian hoare logic for verifying k-safety properties.
PLDI '16, pages 57–69, New York, NY, USA, 2016. ACM.