Verifying Robustness of Programs Under Structural Perturbations

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Motivation

- An attempt to synthesize the max function using PBE:
  - $(13, 15) \mapsto 15$
  - $(-23, 19) \mapsto 19$
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- Synthesized program: `P(a,b):=return b`
- Neither synthesized program, nor synthesizer are *robust*

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  would synthesize very different program
- Synthesize a robust program or develop robust synthesizer

Robustness Properties

- Continuity: small change to input $\Rightarrow$ small change to output

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Sort([1,4,3,6])=[1,3,4,6]
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- Permutation: permuting input permutes the output

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Find([1,4,3,6], 4)=1
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$$\sigma = (0 \mapsto 2, 1 \mapsto 3, 2 \mapsto 1, 3 \mapsto 0)$$

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Find([6,3,1,4], 4)=3
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- Permutation: permuting input leaves output invariant

```
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Verifying Continuity

- Consider

if $x \geq 0$ **then**
    $r := y$
else
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- Proof rule:

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- Only applicable to numerical perturbations

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- Given `if-elsif-elsif-else`,
  - Reason about each branch (4 branches)

If-Elsif-Elsif-Else Cost

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- Given `if-elsif-elsif-else`,
  - Reason about each branch (4 branches)
  - Check equivalence (3 comparisons)

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  - Determinism:

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- Cartesian Hoare Triple examples:
  - Determinism:

$$\|\vec{x_1} = \vec{x_2}\| f(\vec{x}) \| ret_1 = ret_2\|$$

  - Symmetry:

$$\|x_1 = y_2 \wedge x_2 = y_1\| f(x, y) \| ret_1 = ret_2\|$$

If-Elsif-Elsif-Else Cost

- Given `if-elsif-elsif-else`, must reason about product of each pair of branches (16 pairs):

$$P_1 \circledast P_1$$
$$\vdots$$
$$P_1 \circledast P_4$$
$$P_2 \circledast P_1$$
$$\vdots$$
$$P_4 \circledast P_4$$

Structural invariants – Motivation

- Sanity checks and bug finding
- Functions invariant under order of list
  - max, sum, sort
- Data structures with representing something else
- Invariance under value it represents
  - binary search trees, heaps, hash sets

Lists – Invariance under order

Given an array a

- Let a_{swap} be a with its first and second entry swapped
  - $[a[1], a[0], a[2], a[3], \ldots, a[n]]$
- Let a_{rot} be a rotated by 1
  - $[a[1], a[2], a[3], \ldots a[n], a[0]]$

Lemma: If for any a, $P(a) = P(a_{swap}) = P(a_{rot})$, then for any permutation a' of a, we have $P(a) = P(a')$.
Proof: Math

Programs – Invariance under order

- maxList([x]) = x
- maxList([x, ...xs...]) = max(x, maxList(xs))

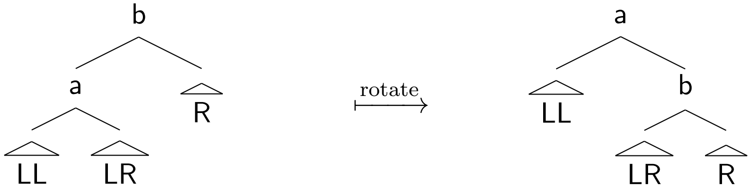
- Verifying maxList(a) = maxList(a_{swap}) has one case:

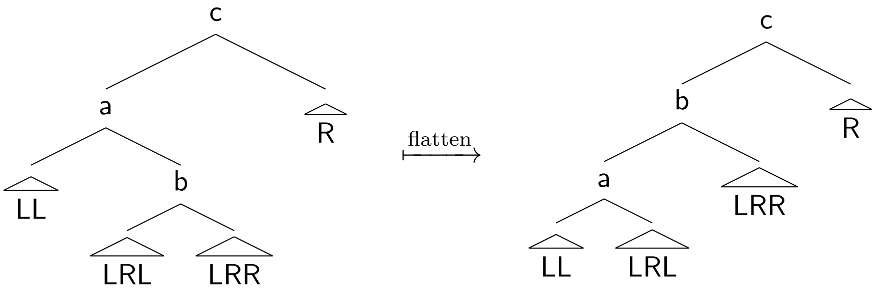
$$\begin{array}{rcl}
maxList([x, y, ...xs...]) & \stackrel{?}{=} & maxList([y, x, ...xs...]) \\
\| & & \| \\
max(x, maxList([y, ...xs...])) & & max(y, maxList([x, ...xs...])) \\
\| & & \| \\
max(x, max(y, maxList(xs))) & & max(y, max(x, maxList(xs))) \\
\| & & \| \\
max(x, max(y, z)) & & max(y, max(x, z))
\end{array}$$

Binary Search Trees

- For lists, two simple permutations generated all permutations
- Goal: similar permutations for BSTs

Binary Search Trees

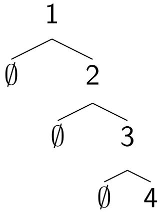




Binary Search Trees

It suffices to show

- Every tree can be transformed into a "normal form" (i.e. list)
  - "flatten" straightens out the tree
  - "rotate" lets you straighten all the parts
- Every operation is reversable



Lists and Binary Search Trees

- Can check robustness under ALL permutations by checking just TWO permutations

More General Procedure

- Sets of permutations are case-by-case
- Goal: formulation of invariance
    - Useful
    - Easy to code/express
    - Checkable

More General Procedure

Invariance of a program $P : T \to Z$ relative to a function $f : T \to T'$

- $f(t)$ gives a "canonical representative" of t
- For concreteness, $f = \mathit{list} : \mathit{BST} \to \mathit{List}$

Observation: The following are equivalent:

- $\mathit{list}(x) = \mathit{list}(y) \implies P(x) = P(y)$
- There exists a program $\widetilde{P} : \mathit{Lists} \to Z$ such that $P(t) = \widetilde{P}(\mathit{list}(t))$

$$\begin{tikzcd}
\mathit{BSTs} \arrow[rr, "P"] \arrow[dr, "\mathit{list}"'] & & Z \\
& \mathit{Lists} \arrow[ur, dashed, "\widetilde{P}"'] &
\end{tikzcd}$$

More General Procedure

- Idea: Synthesize a witness to the invariance
  - A function $\widetilde{P} : \mathit{Lists} \to Z$
- P and *list* provide a *full specification* of $\widetilde{P}$
- Counterexample guided inductive synthesis

$$
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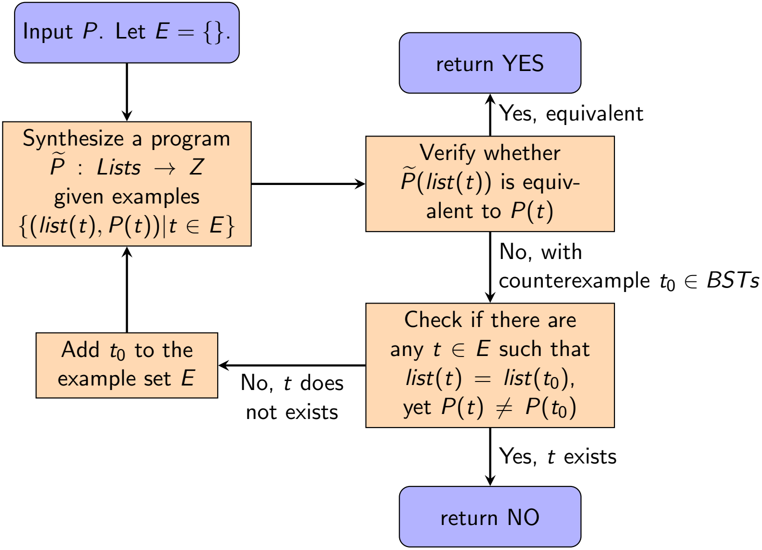

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