Lecture 7: Communication Lower Bounds

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These notes may contain errors, and are not intended for use as a primary source. I have included references at the end of the notes for those who wish detail surrounding skipped steps.

1 Logistics

First homework will be posted soon.

2 Outline

This class will just present a list of constructions proving communication lower bounds in various settings. Many of them are simple, few of them are advanced and may skip details. Just in case some people haven't seen communication complexity before, I'll quickly go through the basic arguments to start.

DEFINITION 1 *A two-party communication problem consists of a boolean function* $f : \{0,1\}^a \times \{0,1\}^b \to \{0,1\}$ *(for this entire lecture, let* $a = b = k$*). Alice is given some input* $A \in \{0,1\}^a$ *(but doesn't know anything about* B*), and Bob is given some input* $B \in \{0,1\}^b$ *(but doesn't know anything about* A*). Their goal is to compute* $f(A, B)$*.*

DEFINITION 2 *A* deterministic communication protocol *for* f *specifies for Alice, as a function of her input* A *and all messages sent so far* $a_1, b_1, \ldots, a_\ell, b_\ell$*, what message* $a_{\ell+1}$ *to send next. It also specifies for Bob, as a function of his input* B *and all messages sent so far what message* $b_{\ell+1}$ *to send next. We'll be interested in the* communication complexity *of a protocol: the maximum number of bits ever sent in total between Alice and Bob.*

You can take the final output of a protocol as a function of the entire transaction history $a_1, b_1, \ldots, a_\ell, b_\ell$. Alternatively, you can stop communication once either party knows the answer (or communicate that answer back with one extra bit). The exact details on this point don't matter.

EXAMPLE 1 (EQUALITY) Let $f(A, B) = 1$ if and only if $A = B$. One protocol to solve this is the following: Alice and Bob both output their entire string in the first round, and declare 1 if and only if their outputs match. A slightly more clever protocol might have Alice and Bob each output the first bit of their string and see if they match. If so, continue. If not, output 0. If they make it all the way to the end without stopping, output 1.

EXAMPLE 2 (DISJOINTNESS) For this problem, it will make more sense to think of A as a subset of $[k]$ (the coordinates which are 1), and B as a subset of $[k]$. Then consider $f(A, B) = 1$ if and only if $A \cap B = \emptyset$. One protocol to solve this is the following: Alice and Bob both output their entire set, and declare 1 if and only if their outputs match.

Both examples are core problems for communication complexity. Interestingly, the natural procedures we posed seem to basically communicate the entire input in the worst case. It turns out that this is because the protocols are (almost) optimal (among deterministic protocols). We'll prove this using a generic type of argument, called a *fooling set argument*. The main idea is the following: Imagine for instance that the complete transcript (that is, all messages sent by Alice and all messages sent by Bob, in order) are identical for inputs (A_1, B_1) and (A_2, B_2). Then we claim that the transcripts must also be identical for (A_1, B_2) and (A_2, B_1).

Observation 1 (Rectangles)
Let $T(A, B)$ denote the complete transcript (all messages sent, in order, and their final answers) produced by Alice and Bob using a communication protocol on input (A, B). Suppose $T(A_1, B_1) = T(A_2, B_2) = T$. Then $T(A_2, B_1) = T = T(A_1, B_2)$ as well, and thus the protocol gives the same answer on all four input pairs.

Proof: Assume for contradiction that this were not the case, and w.l.o.g. say that the different transcript is $T(A_1, B_2) \neq T(A_2, B_2)$. Because the protocol is deterministic, and Bob has the same input in both instances, Bob will continue outputting the same message every round until Alice sends a different message. Therefore, in order for $T(A_1, B_2)$ to possibly not equal $T(A_2, B_2)$, it must be the case that Alice sends the first different message. Let T be the transcript prior to the first different message. Because $T(A_2, B_2) = T(A_1, B_1)$, and Alice's next message can depend only on T and her own input, Alice must send the same next message if her input is A_1 or A_2. This is a contradiction, so the transcripts must indeed be equal. $\Box$

The above is called a "rectangle argument" because if we were to draw a matrix where the rows are Alice's input and columns are Bob's, it means that entries that share the same transcript must form a rectangle (okay, not exactly, but for every transcript T, there exists a relabeling of the rows/columns so that the pairs that induce this transcript are a rectangle). It seems quite simple but is surprisingly powerful because if protocols use limited communication, it must be the case that multiple inputs have the same transcript (by the pigeonhole principle), and the rectangle argument lets us conclude that other inputs necessarily use the same transcript as well.

Definition 3 *A* fooling set *for the function f is a set $F \subseteq \{0,1\}^k \times \{0,1\}^k$ such that:*

- *$f(A, B) = 1$ for all $(A, B) \in F$.*
- *For every $(A_1, B_1), (A_2, B_2) \in F$, either $f(A_1, B_2) \neq 1$ or $f(A_2, B_1) \neq 1$ (or both).*

Proposition 1
Let F be a fooling set for f. Then the deterministic communication complexity of f is at least $\log_2(|F|)$.

Proof: Assume for contradiction that there exists a deterministic protocol with communication $< \log_2 |F|$. There are less than $2^{\log_2 |F|} = |F|$ possible transactions such a protocol can use. Thus, by the pigeonhole principle, there exists some $(A_1, B_1), (A_2, B_2) \in F$ such that $T(A_1, B_1) = T(A_2, B_2)$. By the rectangle argument, this necessarily implies

$T(A_1, B_2) = T(A_2, B_1) = T(A_1, B_1)$ as well. Because f is completely determined by the transaction history, we must have $f(A_1, B_2) = f(A_2, B_1) = 1$ as well.

However, this directly contradicts the second property of fooling sets. $\square$

So if a function admits a large fooling set, it is provably impossible to have a short deterministic communication protocol. The remaining work is just to show that both Equality and Disjointness admit large fooling sets.

PROPOSITION 2
Equality admits a fooling set of size 2^k.

PROOF: Let $F = \{(X, X) \mid X \in \{0, 1\}^k\}$. Then $f(A, B) = 1$ for all $(A, B) \in F$. But $F(A_1, B_2) = 0$ for all $A_1 \neq B_2$. So the second property of the fooling set is satisfied. $\square$

Notice that after unraveling all the definitions, the proof is essentially saying the following: if you had a deterministic protocol that used $< k$ bits, there are two inputs of the form (X, X), (Y, Y) that are treated exactly the same. This protocol must therefore also treat (X, Y) and (Y, X) the same as (X, X), even though the difference affects the value of f. That is, the protocol is "fooled" into thinking the input is (X, X) or (Y, Y) when it really could be (X, Y) or (Y, X), and this affects the output of f.

PROPOSITION 3
Disjointness admits a fooling set of size 2^k.

PROOF: Let $F = \{(X, \bar{X}) \mid X \subseteq [k]\}$. Then $f(A, B) = 1$ for all $(A, B) \in F$. But now consider any $(X, \bar{X}), (Y, \bar{Y}) \in F$. It cannot be the case that $X \cap \bar{Y} = \emptyset$ **and** $Y \cap \bar{X} = \emptyset$ unless $Y = X$. The former implies that $X \subseteq Y$ and the latter implies that $Y \subseteq X$. Therefore, either $f(X, \bar{Y}) = 0$ or $f(Y, \bar{X}) = 0$. $\square$

So both protocols necessarily require communication k to solve deterministically. It turns out that both problems are also hard to solve *nondeterministically*. Disjointness is also hard to solve with a randomized protocol, but Equality can be solved with high probability with very low randomized communication.

2.1 Multi-party communication

We'll also talk about multi-party communication instead of just two parties - this will be relevant when figuring out how good combinatorial auctions we should possibly hope for. Here, there's again a boolean function f but it takes as input $X_1, \ldots, X_n$. Various input models are studied, but we'll only talk about the *number-in-hand* model, where player i knows only X_i and nothing else.

We'll also want a canonical hard problem to start with, multi-disjointness. It also introduces a new concept, called a *promise problem.*

EXAMPLE 3 (MULTI-DISJOINTNESS) Let $f(X_1, \ldots, X_n) = 1$ iff $X_i \cap X_j = \emptyset$ for all $i \neq j$. Moreover, we'll be interested in the *promise problem*, where the protocol is promised that one of the following is true about the input:

- $X_i \cap X_j = \emptyset$ for all $i \neq j$.

- $\cap_i X_i \neq \emptyset$ (there exists an ℓ in all X_i).

More clearly, we say that a communication protocol solves Multi-Disjointness if it outputs 1 whenever $X_i \cap X_j = \emptyset$ for all $i \neq j$, outputs 0 whenever $\cap_i X_i \neq \emptyset$, and is allowed to behave arbitrarily if neither condition holds.

THEOREM 1 ([NIS02], THEOREM 2)
Every deterministic protocol for multi-disjointness has communication complexity $\Omega(k/n)$.

PROOF: First, observe that an analog of the rectangles argument holds in multi-party communication as well: if $T(X_1, \ldots, X_n) = T(Y_1, \ldots, Y_n)$, then $T(Z_1, \ldots, Z_n) = T(X_1, \ldots, X_n)$ whenever $Z_i \in \{X_i, Y_i\}$. The proof is the same: if not, there must be some player who is the first to send a different message. But their input matches their input in one of the two cases $(X_1, \ldots, X_n), (Y_1, \ldots, Y_n)$, so this is a contradiction. The fooling set argument also extends, but we'll use the same concepts slightly differently below.

First, we count the total number 1 instances (i.e. cases where the input sets are disjoint). Observe that each such instance can be selected by choosing, for each item, a player to receive that item, or no player at all. There are $n+1$ choices, independently for all items, giving $(n+1)^k$ possible instances.

Second, we'll first show that no protocol can use the same transcript for more than n^k inputs which evaluate to 1. Let S be a set of inputs, say $\{(X_1^{(j)}, \ldots, X_n^{(j)})\}_{j=1}^{|S|}$ that share the same transcript and are all 1 instances.

CLAIM: For each $\ell \in [k]$, there exists a player i such that player i never gets ℓ in S (i.e. for every input $(X_1^{(j)}, \ldots, X_n^{(j)}) \in S$, we have $\ell \notin X_i^{(j)}$)

PROOF: The negation of the above statement is exactly the following: There exists an $\ell \in [k]$ such that for all players i, i get ℓ in some input from S, say $\ell \in X_i^{j(\ell)}$. But by the rectangle argument, the input $(X_1^{j(\ell)}, \ldots, X_n^{j(\ell)})$ has the same transcript as all the elements of S, and thus should be a 1 instance. However, $\ell \in X_1^{j(\ell)} \cap \ldots \cap X_n^{j(\ell)}$, a contradiction. $\Box$

With this claim in mind, we bound the total possible number of inputs in S. Consider all the ways to put an item $\ell \in [k]$ into an input tuple: it may go to no bidder, or it may go to exactly one bidder (because the sets of an input tuple are disjoint). But, by the above claim, each item has one bidder which it *cannot* go to. This gives each item exactly n choices, for a total size of S of at most n^k.

Finally, observe that the above two results mean that there are at least $(n+1)^k/n^k = (1+1/n)^k$ different transcripts are needed simply to handle the 1-instances of the problem. This gives communication complexity at least $k \log_2(1+1/n) = \Theta(k/n)$. $\Box$

3 Monotone Valuations

First, we'll present a lower bound for monotone valuations, by reduction from MultiDisjointness.

Theorem 2 ([Nis02] Theorem 3)
The communication required to guarantee strictly better than an n-approximation for welfare maximization with m items on all monotone valuations is $2^{\Omega(m/n^2)}$. Note that this implies that exponential communication among the bidders is necessary in order to guarantee that an allocation is within a factor of n of optimal when $n = m^{1/2-\varepsilon}$.

Proof: Think of $m \gg n$ and let k be an integer we'll tune later. First, assume that we have a k-size family of length-n partitions of items M, say $\{(T_1^\ell, \ldots, T_n^\ell)\}_{\ell \in [k]}$, where for all ℓ, $T_i^\ell \cap T_j^\ell = \emptyset$ (note[1]). Additionally, we require that for all $\ell \neq \ell'$, $T_j^\ell \cap T_i^{\ell'} \neq \emptyset$; in other words the disjointness property does not ever mix and match across different partitions. We will associate bidder i to the (some sub collection of) the sets $\{T_i^\ell\}$.

Now, consider the following class of valuations, called *multi-minded*, or *coverage* valuations: bidder i has some subset $X_i \subseteq [k]$, and $v_i(S) = 1$ if and only if $T_i^\ell \subseteq S$ for some $\ell \in X_i$ (and $v_i(S) = 0$ otherwise). That is, bidder i "likes" all the sets T_i^ℓ, for $\ell \in X_i$, and gets value 1 as long as they get some set they like.

Observe that if there exists a single $\ell \in \cap_i X_i$, then it is possible to achieve welfare n: simply give each bidder set T_i^ℓ for the $\ell \in \cap_i X_i$. Also, observe that if it is possible for both bidder i and bidder j to get value 1, then the sets that they receive and are interested in must be disjoint. For desired sets T_i^ℓ and $T_j^{\ell'}$, this can occur only if $\ell = \ell'$. So if for all i, j, $X_i \cap X_j = \emptyset$, then there is no way for both bidder i and bidder j to get non-zero welfare, and the maximum possible welfare is 1. So by a direct reduction to MultiDisjointness, the communication complexity of determining whether the welfare is $\geq n$ or ≤ 1 is at least $\Omega(k/n)$.

Now, we just need to figure out how big of a k we can have while still guaranteeing a "collection of partitions" construction of the desired form. Consider k random partitions of $[m]$. That is, each item is independently and uniformly awarded to a bidder, and T_i^ℓ is the set of items awarded to bidder i in partition ℓ. Clearly, this satisfies the first property: for all ℓ, the T_i^ℓs indeed form a partition, by definition. We just need to make sure we don't take too many random partitions so that we have bad luck and some $T_i^\ell \cap T_j^{\ell'} = \emptyset$.

So consider a fixed $T_i^\ell, T_j^{\ell'}$ for $\ell \neq \ell'$. What is the probability that they don't intersect? This is the probability that for each item x, it is either not put into T_i^ℓ or not put into $T_j^{\ell'}$. Because partitions are selected independently, we get

$$\mathbb{P}\left[x \notin T_i^\ell \vee x \notin T_j^{\ell'}\right] = 1 - \mathbb{P}\left[x \in T_i^\ell \wedge x \in T_j^{\ell'}\right] = 1 - 1/n^2$$

And because items are allocated independently, we get

$$\mathbb{P}\left[T_i^\ell \cap T_j^{\ell'} = \emptyset\right] = \prod_{x \in M} \mathbb{P}\left[x \notin T_i^\ell \vee x \notin T_j^{\ell'}\right] = (1 - 1/n^2)^m \approx e^{-m/n^2}$$

Now just take a union bound over all $kn \cdot (k-1)n \leq (kn)^2$ pairs of sets which we need to have disjoint. We get that the probability that any two sets happen to be disjoint by bad luck is $\approx k^2 n^2 e^{-m/n^2}$. Therefore, we can take $k = 2^{\Omega(m/n^2)}$ and get a non-zero probability

[1]Incidentally, when we actually construct this collection, we'll get *actual* partitions, i.e. $\bigcup_i T_i^\ell = M$ for each ℓ. However, this is unimportant for our reduction.

of success (even if we can't find this instance efficiently, it exists, so we can use that for the construction and proof).

So to wrap up: we need a construction of the form given at the beginning of the proof for large k. The argument above provides a construction for $k = 2^{\Omega(m/n^2)}$, which completes the proof. □

Recall that we have protocols matching this: a $\sqrt{m}$-approximation due to [LS05, DNS05], or the trivial n-approximation which just allocates randomly (both of the previous results would also guarantee an n-approximation).

4 Subadditive Valuations

Theorem 3 ([DNS05])
The communication required to guarantee strictly better than a $(1/2+1/(2n))$-approximation for welfare maximization with m items on all subadditive valuations is $2^{\Omega(m/n^2)}$.

Proof: Again, assume that we have the following construction of partitions: $\{(T_1^\ell, \ldots, T_n^\ell)\}_{\ell \in [k]}$, where for all ℓ, $T_i^\ell \cap T_j^\ell = \emptyset$, but for all $\ell \neq \ell'$, $T_j^\ell \cap T_i^{\ell'} \neq \emptyset$.

Observe that the previous construction is not subadditive. In particular, player's have value 1 for some sets, while having value 0 for every proper subset. The only change we'll make is to bump up the players' values for every non-empty subset by 1. Specifically, bidder i has some subset $X_i \subseteq [k]$, and $v_i(S) = 2$ if $T_i^\ell \subseteq S$ for some $\ell \in X_i$ (and $v_i(S) = 1$ otherwise). This function is now clearly subadditive: any two non-trivial sets have value one, and no set has value more than two.

The only difference is in the welfare guarantee we can claim. Now, if we're in the disjoint case, we can certainly get welfare $2n$. But even in the non-disjoint case, we can get welfare $n+1$ (one player can get a set they like, and everyone else gets a single item). So our lower bound is now only $2n/(n+1)$ instead of n. □

Recall that we have a 2-approximation (that we didn't see) due to [Fei06]. This is asymptotically tight as $n \to \infty$, but not for small n (actually, the algorithm is tight for small n, and the lower bound can be improved, see later section).

5 XOS Valuations

Theorem 4 ([DNS05])
The communication required to guarantee strictly better than a $1-(1-1/n)^n$-approximation for welfare maximization with m items on all XOS valuations is $2^{\Omega(m/n^2)}$.

Proof: This time, we'll need a slightly stronger construction (actually, the same construction works, we just need to appeal to a stronger property). This time, assume that we have the following construction of partitions: $\{T_1^\ell, \ldots, T_n^\ell\}_{\ell \in [k]}$, where for all ℓ, $T_i^\ell \cap T_j^\ell = \emptyset$, but for all $\ell_1, \ldots, \ell_n$ (all distinct) we have $|\cup_i T_i^{\ell_i}| \leq m \cdot (1-(1-1/n)^n)$.

Now, consider the following class of valuations. First, for all ℓ define an additive valuation $v_i^\ell(S) := |S \cap T_i^\ell|$ (that is, value one per item in T_i^ℓ). Next, give each bidder some set X_i of indices, and define $v_i(S) := \max_{\ell \in X_i} v_i^\ell(S)$. This is clearly XOS (even Binary XOS).

Observe again that if there exists a single $\ell \in \cap_i X_i$, then it is possible to achieve welfare m: simply give each bidder set T_i^ℓ for the $\ell \in \cap_i X_i$. But now the welfare might be better in the "no" case, because the bidders will still get value 1 per item even if they barely overlap with their sets. Specifically, if each X_i is disjoint, in any allocation there will be at most $|\cup_i T_i^{\ell_i}|$ items which are valued at 1 (for some $\ell_1, \dots, \ell_n$ all distinct). Thus, the welfare can only be as large as $m \cdot (1-(1-1/n)^n)$.

Again, we just need to figure out how big of a k we can have while still guaranteeing a construction of the desired form. Again consider k random partitions of $[m]$. That is, each item is independently and uniformly awarded to a bidder, and T_i^ℓ is the set of items awarded to bidder i in partition ℓ. Clearly, this satisfies the first property: for all ℓ, the T_i^ℓs indeed form a partition, by definition. We just need to make sure we don't take too many random partitions so that we have bad luck and some $T_i^\ell \cap T_j^{\ell'} = \emptyset$.

Consider a fixed set of distinct indices $\ell_1, \dots, \ell_n$. Then the expected number of items in their union is exactly $1-(1-1/n)^n \cdot m$ (because for any item to miss the union, it needs to be missed independently n times). The probability that it deviates from its expectation by more than $\varepsilon \cdot m$ is upper bounded by $e^{-\varepsilon^2 m/2}$. So taking a union bound over all $\leq k^n$ distinct indices, we get that this holds for all indices with probability $e^{-\varepsilon^2 m/2 + n\log k}$. Setting $n \log k = \varepsilon^2 m/8$ results in $k \leq e^{O(m\varepsilon^2/n)}$. We could, for instance, let $\varepsilon = 1/\sqrt{n}$, giving the bound in the theorem statement (but there are other interesting choices of ε too). $\square$

6 XOS Simultaneous Lower Bound

Consider the following construction:

1. S is drawn uniformly at random from sets of size $m/2$.
2. Let $a_1, \dots, a_k$ be uniformly random sets of size $m/2$, conditioned on $|S \cap a_i| = m/3$.
3. Alice's initial BXOS function has $A(X) = \max_i\{|a_i \cap X|\}$.
4. T is drawn uniformly at random from sets of size $m/2$, conditioned on $|S \cap T| = m/3$.
5. Let $b_1, \dots, b_k$ be uniformly random sets of size $m/2$, conditioned on $|T \cap a_i| = m/3$.
6. Bob's initial BXOS function has $B(X) = \max_i\{|b_i \cap X|\}$.
   - Note: if we used BXOS evaluations A and B, the maximum achievable welfare would be at most $(3/4 - 1/108)m$
7. Let Y be a uniformly random set satisfying $|Y \cap S| = |\bar{Y} \cap T| = m/3$.
   Let Z be a uniformly random set satisfying $|Z \cap S| = |Z \cap T| = m/3$.
   (a) With probability 1/2, set $v^A(X) = \max\{A(X), |Y \cap X|\}$
   and $v^B(X) = \max\{B(X), |\bar{Y} \cap X|\}$
   (in this case, the welfare shoots up to m).
   (b) With probability 1/2 set $v^A(X) = \max\{A(X), |Z \cap X|\}$
   and $v^B(X) = \max\{B(X), |Z \cap X|\}$
   (in this case, the welfare remains approximately the same).

The following example sets give some intuition for why such sets S, T, Y, Z should exist:

	a	b	c	d	e	f
S:	1	1	1			
T:		1	1	1		
Y:	1	1			1	
Z:		1	1		1	

THEOREM 5 ([BMW18])
No simultaneous protocol with subexponential in m communication can beat a $(3/4-1/108)$ *approximation on the above construction with probability* $> 2/3$ *for decision.*

We'll present the following steps, without proof.

1. For sufficiently small k, but $k = \exp(m)$, the maximum welfare between A and B $(3/4 - 1/108)m$. This is basically because the union of any two sets constructed as in steps 1 thru 6 is $(3/4 - 1/108)m$.

2. If we do case (a) in the last step, the maximum achievable welfare between v^A and v^B is m, by giving Y to Alice and $\bar{Y}$ to Bob.

3. In case (b), the maximum achievable welfare between v^A and v^B is $(3/4 - 1/108)m$. This is essentially because the additional sets are no better than another random set.

4. In either case, it is impossible for Alice to distinguish which of her sets is this special one (i.e. Y in case (a) or Z in case (b)). This is because the conditioning related to S is the same as for each of the sets a_i, and only the conditioning related to T is different. Ditto for Bob.

5. Therefore, in order to possibly solve the decision problem, Alice or Bob would have to convey some information about every single one of their sets, which is exponential. (This can be made into a formal statement, but we won't prove that in this class).

Observe that with two rounds, Alice could first send Bob the set S (which is painfully obvious by her construction – each a_i intersects S way more than it should). Bob could then determine which set he has that is "special." (in case (a), his set $\bar{Y}$ should intersect S way less than average, where in case (b), his set Z should intersect S way more than average).

Also observe that for search, Alice will just pick a random i and keep the items in a_i and give the rest to Bob. This will get the right welfare approximation in both cases (it will just get different absolute welfare in both).

7 Subadditive Lower Bound for Two Players

Consider the following construction, which is well-defined for any collection S of subsets: for a given set T, say that T is covered by collection $\mathcal{Z}$ of sets if $T \subseteq \cup_{z \in \mathcal{Z}} z$. Let $f_S(T)$ be the minimum number of sets in S that cover T.

Lemma 4
$f_S(\cdot)$ is subadditive for all S.

Proof: Let Z cover T and Y cover U. Then clearly $Z \cup Y$ covers $T \cup U$. Therefore, if Z and Y are the minimum covers of T and U respectively, we have that $f_S(T \cup U) \leq |Z| + |Y| = f_S(T) + f_S(U)$. $\square$

Now, we wish to further modify $f_S(\cdot)$ into $f_S^\ell(\cdot)$ as follows:

- If $f_S(X) < \ell/2$, $f_S^\ell(X) = f_S(X)$.
- If $f_S(\bar{X}) < \ell/2$, $f_S^\ell(X) = \ell - f_S(\bar{X})$.
- If neither, then $f_S^\ell(X) = \ell/2$.

Observe that $f_S^\ell(\cdot)$ might not be well-defined (the first two bullets might conflict). But for S with the right properties, it is always well-defined. Also observe that $f_S^\ell(X) + f_S^\ell(\bar{X}) = \ell$ for any S and X.

Definition 4 (ℓ-sparse) *We say that $\mathcal{S}$ is* ℓ-sparse *if for all $T_1, \ldots, T_{\ell-1} \in \mathcal{S}$, $\cup_j T_j \neq M$.*

That is, $\mathcal{S}$ is ℓ-sparse if there does not exist $\ell - 1$ elements of $\mathcal{S}$ whose union is the entire ground set M.

Lemma 5
If $\mathcal{S}$ is ℓ-sparse, then $f_{\mathcal{S}}^\ell(\cdot)$ is well-defined.

Proof: It is clear that $f_{\mathcal{S}}^\ell(X)$ is always defined at least once. The only way in which $f_{\mathcal{S}}^\ell(X)$ could be defined multiple times is if $f_{\mathcal{S}}(X) < \frac{\ell}{2}$ (in which case $f_{\mathcal{S}}^\ell(X) = f_{\mathcal{S}}(X)$) and $f_{\mathcal{S}}(\overline{X}) < \frac{\ell}{2}$ (in which case $f_{\mathcal{S}}^\ell(X) = \ell - f_{\mathcal{S}}(\overline{X})$).

So assume for contradiction that both events hold, and let $X \subseteq \cup_{i=1}^{\ell/2-1} T_i$, and $\overline{X} \subseteq \cup_{i=1}^{\ell/2-1} U_i$, where each $T_i, U_i \in \mathcal{S}$ (such sets exist by the definition of $f_{\mathcal{S}}$ and the assumption that $f_{\mathcal{S}}(X), f_{\mathcal{S}}(\overline{X}) < \ell/2$). But now consider that we can write $M = X \cup \overline{X}$ as a union of $\leq \ell - 2$ elements of $\mathcal{S}$, contradicting that $\mathcal{S}$ is ℓ-sparse. $\square$

Proposition 6
If $\mathcal{S}$ is ℓ-sparse, then $f_{\mathcal{S}}^\ell(\cdot)$ is monotone and subadditive.

(Proof omitted). So now we have a class of subadditive functions. We now wish to come up with a hard instance. To this effect, we need one further definition.

Definition 5 ([**?**]) *A collection $\mathcal{S} = \{S_1, \ldots, S_k\}$ is* ℓ-independent *if $\{T_1, \ldots, T_k\}$ is ℓ-sparse whenever $T_i \in \{S_i, \overline{S_i}\}$.*

Proposition 7
Let $\mathcal{S}$ be an ℓ-independent collection with $|\mathcal{S}| = k$. Then any deterministic communication protocol that guarantees a $(\frac{1}{2} + \frac{1}{2\ell-3})$-approximation to the optimal welfare for two monotone subadditive bidders requires communication at least k.

Proof: Let $\mathcal{S} = \{S_1, \dots, S_k\}$ be ℓ-independent. For each i, define $S_i^1 := S_i$, and $S_i^0 := \overline{S_i}$. Now, consider an instance of EQUALITY where Alice is given a and Bob is given b. Alice will create the valuation function $f^\ell_{\mathcal{A}}$, where $\mathcal{A} := \{S_1^{a_1}, \dots, S_k^{a_k}\}$ (i.e. Alice builds $\mathcal{A}$ by taking either S_i^1 or S_i^0, depending on a_i). Bob will create the valuation function $f^\ell_{\mathcal{B}}$, where $\mathcal{B} := \{S_1^{b_1}, \dots, S_k^{b_k}\}$. Observe first that $f^\ell_{\mathcal{A}}(\cdot)$ and $f^\ell_{\mathcal{B}}(\cdot)$ are indeed well-defined, monotone, and subadditive as $\mathcal{S}$ is ℓ-independent (and therefore $\mathcal{A}$ and $\mathcal{B}$ are both ℓ-sparse).

Observe that if $a = b$, then $\mathcal{A} = \mathcal{B}$ and moreover $f^\ell_{\mathcal{A}}(\cdot) = f^\ell_{\mathcal{B}}(\cdot)$. Observe that for any S, ℓ we have $f^\ell_S(X) + f^\ell_S(\bar{X}) = \ell$, so the maximum achievable welfare is ℓ.

On the other hand, if there exists an i such that $a_i \neq b_i$ (without loss of generality say that $a_i = 1$ and $b_i = 0$), we claim that welfare $2\ell - 2$ is achievable. To see this, consider the allocation which awards $\overline{S_i}$ to Alice and S_i to Bob. Indeed, $f_{\mathcal{A}}(S_i) = 1$ (as $S_i \in \mathcal{A}$), so $f^\ell_{\mathcal{A}}(\overline{S_i}) = \ell - f_{\mathcal{A}}(S_i) = \ell - 1$. Similarly, $f_B(\overline{S_i}) = 1$, so $f_{\mathcal{B}}(S_i) = \ell - f_{\mathcal{B}}(\overline{S_i}) = \ell - 1$, achieving total welfare $2(\ell - 1)$. (note[2])

So assume for contradiction that a deterministic $\frac{1}{2} + \frac{1}{2\ell - 3} > \frac{\ell}{2\ell - 2}$-approximation exists to the optimal welfare for two monotone subadditive bidders with communication $< k$. Then such a protocol would solve EQUALITY with communication $< k$ by the reduction above, a contradiction. □

Lemma 8
For all m, $x > 1$, and $\ell = \log_2(m) - \log_2(x)$, there exists a ℓ-independent collection of subsets of $[m]$ of size $k = e^{x/\ell}$.

8 Possible Project Topics

- See if the presented subadditive construction "separates" XOS from subadditive in other cases where good algorithms for subadditive aren't known but good ones for XOS are (ask me for examples).

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[2] As an aside, note that welfare exceeding $2\ell - 2$ is not possible. Proof: for all nonempty X, $f_S(X)$ is at least 1, and therefore $f^\ell_S(X)$ is at most $\ell - 1$ by the second bullet point in the definition of f^ℓ_S. Therefore, the only way Alice or Bob could have value exceeding $\ell - 1$ is to get all of M, meaning that the other player receives value 0.

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