Preferences Resulting From Weighted Sums

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1 Motivation

Suppose students are told to rank different schools they would like to get into. The preferences of the students are likely correlated in some way due to an inherent notion of the quality of different schools. One way to define such a correlation is to assume there is some underlying space of quality along different attributes (e.g. STEM education vs liberal arts education) and the students' preference actually corresponds to a preference among the underlying attributes. The simplest instance of this is for each student to rank schools according to a weighted sum of the different attributes of the school.

We want to study the inherent complexity of the collection of preferences that result from these procedure, as a function of the number of attributes the schools have. In other words, what sort of correlation arises in the preferences of students in this model?

2 Definitions

Let $\{x_1, \ldots, x_n\} = X \subseteq \mathbb{R}^d_{\geq 0}$ be any set of points with nonnegative coordinates. Given any $a \in \mathbb{R}^d_{\geq 0}$, define a relation $>_a$ on X as follows: $x_i >_a x_j$ if and only if $\langle a, x_i \rangle > \langle a, x_j \rangle$, or $\langle a, x \rangle = \langle a, y \rangle$ and $i > j$ (note[1]). Let $P(X) = \{>_a | a \in \mathbb{R}^d_{\geq 0}\}$. Thus, X represents the set of schools, the vectors a represent the preference weights of students, and $P(X)$ denotes the set of all possible preferences of schools.

Question. *As a function of d and n, how "rich" can $P(X)$ be (and what is the right notion of "richness")?*

Some observations:

- If $d = 1$, then $|P(X)| = 1$, i.e. students preferences are completely determined by the underlying set X.
- If $d = n$, then every linear preference on X can occur in $P(X)$.

[1] It's subtly important that ties are broken in the same way by every a. Otherwise, if X was n copies of the same point, any preference would be possible.

Proof. Let $X = \{e_i\}$ simply be the standard basis vectors. To induce an ordering $i_1, i_2, \ldots, i_n$, just create a preference vector a which gives weight $1/k$ to coordinate i_k. $\square$

The above hints that there should be some sort of continuum between $d = 1$ and $d = n$ of how complex the set $P(X)$ can be.

Here are some simple ways that the structure of X induces structure on $P(X)$:

Proposition. *Let $x, y \in X$. If $x[k] > y[k]$ for each $k = 1, \ldots, d$, then $x >_a y$ for any $a \in \mathbb{R}^d_{\geq 0}$.*

Definition. *For points $x_1, \ldots, x_n \in \mathbb{R}^d$, let $\mathrm{hull}(x_1, \ldots, x_n) = \{u_1 x_1 + \ldots + u_n x_n | 0 \leq u_i \leq 1, \sum_{i=1}^n u_i = 1\}$ denote the convex hull of $x_1, \ldots, x_n$.*

Lemma. *Let $z, x_1, \ldots, x_k \in \mathbb{R}^d_{\geq 0}$. If $z >_a x_i$ for $i = 1, \ldots, k$, then $z >_a w$ for any $w \in \mathrm{hull}(x_1, \ldots, x_k)$.*

Proposition. *Let $z, x_1, \ldots, x_k \in \mathbb{R}^d_{\geq 0}$. Suppose that there exists $w \in \mathrm{hull}(x_1, \ldots, x_n)$ such that $z[i] < w[i]$ for $i = 1, \ldots, d$. Then no a satisfies $z >_a x_i$ for each $i = 1, \ldots, k$.*

Proof. For contradiction, suppose such an a exists. Then $z >_a w$ as well. However, because we have $z[i] < w[i]$ for each i, this is a contradiction. $\square$

Let's look at how preferences are related when different weight vectors a are related.

Definition. *For any vectors $a_1, \ldots, a_i \in \mathbb{R}^d_{\geq 0}$, define $\mathrm{cone}(a_1, \ldots, a_i) = \{u_1 a_1 + \ldots + u_n a_n | u_j \geq 0 \forall j\}$. That is, the convex cone of vectors defined by $a_1, \ldots, a_i$.*

Proposition. *Suppose that for preference weights $a_1, \ldots, a_i$, we have $x >_{a_1} y, \ldots, x >_{a_i} y$. Then for any $b \in \mathrm{cone}(a_1, \ldots, a_i)$, $x >_b y$ as well.*

3 Cycles

I think I have a "forbidden subset" type of theorem for $P(X)$, although I can only prove it when $d = 2$.

Proposition. *If $d = 2$, then the preferences*

$$\begin{gathered} x >_a y >_a z \\ y >_b z >_b x \\ z >_c x >_c y \end{gathered}$$

cannot occur for $x, y, z \in X \subseteq \mathbb{R}^2_{\geq 0}$.

Proof. Suppose the above preferences exist. Because $d = 2$, one of a, b, c must be contained in the convex cone of the other two, i.e. $d \in \{a, b, c\}$ with $d \in \text{cone}(e, f)$ where $\{e, f\} = \{a, b, c\} \setminus \{d\}$. (This is visually very obvious: given three different vectors in two dimensions, one must lie between the other two. You can prove it by considering the angle of the vectors from the x axis). Any pair of the above preferences has exactly one pair among x, y, z which they rank the same, so $u >_e v$ and $u >_f v$ for some u, v. However, this pair is reversed in the remaining preference, i.e. $u <_d v$. $\square$

The above set of preferences intuitively forms a "cycle", where each preference is "shifted" over by one. Formally,

Definition. *A k-cycle is a set of points $x_1, \ldots, x_k \in \mathbb{R}^d_{\geq 0}$ and weights $a_1, \ldots, a_k \in \mathbb{R}^d_{\geq 0}$ such that for each $i = 1, \ldots, k$,*

$$x_i >_{a_i} x_{i+1} >_{a_i} x_{i+2} >_{a_i} \ldots >_{a_i} x_{i-1}$$

(where indices wrap around after k).

Conjecture. *If $X \subseteq \mathbb{R}^d_{\geq 0}$, then a $(d+1)$-cycle can never appear in $P(X)$.*

The second part of the proof from $d = 2$ generalizes: in a $d + 1$ cycle, any subset of d preferences has a consistent preference on some pair, and this preference is broken in the remaining element of the cycle. However, the first part is not true: it is easy to construct 4 vectors in 3 dimensions such that none lies in the convex cone of the others.

This conjecture has been checked by randomly placing $d+1$ points in d dimensional space many times, for $d = 3, 4$.

It may be that in some sense, $(d + 1)$-cycles are the *only* set of preferences that are impossible with d dimensions. For example, there are two different (up to relabeling) ways for a collection of preferences on three items to not contain a cycle. Writing a linear preference as a simple ordered (most preferable to least), the preference sets are $\{123, 132, 312, 321\}$ and $\{123, 231, 213, 321\}$. Both are realizable with $d = 2$ (the first via points $\{(1, 0), (0.4, 0.4), (0, 1)\}$ and the second with $\{(1, 0), (0.6, 0.6), (0, 1)\}$).

When $d = 3$, randomly searching over X for the largest $|P(X)|$ yield a maximum size of 18, and those sets exactly correspond to removing one element from each of the 6 cycles on 4 points. A variety of different ways of doing this are possible, but I'm not yet sure if all of them result.

4 Experimental Counts

The following gives upper bounds on the maximal number of preferences $|P(X)|$ as a function of n and d. I've only included entries if those points seem relatively close to the true maximum (i.e. when things seem pretty stable to checking more preference vectors or doing more trials).

n	d=2	d=3	d=4
3	4	6	6
4	7	18	24
5	10	41	85
6	15	87	
7	20	121	
8	24		
9	28		
10	39		
11	43		
12	48		

5 An (Almost) Semidefinite Program Formulation

In this section, we formalize a decision problem for whether a set of preferences can arise from a set of points in $\mathbb{R}^d_{\geq 0}$, then show how this can be reduced to checking the feasibility of a semidefinite program.

Consider the decision problem given by:
Input: d, n, and a collection $P = \{>_1, >_2, \ldots, >_k\}$ of linear orders on $[n]$
Output: YES if there exists a set of n points $X \subseteq \mathbb{R}^d_{\geq 0}$ in d dimensions such that $P \subseteq P(X)$ (according to some labeling of points in X with $[n]$), and NO otherwise.

This is equivalent to asking whether there exist vectors