Condorcet Domains

Clay Thomas
claytont@cs.princeton.edu

Corey Sinnamon
sinnamon@cs.princeton.edu

June 9, 2019

1 Introduction

Arrow's impossibility theorem says voting doesn't work out like you want it to. Condorcet domains work around this.

Theorem 1. *Let $\mathcal{C} \subseteq S_n$ be some set of linear orders. The following are equivalent:*

1. *Simple majority voting over any profile of voters is always acyclic.*
2. *Simple majority voting over any odd-size profile of voters always yields a total order.*
3. *Every triple of outcomes $i, j, k \in [n]$ is* acyclic, *meaning that $\mathcal{C}$ restricted to $\{i, j, k\}$ does not contain the orders ijk, jki, and kij.*
4. *Among ever triple $i, j, k \in [n]$ of distinct outcomes in $[n]$, one of them is either never first, never last, or never in the middle.*

Definition 2. *A Condorcet domain $\mathcal{C} \subseteq S_n$ is any set of orders satisfying any of the above conditions.*

2 Preliminaries

If $\mathcal{C}$ is a Condorcet domain, theorem 1 part 4 tells us that every triple of outcomes falls into some "value restriction case": one of i, j, or k is either never first, never second, or never third among $\{i, j, k\}$. In those cases, if u is the "restricted" outcome among $\{i, j, k\}$, we write uN_1, uN_2, or uN_3, respectively. If the triple $\{i, j, k\}$ is not clear from the context, we will write uN_ivw for $\{v, w\} = \{i, j, k\} \setminus \{u\}$.

Definition 3. *A* value restriction (VR) casting *on $[n]$ is a function ψ from triples of outcomes $(i, j, k) \in \binom{[n]}{3}$ to one of the nine "value restriction cases" uN_v for $u \in \{i, j, k\}$ and $v \in \{1, 2, 3\}$.*

The corresponding domain $\mathcal{D}(\psi)$ of some value restriction is the collection of all orders $\sigma \in S_n$ which satisfy ψ on every triple. That is, $\sigma \in \mathcal{D}(\psi)$ if for all $i, j, k \in [n]$ with $\psi(i, j, k) = uN_v$, $\sigma|_{i,j,k}$ does not have u in position v.

CDs, even closed CDs, are not in general defined by the VR castings they satisfy[1]. However, maximal CDs are always the corresponding domain of a VR casting:

[1] For example, single crossing domains $\mathcal{C}$ are closed but in general much smaller than $\mathcal{D}(\psi(\mathcal{C}))$.

Proposition 4. *If $\mathcal{C}$ is a maximal CD, then $\mathcal{C} = \mathcal{D}(\psi)$ for some ψ.*

Proof. By theorem 1 part 4, every CD is contained in $\mathcal{D}(\psi)$ for some ψ. If $\mathcal{C} \subsetneq \mathcal{D}(\psi)$, then $\mathcal{C}$ is not maximal. $\square$

Definition 5. *We distinguish the two natural orders $\alpha = 12\ldots n$ and $\omega = n(n-1)\ldots 1$.*

The reversed order of any $\sigma \in S_n$ is given by $\operatorname{rev}(\sigma) = \sigma(n)\sigma(n-1)\ldots\sigma(1)$

A domain $\mathcal{C}$ is normal *if $\alpha \in \mathcal{C}$ and $\omega \in \mathcal{C}$.*

3 Peak/Pit Domains

VR castings are especially useful when studying restricted classes of Condorcet domains. In special cases, some simple observations about which VR cases are possible allow us to reduce the complexity of the set of VR castings considerably.

4 Bad News – Counterexamples

The property of maximality gives a lot of structure to Condorcet domains. However, this structure is complicated.

Example 6. *There are maximal Condorcet domains without full projection onto some triple. Moreover, maximality is not preserved under restrictions.*