Voting With d-Dimensional Preferences

Clay Thomas
claytont@princeton.edu

Yufei Zheng
yufei@cs.princeton.edu

Social choice theory is the field of study concerned with making collective decisions. A typical formulation is to study "voting schemes", i.e. functions f which take as input a list of *linear preferences* on $[n]$ (that is, total orders on $[n]$) and outputs a "group decision" $f(>_1, \ldots, >_m) \in [n]$. When $n = 2$, majority rule constitutes an excellent choice of f. Unfortunately when $n > 2$, social choice functions are plagued by impossibility results, such as Arrow's Impossibility Theorem (every "unanimous" voting scheme satisfying "independence of irrelevant alternatives" is a "dictatorship") and the Gibbard-Satterthwaite Theorem (every surjective "strategy-proof" voting scheme is a dictatorship).

$$1 > 2 > 3$$
$$2 > 3 > 1$$
$$3 > 1 > 2$$

Figure 1: Cycle

$$1 > 2 > 3$$
$$1 > 3 > 2$$
$$3 > 2 > 1$$
$$2 > 3 > 1$$

Figure 2: Sandwich

$$1 > 2 > 3$$
$$1 > 3 > 2$$
$$3 > 2 > 1$$

Figure 3: LeastFavorite

$$1 > 2 > 3 > 4$$
$$1 > 2 > 4 > 3$$
$$2 > 1 > 3 > 4$$
$$2 > 1 > 4 > 3$$

Figure 4: FlipFlop

$$1 > 2 > 3 > 4$$
$$4 > 3 > 2 > 1$$

Figure 5: Reverse

To get around the issues with general voting schemes, theorists consider *restricted classes of preferences*, for example *single peaked preferences*. Consider a set $P = \{>_i\}_i$ of linear preferences on $[n]$. We say P is single peaked when there exists a set $Y = \{y_1, \ldots, y_n\} \subseteq [0, 1]$ of distinct numbers such that, for each $>_i \in P$, there exists a "peak" $p_i \in [0, 1]$ where: if $p_i < y_k < y_\ell$, then $k >_i \ell$, and if $y_k < y_\ell < p_i$, then $k <_i \ell$. This corresponds to voters having a "favorite point" (i.e. their peak), and their preference for an alternative decreases as you move away from the peak. Single peaked preferences are dramatically better behaved than general preferences. If P is single peaked, then there exist voting schemes $f : P^m \to [n]$ which are "strategy-proof", "Pareto-optimal" (a stronger version of unanimity), and "anonymous" (a stronger version of not being a dictatorship). This voting scheme essentially takes the medians of the "peaks".

We say P is *d-dimensional* if there exists a set $X = \{x_1, \ldots, x_n\} \subseteq \mathbb{R}^d_{\geq 0}$ such that, for each $>_i \in P$, there exists a "preference weight" a_i such that $k >_i \ell$ if and only if $\langle a_i, x_k \rangle > \langle a_i, x_\ell \rangle$. This corresponds to voters choosing options based on a simple weighted sum of the attributes of these options. **The goal of our project is to study voting schemes for d-dimensional preferences**.

When $d = 1$, the 1-dimensional preference sets are trivial, i.e. they contain only one preference. The class of single peaked preferences and 2-dimensional preferences seem vaguely related. For example, both preference sets Cycle (a common counterexample for certain voting schemes having nice properties) and Sandwich, shown to the right, are neither single peaked nor 2-dimensional. However, we note that neither class is a subset of the other: for example, LeastFavorite is 2-dimensional but not single peaked, and FlipFlop is single peaked but not two dimensional. Finally, Reverse gives a nontrivial preference set which is both single peaked and 2-dimensional. The proofs that certain preference sets fail to be d-dimensional are interesting and novel, and we regard the combinatorial classification of d-dimensional preference sets to be of independent interest.

We conjecture good voting schemes are possible 2-dimensional preferences. For $d \geq 3$, the situation appears less nice. Any set of preferences on 3 options is 3-dimensional, and 3 options is already sufficient to prove negative results such as Arrow's Theorem. However, there still may be interesting things to say in this case. For all values of d, the rich space of questions relating to voting schemes will almost certainly lead to good avenues of investigation, if not especially impressive results [**?**].