Dim Prefs

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1 Lemmas

In this section, we set up the tools needed to reason about d-dimensional preferences.

1.1 Lemmas based on the structure of X

These first few lemmas relate geometric properties of X to limitations on the structure of $>_a$ for a specific, fixed a.

The most simple way X gives structure to preferences is if one outcome is better in all attributes.

Definition. *Let $x, y \in X \subseteq \mathbb{R}^d_{\geq 0}$. We say x* dominates *y, denoted $x \gg y$, if $x[k] > y[k]$ for each $k = 1, \ldots, d$.*

Proposition 1. *If $x \gg y$, then $x >_a y$ for any nonzero $a \in \mathbb{R}^d_{\geq 0}$.*

When comparing different outcomes, a useful tool is the familiar geometric notion of a convex hull. Intuitively, if a preference weight does not like any of a set of options, it will not like any outcome in the hull of those options either. Thus, a point "dominated by the hull" of a set of options cannot be preferred to all those options.

Definition. *For points $x_1, \ldots, x_n \in \mathbb{R}^d$, let $\mathrm{hull}(x_1, \ldots, x_n) = \{u_1 x_1 + \ldots + u_n x_n | 0 \leq u_i \leq 1, \sum_{i=1}^n u_i = 1\}$ denote the convex hull of $x_1, \ldots, x_n$.*

Lemma 2. *Let $z, x_1, \ldots, x_k \in \mathbb{R}^d_{\geq 0}$ and $a \in \mathbb{R}^d_{\geq 0} \setminus \{0\}$. If $z >_a x_i$ for $i = 1, \ldots, k$, then $z >_a w$ for any $w \in \mathrm{hull}(x_1, \ldots, x_k)$.*

Proof. We have $\langle a, z \rangle > \langle a, x_i \rangle$ for each $i = 1, \ldots, k$. If $w = u_1 x_1 + \ldots + u_n x_n$ and $\sum_i u_i = 1$, then $\langle a, w \rangle = u_1 \langle a, x_1 \rangle + \ldots + u_n \langle a, x_n \rangle < u_1 \langle a, z \rangle + \ldots + u_n \langle a, z \rangle = \langle a, z \rangle$. □

Proposition 3. *Let $U, V \subseteq \mathbb{R}^d_{\geq 0}$ be finite sets of points. Suppose that there exists $u \in \mathrm{hull}\, U$ and $v \in \mathrm{hull}\, V$ such that $u \ll v$. Then no a satisfies $u_i >_a v_i$ for each $i = 1, \ldots, k$.*

Proof. For contradiction, suppose such an a exists. First, note $u_i >_a v \in \mathrm{hull}\, V$, then see that $\mathrm{hull}\, U \ni u >_a v$ as well. But $u \ll v$, so this is a contradiction. □

The converse of the last theorem also holds:

Proposition 4. *Suppose that no $a \in \mathbb{R}^d_{>0}$ satisfies $u_i \geq_a v_j$ for each $u_i \in U$ and $v_j \in V$. Then there exist $u \in \mathrm{hull}(U)$ and $v \in \mathrm{hull}(V)$ with $u \ll v$. NOTE: SHOULD USE THE "SEMISTRICT" DEFINITION OF ¡¡*

Proof. If no a has this property, then in particular the following linear program is infeasible:

$$\begin{aligned}
\text{min:} \quad & \sum_{k=0}^{d} 0 \cdot a_k \\
\text{s.t.} \quad & \sum_{k=1}^{d} a_k (u_k^i - v_k^j) \geq 0 \quad \forall i \in U, j \in V \\
& a_k \geq 1 \quad \forall k
\end{aligned}$$

The dual of this linear program is

$$\begin{aligned}
\text{max:} \quad & \sum_{i \in U, j \in V} 0 \cdot b_{ij} + 1 \cdot c_k \\
\text{s.t.} \quad & \sum_{i \in U, j \in V} b_{ij}(u_k^i - v_k^j) + c_k \leq 0 \quad \forall k \\
& b_{ij} \geq 0 \quad \forall i, j \\
& c_k \geq 0 \quad \forall i, j
\end{aligned}$$

This dual program is always feasible (with the all zeros solution) so by strong duality it must be unbounded. Take some solution and some k with $c_k > 1$. At this solution, b_{ij} are not all zero. Thus, take $\sum_{i,j} b_{ij} = B$, and consider:

$$\sum_{i \in U} \left(\sum_{j \in V} b_{ij}/B \right) u_k^i \leq -\frac{c_k}{B} + \sum_{j \in V} \left(\sum_{i \in U} b_{ij}/B \right) v_k^j < \sum_{j \in V} \left(\sum_{i \in U} b_{ij}/B \right) v_k^j$$

NOTE: THIS CANNOT BE CORRECT BECAUSE IT IMPLIES EG THAT ALL POINTS ARE MORE THAN 1 APART ∎

This motivates the following definition:

Definition. *A set of points dominates, denoted $U \ll V$, when DEFINITION*