# Voting with 2-dimensional preferences

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**Abstract**

The study of voting rules often restricts attention to well-behaved classes of possible preferences of voters. We define a new class: 2-dimensional preferences, for which very good voting rules are possible. We argue that 2-dimensional preferences are in some ways more natural and expressive than more traditional classes such as single-peaked preferences. Furthermore, we give an almost-complete combinatorial classification of 2-dimensional preferences, and provide some additional results about the natural extension of $d$-dimensional preferences.

# 1 Introduction

# Part I
# Voting

# 2 Definitions

## 2.1 Voting

Consider a set of preferences $P$ (i.e. linear orders) on a set of outcomes $M$. We follow the convention that $m$ is the number of outcomes and $n$ is the number of voters. We have the following definitions:

- A *social welfare function on* $P$ is a function $F : P^n \to P$.
- A *social choice function on* $P$ is a function $f : P^n \to M$.
- A welfare function $F$ is *unanimous* if, for any $\succ \in P$, we have $F(\succ, \ldots, \succ) = \succ$.
- A choice function $f$ is *unanimous* if, whenever there exists a fixed $a$ with $a \succ_i b$ for all $i = 1, \ldots, n$ and $b \in M \setminus \{a\}$, we have $f(\succ_1, \ldots, \succ_n) = a$. I'm not sure if this is a standard notion.
- A welfare function $F$ is a *dictatorship* if there exists an $i$ such that $F(\succ_1, \ldots, \succ_n) = \succ_i$.
- A welfare function $F$ satisfies *in dependence of irrelevant alternatives* if, whenever $a \succ_i b \iff a \succ_i' b$ and $\succ = F(\succ_1, \ldots, \succ_n), \succ' = F(\succ_1', \ldots, \succ_n')$, we get $a \succ b \iff a \succ' b$

- A choice function $f$ is *incentive compatible* if, for any $\succ_1, \ldots, \succ_n, i, \succ_i'$, we have $f(\succ_1, \ldots, \succ_n) \succ_i f(\succ_1, \ldots, \succ_i' \ldots, \succ_n)$

Let $R(M)$ denote the set of all linear orders on $M$. Recall that when $P = R(M)$, there are many known impossibility results, the two most famous of which are:

((Arrow's))

((G-S))

### 2.2 Single-peaked preferences

Our main point of contrast will be the well-understood class of *single peaked* preferences.

Let $S \subseteq [0, 1]$ be a finite set of $m$ points in the unit interval. We call $S$ the set of *outcomes*. Let $R(S)$ denote the set of linear orders on $X$. A preference $\succ \in R(S)$ is called *single peaked* if there exists an outcome $p \in S$ (called the "peak" of $\succ$) such that $x < y < p \implies x \prec y$ and $p < y < x \implies x \prec y$. In other words, the preference has a favorite outcome, and its opinion strictly decreases as you move farther away from the favorite. Note that no assumption is made about outcomes on different sides of the peak. Define

$$P_{sp}(S) = \{\succ \in R(S) | \succ \text{ is single peaked } \}$$

### 2.3 $d$-dimensional preferences

Let $\{x_1, \ldots, x_m\} = X \subseteq \mathbb{R}^d_{\geq 0}$ be any (ordered) set of $m$ distinct points with nonnegative coordinates. We call $X$ the set of *outcomes*. Given any $a \in \mathbb{R}^d_{\geq 0}$, define a order $>_a$ on $X$ as follows: $x_i >_a x_j$ if and only if $\langle a, x_i \rangle > \langle a, x_j \rangle$. Define

$$P_d(X) = \{\succ \in R(X) | \exists a \in \mathbb{R}^d_{\geq 0} : x \succ y \iff x >_a y\}$$

## 3 Single-peaked verses 2-dimensional preferences

When there are exactly three candidates, 2-dimensional preferences have strictly more expressive power than single-peaked preferences.

**Proposition 1.** *Every single-peaked preference set on $m = 3$ outcomes is a 2-dimensional preference set. In particular, up to relabeling it is a subset of $P_2(X)$, for $X = \{(3, 0), (2, 2), (0, 3)\}$.*

*Proof.* Without loss of generality, relabel the outcomes of the single peaked preferences as $S = \{1, 2, 3\}$. The resulting preferences are given by GoodCompromise. ((INSERT PICTURE OF GOOD COMPROMISE AS A 2D SET)) □

Furthermore, LeastFavorite gives an example of a preference set which is 2-dimensional, but not single-peaked:

**Proposition 2.** *In a single peaked set of preferences, it is not possible for every outcome to be the lowest ranked outcome of some preference.*

*Proof.* The median outcome (via the standard order on $[0, 1]$) cannot be the lowest ranked. □

When $m > 3$, neither single-peaked nor 2-dimensional preferences are a subset of the other. One was to see this to to simply count the number of single-peaked preferences, and see that it grows much faster with $m$ than 2-dimensional preferences do (we'll see later, the maximal size of a 2-dimensional preference set is $O(m^2)$).

**Proposition 3.** *The number of single-peaked preferences for any set $S$ of $m$ outcomes is at least $2^{\Omega(m)}$*

*Proof.* Choose the median outcome of $S$ to be the peak. Given a subset $T \subseteq [m-1]$ of size $|T| = \lfloor m/2 \rfloor$, we can associate a unique single-peaked preference as follows: treat the preference as an array of outcomes, ranked highest to lowest. Put the median of $S$ at index 0. Let the outcomes to the left of the median occupy the indices corresponding to set $T$, and let those to the right occupy the other indices. There are $\binom{m-1}{\lfloor m/2 \rfloor} \geq 2^{\Omega(m)}$ such subsets $T$. □

A cleaner, more constructive way to see this is the following proposition

**Proposition 4.** *The preference set* FLIPFLOP *is single-peaked, but not 2-dimensional.*

*Proof.* To obtain a single-peaked representation, order the outcomes from left to right as follows: $3, 1, 2, 4$. All preferences will then be possible with peak 1 or 2.

((INSERT IMPOSSIBILITY HALF)) □

((DISCUSS SETS WHICH ARE NEITHER SINGLE-PEAKED NOR 2-D))

((ARGUE THAT 2-D IS MORE NATURAL THAN SINGLE-PEAKED, EVEN THOUGH THERE ARE (ASYMPTOTICALLY) MORE SINGLE-PEAKED))

# 4 Voting for 2-dimensional preferences

# 5 2-dim entails a Condorcet winner

# Part II
# Combinatorics

# 6 Bounding the number of 2-dimensional preferences

$1 > 2 > 3$
$2 > 3 > 1$
$3 > 1 > 2$

Cycle

$1 > 2 > 3$
$1 > 3 > 2$
$3 > 2 > 1$
$2 > 3 > 1$

Sandwich

$1 > 2 > 3$
$1 > 3 > 2$
$3 > 2 > 1$

LeastFavorite

$1 > 2 > 3 > 4$
$1 > 2 > 4 > 3$
$2 > 1 > 3 > 4$
$2 > 1 > 4 > 3$

FlipFlop

$1 > 2 > 3 > 4$
$4 > 3 > 2 > 1$

Reverse

$1 > 2 > 3$
$2 > 1 > 3$
$2 > 3 > 1$
$3 > 2 > 1$

GoodCompromise