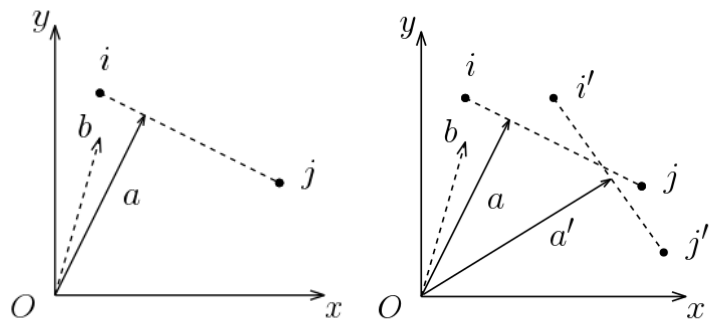


(a) Claim (b) Lemma

Figure 1: Illustrations

**Claim 1.** *For any pair of points $i = (i_1, i_2), j = (j_1, j_2) \in X$, where neither dominates the other, let $a = (a_1, a_2) \in \mathbb{R}^2_{\geq 0}$ be the preference weight vector that gives $i =_a j$, then for any preference weight vector $b = (b_1, b_2) \in \mathbb{R}^2_{\geq 0}$,*

1. $\frac{a \times b}{\|a \times b\|} > 0$ *if and only if* $i >_b j$.
2. $\frac{a \times b}{\|a \times b\|} < 0$ *if and only if* $i <_b j$.

*Proof.* 1 and 2 are completely analogous, here we only prove 1.

- " $\Longrightarrow$ ": On the one hand, from $\frac{a \times b}{\|a \times b\|} > 0$ we have

$$a_1 b_2 - a_2 b_1 < 0 \implies a_1 < \frac{b_1}{b_2} a_2. \tag{1}$$

  On the other hand, by $i =_a j$,

$$a_1 i_1 + a_2 i_2 = a_1 j_1 + a_2 j_2 \implies a_1(i_1 - j_1) = a_2(j_2 - i_2). \tag{2}$$

  Plug (1) into (2), $a_2(j_2 - i_2) < \frac{b_1}{b_2} a_2(i_1 - j_1)$, this gives

$$b_1 i_1 + b_2 i_2 > b_1 j_1 + b_2 j_2 \implies i >_b j.$$

- " $\Longleftarrow$ ": From (2) we get

$$\frac{j_2 - i_2}{i_1 - j_1} = \frac{a_1}{a_2}. \tag{3}$$

  By $i >_b j$ and (3),

$$b_1 i_1 + b_2 i_2 > b_1 j_1 + b_2 j_2 \implies \frac{b_1}{b_2} > \frac{j_2 - i_2}{i_1 - j_1} = \frac{a_1}{a_2} \implies a_1 b_2 - a_2 b_1 < 0 \implies \frac{a \times b}{\|a \times b\|} > 0.$$

$\square$

Using Claim 1 we are able to show the following lemma.

**Lemma 2.** *Let $i, j, i', j'$ be two pairs of points in $X$ satisfying*

- *neither of $i, j$ dominates the other: $i_1 < j_1$ and $i_2 > j_2$,*

- *neither of $i', j'$ dominates the other: $i'_1 < j'_1$ and $i'_2 > j'_2$,*
- *the slope of line $ij$ is less than the slope of line $i'j'$: $\frac{j_2-i_2}{j_1-i_1} < \frac{j'_2-i'_2}{j'_1-i'_1}$.*

*Then for any preference weight vector $b \in \mathbb{R}^2_{\geq 0}$,*

1. *If $i >_b j$, $i' >_b j'$.*
2. *If $j' >_b i'$, $j >_b i$.*

*Proof.* Again 1 and 2 are symmetric, we only prove 1. Let $a, a'$ be the preference weight vectors that gives $i =_a j$ and $i' =_{a'} j'$ respectively. By definition of $a'$,

$$a'_1 i'_1 + a'_2 i'_2 = a'_1 j'_1 + a'_2 j'_2 \implies \frac{a'_1}{a'_2} = \frac{j'_2 - i'_2}{i'_1 - j'_1}. \tag{4}$$

Since $i >_b j$,

$$b_1 i_1 + b_2 i_2 > b_1 j_1 + b_2 j_2 \implies \frac{b_1}{b_2} > \frac{j_2 - i_2}{i_1 - j_1}. \tag{5}$$

From the relation of slopes we have

$$\frac{j_2 - i_2}{i_1 - j_1} > \frac{j'_2 - i'_2}{i'_1 - j'_1}. \tag{6}$$

Combine (4)(5)(6) we get

$$\frac{b_1}{b_2} > \frac{j_2 - i_2}{i_1 - j_1} > \frac{j'_2 - i'_2}{i'_1 - j'_1} = \frac{a'_1}{a'_2} \implies a'_1 b_2 - a'_2 b_1 < 0 \implies \frac{a' \times b}{\|a' \times b\|} > 0.$$

By Proposition **??**, this implies $i' >_b j'$. □

Let $P_2$ be a set of 2-dimensional preferences on $[n]$.

**Theorem 3.** $|P_2| \leq \binom{n}{2} + 1$.

*Proof.* We prove this upper bound algorithmically. Consider the following algorithm of generating a preference in $P_2$ given $X$:

**Algorithm 1** Preference Generator for $P_2$

Relabel all points in $X$ such that $y_1 \geq y_2 \geq \cdots \geq y_n$
**for all** $i, j \in X, i < j$ **do**
  Compute slope $s_{ij}$
  **if** $s_{ij} < 0$ **then**
    $L \leftarrow s_{ij}$   ▷ Suppose that $L[1]$ is the first element in the list $L$
Sort $L$ in ascending order
Choose a pivot $p \in \{0, \ldots, |L|\}$
**if** $p \geq 1$ **then**
  Let $L[p] = s_{ij}$, set $i >_a j$
**if** $p + 1 \leq |L|$ **then**
  Let $L[p+1] = s_{i'j'}$, set $i' <_a j'$
Output $>_a$

Lemma 2 ensures that by setting $i <_a j$ and $i' >_a j'$ in steps 8-11, all pairs $k, l$ before $i, j$ in list $L$ are set to $k <_a l$, and all pairs $k', l'$ after $i', j'$ in $L$ are set to $k' <_a l'$. Moreover, there is a one-to-one correspondence between the actual preference, and choosing a pivot to flip from $<_a$ to $>_a$. Since $|L| \leq \binom{n}{2}$, and the number of pivots that we can choose from is $|L| + 1$, we get that $|P_2| \leq \binom{n}{2} + 1$.

The upper bound is tight. ADD the example here! $\square$

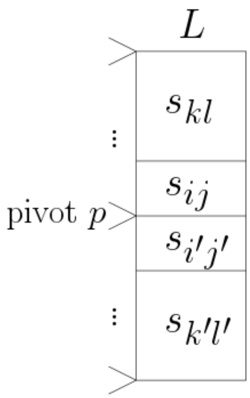


Figure 2: List $L$