Combinatorics and Voting With Dimensional Preferences

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Abstract

((TODO: REWRITE)) The study of voting rules often restricts attention to well-behaved classes of possible preferences of voters. We define a new class: 2-dimensional preferences, for which very good voting rules are possible. We argue that 2-dimensional preferences are in some ways more natural and expressive than more traditional classes such as single-peaked preferences. Furthermore, we give an almost-complete combinatorial classification of 2-dimensional preferences, and provide some additional results about the natural extension of d-dimensional preferences.

1 Introduction

In this project, we study *ordinal preferences*, i.e. preferences which prefer certain outcomes over others, but do not have a quantitative notion of the quality of the outcome. Formally, these are simply total orders. As computer scientists, our first instinct is to make no assumptions, e.g. to assume preferences can be any of the $m!$ distinct total orders on m outcomes. However, this typically does not correspond to reality: the preferences of an agent are typically *caused by something*, i.e. there is an underlying structure explaining the preferences. This is one way to model the idea the preferences are *correlated*.

Consider the following motivating example. Suppose there are m school, and each has two attributes which we assume are objective across all preferences: quality of STEM education and quality of liberal arts education. We can model each school as a point $x \in \mathbb{R}^2_{\geq 0}$, where the x coordinate gives STEM quality and y coordinate gives liberal arts quality. Then it is possible that the school preferences of students actually boil down to a simple preference over STEM and liberal arts, which we can model as a *weighted sum* over the two attributes. For example, a student who prefers 30% STEM and 70% liberal arts would "score" a school $x = (x_1, x_2)$ via the sum $0.3x_1 + 0.7x_2$, and that student prefers school x over school y if $0.3x_1 + 0.7x_2 > 0.3y_1 + 0.7y_2$.

Social choice theory, the study of *voting schemes*, has found much theoretical success in studying restricted classes of preferences. Restricting preferences is a way around various impossibility theorems of social choice theory, which state that certain "good" voting schemes do not exist in general. Given how natural and well-motivated d-dimensional preference seem, a natural question is whether good voting schemes are possible for d-dimensional preferences.

1.1 Outline and highlights of results

We formalize this notion for any number of attributes d, and call the resulting preference model d-dimensional preferences. We setup a framework for studying d-dimensional preferences, identifying the correct geometric objects to use when reasoning about them. These objects give interesting geometric explanations for concepts such as opinion and agreement. Using these tools, we are able to prove a general impossibility result, theorem 8, which shows a qualitative way in which preferences get more complicated as d increases.

When $d = 2$, we achieve a good handle on the combinatorial structure of 2-dimensional preferences, including a tight upper bound on the maximal number of distinct linear preferences corresponding 2-dimensional preferences (theorem 7). We find that $d = 2$ represents a very nice point in the trade-off between expressibility and complexity, with enough preferences to be realistic while still having strong structure.

Next, we turn to study the social choice theory of 2-dimensional preferences. We compare and contrast 2-dimensional preferences with *single-peaked preferences*, the classic restricted class used in social choice theory. Each model different phenomenon, but we argue that 2-dimensional preferences are more expressive in some ways, and certainly more natural in some cases. We find that many positive results which holds for single-peaked preferences hold for 2-dimensional preferences.

Part I
The Combinatorics of d Dimensional Preferences

2 Definition of d-dimensional preferences

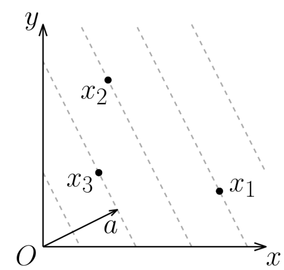


Figure 1: The "level sets" of the preference $>_a$, which ranks outcomes according to distance from the origin along direction a.

Let $\{x_1, \ldots, x_m\} = X \subseteq \mathbb{R}^d_{\geq 0}$ be any set of m distinct points with nonnegative coordinates. We call X the set of *outcomes*. Given any $a \in \mathbb{R}^d_{\geq 0}$, which we call a *preference weight*, define a order $>_a$ on X as follows: $x_i >_a x_j$ if and only if $\langle a, x_i \rangle > \langle a, x_j \rangle$. Let $R(X)$ denote the set of total orders on X. Define

$$P_d(X) = \{\succ \in R(X) | \exists a \in \mathbb{R}^d_{\geq 0} : x \succ y \iff x >_a y\}$$

Note that we consider only total orders, in particular, ties are not allowed.

Some first observations:

- If $d = 1$, then $|P(X)| = 1$, i.e. preferences are completely determined by the underlying set X.
- If $d = n$, then every linear preference on X can occur in $P(X)$.

  Proof. Let $X = \{e_i\}$ simply be the standard basis vectors. To induce an ordering $i_1, i_2, \ldots, i_n$, just create a preference vector a which gives weight $1/k$ to coordinate i_k. □

The above hints that there should be some sort of continuum between $d = 1$ and $d = n$ of how complex a preference set $P_d(X)$ can be. We find that $d = 2$ is a particular sweet spot between expressibility and simplicity.

3 Lemmas

In this section, we set up the tools needed to reason about d-dimensional preferences.

3.1 Lemmas based on the structure of X

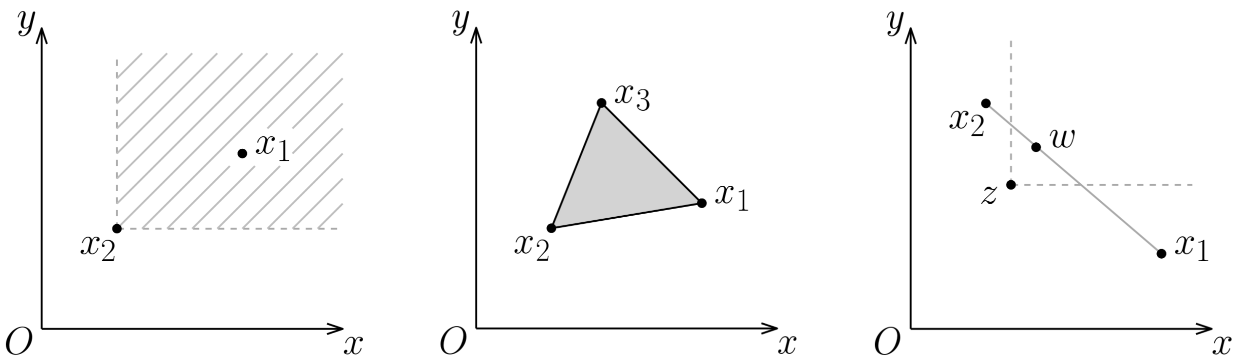
These first few lemmas relate geometric properties of X to limitations on the structure of $>_a$ for a specific, fixed a.

The most simple way X gives structure to preferences is if one outcome is better in all attributes.

Definition. *Let $x, y \in X \subseteq \mathbb{R}^d_{\geq 0}$. We say x* dominates *y, denoted $x \gg y$, if $x[k] > y[k]$ for each $k = 1, \ldots, d$.*

Proposition 1. *If $x \gg y$, then $x >_a y$ for any nonzero $a \in \mathbb{R}^d_{\geq 0}$.*

Proof. Simply observe $a_i x_i \geq a_i y_i$ for each $i = 1, \ldots, d$. Because $a \neq 0$, there is also some i where $a_i x_i > a_i y_i$. □



(a) x_1 dominates x_2 ($x_1 \gg x_2$) (b) The convex hull (c) Proposition 3

Figure 2

When comparing different outcomes, a useful tool is the familiar geometric notion of a convex hull. Intuitively, if a preference weight does not like any of a set of options, it will not like any outcome in the hull of those options either. Thus, a point "dominated by the hull" of a set of options (as in figure 2c) cannot be preferred to all those options.

Definition. *For points $x_1, \ldots, x_n \in \mathbb{R}^d$, let* $\mathrm{hull}(x_1, \ldots, x_n) = \{u_1 x_1 + \ldots + u_n x_n | 0 \leq u_i \leq 1, \sum_{i=1}^{n} u_i = 1\}$ *denote the convex hull of $x_1, \ldots, x_n$.*

Lemma 2. *Let $z, x_1, \ldots, x_k \in \mathbb{R}^d_{\geq 0}$ and $a \in \mathbb{R}^d_{\geq 0} \setminus \{0\}$. If $z >_a x_i$ for $i = 1, \ldots, k$, then $z >_a w$ for any $w \in \mathrm{hull}(x_1, \ldots, x_k)$.*

Proof. We have $\langle a, z \rangle > \langle a, x_i \rangle$ for each $i = 1, \ldots, k$. If $w = u_1 x_1 + \ldots + u_n x_n$ and $\sum_i u_i = 1$, then $\langle a, w \rangle = u_1 \langle a, x_1 \rangle + \ldots + u_n \langle a, x_n \rangle < u_1 \langle a, z \rangle + \ldots + u_n \langle a, z \rangle = \langle a, z \rangle$. □

Proposition 3. *Let $z, x_1, \ldots, x_k \in \mathbb{R}^d_{\geq 0}$. Suppose that there exists $w \in \mathrm{hull}(x_1, \ldots, x_k)$ such that $w \gg z$. Then no a satisfies $z >_a x_i$ for each $i = 1, \ldots, k$.*

Proof. For contradiction, suppose such an a exists. Then $z >_a w$ as well. However, because $w \gg z$, this is a contradiction. □

Note that lemma 2 and proposition 3 both hold when you reverse all the inequalities in their statements, via the same arguments.

3.2 Lemma relating different preference vectors

When comparing different preference weights, a useful tool is the familiar geometric notion of a convex cone. Intuitively, if a set of weights agree about a certain preference, so does every weight in their cone.

Definition. *For any vectors $a_1, \ldots, a_k \in \mathbb{R}^d_{\geq 0}$, define*

$$\mathrm{cone}(a_1, \ldots, a_k) = \{u_1 a_1 + \ldots + u_k a_k | u_i \geq 0 \forall j\}$$

Proposition 4. *Suppose that for preference weights $a_1, \ldots, a_k$, we have $x >_{a_1} y, \ldots, x >_{a_k} y$. Then for any nonzero $b \in \mathrm{cone}(a_1, \ldots, a_k)$, $x >_b y$ as well.*

Proof. Let $b = u_1 a_1 + \ldots + u_k a_k$. We get $\langle b, x \rangle = u_1 \langle a_1, x \rangle + \ldots + u_k \langle a_k, x \rangle > u_1 \langle a_1, y \rangle + \ldots + u_k \langle a_k, y \rangle = \langle b, y \rangle$, as desired. □

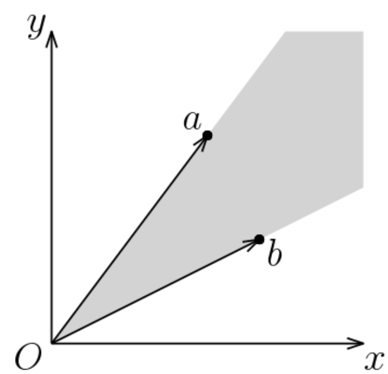


Figure 3: The convex cone

3.3 Lemmas for 2 dimensional preferences

This section turns to our main focus: 2 dimensional preferences. Given two points x, y in our set of outcomes X, we give a simply classification of which preference weight prefer x and which prefer y. To do so, we introduce the notion of a weight "indifferent between x and y".

Lemma 5. *Let $x, y \in \mathbb{R}^2_{\geq 0}$ be distinct points where neither x nor y dominates the other. There exists a unique unit vector $b \in \mathbb{R}^2_{\geq 0}$ such that $\langle b, x\rangle = \langle b, y\rangle$.*

Proof. Without loss of generality let $x_1 < y_1$, $x_2 > y_2$ (if either of these are equalities, the standard basis vectors e_1 or e_2 will do). There exists exactly one unit vector b in $\mathbb{R}^2_{\geq 0}$ such that $b_1/b_2 = (x_2 - y_2)/(y_1 - x_1)$. It's easy to check that this is a necessary and sufficient condition for having $\langle b, x\rangle = \langle b, y\rangle$. □

Definition. *For $x, y \in \mathbb{R}^2_{\geq 0}$ where neither dominates the other, let the "indifferent vector of x and y", denoted* $\mathrm{indif}(x, y)$*, be the unit vector $b \in \mathbb{R}^2_{\geq 0}$ such that $\langle b, x\rangle = \langle b, y\rangle$.*

This next lemma says that the indifference vector of x and y divides the space of preferences into two halves: one that prefers x, one that prefers y. This division is given by another very natural ordering, this one for preference weights. The following definition has an obvious interpretation by considering $\theta(a)$ to be the angle of a from the positive x axis, but we use an equivalent algebraic definition to simply our proofs.

Definition. *We say a has a larger angle than b, denoted $\theta(a) > \theta(b)$, if $a_2/a_1 > b_2/b_1$.*

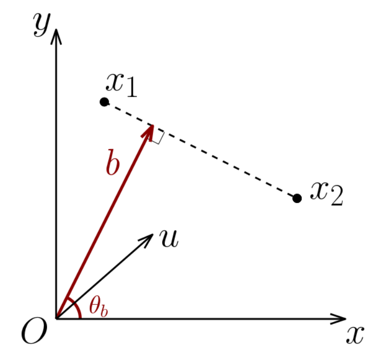


Figure 4: Proposition 6. $b = \mathrm{indif}(x, y)$. Preferences above b (those with higher angle) all prefer x, and those below b (those with lower angle) all prefer y

Proposition 6. *Consider any pair of points $x, y \in \mathbb{R}^2_{\geq 0}$, where $x_1 < y_1$ and $x_2 > y_2$, and let $b =$ $\mathrm{indif}(x, y)$. Then for any preference weight vector $a \in \mathbb{R}^2_{\geq 0}$,*

1. *$\theta(a) > \theta(b)$ if and only if $x >_a y$.*
2. *$\theta(a) < \theta(b)$ if and only if $y >_a x$.*

Proof. Except for the edge case where one vector is a multiple of the other, the two items are exactly the same statement. We prove the first one:

$$\begin{aligned} x >_a y &\iff a_1x_1 + a_2x_2 > a_1y_1 + a_2y_2 \\ &\iff \frac{a_2}{a_1} > \frac{y_1 - x_1}{x_2 - y_2} = \frac{b_2}{b_1} \\ &\iff 0 > a_1b_2 - a_2b_1 \\ &\iff \theta(a) > \theta(b) \end{aligned}$$

□

4 Bounding the number of 2-dimensional preferences

We are able to obtain a tight upper bound on the number of distinct linear preferences which arise from a given two-dimensional outcome set X. In particular, note that $|P_2(X)|$ is much less than $m!$, the total number of linear orders on X.

Theorem 7. *Let $X \subseteq \mathbb{R}^2_{\geq 0}$ with $|X| = m$. We have*

$$|P_2(X)| \leq \binom{m}{2} + 1$$

Proof. Consider the following set of preference weights:

$$B = \{\text{indif}(x, y) | x, y \in X \text{ and neither dominates the other}\}$$

Consider the different regions of $\mathbb{R}^2_{\geq 0}$ separated by elements of B. That is, consider the equivalence classes of the relation $\sim$ on $\mathbb{R}^2_{\geq 0} \setminus B$, where $a \sim a'$ means that $\forall b \in B : \theta(a) < \theta(b) \iff \theta(a') < \theta(b)$. We claim that if $a \sim a'$, then $x >_a y \iff x >_{a'} y$ for all $x, y \in X$. Proof: If $x \gg y$ or $y \gg x$, the order between x and y fixed for any a. Otherwise, $\text{indif}(x, y) \in B$, so by proposition 6, this means $x >_a y \iff x >_{a'} y$.

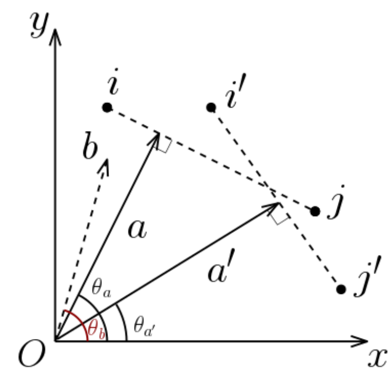


Figure 5: Theorem 7. Indifference vectors b and b' divide preference weight into three regions, which each have the same preferences for x vs. y and x' vs. y'.

Thus, the mapping from $\mathbb{R}^2_{\geq 0} \to R(X)$ given by $a \mapsto (>_a)$ is constant on equivalence classes of $\sim$. This map is surjective by definition. Furthermore, a, a' in distinct equivalence classes of $\sim$ give distinct preferences $>_a, >_{a'} \in R(X)$, again by proposition 6, because a and a' are on a different side of some $b \in B$ under the θ order. Thus, there is a bijection between equivalence classes of $\sim$ and preferences in $P_2(X)$.

Note that $|D| \leq \binom{m}{2}$, so $\sim$ divides the $\mathbb{R}^2_{\geq 0}$ into at most $\binom{m}{2} + 1$ regions based on the θ order. Thus $|P_2(X)| \leq \binom{m}{2} + 1$, as desired. $\square$

Furthermore, this upper bound is tight. Notice that the proof of the upper bound gives a simple condition for the upper bound to be tight: no point of X should dominate another, and every pair of points $x, y \in X$ should give a distinct indifference vector $\text{indif}(x, y)$. To construct such an X with m outcomes, consider $X' = \{(i, m + 1 - i) | i = 1, \ldots, m\}$, and randomly perturb each point a bit such that all $\text{indif}(x, y)$ vectors are distinct[1].

[1] If you want to get really abstract and fancy, let $\alpha_1, \ldots, \alpha_n$ be real numbers which are algebraically independent over $\mathbb{Q}$, and such that $|\alpha_i| < 1/3$. Then the points $\{(i, m+1-i+\alpha_i)\}_{i=1}^m$ cannot possibly yield a repeated indifference vector, by algebraic considerations.

5 Impossibility Results

5.1 Large preference cycles are impossible in any dimension

The following preference set may be important in high dimension. We call it k-Cycle:

$$\begin{array}{ccccccccccccc}
1 & > & 2 & > & 3 & > & \ldots & > & k-2 & > & k-1 & > & k \\
2 & > & 3 & > & 4 & > & \ldots & > & k-1 & > & k & > & 1 \\
3 & > & 4 & > & 5 & > & \ldots & > & k & > & 1 & > & 2 \\
 & & & & & & \vdots & & & & & & \\
k-1 & > & k & > & 1 & > & \ldots & > & k-4 & > & k-3 & > & k-2 \\
k & > & 1 & > & 2 & > & \ldots & > & k-3 & > & k-2 & > & k-1
\end{array}$$

Theorem 8. $(d+1)$-Cycle *is not d-dimensional for any d.*

Proof. Suppose for contradiction there existed $a_1, \ldots, a_{d+1} \in \mathbb{R}^d_{\geq 0}$ such that $>_{a_i}$ yielded the ith line of $(d+1)$-Cycle (in particular, the line whose favorite point is i). The vectors $\{a_i\}$ cannot be linearly independent. Thus, there exists a linear combination of vectors $u_1 a_1 + \ldots + u_{d+1} a_{d+1} = 0$. Let $S = \{a_i | u_i > 0\}$ and let $T = \{a_i | u_i < 0\}$. Note that, because every coordinate of each a_i is nonnegative (and no a_i is zero) neither S nor T are empty. We have

$$\sum_{a_i \in S} u_i a_i = \sum_{a_i \in T} -u_i a_i$$

Denote this above vector by b, and note that $b \in \mathbb{R}^d_{\geq 0} \setminus \{0\}$ and $b \in \text{cone}(S) \cap \text{cone}(T)$.

We claim that b satisfies the following:

$$1 >_b 2 >_b 3 >_b \ldots >_b d >_b d+1 >_b 1$$

Proof: For each pair $i > i+1$, observe that the inequality is satisfied for all the vectors a_i except for one of them (namely, a_{i+1} which ranks $i+1$ highest). Thus, for either S or T, every vector a_i in the set has the opinion $i >_{a_i} i+1$. Thus, every vector in the cone of that set has the opinion $i > i+1$ as well, by proposition 4. In particular, $i >_b i+1$. Of course, the above argument works for the pair $d > 1$ as well.

Thus, transitivity gives us $1 >_b 1$, a contradiction. □

This is one of our few results which work for any dimension. We'll see that 3-Cycle being impossible for 2 dimensional preferences is a necessary condition for many of the nice properties of 2 dimensional preferences (especially theorem 19). Intuitively, it is possible that "cyclic disagreements" are a core issue for different applications with ordinal preferences (we consider voting here, but other cases include matching or resource allocation). Put another way, the imposibility of the $d+1$ cycle in d dimensions gives a very nicely parametrized example of preferences getting more complicated as dimension increases.

5.2 Combinatorial impossibilities for 2 dimensional preferences

Proposition 9. *Let $P = P_2(X)$ be any set of 2 dimensional preferences, where $|X| \geq 3$. It is not possible for every $x \in X$ to be both the favorite and least favorite outcome of some preverence in P.*

Proof. If any point dominates another, then the dominated point cannot be favorite. Thus we may assume that no point in X is dominated by any other point. Pick two points $x \neq y$ such that x has the largest first coordinate (i.e. x_1) among all of X, and y has the highest second coordinate (i.e. y_2). Consider the following figure:

```
y-------------
|''--.._    2|
|1      ''-._|
|------------x
```

Any remaining point $z \neq x, y$ must lie in region 1 or 2, as all other regions either violate the assumption that $x_1 > z_1$ and $y_2 > z_2$ or cause one point to dominate another. If z lies in region 1, it can never be the favorite, by proposition 3, because it is dominated by some point in $\text{hull}(x, y)$. If z lies in region 2, it can never be the least favorite, this time by the opposite but completely analogous version of proposition 3, because z dominates a point in $\text{hull}(x, y)$. For the corner case where $z \in \text{hull}(x, y)$, observe that z is *always* the middle option among x and y. $\square$

This proposition, combined with Theorem 8, gives a complete combinatorial classification of two dimensional preferences on exactly three outcomes. Consider all preferences on three outcomes, organized with the two copies of 3-Cycle in different columns.

$1 > 2 > 3$	$1 > 3 > 2$
$2 > 3 > 1$	$3 > 2 > 1$
$3 > 1 > 2$	$2 > 1 > 3$

In order to make our preference set 2 dimensional, we need to exclude one preference from each cycle. By relabeling, we can without loss of generality assume that the preference with $1 > 2 > 3$ is not in P, and consider removing a preference from the other cycle. If $1 > 3 > 2$ or $2 > 1 > 3$ are removed from P, then P will not violate proposition 9[2]. If we only remove $3 > 2 > 1$, then every outcome is some preference's favorite and some other preference's least favorite[3]. So this P is not two dimensional.

[2]We will see later in section 7 that these preference sets are 2 dimensional. Up to relabeling, they are BadCompromise and GoodCompromise, respectively.

[3]This gives FlipFlop from section 7.

Part II
Voting With 2 Dimensional Preferences

6 Definitions

6.1 Voting

Consider a set of preferences P (i.e. linear orders) on a set of outcomes M. We follow the convention that m is the number of outcomes and n is the number of voters. We have the following definitions:

- A *social welfare function on* P is a function $F : P^n \to P$
- A *social choice function on* P is a function $f : P^n \to M$
- A welfare function F is *unanimous* if, for any $\succ \in P$, we have $F(\succ, \ldots, \succ) = \succ$
- A welfare function F is a *dictatorship* if there exists an i such that $F(\succ_1, \ldots, \succ_n) = \succ_i$
- A choice function f is a *dictatorship* if there exists an i such that $f(\succ_1, \ldots, \succ_n) = a_i$, where a_i is the favorite outcome of $\succ_i$
- A welfare function F satisfies *independence of irrelevant alternatives* (IIA) if, whenever $a \succ_i b \iff a \succ_i' b$ and $\succ = F(\succ_1, \ldots, \succ_n), \succ' = F(\succ_1', \ldots, \succ_n')$, we get $a \succ b \iff a \succ' b$
- A choice function f is *incentive compatible* if, for any $\succ_1, \ldots, \succ_n, i, \succ_i'$, we have $f(\succ_1, \ldots, \succ_i, \ldots, \succ_n) \succeq_i f(\succ_1, \ldots, \succ_i', \ldots, \succ_n)$
- A collection of preferences $\succ_1, \ldots, \succ_n$, has a Condorcet winner $x \in M$ if, for any other $y \in M$, we have $x \succ_i y$ for more than half of the indices $i = 1, \ldots, n$

Let $R(M)$ denote the set of all linear orders on M. Recall that when $P = R(M)$, there are many known impossibility results, the two most famous of which are:

Theorem 10 (Arrow's Impossibility Theorem [Nis07]). *If $|M| \geq 3$, then every unanimous social welfare function on M which satisfies independence of irrelevant alternatives is a dictatorship.*

Theorem 11 (Gibbard Satterthwaite [SV07]). *If $|M| \geq 3$, then every surjective, incentive compatible social choice function on M is a dictatorship.*

6.2 Single-peaked preferences

Our main point of contrast will be the well-understood class of *single peaked* preferences.

Let $S \subseteq [0, 1]$ be a finite set of m points in the unit interval. We call S the set of *outcomes*. Let $R(S)$ denote the set of linear orders on S. A preference $\succ \in R(S)$ is called *single peaked* if there exists an outcome $p \in S$ (called the "peak" of $\succ$) such that $x < y < p \implies x \prec y$ and $p < y < x \implies x \prec y$. In other words, the preference has a favorite outcome, and its opinion strictly decreases as you move farther away from the favorite. Note that no assumption is made about outcomes on different sides of the peak. Define

$$P_{sp}(S) = \{\succ \in R(S) | \succ \text{ is single peaked } \}$$

7 Single-peaked verses 2-dimensional preferences

When there are exactly three candidates, 2-dimensional preferences have strictly more expressive power than single-peaked preferences.

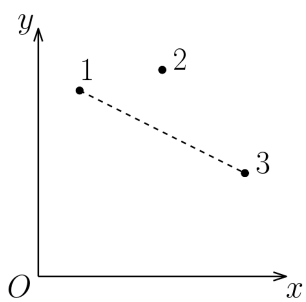
Proposition 12. *Every single-peaked preference set on $m = 3$ outcomes is a 2-dimensional preference set. In particular, up to relabeling it is a subset of $P_2(X)$, for $X = \{(3,0),(2,2),(0,3)\}$.*

Proof. Without loss of generality, relabel the outcomes of the single peaked preferences as $S = \{1,2,3\}$. The resulting preferences are given by GoodCompromise. $\square$

Furthermore, BadCompromise gives an example of a preference set which is 2-dimensional, but not single-peaked:

Proposition 13. *In a single peaked set of preferences, it is not possible for every outcome to be the lowest ranked outcome of some preference.*

Proof. The median outcome (via the standard order on $[0,1]$) cannot be the lowest ranked. $\square$



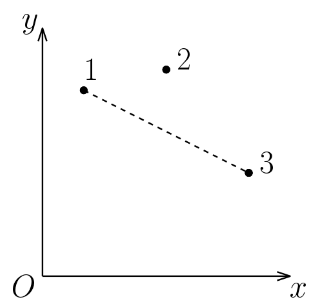
$1 > 2 > 3$

$2 > 1 > 3$

$2 > 3 > 1$

$3 > 2 > 1$

(a) GoodCompromise



$1 > 2 > 3$

$1 > 3 > 2$

$3 > 1 > 2$

$3 > 2 > 1$

(b) BadCompromise

When $m > 3$, neither single-peaked nor 2-dimensional preferences are a subset of the other. One was to see this to to simply count the number of single-peaked preferences, and see that it grows much faster with m than 2-dimensional preferences do (recall that we showed in theorem 7 that the maximal size of a 2-dimensional preference set is $O(m^2)$).

Proposition 14. *The number of single-peaked preferences for any set S of m outcomes is at least $2^{\Omega(m)}$*

Proof. Choose the median outcome of S to be the peak. Given a subset $T \subseteq [m-1]$ of size $|T| = \lfloor m/2 \rfloor$, we can associate a unique single-peaked preference as follows: treat the preference as an array of outcomes, ranked highest to lowest. Put the median of S at index 0. Let the outcomes to the left of the median occupy the indices corresponding to set T, and let those to the right occupy the other indices. There are $\binom{m-1}{\lfloor m/2 \rfloor} \geq 2^{\Omega(m)}$ such subsets T. $\square$

A cleaner, more constructive way to see this is the following proposition

Proposition 15. *The preference set* FlipFlop *is single-peaked, but not 2-dimensional.*

FlipFlop

$1 > 2 > 3 > 4$

$1 > 2 > 4 > 3$

$2 > 1 > 3 > 4$

$2 > 1 > 4 > 3$

Proof. To obtain a single-peaked representation, order the outcomes from left to right as follows: $3, 1, 2, 4$. All preferences will then be possible with peak 1 or 2.

Now we show that FlipFlop is not 2-dimensional. Note that neither 1 nor 2 can dominate the other, and the same holds for 3 and 4. So let $a = \text{indif}(1, 2)$ and $b = \text{indif}(3, 4)$. Without knowing the outcome space $X \subseteq \mathbb{R}^2_{\geq 0}$, we can't tell which side of a corresponds to $1 > 2$ verses $2 > 1$ (same with b). However, we do know that at least one of the following implications should hold for any preference $>$ over X:

$$1 > 2 \implies 3 > 4 \qquad 1 > 2 \implies 4 > 3 \qquad 2 > 1 \implies 3 > 4 \qquad 2 > 1 \implies 4 > 3$$

For example, if $\theta(a) < \theta(b)$ and preference weights with smaller angle than a (resp. b) favor $1 > 2$ (resp. $3 > 4$), then $1 > 2 \implies 3 > 4$.

However, none of these implications are satisfied by the entire preference set FlipFlop. ((POSSIBLY UNCLEAR)) □

When considering restricted domains of preference, it is perhaps more important to consider which preference sets *do not* fall into that domain. Indeed, if too many preferences are possible, then the impossibility results which hold for arbitrary preferences will apply.

As we've directly shown, 3-Cycle is not 2 dimensional. This is already a good sign: 3-Cycle is a classic example of the so-called Condorcet paradox. Despite each preference being a linear order, the majority of voters favor 1 over 2, 2 over 3, and 3 over 1. That is, "collective preference" is cyclic. Indeed, 3-Cycle is a preference set with no Condorcet winner, and is often used as a basic counterexample for the existence of good voting schemes. The fact that cycle is impossible already seems like good news for voting schemes on two dimensional preferences.

Here's another case:

Proposition 16. Sandwich *is neither single-peaked nor* 2 *dimensional*

Proof. Sandwich is not single-peaked by proposition 13. It's not 2-dimensional by proposition 9. □

Reverse	3-Cycle	Sandwich
$1 > 2 > 3 > 4$	$1 > 2 > 3$	$1 > 2 > 3$
$4 > 3 > 2 > 1$	$2 > 3 > 1$	$1 > 3 > 2$
	$3 > 1 > 2$	$3 > 2 > 1$
		$2 > 3 > 1$

2-dimensional	BadCompromise
single peaked	FlipFlop
both	Reverse, GoodCompromise
neither	3-Cycle, Sandwich

((ARGUE THAT 2-D IS MORE NATURAL AND POWERFUL THAN SINGLE-PEAKED, EVEN THOUGH THERE ARE (ASYMPTOTICALLY) MORE SINGLE-PEAKED))

8 Voting for 2-dimensional preferences

Define median-angle voting scheme. ((TODO: DEFINE; STRING TOGETHER THESE RESULTS WITH SOME MOTIVATION))

Theorem 17. *The median-angle voting rule for* 2*-dimensional preferences is incentive compatible.*

Proof. Consider any voter i and suppose the voters true preferences are given by weights $a_1, \ldots, a_n$. Let u denote the median-angled preference vector chosen when all agents report their true preference. Suppose $\theta(a_i) < \theta(u)$ (the other case is symmetric). The only way in which i can change the chosen preference vector by misreporting his value for a_i is to *increase* its angle, say to some ν. Note that because $\theta(a) < \theta(u) < \theta(\nu)$, we have $u \in \text{cone}(\nu, a)$.

For the sake of contradiction, assume i could profit from this manipulation, i.e. he could get an outcome he favored chosen by ν. Let y be the outcome selected by u, and let x be selected by ν. We have $x >_\nu y$ and $x >_{a_i} y$. By proposition 4, we get $x >_u y$ as well. However, this contradicts the assumption that u selects y, and completes our proof. □

Theorem 18. *The median-angle social welfare rule satisfies independence of irrelevant alternatives*

Proof. Consider any pair of outcomes x, y. If either dominates the other, the IIA axiom becomes easy to verify. So suppose neither dominates the other, without loss of generality take $x_1 < y_1$, $x_2 > y_2$, and let $b = \text{indif}(x, y)$.

Suppose $\{a_i\}$ and $\{a'_i\}$ are collections of preference weights such that $x >_{a_i} y \iff x >_{a'_i} y$ for each i. By proposition 6, this means that $\theta(a_i) > \theta(b) \iff \theta(a'_i) > \theta(b)$. Let u denote the median-angled vector among $\{a_i\}$, and u' for $\{a'_i\}$. Because each pair a_i, a'_i lies on the same side of b, we must have u and u' on the same side of b as well. Thus, $x >_u y \iff x >_{u'} y$, as desired. □

Theorem 19. *Among an odd number of 2-dimensional preferences, there is always a Condorcet winner, which is selected by the median angle voting scheme*

Proof. Let u be the median-angled preference vector, and let x be the favorite outcome of u. Given some outcome $y \neq x$, if $y \ll x$, then y is never preferred to x. So suppose neither x nor y dominate each other, and let $b = \text{indif}(x, y)$. Let S be the set of preference weights a such that $x >_a y$. By proposition 6, either consists of all vectors a with $\theta(a) > \theta(b)$ or all a with $\theta(a) < \theta(b)$. The median angled preference weight u must lie in S. Thus, more than half of all the input preference weights must lie in S. Thus, more than half of the input preferences agree that $x > y$. Because y was arbitrary, this means x is the Condorcet winner. □

We note the following important consequence of the previous theorem: if preferences are known to lie in $P_2(X)$ for some X, but no details about X are known, then a voting protocol can simply ask voters for their list of preferences and look for the Condorcet winner (note that there is at most one Condorcet winner). Moreover, it hints that when agents act approximately via a weighted sum of two attributes, it is more likely that there is a Condorcet winner. ((NOTE: CONTEMPLATE THE FOLLOWING HOLE IN THE ABOVE LOGIC: WHAT IF AN AGENT STRATEGICALLY REPORTS A PREFERENCE THAT IS NOT TWO DIMENSIONAL FOR THE UNDERLYING SET X. FOR ONE, THE AGENT MY FORCE THE PROCEDURE TO FAIL BECAUSE IT WON'T HAVE A CONDORCET WINNER. BUT AD HOC WE DON'T KNOW HE CAN'T MANIPULATE THE CONDORCET WINNER))

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