3.9 Derivatives of General Exponential and Logarithmic Functions

In this section, we develop derivative formulas for functions involving exponentials and logarithms, including the natural logarithm function $f(x) = \ln\ x$, the general exponential function $f\left(x\right) = b^{x}$, the general logarithm function $f\left(x\right) = \log_{b}\ x$, and the hyperbolic trigonometric functions and their inverses. We will see that all of the resulting derivative formulas arise from the derivative formula $\frac{d}{dx}\ e^{x} = e^{x}$ via the rules for differentiation that we presented previously.

THEOREM 1

Derivative of the Natural Logarithm

$$\frac{d}{dx} \ln x = \frac{1}{x} \quad \text{for } x > 0 \tag{1}$$

The two most important calculus facts about exponentials and logs are

$$\boxed{\frac{d}{dx} e^x = e^x, \quad \frac{d}{dx} \ln x = \frac{1}{x}}$$

EXAMPLE 1

$f'g + g'f$

$y' = (1)\ln x + \frac{1}{\cancel{x}} \cdot \cancel{x}$

Differentiate $\boxed{y' = \ln x + 1}$ ✓

a. $y = \overset{f}{x}\ \overset{g}{\ln x}$ and

b. $y = (\underbrace{\ln x}_{u})^2$.

$y = u^2 \longrightarrow y' = 2u \cdot u'$

$y' = 2\ln x \cdot \frac{1}{x}$

$\boxed{y' = \frac{2\ln x}{x}}$ ✓

Solution

a. Use the Product Rule:

$$\begin{aligned}\frac{d}{dx}(x\ \ln\ x) &= x \cdot (\ln\ x)' + (x)' \cdot \ln\ x \\ &= x \cdot \frac{1}{x} + \ln\ x = 1 + \ln\ x\end{aligned}$$

b. Use the General Power Rule:

$$\frac{d}{dx}(\ln\ x)^2 = 2\ \ln\ x \cdot \frac{d}{dx}\ \ln\ x = \frac{2\ \ln\ x}{x}$$

100% in Exams!

$$\boxed{(\ln u)' = \frac{u'}{u}}$$

$$\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$$

2

EXAMPLE 2

$y = \ln u \longrightarrow y' = \frac{u'}{u}$

Differentiate $\boxed{y' = \frac{3x^2}{x^3+1}}$ ✓

a. $y = \ln\left(\overset{u}{x^3 + 1}\right)$ and

b. $y = \ln\left(\underset{u}{\sqrt{\sin\ x}}\right)$.

$y = \ln u \longrightarrow y' = \frac{u'}{u}$

$u = \sqrt{\sin x} \longrightarrow u' = \frac{1}{2\sqrt{\sin x}} \cdot \cos x$

$y' = \frac{\left(\frac{\cos x}{2\sqrt{\sin x}}\right)}{\left(\frac{\sqrt{\sin x}}{1}\right)} \Rightarrow y' = \frac{\cos x}{2\ \sin x} \Rightarrow \boxed{y' = \frac{1}{2}\cot x}$ ✓

Solution

Use Eq. (2):

a. $\frac{d}{dx}\ \ln\left(x^3+1\right)=\frac{\left(x^3+1\right)'}{x^3+1}=\frac{3x^2}{x^3+1}$

b. The algebra is simpler if we write

$$\ln\left(\sqrt{\sin\ x}\right)=\ln\ \left(\left(\sin\ x\right)^{1/2}\right)=\frac{1}{2}\ \ln\left(\sin\ x\right):$$

$$\begin{aligned}\frac{d}{dx}\ \ln\left(\sqrt{\sin\ x}\right) &= \frac{1}{2}\ \frac{d}{dx}\ \ln\left(\sin\ x\right)\\ &= \frac{1}{2}\ \frac{\left(\sin\ x\right)'}{\sin\ x}=\frac{1}{2}\ \frac{\cos\ x}{\sin\ x}=\frac{1}{2}\ \cot\ x\end{aligned}$$

The Derivative of $f(x) = b^x$ and $f(x) = \log_b x$

From the change-of-base formulas for exponential and logarithmic functions (see Section 1.6), we have $b^x = e^{x \ln b}$ and $\log_b x = \frac{\ln x}{\ln b}$. Differentiating these equations, using the Chain Rule on $e^{x \ln b}$ and the Constant Multiple Rule on $\frac{\ln x}{\ln b}$, we obtain

$$\frac{d}{dx} b^x = \frac{d}{dx} e^{x \ln b} = (\ln b) e^{x \ln b} = (\ln b) b^x$$
$$\frac{d}{dx} \log_b x = \frac{d}{dx} \frac{\ln x}{\ln b} = \left(\frac{1}{\ln b}\right) \frac{1}{x} = \frac{1}{x \ln b}$$

Therefore, we have the following theorem:

$$y = e^x \longrightarrow y' = e^x$$

$$y = b^x \longrightarrow \boxed{y' = b^x \cdot \ln b}$$

Ex: $y = 3^x \longrightarrow y' = 3^x \cdot \ln 3 = (\ln 3)\, 3^x$

$y = 10^x \longrightarrow y' = 10^x \cdot \ln 10$

$$y = \log_b x$$

$$\boxed{y' = \frac{1}{x \ln b}}$$

Ex: $y = \log_3 x \longrightarrow y' = \dfrac{1}{x \ln 3}$

$y = \log_{10} x \longrightarrow y' = \dfrac{1}{x \ln 10}$

THEOREM 2

Derivative of $f(x) = b^x$ and $f(x) = \log_b\ x$

$$\frac{d}{dx}\, b^x = (\ln\ b)\, b^x \qquad \frac{d}{dx} \log_b\ x = \frac{1}{x\ \ln\ b} \tag{3}$$

For example, $(10^{\ x})' = (\ln\ 10)\, 10^{\ x}$ and $(\log_{10}\ x)' = \dfrac{1}{x\ \ln\ 10}$

$y = b^x \longrightarrow y' = b^x \cdot \ln b$

EXAMPLE 4 $y = b^u \longrightarrow y' = b^u \cdot \ln b \cdot u'$

a) $f'(x) = 4^{3x} \cdot \ln 4 \cdot 3$

Differentiate $\boxed{f'(x) = (3\ln 4) \cdot 4^{3x}}$

b) $y = \log_b x \longrightarrow y' = \frac{1}{x \ln b}$

a. $f(x) = 4^{\overset{u}{3x}}$ and

b. $f(x) = \log_2 (\overset{u}{x^2 + 1})$. $y = \log_b u \rightarrow y' = \frac{1}{u \ln b} \cdot u'$

$f'(x) = \frac{1}{(x^2+1)\ln 2} \cdot 2x$

$\Rightarrow \boxed{f'(x) = \frac{2x}{(x^2+1)\ln 2}}$

Solution

a. The function $f(x) = 4^{3x}$ is a composite of $f(u) = 4^u$ and $u = 3x$:

$$\frac{d}{dx}4^{3x} = \left(\frac{d}{du}4^u\right)\frac{du}{dx} = (\ln\ 4)\,4^u\,(3x)' = (\ln\ 4)\,4^{3x}\,(3) = (3\ \ln\ 4)\,4^{3x}$$

b. The function $f(x) = \log_2(x^2+1)$ is a composite of $f(u) = \log_2 u$ and $u = x^2+1$:

$$\frac{d}{dx}\log_2(x^2+1) = \left(\frac{d}{du}\log_2 u\right)\frac{du}{dx} = \frac{1}{u\ \ln\ 2}(x^2+1)' = \frac{2x}{(x^2+1)\ \ln\ 2}$$

$$\log_c ab = \log_c a + \log_c b$$

$$\log_c \frac{a}{b} = \log_c a - \log_c b$$

$$\log_c a^b = b \log_c a$$

$$\log_a a = 1$$

$$\ln ab = \ln(a) + \ln(b)$$

$$\ln \frac{a}{b} = \ln(a) - \ln(b)$$

$$\ln a^b = b \ln a$$

Logarithmic Differentiation

The next example illustrates **logarithmic differentiation**. This technique saves work when the function is a product or quotient with several factors.

$$Ex: \ln y = \ln \frac{(x^2-3)^5 \cdot (x^3-5x)}{(5x^4-x+1)^{100}} \quad , \quad y' = ?$$

$$\ln y = 5\ln(x^2-3) + \ln(x^3-5x) - 100\ln(5x^4-x+1)$$

$$\cancel{y} \cdot \frac{y'}{\cancel{y}} = \left(5 \cdot \frac{2x}{x^2-3} + \frac{3x^2-5}{x^3-5x} - 100 \cdot \frac{20x^3-1}{5x^4-x+1}\right)\left(\frac{(x^2-3)^5 \cdot (x^3-5x)}{(5x^4-x+1)^{100}}\right)$$

Step 1 : Take Ln of both sides

Step 2 : Apply properties of Ln

Step 3 : Take derivative

Step 4 : Multiply by (y)

Step 5 : Replace y with original equatio

EXAMPLE 5

Find the derivative of

$$\ln \overset{y}{\cancel{f(x)}} = \ln \frac{(x+1)^2\,(2x^2-3)}{\sqrt{x^2+1}} \qquad \sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}}$$

$$\ln y = 2\ln(x+1) + \ln(2x^2-3) - \frac{1}{2}\ln(x^2+1)$$

$$\cancel{y}\cdot\frac{y'}{\cancel{y}} = \left(\frac{2}{x+1} + \frac{4x}{2x^2-3} - \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}x}{x^2+1}\right)\left(\frac{(x+1)^2(2x^2-3)}{\sqrt{x^2+1}}\right) \checkmark$$

$$f'g+g'f$$

$$\frac{d}{dx}(\underbrace{\overbrace{6x}^{f}\,\overbrace{\ln(x)}^{g}}_{} - 3x) =$$

$$= 6\ln x + \frac{1}{\cancel{x}} \cdot 6\cancel{x} - 3$$

$$= 6\ln x + 6 - 3 \implies \boxed{y' = 6\ln x + 3}$$

derivative of $y = 5^{x^8}$ u

$y = b^x \longrightarrow y' = b^x \cdot \ln b$

$y = b^u \longrightarrow y' = b^u \cdot \ln b \cdot u'$

$y' = 5^{x^8} \cdot \ln 5 \cdot 8x^7$

$$\boxed{y' = (8x^7 \ln 5) \cdot 5^{x^8}}$$

$y = \ln u \longrightarrow \boxed{y' = \frac{u'}{u}}$

derivative of $y = 17 \ln(\underbrace{\ln(6x)}^{u})$.

$$y' = 17 \frac{\frac{1}{x}}{\ln(6x)}$$

$u = \ln(\underset{u}{6x})$

$$u' = \frac{\cancel{6}}{\cancel{6}x} = \frac{1}{x}$$

$$\boxed{y' = \frac{17}{x\ln(6x)}}$$