6.8 The Pigeonhole Principle

The **Pigeonhole Principle** (also known as the *Dirichlet Drawer Principle* or the *Shoe Box Principle*) is sometimes useful in answering the question: Is there an item having a given property? **When the Pigeonhole Principle is successfully applied, the principle tells us only that the object exists; the principle will not tell us how to find the object or how many there are.**

The first version of the Pigeonhole Principle that we will discuss asserts that if n pigeons fly into k pigeonholes and $k<n$, some pigeonhole contains at least two pigeons. The reason this statement is true can be seen by arguing by contradiction. If the conclusion is false, each pigeonhole contains at most one pigeon and, in this case, we can account for at most k pigeons. Since there are n pigeons and $n > k$, we have a contradiction.

Pigeonhole Principle

(First Form)

If n pigeons fly into k pigeonholes and

$k < n$, some pigeonhole contains at least two pigeons.

Example 6.8.1

Ten persons have first names Alice, Bernard, and Charles and last names Lee, McDuff, and Ng. Show that at least two persons have the same first and last names.

SOLUTION: There are nine possible names for the 10 persons. If we think of the persons as pigeons and the names as pigeonholes, we can consider the assignment of names to people to be that of assigning pigeonholes to the pigeons. By the Pigeonhole Principle, some name (pigeonhole) is assigned to at least two persons (pigeons)

Example 6.8.2

Give another proof of the result in Example 6.8.1 using proof by contradiction.

SOLUTION: Suppose by way of contradiction, that no two of the 10 people in Example 6.8.1 have the same first and last names. Since there are three first names and three last names, there are at most nine people.

This contradiction shows that there are at least two people having the same first and last names.

■

We next restate the Pigeonhole Principle in an alternative form.

Pigeonhole Principle (Second Form)

If f is a function from a finite set X to a finite set Y and $|X| > |Y|$, then

$$f(x_1) = f(x_2)$$

for some $x_1, x_2 \in X, x_1 \neq x_2$.

Our next examples illustrate the use of the second form of the Pigeonhole Principle.

Example 6.8.3 If 20 processors are interconnected, show that at least 2 processors are directly connected to the same number of processors.

SOLUTION :

Designate the processors 1, 2, . . . , 20. Let a_i be the number of processors to which processor $\boldsymbol{i}$ is directly connected.

We are to show that $\boldsymbol{a_i = a_j}$ for some $i \neq j$.

The domain of the function $\boldsymbol{a}$ is

$$X = \{1, 2, \ldots, 20\}$$

and the range Y is some subset of

$$\{0, 1, \ldots, 19\}.$$

Unfortunately, $|X| = |\{0, 1, \ldots, 19\}|$

and we cannot immediately use the second form of the Pigeonhole Principle.

Let us examine the situation more closely.

$$\boldsymbol{a_i = 0}$$

for some *i*,

and

$$a_j = \mathbf{19}\text{, for some } j,$$

Thus the range Y is a subset of either

$$\{0, 1, \ldots, 18\} \text{ or } \{1, 2, \ldots, 19\}.$$

In either case,

$$|Y| < 20 = |X|.$$

By the second form of the Pigeonhole Principle, $ai = aj$, for some $i \neq j$, as desired. ■

Example 6.8.4

Show that if we select 151 distinct computer science courses numbered between 1 and 300 inclusive, at least two are consecutively numbered.

SOLUTION :

Let the selected course numbers be

$$\boldsymbol{c_1, c_2, \ldots, c_{151}} \qquad (1)$$

The 302 numbers consisting of (1) together with

$$\boldsymbol{c_1 + 1, c_2 + 1, \ldots, c_{151} + 1} \qquad (2)$$

Thus we have $\boldsymbol{c_i} = \boldsymbol{c_j} + 1$ and course $\boldsymbol{c_i}$ follows course $\boldsymbol{c_j}$. ■

Example 6.8.5

An inventory consists of a list of 89 items, each marked “available” or “unavailable.” There are 50 available items. Show that there are at least two available items in the list exactly nine items apart. (For example, available items at positions 13 and 22 or positions 69 and 78 satisfy the condition.)

SOLUTION:

Let a_i denote the position of the ith available item. We must show that

$$a_i - a_j = 9$$

for some *i* and *j*.

Consider the numbers

$$a_1, a_2, \ldots, a_{50} \qquad \textbf{(3)}$$

and

$$a_1 + 9, a_2 + 9, \ldots, a_{50} + 9 \qquad \textbf{(4)}$$

The 100 numbers in (3) and (4) have possible values from 1 to 98. By the second form of the Pigeonhole Principle, two of the numbers must coincide. We cannot have two of (3) or two of (4) identical; thus some number in (3) is equal to some number in (4). Therefore, $ai - aj = 9$ for some i and j, as desired.

■

We next state yet another form of the Pigeonhole Principle.

Pigeonhole Principle

(Third Form)

Let f be a function from a finite set X into a finite set Y. Suppose that $|X| = n$ and $|Y| = m$. Let $k = \lceil \frac{n}{m} \rceil$. Then there are at least k distinct values

$$a_1, a_2, \ldots, a_k \in X$$

Such that

$$f(a_1) = f(a_2) = \cdot \quad \cdot \quad \cdot = f(a_k).$$

Proof:

We argue by contradiction. Let

$$Y = \{y_1, \ldots, y_m\}.$$

Suppose that the conclusion is false. Then there are at most $k - 1$ distinct values $x \in X$ with $f(x) = y_1$; there are at most $k - 1$ distinct values $x \in X$ with $f(x) = y_2$; . . . ; there are at most $k - 1$ distinct values $x \in X$ with

$$f(x) = y_m .$$

Thus there are at most $m(k - 1)$ members in the domain of f. But

$$m(k - 1) < m\frac{n}{m} = n$$

which is a contradiction. Therefore, there are at least k distinct values,

$$a_1, a_2, \ldots, a_k \in X$$

such that

$$f(a_1) = f(a_2) = \cdot \cdot \cdot = f(a_k).$$

■

Our last example illustrates the use of the third form of the Pigeonhole Principle.

Example 6.8.6

A useful feature of black-and-white pictures is the average brightness of the picture. Let us say that two pictures are similar if their average brightness differs by no more than some fixed value. Show that among six pictures, there are either three that are mutually similar or three that are mutually dissimilar.

SOLUTION :

Denote the pictures $P_1, P_2, \ldots, P_6$. Each of the five pairs

$(P_1, P_2), (P_1, P_3), (P_1, P_4), (P_1, P_5), (P_1, P_6),$

has the value "similar" or "dissimilar".

By the third form of the Pigeonhole Principle, there are at least

$$\left\lceil \frac{5}{2} \right\rceil = 3 \text{ pairs}$$

with the same value; that is, there are three pairs

$$(P_1, P_i), (P_1, P_j), (P_1, P_k), \qquad (5)$$

all similar or all dissimilar. Suppose that each pair is similar. (The case that each pair is dissimilar is Exercise 14.) If any pair

$$(P_i, P_j), (P_i, P_k), (P_j, P_k),$$

is similar, then these two pictures together with P_1 are mutually similar and we have found three mutually similar pictures. Otherwise, each of the pairs (5) is dissimilar and we have found three mutually dissimilar pictures.