2.7 Limits at Infinity

$\lim f(x)$

$x \to 0$

$x \to 2$

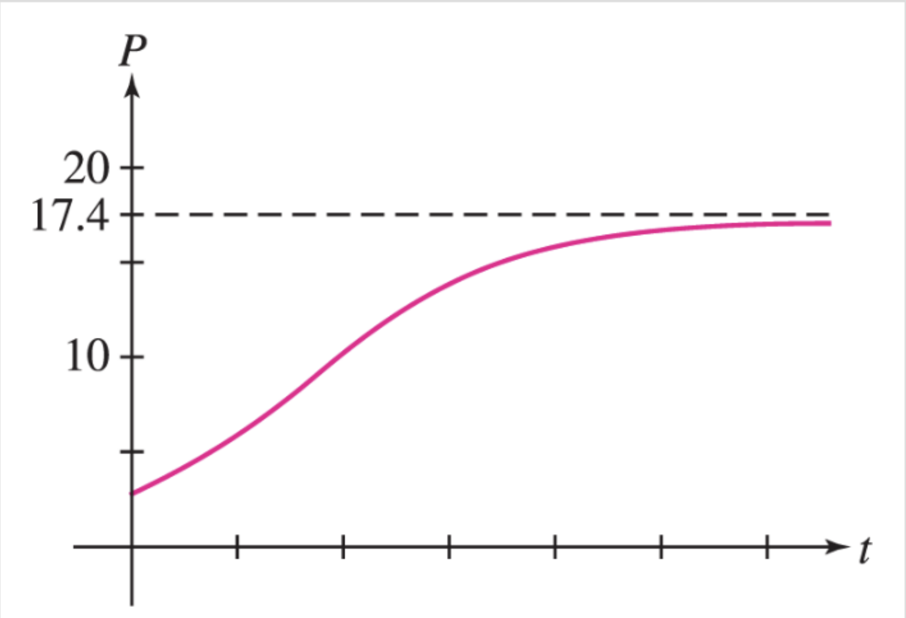
$x \to -3$

$x \to 5^+$

$x \to \infty$

So far we have considered limits as x approaches a number c. It is also important to consider limits where x approaches ∞ or $-\infty$, which we refer to as **limits at infinity**. In applications, limits at infinity arise naturally when we describe the "long-term" behavior of a system as in Figure 1.

$\lim_{x \to \infty} f(x) = 17.4$

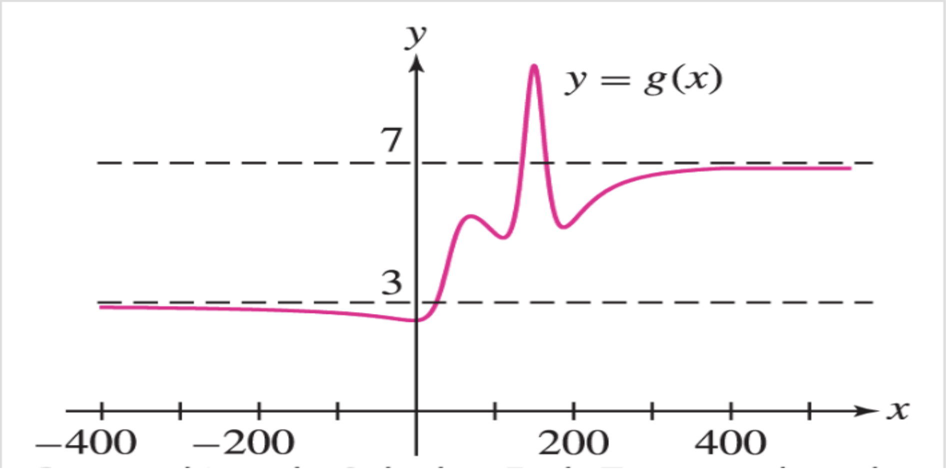


$y = 17.4$

Horizontal asy.

EXAMPLE 1

Discuss the asymptotic behavior in Figure 2.



$$\lim_{x\to\infty} e^x = e^{\infty} = \boxed{\infty}$$

$$\lim_{x\to -\infty} e^x = e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = \boxed{0}$$

Solution

The function g approaches $L = 7$ as we move to the right and it approaches $L = 3$ as we move to left, so

$$\lim_{x\to\infty} g(x) = 7 \qquad \text{and} \qquad \lim_{x\to-\infty} g(x) = 3$$

Accordingly, the lines $y = 7$ and $y = 3$ are horizontal asymptotes of g.

$$\lim_{x\to\infty} e^x = \infty \quad \text{and} \quad \lim_{x\to-\infty} e^x = 0$$

$$\lim_{x\to\infty} x^n = (\infty)^n = \infty$$

$$\lim_{x\to-\infty} x^n = (-\infty)^n = \begin{cases} \infty & \text{even} \\ -\infty & \text{odd} \end{cases}$$

THEOREM 1

For all $n > 0$,

$$\boxed{\lim_{x\to\infty} x^n = \infty} \quad \text{and} \quad \boxed{\lim_{x\to\infty} x^{-n} = \lim_{x\to\infty} \frac{1}{x^n} = 0}$$

If n is a positive whole number,

$$\boxed{\lim_{x\to-\infty} x^n = \begin{cases} \infty & \text{if } n \text{ is even} \\ -\infty & \text{if } n \text{ is odd} \end{cases}} \quad \text{and} \quad \boxed{\lim_{x\to-\infty} x^{-n} = \lim_{x\to-\infty} \frac{1}{x^n} = 0}$$

Calculate $\lim_{x\to\infty} \frac{20x^2 - 3x}{3x^5 - 4x^2 + 5}$.

$$= \lim_{x\to\infty} \frac{20x^2}{3x^5} = \lim_{x\to\infty} \frac{20}{3x^3} = \frac{20}{\infty} = \boxed{0}$$

$$\lim_{x\to\infty} \frac{\frac{20x^2}{x^5} - \frac{3x}{x^5}}{\frac{3x^5}{x^5} - \frac{4x^2}{x^5} + \frac{5}{x^5}} = \lim_{x\to\infty} \frac{\frac{20}{x^3} - \frac{3}{x^4}}{3 - \frac{4}{x^3} + \frac{5}{x^5}} = \frac{\frac{20}{\infty} - \frac{3}{\infty}}{3 - \frac{4}{\infty} + \frac{5}{\infty}} = \frac{0}{3} = \boxed{0}$$

THEOREM 2

Limits at Infinity of a Rational Function

The asymptotic behavior of a rational function depends only on the leading terms of its numerator and denominator. If $a_n, b_m \neq 0,$ then

$$\lim_{x\to\pm\infty} \frac{a_n x^n + \cancel{a_{n-1}x^{n-1} + \cdots + a_0}}{b_m x^m + \cancel{b_{m-1}x^{m-1} + \cdots + b_0}} = \frac{a_n}{b_m} \lim_{x\to\pm\infty} x^{n-m}$$

Here are some examples:

- $n = m:$ $$\lim_{x\to\infty} \frac{3x^4 - 7x + 9}{7x^4 - 4} = \frac{3}{7} \lim_{x\to\infty} x^0 = \frac{3}{7}$$
- $n < m:$ $$\lim_{x\to\infty} \frac{3x^3 - 7x + 9}{7x^4 - 4} = \frac{3}{7} \lim_{x\to\infty} x^{-1} = 0$$
- $n > m,\ n - m$ odd: $$\lim_{x\to-\infty} \frac{3x^8 - 7x + 9}{7x^3 - 4} = \frac{3}{7} \lim_{x\to-\infty} x^5 = -\infty$$
- $n > m,\ n - m$ even: $$\lim_{x\to-\infty} \frac{3x^7 - 7x + 9}{7x^3 - 4} = \frac{3}{7} \lim_{x\to-\infty} x^4 = \infty$$

EXAMPLE 4

Calculate the limits

a. $\lim_{x\to\infty} \frac{3x^{7/2}+7x^{-1/2}}{x^2-x^{1/2}}$

b. $\lim_{x\to\pm\infty} \frac{4x}{\sqrt{x^2+1}}$.

2.7 SUMMARY

- *Limits at infinity*:
  - $\lim_{x \to \infty} f(x) = L$ if $|f(x) - L|$ becomes arbitrarily small as x increases without bound.
  - $\lim_{x \to -\infty} f(x) = L$ if $|f(x) - L|$ becomes arbitrarily small as x decreases without bound.
  - $\lim_{x \to \infty} e^x = \infty \qquad \text{and} \qquad \lim_{x \to -\infty} e^x = 0$
- A horizontal line $y = L$ is a *horizontal asymptote* of f if

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

  A function can have 0, 1 or 2 horizontal asymptotes.

- If $n > 0$, then $\lim_{x \to \pm\infty} x^{-n} = 0$.
- If $n > 0$ is a whole number, then

$$\lim_{x \to \infty} x^n = \infty \quad \text{and} \quad \lim_{x \to -\infty} x^n = \begin{cases} \infty & \text{if } n \text{ is even} \\ -\infty & \text{if } n \text{ is odd} \end{cases}$$

- If $f(x) = \dfrac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0}$ with $a_n, b_m \neq 0$, then

$$\lim_{x \to \pm\infty} f(x) = \frac{a_n}{b_m} \lim_{x \to \pm\infty} x^{n-m}$$