3.3 Product and Quotient Rules

This section covers the **Product Rule** and **Quotient Rule** for computing derivatives. These two rules, together with the Chain Rule and implicit differentiation (covered in later sections), make up an extremely effective differentiation toolkit.

THEOREM 1

Product Rule

If f and g are differentiable functions, then fg is differentiable and

$$(fg)'(x) = f'(x)\,g(x) + f(x)\,g'(x)$$

It may be helpful to remember the Product Rule in words: The derivative of a product of terms is equal to *the derivative of the first term times the second plus the first term times the derivative of the second*:

$$(\text{first})' \cdot \text{second} + \text{first} \cdot (\text{second})'$$

$(fg)' = f'g + g'f \quad = \underbrace{2x}_{f'} \cdot \underbrace{(9x+2)}_{g} + \underbrace{9}_{g'} \cdot \underbrace{x^2}_{f}$

EXAMPLE 1

$= 18x^2 + 4x + 9x^2 = \boxed{27x^2 + 4x}$ ✓

Find the derivative of $h(x) = \overbrace{x^2}^{f}\,\overbrace{(9x+2)}^{g}$.

Solution

This function is a product:

$$h(x) = \overbrace{x^2}^{\text{First}}\ \overbrace{(9x+2)}^{\text{Second}}$$

By the Product Rule (in Leibniz notation),

$$\begin{aligned} h'(x) &= \overbrace{\frac{d}{dx}\left(x^2\right)}^{(\text{First})'}\ \overbrace{(9x+2)}^{\text{Second}} + \overbrace{\left(x^2\right)}^{\text{First}}\ \overbrace{\frac{d}{dx}(9x+2)}^{(\text{Second})'} \\ &= (2x)(9x+2) + \left(x^2\right)(9) = 27x^2 + 4x \end{aligned}$$

EXAMPLE 2

$(f \cdot g)' = f'g + g'f$

Find the derivative of $y = \underbrace{(2 + x^{-1})}_{f} \underbrace{(x^{3/2} + 1)}_{g}$.

$$y' = \underbrace{-x^{-2}}_{f'} \cdot \underbrace{(x^{3/2} + 1)}_{g} + \underbrace{\frac{3}{2}x^{\frac{1}{2}}}_{g'} \cdot \underbrace{(2 + x^{-1})}_{f} \checkmark$$

$$-2 + \frac{3}{2} = -\frac{4}{2} + \frac{3}{2}$$

$$= -\frac{1}{2}$$

EXAMPLE 3

$(f \cdot g)' = f'g + g'f$

Calculate $\frac{d}{dt}\left(t^2 e^t\right)$. $= 2t \cdot e^t + e^t \cdot t^2$ ✓

$t^2 \to f$, $e^t \to g$

$= e^t(2t + t^2)$ ✓

Solution

Use the Product Rule and the formula $\dfrac{d}{dt}\, e^t = e^t$:

Ex: $\frac{d}{dx} x^3 \cdot e^2 = ?$ $e^2 \cdot 3x^2$ constant

$= 3e^2 x^2$

Ex: $\frac{d}{dx} x^5 \cdot e^x = ?$ $(fg)' = f'g + g'f$

$x^5 \to f$, $e^x \to g$

$= 5x^4 \cdot e^x + e^x \cdot x^5$

$\boxed{= e^x(5x^4 + x^5)}$ ✓

EXAMPLE 4

Figure 1 depicts a rectangle whose length $L(t)$ and width $W(t)$ (measured in inches) are varying in time (t, in minutes). At $t = 5$, the length is 8, the width is 5, and they are changing according to $L'(5) = -4$ and $W'(5) = 3$. Compute $A'(5)$.

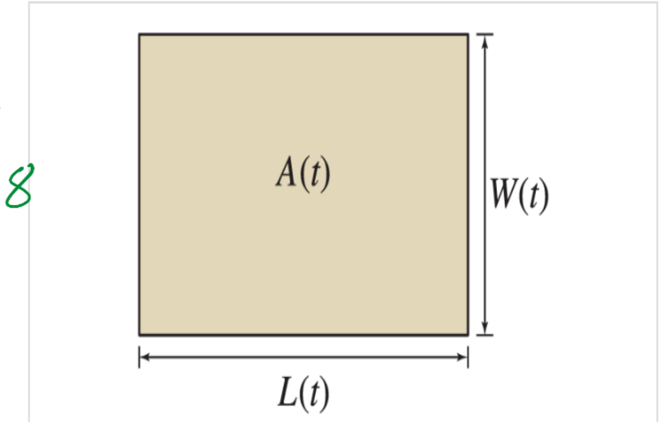
$$A = L \cdot W$$

$$A' = L' \cdot W + W' L$$

$$= -4 \cdot 5 + 3 \cdot 8$$

$$= -20 + 24$$

$$= \boxed{4}$$



Solution

Since the area is given by $A(t) = L(t)\,W(t)$, we can use the product rule to compute $A'(t)$. We have $A'(t) = L'(t)\,W(t) + L(t)\,W'(t)$. Therefore,

$$A'(5) = (-4)(5) + (8)(3) = 4.$$

THEOREM 2

Quotient Rule

$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$

If f and g are differentiable functions, then f/g is differentiable for all x such that $g(x) \neq 0$, and

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)\,f'(x) - f(x)\,g'(x)}{g(x)^2}$$

The numerator in the Quotient Rule is *the bottom times the derivative of the top minus the top times the derivative of the bottom*. The denominator is *the bottom squared*:

$$\frac{\text{bottom} \cdot (\text{top})' - \text{top} \cdot (\text{bottom})'}{\text{bottom}^2}$$

EXAMPLE 5

Compute the derivative of $f(x) = \dfrac{x}{1+x^2}$.

$x \to f$, $1+x^2 \to g$

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2} = \frac{(1)(1+x^2) - 2x \cdot x}{(1+x^2)^2}$$

$$= \frac{1+x^2-2x^2}{(1+x^2)^2} = \boxed{\frac{1-x^2}{(1+x^2)^2}}$$

EXAMPLE 6

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

Calculate $\dfrac{d}{dt}\left(\dfrac{\overset{f}{e^t}}{e^t + t}\right)$.

$$= \frac{e^t(e^t+t) - (e^t+1)\cdot e^t}{(e^t+t)^2} = \frac{e^t(\cancel{e^t}+t-\cancel{e^t}-1)}{(e^t+t)^2}$$

$$\boxed{= \frac{e^t(t-1)}{(e^t+t)^2}}$$

✓

EXAMPLE 7

$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$

Point: $\left(1, \frac{2}{5}\right)$

(x, y)

$(1, f(1))$

Find the tangent line to the graph of $f(x) = \frac{\overbrace{3x^2 + x - 2}^{f}}{\underbrace{4x^3 + 1}_{g}}$ at $x = 1$.

Slope:

$$= \frac{(6x+1)(4x^3+1) - (12x^2)(3x^2+x-2)}{(4x^3+1)^2}$$

$$\xrightarrow{x=1} = \frac{(\cancel{6+1}^{\,7})(\cancel{4+1}^{\,5}) - (12)(\cancel{3+1-2}^{\,2})}{(4+1)^2} = \frac{35-24}{25} = \boxed{\frac{11}{25}} \checkmark \text{ slope}$$

$$y - y_1 = m(x - x_1) \Rightarrow y - \frac{2}{5} = \frac{11}{25}(x - 1) \Rightarrow y = \frac{11}{25}x - \frac{11}{25} + \frac{2\cdot 5}{5\cdot 5} \Rightarrow y = \frac{11}{25}x - \frac{1}{2}$$

3.3 SUMMARY

- Two basic rules of differentiation:

$$\text{Product Rule:} \qquad (fg)' = f'g + fg'$$

$$\text{Quotient Rule:} \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

- Remember: The derivative of fg is *not* equal to $f'g'$. Similarly, the derivative of f/g is *not* equal to f'/g'.

$$(f+g)' = f' + g'$$

$$(f-g)' = f' - g'$$

$$(f \cdot g)' \neq f' \cdot g'$$ 

$$(f \cdot g)' = f'g + g'f$$

$$\left(\frac{f}{g}\right)' \neq \frac{f'}{g'}$$  $$\longrightarrow \left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

Use the Product Rule to calculate the derivative.

$$f(x) = \overbrace{\left(6x^2 + 5\right)}^{f} \overbrace{e^x}^{g}$$

$(f \cdot g)' = f'g + g'f$

$= (12x)\, e^x + e^x \cdot (6x^2 + 5)$

$= e^x (12x + 6x^2 + 5)$

Calculate the derivative for $f(x) = e^x\left(x^2 + 3\right)(x + 8)$.

$f'g + g'f$

$$f(x) = \underbrace{e^x}_{f} \cdot \underbrace{\left(x^3 + 8x^2 + 3x + 24\right)}_{g}$$

$$f'(x) = e^x\left(x^3 + 8x^2 + 3x + 24\right) + \left(3x^2 + 16x + 3\right) \cdot e^x$$

$$= e^x\left(x^3 + 8x^2 + 3x + 24 + 3x^2 + 16x + 3\right)$$

$$\boxed{f'(x) = e^x\left(x^3 + 11x^2 + 19x + 27\right)}$$