6.2 Permutations and Combinations

Four candidates, Zeke, Yung, Xeno, and Wilma, are running for the same office. So that the positions of the names on the ballot will not influence the voters, it is necessary to print ballots with the names listed in every possible order. How many distinct ballots will there be?

$$\mathbf{4 \cdot 3 \cdot 2 \cdot 1 = 24}$$

An ordering of objects, such as the names on the ballot, is called a **permutation.**

Definition 6.2.1 A *permutation* of n distinct elements $x_1, \ldots, x_n$ is an ordering of the n elements $x_1, \ldots, x_n$.

Example 6.2.2 There are six permutations of three elements. If the elements are denoted A, B, C, the six permutations are

ABC, ACB, BAC, BCA, CAB, CBA.

We found that there are 24 ways to order four candidates on a ballot; thus there are 24 permutations of four objects.

Theorem 6.2.3 There are $n!$ permutations of n elements.

Proof:

We use the Multiplication Principle. A permutation of n elements can be constructed in n successive steps.

$$n(n-1)(n-2) \cdot \cdot \cdot 2 \cdot 1 = n!$$

Example 6.2.4 There are

$$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 =$$

3,628,800

permutations of 10 elements.

Example 6.2.5.

How many permutations of the letters *ABCDEF* contain the substring *DEF*?

SOLUTION:

$$4! = 24$$

Example 6.2.6

How many permutations of the letters *ABCDEF* contain the letters *DEF* together in any order?

SOLUTION:

$$6 \cdot 24 = 144$$

Example 6.2.7 In how many ways can six persons be seated around a circular table? If a seating is obtained from another seating by having everyone move *n* seats clockwise, the seatings are considered identical.

SOLUTION:

$$5! = 120$$

The same argument can be used to show that there are $(n-1)!$ ways that *n* persons can be seated around a circular table.

Sometimes we want to consider an ordering of *r* elements selected from *n* available elements. Such an ordering is called an *r*-permutation.

Definition 6.2.8 An ***r-permutation*** of n (distinct) elements $x_1, \ldots, x_n$ is an ordering of an r-element subset of

$$\{x_1, \ldots, x_n\}.$$

The number of ***r*-permutations** of a setof n distinct elements is denoted

P(n, r).

Example 6.2.9 Examples of 2-permutations of a, b, c are ab, ba, and ca.

Theorem 6.2.10 The number of ***r*-permutations** of a set of n distinct objects is

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

$$= \frac{n!}{(n-r)!} \qquad r \leq n.$$

Proof: By the Multiplication Principle, the number of r-permutations of a set of n distinct objects is

$$n(n-1)(n-2)\cdots(n-r+1) =$$

$$\frac{[n(n-1)\cdots(n-r+1)][(n-r)(n-r-1)\cdots 2\cdot 1]}{(n-r)(n-r-1)\cdots 2\cdot 1}$$

$$= \frac{\boldsymbol{n}!}{(\boldsymbol{n}-\boldsymbol{r})!}$$

Example 6.2.11 According to Theorem 6.2.10, the number of 2-permutations of $X = \{a, b, c\}$ is

$$P(3, 2) = 3 \cdot 2 = 6.$$

These six 2-permutations are

ab, ac, ba, bc, ca, cb.

Example 6.2.12 In how many ways can we select a chairperson, vice-chairperson, secretary, and treasurer from a group of 10 persons?

SOLUTION

$$P(10, 4) = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$$

Example 6.2.13 In how many ways can seven distinct Martians and five distinct Jovians wait in line if no two Jovians stand together?

SOLUTION:

$$7! = 5040 \text{ ways}$$

$$-M1-M2-M3-M4-M5-M6-M7- \,.$$

$$P(8, 5) = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6720 \text{ ways}$$

$$5040 \cdot 6720 = 33{,}868{,}800$$

We turn next to combinations. A selection of objects without regard to order is called a **combination.**

Definition 6.2.14 Given a set

$$\mathbf{X} = \{\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}\}$$

containing *n* (distinct) elements,

(a) An *r-combination* of *X* is an unordered selection of *r*-elements of *X* (i.e., an *r*-element subset of *X*).

(b) The number of *r*-combinations of a set of *n* distinct elements is denoted

C(n, r) or $\binom{n}{r}$

Example 6.2.15 A group of five students, Mary, Boris, Rosa, Ahmad, and Nguyen, has decided to talk with the Mathematics Department chairperson about having the Mathematics Department offer more courses in discrete mathematics. The chairperson has said that she will speak with three of the students. In how many ways can these five students choose three of their group to talk with the chairperson?

SOLUTION:

***MBR*, *MBA*, *MRA*, *BRA*, *MBN*, *MRN*, *BRN*, *MAN*, *BAN*, *RAN*.**

$$C(5, 3) = 10. \qquad \blacksquare$$

We next derive a formula for *C(n, r)* by counting the number of *r* -permutations of an *n*-element set in two ways. The first way simply uses the formula *P(n, r)*. **The second** way of

counting the number of *r*-permutations of an *n*-element set involves *C(n, r)*. Equating the two values will enable us to derive a formula for *C(n, r)*.

We can construct *r*-permutations of an *n*-element set *X* in two successive steps: **First**, select an *r*-combination of *X* (an unordered subset of *r* items); second, order it. **For example**, to construct a 2-permutation of {*a*, *b*, *c*, *d*}, we can first select a 2-combination and then order it.

{*a*, *b*}
ab ba
{*a*, *c*}

ac ca

{*a*, *d*}

ad da

{*b*, *c*}

bc cb

{*b*, *d*}

bd db

{*c*, *d*}

cd dc

2-permutations of $\{a, b, c, d\}$.

$$\mathbf{P(n, r) = C(n, r)r!.}$$

$$\mathbf{C(n, r) = \frac{P(n,r)}{r!}}$$

Theorem 6.2.16 The number of *r*-combinations of a set of *n* distinct objects is

$$\mathbf{C(n, r)} = \frac{\mathbf{P(n,r)}}{\mathbf{r!}} = \frac{n(n-1)\ \cdot\ \cdot\ \cdot\ (n-r+1)}{r!}$$

$$= \frac{n!}{(n-r)!\,r!} \qquad r \le n.$$

Proof The proof of the first equation is given before the statement of the theorem.

The other forms of the equation follow from Theorem 6.2.10.

Example 6.2.17 In how many ways can we select a committee of three from a group of 10 distinct persons?

SOLUTION Since a committee is an unordered group of people, the answer is

$$C(10, 3) = \frac{10 \cdot 9 \cdot 8}{3!} = 120.$$

Example 6.2.18 In how many ways can we select a committee of two women and three men from a group of five distinct women and six distinct men?

SOLUTION:

$$C(5, 2) = 10 \text{ ways}$$

$$C(6, 3) = 20 \text{ ways}$$

the total number of committees is $\mathbf{10 \cdot 20 = 200}$.

Example 6.2.19 How many eight-bit strings contain exactly four 1's?

SOLUTION:

$$\boldsymbol{C(8,\ 4) = \ 70 \text{ ways}}$$

Example 6.2.20 An ordinary deck of 52 cards consists of four suits, clubs, diamonds, hearts, spades of 13 denominations each ace, 2–10, jack, queen, king.

(a) **How many (unordered) five-card poker hands, selected from an ordinary 52-card deck, are there?**

(b) **How many poker hands contain cards all of the same suit?**

(c) **How many poker hands contain three cards of one denomination and two cards of a second denomination?**

SOLUTION:

(a) $\mathbf{C(52, 5) = 2{,}598{,}960.}$

(b) $\mathbf{4 \cdot \mathit{C}(13, 5) = 5148}$

(c) $\mathbf{13 \cdot 12 \cdot \mathit{C}(4, 3) \cdot \mathit{C}(4, 2) = 3744}$.

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