Solve each of the following equations.

(a) $x^{2/3} - 2x^{1/3} = 0$

(b) $\ln x - e^x \ln x = 0$

(c) $2\sin x = \sqrt{3}$ on $[0, 2\pi]$

(d) $\sin x \cos x + \cos x = 0$ on $[0, 2\pi]$

(e) $e^{2x} - 2e^x = -1$

(f) $3(x+2)^2(3x-4)^{-2} + (x+2)^3(-2)(3x-4)^{-3}(3) = 0$

(g) $3(x+1)^2(2x-1)^{1/2} - (x+1)^3\left(\frac{1}{2}\right)(2x-1)^{-1/2}(2) = 0$

Solutions on the next page

(a) $x^{2/3} - 2x^{1/3} = 0$ Answer: $x = 0,\ x = 8$

First we factor, noticing that $x^{1/3}$ is the smallest power of x contained in both terms, and $x^{1/3} \cdot x^{1/3} = x^{2/3}$.

$$x^{2/3} - 2x^{1/3} = x^{1/3}(x^{1/3} - 2) = 0$$

Now we apply the zero-product property:

$$x^{1/3} = 0 \qquad\qquad (x^{1/3} - 2) = 0$$

At this point, it may be easier to think of these as roots:

$$\sqrt[3]{x} = 0 \qquad\qquad (\sqrt[3]{x} - 2) = 0$$

The first one is easy to solve by cubing both sides, giving us our first solution:

$$\sqrt[3]{x} = 0$$

$$x = 0^3 = 0$$

For the second one, we first move the 2 to the other side, and then again cube both sides.

$$\sqrt[3]{x} - 2 = 0$$

$$\sqrt[3]{x} = 2$$

$$x = 2^3 = 8$$

(b) $\ln x - e^x \ln x = 0$; Answer: $x = 1$

First we factor, since each term has a factor of $\ln x$:

$$\ln x - e^x \ln x = (\ln x)(1 - e^x) = 0$$

Now we apply the zero-product property:

$$\ln x = 0 \qquad\qquad 1 - e^x = 0$$

For the first equation, we know that $x = 1$ is the only solution (because $e^0 = 1$).
For the second equation, we rearrange:

$$1 - e^x = 0$$

$$e^x = 1$$

But here the solution is $x = 0$, and $x = 0$ is outside the domain of the natural log function, so it isn't a solution. That leaves us with $x = 1$ as the only solution.

(c) $2 \sin x = \sqrt{3}$ on $[0, 2\pi]$ Answer: $x = \pi/3,\ x = 2\pi/3$

Here we start by isolating the trig function:

$$\sin x = \frac{\sqrt{3}}{2}$$

And now we want to know which angles between 0 and 2π have a sine value of $\frac{\sqrt{3}}{2}$. Certainly $\pi/3$ is one, in QI. The other quadrant where sine if positive is QII. When the reference angle in QII is $\pi/3$ we have a solution, and $2\pi/3$ is the angle that leads to that reference angle.

(d) $\sin x \cos x + \cos x = 0$ on $[0, 2\pi]$ Answer: $x = \pi/2$ and $x = 3\pi/2$

Start by factoring:

$$\sin x \cos x + \cos x = (\cos x)(\sin x + 1) = 0$$

Now apply the zero-product property:

$$\cos x = 0 \qquad\qquad \sin x + 1 = 0$$

For the first equation, thinking about the graph of $y = \cos x$, the x-intercepts between 0 and 2π are at $\pi/2$ and $3\pi/2$, giving the solutions $x = \pi/2$ and $x = 3\pi/2$.
For the second equation, we isolate the trig function:

$$\sin x = -1$$

Again thinking about the graph of $y = \sin x$, we see that $\sin x = -1$ when $x = 3\pi/2$; we already found that solution.
Hence the solutions are $x = \pi/2$ and $x = 3\pi/2$.

(e) $e^{2x} - 2e^x = -1$ Answer: $x = 0$

The key here is to notice that the equation is in quadratic form once we move the -1 to the other side, because $e^{2x} = (e^x)^2$:

$$e^{2x} - 2e^x + 1 = 0$$
$$(e^x)^2 - 2e^x + 1 = 0$$

Now we can factor this equation (using the fact that $u^2 - 2u + 1 = (u - 1)(u - 1)$):

$$(e^x - 1)(e^x - 1) = 0$$

The zero-product property now yields

$$e^x - 1 = 0$$

and $e^x = 1$ when $x = 0$.
Note that even if you see from the beginning that $x = -0$ is a solution, simply verifying that isn't sufficient, because you need to show there are no OTHER solutions.

(f) $3(x + 2)^2(3x - 4)^{-2} + (x + 2)^3(-2)(3x - 4)^{-3}(3) = 0$ Answer: $x = -2,\ x = 8$

From the two monster terms, factor out a 3, the smallest power of $x + 2$ (which is 2) and the smallest power of $3x - 4$ (which is -3):

$$3(x + 2)^2(3x - 4)^{-3}\left[(3x - 4)^1 + (-2)(x + 2)^1\right] = 0$$

Expand what is in brackets:

$$3(x + 2)^2(3x - 4)^{-3}\left[(3x - 4 - 2x - 4\right] = 0$$

Combine:

$$3(x + 2)^2(3x - 4)^{-3}(x - 8) = 0$$

Rewrite as a fraction:

$$\frac{3(x + 2)^2(x - 8)}{(3x - 4)^3} = 0$$

A fraction is equal to 0 when its numerator is equal to 0:

$$3(x+2)^2(x-8) = 0$$

Now by the zero-product property,

$$x+2=0 \qquad\qquad x-8=0$$

So $x=-2$ and $x=8$. (Note that both of those values ARE in the domain of the function.)

(g) $3(x+1)^2(2x-1)^{1/2} - (x+1)^3\left(\frac{1}{2}\right)(2x-1)^{-1/2}(2) = 0$ Answer: No solutions

First notice that the 2 and the 1/2 in the second term multiply to 1, so they simplify away. Now from the two monster terms, factor out the smallest power of $x+1$ (which is 2) and the smallest power of $2x-1$ (which is $-1/2$):

$$(x+1)^2(2x-1)^{-1/2}\left[3(2x-1)^1 + (x+1)\right] = 0$$

Expand what is in brackets:

$$(x+1)^2(2x-1)^{-1/2}\left[(6x-3+x+1\right] = 0$$

Combine:

$$(x+1)^2(2x-1)^{-1/2}(7x-2) = 0$$

Rewrite as a fraction and root:

$$\frac{(x+1)^2(7x-2)}{\sqrt{2x-1}} = 0$$

A fraction is equal to 0 when its numerator is equal to 0:

$$(x+1)^2(7x-2) = 0$$

Now by the zero-product property,

$$x+1=0 \qquad\qquad 7x-2=0$$

So $x=-1$ and $x=2/7$. However, the domain of the function requires $2x-1>0$ (strict inequality because the denominator can't be 0), so $x>1/2$. Since neither of the solutions satisfy this inequality, there are no solutions to the equation.