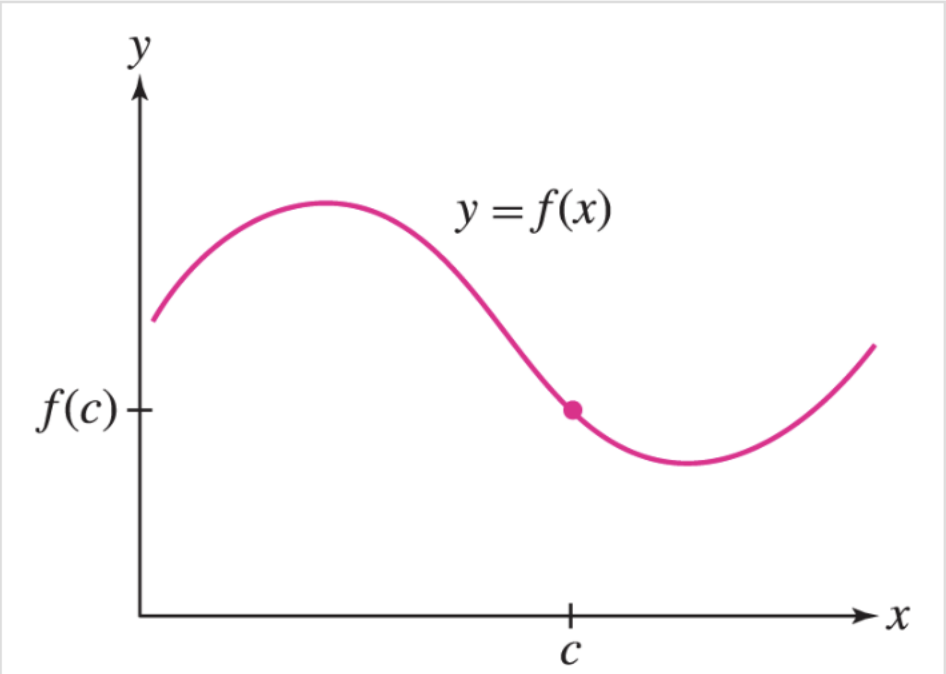
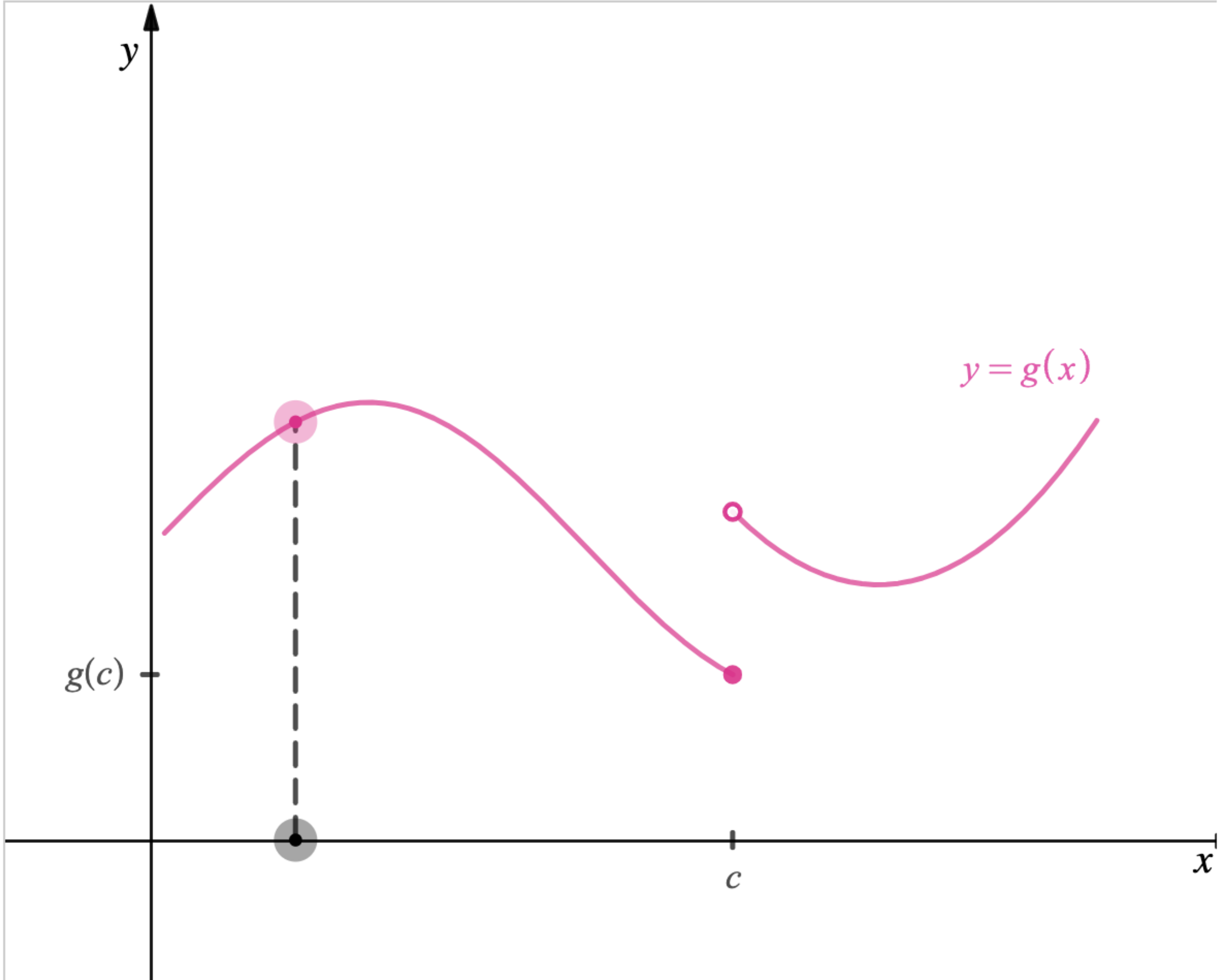
2.4 Limits and Continuity

In everyday speech, the word “continuous” means having no breaks or interruptions. In calculus, continuity is used to describe functions whose graphs have no breaks. If we imagine the graph of a function f as a wavy metal wire, then f is continuous if its graph consists of a single piece of wire as in Figure 1.



A break in the wire as in Figure 2 is called a **discontinuity**. Observe in Figure 2 that the break in the graph occurs because the left- and right-hand limits as x approaches c are not equal and thus $\lim_{x \to c} g(x)$ does not exist. By contrast, in Figure 1, $\lim_{x \to c} f(x)$ exists and is equal to the function value $f(c)$. This suggests the following definition of continuity in terms of limits.



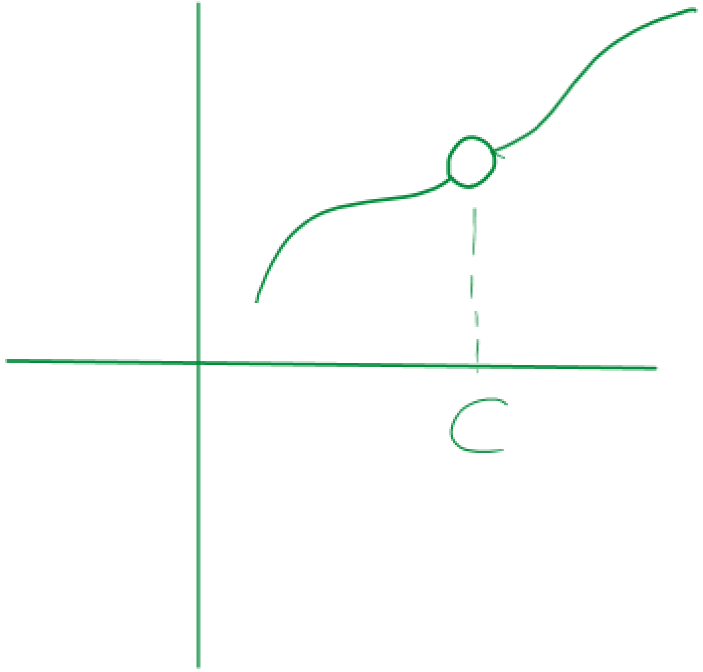
DEFINITION

Continuity at a Point

Assume that $f(x)$ is defined on an open interval containing $x = c$. Then f is continuous at $x = c$ if

$$\lim_{x \to c} f(x) = f(c)$$

If the limit does not exist, or if it exists but is not equal to $f(c)$, we say that f has a **discontinuity** (or is discontinuous) at $x = c$.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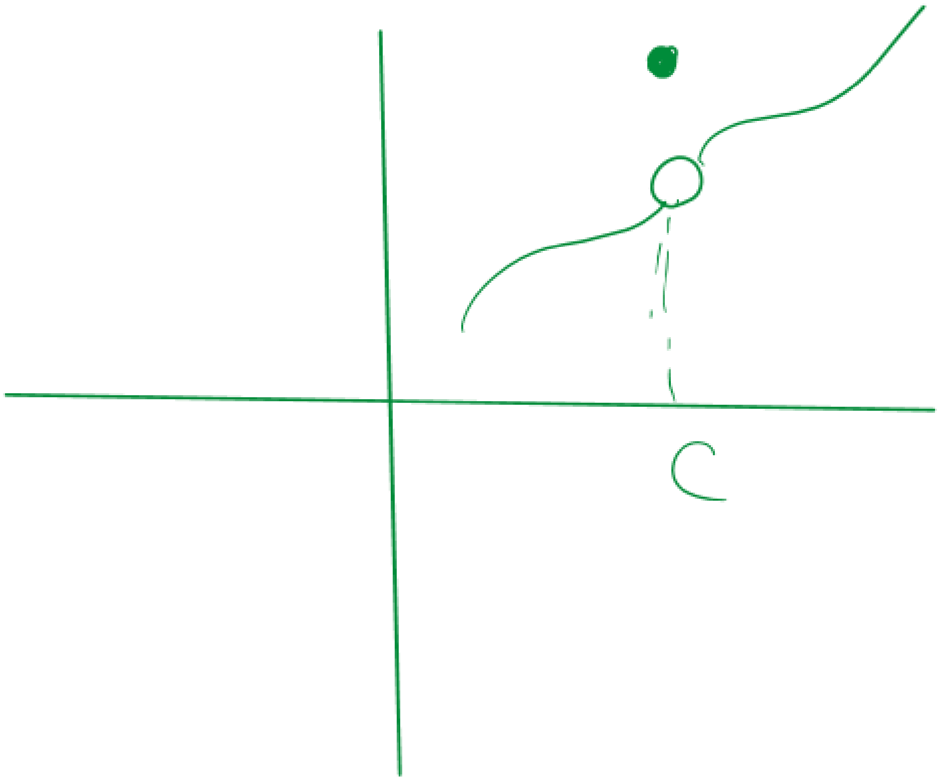
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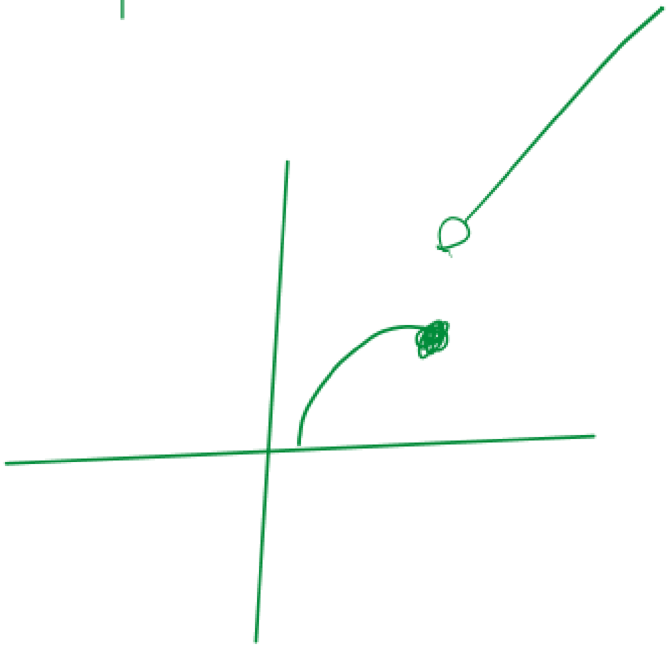
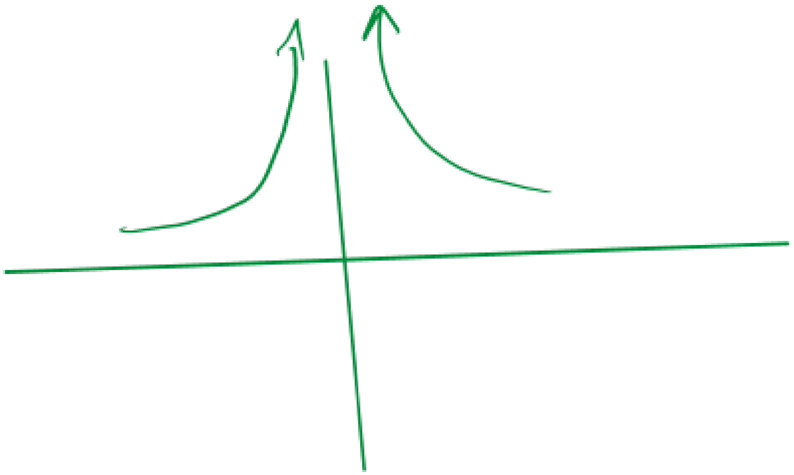




Note that for f to be continuous at c, three conditions must hold:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. They are equal.

important!

A function f may be continuous at some points and discontinuous at others. If f is continuous at all points in its domain, then f is simply called continuous.

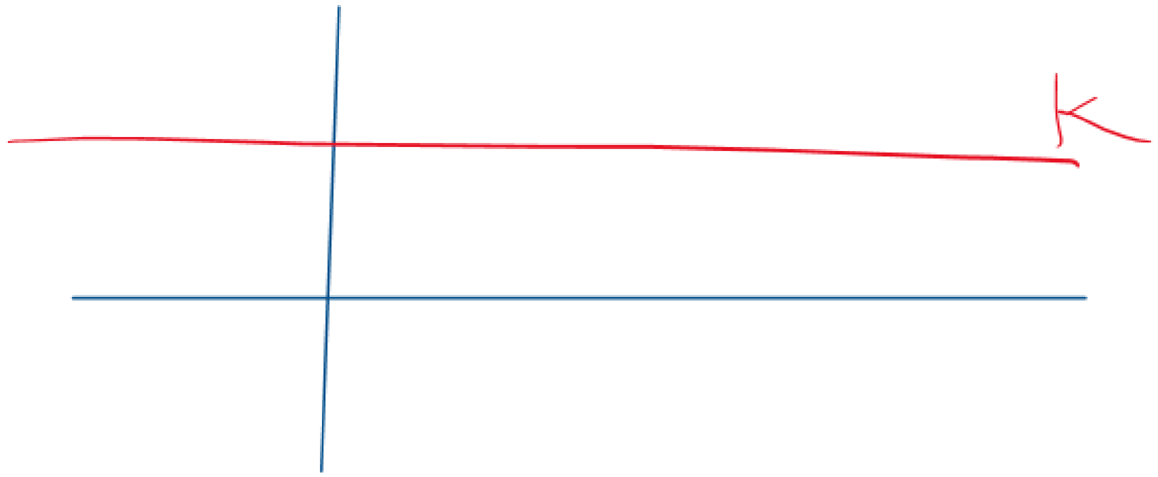
EXAMPLE 1

Show that the following functions are continuous:

a. $f(x) = k \qquad (k \text{ any constant})$

b. $g(x) = x^n \qquad (n \text{ a whole number})$

a) $f(x) = K$ Cont at $x = c$?



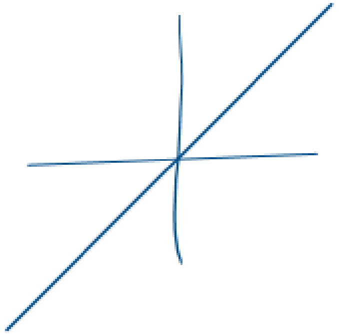
1) $f(c) = K$

2) $\lim_{x \to c} f(x) = \lim_{x \to c} K = K$

3) $\lim_{x \to c} f(x) = f(c) \longrightarrow$ f is Cont. at $x = c$ ✓ ☺

b) $g(x) = x^n$

$n=1 \Rightarrow g(x)=x$



1) $g(c)=c$

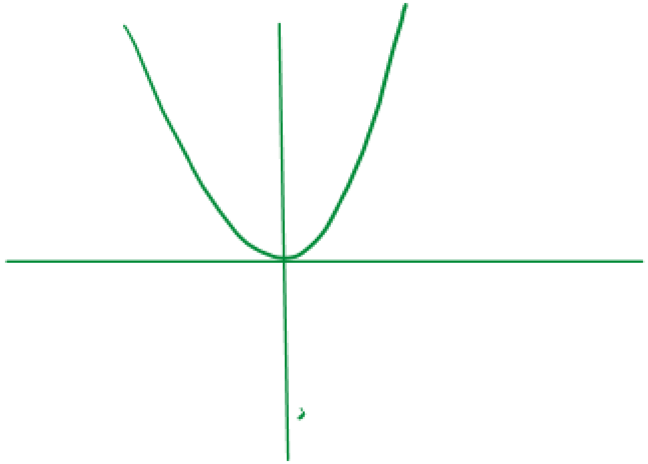
2) $\lim\limits_{x\to c} g(x) = \lim\limits_{x\to c} x = c$

3) $\lim\limits_{x\to c} g(x) = g(c)$

$\Rightarrow g$ is cont. at $x=c$ ☺

$n=2$

$g(x) = x^2$



1) $g(c) = c^2$

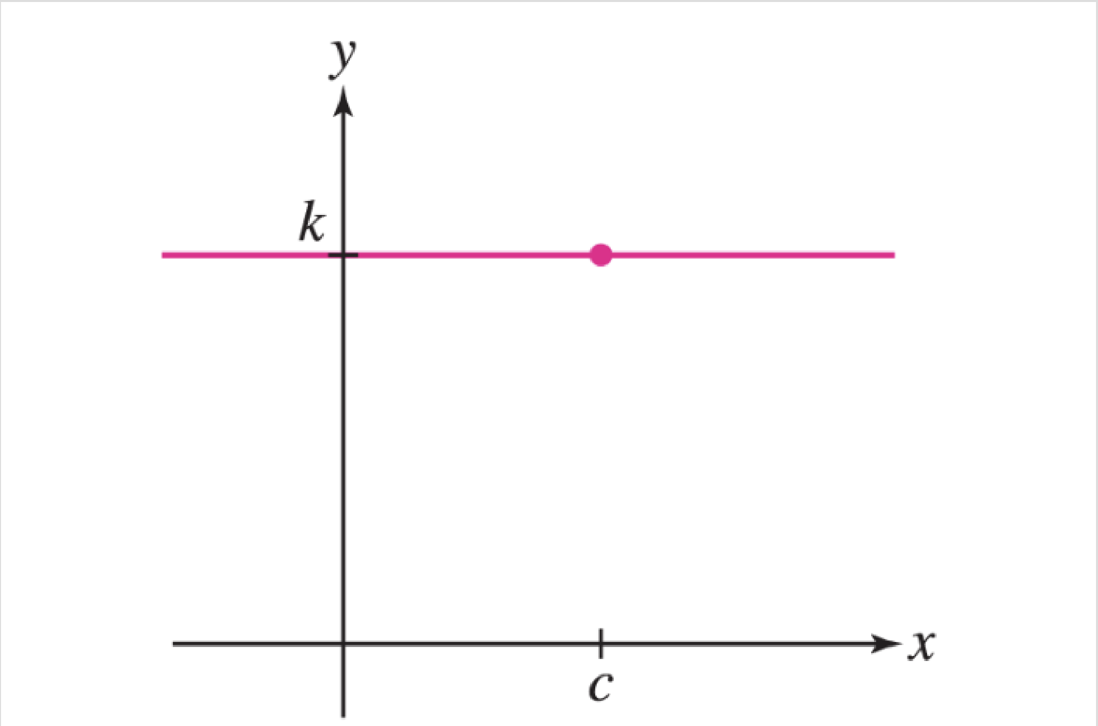
2) $\lim\limits_{x\to c} g(x) = \lim\limits_{x\to c} x^2 = c^2$

3) $\lim\limits_{x\to c} g(x) = g(c)$

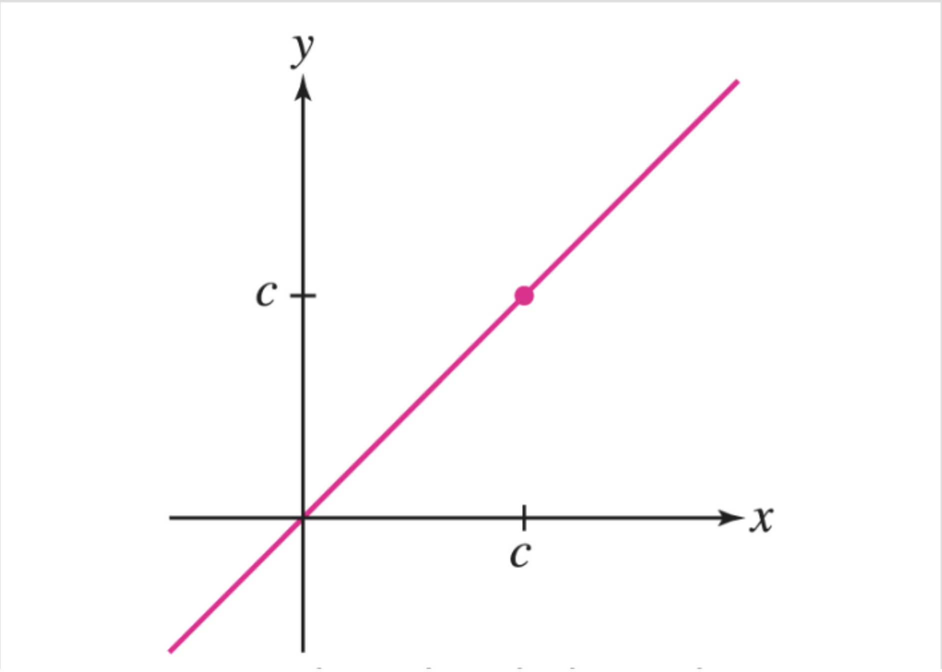
$\Rightarrow g$ is cont. at $x=c$ ☺

Solution

a. We have $\lim_{x \to c} f(x) = \lim_{x \to c} k = k$ and $f(c) = k$. The limit exists and is equal to the function value for all c, so f is continuous (Figure 3).



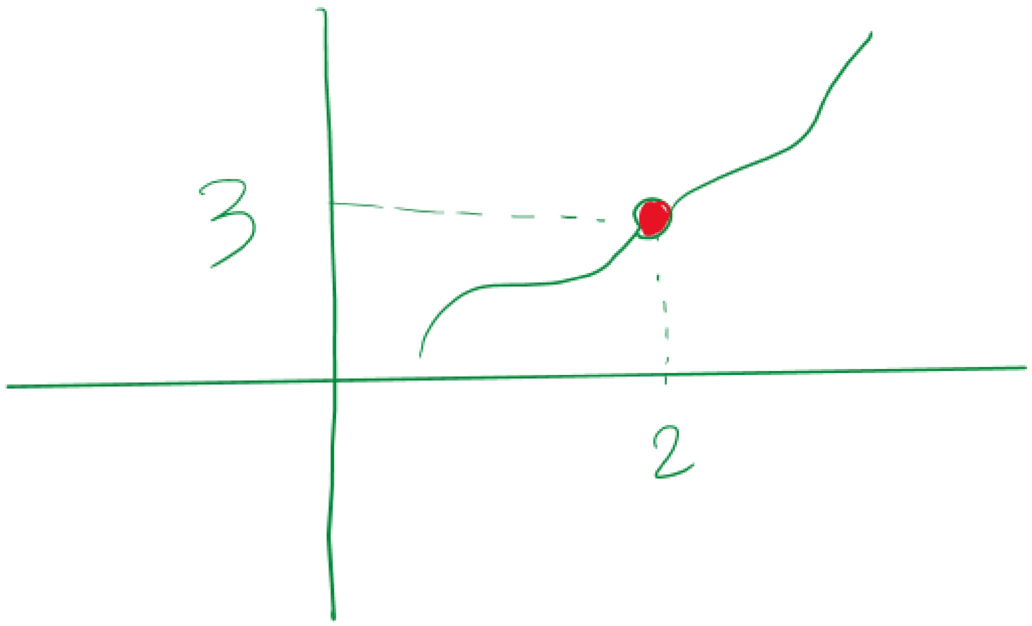
b. By Eq. (1) in Section 2.3, $\lim_{x \to c} g(x) = \lim_{x \to c} x^n = c^n$ for all c. Also $g(c) = c^n$, so again, the limit exists and is equal to the function value. Therefore, g is continuous. (Figure 4 illustrates the case $n = 1$.)

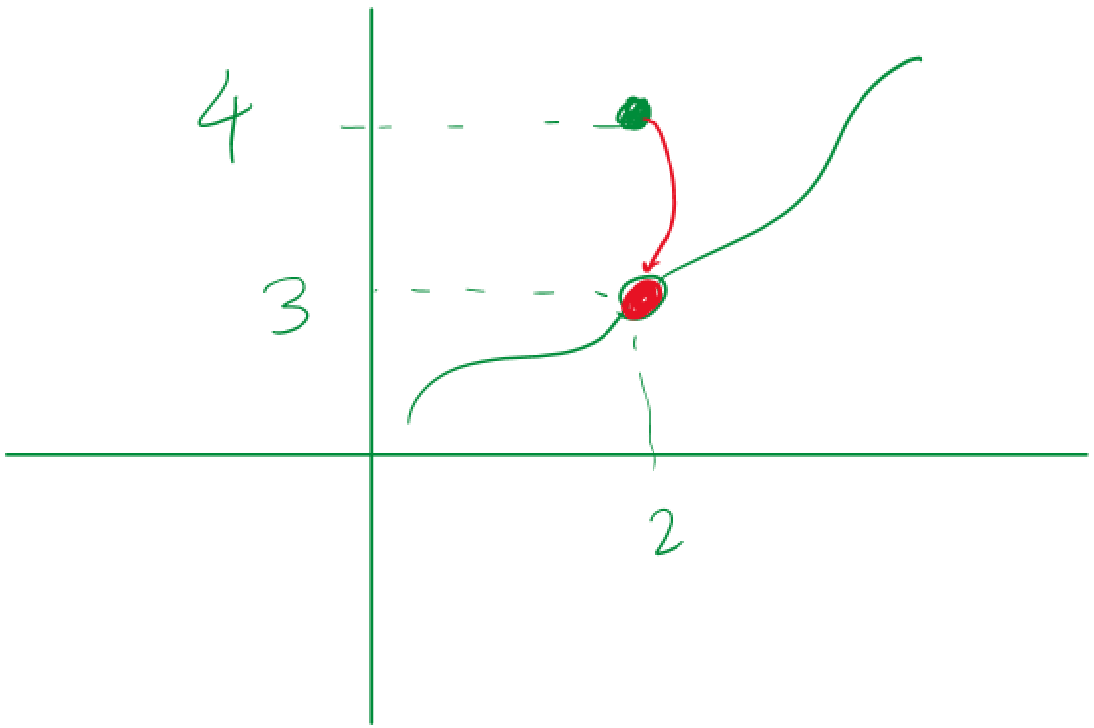


Examples of Discontinuities

To understand continuity better, let's consider some ways in which a function can fail to be continuous. Keep in mind that continuity at a point $x = c$ requires that:

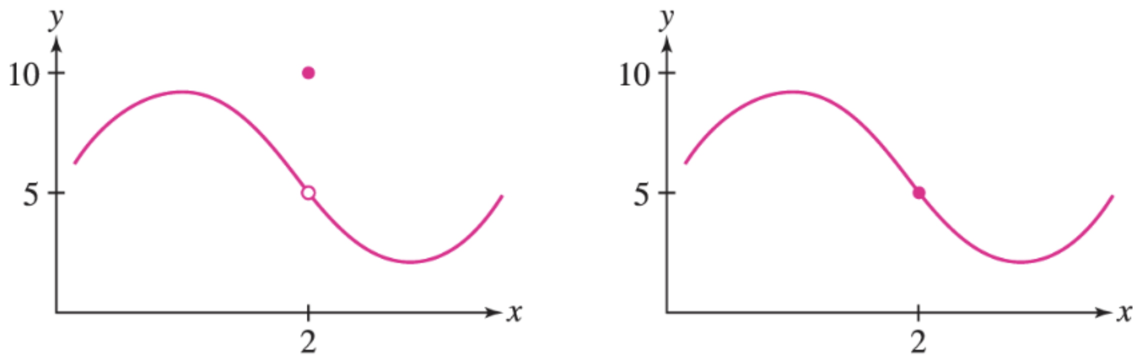
1. $f(c)$ is defined.
2. $\lim\limits_{x \to c} f(x)$ exists.
3. They are equal.


3
2


4
3
2

If $\lim_{x \to c} f(x)$ exists, but either the limit is not equal to $f(c)$, or $f(c)$ is not defined, then we say that f has a **removable discontinuity at $x = c$**. The function in Figure 5(A) has a removable discontinuity at $c = 2$ because

$$\underbrace{f(2) = 10 \qquad \text{but} \qquad \lim_{x \to 2} f(x) = 5}_{\text{Limit exists but is not equal to function value}}$$



(A) Removable discontinuity at $x = 2$ (B) Function redefined at $x = 2$

Removable discontinuities are mild in the following sense: We can make f continuous at $x = c$ by redefining $f(c)$ [in the case $\lim_{x \to c} f(x) \neq f(c)$] or defining $f(c)$ [in the case $f(c)$ is not defined] so that $f(c) = \lim_{x \to c} f(x)$. In Figure 5(B), $f(2)$ has been redefined as $f(2) = 5$, and this makes f continuous at $x = 2$.

Ex: $f(x) = \dfrac{x^3 - 8}{x - 2 \neq 0}$

a) Cont. at $x = 2$?

b) Type of discont.?

c) How you can make cont. if possible?

a)

1) $f(2)$ $\quad x - 2 \neq 0$

undefined $\quad x \neq 2$

$D_f = (-\infty, 2) \cup (2, \infty)$

f is Not cont. at $x = 2$.

b) Removable

c) $\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2} \dfrac{x^3 - 8}{x - 2} = \dfrac{0}{0}$ indeterminate $\to \lim\limits_{x \to 2} \dfrac{\cancel{(x-2)}(x^2 + 2x + 4)}{\cancel{x - 2}} = 2^2 + (2 \cdot 2) + 4$

$= 12$

if you redefine $f(c) = 12$ then f Cont at $x = 2$ ✓

EXAMPLE 2

Show that $g\left(x\right)=\dfrac{x^3-8}{x-2}$ has a removable discontinuity at $x=2$. How should $g\left(2\right)$ be defined so that g is continuous at $x=2$?

Solution

First note that g is not defined at $x=2$ since evaluating g at 2 involves division by 0. Also,

$$\lim_{x\to 2} g\left(x\right)=\lim_{x\to 2}\frac{x^3-8}{x-2}=\lim_{x\to 2}\frac{\left(x-2\right)\left(x^2+2x+4\right)}{x-2}=\lim_{x\to 2}\left(x^2+2x+4\right)=12$$

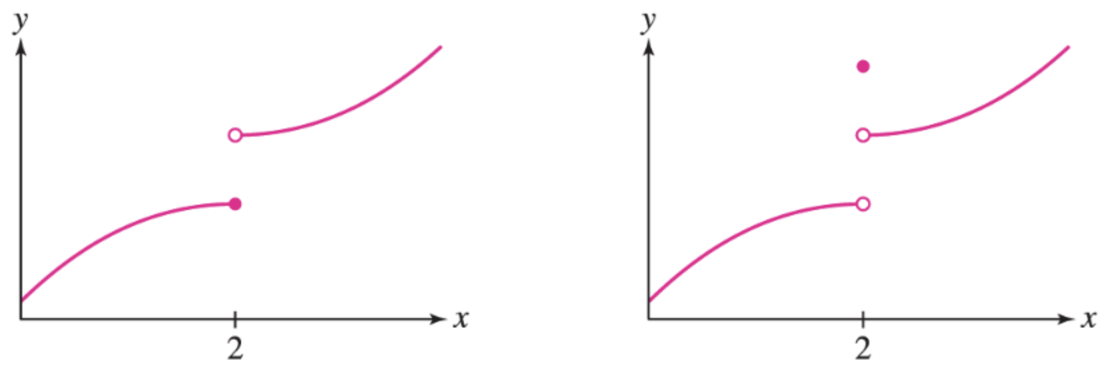
Solution

First note that g is not defined at $x = 2$ since evaluating g at 2 involves division by 0. Also,

$$\lim_{x\to 2} g(x) = \lim_{x\to 2} \frac{x^3 - 8}{x - 2} = \lim_{x\to 2} \frac{(x-2)(x^2 + 2x + 4)}{x - 2} = \lim_{x\to 2} (x^2 + 2x + 4) = 12$$

where the Basic Limit Laws are used to determine the value of the limit. Since $\lim_{x\to 2} g(x)$ exists, but $g(2)$ is not defined, g has a removable discontinuity at $x = 2$. If we define $g(2) = 12$, then g would be continuous at $x = 2$.

A worse type of discontinuity is a **jump discontinuity**, which occurs if the one-sided limits $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist but are not equal. In this case f is not continuous at c because $\lim_{x\to c} f(x)$ does not exist. Figure 6 shows two functions with jump discontinuities at $c = 2$. Unlike the removable case, we cannot make f continuous simply by redefining f at the single point c.



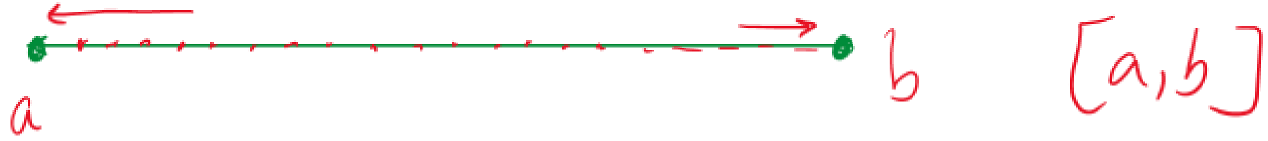
(A) (B)

DEFINITION

One-Sided Continuity

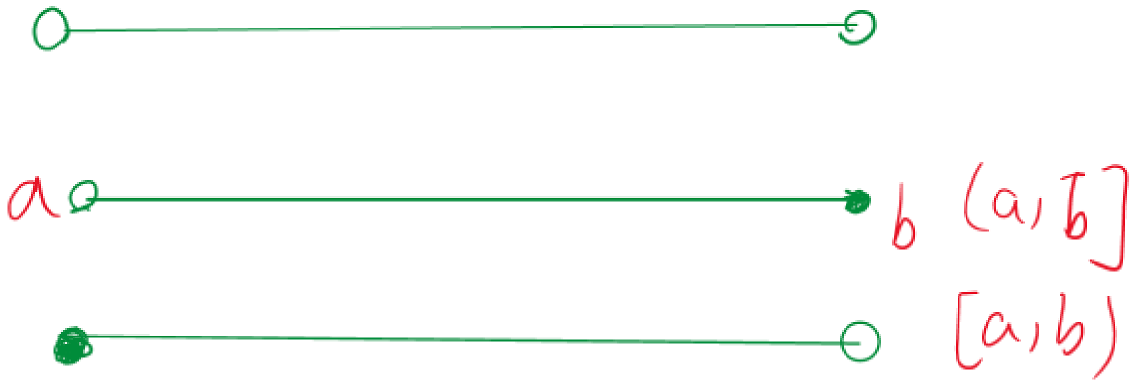
A function f is called

- Left-continuous at $x = c$ if $\lim\limits_{x \to c^-} f(x) = f(c)$
- Right-continuous at $x = c$ if $\lim\limits_{x \to c^+} f(x) = f(c)$



DEFINITION

Continuity on an Interval I



Assume that I is an interval in the form (a, b), $[a, b)$, $(a, b]$, or $[a, b]$. Then f is **continuous on** I if f is continuous at each point in (a, b), f is right-continuous at a (if a is in I), and f is left-continuous at b (if b is in I).

EXAMPLE 3

Piecewise-Defined Function

Discuss the continuity of

$$F(x) = \begin{cases} x & \text{for } x < 1 \\ 3 & \text{for } 1 \le x \le 3 \\ x & \text{for } x > 3 \end{cases}$$

$x = 1$

1) $f(1) = 3$

2) $\lim_{x \to 1} f(x) = DNE$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 3 = 3$

$1 \neq 3 \Rightarrow DNE \Rightarrow f$ is discont. at $x = 1$ ☹

$x = 3$

1) $f(3) = 3$

2) $\lim_{x \to 3} f(x) = 3$

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} 3 = 3$

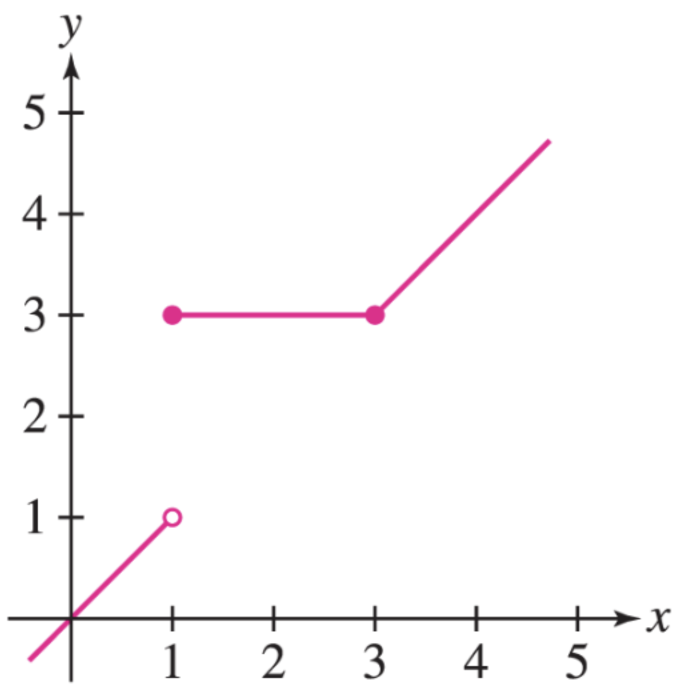
$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} x = 3$

3) $\lim_{x \to 3} f(x) = f(3) \Rightarrow$

f is cont. at $x = 3$ ☺

Solution

The functions $f(x) = x$ and $g(x) = 3$ are continuous, so F is also continuous, except possibly at the transition points $x = 1$ and $x = 3$, where the formula for $F(x)$ changes (Figure 7).



- At $x = 1,$ the one-sided limits exist but are not equal:

$$\lim_{x \to 1^-} F(x) = \lim_{x \to 1^-} x = 1, \qquad \lim_{x \to 1^+} F(x) = \lim_{x \to 1^+} 3 = 3$$

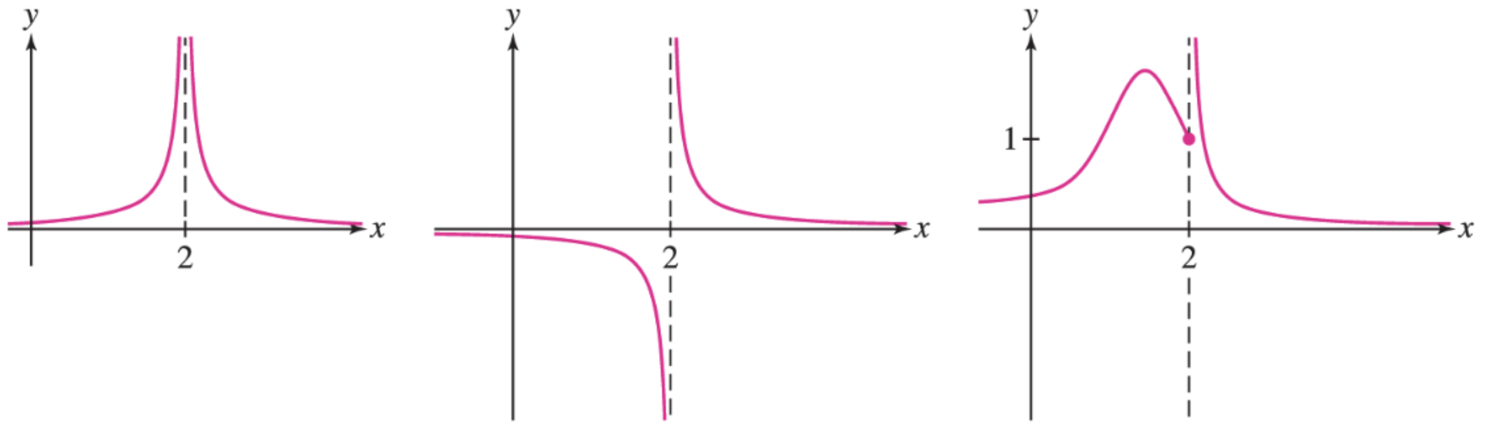
  Thus, F has a jump discontinuity at $x = 1.$ However, the right-hand limit is equal to the function value $F(1) = 3,$ so F is *right-continuous* at $x = 1.$

- At $x = 3,$ the left- and right-hand limits exist and both are equal to $F(3),$ so F is *continuous* at $x = 3$:

$$\lim_{x \to 3^-} F(x) = \lim_{x \to 3^-} 3 = 3, \qquad \lim_{x \to 3^+} F(x) = \lim_{x \to 3^+} x = 3$$

We say that f has an infinite discontinuity at $x = c$ if one or both of the one-sided limits are infinite [even if $f(x)$ itself is not defined at $x = c$]. Like with a jump discontinuity, in this case f is not continuous at c because $\lim\limits_{x \to c} f(x)$ does not exist.

Figure 8 illustrates three types of infinite discontinuities occurring at $x = 2$. Notice that $x = 2$ does not belong to the domain of the function in cases (A) and (B).

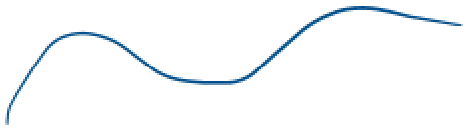


(A) (B) (C)

THEOREM 1

$$f(x) = 5x^4 - 3x^2 + 7x - \sqrt{10}$$

Basic Laws of Continuity



If f and g are continuous at $x = c$, then the following functions are also continuous at $x = c$:

i. $f + g$ and $f - g$

ii. kf for any constant k

iii. fg

iv. f/g if $g(c) \neq 0$

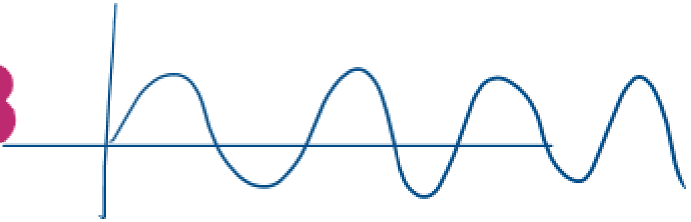
THEOREM 2

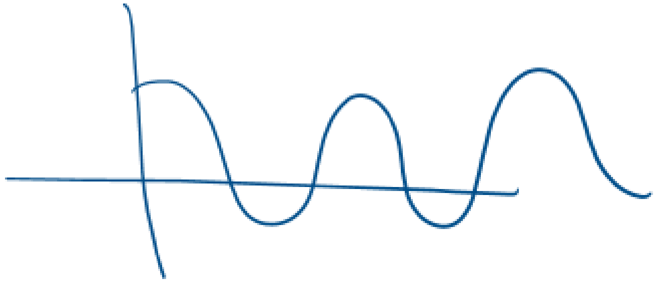
Continuity of Polynomial and Rational Functions

Let P and Q be polynomials. Then:

- P and Q are continuous on the real line.
- P/Q is continuous on its domain [at all values $x = c$ such that $Q(c) \neq 0$].

THEOREM 3





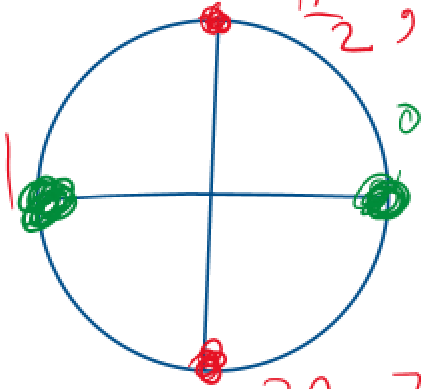
Continuity of Some Basic Functions

- $y = x^{1/n}$ is continuous on its domain for n a natural number.
- $y = \sin\ x$ and $y = \cos\ x$ are continuous on the real line.
- $y = b^x$ is continuous on the real line (for $b > 0,\ b \neq 1$).
- $y = \log_b x$ is continuous for $x > 0$ (for $b > 0,\ b \neq 1$).

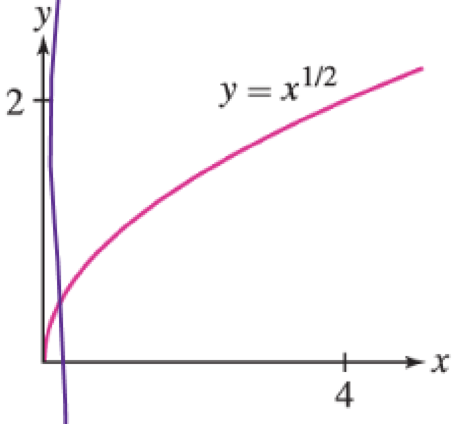
REMINDER

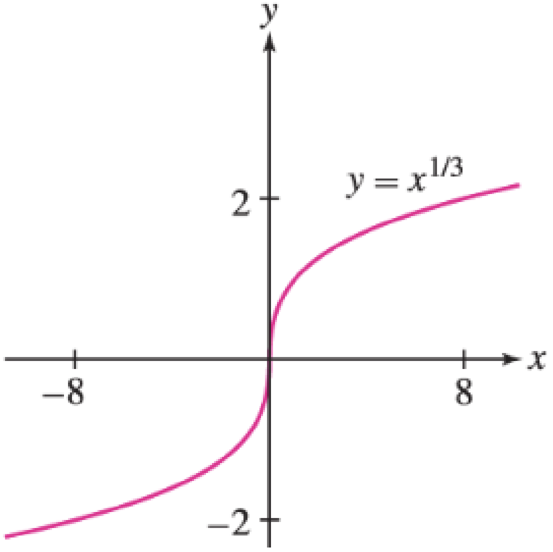
The domain of $y = x^{1/n}$ is the real line if n is odd and the half-line $[0, \infty)$ if n is even.

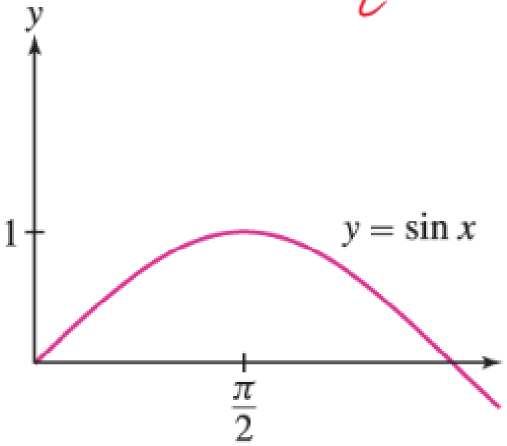
$$\tan x = \frac{\sin x}{\cos x \neq 0}$$

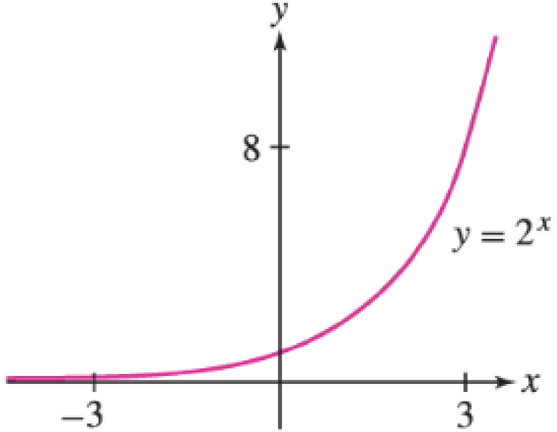


$$y = \cot x = \frac{\cos x}{\sin x \neq 0}$$



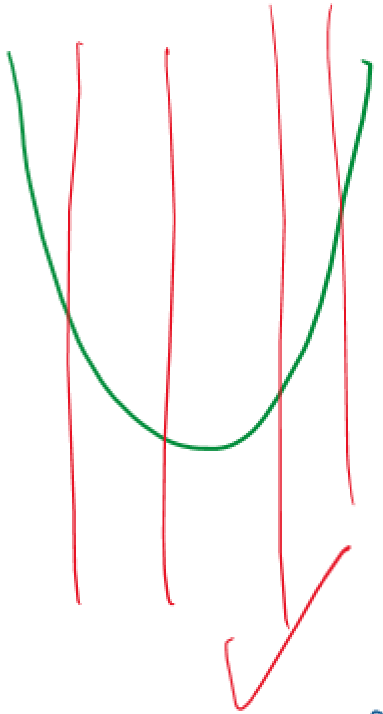


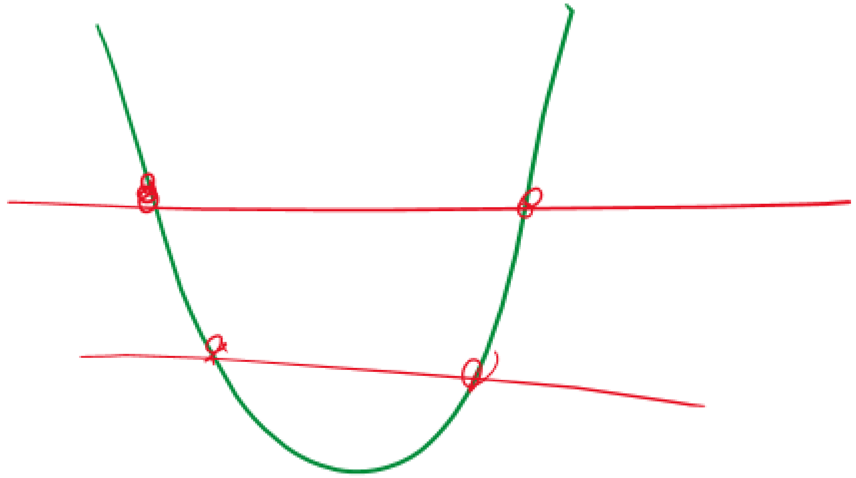




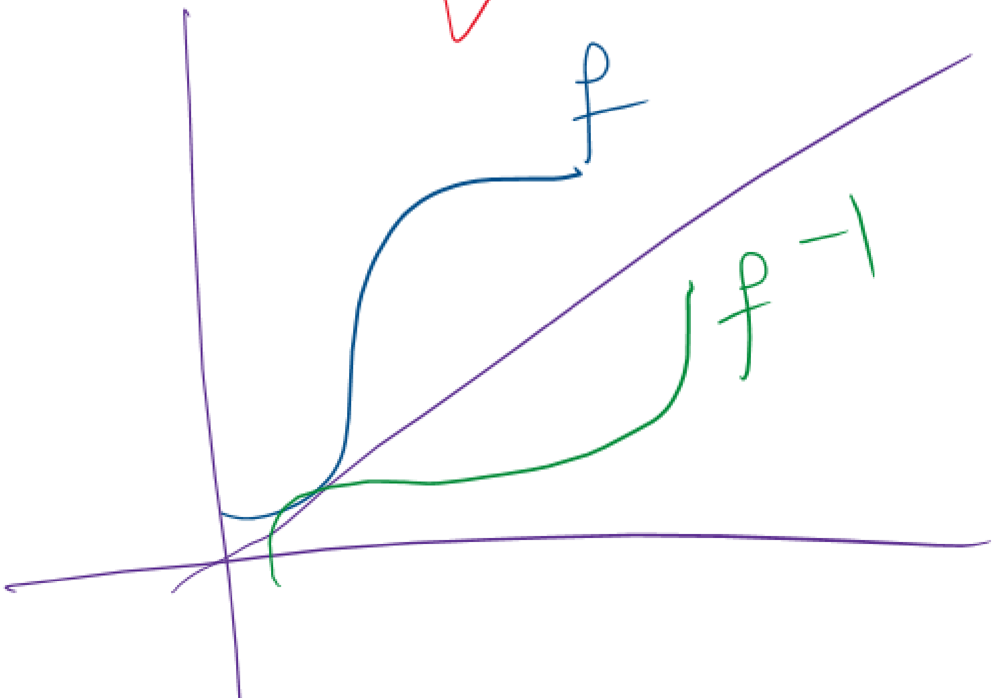
$$\sec x = \frac{1}{\cos x \neq 0}$$

$$\csc x = \frac{1}{\sin x \neq 0}$$





Not one-to-one

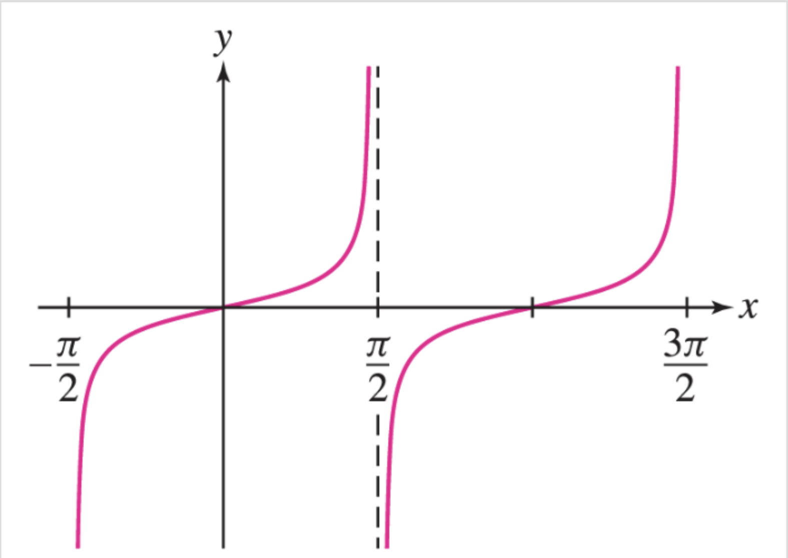


Because $f(x) = \sin\ x$ and $f(x) = \cos\ x$ are continuous, the Continuity Law (iv) for Quotients implies that the other standard trigonometric functions are continuous on their domains, consisting of the values of x where the denominators, in the following quotient expressions for them, are nonzero:

$$\tan\ x = \frac{\sin\ x}{\cos\ x}, \qquad \cot\ x = \frac{\cos\ x}{\sin\ x}, \qquad \sec\ x = \frac{1}{\cos\ x}, \qquad \csc\ x = \frac{1}{\sin\ x}$$

They have infinite discontinuities at points where the denominators are zero. For example, as illustrated in Figure 12, $f(x) = \tan\ x$ has infinite discontinuities at the points

$$x = \pm\frac{\pi}{2}, \quad \pm\frac{3\pi}{2}, \quad \pm\frac{5\pi}{2}, \ldots$$



The next theorem states that the inverse f^{-1} of a continuous function f is continuous. This is to be expected because the graph of f^{-1} is the reflection of the graph of f through the line $y = x$. If the graph of f has "no breaks," the same ought to be true of the graph of f^{-1}.

THEOREM 4

Continuity of the Inverse Function

If f is continuous on an interval I with range R, and if f^{-1} exists, then f^{-1} is continuous with domain R.

Finally, it is important to know that a composition of continuous fu
continuous. The following theorem is proved in Appendix D.

THEOREM 5

Continuity of Composite Functions

If g is continuous at $x = c,$ and f is continuous at $x = g\left(c\right),$ then the
function $F\left(x\right) = f\left(g\left(x\right)\right)$ is continuous at $x = c.$

$$f \circ g = f(g(x))$$

More generally, an **elementary function** is a function that is constructed basic functions using the operations of addition, subtraction, multiplication, and composition. Since the basic functions are continuous (on their domains elementary function is also continuous on its domain by the Laws of Continu example of an elementary function is

$$F(x) = \tan^{-1}\left(\frac{x^2 + \cos(2^x + 9)}{x - 8}\right)$$

This function is continuous on its domain $\{x : x \neq 8\}$.

EXAMPLE 5

Evaluate

a. $\lim_{y \to \frac{\pi}{3}} \sin y$ and

b. $\lim_{x \to -1} \dfrac{3^x}{\sqrt{x+5}}$.

Solution

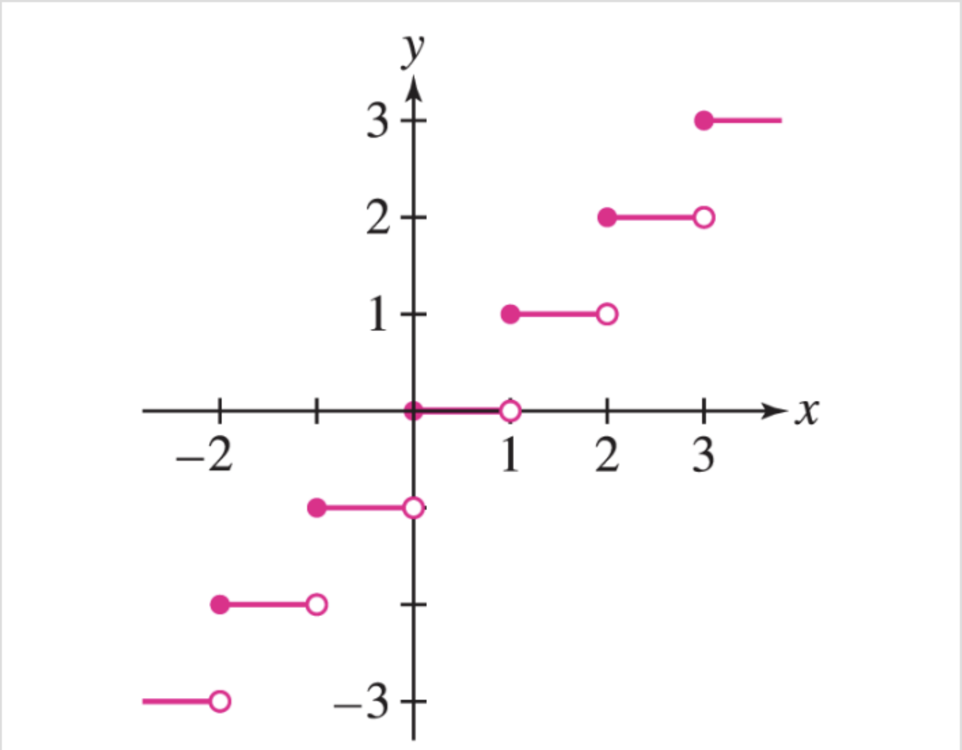
a. We can use substitution because $f(y) = \sin\ y$ is continuous.

$$\lim_{y \to \frac{\pi}{3}} \sin\ y = \sin\ \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

b. The function $f(x) = 3^x / \sqrt{x+5}$ is continuous at $x = -1$ because the numerator and denominator are both continuous at $x = -1$ and the denominator $\sqrt{x+5}$ is nonzero at $x = -1$. Therefore, we can use substitution:

$$\lim_{x \to -1} \frac{3^x}{\sqrt{x+5}} = \frac{3^{-1}}{\sqrt{-1+5}} = \frac{1}{6}$$

The **greatest integer function** $f(x) = \lfloor x \rfloor$ is the function defined by $\lfloor x \rfloor = n$, where n is the unique integer such that $n \leq x < n + 1$ (Figure 13). For example, $\lfloor 4.7 \rfloor = 4$ and $\lfloor -2.3 \rfloor = -3$. This function is called the greatest integer function because $\lfloor x \rfloor = n$ represents the greatest integer less than or equal to x.



EXAMPLE 6

Assumptions Matter

Can we evaluate $\lim_{x \to 2} \lfloor x \rfloor$ using substitution?

Solution

Substitution cannot be applied because $f(x) = \lfloor x \rfloor$ is not continuous at $x = 2$. In fact, $\lim_{x \to 2} \lfloor x \rfloor$ does not exist, and that follows because the one-sided limits are not equal:

$$\lim_{x \to 2^+} \lfloor x \rfloor = 2 \qquad \text{and} \qquad \lim_{x \to 2^-} \lfloor x \rfloor = 1$$

2.4 SUMMARY

- Definition: f is *continuous* at $x = c$ if $\lim_{x \to c} f(x) = f(c)$. This means that $f(c)$ exists, $\lim_{x \to c} f(x)$ exists, and they are equal.
- If $\lim_{x \to c} f(x)$ does not exist, or if it exists but does not equal $f(c)$, then f is *discontinuous* at $x = c$.
- If f is continuous at all points in its domain, f is simply called *continuous*.
- *Right-continuous* at $x = c$: $\lim_{x \to c^+} f(x) = f(c)$.
- *Left-continuous* at $x = c$: $\lim_{x \to c^-} f(x) = f(c)$.
- Three common types of discontinuities:

  – *Removable discontinuity:* $\lim_{x \to c} f(x)$ exists, but either the limit does not equal $f(c)$ or $f(c)$ is not defined.

  – *Jump discontinuity:* The one-sided limits both exist but are not equal.

  – *Infinite discontinuity:* The limit is infinite as x approaches c from one or both sides.
- Laws of Continuity: Sums, products, multiples, inverses, and composites of continuous functions are continuous. The same holds for a quotient f/g at points where $g(x) \neq 0$.
- The basic functions are continuous on their domains where the basic functions are polynomials, rational functions, nth-root and algebraic functions, trigonometric functions and their inverses, and exponential and logarithmic functions.
- Substitution Method: If f is known to be continuous at $x = c$, then the value of the limit $\lim_{x \to c} f(x)$ is $f(c)$.