# Chapter 1-5

## 2.4 ◆ Mathematical Induction

**Example:** let $S_n$ denote the sum of the first n positive integers:

$$S_n = 1 + 2 + \cdots + n. \qquad (2.4.3)$$

Suppose that someone claims that

$$S_n = \frac{n(n+1)}{2} \text{ for all } n \geq 1. \quad (2.4.4)$$

A sequence of statements is really being made, namely,

$$S_1 = \frac{1(2)}{2} = 1,\ S_2 = \frac{2(3)}{2} = 3,\ S_3 = \frac{3(4)}{2} = 6, \ldots.$$

Suppose that each true equation has an "×" placed beside it (see Figure 2.4.2).

$$
\begin{aligned}
S_1 &= \frac{1(2)}{2} && \times \\
S_2 &= \frac{2(3)}{2} && \times \\
&\vdots \\
S_{n-1} &= \frac{(n-1)n}{2} && \times \\
S_n &= \frac{n(n+1)}{2} && \times \\
S_{n+1} &= \frac{(n+1)(n+2)}{2} && ? \\
&\vdots
\end{aligned}
$$

**Figure 2.4.2** A sequence of statements. True statements are marked with $\times$.

Since the first equation is true, it is marked. Now suppose we can show that for all **n**, if equation **n** is marked, then equation **n + 1** is also marked.

We must show that for all **n**, if equation **n** is true, then equation **n + 1** is also true.

Equation **n** is

$$S_n = \frac{n(n+1)}{2} \; . \qquad (2.4.5)$$

Assuming that this equation is true, we must show that equation **n + 1**

$$S_n = \frac{(n+1)(n+2)}{2}$$

is true.

According to definition (2.4.3),

$$S_{n+1} = 1 + 2 + \cdots + n + (n + 1).$$

We note that $S_n$ is contained within $S_{n+1}$, in the sense that

$$S_{n+1} = 1 + 2 + \cdots + n + (n + 1) = S_n + (n + 1). \qquad (2.4.6)$$

Because of (2.4.5) and (2.4.6), we have

$$S_{n+1} = S_n + (n+1) = \frac{n(n+1)}{2} + (n+1).$$

Since

$$\begin{aligned}\frac{n(n+1)}{2} + (n+1) &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2},\end{aligned}$$

we have

$$S_{n+1} = \frac{(n+1)(n+2)}{2}.$$

Therefore, assuming that equation **n** is true, we have proved that equation **n + 1** is true. We conclude that all of the equations are true.

**Our proof using mathematical induction consisted of two steps. First, we verified that the statement corresponding to n=1 was true. Second, we assumed that statement n was true and then proved**

**that statement n + 1 was also true. In proving statement n + 1, we were permitted to make use of statement n; indeed, the trick in constructing a proof using mathematical induction is to relate statement n to statement n + 1.**

**We next formally state the Principle of Mathematical Induction.**

Suppose that we have a propositional function S(n) whose domain of discourse is the set of positive integers. Suppose that

$S(1)$ is true; (2.4.7)

for all n ≥ 1,

if $S(n)$ is true, then S(n + 1) is true. (2.4.8)

Then S(n) is true for every positive integer n.

Condition (2.4.7) is sometimes called the **Basis Step** and condition (2.4.8) is sometimes called the **Inductive Step**. Hereafter, "induction" will mean "**mathematical induction**."

**Example 2.4.3**

Use induction to show that

$$n! \geq 2^{n-1} \text{ for all } n \geq 1. \qquad (2.4.9)$$

**SOLUTION**

**Basis Step** (n = 1)

[Condition (2.4.7)] We must show that (2.4.9) is true if n = 1. This is easily accomplished, since $1! = 1 \geq 1 = 2^{1-1}$. **Inductive Step** [Condition (2.4.8)] We assume that the inequality is true for n≥1; that is, we assume that

$$\mathbf{n! \geq 2^{n-1}} \tag{2.4.10}$$

is **true**. We must then prove that the inequality is true for n + 1; that is, we must prove that

$$\mathbf{(n + 1)! \geq 2^{n}} \tag{2.4.11}$$

is true. We can relate (2.4.10) and (2.4.11) by observing that

$$(n + 1)! = (n + 1)(n!).$$

Now

$$
\begin{aligned}
(n+1)! &= (n+1)(n!) \\
&\geq (n+1)2^{n-1} && \text{by (2.4.10)} \\
&\geq 2 \cdot 2^{n-1} && \text{since } n+1 \geq 2 \\
&= 2^n.
\end{aligned}
$$

Therefore, (2.4.11) is true. We have completed the Inductive Step. Since the Basis Step and the Inductive Step have been verified, the Principle of Mathematical Induction tells us that (2.4.9) is true for every positive integer n.

## Example 2.4.5

Use induction to show that $5^n$ – 1 is divisible by 4 for all n ≥ 1.

## Solution

**Basis Step** (n = 1)

If n = 1, $5^n - 1 = 5^1 - 1 = 4$, which is divisible by 4.

**Inductive Step**

We assume that $5^n - 1$ is divisible by 4. We must then show that $5^{n+1} - 1$ is divisible by 4. We use the fact that if p and q are each divisible by k, then p + q is also divisible by k. In our case, k = 4. We leave the proof of this fact to the exercises (see Exercise 74).

We relate the (n + 1)st case to the nth case by writing

$5^{n+1} - 1 = 5^n - 1 +$ to be determined.

Now, by the inductive assumption, $5^n - 1$ is divisible by 4. If "to be determined" is also divisible by 4, then the preceding sum, which is equal to $5^{n+1} - 1$ , will also be divisible by 4, and the Inductive Step will be complete. We must find the value of “to be determined.”

Now

$5^{n+1} - 1 = 5 \cdot 5^{n} - 1 = 4 \cdot 5^{n} + 1 \cdot 5^{n} - 1.$

Thus, “to be determined” is $4 \cdot 5^{n}$, which is divisible by 4. Formally, we could write the Inductive Step as follows.

By the inductive assumption, $5^{n} - 1$ is divisible by 4 and, since $4 \cdot 5^{n}$ is divisible by 4, the sum

$$(5^{n} - 1) + 4 \cdot 5^{n} = 5^{n+1} - 1$$

is divisible by 4.

Since the Basis Step and the Inductive Step have been verified, the Principle of

Mathematical Induction tells us that $5^n$-1 is divisible by 4 for all n ≥ 1.

# 3.3 Relation

A relation from one set to another can be thought of as a table that lists which elements of the first set relate to which elements of the second set (see Table 3.3.)

**TABLE 3.3.1** ■ Relation of Students to Courses

| *Student* | *Course* |
|---|---|
| Bill | CompSci |
| Mary | Math |
| Bill | Art |
| Beth | History |
| Beth | CompSci |
| Dave | Math |

**Definition 3.3.1** A (binary) relation R from a set X to a set Y is a subset of the Cartesian product $X \times Y$. If $(x, y) \in R$, we write xRy and say that x is related to y. If $X = Y$, we call R a (binary) relation on X.

**Example 3.3.2** If we let X = {Bill, Mary, Beth, Dave} and Y = {CompSci, Math, Art, History}, our relation R of Table 3.3.1 can be written R = {(Bill, CompSci), (Mary, Math), (Bill, Art),(Beth, History), (Beth, CompSci),(Dave, Math)}. Since (Beth, History) ∈ R, we may write

**Beth R History.**

**Example 3.3.3** Let X = {2, 3, 4} and Y = {3, 4, 5, 6, 7}. If we define a relation R from X to Y by $(x, y) \in R$ if x divides y, we obtain R = {(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)}. If we rewrite R as a table, we obtain

| $X$ | $Y$ |
|---|---|
| 2 | 4 |
| 2 | 6 |
| 3 | 3 |
| 3 | 6 |
| 4 | 4 |

**Example 3.3.4** Let R be the relation on X = {1, 2, 3, 4} defined by (x, y) ∈ R if x ≤ y, x, y ∈ X. Then R = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)}.

An informative way to picture a relation on a set is to draw its **digraph.**

To draw the digraph of a relation on a set X, we first draw dots or vertices to represent the elements of X.

Next, if the element (x, y) is in the relation, we draw an arrow (called a **directed edge**) from x to y.

Notice that an element of the form (x, x) in a relation corresponds to a directed edge from x to x. Such an edge is called a **loop.**

There is a loop at every vertex in Figure 3.3.1.

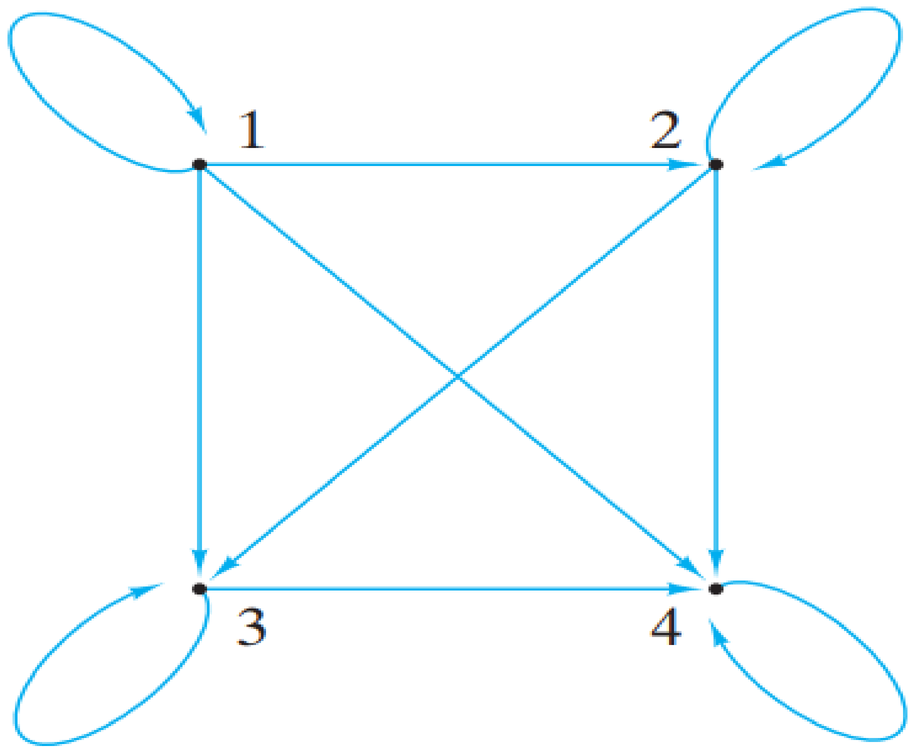


**Figure 3.3.1** The digraph of the relation of Example 3.3.4.

**Example 3.3.5** The relation R on X = {a, b, c, d} given by the digraph of Figure 3.3.2 is R = {(a, a), (b, c), (c, b), (d, d)}.

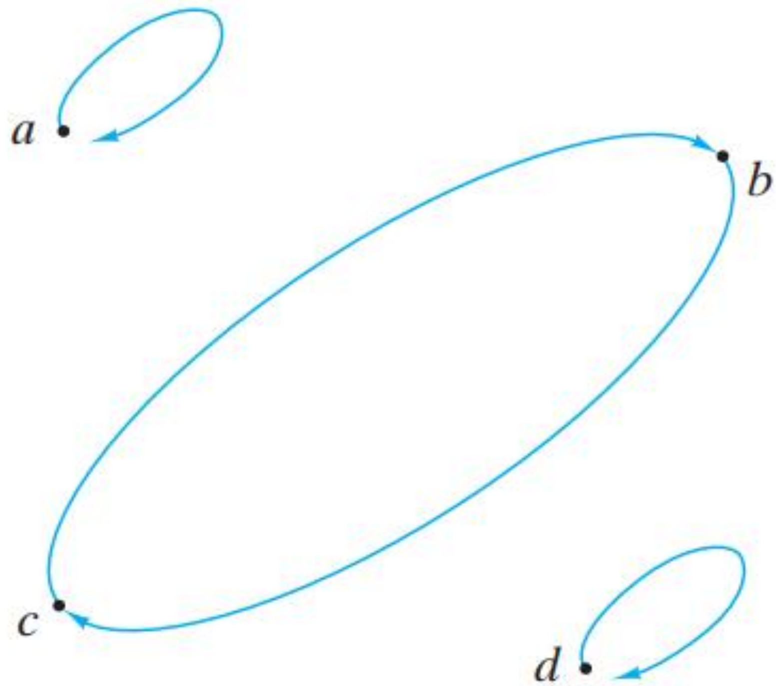


Figure 3.3.2 The digraph of the relation of Example 3.3.5.

# Several properties that relations may have

**Definition 3.3.6** A relation R on a set X is **reflexive** if $(x, x) \in R$ for every $x \in X$.

**Example 3.3.7** The relation R on X = {1, 2, 3, 4} defined by (x, y) ∈ R if $x \le y$, x, y ∈ X, is reflexive because for each element x ∈ X, (x, x) ∈ R; specifically, (1, 1), (2, 2), (3, 3), and (4, 4) are each in R.

**The digraph of a reflexive relation has a loop at every vertex.** Notice that the digraph of this relation (see Figure 3.3.1) has a loop at every vertex.

**a relation R on X is not reflexive if there exists x ∈ X such that (x, x) ∉ R.**

**Example 3.3.8** The relation R = {(a, a), (b, c), (c, b), (d, d)} on X = {a, b, c, d} is not reflexive. For example, b ∈ X, but (b,b)∉R. That this relation is not reflexive can also be seen by looking at its digraph (see Figure 3.3.2); **vertex b does not have a loop.**

**Definition 3.3.9** A relation R on a set X is symmetric if for all x, y ∈ X, if (x, y) ∈ R, then (y, x) ∈ R.

**Example 3.3.10** The relation R = {(a, a), (b, c), (c, b), (d, d)} on X = {a, b, c, d} is **symmetric** because for all x, y, if (x, y) ∈ R, then (y, x) ∈ R, (see Figure 3.3.2).

**Example 3.3.11** The relation R on X = {1, 2, 3, 4} defined by (x, y) ∈ R if x ≤ y, x, y ∈ X, is not symmetric. For example, (2, 3) ∈ R, but (3, 2) ∉ R. The digraph of this relation (see Figure 3.3.1) has a directed edge from 2 to 3, but there is no directed edge from 3 to 2.

**Definition 3.3.12** A relation R on a set X is **antisymmetric** if for all $x, y \in X$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.

**Example 3.3.13** The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y$, $x, y \in X$, is antisymmetric because for all x, y, if $(x, y) \in R$ (i.e., $x \leq y$) and $(y, x) \in R$ (i.e., $y \leq x$), then $x = y$.

**to obtain a logically equivalent characterization of "antisymmetric":**

A relation R on a set X is antisymmetric if for all $x, y \in X$, if $x \neq y$, then $(x, y) \notin R$ or $(y, x) \notin R$.

**Example 3.3.15** If a relation has no members of the form (x, y), x $\neq$ y, we see that the equivalent characterization of "antisymmetric" for all x, y $\in$ X, if x $\neq$ y, then (x, y) $\notin$ R or (y, x) $\notin$ R (see Example 3.3.14) is trivially true (since the hypothesis x $\neq$ y is always false).

Thus if a relation R has no members of the form (x, y), x $\neq$ y, **R is antisymmetric**. For example, R = {(a, a), (b, b), (c, c)} on X = {a, b, c} is antisymmetric. The digraph of R shown in Figure 3.3.3 has at most one directed edge between each pair of distinct vertices.


**Figure 3.3.3** The digraph of the relation of Example 3.3.15.

**Notice that R** is also **reflexive** and **symmetric**. This example shows that "**antisymmetric**" is not the same as "**not symmetric**" because this relation is in fact both symmetric and antisymmetric.

**Example 3.3.16** The relation R = {(a, a), (b, c), (c, b), (d, d)} on X = {a, b, c, d} is not antisymmetric because both (b, c) and (c, b) are in R. Notice that in the digraph of this relation (see Figure 3.3.2) there are two directed edges between b and c.

**Definition 3.3.17** A relation R on a set X is **transitive** if for all x, y, z ∈ X, if (x, y) and (y,z) ∈ R, then (x,z) ∈ R.

**Example 3.3.18** The relation R on X = {1, 2, 3, 4} defined by (x, y) ∈ R if x ≤ y, x, y ∈ X, is transitive because for all x, y, z, if (x, y) and (y,z) ∈ R, then (x,z) ∈ R. To formally verify that this relation satisfies Definition 3.3.17, we can list all pairs of the form (x, y) and (y,z) in R and then verify that in every case, (x,z) ∈ R:

| *Pairs of Form* $(x, y)$ | $(y, z)$ | $(x, z)$ | *Pairs of Form* $(x, y)$ | $(y, z)$ | $(x, z)$ |
|---|---|---|---|---|---|
| (1, 1) | (1, 1) | (1, 1) | (2, 2) | (2, 2) | (2, 2) |
| (1, 1) | (1, 2) | (1, 2) | (2, 2) | (2, 3) | (2, 3) |
| (1, 1) | (1, 3) | (1, 3) | (2, 2) | (2, 4) | (2, 4) |
| (1, 1) | (1, 4) | (1, 4) | (2, 3) | (3, 3) | (2, 3) |
| (1, 2) | (2, 2) | (1, 2) | (2, 3) | (3, 4) | (2, 4) |
| (1, 2) | (2, 3) | (1, 3) | (2, 4) | (4, 4) | (2, 4) |
| (1, 2) | (2, 4) | (1, 4) | (3, 3) | (3, 3) | (3, 3) |
| (1, 3) | (3, 3) | (1, 3) | (3, 3) | (3, 4) | (3, 4) |
| (1, 3) | (3, 4) | (1, 4) | (3, 4) | (4, 4) | (3, 4) |
| (1, 4) | (4, 4) | (1, 4) | (4, 4) | (4, 4) | (4, 4) |

**Actually, some of the entries in the preceding table were unnecessary. If x = y or y = z, we need not explicitly verify that the condition if (x, y) and (y,z) ∈ R, then (x,z) ∈ R is satisfied since it will automatically be true.**

Suppose, for example, that x = y and (x, y) and (y,z) are in R. Since x = y, (x,z) = (y,z) is in R and the condition is satisfied. Eliminating the cases x = y and y = z leaves only the following to be explicitly checked to verify that the relation is transitive:

| *Pairs of Form* $(x, y)$ | $(y, z)$ | $(x, z)$ |
|---|---|---|
| (1, 2) | (2, 3) | (1, 3) |
| (1, 2) | (2, 4) | (1, 4) |
| (1, 3) | (3, 4) | (1, 4) |
| (2, 3) | (3, 4) | (2, 4) |

The digraph of a transitive relation has the property that whenever there are directed edges from x to y and from y to z, there is also a directed edge from x to z. Notice that the digraph of this relation (see Figure 3.3.1) has this property.

A relation R **is not transitive** if there exist x, y, and z such that (x, y) and (y,z) are in R, but (x,z) is not in R.

**Example 3.3.19** The relation R = {(a, a), (b, c), (c, b), (d, d)} on X = {a, b, c, d} is not transitive. For example, (b, c) and (c, b) are in R, but (b, b) is not in R. Notice that in the digraph of this relation (see

Figure 3.3.2) there are directed edges from b to c and from c to b, but there is no directed edge from b to b.

**Definition 3.3.20** A relation R on a set X is a **partial order** if R is **reflexive, antisymmetric, and transitive**.

**Example 3.3.21** Since the relation R defined on the positive integers by (x, y) ∈ R if x divides y is reflexive, antisymmetric, and transitive, R is a partial order.