3.2 The Derivative as a Function

In the previous section, we computed the derivative $f'(a)$ for specific values of a. It is also useful to view the derivative as a function f' whose values $f'(x)$ are defined by the limit definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

3.2 The Derivative as a Function

If $y = f(x)$, we also write y' or $y'(x)$ for $f'(x)$.

The domain of f' consists of all values of x in the domain of f for which the limit in Eq. (1) exists. We say that f is **differentiable** on (a, b) if $f'(x)$ exists for all x in (a, b). When $f'(x)$ exists for all x in the interval or intervals on which $f(x)$ is defined, we say simply that f is differentiable.

EXAMPLE 1

Prove that $f(x) = x^3 - 12x$ is differentiable. Compute $f'(x)$ and find $f'(-3)$, $f'(0)$, $f'(2)$, and $f'(3)$.

Solution

We compute $f'(x)$ in three steps as in the previous section.

Step 1. **Write out the numerator of the difference quotient.**

$$\begin{aligned} f(x+h) - f(x) &= \left((x+h)^3 - 12(x+h)\right) - (x^3 - 12x) \\ &= (x^3 + 3x^2h + 3xh^2 + h^3 - 12x - 12h) - (x^3 - 12x) \\ &= 3x^2h + 3xh^2 + h^3 - 12h \\ &= h(3x^2 + 3xh + h^2 - 12) \qquad \text{(factor out } h\text{)} \end{aligned}$$

Step 2. **Divide by h and simplify.**

$$\frac{f(x+h) - f(x)}{h} = \frac{h(3x^2 + 3xh + h^2 - 12)}{h} = 3x^2 + 3xh + h^2 - 12 \quad (

***Step 3.* Compute the limit.**

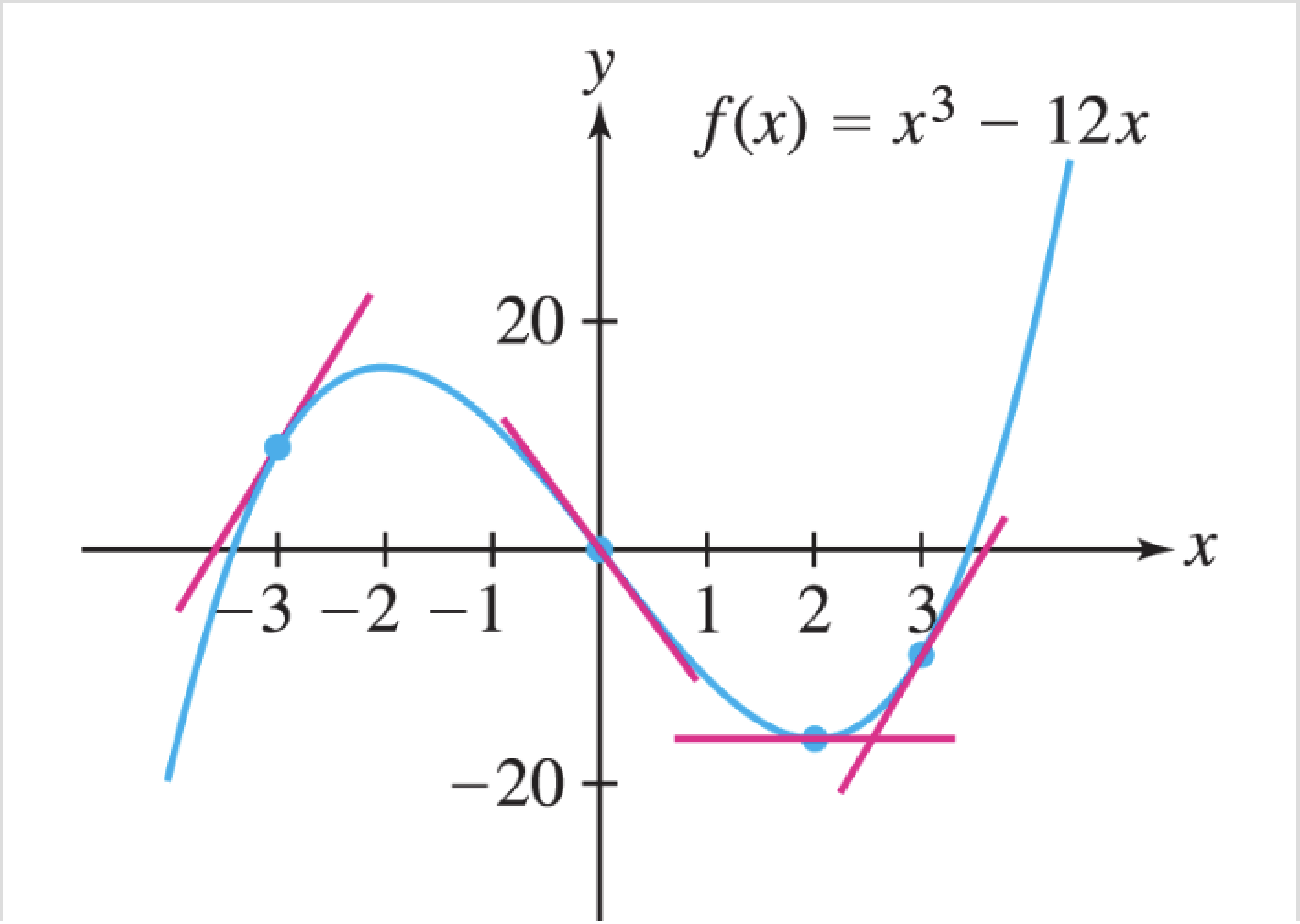
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2 - 12) = 3x^2 - 12$$

In this limit, x is treated as a constant because it does not change as $h \to 0$. We see that the limit exists for all x, so f is differentiable and $f'(x) = 3x^2 - 12$.

Now evaluate:

- $f'(-3) = 3(-3)^2 - 12 = 15$
- $f'(0) = 3(0)^2 - 12 = -12$
- $f'(2) = 3(2)^2 - 12 = 0$
- $f'(3) = 3(3)^2 - 12 = 15$

These derivatives indicate the slope of the graph of f (and the tangent line to the graph) at the corresponding points, as shown in Figure 1.



EXAMPLE 2

Prove that $y = x^{-2}$ is differentiable and calculate y'.

Solution

The domain of $f(x) = x^{-2}$ is $\{x : x \neq 0\}$, so assume that $x \neq 0$. We compute $f'(x)$ directly, without the separate steps of the previous example:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{x^2 - (x+h)^2}{x^2(x+h)^2}}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{x^2 - (x+h)^2}{x^2(x+h)^2}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{-h(2x+h)}{x^2(x+h)^2}\right) = \lim_{h\to 0} -\frac{2x+h}{x^2(x+h)^2} \quad (\text{cancel } h) \\ &= -\frac{2x+0}{x^2(x+0)^2} = -\frac{2x}{x^4} = -2x^{-3} \end{aligned}$$

The limit exists for all $x \neq 0$, so $y = x^{-2}$ is differentiable and $y' = -2x^{-3}$.

$y = f(x)$

$y' = f'(x) = \frac{df}{dx}$

$\frac{dy}{dx}$

$g'(x) = \frac{dg}{dx}$

$h'(z) = \frac{dh}{dz}$

Leibniz Notation

The "prime" notation y' and $f'(x)$ was introduced by the French mathematician Joseph Louis Lagrange (1736–1813). There is another standard notation for the derivative that we owe to Gottfried Wilhelm Leibniz:

$$\frac{df}{dx} \quad \text{or} \quad \frac{dy}{dx}$$

$f'(2) = \left.\frac{df}{dx}\right|_{x=2}$

In Example 2, we showed that the derivative of $y = x^{-2}$ is $y' = -2x^{-3}$. In Leibniz notation, we would write

$$\frac{dy}{dx} = -2x^{-3} \qquad \text{or} \qquad \frac{d}{dx}x^{-2} = -2x^{-3}$$

To specify the value of the derivative for a fixed value of x, say, $x = 4$, we write

$$\left.\frac{df}{dx}\right|_{x=4} \quad \text{or} \quad \left.\frac{dy}{dx}\right|_{x=4}$$

You should not think of dy/dx as the fraction "dy divided by dx." Separately, the expressions dy and dx are called **differentials**. They play a role in linear approximation (Section 4.1), and relationships between them are used as a guide for "substitutions" we do later when working with integrals.

CONCEPTUAL INSIGHT

Leibniz notation is widely used for several reasons. First, it reminds us that the derivative df/dx, although not itself a ratio, is in fact a *limit* of ratios $\Delta f/\Delta x$. Second, the notation specifies the independent variable. This is useful when variables other than x are used. For example, if the independent variable is t, we write df/dt. Third, we often think of d/dx as an "operator" that performs differentiation on functions. In other words, we apply the operator d/dx to f to obtain the derivative df/dx. We will see other advantages of Leibniz notation when we discuss the Chain Rule in Section 3.7.

We read $\dfrac{dy}{dx}$ *and* $\dfrac{d}{dx}\,y$ *as "the derivative of* y *with respect to* x*."*

$$\frac{dx}{dx} = \frac{d}{dx} x$$

$$\frac{\cancel{dx}}{\cancel{dx}} = 1$$

Now we are ready to start assembling a collection of derivative rules and formulas that will enable us to compute derivatives of the most common functions in mathematics, the sciences, and engineering. We begin with two simple formulas that are consequences of Theorem 1 in the previous section:

$$\frac{d}{dx} x = 1 \quad \text{and} \quad \frac{d}{dx} c = 0 \quad \text{for any constant } c$$

The first indicates that the derivative with respect to x of x is 1, reflecting that the slope of the line $y = x$ is 1. The second, known as **the Constant Rule**, indicates that the derivative of a constant is 0. This makes sense, of course, since a constant does not change and therefore has a rate of change of zero. As simple as the latter is, we will find it quite useful as we work with derivatives throughout the book.

THEOREM 1

The Power Rule

For all exponents n:

$$\frac{d}{dx}x^n = nx^{n-1}$$

$(x^3)' = 3x^{3-1} = 3x^2$

$(x^{-3})' = -3x^{-3-1} = -3x^{-4}$

$(x^{\frac{2}{3}})' = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}}$

$(x^{\sqrt{2}})' = \sqrt{2}\,x^{\sqrt{2}-1}$

$(x^n)' = nx^{n-1}$

- The Power Rule in words: To differentiate x to a power, multiply by the power and reduce the power by one.

$$\boxed{\frac{d}{dx}\, x^{\text{power}} = (\text{power})\ x^{\text{power}-1}}$$

- The Power Rule is valid for all exponents, whether positive, negative, fractional, or irrational:

$$\frac{d}{dx}\, x^{-3/5} = -\frac{3}{5}\, x^{-8/5}, \qquad \frac{d}{dx}\, x^{\sqrt{2}} = \sqrt{2}x^{\sqrt{2}-1}$$

- The Power Rule can be applied with any variable, not just x. For example,

$$\frac{d}{dz}z^2 = 2z, \qquad \frac{d}{dt}t^{20} = 20t^{19}, \qquad \frac{d}{dr}r^{1/2} = \frac{1}{2}r^{-1/2}$$

THEOREM 2

Linearity Rules

Assume that f and g are differentiable. Then

Sum and Difference Rules: $f + g$ and $f - g$ are differentiable, and

$$(f+g)' = f' + g', \qquad (f-g)' = f' - g'$$

Constant Multiple Rule: For any constant c, cf is differentiable, and

$$(cf)' = cf'$$

Ex $\frac{d}{dx} x^4 \longrightarrow 4x^3$

$\frac{d}{dx} 7x^4 \longrightarrow 7(4x^3) = 28x^3$

$\frac{d}{dx} \sqrt{3}\, x^2 \longrightarrow 2\sqrt{3}\, x$

EXAMPLE 3

Find the points on the graph of $f(t) = t^3 - 12t + 4$ where the tangent line is horizontal.

Solution

We calculate the derivative:

$$\begin{aligned}
\frac{df}{dt} &= \frac{d}{dt}\left(t^3 - 12t + 4\right) \\
&= \frac{d}{dt}t^3 - \frac{d}{dt}(12t) + \frac{d}{dt}4 && \text{(Sum and Difference Rules)} \\
&= \frac{d}{dt}t^3 - 12\frac{d}{dt}t + 0 && \text{(Constant Multiple Rule and Constant R} \\
&= 3t^2 - 12 && \text{(Power Rule)}
\end{aligned}$$

Ex: $\frac{d\sqrt{x}}{dx} = ?$

$\sqrt{t}' = \frac{1}{2\sqrt{t}}$

$\sqrt{x} = x^{\frac{1}{2}} \longrightarrow \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$

$\sqrt{z}' = \frac{1}{2\sqrt{z}}$

$$\boxed{\sqrt{x}' = \frac{1}{2\sqrt{x}}}$$

$= \frac{1}{2x^{1/2}}$

$\sqrt{3}' = 0$

$= \frac{1}{2\sqrt{x}}$

$\frac{d\sqrt{5}}{dx} = 0$

EXAMPLE 4

Calculate $\left.\dfrac{dg}{dt}\right|_{t=1}$, where $g(t) = t^{-3} + 2\sqrt{t} - t^{-4/5}$.

$$\frac{dg}{dt} = -3t^{-4} + \cancel{2}\cdot\frac{1}{\cancel{2}\sqrt{t}} \overset{+}{\cancel{-}}\left(-\frac{4}{5}t^{-\frac{9}{5}}\right)$$

$$= \frac{-3}{t^4} + \frac{1}{\sqrt{t}} + \frac{4}{5t^{9/5}}$$

$$\left.\frac{dg}{dt}\right|_{t=1} = \frac{-3}{\underset{1}{\cancel{(1)^4}}} + \frac{1}{\underset{1}{\cancel{\sqrt{1}}}} + \frac{4}{5\underset{1}{\cancel{(1)^{9/5}}}}$$

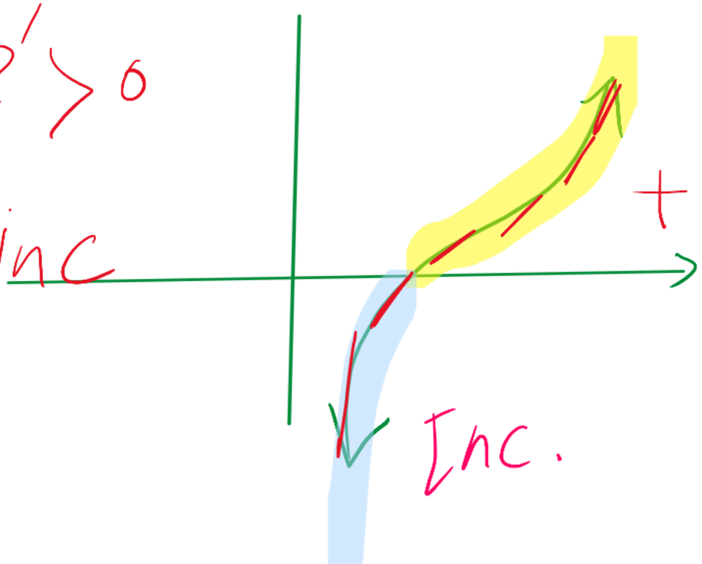
$$= \frac{-3\cdot 5}{5} + \frac{1\cdot 5}{5} + \frac{4}{5} = \frac{-15+5+4}{5} = \boxed{\frac{-6}{5}}$$

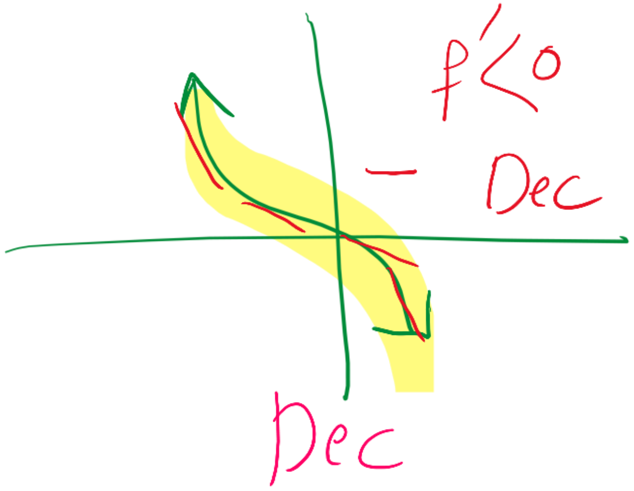
The Derivative and Behavior of the Graph

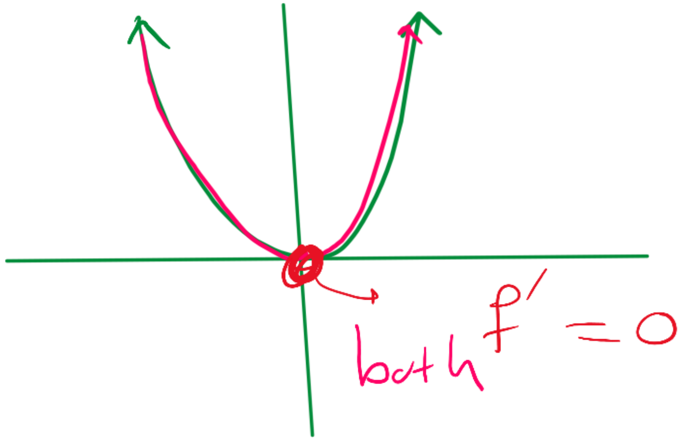
The derivative f' gives us important information about the graph of f. For example, the sign of $f'(x)$ tells us whether the tangent line has positive or negative slope. When the tangent line has positive slope, it slopes upward and the graph must be increasing. When the tangent line has negative slope, it slopes downward and the graph must be decreasing. The magnitude of $f'(x)$ reveals how steep the slope is.

$f' > 0$

inc







+   −   0   undefined

$\boxed{\begin{array}{l} f' > 0 \implies f \text{ increasing} \\ f' < 0 \implies f \text{ Decr...} \\ f' = 0 \implies f \text{ has a Max or Min} \end{array}}$

EXAMPLE 6

$f' + \Rightarrow f \uparrow$

$f' - \Rightarrow f \searrow$

$f'\ 0 \Rightarrow$ Max or Min

f' and the Graph of f

How is the graph of $\boxed{f(x) = x^3 - 12x^2 + 36x - 16}$ related to the derivative $f'(x) = 3x^2 - 24x + 36$?

$3x^2 - 24x + 36 = 0$

$3(x^2 - 8x + 12) = 0$

12, -6, -2, -8

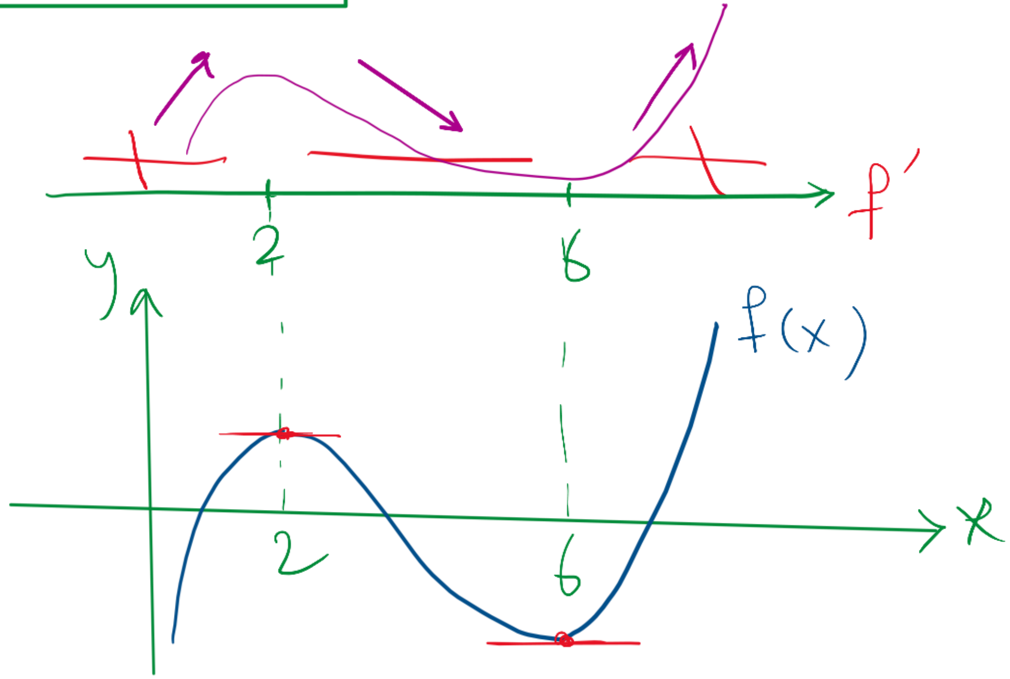
$3(x-6)(x-2) = 0 \rightarrow 3(1-6)(1-2)$ $\quad - \quad -$

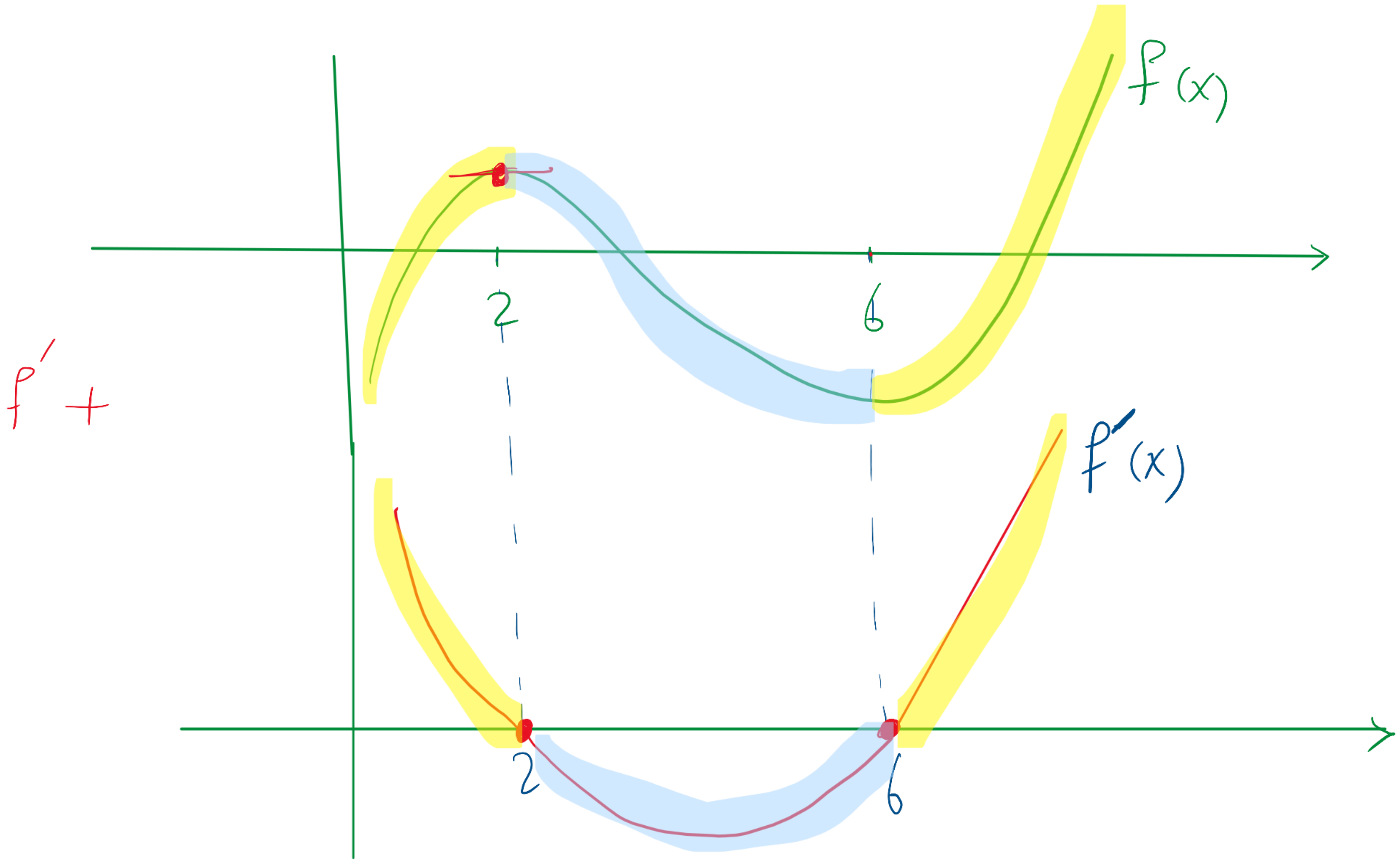
$\Downarrow \quad \Downarrow$

$x = 6 \qquad x = 2$

$3(3-6)(3-2)$ $\quad - \quad +$

$3(7-6)(7-2)$ $\quad + \quad +$

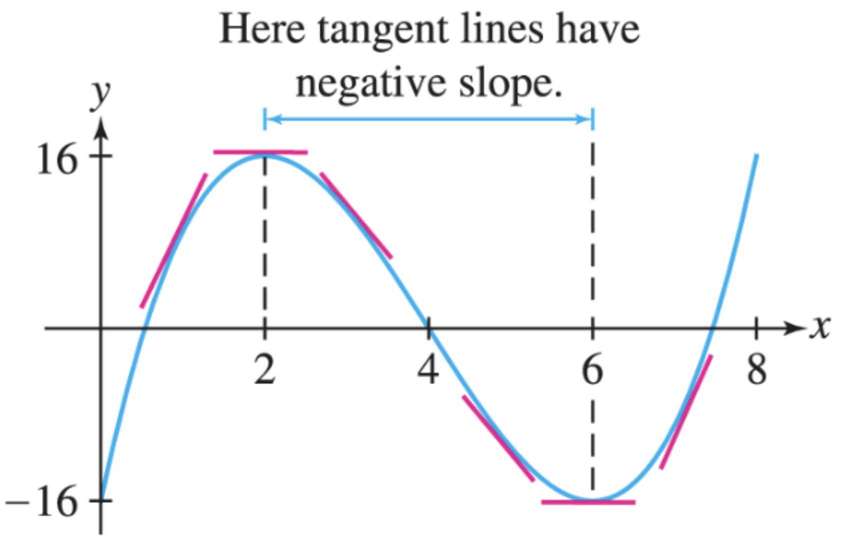



f(x)
2
6
f' +
f'(x)
2
6

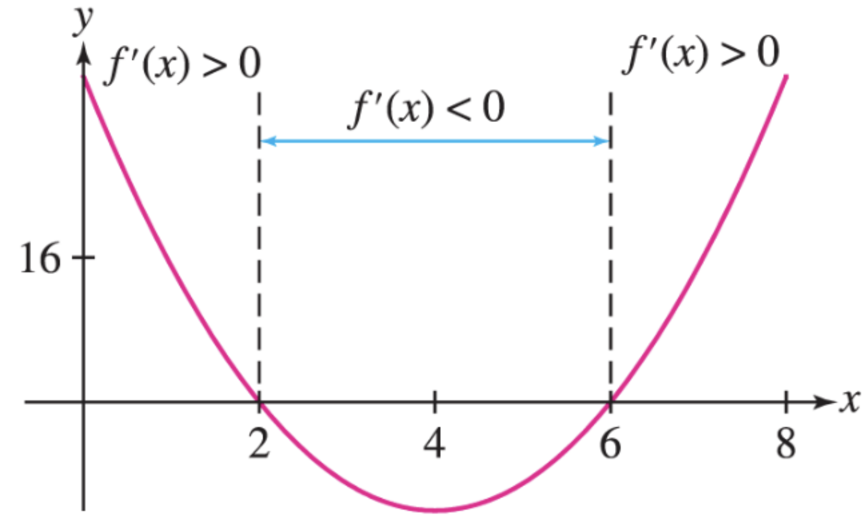
Solution

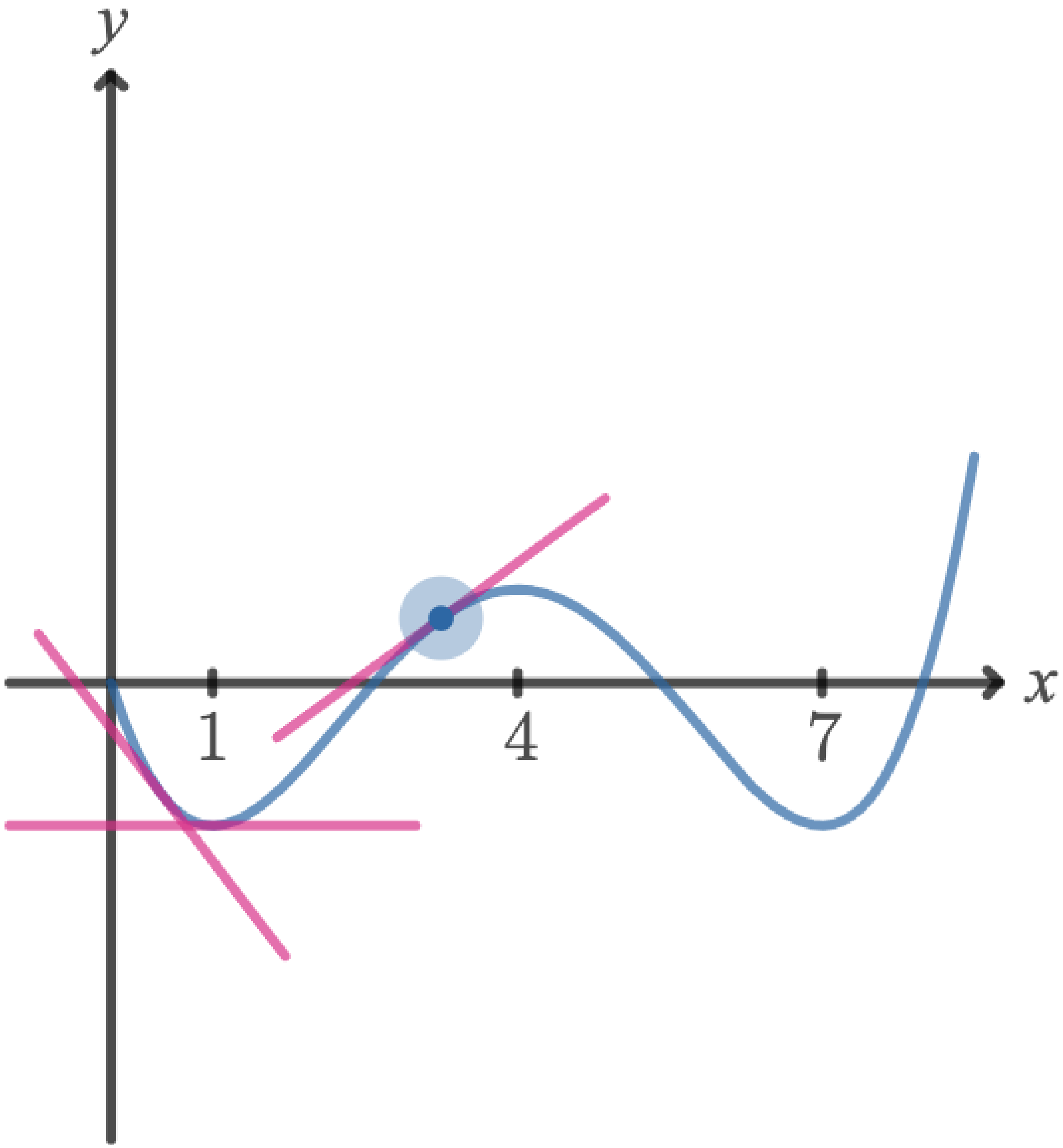
The derivative $f'(x) = 3x^2 - 24x + 36 = 3(x-6)(x-2)$ is negative for $2 < x < 6$ and positive for $x < 2$ and $x > 6$ (Figure 4). The following table summarizes this sign information:

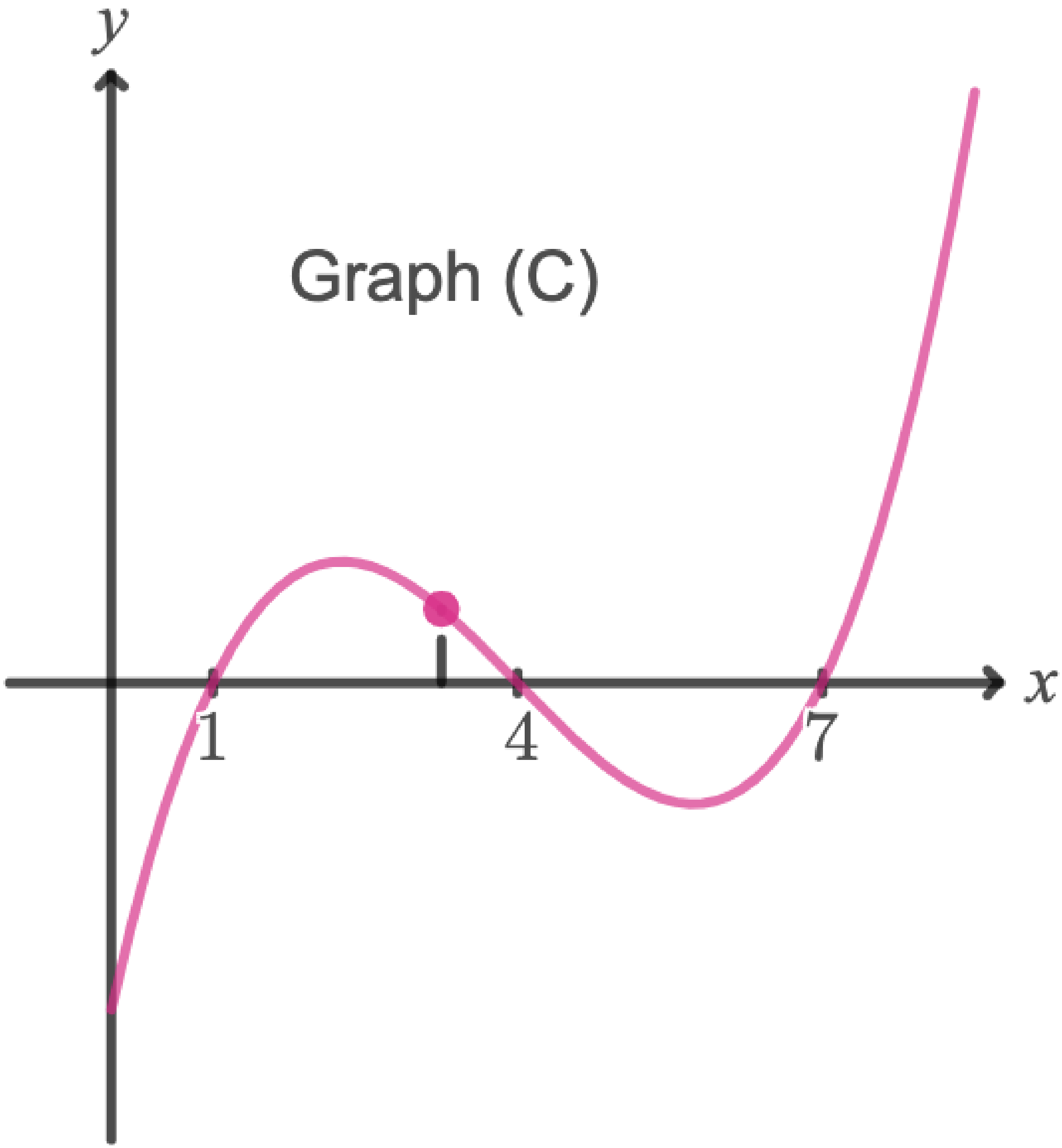
Property of $f'(x)$	Property of the Graph of f
$f'(x) < 0$ for $2 < x < 6$	Tangent has negative slope for $2 < x < 6$ (graph is decreasing).
$f'(2) = f'(6) = 0$	Tangent is horizontal at $x = 2$ and $x = 6$.
$f'(x) > 0$ for $x < 2$ and $x > 6$	Tangent has positive slope for $x < 2$ and $x > 6$ (graph is increasing).



(A) Graph of $f(x) = x^3 - 12x^2 + 36x - 16$




y
x
1
4
7


y
x
Graph (C)
1
4
7

d

THEOREM 3

$(3e^x)' = 3e^x$

Ex: $(e^x)' \longrightarrow e^x$

$(e^t)' \longrightarrow e^t$

$\dfrac{de^m}{dm} \longleftarrow (e^m)' \longrightarrow e^m$

$(e^3)' = 0$

The Exponential Function Rule

$$\frac{d}{dx}e^x = e^x$$

2

$(e^\pi)' = 0$

$(e^{\sqrt{2}})' = 0$

Point : $(2, f(2)) = (2, 2.17)$

slope : $f'(2) = 2.17$

$y - y_1 = m(x - x_1)$

$y - 2.17 = 2.17(x - 2)$

$y = 2.17x - 4.34 + 2.17$

$\boxed{y = 2.17x - 2.17}$ ✓

EXAMPLE 8

Find the tangent line to the graph of $f(x) = 3e^x - 5x^2$ at $x = 2$.

$f(2) = 3e^2 - 5(2)^2 = 3e^2 - 20 \approx 2.17$

Solution

$\left.\frac{df}{dx}\right|_{x=2}$ $= 3e^x - 10x$

$= 3e^2 - 10(2)$

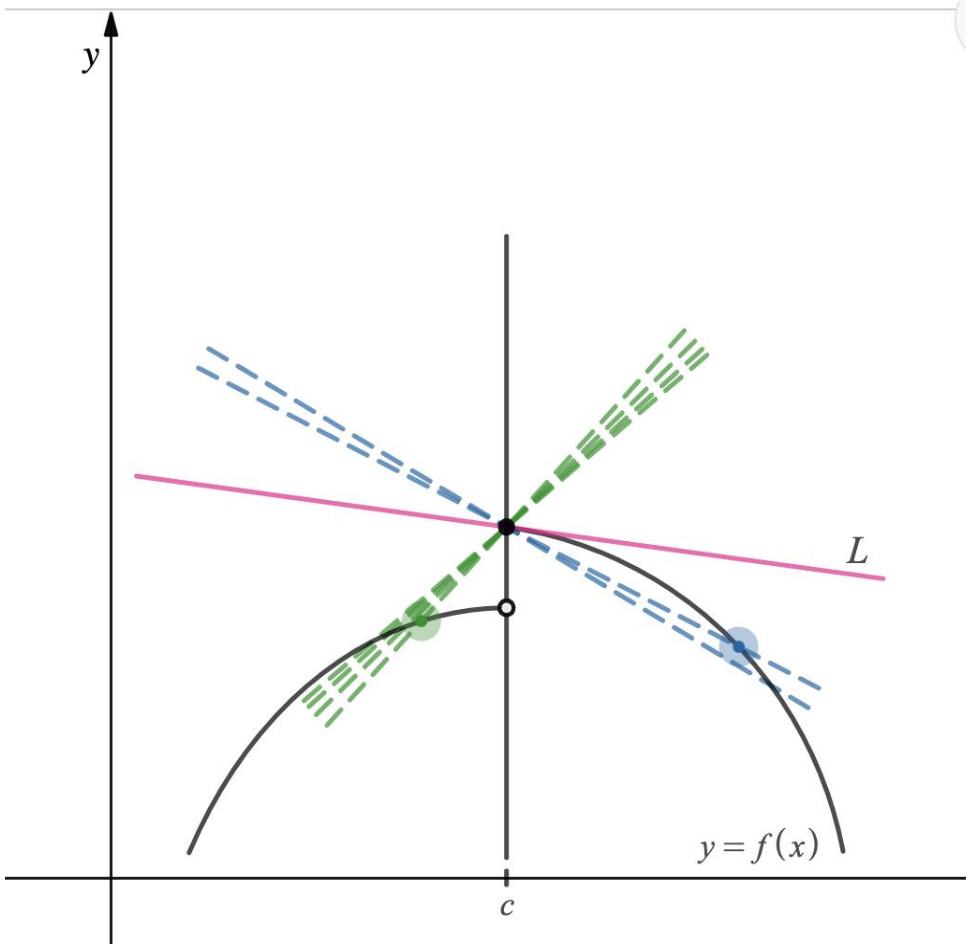
$= 3e^2 - 20$

≈ 2.17

We compute both $f'(2)$ and $f(2)$:

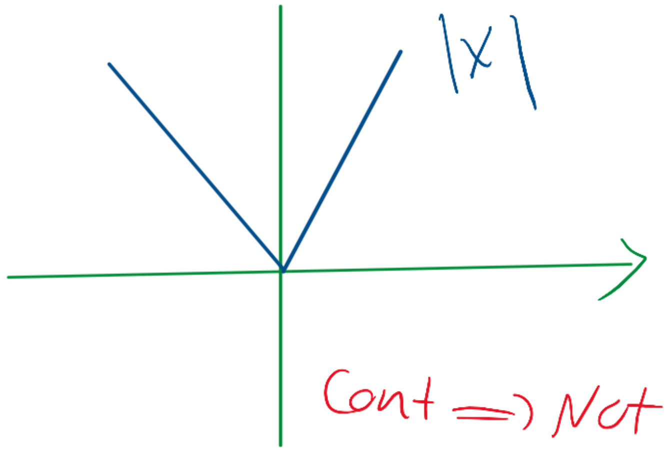
$$
\begin{aligned}
f'(x) &= \frac{d}{dx}(3e^x - 5x^2) = 3\frac{d}{dx}e^x - 5\frac{d}{dx}x^2 = 3e^x - 10x \\
f'(2) &= 3e^2 - 10(2) \approx 2.17 \\
f(2) &= 3e^2 - 5(2^2) \approx 2.17
\end{aligned}
$$

Differentiability, Continuity, and Local Linearity



THEOREM 4

Differentiability Implies Continuity



Cont $\Rightarrow$ Not diff.

If f is differentiable at $x = c$, then f is continuous at $x = c$.

?

Most of the functions encountered in this text are differentiable, but exceptions exist. As we saw in Example 5 in Section 3.1, the functions $f(x) = |x|$ and $g(x) = x^{1/3}$ are not differentiable at $x = 0$. Note that both of these functions are continuous at $x = 0$, and therefore they demonstrate that continuity at a point does not imply differentiability at the point (i.e., the converse of Theorem 4 does not hold).

3.2 SUMMARY

- The derivative f' is the function whose value at x is the derivative $f'(x)$.
- We have several different notations for the derivative of $y = f(x)$:

$$y', \quad y'(x), \quad f'(x), \quad \frac{dy}{dx}, \quad \frac{df}{dx}$$

The value of the derivative at a particular point $x = a$ is written

$$y'(a), \quad f'(a), \quad \left.\frac{dy}{dx}\right|_{x=a}, \quad \left.\frac{df}{dx}\right|_{x=a}$$

- Derivative Rules

The Constant Rule: $\frac{d}{dx}c = 0$ The Power Rule: $\frac{d}{dx}x^n = nx^{n-1}$

The Exponential Function Rule: $\frac{d}{dx}e^x = e^x$

The Linearity Rules: $(f+g)' = f' + g'$ and $(cf)' = cf'$

- Differentiability implies continuity: If f is differentiable at $x = a$, then f is continuous at $x = a$. However, there exist continuous functions that are not differentiable.
- If $f'(a)$ exists, then f is locally linear in the following sense: As we zoom in on the point $(a, f(a))$, the graph becomes nearly indistinguishable from its tangent line.