Chapter 7
RECURRENCE RELATIONS

This chapter offers an introduction to recurrence relations. Recurrence relations are useful in certain counting problems. A recurrence relation relates the *n*th element of a sequence to its predecessors. Because recurrence relations are closely related to recursive algorithms, recurrence relations arise naturally in the analysis of recursive algorithms.

7.1 Introduction

Consider the following instructions for generating a sequence:

1. Start with 5.
2. Given any term, add 3 to get the next term.

If we list the terms of the sequence, we obtain

$$5, 8, 11, 14, 17, \ldots \qquad \textbf{(7.1.1)}$$

If we denote the sequence (7.1.1) as a_1, a_2, . . . , we may rephrase instruction 1 as

$$a_1 = 5 \qquad \textbf{(7.1.2)}$$

$$a_n = a_{n-1} + 3, \quad n \geq 2. \qquad \textbf{(7.1.3)}$$

Taking $n = 2$ in (7.1.3), we obtain

$$a_2 = a_1 + 3.$$

By (7.1.2), $a_1 = 5$; thus

$$a_2 = a_1 + 3 = 5 + 3 = 8.$$

Taking $n = 3$ in (7.1.3), we obtain

$$a_3 = a_2 + 3.$$

Since $a_2 = 8$,

$$a_3 = a_2 + 3 = 8 + 3 = 11.$$

These start-up values are called **initial conditions.** The formal definitions follow.

Definition 7.1.1 A *recurrence relation* for the sequence $a_0, a_1, \ldots$ is an equation that relates a_n to certain of its predecessors $a_0, a_1 \ldots, a_{n-1}$.

Initial conditions for the sequence

$$a_0, a_1 \ldots$$

are explicitly given values for a finite number of the terms of the sequence.

Example 7.1.2 The Fibonacci sequence (see the discussion following Algorithm 4.4.6) is defined by the recurrence relation

$$f_n = f_{n-1} + f_{n-2}, \qquad n \geq 3,$$

and initial conditions

$$f_1 = 1, f_2 = 1.$$

■

Example 7.1.3

A person invests $1000 at 12 percent interest compounded annually. If A_n represents the amount at the end of n years, find a recurrence relation and initial conditions that define the sequence $\{A_n\}$.

SOLUTION:

$$\mathbf{A_n = A_{n-1} + (0.12)\, A_{n-2} = (1.12)\, A_{n-2}, \; n \geq 1.}$$

(7.1.4)

n =1

$\mathbf{A_0} = ?$

$$\mathbf{A_0 = 1000} \qquad (7.1.5)$$

For example,

$$A_3 = (1.12)A_2 = (1.12)(1.12)A_1$$

$$= (1.12)(1.12)(1.12)A_0 = (1.12)^3(1000)$$

$$= 1404.93. \qquad (7.1.6)$$

■

The computation (7.1.6) can be carried out for an arbitrary value of n to obtain

$$A_n = (1.12)A_{n-1}$$

$$\cdots$$

$$= (1.12)^n(1000).$$

■

Algorithm 7.1.4

Computing Compound Interest

This recursive algorithm computes the amount of money at the end of *n* years assuming an initial amount of $1000 and an interest rate of 12 percent compounded
annually.

Input: *n*, the number of years

Output: The amount of money at the end of *n* years

1. *compound interest(n)* {
2. if $(n == 0)$
3. return 1000
4. return $1.12 * \textit{compound interest}(n - 1)$
5. }

■

Example 7.1.5

Let S_n denote the number of subsets of an n-element set. Since going from an $(n-1)$-element set to an n-element set doubles the number of subsets (see Theorem 2.4.6), we obtain the recurrence relation $S_n = 2S_{n-1}$. The initial condition is S_0=1.

■

Example 7.1.6 Let S_n denote the number of n-bit strings that do not contain the pattern 111. Develop a recurrence relation for $S_1, S_2, \ldots$ and initial conditions that define the sequence S.

SOLUTION:

We will count the number of n-bit strings that do not contain the pattern 111

(a) that begin with 0;

(b) that begin with 10;

(c) that begin with 11.

Thus

$$S_n = S_{n-1} + S_{n-2} + S_{n-3} \qquad n \geq 4.$$

By inspection, we find the initial conditions $S_1 = 2$, $S_2 = 4$, $S_3 = 7$.

7.2 Solving Recurrence Relations

In this section we discuss two methods of solving recurrence relations: **iteration** and a special method that applies to **linear homogeneous recurrence relations with constant coefficients.**

To solve a recurrence relation involving the sequence $a_0, a_1, . . .$ by iteration,
we
use the recurrence relation to write the nth term a_n in terms of certain of its predecessors $a_{n-1}, . . . , a_0$.

Example 7.2.1 Solve the recurrence relation

$$a_n = a_{n-1} + 3, \qquad (7.2.1)$$

subject to the initial condition, $a_1 = 2$, by iteration.

SOLUTION Replacing n by $n - 1$ in (7.2.1), we obtain

$$a_{n-1} = a_{n-2} + 3.$$

If we substitute this expression for a_{n-1} into (7.2.1), we obtain

$$a_n = a_{n-1} + 3$$

$$\downarrow$$

$$= a_{n-2} + 3 + 3$$

$$= a_{n-2} + 2 \cdot 3. \qquad (7.2.2)$$

Replacing n by $n - 2$ in (7.2.1), we obtain

$$a_{n-2} = a_{n-3} + 3.$$

If we substitute this expression for a_{n-2} into (7.2.2), we obtain

$$a_n = a_{n-2} + 2 \cdot 3$$

$$\downarrow$$

$$= a_{n-3} + 3 + 2 \cdot 3$$

$$= a_{n-3} + 3 \cdot 3.$$

In general, we have

$$a_n = a_{n-k} + k \cdot 3$$

If we set $k = n - 1$ in this last expression, we have

$$a_n = a_1 + (n-1) \cdot 3$$

Since $a_1 = 2$, we obtain the explicit formula

$$a_n = 2 + 3(n-1)$$

for the sequence a.

■

Example 7.2.2 Solve the recurrence relation $S_n = 2S_{n-1}$ of Example 7.1.5, subject to the initial condition,

$S_0 = 1$, by iteration.

SOLUTION:

$$S_n = 2S_{n-1} = 2(2S_{n-2}) = \cdot \cdot \cdot = 2^n S_0 = 2^n$$

Example 7.2.3 Population Growth Assume that the deer population of Rustic County is 1000 at time $n = 0$ and that the increase from time $n - 1$ to time n is 10 percent of the size at time $n-1$. Write a recurrence relation and an initial condition that define the deer population at time n and then solve the recurrence relation.

SOLUTION Let d_n denote the deer population at time n.

$$d_0 = 1000$$

$$d_n - d_{n-1}$$

$$d_n - d_{n-1} = 0.1\, d_{n-1}$$

$$d_n = 1.1\, d_{n-1}$$

$d_n = 1.1 dn{-}1 = 1.1(1.1 d_{n-2}) =$

$(1.1)^2 (d_{n-2})$

$= \cdot \cdot \cdot = (1.1)^n d_0 = (1.1)^n 1000.$

■

Example 7.2.4 Find an explicit formula for c_n, the minimum number of moves in which the *n*-disk Tower of Hanoi puzzle can be solved (see Example 7.1.8).

SOLUTION In Example 7.1.8 we obtained the recurrence relation

$$c_n = 2c_{n-1} + 1 \qquad (7.2.3)$$

and initial condition $c_1 = 1$. Applying the iterative method to (7.2.3), we obtain

$c_n = 2c_{n-1} + 1$

$= 2(2c_{n-2} + 1) + 1$

$= 22c_{n-2} + 2 + 1$

$= 22(2c_{n-3} + 1) + 2 + 1$

$= 2^3 c_{n-3} + 2^2 + 2 + 1$

...

$= 2^{n-1}c_1 + 2^{n-2} + 2^{n-3} + \cdot\cdot\cdot + 2 + 1$

$= 2^{n-1} + 2^{n-2} + 2^{n-3} + \cdot\cdot\cdot + 2 + 1$

$= 2^n - 1.$

The last step results from the formula for the geometric sum (see Example 2.4.4).

Example 7.2.5

Solve the recurrence relation

$p_n = a - \frac{b}{k}\, p_{n-1}$

for the price p_n in the economics model of Example 7.1.9 by iteration.

SOLUTION:

To simplify the notation, we set $s = -\frac{b}{k}$.

$$p_n = a + sp_{n-1}$$

$$= a + s(a + sp_{n-2})$$

$$= a + as + s^2(a + sp_{n-3})$$

$$= a + as + as^2 + s^3 p_{n-4}$$

$$\vdots$$

$$= a + as + as^2 + \cdots + a\, s^{n-1} + s^n p_0$$

$$= a - \frac{a-as^n}{1-s} + s^n p_0$$

$$= s^n\left(\frac{-a}{1-s}+p_0\right)+\frac{a}{1-s}$$

$$= \left(\frac{-b}{k}\right)^n\left(\frac{-ak}{k+b}+p_0\right)+\frac{ak}{k+b} \qquad (7.2.4)$$

We see that if *b/k* < 1, the term

$$\left(\frac{-b}{k}\right)^n\left(\frac{-ak}{k+b}+p_0\right)$$

becomes small as *n* gets large so that the price tends to stabilize at approximately *ak/(k + b)*. If *b/k* = 1, (7.2.4) shows that p_n oscillates between p_0 and p_1. If *b/k*>1,

(7.2.4) shows that the differences between successive prices increase.

We turn next to a special class of recurrence relations.

Definition 7.2.6 A *linear homogeneous recurrence relation of order k with constant coefficients* is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}, \quad c_k \neq 0, \qquad (7.2.5)$$

Notice that a linear homogeneous recurrence relation of order k with constant coefficients (7.2.5), together with the k initial conditions

$$a_0 = C_0,\ a_1 = C_1, \ldots,\ a_{k-1} = c_{k-1},$$

uniquely defines a sequence $a_0, a_1, \ldots$.

Example 7.2.7

The recurrence relations

$$S_n = 2S_{n-1} \qquad (7.2.6)$$

of Example 7.2.2 and

$$f_n = f_{n-1} + f_{n-2}, \qquad (7.2.7)$$

which defines the Fibonacci sequence, are both linear homogeneous recurrence relations with constant coefficients. The recurrence relation (7.2.6) is of order 1 and (7.2.7) is of order 2.

Example 7.2.8

The recurrence relation

$$a_n = 3a_{n-1}a_{n-2} \qquad (7.2.8)$$

is not a linear homogeneous recurrence relation with constant coefficients. In a linear homogeneous recurrence relation with constant coefficients, each term is of the form ca_k. Terms such as $a_{n-1}a_{n-2}$ are not permitted. Recurrence relations such as (7.2.8) are said to be **nonlinear.**

Example 7.2.10

The recurrence relation $a_n = 3n\, a_{n-1}$is not a linear homogeneous recurrence relation with constant coefficients because the coefficient $3n$ is not constant. It is a linear homogeneous recurrence relation with nonconstant coefficients.

We will illustrate the general method of solving linear homogeneous recurrence relations with constant coefficients by finding an explicit formula for the sequence defined by the recurrence relation

$$a_n = 5\, a_{n-1} - 6\, a_{n-2} \qquad (7.2.9)$$

and initial conditions

$$a_0 = 7,\ a_1 = 16 \qquad (7.2.10)$$

we will search for a solution of the form

$$\boldsymbol{V_n = t^n}.$$

If $\boldsymbol{V_n = t^n}$ is to solve (7.2.9), we must have

$$V_n = 5V_{n-1} - 6V_{n-2}$$

or

$$t^n = 5t^{n-1} - 6t^{n-2}$$

or

$$t^n - 5t^{n-1} + 6t^{n-2} = 0$$

Dividing by t^{n-2}, we obtain the equivalent equation

$$t^2 - 5t + 6 = 0 \qquad (7.2.11)$$

Solving (7.2.11), we find the solutions

$$t = 2, t = 3.$$

At this point, we have two solutions *S* and *T* of (7.2.9), given by

$$S_n = 2^n,\ T_n = 3^n \qquad (7.2.12)$$

We can verify (see Theorem 7.2.11) that if S and T are solutions of (7.2.9), then $bS + dT$, where b and d are any numbers whatever, is also a solution of (7.2.9). In our case, if we define the sequence U by the equation

$$U_n = bS_n + dT_n = b2^n + d3^n,$$

U is a solution of (7.2.9).

To satisfy the initial conditions (7.2.10), we must have

$$7 = U_0 = b2^0 + d3^0 = b + d,$$

$$16 = U_1 = b2^1 + d3^1 = 2b + 3d.$$

Solving these equations for b and d, we obtain $b = 5$ and $d = 2$. Therefore, the sequence U defined by

$$U_n = 5 \cdot 2^n + 2 \cdot 3^n$$

satisfies the recurrence relation (7.2.9) and the initial conditions (7.2.10). We conclude that

$\boldsymbol{a_n = U_n = 5 \cdot 2^n + 2 \cdot 3^n}$ for $n = 0, 1, \ldots$

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At this point we will summarize and justify the techniques used to solve the preceding recurrence relation.

Theorem 7.2.11

Let

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \qquad (7.2.13)$$

be a second-order, linear homogeneous recurrence relation with constant coefficients.

If S and T are solutions of *(*7.2.13*)*, then $U = bS + dT$ is also a solution of *(*7.2.13*)*.

If r is a root of

$$t^2 - c_1 t - c_2 = 0, \qquad (7.2.14)$$

then the sequence r^n, $n = 0, 1, \ldots,$ is a solution of *(*7.2.13*)*.

If a is the sequence defined by *(*7.2.13*)*,

$$a_0 = C_0,\ a_1 = C_1, \qquad (7.2.15)$$

and r_1 and r_2 are roots of (7.2.14) with $r_1 \neq r_2$, then there exist constants b and d

such that

$$a_n = br_1^n + dr_2^n \qquad n = 0, 1, \ldots$$

Example 7.2.12

More Population Growth

Assume that the deer population of Rustic County is 200 at time $n = 0$ and 220 at time $n = 1$ and that the increase from time $n - 1$ to time n is twice the increase from time $n-2$ to time $n - 1$. Write a recurrence relation and an initial condition that define the deer population at time n and then solve the recurrence relation.

SOLUTION Let d_n denote the deer population at time n. We have the initial conditions

$$d_0 = 200, d_1 = 220.$$

The increase from time $n-1$ to time n is $d_n - d_{n-1}$, and the increase from time $n-2$ to time $n-1$ is $d_{n-1} - d_{n-2}$. Thus we obtain the recurrence relation

$$d_n - d_{n-1} = 2\,(d_{n-1} - d_{n-2})\,,$$

which may be rewritten

$$d_n = 3d_{n-1} - 2d_{n-2}.$$

To solve this recurrence relation, we first solve the quadratic equation

$$t^2 - 3t + 2 = 0$$

to obtain roots 1 and 2. The sequence d is of the form

$$d_n = b \cdot 1^n + c \cdot 2^n = b + c2^n.$$

To meet the initial conditions, we must have

$$200 = d_0 = b + c, \qquad 220 = d_1 = b + 2c$$

Solving for b and c, we find that $b = 180$ and $c = 20$. Thus d_n is given by

$$d_n = 180 + 20 \cdot 2n.$$

As in Example 7.2.3, the growth is exponential

Example 7.2.13

Find an explicit formula for the Fibonacci sequence.

SOLUTION The Fibonacci sequence is defined by the linear homogeneous, second-order recurrence relation

$$f_n - f_{n-1} - f_{n-2} = 0 \quad n \geq 3,$$

and initial conditions $f_1 = 1$ and $f_2 = 1$. We begin by using the quadratic formula to solve $t^2 - t - 1 = 0$. The solutions are

$$t = \frac{1 \pm \sqrt{5}}{2}.$$

Thus the solution is of the form

$$f_n = b\left(\frac{1+\sqrt{5}}{2}\right)^n + d\left(\frac{1-\sqrt{5}}{2}\right)^n$$

To satisfy the initial conditions, we must have

$$b\left(\frac{1+\sqrt{5}}{2}\right)^1 + d\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$$

$$b\left(\frac{1+\sqrt{5}}{2}\right)^2 + d\left(\frac{1-\sqrt{5}}{2}\right)^2 = 1$$

Solving these equations for *b* and *d*, we obtain

$$b = \frac{1}{\sqrt{5}}, \; d = -\frac{1}{\sqrt{5}}$$

Therefore, an explicit formula for the Fibonacci sequence is

$$\boldsymbol{f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

■

Theorem 7.2.14

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ (7.2.16)

be a second-order linear homogeneous recurrence relation with constant coefficients. Let **a** be the sequence satisfying (7.2.16) and $a_0 = C_0$, $a_1 = C_1$.

If both roots of

$$\boldsymbol{t^2 - c_1 \mathrm{t} - c_2 = 0} \qquad (7.2.17)$$

are equal to **r**, then there exist constants b and d such that

$$\boldsymbol{a_n = \mathrm{b}r^n + \mathrm{dn}r^n \qquad \mathrm{n} = 0, 1, \ldots.}$$

■

Example 7.2.15:

Solve the recurrence relation

$$\boldsymbol{d_n = 4(d_{n-1} - d_{n-2})} \qquad (7.2.18)$$

subject to the initial conditions

$$d_0 = 1 = d_1.$$

Solution:

$$d_n - 4(d_{n-1} - d_{n-2}) = 0$$

$$t^2 - 4t + 4 = 0$$

$$(t-2)^2 = 0$$

$$t = 2$$

Thus we obtain the solution

$d_n = a\,2^n + bn2^n$

Since $d_0 = 1 = d_1$, we have

$1 = a\,2^0 + b(0)\,2^0 \rightarrow a = 1$

and

$1 = a\,2^1 + b\,(1)\,2^1 \rightarrow b = -\frac{1}{2}$

Therefore, the solution of (7.2.18) is

$$d_n = 2^n - \frac{1}{2}\mathbf{n}2^n$$

$\blacksquare$