Factor each of the following expressions.

(a) $x^{2/3} - 2x^{1/3}$

(b) $\ln x - e^x \ln x$

(c) $\sin x \cos x + \cos x$

(d) $e^{2x} - 2e^x + 1$

(e) $3(x+2)^2(3x-4)^{-2} + (x+2)^3(-2)(3x-4)^{-3}(3)$

Solutions on the next page

(a) $x^{2/3} - 2x^{1/3} = x^{1/3}(x^{1/3} - 2)$

This expression has two terms. We first look for common factors, and here we see an x in each term.
We want to factor out the smallest power of x. In this case, $1/3 < 2/3$, so we factor out $x^{1/3}$.

$$x^{2/3} - 2x^{1/3} = x^{1/3}(x^{?} - 2)$$

To determine what goes in place of ?, we must calculate $\frac{x^{2/3}}{x^{1/3}}$, which simplifies to $x^{1/3}$ using laws of exponents. is the smallest power of x contained in both terms, and $x^{1/3} \cdot x^{1/3} = x^{2/3}$.

$$x^{2/3} - 2x^{1/3} = x^{1/3}(x^{1/3} - 2)$$

Note that if we distribute $x^{1/3}$ to both terms, we get back to where we started.

(b) $\ln x - e^x \ln x = (\ln x)(1 - e^x)$

Here, we just notice that both terms contain a factor of $\ln x$, and we factor it out of both terms:

$$\ln x - e^x \ln x = (\ln x)(1 - e^x)$$

Again, distributing would result in the expression we started with.

(c) $\sin x \cos x + \cos x = (\cos x)(\sin x + 1)$

Again, here we just see the common factor of $\cos x$ in each term.

$$\sin x \cos x + \cos x = (\cos x)(\sin x + 1)$$

(d) $e^{2x} - 2e^x + 1 = (e^x - 1)(e^x - 1) = (e^x - 1)^2$

The expression is in quadratic form, because $e^{2x} = (e^x)^2$. If we substitute $u = e^x$, we get

$$e^{2x} - 2e^x + 1 = (e^x)^2 - 2e^x + 1 = u^2 - 2u + 1$$

Now we can factor the u version of the expression:

$$u^2 - 2u + 1 = (u - 1)(u - 1)$$

Substituting $u = e^x$ back in, we see

$$e^{2x} - 2e^x + 1 = (u - 1)(u - 1) = (e^x - 1)(e^x - 1)$$

(e) $3(x+2)^2(3x-4)^{-2} + (x+2)^3(-2)(3x-4)^{-3}(3) = 3(x+2)^2(3x-4)^{-3}(x-8)$

This expression has two monster terms in it. Each of the two monster terms contains a factor of 3, some power of $x + 2$, and some power of $3x - 4$.
The smallest exponent on $x + 2$ is 2, so we factor out $(x + 2)^2$.
The smallest exponent on $3x - 4$ is -3 (remember $-3 < -2$), so we factor out $(3x - 4)^{-3}$.
We will be left with two terms.

$$3(x+2)^2(3x-4)^{-2} + (x+2)^3(-2)(3x-4)^{-3}(3) = 3(x+2)^2(3x-4)^{-3}\left[\qquad + \qquad\right]$$

To determine what goes in brackets, we take each initial term and divide by what we have factored out, using laws of exponents to simplify.

$$\frac{3(x+2)^2(3x-4)^{-2}}{3(x+2)^2(3x-4)^{-3}} = \frac{(3x-4)^{-2}}{(3x-4)^{-3}} = (3x-4)^{-2-(-3)} = (3x-4)^1 = 3x-4$$

$$\frac{(x+2)^3(-2)(3x-4)^{-3}(3)}{3(x+2)^2(3x-4)^{-3}} = \frac{(x+2)^3(-2)}{(x+2)^2} = (-2)(x+2)^1 = -2x-4$$

Filling in the brackets:

$$3(x+2)^2(3x-4)^{-2} + (x+2)^3(-2)(3x-4)^{-3}(3) = 3(x+2)^2(3x-4)^{-3}\left[(3x-4) + (-2x-4)\right]$$

We can check that if we expand the expression on the right, we get back to where we started.
And now combining what is in the brackets:

$$3(x+2)^2(3x-4)^{-2} + (x+2)^3(-2)(3x-4)^{-3}(3) = 3(x+2)^2(3x-4)^{-3}(x-8)$$