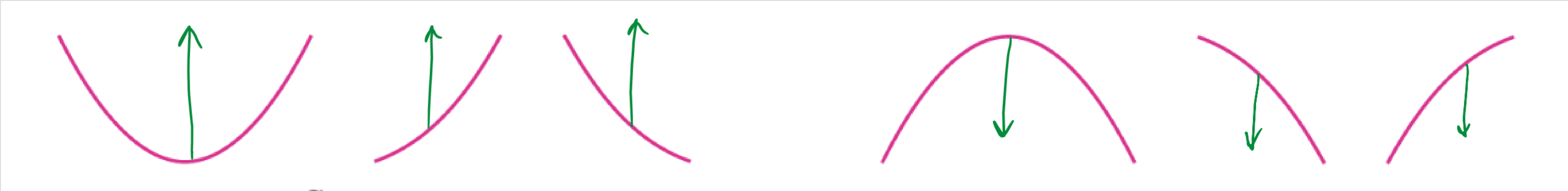
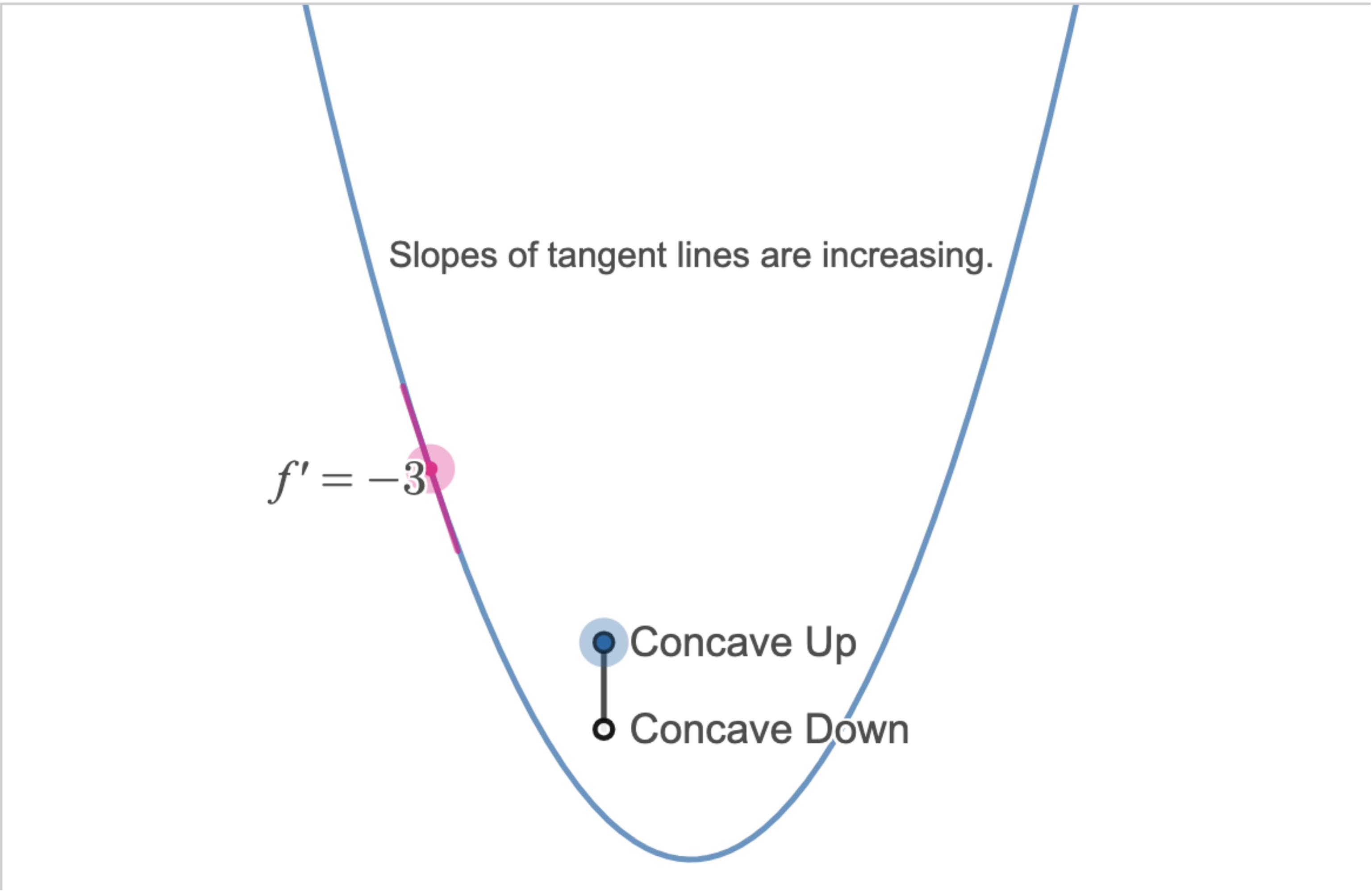
4.4 The Second Derivative and Concavity

In the previous section, we studied the increasing/decreasing behavior of a function, as determined by the sign of the derivative. Another important property is concavity, which refers to the way the graph bends. Informally, a curve is *concave up* if it bends up and *concave down* if it bends down (Figure 1).



To analyze concavity in a precise fashion, let's examine how concavity is related to tangent lines and derivatives. Observe in Figure 2 that when f is concave up, f' is increasing (the slopes of the tangent lines increase as we move to the right). Similarly, when f is concave down, f' is decreasing. This suggests the following definition.



DEFINITION

Concavity

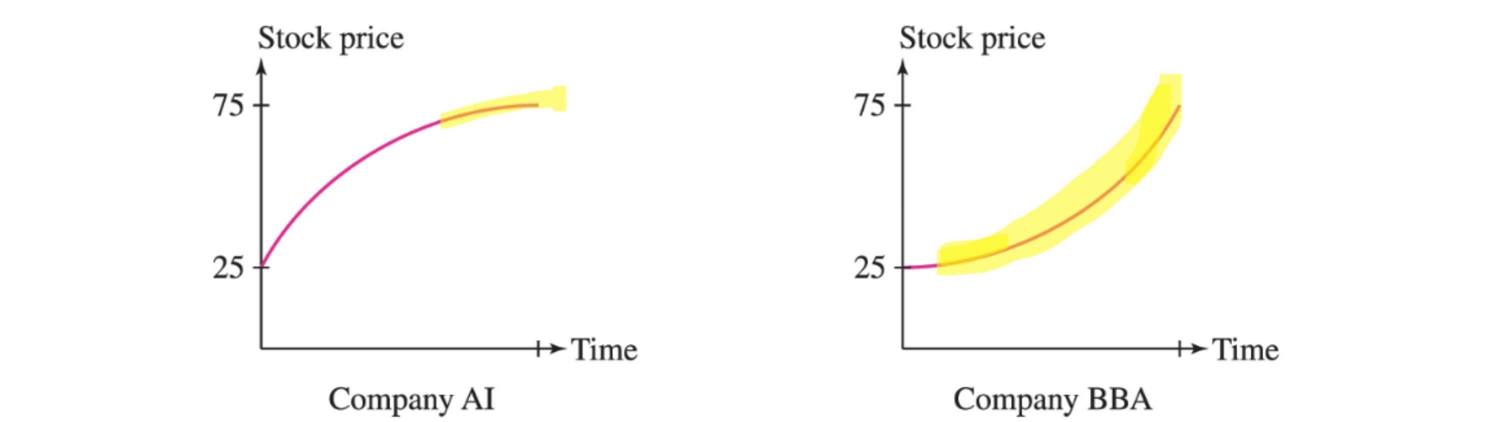
Let f be a differentiable function on an open interval (a, b). Then

- f is **concave up** on (a, b) if f' is increasing on (a, b).
- f is **concave down** on (a, b) if f' is decreasing on (a, b).

EXAMPLE 1

Concavity and Stock Prices

The stocks of two companies, Arenot Industries (AI) and Blurbenthal Business Associates (BBA), went up in value, and both currently sell for $75 (Figure 3). However, one is clearly a better investment than the other, assuming these trends continue in the same manner. Explain in terms of concavity.



Solution

The graph of Stock AI is concave down, so its growth rate (first derivative) is declining as time goes on. The graph of Stock BBA is concave up, so its growth rate is increasing. If these trends continue, Stock BBA is the better investment.


Max
Min

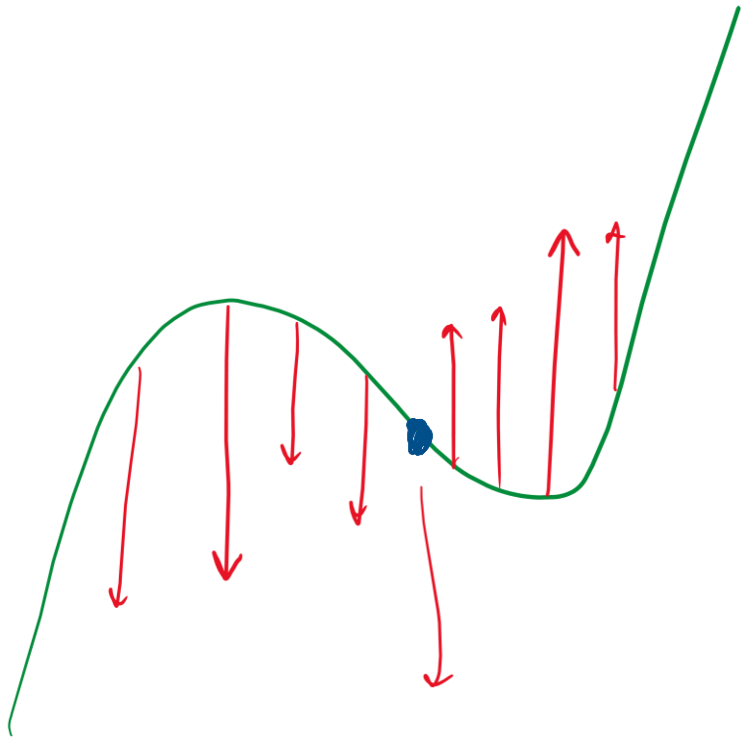
The concavity of a function is determined by the *sign* of its second derivative. Indeed, if $f''(x) > 0$, then f' is increasing and hence f is concave up. Similarly, if $f''(x) < 0$, then f' is decreasing and f is concave down.

THEOREM 1

Test for Concavity

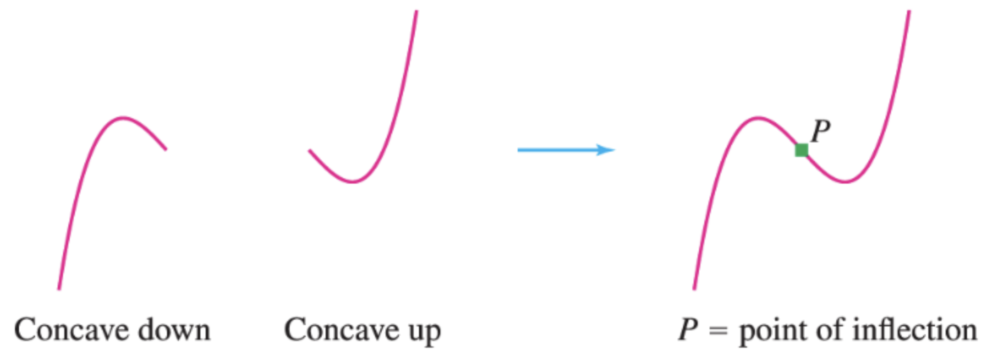
Assume that $f''(x)$ exists for all $x \in (a, b)$.

- If $f''(x) > 0$ for all $x \in (a, b)$, then f is concave up on (a, b).
- If $f''(x) < 0$ for all $x \in (a, b)$, then f is concave down on (a, b).



Point of inflection

Of special interest are the points on the graph where the concavity changes. We say that $P = (c, f(c))$ is a **point of inflection** of f if the concavity changes from up to down or from down to up at $x = c$. Figure 4 shows a curve made up of two arcs—one is concave down and one is concave up (the word "arc" refers to a piece of a curve). The point P where the arcs are joined is a point of inflection. We will denote points of inflection in graphs by a solid square ■.

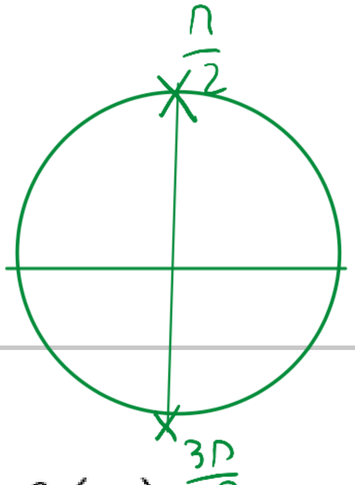


THEOREM 2

Test for Inflection Points

If $f''(c) = 0$ or $f''(c)$ does not exist and $f''(x)$ changes sign at $x = c$, then f has a point of inflection at $x = c$.

EXAMPLE 2



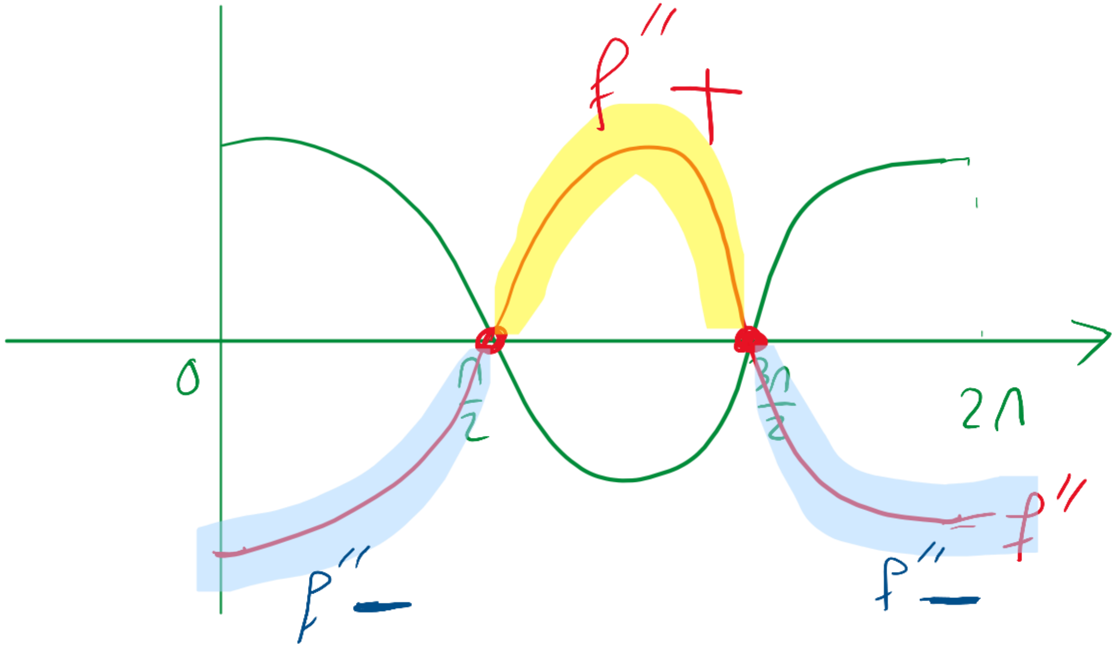
Find the points of inflection of $f(x) = \cos x$ on $[0, 2\pi]$.

$f'(x) = -\sin x$

$f''(x) = -\cos x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

points of inflection

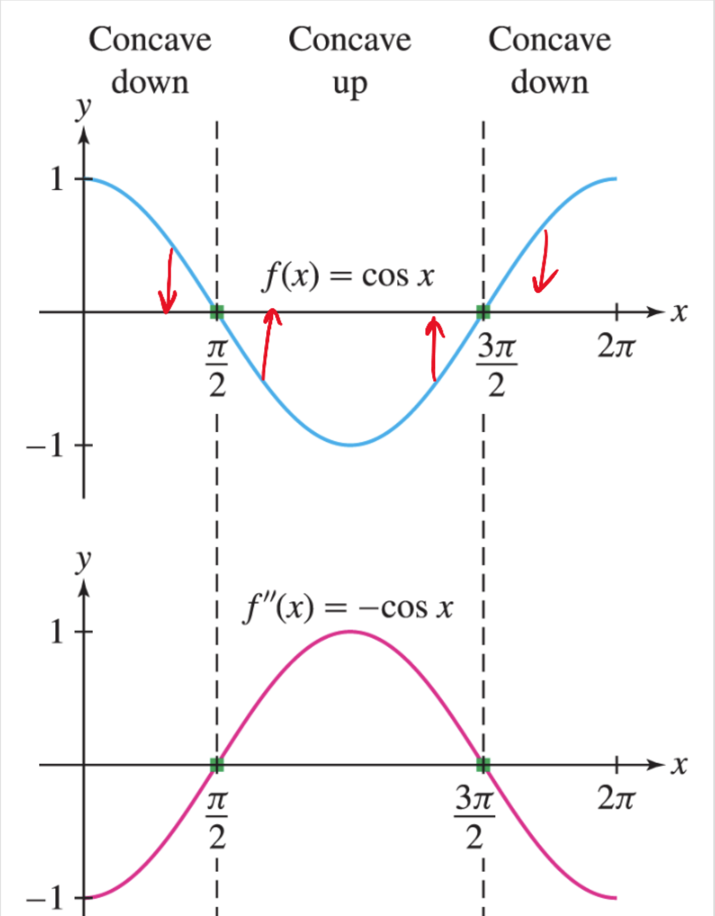


Solution

We have

$$f''(x) = -\cos\ x, \quad \text{and} \quad f''(x) = 0 \quad \text{for} \quad x = \frac{\pi}{2}, \frac{3\pi}{2}.$$

Figure 5 shows that $f''(x)$ changes sign at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, so f has a point of inflection at both points.



$f'(x) = 15x^4 - 20x^3$

$f''(x) = 60x^3 - 60x^2 = 0$

$60x^2(x-1) = 0$

$\boxed{x=0}$ $\boxed{x=1}$

$\frac{60x^2}{60} = \frac{0}{60}$

$\sqrt{x^2} = \sqrt{0}$

$x = 0$

EXAMPLE 3

Points of Inflection and Intervals of Concavity

Find the points of inflection and the intervals on which $f(x) = 3x^5 - 5x^4 + 1$ is concave up and concave down.

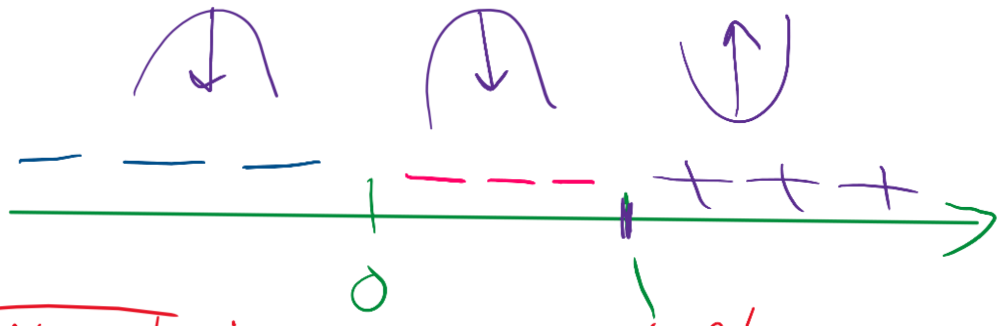
$f''(2) = 60(2)^3 - 60(2^2) = +$

$f''(-1) = 60(-1)^3 - 60(-1)^2 =$

$= -60 - 60$

$= -120$

$f''\left(\frac{1}{2}\right) = \overset{30}{\cancel{60}}\left(\frac{1}{\cancel{2}}\right)^3 - 60\left(\frac{1}{2}\right)^2 = \frac{15}{2} - 15$



$\boxed{x=1}$ is a point of inflection.

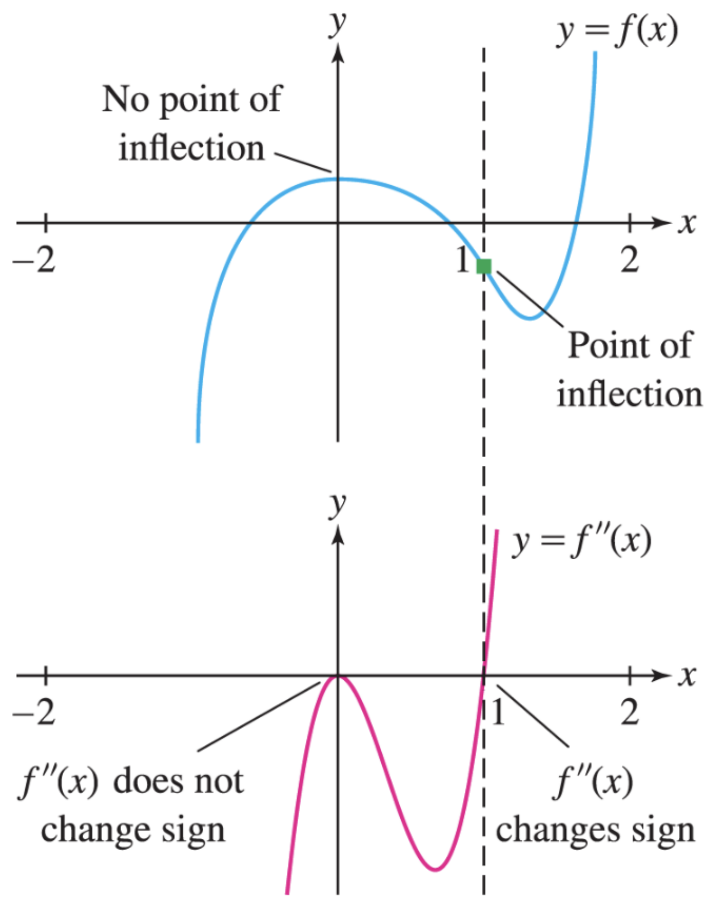
Solution

The first derivative is $f'(x) = 15x^4 - 20x^3$ and

$$f''(x) = 60x^3 - 60x^2 = 60x^2(x-1)$$

The zeros of $f''(x) = 60x^2(x-1)$ are $x = 0$ and $x = 1$. They divide the x-axis into three intervals: $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$. We determine the sign of $f''(x)$ and the concavity of f by computing test values within each interval (Figure 6):

Interval	Test value	Sign of $f''(x)$	Behavior of $f(x)$
$(-\infty, 0)$	$f''(-1) = -120$	$-$	Concave down
$(0, 1)$	$f''\left(\frac{1}{2}\right) = -\dfrac{15}{2}$	$-$	Concave down
$(1, \infty)$	$f''(2) = 240$	$+$	Concave up



Rogawski et al., *Calculus: Early Transcendentals*, 4e, © 2019 W. H. Freeman and Company

FIGURE 6 Graph of $f(x) = 3x^5 - 5x^4 + 1$ and its second derivative.

Since the concavity changes at $x = 1$ there is an inflection point there. The inflection point is $(1, -1)$. Note that, even though $f''(0) = 0$, there is not an inflection point at $x = 0$ because the concavity does not change at $x = 0$.

Usually, we find the inflection points by solving $f''(x) = 0$. However, an inflection point can also occur at a point $(c, f(c))$, where $f''(c)$ does not exist.

Ex: Find point of inflection $f(x) = x^{5/3}$.

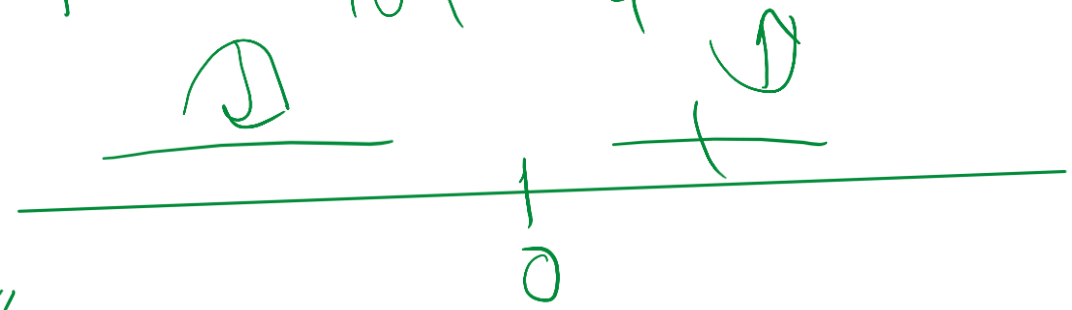
$$f'(x) = \frac{5}{3}x^{\frac{2}{3}} \longrightarrow f''(x) = \frac{10}{9}x^{-\frac{1}{3}} = \frac{10}{9x^{\frac{1}{3}}} = \frac{10}{9\sqrt[3]{x}}$$

$$f''(x) = \frac{10}{9x^{\frac{1}{3}}} = 0$$

$10 \neq 0$ No solution

at $\boxed{x=0}$ f'' Does Not Exist.

$$f''(1) = \frac{10}{9\sqrt[3]{1}} = \frac{10}{9}$$



$$\frac{10}{9x^{\frac{1}{3}}} \begin{cases} + & x>0 \\ - & x<0 \end{cases}$$

$$f''(-1) = \frac{10}{9\sqrt[3]{-1}} = \frac{10}{-9} \qquad f''(1) = \frac{10}{\ }$$

$\boxed{x=0}$ is an I.P

EXAMPLE 4

A Case Where the Second Derivative Does Not Exist

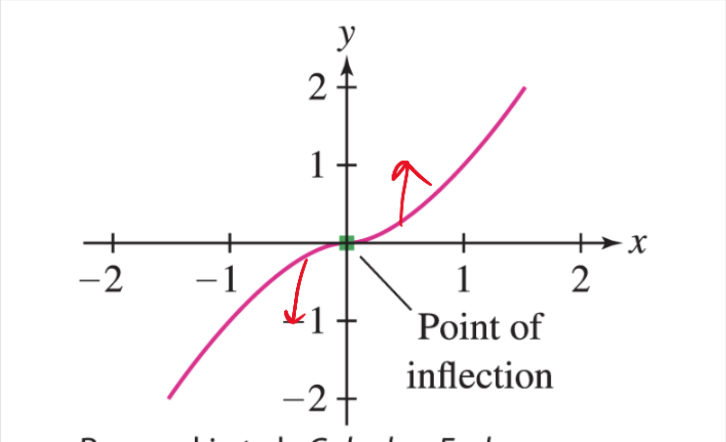
Find the points of inflection of $f(x) = x^{5/3}$.

Solution

In this case, $f'(x) = \frac{5}{3}x^{2/3}$ and $f''(x) = \frac{10}{9}x^{-1/3}$. Although $f''(0)$ does not exist, $f''(x)$ does change sign at $x = 0$:

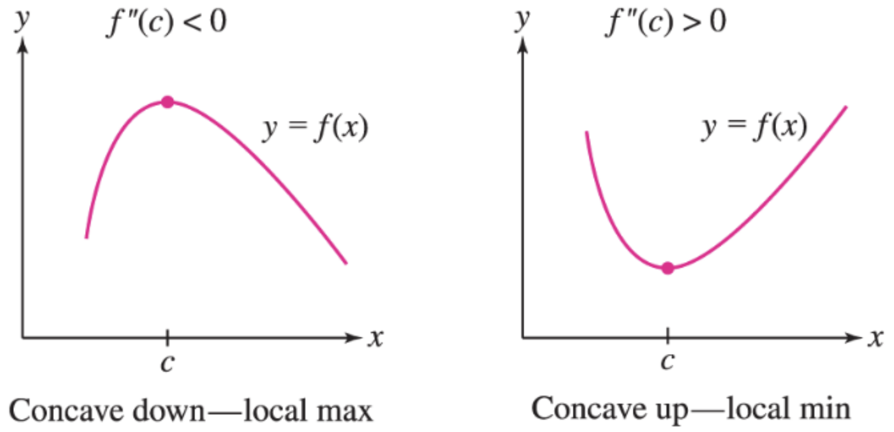
$$f''(x) = \frac{10}{9x^{1/3}} \begin{cases} > 0 & \text{for } x > 0 \\ < 0 & \text{for } x < 0 \end{cases}$$

Therefore, the concavity of f changes at $x = 0$, and $(0, 0)$ is a point of inflection (Figure 7).



Second Derivative Test for Critical Points

There is a simple test for critical points based on concavity. Suppose that $f'(c) = 0$. As we see in Figure 9, $f(c)$ is a local max if f is concave down, and it is a local min if f is concave up. Concavity is determined by the sign of $f''(x)$, so we obtain the Second Derivative Test in Theorem 3. (See Exercise 73 for a detailed proof.)



THEOREM 3

Second Derivative Test

Let c be a critical point of $f(x)$. If $f''(c)$ exists, then

- $f''(c) > 0 \;\Rightarrow\; f(c)$ is a local minimum.
- $f''(c) < 0 \;\Rightarrow\; f(c)$ is a local maximum.
- $f''(c) = 0 \;\Rightarrow\;$ inconclusive: $f(c)$ may be a local min, a local max, or neither.

Mnemonic Device:

$f''(c) > 0 \Rightarrow$ local min

$f''(c) < 0 \Rightarrow$ local max

EXAMPLE 6

Second Derivative Test Inconclusive

Analyze the critical points of $f(x) = x^5 - 5x^4$.

$f'(x) = 5x^4 - 20x^3 = 0 \longrightarrow 5x^3(x-4) = 0 \longrightarrow \boxed{x=0}, \ \boxed{x=4}$

$f''(x) = 20x^3 - 60x^2 \longrightarrow f''(0) = 20(0)^3 - 60(0)^2 = 0$

$f''(4) = 20(4)^3 - 60(4)^2 = 320 > 0$

$x=0$ is a local max & $x=4$ is a local Min

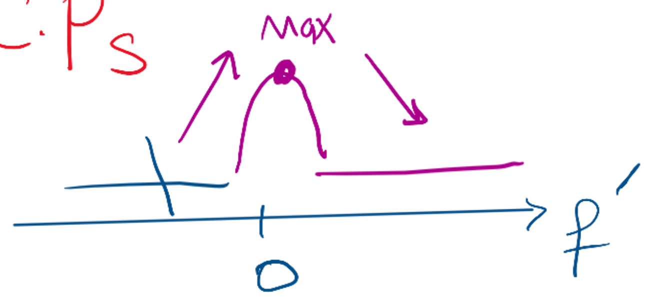
$f'(-1) = \cancel{5(-1)^3}\,\cancel{(-1-4)}$

$\quad - \qquad -$

$f'(1) = 5(1)^3(1-4)$

$\quad + \qquad -$

C.Ps



Solution

The first two derivatives are

$$\begin{aligned} f'(x) &= 5x^4 - 20x^3 = 5x^3(x-4) \\ f''(x) &= 20x^3 - 60x^2 \end{aligned}$$

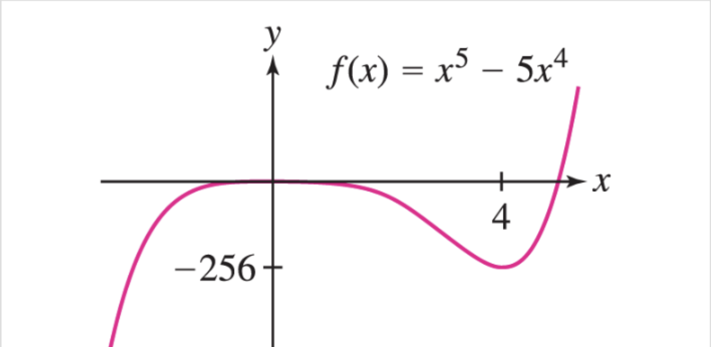
The critical points are $c = 0, 4$, and the Second Derivative Test yields

$$\begin{aligned} f''(0) &= 0 && \Rightarrow \text{ Second Derivative Test fails} \\ f''(4) &= 320 > 0 && \Rightarrow f(4) \text{ is a local min} \end{aligned}$$

The Second Derivative Test fails at $x = 0$, so we fall back on the First Derivative Test. Choosing test points to the left and right of $x = 0$, we find

$$\begin{aligned} f'(-1) &= 5 + 20 = 25 > 0 && \Rightarrow f'(x) \text{ is positive on } (-\infty, 0) \\ f'(1) &= 5 - 20 = -15 < 0 && \Rightarrow f'(x) \text{ is negative on } (0, 4) \end{aligned}$$

Since $f'(x)$ changes from $+$ to $-$ at $x = 0$, $f(0)$ is a local max (Figure 12).



4.4 SUMMARY

- A differentiable function f is *concave up* on (a, b) if f' is increasing and *concave down* if f' is decreasing on (a, b).
- The signs of the first two derivatives provide the following information:

First derivative	Second derivative
$f' > 0 \quad \Rightarrow \quad f$ is increasing	$f'' > 0 \quad \Rightarrow \quad f$ is concave up
$f' < 0 \quad \Rightarrow \quad f$ is decreasing	$f'' < 0 \quad \Rightarrow \quad f$ is concave down

- A *point of inflection* is a point $(c, f(c))$ where the concavity changes from concave up to concave down, or vice versa.
- Second Derivative Test: If $f'(c) = 0$ and $f''(c)$ exists, then
  - $f(c)$ is a local maximum value if $f''(c) < 0$
  - $f(c)$ is a local minimum value if $f''(c) > 0$
  - The test fails if $f''(c) = 0$

  If this test fails, use the First Derivative Test.