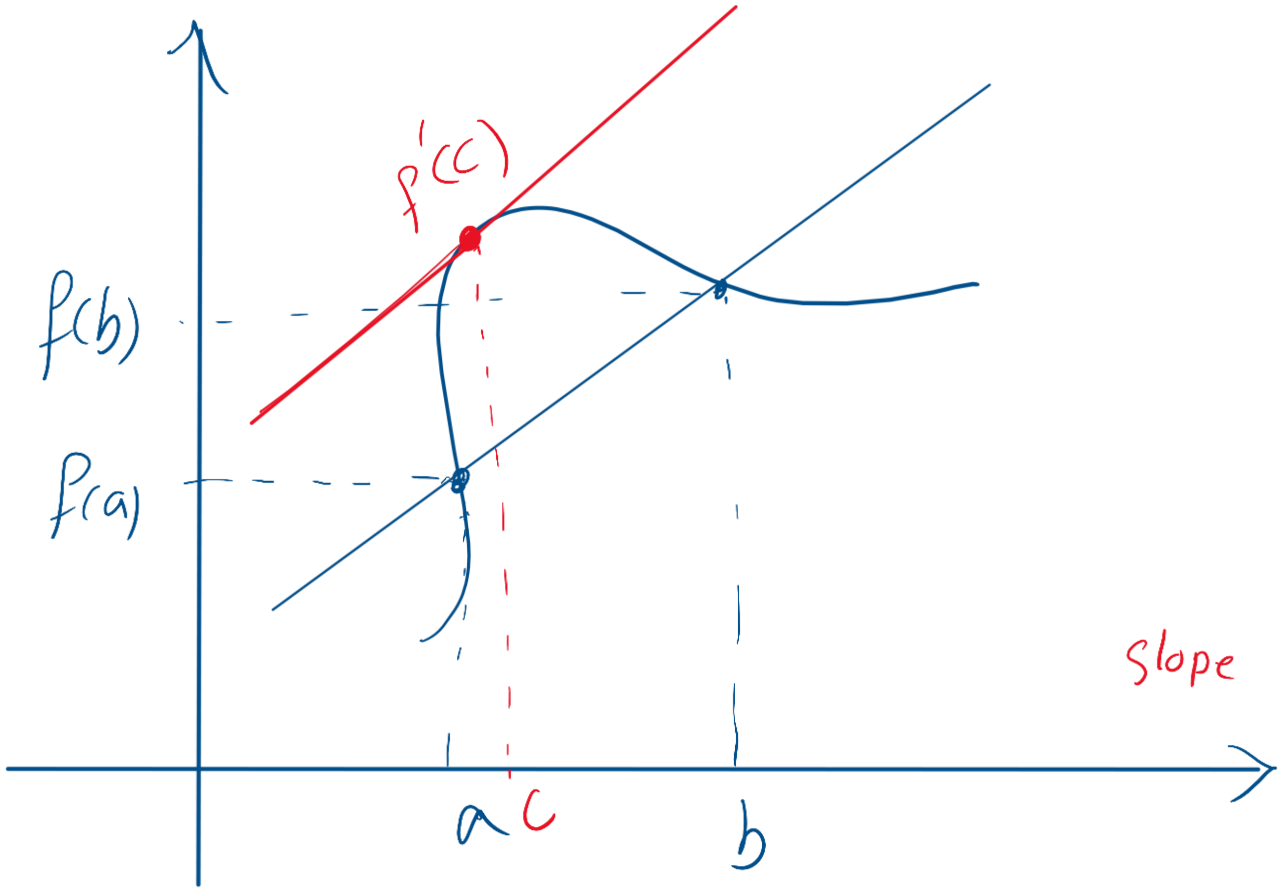
4.3 The Mean Value Theorem and Monotonicity

MVT

We have taken for granted that if $f'(x)$ is positive, the function f is increasing, and if $f'(x)$ is negative, f is decreasing. In this section, we prove this rigorously using an important result called the Mean Value Theorem (MVT). Then we develop a method for "testing" critical points—that is, for determining whether they correspond to local maxima, local minima, or neither.



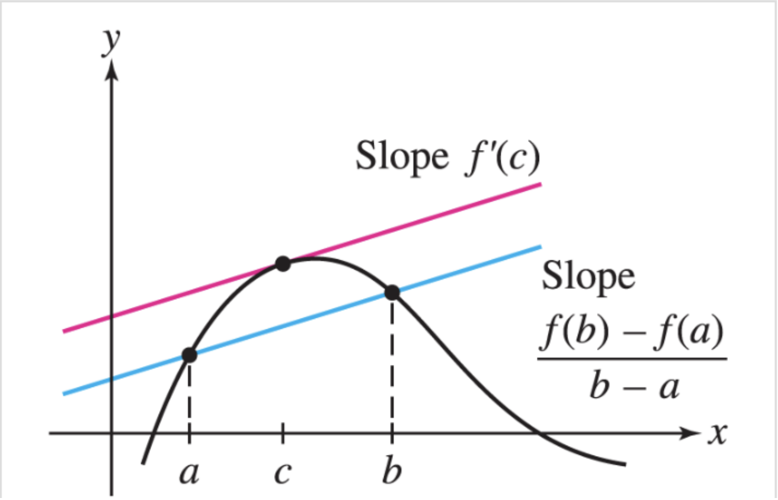
$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

slope of the Tangent line

slope of the Secant line

The MVT says that a secant line between two points $(a, f(a))$ and $(b, f(b))$ on a graph is parallel to at least one tangent line in the interval (a, b) (Figure 1). Since the secant line between $(a, f(a))$ and $(b, f(b)$ has slope $\dfrac{f(b) - f(a)}{b - a}$ and since two lines are parallel if they have the same slope, the MVT is claiming that there exists a point c between a and b such that

$$\underbrace{f'(c)}_{\text{Slope of tangent line}} = \underbrace{\frac{f(b) - f(a)}{b - a}}_{\text{Slope of secant line}}$$



THEOREM 1

The Mean Value Theorem

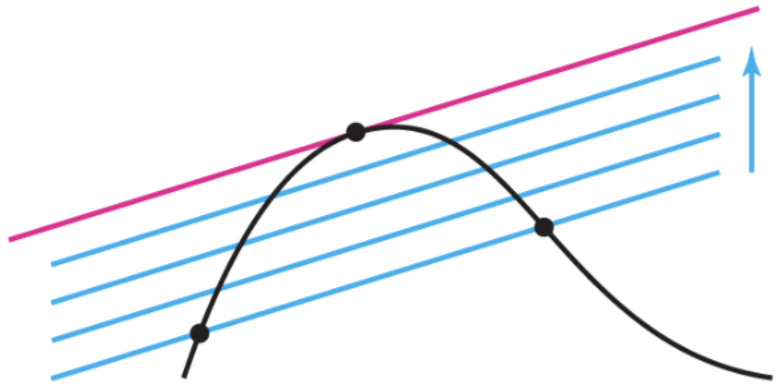
Assume that f is continuous on the closed interval $[a, b]$ and differentiable on (a, b). Then there exists at least one value c in (a, b) such that

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

Rolle's Theorem (Section 4.2) is the special case of the MVT in which $f(a) = f(b)$. In this case, the conclusion is that $f'(c) = 0$.

GRAPHICAL INSIGHT

Imagine what happens when a secant line is moved parallel to itself. Eventually, it becomes a tangent line, as shown in Figure 2. This is the idea behind the MVT. We present a formal proof at the end of this section.



$$\frac{f(b)-f(a)}{b-a} = \frac{f(9)-f(1)}{9-1} = \frac{\cancel{\sqrt{9}}^{\,3} - \cancel{\sqrt{1}}^{\,1}}{8} = \frac{3-1}{8} = \frac{2}{8} = \boxed{\frac{1}{4}}$$

EXAMPLE 1

$$f'(x) = \frac{1}{2\sqrt{x}} \to f'(c) = \frac{1}{2\sqrt{c}} \overset{MVT}{=} \frac{1}{4} \Rightarrow \frac{2\sqrt{c}}{\cancel{2}} = \frac{4}{2} \to \sqrt{c}^{\,2} = 2^2 \Rightarrow \boxed{c=4}$$

Verify the MVT with $\boxed{f(x) = \sqrt{x}}$, $a = 1$, and $b = 9$.

Solution

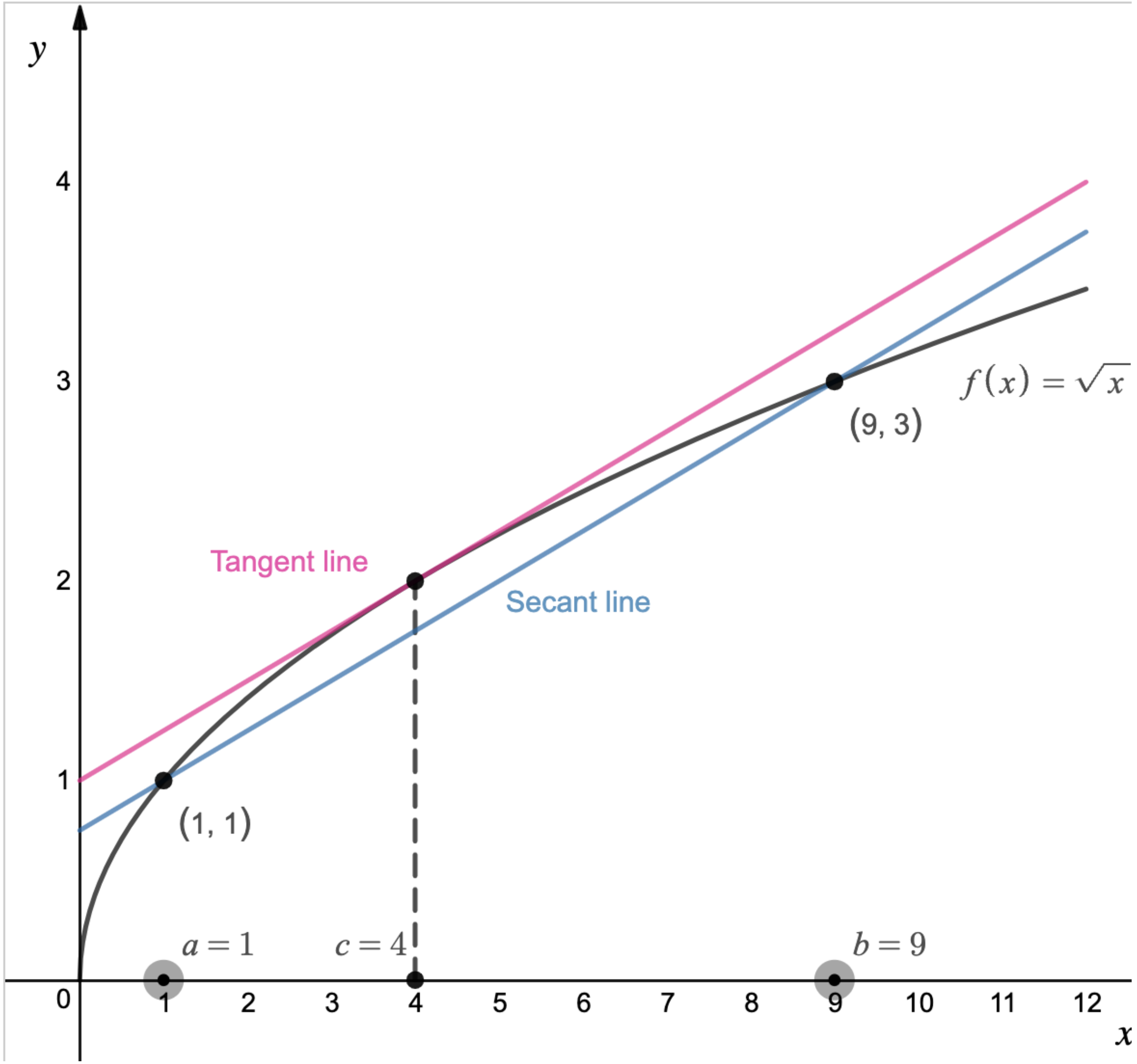
First, compute the slope of the secant line (Figure 3):

$$\frac{f(b) - f(a)}{b - a} = \frac{\sqrt{9} - \sqrt{1}}{9 - 1} = \frac{3 - 1}{9 - 1} = \frac{1}{4}$$

We must find c such that $f'(c) = 1/4$. The derivative is $f'(x) = \frac{1}{2}x^{-1/2}$, and

$$f'(c) = \frac{1}{2\sqrt{c}} = \frac{1}{4} \quad \Rightarrow \quad 2\sqrt{c} = 4 \quad \Rightarrow \quad \boxed{c = 4}$$

The value $c = 4$ lies in $(1, 9)$ and satisfies $f'(4) = \frac{1}{4}$. This verifies the MVT.

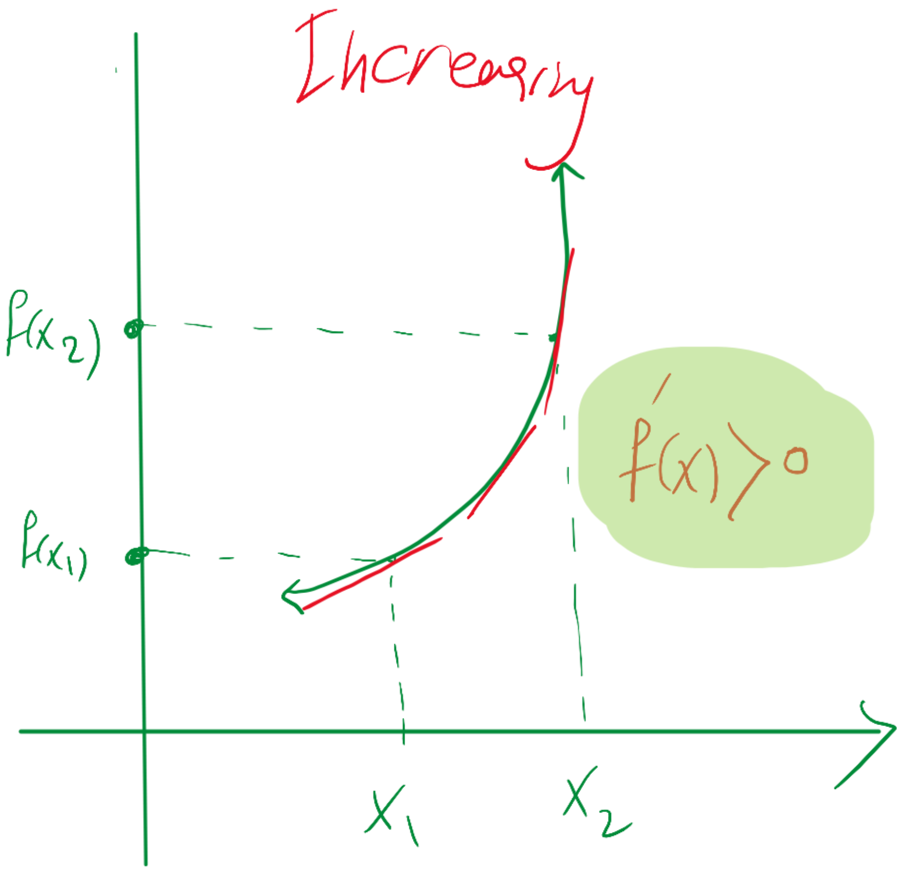

y
4
3
2
1
0
Tangent line
Secant line
(1, 1)
(9, 3)
$f(x) = \sqrt{x}$
$a = 1$
$c = 4$
$b = 9$
1
2
3
4
5
6
7
8
9
10
11
12
x

$f(x) = C \longrightarrow f'(x) = 0$

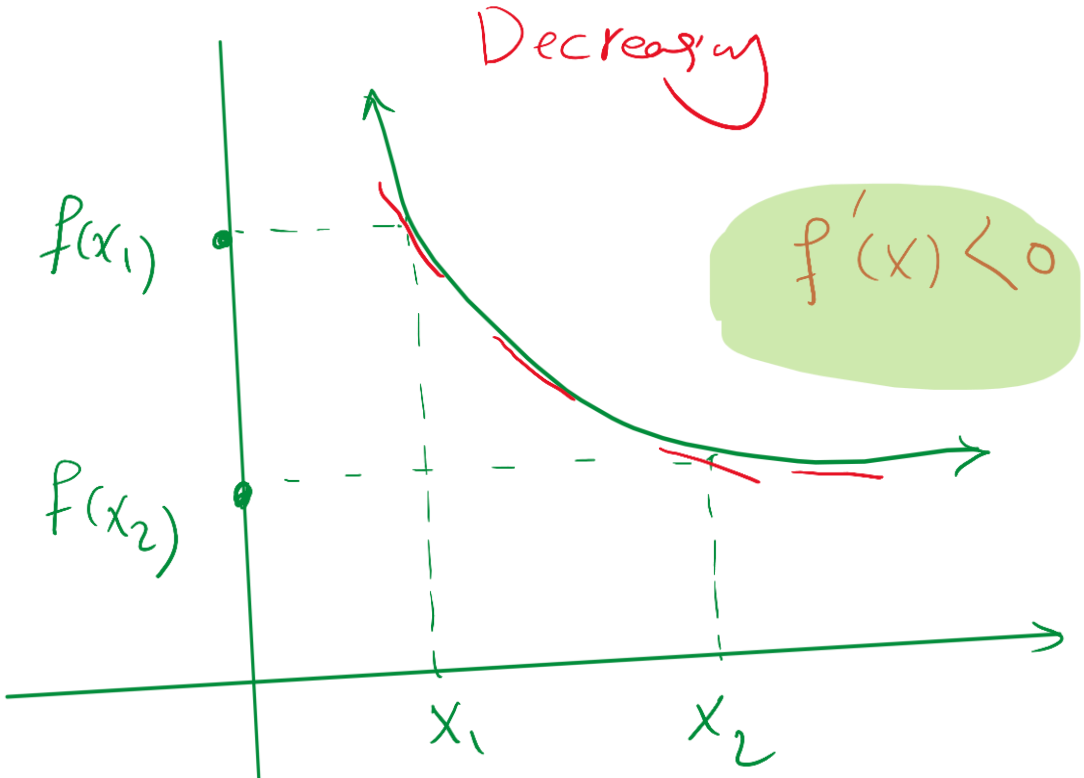
As a first application, we prove that a function with zero derivative is constant.

COROLLARY

If f is differentiable and $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b). In other words, $f(x) = C$ for some constant C.



$f(x_1) < f(x_2) \to x_1 < x_2$



$f(x_1) > f(x_2) \to x_1 < x_2$

Increasing / Decreasing Behavior of Functions

We prove now that the sign of the derivative determines whether a function f is increasing or decreasing. Recall that f is

- **Increasing on** (a, b) if $f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$ such that $x_1 < x_2$.
- **Decreasing on** (a, b) if $f(x_1) > f(x_2)$ for all $x_1, x_2 \in (a, b)$ such that $x_1 < x_2$.

We say that f is **monotonic** on (a, b) if it is either increasing or decreasing on (a, b).

THEOREM 2

100% in the Exams!

The Sign of the Derivative

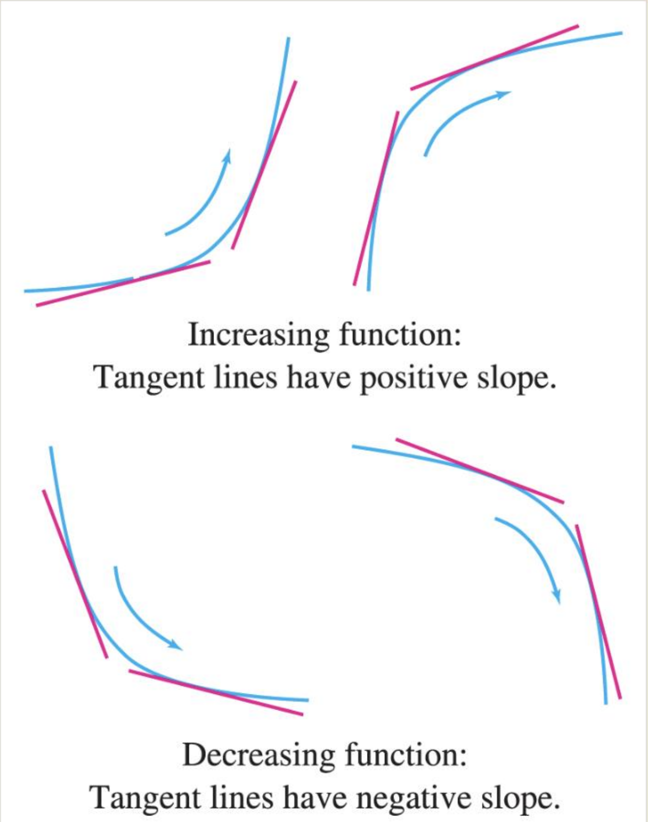
Let f be a differentiable function on an open interval (a, b).

- If $f'(x) > 0$ for $x \in (a, b)$, then f is increasing on (a, b).
- If $f'(x) < 0$ for $x \in (a, b)$, then f is decreasing on (a, b).

GRAPHICAL INSIGHT

Theorem 2 confirms our graphical intuition (Figure 4):

- $f'(x) > 0 \quad \Rightarrow \quad$ tangent lines have positive slope $\quad \Rightarrow \quad f$ increasing
- $f'(x) < 0 \quad \Rightarrow \quad$ tangent lines have negative slope $\quad \Rightarrow \quad f$ decreasing

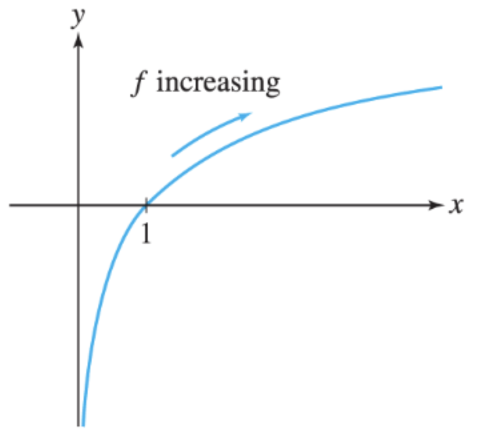


EXAMPLE 2

Show that $f(x) = \ln x$ is increasing.

Solution

The derivative $f'(x) = x^{-1}$ is positive on the domain $\{x : x > 0\}$, so $f(x) = \ln x$ is increasing (Figure 5).



$f'(x) = 2x - 2 = 0$

$2(x-1) = 0$

$x = 1$

$f'(0) = 2(0) - 2 = -2$

$f'(2) = 2(2) - 2 = 2$

$- \; - \; - \;|\; + \; + \; + \; + \quad f'$

$0 \quad 1 \quad 2$

EXAMPLE 3

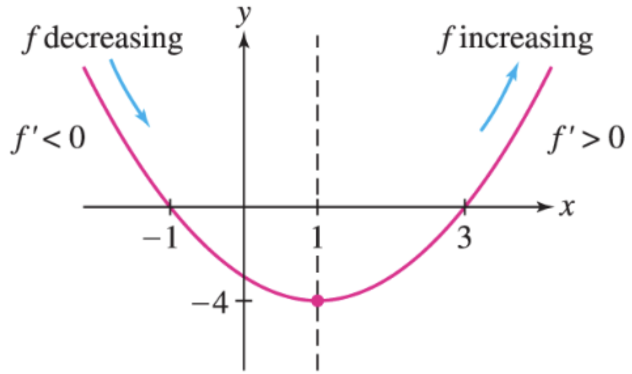
Dec: $(-\infty, 1)$

Inc: $(1, \infty)$

Find the intervals on which $f(x) = x^2 - 2x - 3$ is monotonic.

Solution

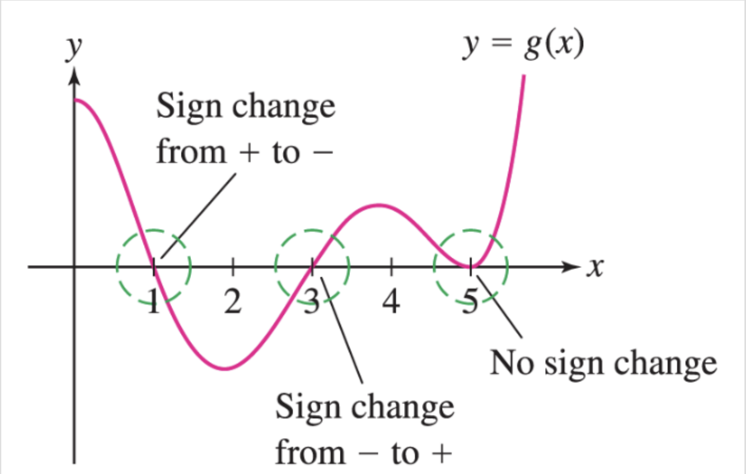
The derivative $f'(x) = 2x - 2 = 2(x-1)$ is positive for $x > 1$ and negative for $x < 1$. By Theorem 2, f is decreasing on the interval $(-\infty, 1)$ and increasing on the interval $(1, \infty)$, as confirmed in Figure 6.

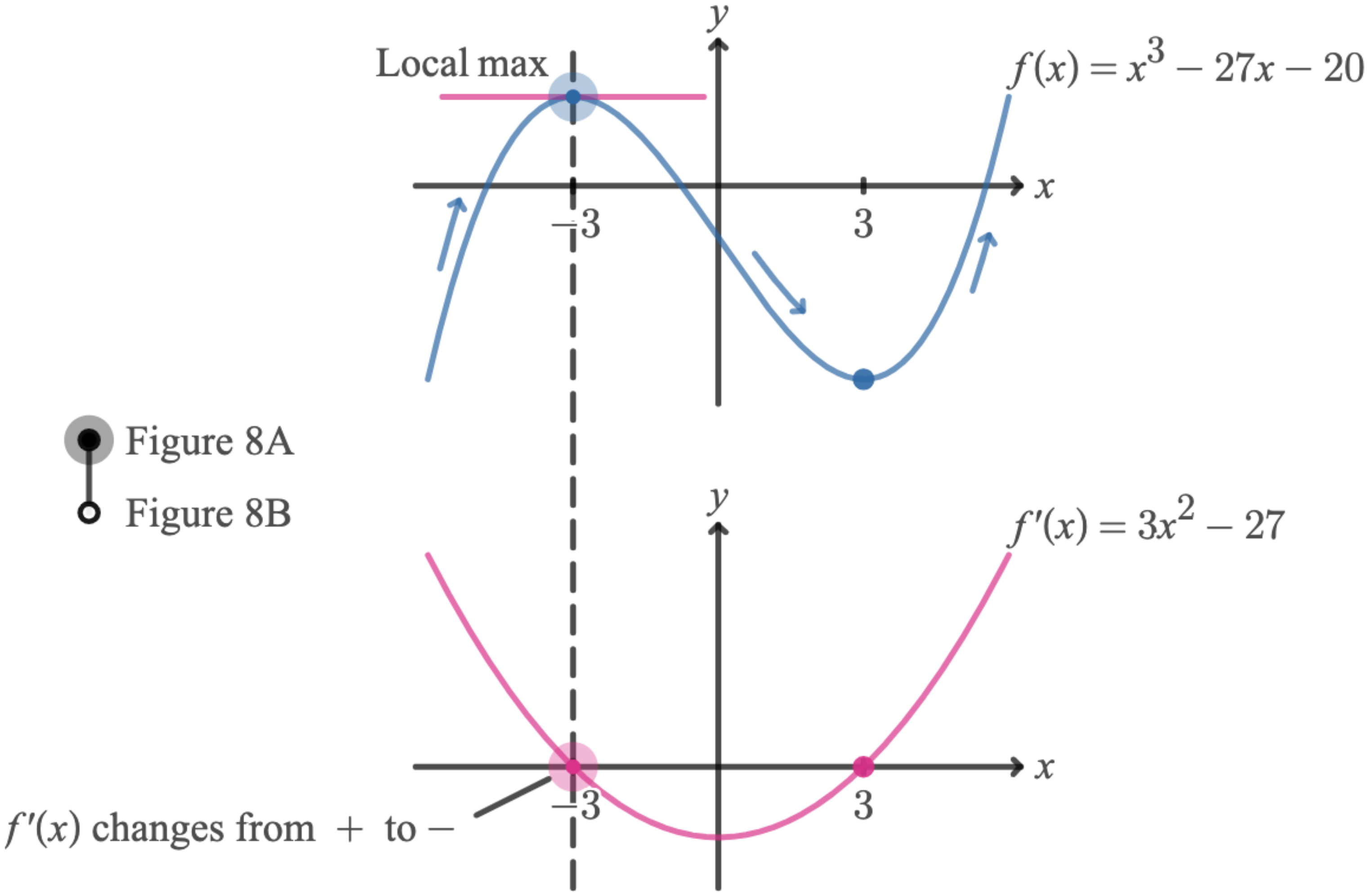


Testing Critical Points

There is a useful test for determining whether a critical point yields a min or max (or neither) based on the *sign change* of the derivative $f'(x)$.

To explain the term “sign change,” suppose that a function g satisfies $g(c) = 0$. We say that $g(x)$ *changes from positive to negative* at $x = c$ if $g(x) > 0$ to the left of c and $g(x) < 0$ to the right of c for x within a small open interval around c (Figure 7). A sign change from negative to positive is defined similarly. Observe in Figure 7 that $g(5) = 0$ but $g(x)$ does not change sign at $x = 5$.

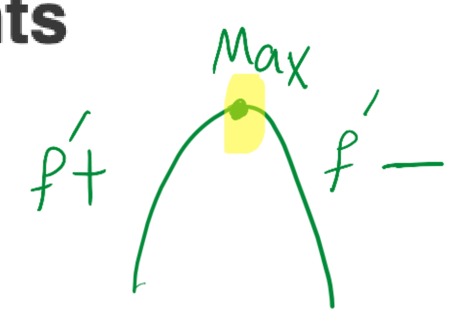




Local max
y
$f(x) = x^3 - 27x - 20$
x
−3
3
Figure 8A
Figure 8B
y
$f'(x) = 3x^2 - 27$
x
$f'(x)$ changes from + to −
−3
3

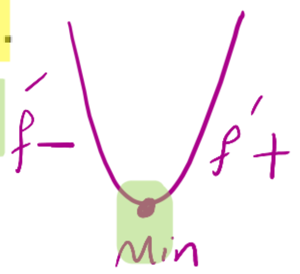
THEOREM 3

First Derivative Test for Critical Points



Let c be a critical point of f. Then

- $f'(x)$ changes from $+$ to $-$ at c $\Rightarrow$ $f(c)$ is a local maximum.
- $f'(x)$ changes from $-$ to $+$ at c $\Rightarrow$ $f(c)$ is a local minimum.



EXAMPLE 4

local Max: $(-3, f(-3)) = (-3, 34)$

local Min $(3, f(3)) = (3, -74)$

Analyze the critical points of $f(x) = x^3 - 27x - 20$.

$f'(x) = 3x^2 - 27 = 0$

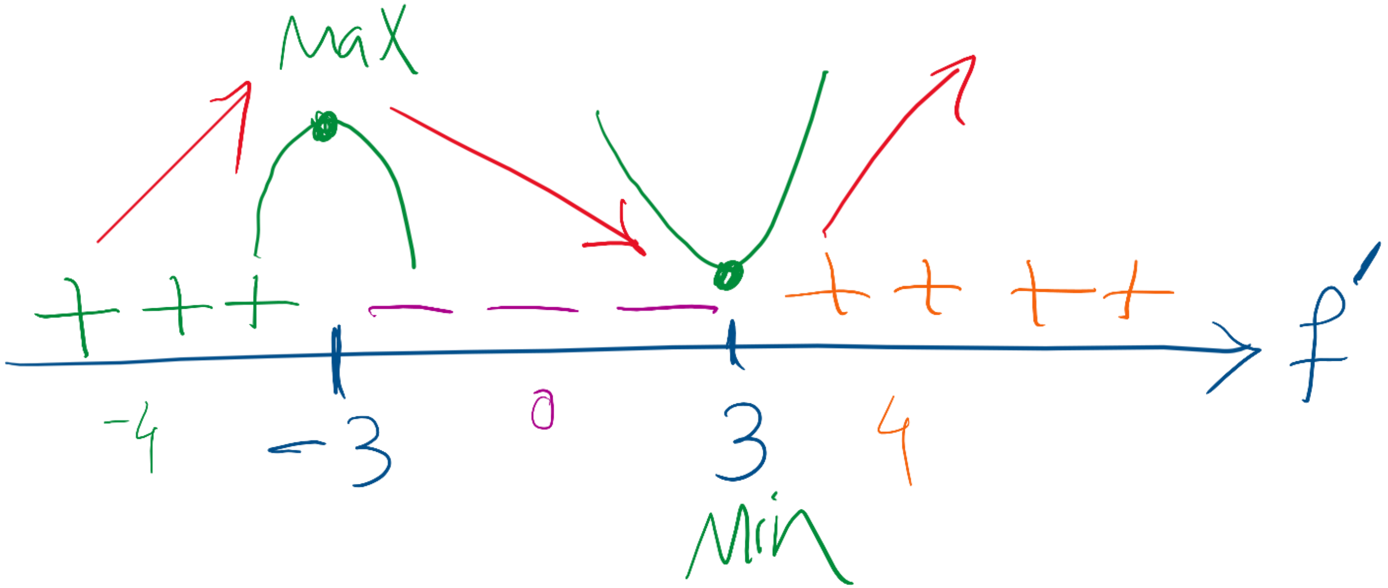
$3(x^2 - 9) = 0$

$3(x-3)(x+3) = 0$

$\Downarrow \qquad \Downarrow$

$x = 3 \qquad x = -3$

CPs



$f'(-4) = 3(-4-3)(-4+3)$ $+$

$f'(0) = 3(0-3)(0+3)$ $-$

$f'(4) = 3(4-3)(4+3)$ $+$

Inc: $(-\infty, -3) \cup (3, \infty)$

Dec: $(-3, 3)$

Solution

Our analysis will confirm the picture in Figure 8(A).

Step 1**. Find the critical points.**

We have $f'(x) = 3x^2 - 27 = 3(x^2 - 9)$. The critical points satisfy $f'(c) = 0$ and therefore are $c = \pm 3$.

Step 2**. Find the sign of $f'(x)$ on the intervals between the critical points.**

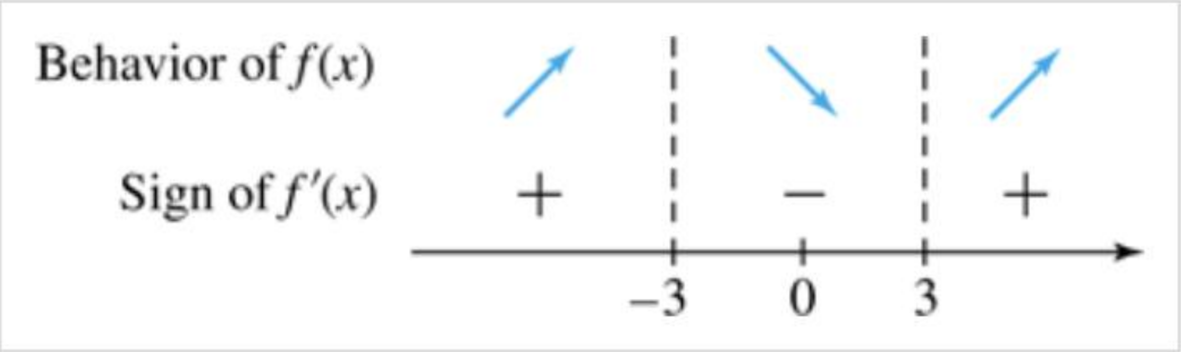
The critical points $c = \pm 3$ divide the real line into three intervals:

$$(-\infty, -3), \qquad (-3, 3), \qquad (3, \infty)$$

To determine the sign of $f'(x)$ on these intervals, we choose a test point inside each interval and evaluate. For example, in $(-\infty, -3)$ we choose $x = -4$. Because $f'(-4) = 21 > 0$, $f'(x)$ is positive on the entire interval $(-3, \infty)$. Taking this result, along with the results from test points at 0 and 4, we have

$$\begin{aligned} f'(-4) &= 21 > 0 &\Rightarrow\ & f'(x) > 0 \ \text{ for all } x \in (-\infty, -3) \\ f'(0) &= -27 < 0 &\Rightarrow\ & f'(x) < 0 \ \text{ for all } x \in (-3, -3) \\ f'(4) &= 21 > 0 &\Rightarrow\ & f'(x) > 0 \ \text{ for all } x \in (3, \infty) \end{aligned}$$

This information is displayed in the following sign diagram:



***Step 3*. Use the First Derivative Test.**

- $c = -3:$ $f'(x)$ changes from $+$ to $- \Rightarrow$ $f(-3) = 34$ is a local maximum value.
- $c = 3:$ $f'(x)$ changes from $-$ to $+ \Rightarrow f(3) = -74$ is a local minimum value.

EXAMPLE 6

$D = (-\infty, 0) \cup (0, \infty)$

Analyze the critical points and the increase/decrease behavior of $f(x) = x^2 + \frac{1}{x^2} = x^2 + x^{-2}$

$f'(x) = 2x - 2x^{-3} = 2x - \frac{2}{x^3} = 0$

$\frac{2x}{1} = \frac{2}{x^3} \to \frac{\cancel{2}x^4}{\cancel{2}} = \frac{2}{2} \to \sqrt[4]{x^4} = \sqrt[4]{1} \to |x| = 1 \Rightarrow \boxed{x = \pm 1}$

$x^4 - 1 = 0$

$(x^2 - 1)(x^2 + 1) = 0$

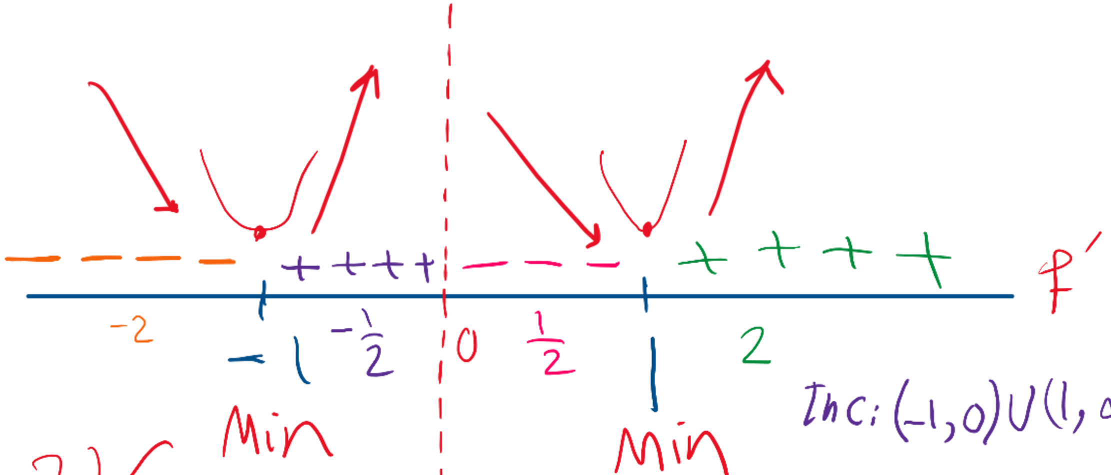
$(x-1)(x+1)(x^2+1) = 0$

$f'(-2) = 2(-2) - \frac{2}{(-2)^3} = -4 + \frac{2}{8} = -4 + \frac{1}{4}$ $-$

$f'(-\frac{1}{2}) = \cancel{2}(-\frac{1}{\cancel{2}}) - \frac{2}{(-\frac{1}{2})^3} = -1 + 16$ $+$

$f'(\frac{1}{2}) = \cancel{2}(\frac{1}{\cancel{2}}) - \frac{2}{(\frac{1}{2})^3} = 1 - 16 = -$

$f'(2) = 2(2) - \frac{2}{8} = 4 - \frac{1}{4}$ $+$



local min $(-1, 2)$

local Min $(1, 2)$

Inc: $(-1, 0) \cup (1, \infty)$

Dec: $(-\infty, -1) \cup (0, 1)$

Solution

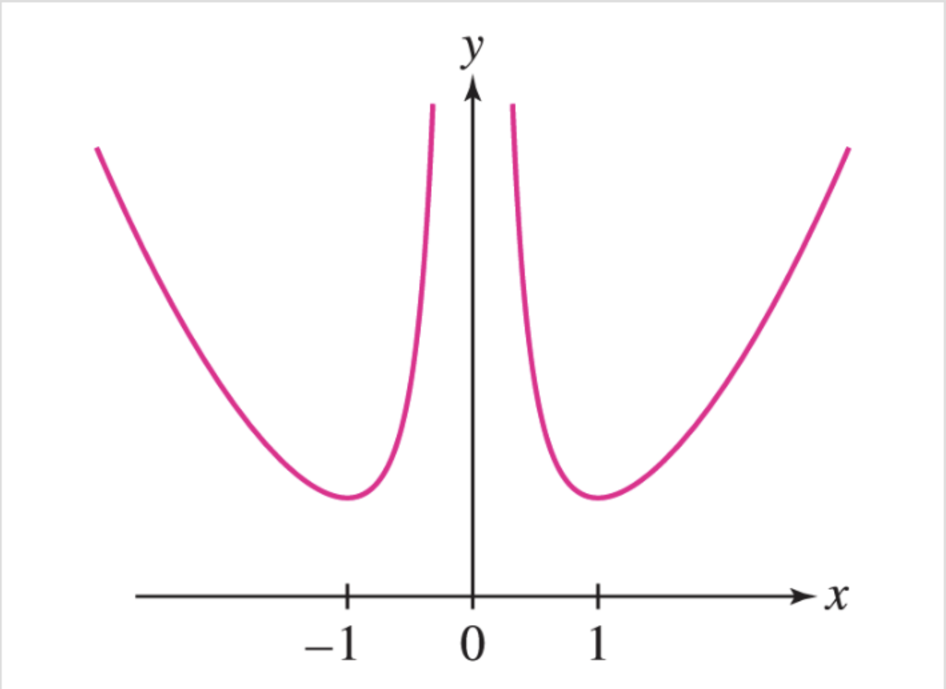
Note that f is undefined at $x = 0$, so we need to analyze f separately on $(-\infty, 0)$ and $(0, \infty)$. We have

$$f'(x) = 2x - \frac{2}{x^3}$$

The critical points are solutions to $x - \frac{1}{x^3} = 0$; that is, to $x^4 - 1 = 0$. They are $c = \pm 1$. Since we need to consider f separately on $(-\infty, 0)$ and $(0, \infty)$, there are four intervals on which we need to examine the sign of $f'(x)$: $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$. We determine the sign of $f'(x)$ by evaluating $f'(x)$ at a test point inside each interval.

Interval	Test value	Sign of $f'(x)$	Behavior of $f(x)$
$(-\infty, -1)$	$f'(-2) = -3.75$	$-$	$\searrow$
$(-1, 0)$	$f'(-0.5) = 15$	$+$	$\nearrow$
$(0, 1)$	$f'(0.5) = -15$	$-$	$\searrow$
$(1, \infty)$	$f'(2) = 3.75$	$+$	$\nearrow$

Applying the First Derivative Test, we see that both critical points are local minima. This is verified in the graph in Figure 10.



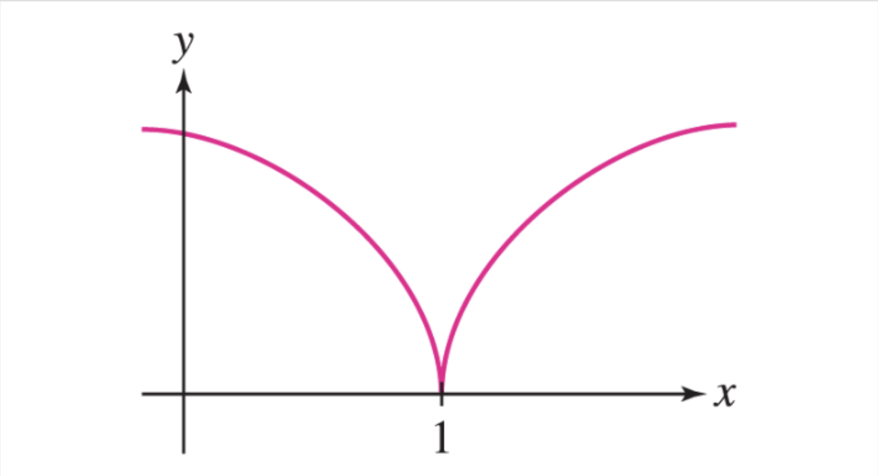
EXAMPLE 7

A Critical Point Where $f'(x)$ Is Undefined

Analyze the critical points of $f(x) = (1-x)^{2/3}$.

Solution

The derivative is $f'(x) = -\frac{2}{3}(1-x)^{-1/3} = \frac{-2}{3(1-x)^{1/3}}$. The only critical point occurs at $c = 1$, when $f'(x)$ is undefined. For $x < 1$, $f'(x)$ is negative. For $x > 1$, $f'(x)$ is positive. So $f'(x)$ changes sign as we pass through $c = 1$, and by the First Derivative Test, $f(c)$ is a local minimum. See Figure 11.



4.3 SUMMARY

- The Mean Value Theorem (MVT): If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists at least one value c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

This conclusion can also be written

$$f(b) - f(a) = f'(c)(b - a)$$

- Important corollary of the MVT: If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).
- The *sign* of $f'(x)$ determines whether f is increasing or decreasing:

$$\begin{aligned} f'(x) > 0 \text{ for } x \in (a,b) \quad &\Rightarrow \quad f \text{ is increasing on } (a,b) \\ f'(x) < 0 \text{ for } x \in (a,b) \quad &\Rightarrow \quad f \text{ is decreasing on } (a,b) \end{aligned}$$

- On an interval over which f is defined, the sign of $f'(x)$ can change only at the critical points, so f is *monotonic* (increasing or decreasing) on the intervals between the critical points.
- On an interval over which f is defined, to find the sign of $f'(x)$ on an interval between two critical points, calculate the sign of $f'(x_0)$ at any test point x_0 in that interval.
- First Derivative Test: If f is differentiable and c is a critical point, then

Sign change of $f'(x)$ at c	Type of critical point
From $+$ to $-$	Local maximum
From $-$ to $+$	Local minimum