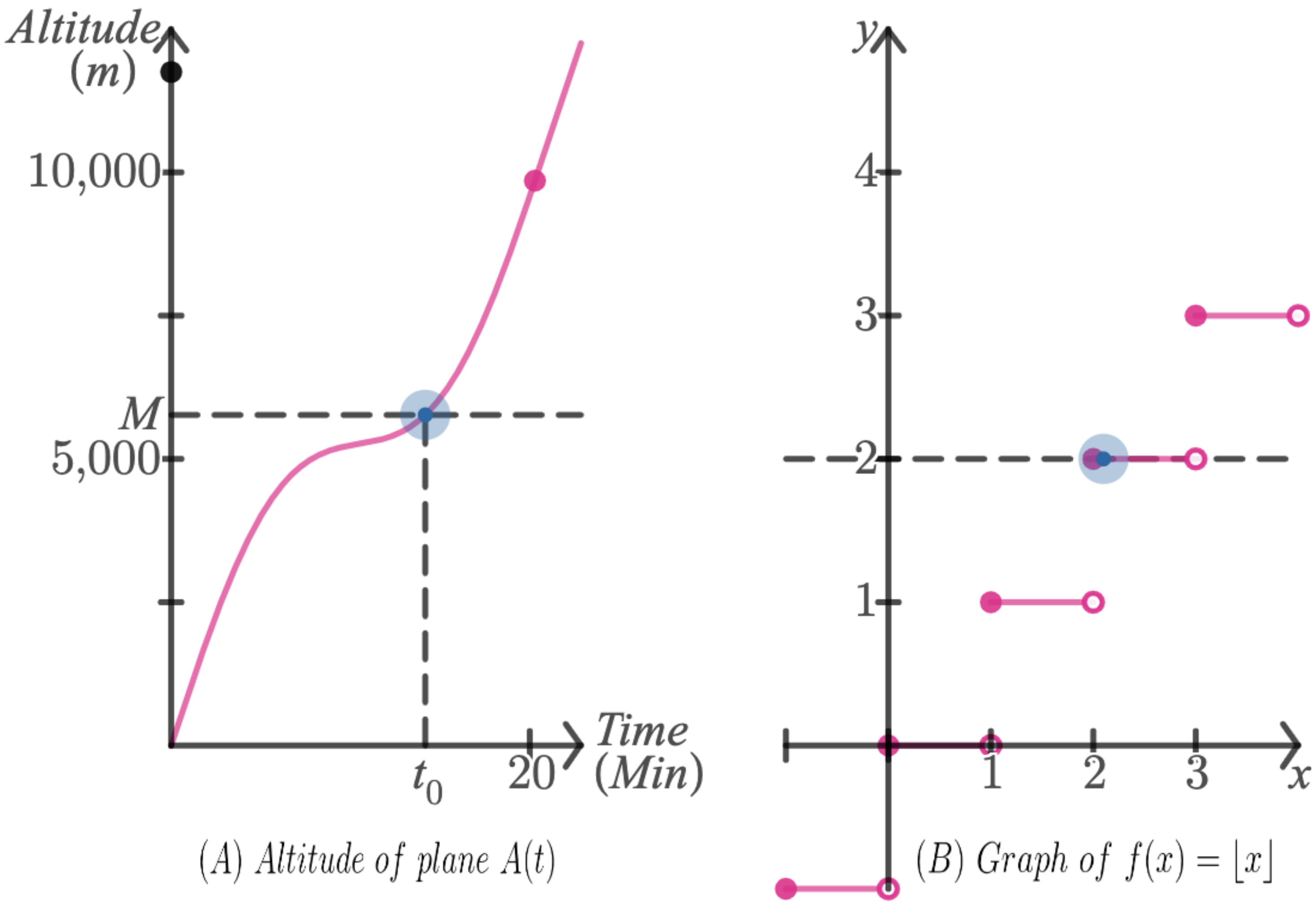
2.8 The Intermediate Value Theorem

The **Intermediate Value Theorem** **(IVT)** says, roughly speaking, that *a continuous function cannot skip values*. Consider a plane that takes off and climbs from 0 to 10,000 m in 20 min. The plane must reach every altitude between 0 and 10,000 m during this 20-min interval. Thus, at some moment, the plane's altitude must have been exactly 8371 m. Of course, this assumes that the plane's motion is continuous, so its altitude cannot jump abruptly from, say, 8000 to 9000 m.

THEOREM 1

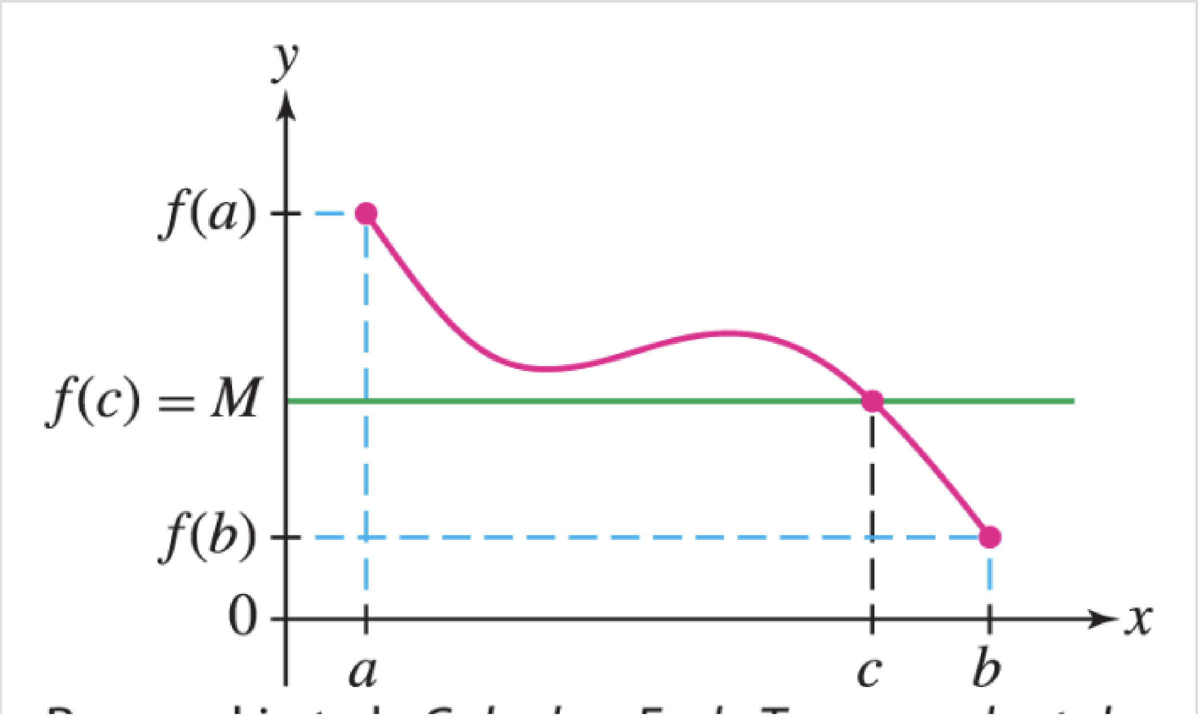
Intermediate Value Theorem

If f is continuous on a closed interval $[a, b]$, then for every value M, strictly between $f(a)$ and $f(b)$, there exists at least one value $c \in (a, b)$ such that $f(c) = M$.



(A) Altitude of plane $A(t)$ (B) Graph of $f(x) = \lfloor x \rfloor$

Graphically, as in Figure 2, the result appears obvious. For a continuous function, every horizontal line at height M between $f(a)$ and $f(b)$ is forced to hit the graph, and therefore there must be at least one value c in (a, b) such that $f(c) = M$. The proof appears in Appendix B.



EXAMPLE 1

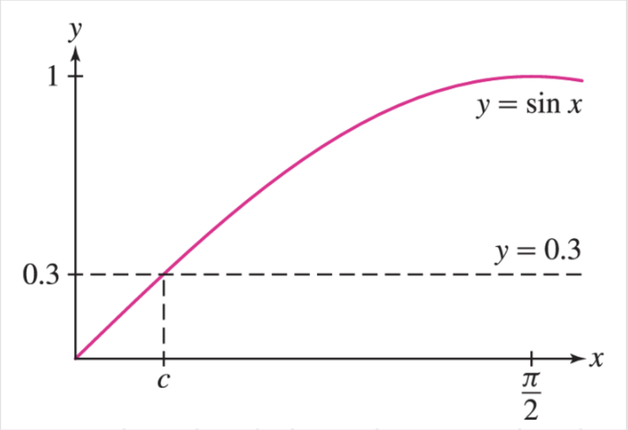
Prove that the equation $\sin\ x = 0.3$ has at least one solution in the interval $\left(0, \frac{\pi}{2}\right)$.

Solution

We may apply the IVT because $f\left(x\right) = \sin\ x$ is continuous. The desired value 0.3 lies between the values of the function at the endpoints of the interval:

$$\sin\ 0 = 0 \qquad \text{and} \qquad \sin\ \frac{\pi}{2} = 1$$

as illustrated in (Figure 3). The IVT tells us that $\sin\ x = 0.3$ has at least one solution in $\left(0, \frac{\pi}{2}\right)$.



The IVT can be used to show the existence of zeros of functions. If f is continuous and takes on both nonpositive and nonnegative values (say, $f(a) \leq 0$ and $f(b) \geq 0$) then the IVT guarantees that $f(c) = 0$ for some c in $[a, b]$. This is extremely useful when we cannot explicitly solve for the zero but would like to know that there is one in the interval.

COROLLARY 2

Existence of Zeros

If f is continuous on $[a, b]$, and if one of $f(a)$ and $f(b)$ is nonnegative and the other is nonpositive, then f has a zero in $[a, b]$.

We can locate zeros of functions to arbitrary accuracy using the **Bisection Method**. The idea is to find an interval $[a, b]$ such that the function has opposite signs at the endpoints. Then Corollary 2 tells us that there is a zero on this interval. To find its location more precisely, we cut the interval into two equal subintervals. Then, check the signs at the endpoints of each of these intervals to see which one Corollary 2 tells us has a zero. (But keep in mind that there may be more than one zero, so both could contain a zero). Next, we repeat the process on this smaller interval. Eventually, we narrow down on a zero. This is illustrated in the next example.

EXAMPLE 2

The Bisection Method

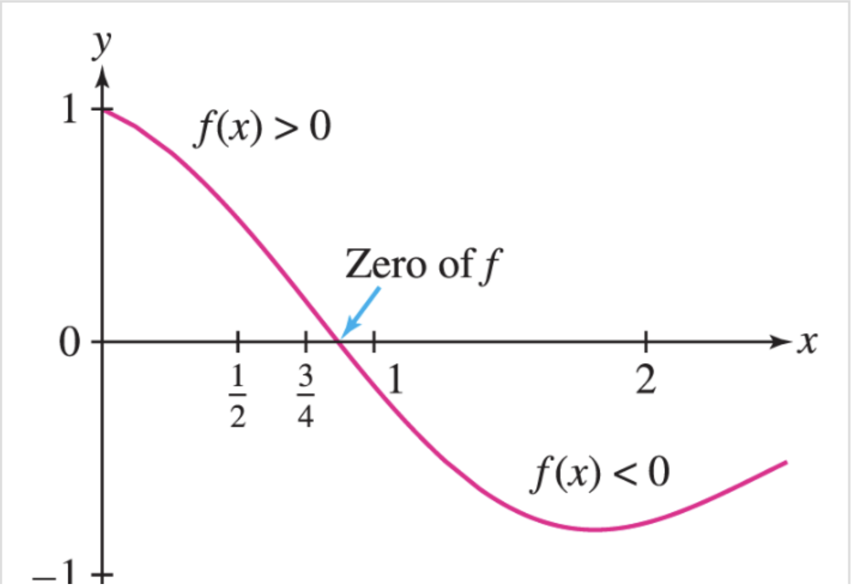
Show that $f\,(x) = \cos^2\,x - 2\,\sin\,\frac{x}{4}$ has a zero in $(0, 2)$. Then, using the Bisection Method, find a subinterval of $(0, 2)$ of length $1/8$ that contains a zero of f.

Solution

To begin, we note that f is continuous on $[0, 2]$. Calculating $f(0)$ and $f(2)$, we find that they have opposite signs:

$$f(0) = 1 > 0, \qquad f(2) \approx -0.786 < 0$$

Corollary 2 guarantees that $f(x) = 0$ has a solution in $(0, 2)$ (Figure 4).



To locate a zero more accurately, divide $[0, 2]$ into two intervals $[0, 1]$ and $[1, 2]$. At least one of these intervals must contain a zero of f. To determine which, we evaluate f at the midpoint $m = 1$, obtaining $f(1) \approx -0.203 < 0$. Since $f(0) = 1$, we see that

$$f(x) \text{ takes on opposite signs at the endpoints of } [0, 1]$$

Therefore, $(0, 1)$ must contain a zero. Note that from the function values $f(1)$ and $f(2)$ alone, we cannot conclude that f *does not* have a zero in the interval $[1, 2]$, but it is clear from the graph in Figure 4 that it does not.

The Bisection Method consists of continuing this process until we narrow down the location of a zero to any desired accuracy. In the following table, the process is carried out three times to find an interval of length $1/8$ containing a zero of f.

Interval	Midpoint of interval	Function values		Conclusion
$[0, 1]$	$\frac{1}{2}$	$f\left(\frac{1}{2}\right) \approx 0.521$	$f(1) \approx -0.203$	A zero lies in $\left(\frac{1}{2}, 1\right)$.
$\left[\frac{1}{2}, 1\right]$	$\frac{3}{4}$	$f\left(\frac{3}{4}\right) \approx 0.163$	$f(1) \approx -0.203$	A zero lies in $\left(\frac{3}{4}, 1\right)$.
$\left[\frac{3}{4}, 1\right]$	$\frac{7}{8}$	$f\left(\frac{7}{8}\right) \approx -0.0231$	$f\left(\frac{3}{4}\right) \approx 0.163$	A zero lies in $\left(\frac{3}{4}, \frac{7}{8}\right)$.

We conclude that f has a zero c satisfying $0.75 < c < 0.875$.

2.8 SUMMARY

- The Intermediate Value Theorem (IVT) says that a continuous function cannot *skip* values.
- More precisely, if f is continuous on $[a, b]$ with $f(a) \neq f(b)$, and if M is a number strictly between $f(a)$ and $f(b)$, then $f(c) = M$ for some $c \in (a, b)$.
- Existence of zeros: If f is continuous on $[a, b]$, and if one of $f(a)$ and $f(b)$ is non-negative and the other is nonpositive, then f has a zero in $[a, b]$.
- Bisection Method: Assume f is continuous and that $f(a)$ and $f(b)$ have opposite signs, so that f has a zero in (a, b). Then f has a zero in $[a, m]$ or $[m, b]$, where $m = (a + b)/2$ is the midpoint of $[a, b]$. A zero lies in (a, m) if $f(a)$ and $f(m)$ have opposite signs and a zero lies in (m, b) if $f(m)$ and $f(b)$ have opposite signs. Continuing the process, we can locate zeros with arbitrary accuracy.