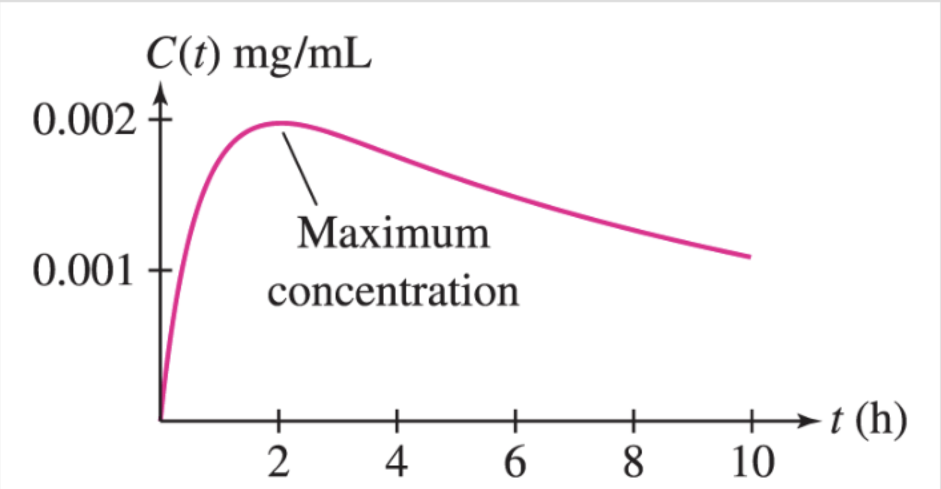
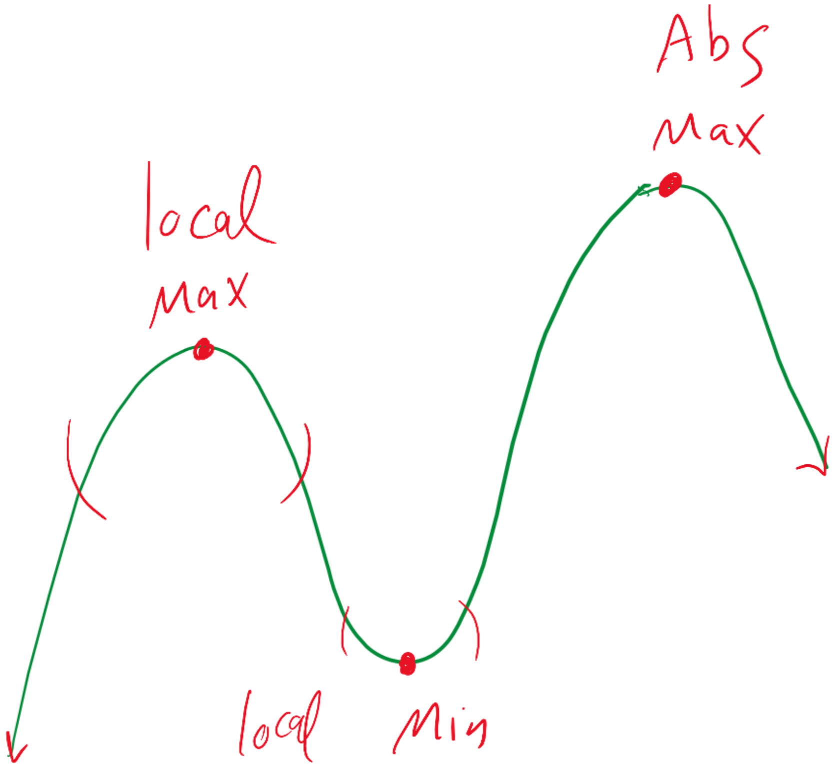
4.2 Extreme Values

In many applications, it is important to find the minimum or maximum value of a function f. For example, a physician needs to know the maximum drug concentration in a patient's bloodstream when a drug is administered. This amounts to finding the highest point on the graph of C, the concentration at time t (Figure 1).




Abs
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We refer to the maximum and minimum values (max and min for short) as extreme values or **extrema** (singular: extremum) and to the process of finding them as **optimization**. Sometimes, we are interested in finding the min or max for x in a particular interval I, rather than on the entire domain of f.

DEFINITION

Extreme Values on an Interval

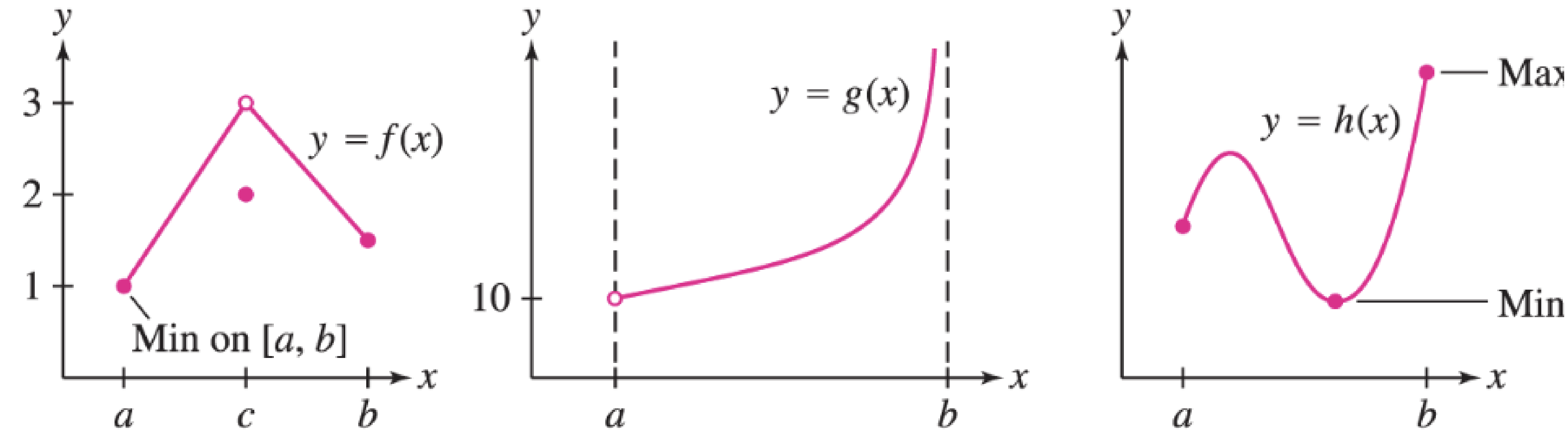
Let f be a function on an interval I and let $a \in I$. We say that $f(a)$ is the

- Absolute minimum of f on I if $f(a) \leq f(x)$ for all $x \in I$.
- Absolute maximum of f on I if $f(a) \geq f(x)$ for all $x \in I$.

Does every function have a minimum or maximum value? Clearly not, as we see by taking $f(x) = x$. Indeed, $f(x) = x$ increases without bound as $x \to \infty$ and decreases without bound as $x \to -\infty$. In fact, extreme values do not always exist even if we restrict ourselves to an interval I. Figure 2 illustrates what can go wrong if I is open or f has a discontinuity.

- **Discontinuity:** (A) shows a discontinuous function with no maximum value. The values of $f(x)$ get arbitrarily close to 3 from below, but 3 is not the maximum value because $f(x)$ never actually takes on the value 3.
- **Open interval:** In (B), $g(x)$ is defined on the *open* interval (a, b). It has no max because it tends to ∞ on the right, and it has no min because it tends to 10 on the left without ever reaching this value.

Fortunately, our next theorem guarantees that extreme values exist when t
is continuous and I is closed [Figure 2(C)].



(A) Discontinuous function with no max on $[a, b]$, and a min at $x = a$.

(B) Continuous function with no min or max on the open interval (a, b).

(C) Every continuous functio[n on a closed] interval $[a, b]$ has both a [min and a] max on $[a, b]$.

THEOREM 1

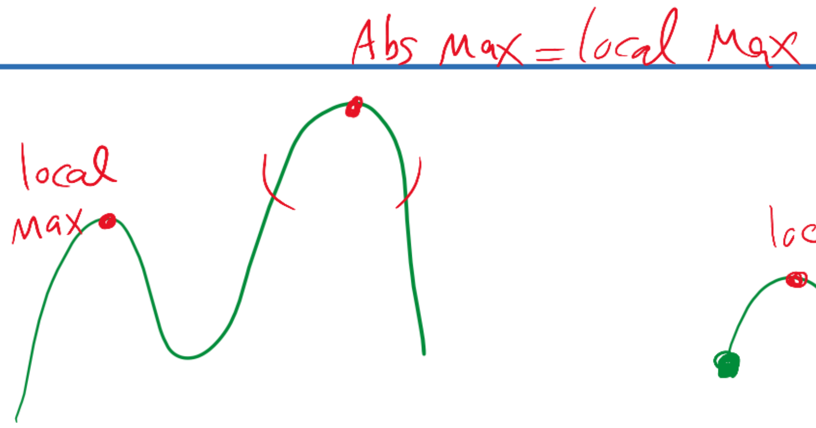
Existence of Extrema on a Closed Interval

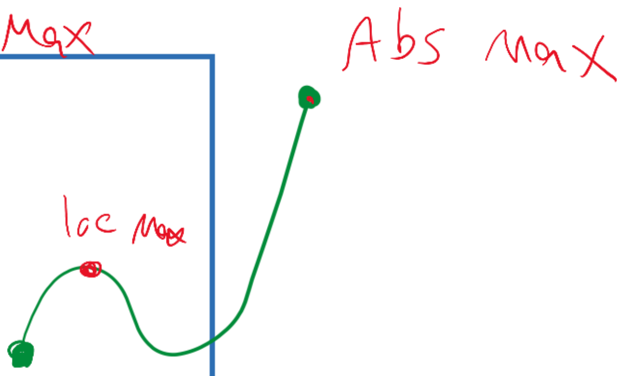
A continuous function f on a closed (bounded) interval $I = [a, b]$ takes on both a minimum and a maximum value on I.

Local Extrema and Critical Points

We focus now on the problem of finding extreme values. A key concept is that of a local minimum or maximum.

DEFINITION

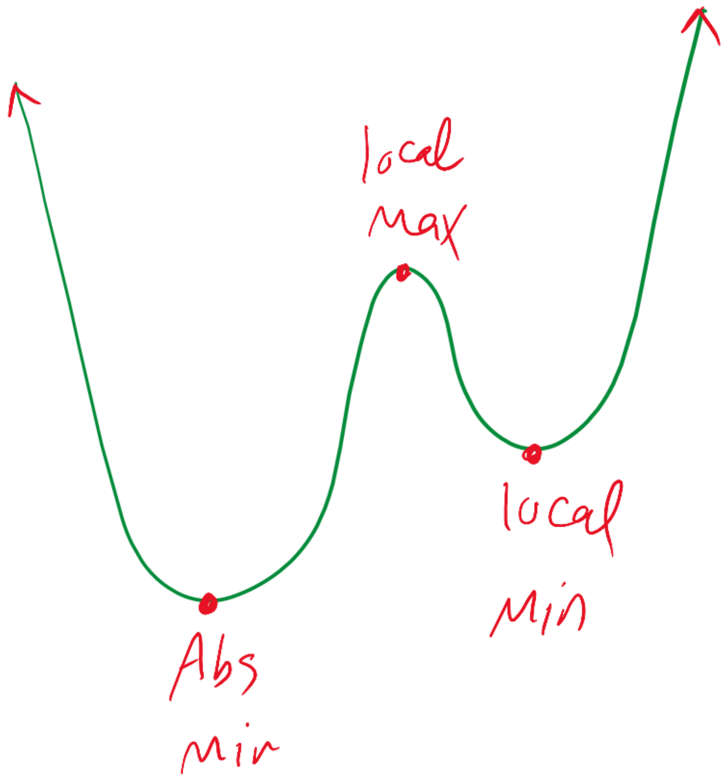




Local Extrema

We say that $f(c)$ is a

- **Local minimum** occurring at $x = c$ if $f(c)$ is the minimum value of f on some open interval (in the domain of f) containing c.
- **Local maximum** occurring at $x = c$ if $f(c)$ is the maximum value of f on some open interval (in the domain of f) containing c.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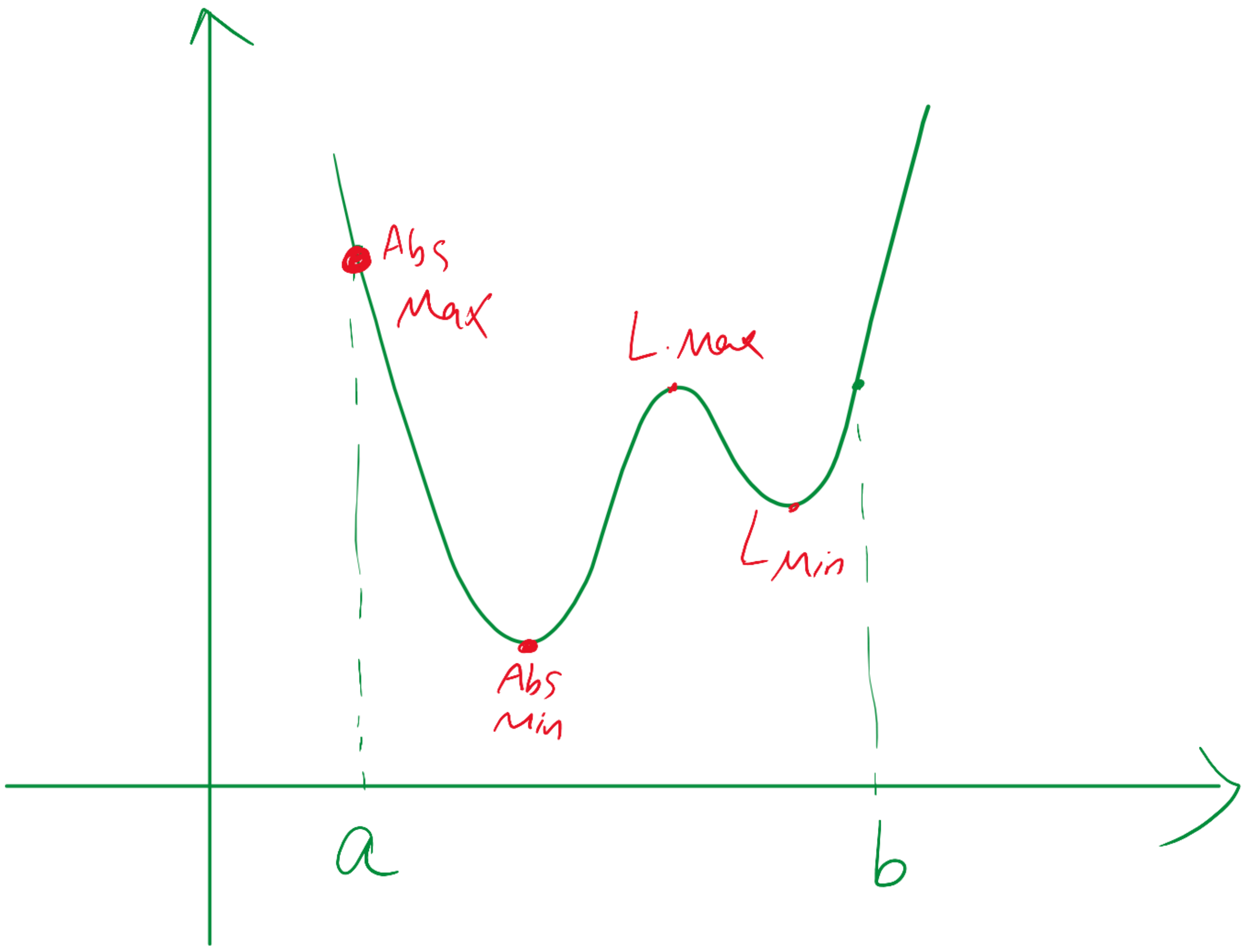
Abs
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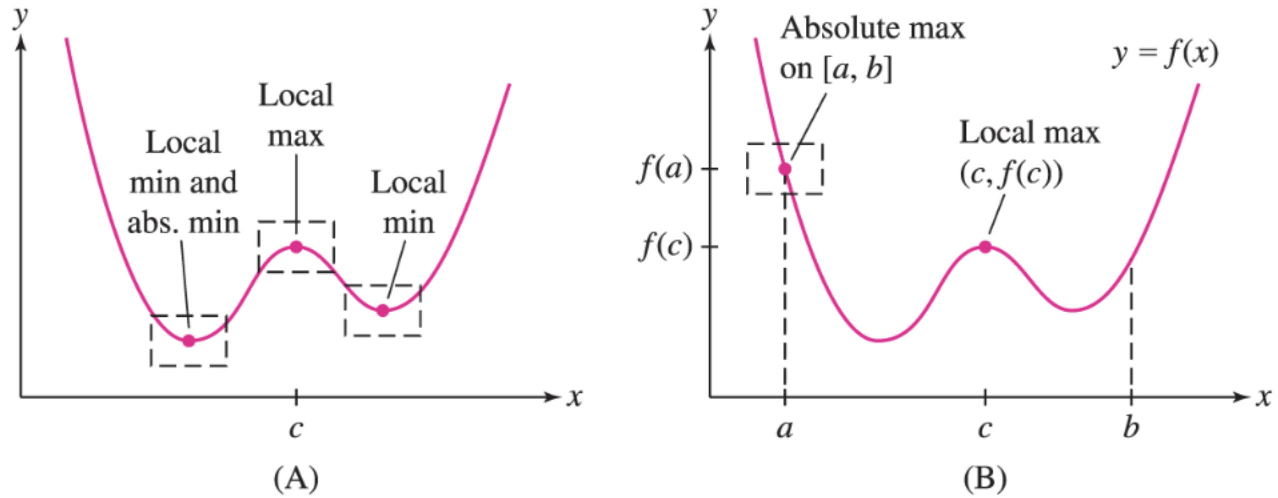
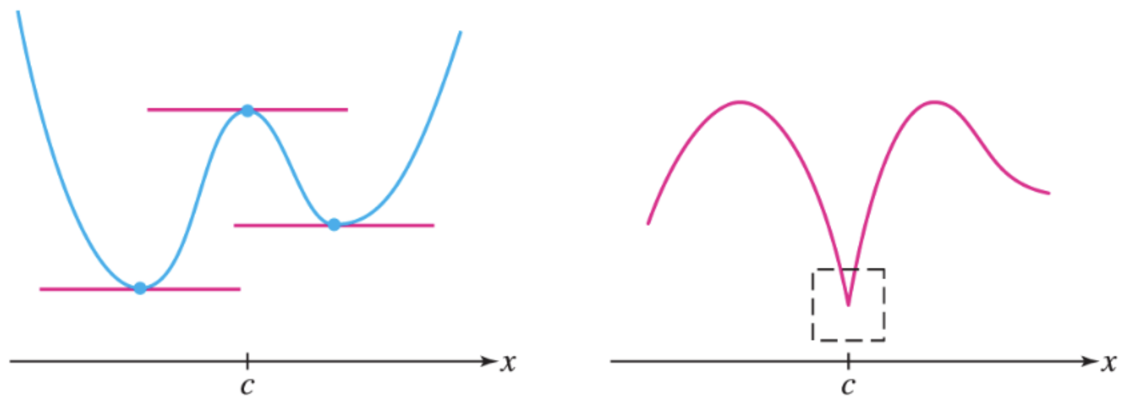
min
a
b

A local max occurs at $x = c$ if $(c, f(c))$ is the highest point on the graph within some small box [Figure 4(A)]. Thus, $f(c)$ is greater than or equal to all other *nearby* values, but it does not have to be the absolute maximum value of f (Figure 3). Local minima are similar. On the other hand, as Figure 4(B) illustrates, an absolute maximum of f on an interval $[a, b]$ need not be a local maximum of f in open intervals containing the point. In the figure, $f(a)$ is the absolute max on $[a, b]$ but is not a local max on open intervals containing a because $f(x)$ takes on greater values to the left of $x = a$.



How do we find the local extrema? The crucial observation is that *the tangent line at a local min or max is horizontal* [Figure 5(A)]. In other words, if $f(c)$ is a local min or max, then $f'(c) = 0$. However, this assumes that f is differentiable. Otherwise, the tangent line may not exist, as in Figure 5(B). To take both possibilities into account, we define the notion of a critical point.



(A) Tangent line is horizontal at the local extrema.

(B) This local minimum occurs at a point where the function is not differentiable.

DEFINITION

1- Domain

2- $f'(x)=0$ or

3- $f'(x)$ DNE

Critical Points

A number c in the domain of f is called a **critical point** if *either* $f'(c) = 0$ or $f'(c)$ does not exist.

EXAMPLE 1

Find the critical points of $f(x) = x^3 - 9x^2 + 24x - 10.$

$$f'(x) = 3x^2 - 18x + 24 = 0$$

$$3(x^2 - 6x + 8) = 0$$

$$3(x-4)(x-2) = 0$$

$$x - 4 = 0 \qquad x - 2 = 0$$

$$x = 4 \qquad x = 2$$

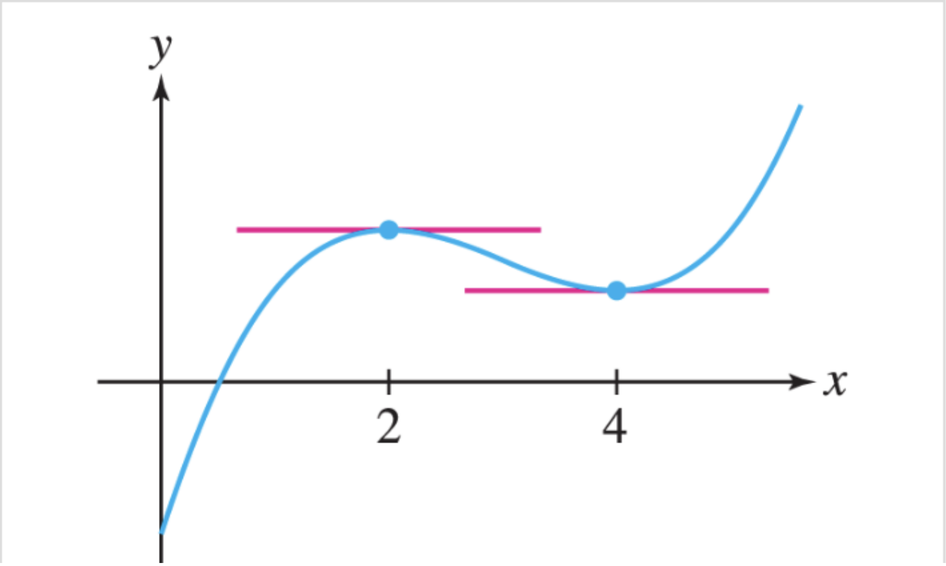
Critical Points

Solution

The function f is differentiable everywhere (Figure 6). Therefore, the critical points are the solutions of $f'(x) = 0$:

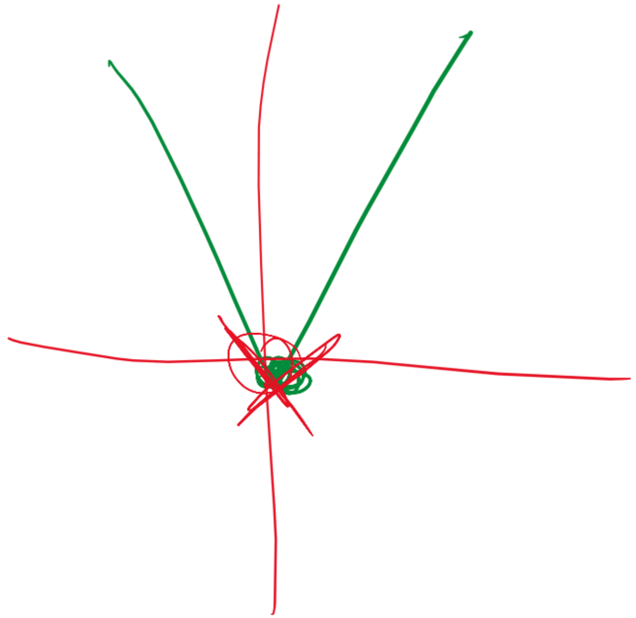
$$f'(x) = 3x^2 - 18x + 24 = 3\left(x^2 - 6x + 8\right) = 3(x-2)(x-4)$$

To find the critical points, we solve $3(x-2)(x-4) = 0$. Thus, they are $x = 2$ and $x = 4$.



Ex : Find C.ps of $f(x) = |x|$.

$x = 0$ C.P



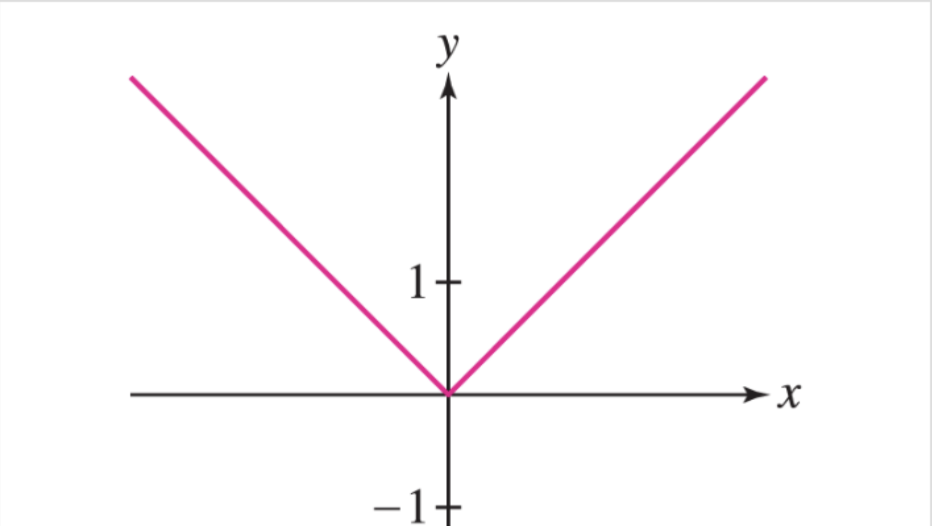
EXAMPLE 2

Nondifferentiable Function

Find the critical points of $f(x) = |x|$.

Solution

As we see in Figure 7, $f'(x) = -1$ for $x < 0$ and $f'(x) = 1$ for $x > 0$. Therefore, $f'(x) = 0$ has no solutions with $x \neq 0$. However, $f'(0)$ does not exist. Thus, $c = 0$ is a critical point.



THEOREM 2

Fermat's Theorem on Local Extrema

If $f(c)$ is a local min or max, then c is a critical point of f.

Optimizing on a Closed Interval

Finally, we have all the tools needed for optimizing a continuous function on a closed interval. Theorem 1 guarantees that the extreme values exist, and the next theorem tells us where to find them, namely among the critical points or endpoints of the interval.

THEOREM 3

Extreme Values on a Closed Interval

Assume that f is continuous on $[a, b]$ and let $f(c)$ be the minimum or maximum value on $[a, b]$. Then c is either a critical point or one of the endpoints a or b.

EXAMPLE 3

Step 1 : Find Critical Points

Step 2 : Evaluate f(x) at C.Ps AND end points

Step 3 : pick the highest value as Abs Max

Lowest value as Abs Min

Find the extrema of $f(x) = 2x^3 - 15x^2 + 24x + 7$ on $[0, 6]$.

$$f'(x) = 6x^2 - 30x + 24 = 0$$

$$6(x^2 - 5x + 4) = 0$$

$$6(x-4)(x-1) = 0$$

$x - 4 = 0 \Rightarrow x = 4$

$x - 1 = 0 \Rightarrow x = 1$

C.Ps

$f(1) = 2 - 15 + 24 + 7 = 18$

$f(4) = 2(4)^3 - 15(4)^2 + 24(4) + 7 = -9 \longrightarrow$ Abs Min $(4, -9)$

$f(0) = 7$

$f(6) = 2(6)^3 - 15(6)^2 + 24(6) + 7 = 43 \longrightarrow$ Abs max $(6, 43)$

Solution

The extreme values occur at critical points or endpoints by Theorem 3, so we can break up the problem neatly into two steps.

***Step 1*. Find the critical points.**

The function f is differentiable, so the critical points are solutions to $f'(x) = 0$.

$$f'(x) = 6x^2 - 30x + 24 = 6(x-1)(x-4)$$

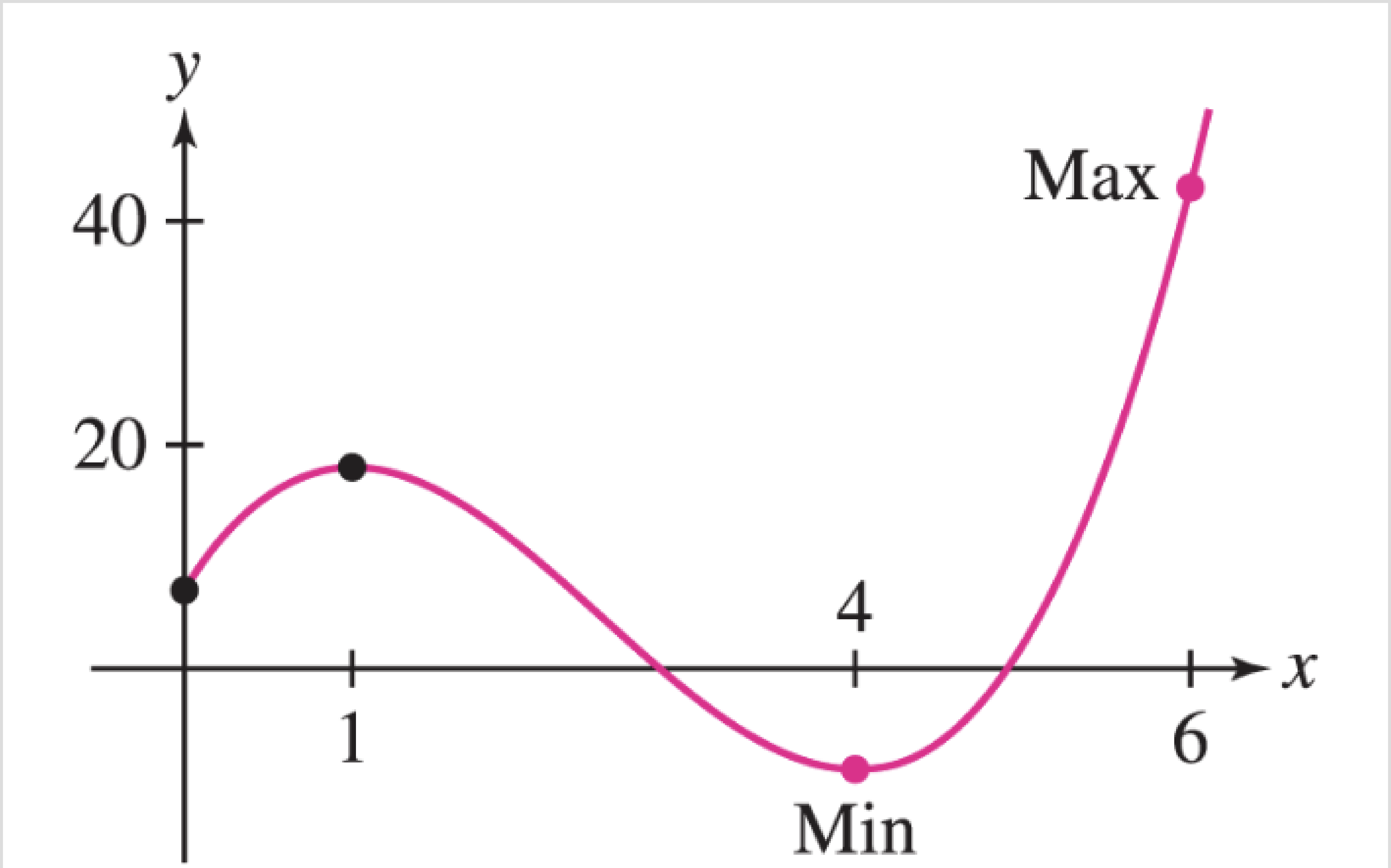
The critical points satisfy $6(x-1)(x-4) = 0,$ and therefore are $x = 1$ and 4

.

***Step 2*. Compare values of $f(x)$ at the critical points and endpoints.**

x -value	Value of $f(x)$
1 (critical point)	$f(1) = 18$
4 (critical point)	$f(4) = -9 \quad \text{min}$
0 (endpoint)	$f(0) = 7$
6 (endpoint)	$f(6) = 43 \quad \text{max}$

The maximum value of $f(x)$ on $[0, 6]$ is the greatest of the values in this table, namely $f(6) = 43$. Similarly, the minimum is $f(4) = -9$. See Figure 10.


y
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20
Max
4
Min
1
6
x

Step 1: $f'(x) = -\frac{2}{3}(x-1)^{-\frac{1}{3}} = \frac{-2}{3(x-1)^{1/3}} = 0$

EXAMPLE 4

$-2 \neq 0 \Rightarrow$ No Solution

$x=1$ is a C.P because $f'(1)$ Does Not exist.

Function with a Cusp

Find the extrema of $f(x) = \cancel{1}^{0} - (\underbrace{x-1}_{u})^{2/3}$ on $[-1, 2]$.

$f(1) = 1 - (\cancel{1-1}^{0})^{\frac{2}{3}} = 1 - 0 = \boxed{1} \rightarrow$ Abs Max $(1, 1)$

$f(-1) = 1 - (\cancel{-1-1}^{-2})^{\frac{2}{3}} \approx \boxed{-0.59} \rightarrow$ Abs Min $(-1, -0.59)$

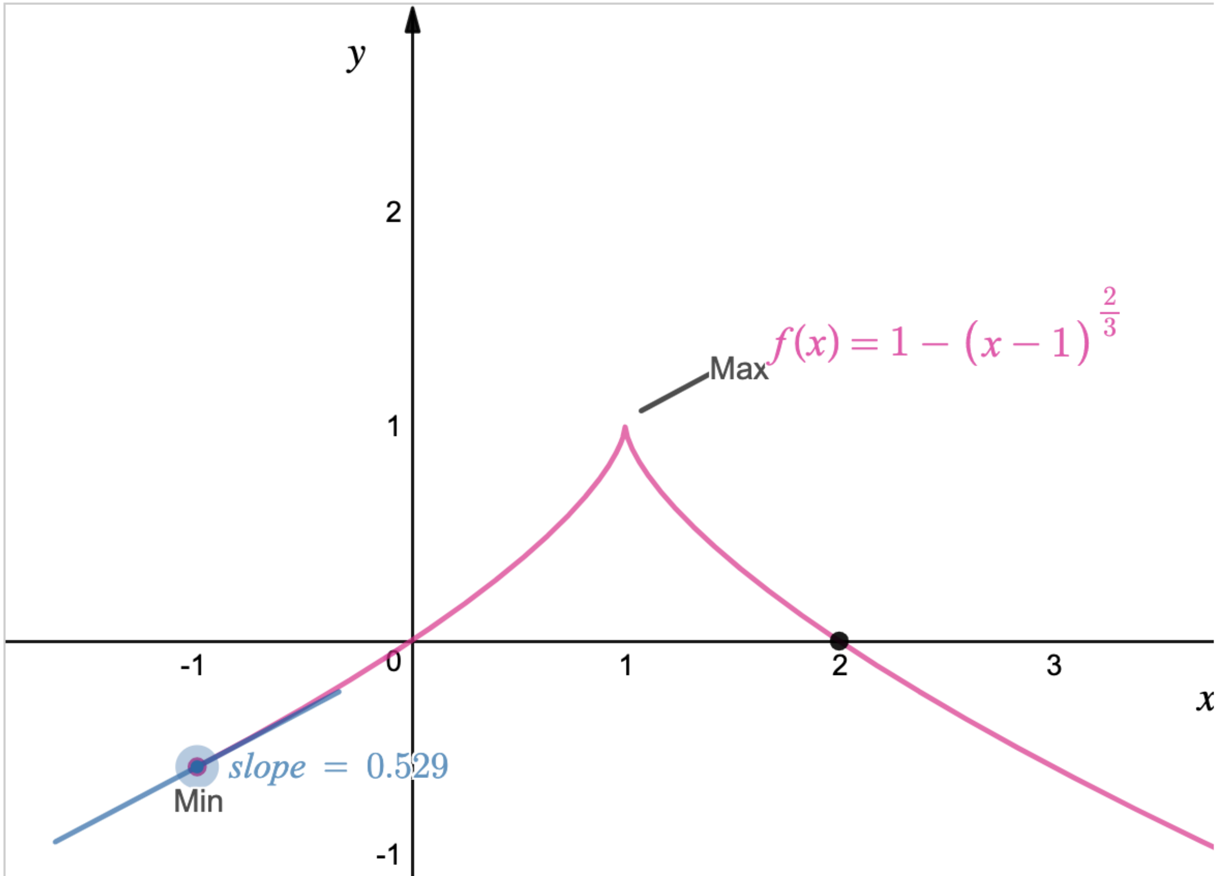
$f(2) = 1 - (\cancel{2-1}_{1})^{\frac{2}{3}} = 1 - 1 = 0$

Solution

First, find the critical points:

$$f'(x) = -\frac{2}{3}(x-1)^{-1/3} = -\frac{2}{3(x-1)^{1/3}}$$

The equation $f'(x) = 0$ has no solutions because $f'(x)$ is never zero. However, $f'(x)$ does not exist at $x = 1$, so there is a critical point there (Figure 11).



Next, compare values of $f(x)$ at the critical points and endpoints:

x -value	Value of $f(x)$
1 (critical point)	$f(1) = 1 \quad \text{max}$
-1 (endpoint)	$f(-1) \approx -0.59 \quad \text{min}$
2 (endpoint)	$f(2) = 0$

So on $[-1, 2]$, the maximum of f is $f(1) = 1$ and the minimum is $f(-1) \approx -0.59.$

EXAMPLE 5

$f'(x) = 2x - 8 \cdot \frac{1}{x} = 2x - \frac{8}{x} = 0$

Logarithmic Example

$\frac{2x}{1} = \frac{8}{x} \rightarrow \frac{\cancel{2}x^2}{\cancel{2}} = \frac{8}{2} \rightarrow \sqrt{x^2} = \sqrt{4} \Rightarrow x = \pm 2$

Find the extreme values of the function $\boxed{f(x) = x^2 - 8 \ln x}$ on $[1, 4]$.

$-2 \notin [1,4] \Rightarrow x = -2$ is Not a C.P

$x = 2$ is a C.P

$f(1) = 1^2 - 8 \cancel{\ln(1)}^{0} = 1$

$f(4) = 4^2 - 8 \ln(4) = 4.9 \longrightarrow$ Abs Max $(4, 4.9)$

$f(2) = 2^2 - 8 \ln(2) = -1.55 \longrightarrow$ Abs Min $(2, -1.55)$

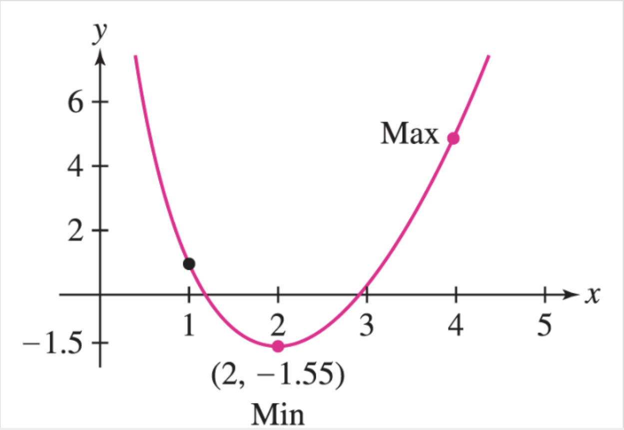
Solution

First, we solve for the critical points. We have $f'(x) = 2x - 8/x$, so we solve

$$2x - \frac{8}{x} = 0 \quad \Rightarrow \quad 2x = \frac{8}{x} \quad \Rightarrow \quad x = \pm 2$$

The only critical point in the interval $[1, 4]$ is $x = 2$. Next, compare the values of $f(x)$ at the critical points and endpoints (Figure 12):

x -value	Value of $f(x)$
2 (critical point)	$f(2) \approx -1.55$ min
1 (endpoint)	$f(1) = 1$
4 (endpoint)	$f(4) \approx 4.9$ max



We see that the minimum on $[1, 4]$ is $f\left(2\right) \approx -1.55$ and the maximum is $f\left(4\right) \approx 4.9$.

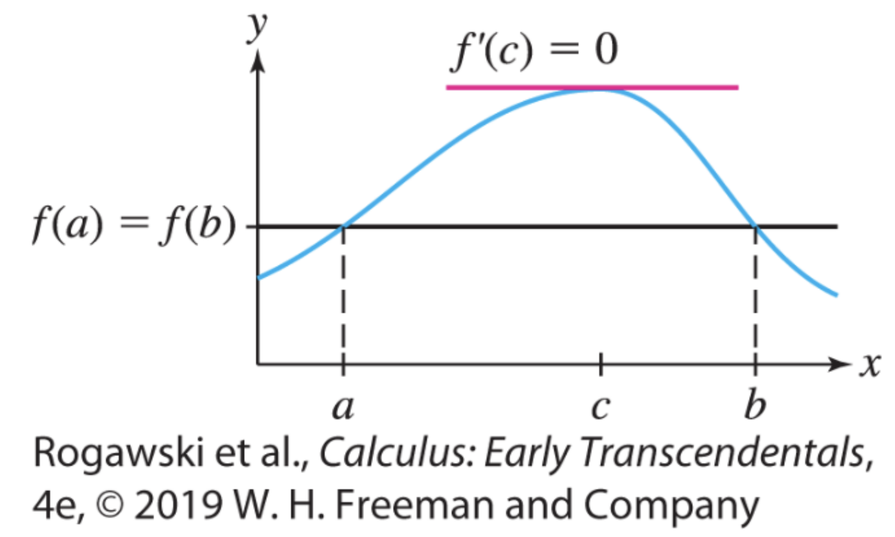


FIGURE 16 Rolle's Theorem: If $f(a) = f(b)$, then $f'(c) = 0$ for some c between a and b.

THEOREM 4

Rolle's Theorem

Assume that f is continuous on $[a, b]$ and differentiable on (a, b). If $\boxed{f(a) = f(b)}$, then there exists a number c between a and b such that $\boxed{f'(c) = 0}$.

$f(a) = f(-2) = (-2)^4 - (-2)^2 = 16 - 4 = 12$

$f(b) = f(2) = (2)^4 - (2)^2 = 16 - 4 = 12$

EXAMPLE 7

Since $f(2) = f(-2) \longrightarrow$ based on Roll's T there is a $c \in [-2, 2]$. S. that $f'(c) = 0$

Illustrating Rolle's Theorem

Verify Rolle's Theorem for

$$f(x) = x^4 - x^2 \qquad \text{on} \qquad [\overset{a}{-2}, \overset{b}{2}]$$

$f'(x) = 4x^3 - 2x = 0$

$2x(2x^2 - 1) = 0$

$2x = 0 \Longrightarrow x = 0$

$2x^2 - 1 = 0 \rightarrow \frac{\cancel{2}x^2}{\cancel{2}} = \frac{1}{2} \rightarrow x^2 = \frac{1}{2} \rightarrow x = \pm\frac{1}{\sqrt{2}}$

$\approx \pm 0.7..$

Solution

The hypotheses of Rolle's Theorem are satisfied because f is differentiable (and therefore continuous) everywhere, and $f(2) = f(-2)$:

$$f(2) = 2^4 - 2^2 = 12, \qquad f(-2) = (-2)^4 - (-2)^2 = 12$$

We must verify that $f'(c) = 0$ has a solution in $(-2, 2)$. Since

$$f'(x) = 4x^3 - 2x = 2x(2x^2 - 1)$$

we need to solve $2x(2x^2 - 1) = 0$. The solutions are $c = 0$ and $c = \pm 1/\sqrt{2} \approx \pm 0.707$. They all lie in $(-2, 2)$, so Rolle's Theorem is satisfied with three values of c.

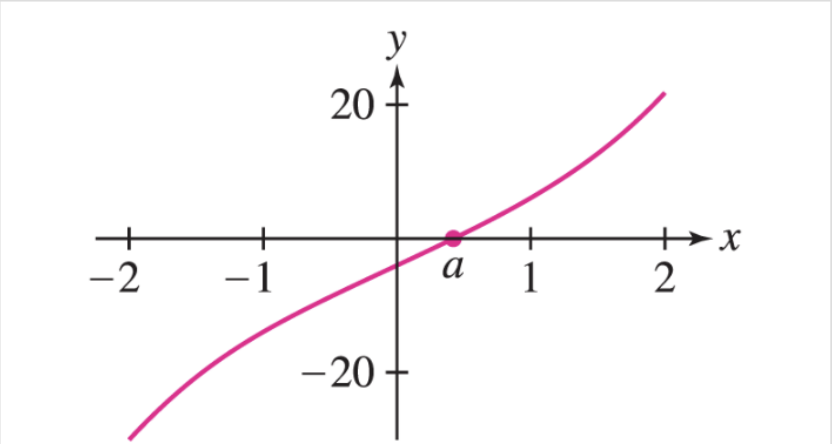
EXAMPLE 8

Using Rolle's Theorem

Show that $f(x) = x^3 + 9x - 4$ has precisely one real root.

Solution

First, we note that $f(0) = -4$ is negative and $f(1) = 6$ is positive. By the Intermediate Value Theorem (Section 2.8), f has *at least* one root a in $[0, 1]$. If f had a second root b, then we would have $f(a) = f(b) = 0$. Rolle's Theorem would then imply that $f'(c) = 0$ for some $c \in (a, b)$. This is not possible because $f'(x) = 3x^2 + 9 > 0$, so $f'(c) = 0$ has no solutions. We conclude that a is *the only* real root of f (Figure 17).



4.2 SUMMARY

- The *extreme values* of f on an interval I are the minimum and maximum values of f for $x \in I$ (also called *absolute extrema* on I).
- Basic Theorem: If f is continuous on a closed interval $[a, b]$, then f has both a min and a max on $[a, b]$.
- $f(c)$ is a *local minimum* if $f(x) \geq f(c)$ for all x in some open interval around c. Local maxima are defined similarly.
- $x = c$ is a *critical point* of f if either $f'(c) = 0$ or $f'(c)$ does not exist.
- Fermat's Theorem on Local Extrema: If $f(c)$ is a local min or max, then c is a critical point.
- To find the extreme values of a continuous function f on a closed interval $[a, b]$:

  Step 1. Find the critical points of f in $[a, b]$.

  Step 2. Calculate $f(x)$ at the critical points in $[a, b]$ and at the endpoints. The min and max on $[a, b]$ are the least and greatest among the values computed in Step 2.

- Rolle's Theorem: If f is continuous on $[a, b]$ and differentiable on (a, b), and if $f(a) = f(b)$, then there exists c between a and b such that $f'(c) = 0$.