# 3.8 Implicit Differentiation

We have developed techniques for calculating a derivative $dy/dx$ when $y$ is given in terms of $x$ by a formula—such as $y = x^3 + 1$; that is, when $y$ is expressed *explicitly* as a function of $x$. But suppose that $y$ is instead related to $x$ by an equation such as

$$y^4 + xy = x^3 - x + 2 \tag{1}$$

In this case, we say that $y$ is defined *implicitly* as a function of $x$. How can we find the slope of the tangent line at a point, such as $(1, 1)$, on the graph (Figure 1)? Although it may be difficult or even impossible to solve for $y$ explicitly as a function of $x$, we can find $dy/dx$ using the method of **implicit differentiation** (see Example 2).

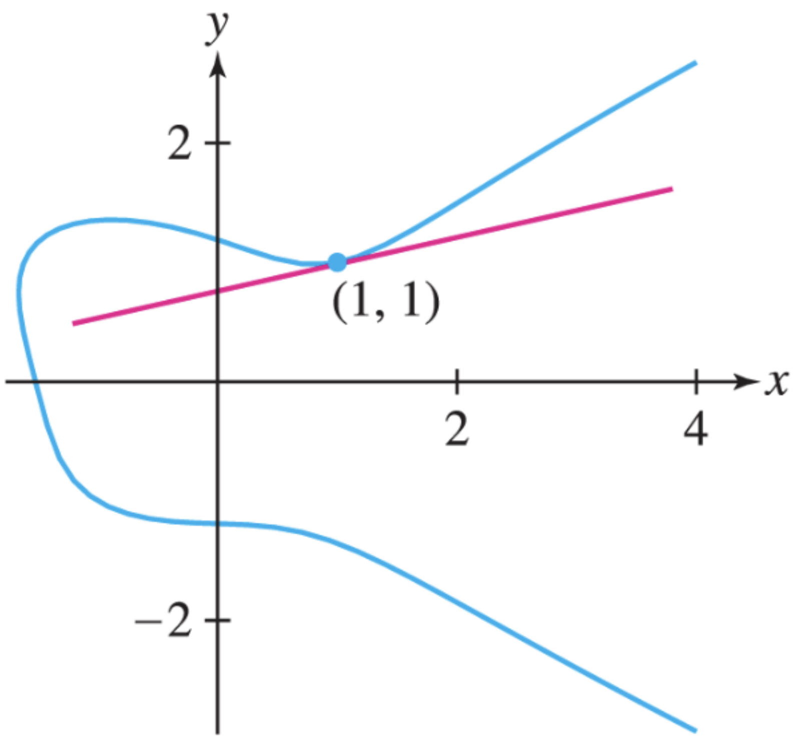





**FIGURE 1** Graph of $y^4 + xy = x^3 - x + 2$.

Ex: write equation of tangent line to

$$x^2+y^2=1 \quad \text{at} \quad \left(\overset{x}{\frac{3}{5}}, \overset{y}{\frac{4}{5}}\right). \qquad y' = ?$$

$$2x + 2y \cdot y' = 0$$

$$-2x \qquad\qquad -2x$$

$$\frac{\cancel{2y}\,y'}{\cancel{2y}} = -\frac{\cancel{2}x}{\cancel{2}y} \Rightarrow \boxed{y' = -\frac{x}{y}} = -\frac{\frac{3}{5}}{\frac{4}{5}} = -\frac{15}{20} = \boxed{-\frac{3}{4}}$$

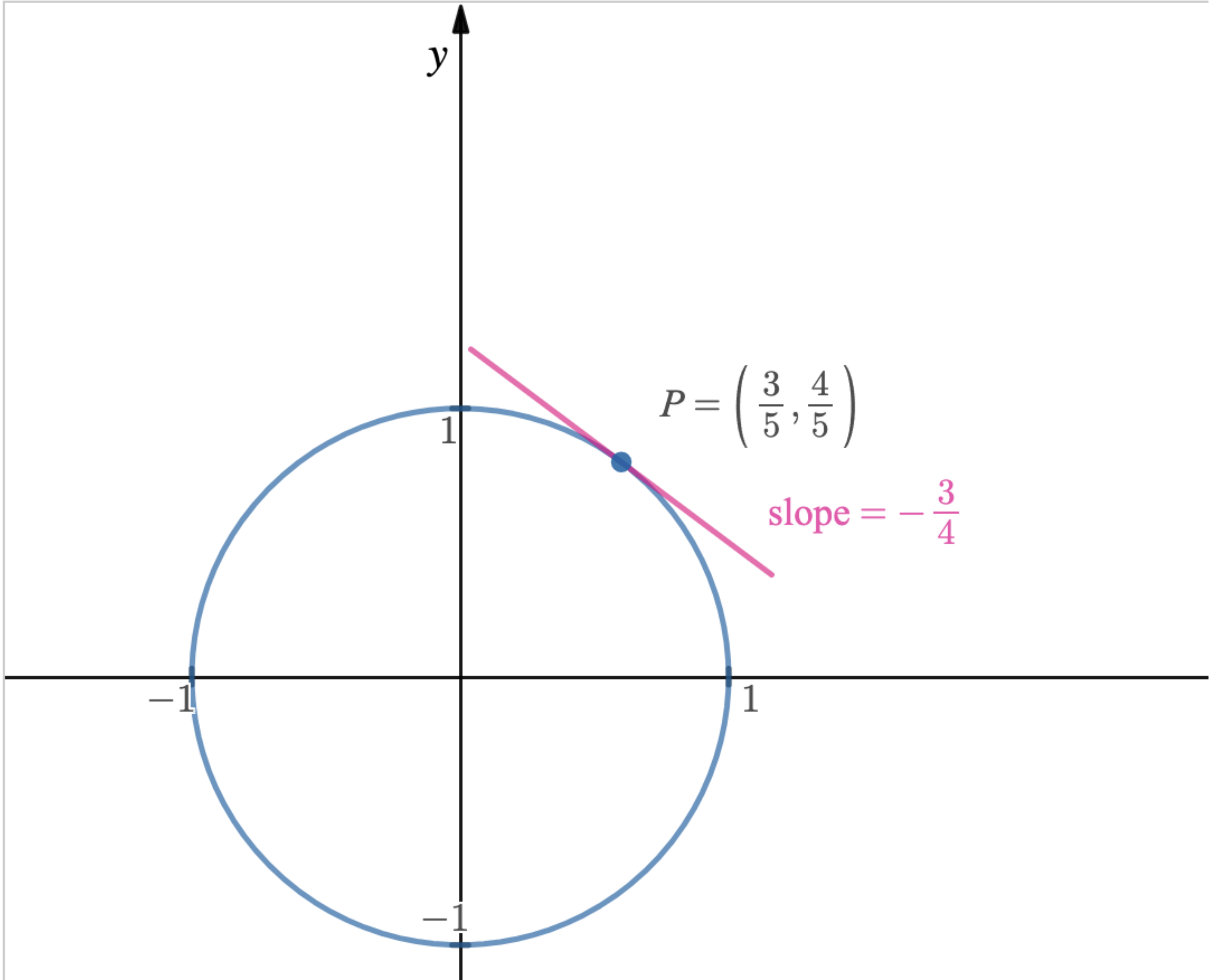
slope

$$y - y_1 = m(x - x_1)$$

$$y - \frac{4}{5} = -\frac{3}{4}\left(x - \frac{3}{5}\right) \Rightarrow y = \cdots$$

To illustrate, we first compute the slope of the tangent line to the unit circle at $\left(\frac{3}{5}, \frac{4}{5}\right)$ (Figure 2). The equation of the unit circle is:

$$x^2 + y^2 = 1$$



Compute $dy/dx$ by taking the derivative of both sides of the equation:

$$\begin{aligned} \frac{d}{dx}\left(x^2+y^2\right) &= \frac{d}{dx}(1) \\ \frac{d}{dx}\left(x^2\right)+\frac{d}{dx}\left(y^2\right) &= 0 \\ 2x+\frac{d}{dx}\left(y^2\right) &= 0 \end{aligned}$$

How do we handle the term $\frac{d}{dx}\left(y^2\right)$? We use the Chain Rule. Think of $y$ as a function $y=y(x)$. Then $y^2=(y(x))^2$ and by the Chain Rule,

$$\frac{d}{dx}y^2=\frac{d}{dx}(y(x))^2=2y(x)\frac{dy}{dx}=2y\frac{dy}{dx}$$

Equation (2) becomes $2x+2y\frac{dy}{dx}=0$, and we can solve for $\frac{dy}{dx}$ if $y\neq 0$:

$$\frac{dy}{dx}=-\frac{x}{y}$$

## EXAMPLE 1

Use Eq. (3) to find the slope of the tangent line at the point $P = \left(\frac{3}{5}, \frac{4}{5}\right)$ on the unit circle.

### Solution

Set $x = \frac{3}{5}$ and $y = \frac{4}{5}$ in Eq. (3):

$$\left.\frac{dy}{dx}\right|_P = -\frac{x}{y} = -\frac{\frac{3}{5}}{\frac{4}{5}} = -\frac{3}{4}$$

Ex: $x^3 + y^3 = 12 \qquad \frac{dy}{dx} = ?$

$$\cancel{3x^2} + 3y^2 \cdot \frac{dy}{dx} = 0$$

$$\cancel{-3x^2} \qquad\qquad -3x^2$$

$$\frac{\cancel{3y^2} \cdot \frac{dy}{dx}}{\cancel{3y^2}} = -\frac{\cancel{3}x^2}{\cancel{3}y^2} \implies \boxed{\frac{dy}{dx} = -\frac{x^2}{y^2}}$$

$$\sin x \longrightarrow \cos x$$

$$\frac{d \sin y}{dy} \rightarrow \cos y \checkmark$$

$$\frac{d\sin y}{dx} \rightarrow \cos y \cdot y'$$

$$\frac{d \sin u}{du} \rightarrow \cos u$$

$$\frac{d \sin u}{dx} \rightarrow \cos u \cdot u'$$

Before presenting additional examples, let's examine again how the factor $dy/dx$ arises when we differentiate an expression involving $y$ with respect to $x$. It would not appear if we were differentiating with respect to $y$. Thus,

$$\frac{d}{dy}\sin y = \cos y \quad \text{but} \quad \frac{d}{dx}\sin y = (\cos y)\frac{dy}{dx}$$

$$\frac{d}{dy}y^4 = 4y^3 \quad \text{but} \quad \frac{d}{dx}y^4 = 4y^3\frac{dy}{dx}$$

*Notice what happens if we apply the Chain Rule to* $\frac{d}{dy}\sin y$ *. The extra derivative factor appears, but it is equal to* $1$ *:*

$$\frac{d}{dy}\sin y = (\cos y)\frac{dy}{dy} = \cos y$$

# EXAMPLE 2

Find an equation of the tangent line at the point $P = (1, 1)$ on the curve in Figure 1 with equation

$f'g + g'f = (1)y + y' \cdot x = y + xy'$

$$y^4 + \overset{f}{x}\overset{g}{y} = x^3 - x + \cancel{2}^{0}$$

1

$4y^3 \cdot y' + y + xy' = 3x^2 - 1 - y$

$-y$

$4y^3 y' + xy' = 3x^2 - 1 - y$

$\dfrac{y'(\cancel{4y^3 + x})}{\cancel{4y^3 + x}} = \dfrac{3x^2 - 1 - y}{4y^3 + x} \Rightarrow \boxed{y' = \dfrac{3x^2 - 1 - y}{4y^3 + x}}$

$(1,1) \longrightarrow m = \dfrac{3 - 1 - 1}{4 + 1} = \dfrac{1}{5}$ slope

2

$y - y_1 = m(x - x_1)$

$y - 1 = \frac{1}{5}(x - 1)$

$y = \frac{1}{5}x - \frac{1}{5} + 1$

$\boxed{y = \frac{1}{5}x + \frac{4}{5}}$ ✓ Tangent line

## Solution

We break up the calculation into two steps.

***Step 1*. Differentiate both sides of the equation with respect to $x$ .**

Note that each occurrence of $y$ in the original equation generates an additional $\dfrac{dy}{dx}$ upon differentiation.

$$\begin{aligned} \frac{d}{dx}\,y^4 + \frac{d}{dx}\,(xy) &= \frac{d}{dx}\,(x^3 - x + 2\,) \\ 4y^3\frac{dy}{dx} + \left(y + x\frac{dy}{dx}\right) &= 3x^2 - 1 \end{aligned}$$

**Step 2.** **Solve for $\frac{dy}{dx}$.**

Move the terms involving $dy/dx$ in Eq. (4) to the left and place the remaining terms on the right:

$$4y^3 \frac{dy}{dx} + x \frac{dy}{dx} = 3x^2 - 1 - y$$

Then factor out $dy/dx$ and divide:

$$\begin{aligned} (4y^3 + x) \frac{dy}{dx} &= 3x^2 - 1 - y \\ \frac{dy}{dx} &= \frac{3x^2 - 1 - y}{4y^3 + x} \end{aligned}$$

To find the derivative at $P = (1, 1)$, apply Eq. (5) with $x = 1$ and $y = 1$:

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{3 \cdot 1^2 - 1 - 1}{4 \cdot 1^3 + 1} = \frac{1}{5}$$

An equation of the tangent line is $y - 1 = \frac{1}{5}(x - 1)$ or $y = \frac{1}{5}x + \frac{4}{5}$.

# EXAMPLE 3

$e^{u} \longrightarrow e^{u} \cdot u'$

Find the slope of the tangent line at the point $P = (1,1)$ on the graph of $e^{x-y} = 2x^2 - y^2$.

$e^{x-y} \cdot (1 - y') = 4x - 2y \cdot y'$

$e^{x-y} - y' \cdot e^{x-y} = 4x - 2y y'$

$-y' \cdot e^{x-y} + 2y y' = 4x - e^{x-y}$

$\cancel{y' (-e^{x-y} + 2y)} = \dfrac{4x - e^{x-y}}{-e^{x-y} + 2y}$

$\cancel{\dfrac{}{-e^{x-y} + 2y}}$

$(1,1)$

$m = \dfrac{4 - e^{\cancel{1-1}\,0}}{-e^{\cancel{1-1}\,0} + 2} = \dfrac{4-1}{-1+2} = \dfrac{3}{1} = 3$

Slope

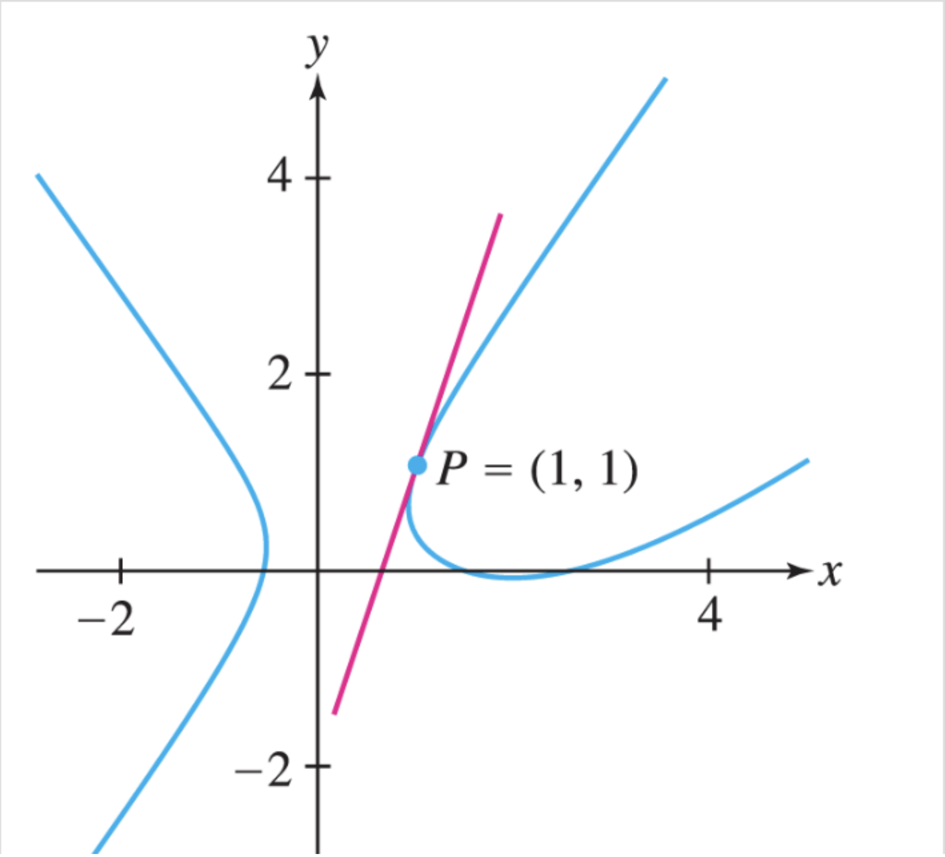
## Solution

We follow the steps of the previous example, this time writing $y'$ for $dy/dx$:

$$\begin{aligned}
\frac{d}{dx}\, e^{x-y} &= \frac{d}{dx}\,(2x^2 - y^2) \\
e^{x-y}\,(1 - y') &= 4x - 2yy' && \text{(Chain Rule applied to } e^{x-y} \text{ and } y^2 \\
e^{x-y} - e^{x-y}\,y' &= 4x - 2yy' \\
(2y - e^{x-y})\,y' &= 4x - e^{x-y} && \text{(place all } y'\text{-terms on left)} \\
y' &= \frac{4x - e^{x-y}}{2y - e^{x-y}}
\end{aligned}$$

The slope of the tangent line at $P = (1, 1)$ is (Figure 4)

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{4\,(1) - e^{1-1}}{2\,(1) - e^{1-1}} = \frac{4-1}{2-1} = 3$$

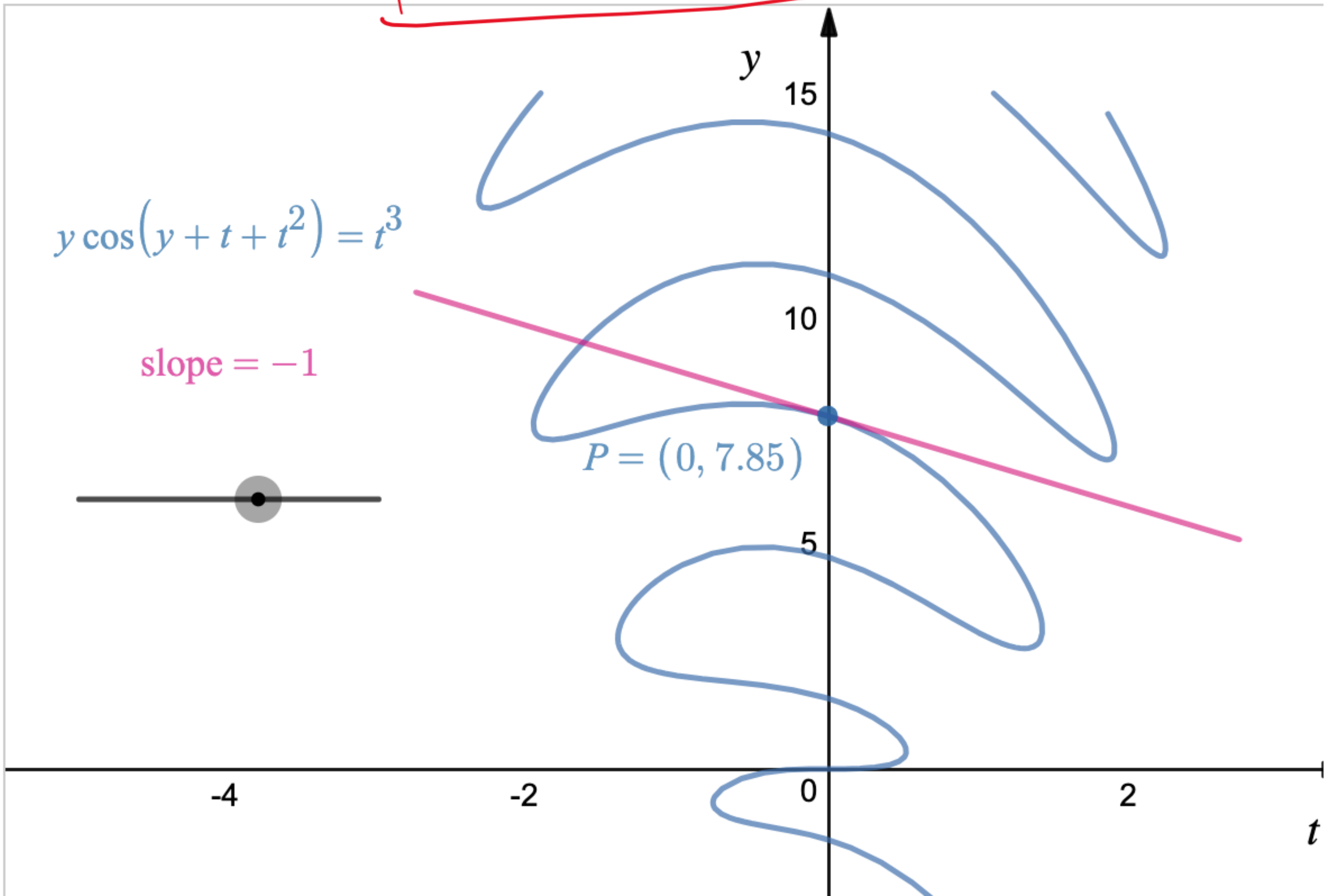


## EXAMPLE 4

### Shortcut to Derivative at a Specific Point

Calculate $\left.\dfrac{dy}{dt}\right|_P$ at the point $P = \left(0, \dfrac{5\pi}{2}\right)$ on the curve (Figure 5):

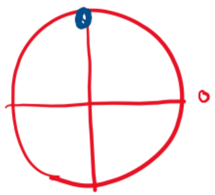
$$y\cos\left(y + t + t^2\right) = t^3$$



$$\underset{f}{y}\cdot\underset{g}{\cos}(\underset{f}{y}+\underset{g}{t}+t^2) = t^3 \qquad (\overset{t}{0}, \overset{y}{\tfrac{5\pi}{2}}) \qquad \frac{dy}{dx}\Big|_p = ?$$

$f'g+g'f$

$$y'\cos(y+t+t^2) + \overbrace{(-\sin(y+t+t^2))\cdot(y'+1+2t)}^{g'}\,y = 3t^2$$



$$y'\cos\left(\frac{5\pi}{2}+0+0\right) + \left(-\sin\left(\frac{5\pi}{2}+0+0\right)\right)(y'+1+0)\cdot\frac{5\pi}{2} = 0$$

$$y'\underbrace{\cancel{\cos\tfrac{5\pi}{2}}}_{0} - \overset{1}{\cancel{\sin\tfrac{5\pi}{2}}}\cdot(y'+1)\cdot\frac{5\pi}{2} = 0$$

$$-\frac{5\pi}{2}(y'+1) = 0 \Rightarrow -\frac{5\pi}{2}y' - \frac{5\pi}{2} = 0 \Rightarrow y' = \frac{\frac{5\pi}{2}}{-\frac{5\pi}{2}} = -1$$

slope

## Solution

As before, differentiate both sides of the equation (we write $y'$ for $dy/dt$):

$$\frac{d}{dt} y \ \cos\left(y + t + t^2\right) = \frac{d}{dt} t^3$$
$$y' \cos\left(y + t + t^2\right) - y \ \sin\left(y + t + t^2\right)\left(y' + 1 + 2t\right) = 3t^2$$

We could continue to solve for $y'$ in terms of $t$ and $y$, but that is not necessary since we are only interested in $dy/dt$ at the point $P$. Instead, we can substitute $t = 0$ $y = \frac{5\pi}{2}$ directly in Eq. (6) and then solve for $y'$:

$$y' \cos\left(\frac{5\pi}{2} + 0 + 0^2\right) - \left(\frac{5\pi}{2}\right) \sin\left(\frac{5\pi}{2} + 0 + 0^2\right)\left(y' + 1 + 0\right) = 0$$
$$0 - \left(\frac{5\pi}{2}\right)(1)\left(y' + 1\right) = 0$$

This gives us $y' + 1 = 0$ or $y' = -1$.

# Derivatives of Inverse Trigonometric Functions

We now apply implicit differentiation to determine the derivatives of the inverse trigonometric functions. An interesting feature of these functions is that their derivatives are not trigonometric. Rather, they involve quadratic expressions and their square roots. Keep in mind the restricted domains of these functions.

## THEOREM 1

### Derivatives of Arcsine and Arccosine

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d\,\sin^{-1}x}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d\,\sin^{-1}u}{dx} = \frac{1\cdot u'}{\sqrt{1-u^2}}$$

$$\frac{d\,\cos^{-1}x}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

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$$\frac{d\,\tan^{-1}x}{dx} = \frac{1}{x^2+1}$$

$$\frac{d\,\cot^{-1}x}{dx} = -\frac{1}{x^2+1}$$

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$$\frac{d\,\sec^{-1}x}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d\,\csc^{-1}x}{dx} = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d \sin^{-1} u}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot u' = \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x = \frac{2x}{\sqrt{1-x^4}}$$

## EXAMPLE 5

Calculate $f'\left(\frac{1}{2}\right)$, where $f(x) = \arcsin\left(\underset{u}{x^2}\right)$. $= \dfrac{\cancel{2}\cdot\frac{1}{\cancel{2}}}{\sqrt{1-\left(\frac{1}{2}\right)^4}}$

### Solution

$$= \frac{1}{\sqrt{\frac{16}{16}-\frac{1}{16}}} = \frac{1}{\sqrt{\frac{15}{16}}} = \frac{1}{\frac{\sqrt{15}}{\sqrt{16}\,4}} = \boxed{\frac{4}{\sqrt{15}}}$$

$$\frac{1}{\sqrt{\frac{15}{16}}} = \frac{1}{\frac{\sqrt{15}}{4}}$$

$$1 \div \frac{\sqrt{15}}{4}$$

$$1 \cdot \frac{4}{\sqrt{15}} = \frac{4}{\sqrt{15}}$$

Recall that $\arcsin\ x$ is another notation for $\sin^{-1} x$. By the Chain Rule,

$$\frac{d}{dx}\arcsin(x^2) = \frac{d}{dx}(\sin^{-1}(x^2)) = \frac{1}{\sqrt{1-(x^2)^2}}\frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1-x^4}}$$

$$f'\left(\frac{1}{2}\right) = \frac{2\left(\frac{1}{2}\right)}{\sqrt{1-\left(\frac{1}{2}\right)^4}} = \frac{1}{\sqrt{\frac{15}{16}}} = \frac{4}{\sqrt{15}}$$

# THEOREM 2

## Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{x^2+1}, \qquad \frac{d}{dx}\cot^{-1}x = -\frac{1}{x^2+1}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}, \qquad \frac{d}{dx}\csc^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}}$$

## EXAMPLE 6 $\frac{d\csc^{-1}x}{dx} = -\frac{1}{|u|\sqrt{u^2-1}} \cdot u'$

Calculate $\frac{dy}{dx}\left(\csc^{-1}(\overset{u}{e^x + 1})\right)\Big|_{x=0}$.

$$= \frac{-1}{|e^x+1|\sqrt{(e^x+1)^2-1}} \cdot e^x$$

$$= \frac{-1}{|\cancel{e^0}+1|\sqrt{(\cancel{e^0}+1)^2-1}} \cdot \cancel{e^0}$$

$$= \boxed{\frac{-1}{2\sqrt{3}}}$$

### Solution

Apply the Chain Rule using the formula $\frac{d}{du}\csc^{-1} u = -\frac{1}{|u|\sqrt{u^2-1}}$: $u'$

$$\begin{aligned}\frac{d}{dx}\csc^{-1}(e^x+1) &= -\frac{1}{|e^x+1|\sqrt{(e^x+1)^2-1}}\frac{d}{dx}(e^x+1)\\ &= -\frac{e^x}{(e^x+1)\sqrt{e^{2x}+2e^x}}\end{aligned}$$

We have replaced $|e^x+1|$ by $e^x+1$ because this quantity is positive. Now we have

$$\frac{dy}{dx}\csc^{-1}(e^x+1)\Big|_{x=0} = -\frac{e^0}{(e^0+1)\sqrt{e^0+2e^0}} = \boxed{-\frac{1}{2\sqrt{3}}}$$

# Finding Higher Order Derivatives Implicitly

By using the implicit derivative process repeatedly, we can find higher order derivatives of a function that is defined implicitly. We do so in the next example.

## EXAMPLE 7

Find $\dfrac{d^2 y}{dx^2}$ for $x^2 + 4y^2 = 7$.

$$x^2 + 4y^2 = 7$$

$$2x + 8y \cdot \frac{dy}{dx} = 0$$

$$-2x \qquad -2x$$

$$\frac{\cancel{8y} \cdot \frac{dy}{dx}}{\cancel{8y}} = \frac{-\overset{1}{\cancel{2}}x}{4\cancel{8}y}$$

$$\frac{dy}{dx} = \frac{-x}{4y}$$

$$\frac{d^2y}{dx^2} = ?$$

$$\frac{f'g - g'f}{g^2}$$

$$\frac{d^2y}{dx^2} = \frac{(-1)(4y) - 4\frac{dy}{dx} \cdot (-x)}{(4y)^2}$$

$$= \frac{-4y + \cancel{4}x \cancel{\frac{dy}{dx}} \frac{-x}{\cancel{4}y}}{16y^2} = \frac{\frac{-4y \cdot y}{y} - \frac{x^2}{y}}{16y^2}$$

$$= \frac{\left(\frac{-4y^2 - x^2}{y}\right)}{16y^2} = \frac{-4y^2 - x^2}{16y^3} = \frac{-(\cancel{4y^2 + x^2})^{7}}{16y^3} = \boxed{\frac{-7}{16y^3}}$$

## Solution

We differentiate with respect to $x$, writing $y'$ for $\frac{dy}{dx}$:

$$2x + 8yy' = 0$$

Solving for $y'$, we obtain

$$y' = \frac{-x}{4y}$$

Differentiating again with respect to $x$, we obtain

$$y'' = \frac{4y(-1) - (-x)(4y')}{16y^2} = \frac{-y + xy'}{4y^2}$$

Substituting in the fact that $y' = \frac{-x}{4y}$ yields

$$y'' = \frac{-y + x(-x/(4y))}{4y^2} = \frac{-4y^2 - x^2}{16y^3} = \frac{-7}{16y^3}$$

The last equality holds since $x^2 + 4y^2 = 7$

## 3.8 SUMMARY

- Implicit differentiation is used to compute $dy/dx$ when $x$ and $y$ are related by an equation.
  *Step 1*. Take the derivative of both sides of the equation with respect to $x$, treating $y$ as a function of $x$.

  *Step 2*. Solve for $dy/dx$ by collecting the terms involving $dy/dx$ on one side and the remaining terms on the other side of the equation.

- Remember to include the factor $dy/dx$ when differentiating expressions involving $y$ with respect to $x$. For instance,
$$\frac{d}{dx}\sin\, y=(\cos\, y)\,\frac{dy}{dx}$$
- Derivative formulas:
$$\frac{d}{dx}\sin^{-1}x=\frac{1}{\sqrt{1-x^2}},\qquad \frac{d}{dx}\cos^{-1}x=-\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1}x=\frac{1}{x^2+1},\qquad \frac{d}{dx}\cot^{-1}x=-\frac{1}{x^2+1}$$
$$\frac{d}{dx}\sec^{-1}x=\frac{1}{|x|\sqrt{x^2-1}},\quad \frac{d}{dx}\csc^{-1}x=-\frac{1}{|x|\sqrt{x^2-1}}$$