6.3 Generalized Permutations and Combinations

In Section 6.2, we dealt with orderings and selections without allowing repetitions. In this section we consider orderings of sequences containing repetitions and unordered selections in which repetitions are allowed.

Example 3.1 How many strings can be formed using the following letters?

$$M \, I \, S \, S \, I \, S \, S \, I \, P \, P \, I$$

SOLUTION:

Because of the duplication of letters, the answer is not 11!, but some number less than 11!.

$$C(11, 2)C(9, 4)C(5, 4) = \frac{11!}{2! \, 9!} \frac{9!}{4! \, 5!} \frac{5!}{4! 1!} =$$

$$\frac{11!}{2! 4! 4! 1!} = 34,650.$$

Theorem 3.2 Suppose that a sequence S of n items has n_1 identical objects of type 1, n_2 identical objects of type 2, . . . , and n_t identical objects of type t. Then the number of orderings of S is

$$\frac{n!}{n_1! \, n_2! \, \cdots \, n_t!}$$

Proof:

$$C(n, n_1)C(n - n_1, n_2)C(n - n_1 - n_2, n_3)$$
$$\cdot \cdot \cdot C(n - n_1 - \cdot \cdot \cdot - n_{t-1}, n_t) =$$

$$\frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdots\frac{(n-n_1-\cdots-n_{t-1})!}{n_t!0!}$$

$$= \frac{n!}{n_1!\,n_2!\cdots n_t!}$$

∎

Example 3.3 In how many ways can eight distinct books be divided among three students if Bill gets four books and Shizuo and Marian each get two books?

SOLUTION:

An example is

B B B S M B M S.

Thus the number of ways of ordering *BBBBSSMM* is the number of ways to distribute the books. By Theorem 3.2, this number is

$$\frac{8!}{4!2!2!} = 420.$$

■

Example 3.4 Consider three books: a computer science book, a physics book, and a history book. Suppose that the library has at least six copies of each of these books. In how many ways can we select six books?

SOLUTION:

The problem is to choose unordered, six-element selections from the set {computer science, physics, history}, **repetitions allowed.**

CS   Physics   History

$\times$ $\times$ $\times$ | $\times$ $\times$ | $\times$

CS   Physics   History

| $\times$ $\times$ $\times$ $\times$ | $\times$ $\times$
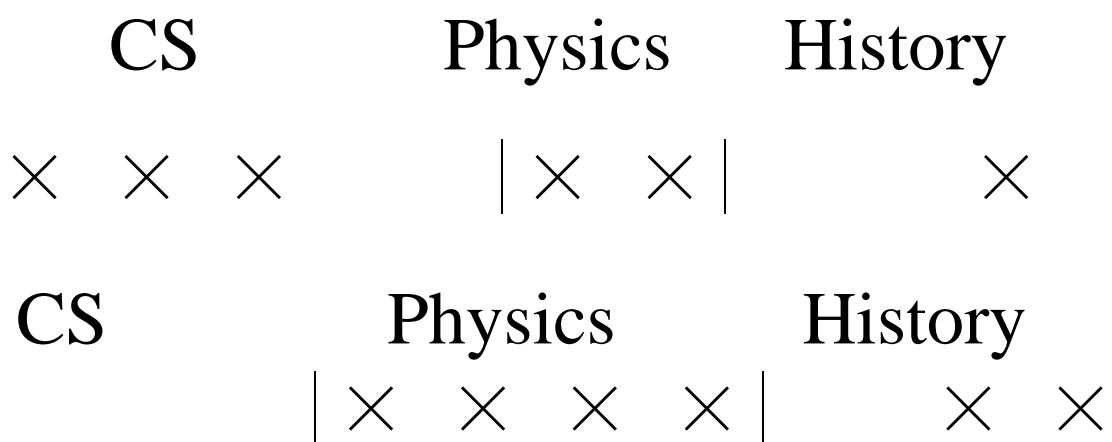
But this is just the number of ways

$$C(8, 2) = 28$$

of selecting two positions for the |'s from eight possible positions. Thus there are **28 ways to select six books**. ■

The method used in Example 3.4 can be used to derive a general result.

Theorem 3.5 If X is a set containing t elements, the number of unordered, k-element selections from X, repetitions allowed, is

C(k + t − 1, t − 1) = C(k + t − 1, k).

Proof: Let $X = \{a_1, \ldots, a_t\}$. Consider the $k + t − 1$ slots

$$\underline{} \cdot \cdot \cdot \underline{}$$

and **k+t−1** symbols consisting of k ✕'s and **t−1** |'s. The number n_1 of ✕'s up to

the first $|$ represents the selection of n_1 a_1's; the number n_2 of ×'s between the first and second $|$'s represents the selection of n_2 a_2's; and so on. Since there are

$$C(k + t - 1, t - 1)$$

ways to select the positions for the $|$'s, there are also

$$C(k + t - 1, t - 1)$$

selections. This is equal to

$$C(k + t - 1, k),$$

the number of ways to select the positions for the ×'s; hence there are

$$C(k + t - 1, t - 1) = C(k + t - 1, k)$$

unordered, *k*-element selections from *X*, repetitions allowed. ∎

Example 3.6 Suppose that there are piles of red, blue, and green balls and that each pile contains at least eight balls.

(a) In how many ways can we select eight balls?

(b) In how many ways can we select eight balls if we must have at least one ball of each color?

Solution:

(a)

$$C(8 + 3 - 1, 3 - 1) = C(10, 2) = 45.$$

(b)

$$C(5 + 3 - 1, 3 - 1) = C(7, 2) = 21$$

■

Example 3.7 In how many ways can 12 identical mathematics books be distributed among the students Anna, Beth, Candy, and Dan?

Solution:

$$C(12 + 4 - 1, 4 - 1) = C(15, 3) = 455.$$

■

Example 3.8

(a) How many solutions in nonnegative integers are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 29 \ ? \qquad\qquad (6.3.1)$$

(b) How many solutions in integers are there to (6.3.1) satisfying $x_1 > 0$, $x_2 > 1$, $x_3 > 2$, $x_4 \geq 0$?

Solution:

(a)

$C(29 + 4 - 1, 4 - 1) = C(32, 3) = 4960.$

(b)

$C(23 + 4 - 1, 4 - 1) = C(26, 3) = 2600$