

3.4 Rates of Change

In this section, we pause from building tools for computing the derivative and instead focus on the derivative as a rate of change, particularly in applied settings.

Recall the notation for the average rate of change of a function $y = f(x)$ over an interval $[x_0, x_1]$:

$$\Delta y = \text{change in } y = f(x_1) - f(x_0)$$

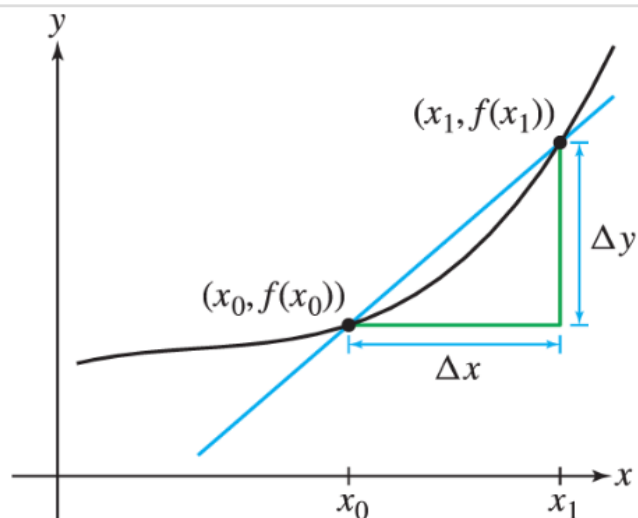
$$\Delta x = \text{change in } x = x_1 - x_0$$

$$\text{average rate of change} = \frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

In our prior discussion in Section 2.1, limits and derivatives had not yet been introduced. Now that we have them at our disposal, we can define the instantaneous rate of change of y with respect to x at $x = x_0$:

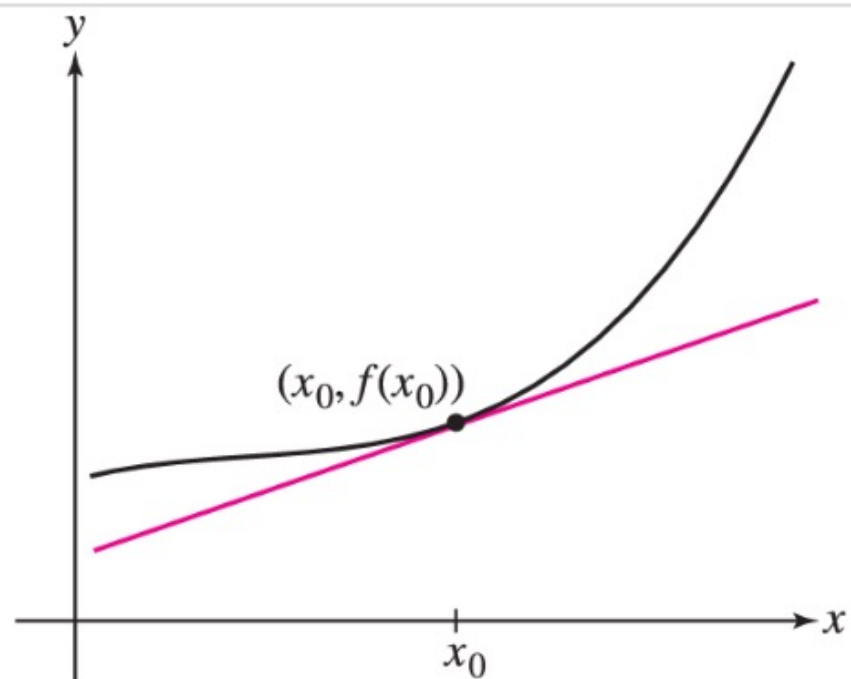
$$\text{instantaneous rate of change} = f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{x_1 \rightarrow x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Keep in mind the geometric interpretations: The average rate of change is the slope of the secant line (Figure 1), and the instantaneous rate of change is the slope of the tangent line (Figure 2).



Rogawski et al., *Calculus: Early Transcendentals*, 4e, © 2019
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FIGURE 1 The average rate of change over $[x_0, x_1]$ is the slope of the secant line.



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FIGURE 2 The instantaneous rate of change at x_0 is the slope of the tangent line.

Leibniz notation dy / dx is particularly convenient because it specifies that we are considering the rate of change of y with respect to the independent variable x . The rate dy / dx is measured in units of y per unit of x . For example, the rate of change of temperature with respect to time has units such as degrees per minute, whereas the rate of change of temperature with respect to altitude has units such as degrees per kilometer. In applications, it is important to be mindful of the units on rates of change and to interpret properly what the rate of change is communicating about the variables.

$$A = \pi r^2$$

EXAMPLE 2

$$\frac{dA}{dr} = \pi \cdot (2r) = 2\pi r$$

Let $A = \pi r^2$ be the area of a circle of radius r .

$$\left. \frac{dA}{dr} \right|_{r=2} = 2\pi(2) = 4\pi \approx 12.56$$

a. Compute dA / dr at $r = 2$ and $r = 5$.

$$\left. \frac{dA}{dr} \right|_{r=5} = 2\pi(5) = 10\pi \approx 31.42$$

b. Explain geometrically why dA / dr is greater at $r = 5$ than at $r = 2$.

Solution

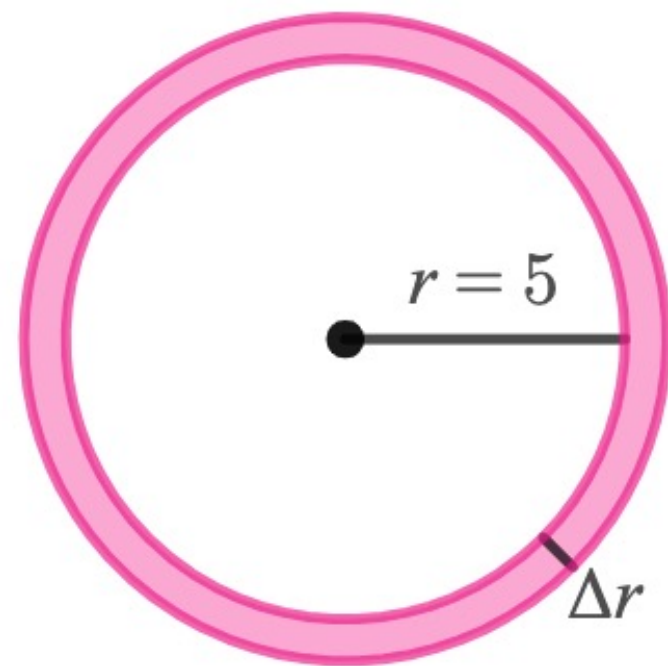
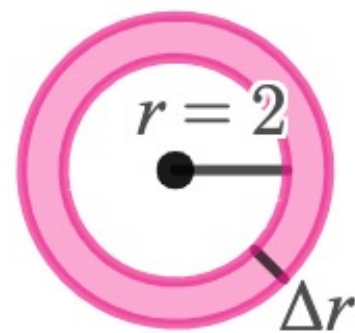
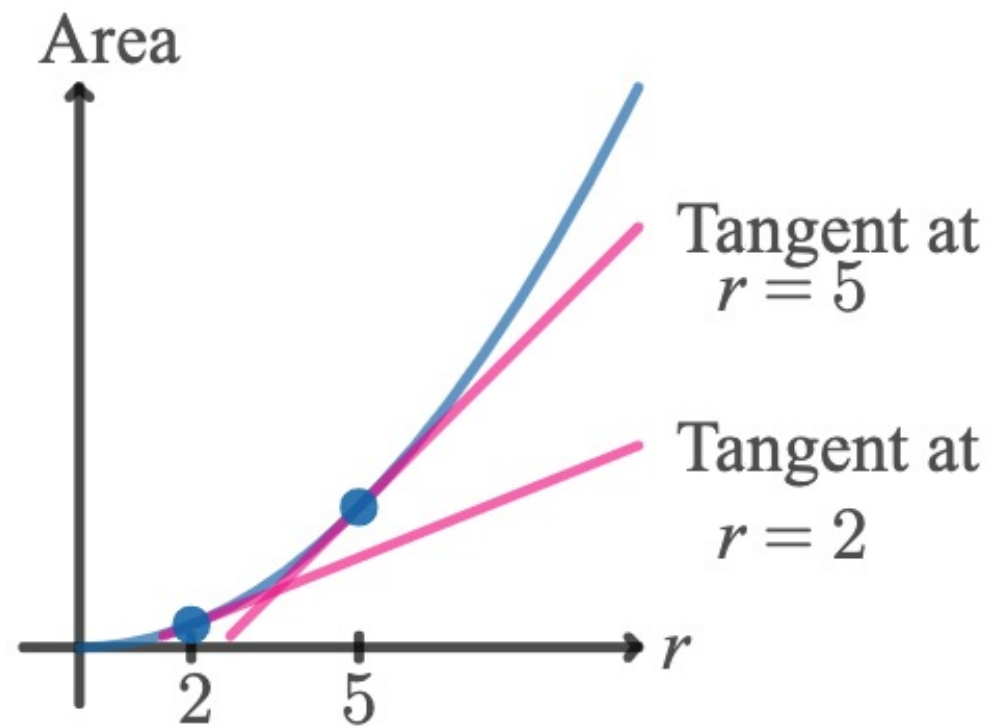
The rate of change of area with respect to radius is the derivative

$$\frac{dA}{dr} = \frac{d}{dr} (\pi r^2) = 2\pi r$$

a. We have

$$\left. \frac{dA}{dr} \right|_{r=2} = 2\pi(2) \approx 12.57 \quad \text{and} \quad \left. \frac{dA}{dr} \right|_{r=5} = 2\pi(5) \approx 31.42$$

b. The derivative dA / dr measures how the area of the circle changes when r increases. Figure 4 shows that when the radius increases by Δr , the area increases by a band of thickness Δr . The area of the band is greater at $r = 5$ than at $r = 2$. Therefore, the derivative is larger (and the tangent line is steeper) at $r = 5$. In general, for a fixed Δr , the change in area ΔA is greater when r is larger.



Marginal Cost in Economics

Let $C(x)$ denote the dollar cost (including labor and parts) of producing x units of a particular product. The number x of units manufactured is called the **production level**. To study the relation between costs and production, economists define the **marginal cost** at production level x_0 as the cost of producing one additional unit:

$$\text{marginal cost} = C(x_0 + 1) - C(x_0)$$

Note that if we use a difference quotient approximation for the derivative $C'(x_0)$ with $h = 1$, we obtain

$$C'(x_0) \approx \frac{C(x_0 + 1) - C(x_0)}{1} = C(x_0 + 1) - C(x_0)$$

and therefore we can use the derivative at x_0 as an approximation to the marginal cost.

Although $C(x)$ is meaningful only when x is a whole number, economists often treat $C(x)$ as a differentiable function of x so that the techniques of calculus can be applied. This is reasonable when the domain of C is large.

EXAMPLE 3

Cost of an Air Flight

Company data suggest that when there are 50 or more passengers, the total dollar cost of a certain flight is approximately $C(x) = 0.0005x^3 - 0.38x^2 + 120x$, where x is the number of passengers (Figure 5).

$$C'(x) = 0.0005 \cdot 3x^2 - 0.38 \cdot 2x + 120$$

- Estimate the marginal cost of an additional passenger if the flight already has 150 passengers.
- Compare your estimate with the actual cost of an additional passenger.
- Is it more expensive to add a passenger when $x = 150$ or when $x = 200$?

Solution

The derivative is $C'(x) = 0.0015x^2 - 0.76x + 120$.

- a. We estimate the marginal cost at $x = 150$ by the derivative

$$C'(150) = 0.0015(150)^2 - 0.76(150) + 120 = 39.75$$

Thus, it costs approximately \$39.75 to add one additional passenger.

- b. The actual cost of adding one additional passenger is

$$\underline{C(151)} - \underline{C(150)} \approx \underline{11,177.10} - \underline{11,137.50} = 39.60$$

Our estimate of \$39.75 is close enough for practical purposes.

- c. The marginal cost at $x = 200$ is approximately

$$C'(200) = 0.0015(200)^2 - 0.76(200) + 120 = 28$$

Since $39.75 > 28$, it is more expensive to add a passenger when $x = 150$ than when $x = 200$.

$$s(t)$$

$$v(t) = s'(t)$$

$$\text{Speed} = |v(t)|$$

$$a = v'(t) = s''(t)$$

Linear Motion

Recall that *linear motion* is motion along a straight line. This includes horizontal motion along a straight highway and vertical motion of a falling object. Let $s(t)$ denote the position on a line, relative to the origin, at time t . Velocity is the rate of change of position with respect to time:

$$v(t) = \text{velocity} = \frac{ds}{dt}$$

The sign of $v(t)$ indicates the direction of motion. For example, if $s(t)$ is the height above ground, then $v(t) > 0$ indicates that the object is rising. Speed is defined as the absolute value of velocity, $|v(t)|$.

EXAMPLE 4

A truck enters the off-ramp of a highway at $t = 0$. Its position on the off-ramp after t seconds is $s(t) = 25t - 0.3t^3$ m for $0 \leq t \leq 5$.

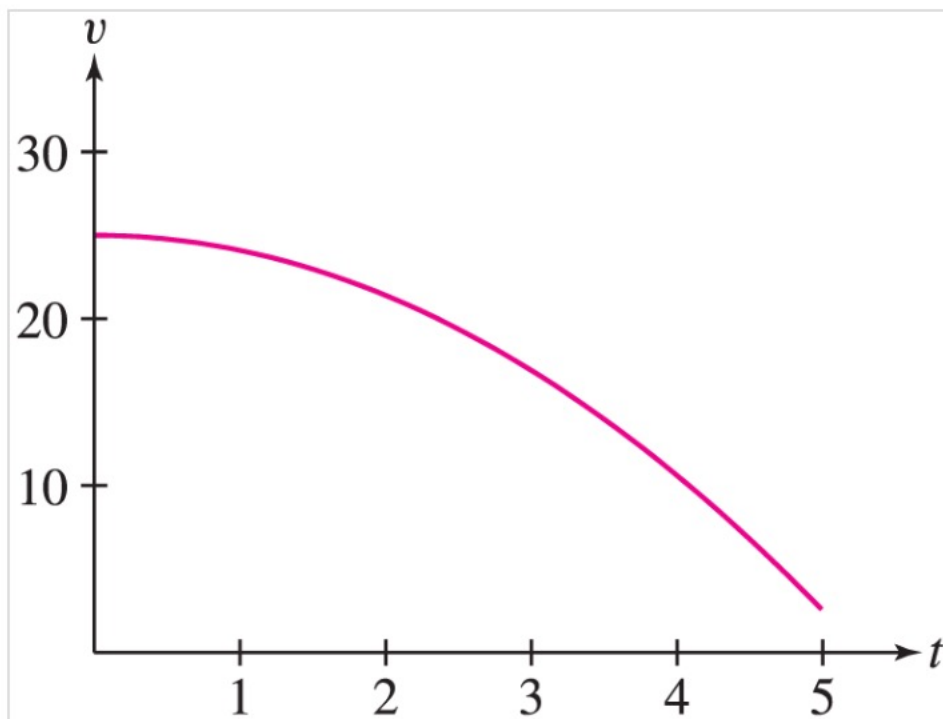
- a. How fast is the truck going at the ^{$t=0$} moment it enters the off-ramp?
- b. Is the truck speeding up or slowing down?

$$v(0) = s'(t) = 25 - 0.9t^2 \xrightarrow{t=0} 25 - 0.9(0) = 25 \rightarrow 25$$
$$v(5) = 25 - 0.9(5)^2 = 2.5 \rightarrow 2.5$$

Solution

The truck's velocity at time t is $v(t) = \frac{d}{dt} (25t - 0.3t^3) = 25 - 0.9t^2$.

- a. The truck enters the off-ramp with velocity $v(0) = 25 \text{ m/s}$.
- b. Since $v(t) = 25 - 0.9t^2$ is decreasing and positive ([Figure 6](#)), the speed is decreasing and the truck is **slowing down**.



When we say “speeding up” or “slowing down” we typically are referring to the *speed* of an object, not its velocity. The relationship between speed and velocity is simple: Speed is the absolute value of velocity. Use care to apply velocity and speed properly when describing an object’s motion. For instance, as the next example demonstrates, an object’s velocity can increase while its speed decreases.

EXAMPLE 5

Velocity and Speed

Figure 7 shows graphs of an object in linear motion whose position s is changing in time t in four different circumstances.

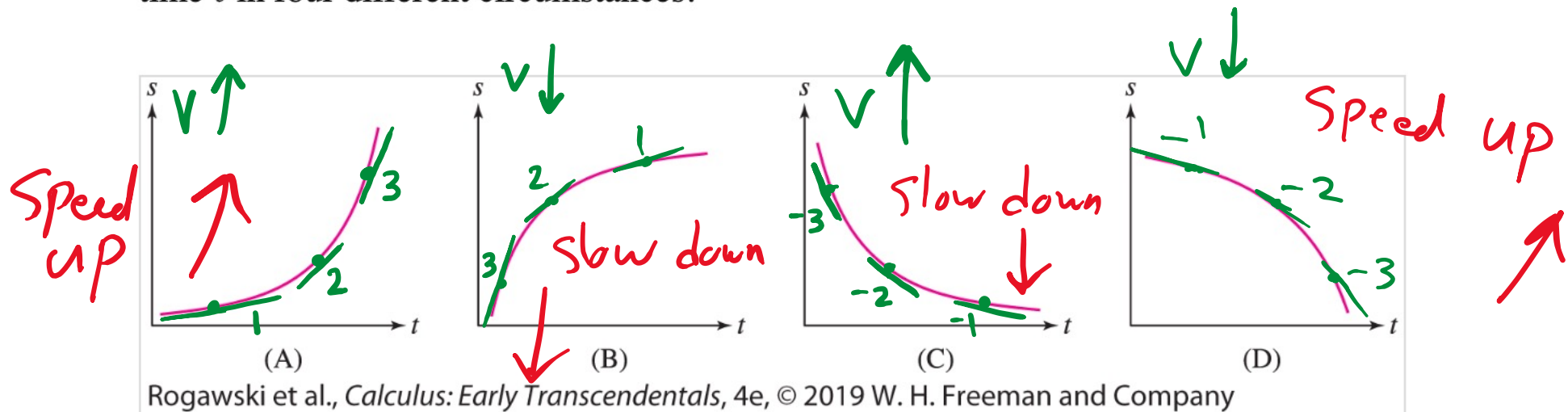


FIGURE 7

- In which cases is the **velocity** increasing? decreasing?
- In which cases is the speed increasing (so the object is speeding up)? decreasing (so the object is slowing down)?

 **REMINDER**

“Larger” means farther from 0 , while “smaller” means closer to 0.

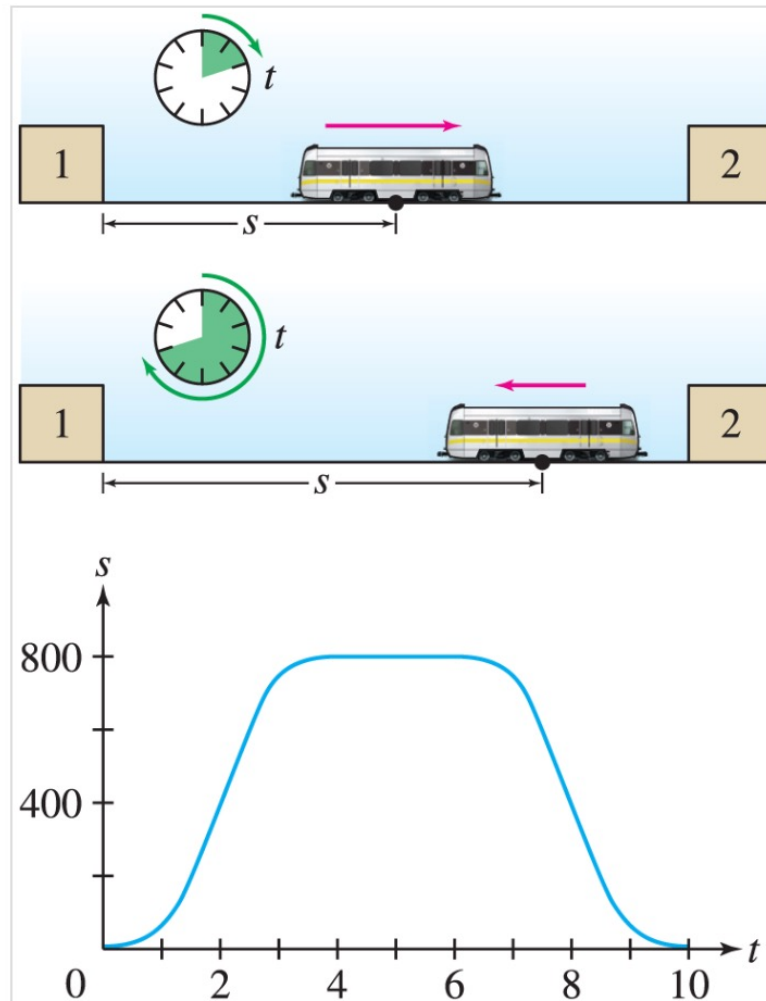
- a.
 - In [Figure 7\(A\)](#), the slope is positive and getting larger, so the velocity (or rate change) is increasing.
 - In [Figure 7\(B\)](#), the slope is positive and getting smaller, so the velocity is decreasing.
 - In [Figure 7\(C\)](#), the slope is negative and getting smaller; that is, getting closer to zero. Since the slope values are negative and approaching zero, the slope is increasing, and therefore the velocity is increasing.
 - In [Figure 7\(D\)](#), the slope is negative and is getting larger in the negative direction, so the velocity is decreasing.
- b. Now we are considering the absolute value of the velocity; that is, the absolute value of the slope of the graph. It (and therefore speed) increases when the slope gets steeper, and that occurs in both [Figures 7\(A\)](#) and [7\(D\)](#). Thus, in both of those cases the object is speeding up. On the other hand, in [Figures 7\(B\)](#) and [7\(C\)](#), the slopes are getting less steep and therefore are getting smaller. Thus, in those cases the speed is decreasing and the object is slowing down.

Notice that [Figure 7\(C\)](#) depicts a situation where the velocity is increasing but the object is slowing down.

Suppose s is the distance between a car and a wall during a crash test, and assume that during the test the car continued to speed up until it hit the wall. Which of the four graphs above best represents $s(t)$? This question, and others like it, are addressed in Exercises [15](#)–[16](#).

EXAMPLE 6

Describe the motion and velocities of a shuttle train that runs on a straight track at the airport, ferrying passengers from Terminal 1 to Terminal 2 according to the graph given in [Figure 8](#). Assume that s represents the distance from Terminal 1 in meters, t represents time in minutes, and the terminals are 800 m apart.



Solution

Note that the graph has portions resembling each of the four graphs in [Example 5](#).

Analyzing the motion:

- The train starts at rest, but then speeds up with increasing positive velocity for the first 2 min.
- Over the interval $[2, 4]$, the velocity remains positive, but begins decreasing as the graph becomes less steep. The train is slowing down as it approaches Terminal 2.
- In the interval $[4, 6]$, the graph is flat with slope 0. In this interval the train is stopped at Terminal 2.
- The train speeds up again at $t = 6$, now with negative velocity since the distance to Terminal 1 is decreasing. Furthermore, since the graph has a negative slope and is getting steeper, the velocity is decreasing and getting larger, indicating that the train is speeding up.
- Over the interval $[8, 10]$, the velocity remains negative, but gets smaller as the graph become less steep. The train is slowing down as it approaches and arrives back at Terminal 1.

Motion Under the Influence of Gravity

Galileo discovered that the height $s(t)$ and velocity $v(t)$ at time t (seconds) of an object tossed vertically in the air near the earth's surface are accurately represented by the formulas

$$s(t) = s_0 + v_0 t - \frac{1}{2}gt^2, \quad v(t) = \frac{ds}{dt} = v_0 - \underline{gt} \quad 2$$

Galileo's formulas are valid only when air resistance is negligible. We assume this to be the case in all examples.

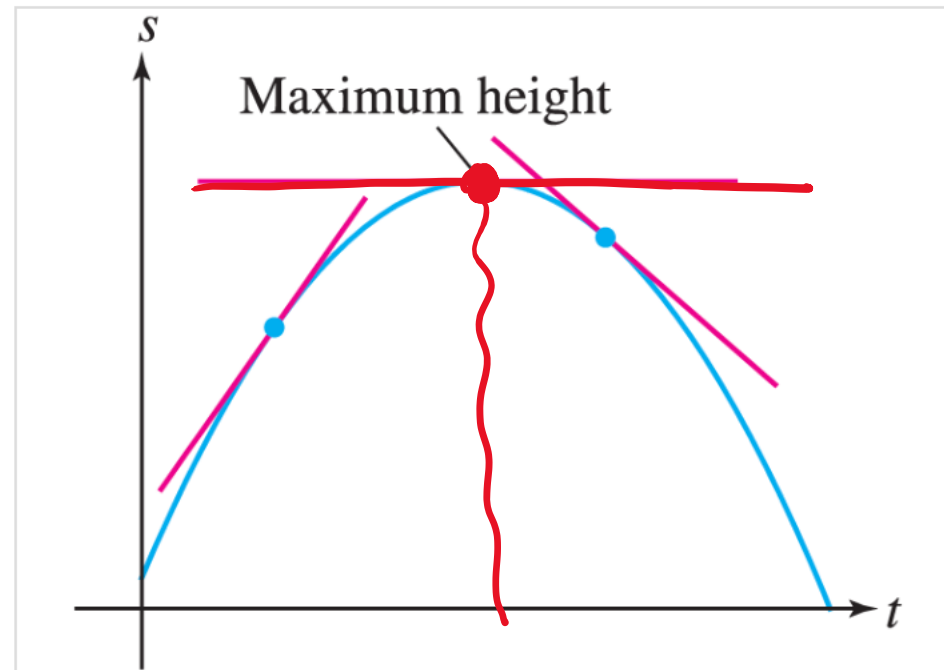
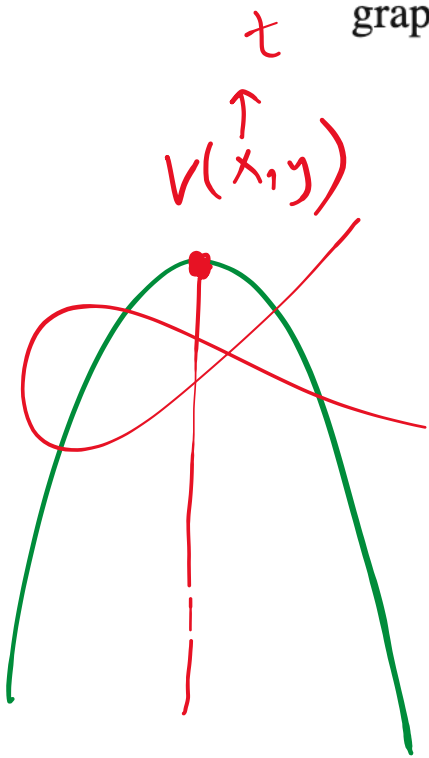
The constants s_0 and v_0 are the *initial values*:

- $s_0 = s(0)$, the position at time $t = 0$.
- $v_0 = v(0)$, the velocity at $t = 0$.
- $-g$ is the acceleration due to gravity on the surface of the earth (negative because the up direction is positive), where

$$g \approx 9.8 \text{ m/s}^2 \quad \text{or} \quad g \approx 32 \text{ ft/s}^2$$

$$-\frac{1}{2}gt^2$$
$$-\cancel{\frac{1}{2}}g \cdot \cancel{2}t = -gt$$

A simple observation enables us to find the object's maximum height. Since velocity is positive as the object rises and negative as it falls back to Earth, the object reaches its maximum height at the moment of transition, when it is no longer rising and has not yet begun to fall. At that moment, its velocity is zero. In other words, *the maximum height is attained when $v(t) = 0$* . At this moment, the tangent line to the graph of s is horizontal ([Figure 9](#)).



$$v(t) = 0$$

$$t = ?$$

$$s(t) = \dots$$

EXAMPLE 7

Finding the Maximum Height

$$s_0 = 0$$

$$v_0 = 30$$

A projectile is launched upward from ground level with an initial velocity of 30 m/s

a. Find the velocity at $t = 2$ and at $t = 4$. Explain the change in sign.

b. What is the projectile's maximum height and when does it reach that height?

$$v(t) = s'(t) = 30 - 9.8t$$

$t=2 \Rightarrow 30 - 9.8(2)$
 $t=4 \Rightarrow 30 - 9.8(4)$

Galileo's formulas:

$$g = 9.8$$

$$s(t) = s_0 + v_0 t - \frac{1}{2} g t^2$$

$$v(t) = \frac{ds}{dt} = v_0 - gt$$

$$s(t) = 30t - 4.9t^2$$

Solution

Apply Eq. (2) with $s_0 = 0$, $v_0 = 30$, and $g = 9.8$:

$$s(t) = 30t - 4.9t^2, \quad v(t) = 30 - 9.8t$$

a. Therefore,

$$v(2) = 30 - 9.8(2) = 10.4 \text{ m/s}, \quad v(4) = 30 - 9.8(4) = -9.2 \text{ m/s}$$

At $t = 2$, the projectile is rising and its velocity $v(2)$ is positive (Figure 9). At $t = 4$, the projectile is on the way down and its velocity $v(4)$ is negative.

b. Maximum height is attained when the velocity is zero, so we solve

$$v(t) = 0 \Rightarrow 30 - 9.8t = 0 \Rightarrow t = \frac{150}{49} \approx 3.06$$

$\frac{30}{9.8} = \frac{9.8}{9.8} t$

The projectile reaches ~~maximum~~ height at $t = 150/49$ s. Its maximum height is

$$s(t) =$$
$$s(150/49) = 30(150/49) - 4.9(150/49)^2 \approx 45.92 \text{ m}$$

Max height

3.4 SUMMARY

- The (instantaneous) rate of change of $y = f(x)$ with respect to x at $x = x_0$ is defined as the derivative

$$f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{x_1 \rightarrow x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

- The rate dy/dx is measured in *units of y per unit of x* .
- Marginal cost is the cost of producing one additional unit. If $C(x)$ is the cost of producing x units, then the marginal cost at production level x_0 is $C(x_0 + 1) - C(x_0)$. The derivative $C'(x_0)$ is often a good estimate for marginal cost.
- For linear motion, velocity $v(t)$ is the rate of change of position $s(t)$ with respect to time—that is, $v(t) = s'(t)$.
- Galileo's formulas for an object rising or falling under the influence of gravity near Earth's surface ignoring air resistance (s_0 = initial position, v_0 = initial velocity):

$$s(t) = s_0 + v_0 t - \frac{1}{2}gt^2, \quad v(t) = v_0 - gt$$

where $g \approx 9.8 \text{ m/s}^2$, or $g \approx 32 \text{ ft/s}^2$. Maximum height is attained when $v(t) = 0$.

$$s(t)$$

$$v = s'(t)$$

$$\text{speed} = |v(t)|$$

$$a(t) = s''(t)$$

$$\text{max hgh } v(t) = 0$$

$$t \rightarrow s(t)$$