

3.3 Relation

A relation from one set to another can be thought of as a table that lists which elements of the first set relate to which elements of the second set (see Table 3.3.)

TABLE 3.3.1 ■ Relation of Students to Courses

<i>Student</i>	<i>Course</i>
Bill	CompSci
Mary	Math
Bill	Art
Beth	History
Beth	CompSci
Dave	Math

Definition 3.3.1 A (binary) relation R from a set X to a set Y is a subset of the Cartesian product $X \times Y$. If $(x, y) \in R$, we write xRy and say that x is related to y . If $X = Y$, we call R a (binary) relation on X .

Example 3.3.2 If we let $X = \{\text{Bill, Mary, Beth, Dave}\}$ and $Y = \{\text{CompSci, Math, Art, History}\}$, our relation R of Table 3.3.1 can be written $R = \{(\text{Bill, CompSci}), (\text{Mary, Math}), (\text{Bill, Art}), (\text{Beth, History}), (\text{Beth, CompSci}), (\text{Dave, Math})\}$. Since $(\text{Beth, History}) \in R$, we may write

Beth R History.

Example 3.3.3 Let $X = \{2, 3, 4\}$ and $Y = \{3, 4, 5, 6, 7\}$. If we define a relation R from X to Y by $(x, y) \in R$ if x divides y , we obtain $R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$. If we rewrite R as a table, we obtain

X	Y
2	4
2	6
3	3
3	6
4	4

Example 3.3.4 Let R be the relation on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y$, $x, y \in X$. Then $R = \{(1, 1), (1, 2), (1, 3),$

$(1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4),$
 $(4, 4)\}.$

An informative way to picture a relation on a set is to draw its **digraph**.

To draw the digraph of a relation on a set X , we first draw dots or vertices to represent the elements of X .

Next, if the element (x, y) is in the relation, we draw an arrow (called a **directed edge**) from x to y .

Notice that an element of the form (x, x) in a relation corresponds to a directed edge from x to x . Such an edge is called a **loop**.

There is a loop at every vertex in Figure 3.3.1.

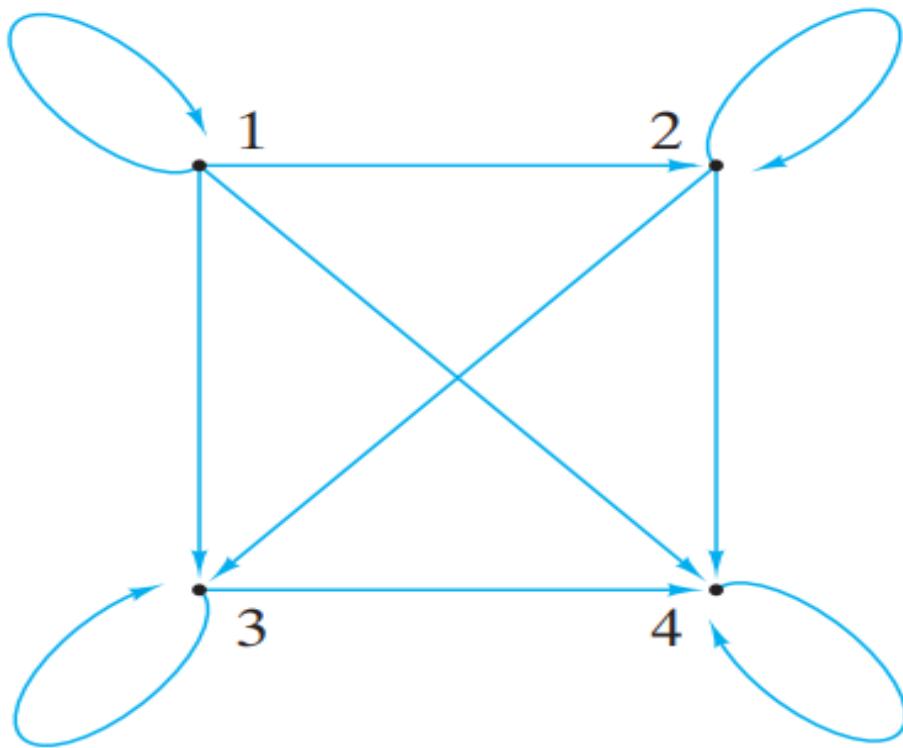


Figure 3.3.1 The digraph of the relation of Example 3.3.4.

Example 3.3.5 The relation R on $X = \{a, b, c, d\}$ given by the digraph of Figure 3.3.2 is $R = \{(a, a), (b, c), (c, b), (d, d)\}$.

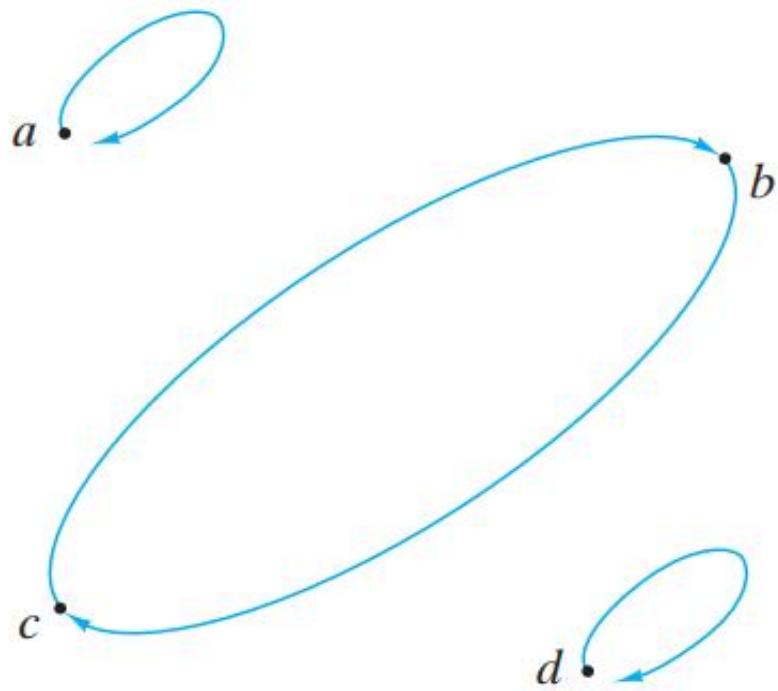


Figure 3.3.2 The digraph of the relation of Example 3.3.5.

Several properties that relations may have

Definition 3.3.6 A relation R on a set X is **reflexive** if $(x, x) \in R$ for every $x \in X$.

Example 3.3.7 The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y$, $x, y \in X$, is reflexive because for each element $x \in X$, $(x, x) \in R$; specifically, $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$ are each in R .

The digraph of a reflexive relation has a loop at every vertex. Notice that the digraph of this relation (see Figure 3.3.1) has a loop at every vertex.

a relation R on X is not reflexive if there exists $x \in X$ such that $(x, x) \notin R$.

Example 3.3.8 The relation $R = \{(a, a), (b, c), (c, b), (d, d)\}$ on $X = \{a, b, c, d\}$ is not reflexive. For example, $b \in X$, but

$(b,b) \notin R$. That this relation is not reflexive can also be seen by looking at its digraph (see Figure 3.3.2); **vertex b does not have a loop.**

Definition 3.3.9 A relation R on a set X is symmetric if for all $x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R$.

Example 3.3.10 The relation $R = \{(a, a), (b, c), (c, b), (d, d)\}$ on $X = \{a, b, c, d\}$ is **symmetric** because for all x, y , if $(x, y) \in R$, then $(y, x) \in R$, (see Figure 3.3.2).

Example 3.3.11 The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y$, $x, y \in X$, is not symmetric. For example, $(2, 3) \in R$, but $(3, 2) \notin R$. The digraph of this relation (see Figure 3.3.1) has a

directed edge from 2 to 3, but there is no directed edge from 3 to 2.

Definition 3.3.12 A relation R on a set X is **antisymmetric** if for all $x, y \in X$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.

Example 3.3.13 The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y$, $x, y \in X$, is antisymmetric because for all x, y , if $(x, y) \in R$ (i.e., $x \leq y$) and $(y, x) \in R$ (i.e., $y \leq x$), then $x = y$.

to obtain a logically equivalent characterization of “antisymmetric”:

A relation R on a set X is antisymmetric if for all $x, y \in X$, if $x \neq y$, then $(x, y) \notin R$ or $(y, x) \notin R$.

Example 3.3.15 If a relation has no members of the form (x, y) , $x \neq y$, we see that the equivalent characterization of “antisymmetric” for all $x, y \in X$, if $x \neq y$, then $(x, y) \notin R$ or $(y, x) \notin R$ (see Example 3.3.14) is trivially true (since the hypothesis $x \neq y$ is always false).

Thus if a relation R has no members of the form (x, y) , $x \neq y$, **R is antisymmetric**. For example, $R = \{(a, a), (b, b), (c, c)\}$ on $X = \{a, b, c\}$ is antisymmetric. The digraph of R shown in Figure 3.3.3 has at most one directed edge between each pair of distinct vertices.



Figure 3.3.3 The digraph of the relation of Example 3.3.15.

Notice that **R** is also **reflexive** and **symmetric**. This example shows that “**antisymmetric**” is not the same as “**not symmetric**” because this relation is in fact both symmetric and antisymmetric.

Example 3.3.16 The relation $R = \{(a, a), (b, c), (c, b), (d, d)\}$ on $X = \{a, b, c, d\}$ is not antisymmetric because both (b, c) and (c, b) are in R . Notice that in the digraph of this relation (see Figure 3.3.2) there are two directed edges between b and c .

Definition 3.3.17 A relation R on a set X is **transitive** if for all $x, y, z \in X$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Example 3.3.18 The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y$, $x, y \in X$, is transitive because for all x, y, z , if (x, y) and $(y, z) \in R$, then $(x, z) \in R$. To formally verify that this relation satisfies Definition 3.3.17, we can list all pairs of the form (x, y) and (y, z) in R and then verify that in every case, $(x, z) \in R$:

Pairs of Form			Pairs of Form		
(x, y)	(y, z)	(x, z)	(x, y)	(y, z)	(x, z)
(1, 1)	(1, 1)	(1, 1)	(2, 2)	(2, 2)	(2, 2)
(1, 1)	(1, 2)	(1, 2)	(2, 2)	(2, 3)	(2, 3)
(1, 1)	(1, 3)	(1, 3)	(2, 2)	(2, 4)	(2, 4)
(1, 1)	(1, 4)	(1, 4)	(2, 3)	(3, 3)	(2, 3)
(1, 2)	(2, 2)	(1, 2)	(2, 3)	(3, 4)	(2, 4)
(1, 2)	(2, 3)	(1, 3)	(2, 4)	(4, 4)	(2, 4)
(1, 2)	(2, 4)	(1, 4)	(3, 3)	(3, 3)	(3, 3)
(1, 3)	(3, 3)	(1, 3)	(3, 3)	(3, 4)	(3, 4)
(1, 3)	(3, 4)	(1, 4)	(3, 4)	(4, 4)	(3, 4)
(1, 4)	(4, 4)	(1, 4)	(4, 4)	(4, 4)	(4, 4)

Actually, some of the entries in the preceding table were unnecessary. If $x = y$ or $y = z$, we need not explicitly verify that the condition if (x, y) and $(y, z) \in R$, then $(x, z) \in R$ is satisfied since it will automatically be true.

Suppose, for example, that $x = y$ and (x, y) and (y, z) are in R . Since $x = y$, (x, z)

$=$ (y,z) is in R and the condition is satisfied. Eliminating the cases $x = y$ and $y = z$ leaves only the following to be explicitly checked to verify that the relation is transitive:

<i>Pairs of Form</i>		
(x, y)	(y, z)	(x, z)
$(1, 2)$	$(2, 3)$	$(1, 3)$
$(1, 2)$	$(2, 4)$	$(1, 4)$
$(1, 3)$	$(3, 4)$	$(1, 4)$
$(2, 3)$	$(3, 4)$	$(2, 4)$

The digraph of a transitive relation has the property that whenever there are directed edges from x to y and from y to z , there is also a directed edge from x to z . Notice that the digraph of this relation (see Figure 3.3.1) has this property.

A relation R is **not transitive** if there exist x , y , and z such that (x, y) and (y, z) are in R , but (x, z) is not in R .

Example 3.3.19 The relation $R = \{(a, a), (b, c), (c, b), (d, d)\}$ on $X = \{a, b, c, d\}$ is not transitive. For example, (b, c) and (c, b) are in R , but (b, b) is not in R .

Notice that in the digraph of this relation (see Figure 3.3.2) there are directed edges from b to c and from c to b , but there is no directed edge from b to b .

Definition 3.3.20 A relation R on a set X is a **partial order** if R is **reflexive**, **antisymmetric**, and **transitive**.

Example 3.3.21 Since the relation R defined on the positive integers by $(x, y) \in R$ if x divides y is reflexive, antisymmetric, and transitive, R is a partial order.