

3.3 Relation

A relation from one set to another can be thought of as a table that lists which elements of the first set relate to which elements of the second set (see Table 3.3.)

TABLE 3.3.1 ■ Relation of Students to Courses

<i>Student</i>	<i>Course</i>
Bill	CompSci
Mary	Math
Bill	Art
Beth	History
Beth	CompSci
Dave	Math

Definition 3.3.1 A (binary) relation R from a set X to a set Y is a subset of the Cartesian product $X \times Y$. If $(x, y) \in R$, we write xRy and say that x is related to y . If $X = Y$, we call R a (binary) relation on X .

Example 3.3.2 If we let $X = \{\text{Bill, Mary, Beth, Dave}\}$ and $Y = \{\text{CompSci, Math, Art, History}\}$, our relation R of Table 3.3.1 can be written $R = \{(\text{Bill, CompSci}), (\text{Mary, Math}), (\text{Bill, Art}), (\text{Beth, History}), (\text{Beth, CompSci}), (\text{Dave, Math})\}$. Since $(\text{Beth, History}) \in R$, we may write

Beth R History.

Example 3.3.3 Let $X = \{2, 3, 4\}$ and $Y = \{3, 4, 5, 6, 7\}$. If we define a relation R from X to Y by $(x, y) \in R$ if x divides y , we obtain $R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$. If we rewrite R as a table, we obtain

X	Y
2	4
2	6
3	3
3	6
4	4

Example 3.3.4 Let R be the relation on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y$, $x, y \in X$. Then $R = \{(1, 1), (1, 2), (1, 3),$

$(1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$.

An informative way to picture a relation on a set is to draw its **digraph**.

To draw the digraph of a relation on a set X , we first draw dots or vertices to represent the elements of X .

Next, if the element (x, y) is in the relation, we draw an arrow (called a **directed edge**) from x to y .

Notice that an element of the form (x, x) in a relation corresponds to a directed edge from x to x . Such an edge is called a **loop**.

There is a loop at every vertex in Figure 3.3.1.

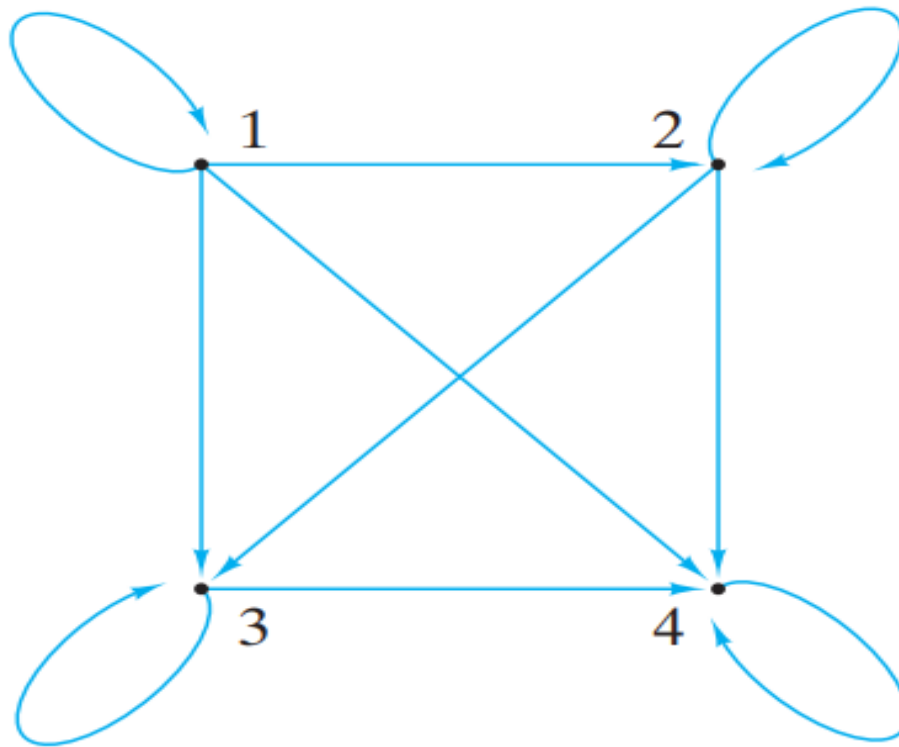


Figure 3.3.1 The digraph of the relation of Example 3.3.4.

Example 3.3.5 The relation R on $X = \{a, b, c, d\}$ given by the digraph of Figure 3.3.2 is $R = \{(a, a), (b, c), (c, b), (d, d)\}$.

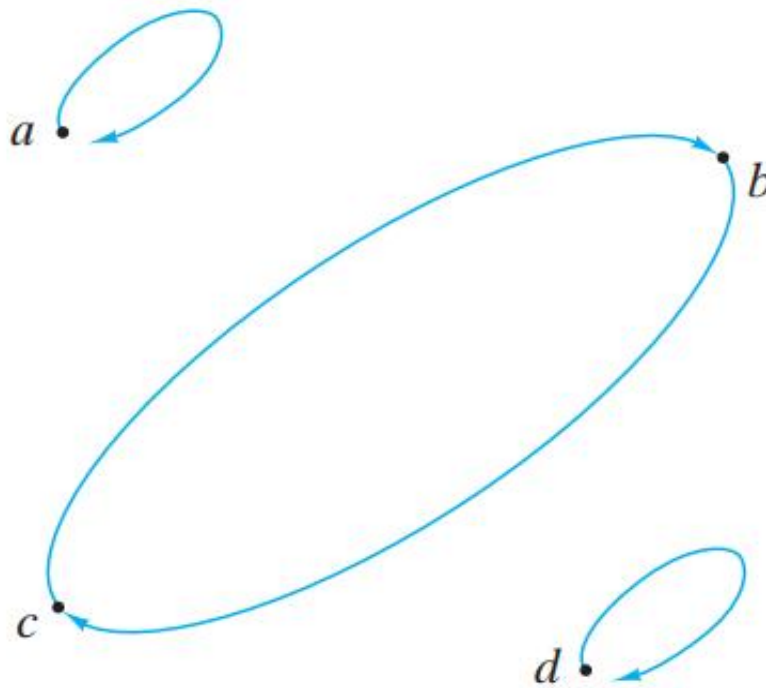


Figure 3.3.2 The digraph of the relation of Example 3.3.5.

Several properties that relations may have

Definition 3.3.6 A relation R on a set X is **reflexive** if $(x, x) \in R$ for every $x \in X$.

Example 3.3.7 The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y$, $x, y \in X$, is reflexive because for each element $x \in X$, $(x, x) \in R$; specifically, $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$ are each in R .

The digraph of a reflexive relation has a loop at every vertex. Notice that the digraph of this relation (see Figure 3.3.1) has a loop at every vertex.

a relation R on X is not reflexive if there exists $x \in X$ such that $(x, x) \notin R$.

Example 3.3.8 The relation $R = \{(a, a), (b, c), (c, b), (d, d)\}$ on $X = \{a, b, c, d\}$ is not reflexive. For example, $b \in X$, but

$(b,b) \notin R$. That this relation is not reflexive can also be seen by looking at its digraph (see Figure 3.3.2); **vertex b does not have a loop.**

Definition 3.3.9 A relation R on a set X is symmetric if for all $x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R$.

Example 3.3.10 The relation $R = \{(a, a), (b, c), (c, b), (d, d)\}$ on $X = \{a, b, c, d\}$ is **symmetric** because for all x, y , if $(x, y) \in R$, then $(y, x) \in R$, (see Figure 3.3.2).

Example 3.3.11 The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y$, $x, y \in X$, is not symmetric. For example, $(2, 3) \in R$, but $(3, 2) \notin R$. The digraph of this relation (see Figure 3.3.1) has a

directed edge from 2 to 3, but there is no directed edge from 3 to 2.

Definition 3.3.12 A relation R on a set X is **antisymmetric** if for all $x, y \in X$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.

Example 3.3.13 The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y$, $x, y \in X$, is antisymmetric because for all x, y , if $(x, y) \in R$ (i.e., $x \leq y$) and $(y, x) \in R$ (i.e., $y \leq x$), then $x = y$.

to obtain a logically equivalent characterization of “antisymmetric”:

A relation R on a set X is antisymmetric if for all $x, y \in X$, if $x \neq y$, then $(x, y) \notin R$ or $(y, x) \notin R$.

Example 3.3.15 If a relation has no members of the form (x, y) , $x \neq y$, we see that the equivalent characterization of “antisymmetric” for all $x, y \in X$, if $x \neq y$, then $(x, y) \notin R$ or $(y, x) \notin R$ (see Example 3.3.14) is trivially true (since the hypothesis $x \neq y$ is always false).

Thus if a relation R has no members of the form (x, y) , $x \neq y$, **R is antisymmetric**. For example, $R = \{(a, a), (b, b), (c, c)\}$ on $X = \{a, b, c\}$ is antisymmetric. The digraph of R shown in Figure 3.3.3 has at most one directed edge between each pair of distinct vertices.



Figure 3.3.3 The digraph of the relation of Example 3.3.15.

Notice that **R** is also **reflexive** and **symmetric**. This example shows that “**antisymmetric**” is not the same as “**not symmetric**” because this relation is in fact both symmetric and antisymmetric.

Example 3.3.16 The relation $R = \{(a, a), (b, c), (c, b), (d, d)\}$ on $X = \{a, b, c, d\}$ is not antisymmetric because both (b, c) and (c, b) are in R . Notice that in the digraph of this relation (see Figure 3.3.2) there are two directed edges between b and c .

Definition 3.3.17 A relation R on a set X is **transitive** if for all $x, y, z \in X$, if (x, y) and $(y, z) \in R$, then $(x, z) \in R$.

Example 3.3.18 The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y$, $x, y \in X$, is transitive because for all x, y, z , if (x, y) and $(y, z) \in R$, then $(x, z) \in R$. To formally verify that this relation satisfies Definition 3.3.17, we can list all pairs of the form (x, y) and (y, z) in R and then verify that in every case, $(x, z) \in R$:

<i>Pairs of Form</i>			<i>Pairs of Form</i>		
(x, y)	(y, z)	(x, z)	(x, y)	(y, z)	(x, z)
(1, 1)	(1, 1)	(1, 1)	(2, 2)	(2, 2)	(2, 2)
(1, 1)	(1, 2)	(1, 2)	(2, 2)	(2, 3)	(2, 3)
(1, 1)	(1, 3)	(1, 3)	(2, 2)	(2, 4)	(2, 4)
(1, 1)	(1, 4)	(1, 4)	(2, 3)	(3, 3)	(2, 3)
(1, 2)	(2, 2)	(1, 2)	(2, 3)	(3, 4)	(2, 4)
(1, 2)	(2, 3)	(1, 3)	(2, 4)	(4, 4)	(2, 4)
(1, 2)	(2, 4)	(1, 4)	(3, 3)	(3, 3)	(3, 3)
(1, 3)	(3, 3)	(1, 3)	(3, 3)	(3, 4)	(3, 4)
(1, 3)	(3, 4)	(1, 4)	(3, 4)	(4, 4)	(3, 4)
(1, 4)	(4, 4)	(1, 4)	(4, 4)	(4, 4)	(4, 4)

Actually, some of the entries in the preceding table were unnecessary. If $x = y$ or $y = z$, we need not explicitly verify that the condition if (x, y) and $(y, z) \in R$, then $(x, z) \in R$ is satisfied since it will automatically be true.

Suppose, for example, that $x = y$ and (x, y) and (y, z) are in R . Since $x = y$, (x, z)

$= (y,z)$ is in R and the condition is satisfied. Eliminating the cases $x = y$ and $y = z$ leaves only the following to be explicitly checked to verify that the relation is transitive:

<i>Pairs of Form</i>		
(x, y)	(y, z)	(x, z)
(1, 2)	(2, 3)	(1, 3)
(1, 2)	(2, 4)	(1, 4)
(1, 3)	(3, 4)	(1, 4)
(2, 3)	(3, 4)	(2, 4)

The digraph of a transitive relation has the property that whenever there are directed edges from x to y and from y to z , there is also a directed edge from x to z . Notice that the digraph of this relation (see Figure 3.3.1) has this property.

A relation R **is not transitive** if there exist x , y , and z such that (x, y) and (y, z) are in R , but (x, z) is not in R .

Example 3.3.19 The relation $R = \{(a, a), (b, c), (c, b), (d, d)\}$ on $X = \{a, b, c, d\}$ is not transitive. For example, (b, c) and (c, b) are in R , but (b, b) is not in R . Notice that in the digraph of this relation (see Figure 3.3.2) there are directed edges from b to c and from c to b , but there is no directed edge from b to b .

Definition 3.3.20 A relation R on a set X is a **partial order** if R is **reflexive**, **antisymmetric**, and **transitive**.

Example 3.3.21 Since the relation R defined on the positive integers by $(x, y) \in R$ if x divides y is reflexive, antisymmetric, and transitive, R is a partial order.