2.3 Basic Limit Laws

In Section 2.2, we relied on graphical and numerical approaches to investigate limits and estimate their values. In the next four sections, we go beyond this intuitive approach and develop tools for computing limits in a precise way. The next theorem provides our first set of tools.

THEOREM 1

Basic Limit Laws

If $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ exist, then

i. **Sum Law:** $\lim\limits_{x\to c} \left(f(x)+g(x)\right)$ exists and

$$\lim_{x\to c} \left(f(x)+g(x)\right) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$

ii. **Constant Multiple Law:** For any number k, $\lim\limits_{x\to c} kf(x)$ exists and

$$\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x)$$

iii. **Product Law:** $\lim\limits_{x\to c} f(x)\,g(x)$ exists and

$$\lim_{x\to c} f(x)\,g(x) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$$

iv. **Quotient Law:** If $\lim\limits_{x\to c} g(x) \neq 0$, then $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)}$ exists and

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}$$

v. **Powers and Roots:** If n is a positive integer, then

$$\lim_{x\to c} \left[f(x)\right]^n = \left(\lim_{x\to c} f(x)\right)^n, \qquad \lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$$

In the second limit, assume that $\lim_{x \to c} f(x) \geq 0$ if n is even.

If $p,\ q$ are integers with $q \neq 0,$ then $\lim_{x \to c} [f(x)]^{p/q}$ exists and

$$\lim_{x \to c} [f(x)]^{p/q} = \left(\lim_{x \to c} f(x)\right)^{p/q}$$

Assume that $\lim_{x \to c} f(x) \geq 0$ if q is even, and that $\lim_{x \to c} f(x) \neq 0$ if $p/q < 0.$

Before proceeding to the examples, we make some useful remarks.

- The Sum and Product Laws are valid for any number of functions. For example,
$$\lim_{x\to c}\left(f_1(x)+f_2(x)+f_3(x)\right)=\lim_{x\to c}f_1(x)+\lim_{x\to c}f_2(x)+\lim_{x\to c}f_3(x)$$
- The Sum Law has a counterpart for differences:
$$\lim_{x\to c}\left(f(x)-g(x)\right)=\lim_{x\to c}f(x)-\lim_{x\to c}g(x)$$

  This follows from the Sum and Constant Multiple Laws (with $k=-1$):

$$\lim_{x\to c}\left(f(x)-g(x)\right)=\lim_{x\to c}f(x)+\lim_{x\to c}\left(-g(x)\right)=\lim_{x\to c}f(x)-\lim_{x\to c}g(x)$$

- Recall two basic limits from Theorem 1 in Section 2.2:
$$\lim_{x\to c}k=k,\qquad \lim_{x\to c}x=c$$

Applying Law (v) to $f(x) = x$, we obtain

$$\lim_{x \to c} x^{p/q} = c^{p/q} \tag{1}$$

for integers p and q such that $q \neq 0$. Note, in Eq. (1) we need to assume that $c \geq 0$ if q is even and that $c \neq 0$ if $p/q < 0$.

EXAMPLE 1

Use the Basic Limit Laws to evaluate:

a. $\lim\limits_{x \to 2} x^3$

b. $\lim\limits_{x \to 2} \left(x^3 + 5x + 7\right)$

c. $\lim\limits_{x \to 2} \sqrt{x^3 + 5x + 7}$

Solution

a. By Eq. (1), $\lim\limits_{x\to 2} x^3 = 2^3 = 8.$

b.
$$\begin{aligned}\lim_{x\to 2}\left(x^3+5x+7\right) &= \lim_{x\to 2} x^3 + \lim_{x\to 2} 5x + \lim_{x\to 2} 7 && \text{(Sum Law)}\\ &= \lim_{x\to 2} x^3 + 5\lim_{x\to 2} x + \lim_{x\to 2} 7 && \text{(Constant Mult}\\ &= 8 + 5\,(2) + 7 = 25\end{aligned}$$

c. By Law (v) for roots and (b),

$$\lim_{x\to 2}\sqrt{x^3+5x+7} = \sqrt{\lim_{x\to 2}\left(x^3+5x+7\right)} = \sqrt{25} = 5$$

EXAMPLE 2

Evaluate

a. $\lim\limits_{t \to -1} \dfrac{t+6}{2t^4}$ and

b. $\lim\limits_{t \to 3} t^{-1/4}(t+5)^{1/3}$.

Solution

a. Use the Quotient, Sum, and Constant Multiple Laws:

$$\lim_{t\to -1} \frac{t+6}{2t^4} = \frac{\lim_{t\to -1}(t+6)}{\lim_{t\to -1} 2t^4} = \frac{\lim_{t\to -1} t + \lim_{t\to -1} 6}{2\lim_{t\to -1} t^4} = \frac{-1+6}{2(-1)^4} = \frac{5}{2}$$

b. Use the Product, Powers, and Sum Laws:

$$\begin{aligned}\lim_{t\to 3} t^{-1/4}(t+5)^{1/3} &= \left(\lim_{t\to 3} t^{-1/4}\right)\left(\lim_{t\to 3}\sqrt[3]{t+5}\right) = \left(3^{-1/4}\right)\left(\sqrt[3]{\lim_{t\to 3} t+5}\right)\\ &= 3^{-1/4}\sqrt[3]{3+5} = 3^{-1/4}(2) = \frac{2}{3^{1/4}}\end{aligned}$$

EXAMPLE 3

Assumptions Matter Show that the Product Law cannot be applied to $\lim_{x \to 0} f(x)\, g(x)$ if $f(x) = x$ and $g(x) = x^{-1}$.

Solution

For all $x \neq 0$, we have $f(x)\, g(x) = x.\, x^{-1} = 1$, so the limit of the product exists:

$$\lim_{x \to 0} f(x)\, g(x) = \lim_{x \to 0} 1 \ = 1$$

However, there is an issue with the product of the limits because $\lim_{x \to 0} x^{-1}$ does not exist (since $g(x) = x^{-1}$ becomes infinite as $x \to 0$). Therefore, the Product Law cannot be applied and its conclusion does not hold even though the limit of the products does exist. Specifically, $\lim_{x \to 0} f(x)\, g(x) = 1$, but the product of the limits is not defined:

$$1 \neq \left(\lim_{x \to 0} f(x)\right) \left(\lim_{x \to 0} g(x)\right) = \left(\lim_{x \to 0} x\right) \underbrace{\left(\lim_{x \to 0} x^{-1}\right)}_{\text{Does not exist}}$$

2.3 SUMMARY

- The Basic Limit Laws: If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ both exist, then
  i. $\lim_{x \to c}\left(f(x) + g(x)\right) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$
  ii. $\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x)$
  iii. $\lim_{x \to c} f(x)\ \ g(x) = \left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right)$
  iv. If $\lim_{x \to c} g(x) \neq 0,$ then $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$
  v. If $p,\ q$ are integers with $q \neq 0,$

$$\lim_{x \to c} [f(x)]^{p/q} = \left(\lim_{x \to c} f(x)\right)^{p/q}$$

  For n a positive integer,

$$\lim_{x \to c} [f(x)]^{n} = \left(\lim_{x \to c} f(x)\right)^{n}, \qquad \lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$$

- If $\lim_{x \to c} f(x)$ or $\lim_{x \to c} g(x)$ does not exist, then the Basic Limit Laws cannot be applied.

$$\lim_{x \to 512} \left(3x^{2/3} - 9x^{-1}\right) =$$

Use the Sum Law, the Constant Multiple Law, and the Powers Law.

$$\begin{aligned}
\lim_{x \to 512} \left(3x^{2/3} - 9x^{-1}\right) &= \lim_{x \to 512} \left(3x^{2/3}\right) - \lim_{x \to 512} \left(9x^{-1}\right) \\
&= 3 \lim_{x \to 512} x^{2/3} - 9 \lim_{x \to 512} x^{-1} \\
&= 3\left(\lim_{x \to 512} x\right)^{2/3} - 9\left(\lim_{x \to 512} x\right)^{-1} \\
&= 3 \cdot 512^{2/3} - 9 \cdot 512^{-1} \\
&= \frac{98295}{512}
\end{aligned}$$

$$\lim_{x \to \frac{1}{16}} (32x + 1)\left(4x^{1/2} + 6\right) =$$

Using the Product Law, the Sum Law, the Constant Multiple Law, and the Powers Law.

$$\begin{aligned}
\lim_{x \to \frac{1}{16}} (32x + 1)\left(4x^{1/2} + 6\right) &= \left(\lim_{x \to \frac{1}{16}} (32x + 1)\right)\left(\lim_{x \to \frac{1}{16}} \left(4x^{1/2} + 6\right)\right) \\
&= \left(32 \lim_{x \to \frac{1}{16}} x + \lim_{x \to \frac{1}{16}} 1\right)\left(4 \lim_{x \to \frac{1}{16}} x^{1/2} + \lim_{x \to \frac{1}{16}} 6\right) \\
&= \left(32 \cdot \frac{1}{16} + 1\right)\left(4 \cdot \frac{1}{4} + 6\right) \\
&= 21
\end{aligned}$$

$$\lim_{x \to 7} \frac{\sqrt{x+2}+1}{\sqrt{x+9}-1} =$$