3.6 Trigonometric Functions

We can use the rules developed so far to differentiate functions involving powers of x, but we cannot yet handle the trigonometric functions. What is missing are the formulas for the derivatives of $\sin x$ and $\cos x$. Fortunately, their derivatives are simple—each is the derivative of the other up to a sign.

THEOREM 1

Derivative of Sine and Cosine

The functions $y = \sin\ x$ and $y = \cos\ x$ are differentiable and

$$\frac{d}{dx}\ \sin\ x = \cos\ x \qquad \text{and} \qquad \frac{d}{dx}\ \cos\ x = -\sin\ x$$

EXAMPLE 1

For $f(x) = \sin\ x$, compute f' at $x = 0, \frac{\pi}{6}, \frac{\pi}{2}$, and $\frac{5\pi}{6}$.

Solution

We have $f'(x) = \cos\ x$. Thus, $f'(0) = \cos(0) = 1$, $f'\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, $f'\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$, and $f'\left(\frac{5\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right) = \frac{-\sqrt{3}}{2}$.

Ex: $f(x) = \overset{f}{x} \cdot \overset{g}{\cos x} \longrightarrow f''(x) = ?$ $\quad f'g + g'f$

$$f'(x) = (1)\cos x + (-\sin x)\cdot x = \cos x - \overset{f}{x} \cdot \overset{g}{\sin x}$$

$$f''(x) = -\sin x - (1 \cdot \sin x + x \cdot \cos x)$$

$$= -\sin x - \sin x - x\cos x$$

$$\boxed{f''(x) = -2\sin x - x \cdot \cos x}$$

EXAMPLE 2

Calculate $f''(x)$, where $f(x) = x \cos x$.

Solution

By the Product Rule,

$$\begin{aligned} f'(x) &= x' \cos x + x(\cos x)' = \cos x - x \sin x \\ f''(x) &= (\cos x - x \sin x)' = -\sin x - \left(x'(\sin x) + x(\sin x)'\right) \\ &= -2 \sin x - x \cos x \end{aligned}$$

THEOREM 2

Derivatives of Standard Trigonometric Functions

$$\frac{d}{dx}\tan x = \sec^2 x, \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\cot x = -\csc^2 x, \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

$$\frac{d\,\tan x}{dx} = \sec^2 x$$

$$\frac{f'g - g'f}{g^2}$$

$$\frac{1}{\cos^2 x} = \sec^2 x$$

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)'$$

$$= \frac{\cos x \cdot \cos x \overset{+}{-} (-\sin x)\sin x}{\cos^2 x}$$

$$= \frac{\overset{1}{\cancel{\cos^2 x + \sin^2 x}}}{\cos^2 x} = \frac{1}{\cos^2 x} = \boxed{\sec^2 x}$$

EXAMPLE 4

Derive the formula $\frac{d}{dx}\ \tan\ x = \sec^2\ x$

Solution

Use the Quotient Rule and the identity $\cos^2 x + \sin^2 x = 1$:

$$
\begin{aligned}
\frac{d}{dx}\tan x &= \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cdot (\sin x)' - \sin x \cdot (\cos x)'}{\cos^2 x} \\
&= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x
\end{aligned}
$$

EXAMPLE 5

Determine y' for $y = \tan\ \theta\ \sec\ \theta$, and find an equation of the tangent line to the graph at $\theta = \frac{\pi}{4}$.

Solution

By the Product Rule,

$$\begin{aligned} y' &= (\tan\ \theta)'\ \sec\ \theta + \tan\ \theta\ (\sec\ \theta)' = \sec^2\ \theta\ \sec\ \theta + \tan\ \theta\ (\sec\ \theta\ \tan\ \theta) \\ &= \sec^3\ \theta + \tan^2\ \theta\ \sec\ \theta \end{aligned}$$

Now use the values $\sec\ \frac{\pi}{4} = \sqrt{2}$ and $\tan\ \frac{\pi}{4} = 1$ to compute

$$\begin{aligned} y\left(\frac{\pi}{4}\right) &= \tan\left(\frac{\pi}{4}\right)\sec\left(\frac{\pi}{4}\right) = \sqrt{2} \\ y'\left(\frac{\pi}{4}\right) &= \sec^3\left(\frac{\pi}{4}\right) + \tan^2\left(\frac{\pi}{4}\right)\sec\left(\frac{\pi}{4}\right) = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2} \end{aligned}$$

3.6 SUMMARY

- The derivatives of the trigonometric functions:

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\cot x = -\csc^2 x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

Calculate the first five derivatives of $\boxed{f(x) = \sin(x)}$. Then determine $f^{(90)}(x)$ and $f^{(13)}(x)$.

(Express numbers in exact form. Use symbolic notation and fractions where needed.)

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{(4)}(x) = \sin x$

$f^{(5)}(x) = \cos x$

$f^{(90)}(x) = ?\ -\sin x$

$$\begin{array}{r} 22 \\ 4 \overline{)\,90} \\ 8 \\ \hline 10 \\ 8 \\ \hline 2 \end{array}$$

$2 \to \cos x$

$2 \to -\sin x$

$f^{(13)}(x) = \cos x$

$f^{(2025)}(x) = \cos x$

$f^{(100)}(x) = \sin x$

$f^{(101)}(x) = \cos x$

$$\begin{array}{r} 506 \\ 4 \overline{)\,2025} \\ 20 \\ \hline 25 \\ 24 \\ \hline 1 \end{array}$$

The height at time t (in seconds) of a mass, oscillating at the end of a spring, is $\boxed{s(t) = 300 + 36\sin(t)}$ cm.

Find the velocity and acceleration at $t = \frac{\pi}{3}$ s.

(Use symbolic notation and fractions where needed.)

$$v(t) = s'(t) = 36\cos(t)$$

$$v\left(\frac{\pi}{3}\right) = 36\cos\frac{\pi}{3} = 36\left(\frac{1}{2}\right) = \boxed{18}$$

$$a(t) = s''(t) = v'(t) = -36\sin t$$

$$a\left(\frac{\pi}{3}\right) = -36\sin\frac{\pi}{3} = -36\cdot\frac{\sqrt{3}}{2} = \boxed{-18\sqrt{3}}$$

Use the Product and Quotient Rules as necessary to find the derivative of

$$f(x) = \overbrace{4x^2}^{f}\ \overbrace{\cos(x)}^{g}$$

$f'g + g'f$

(Use symbolic notation and fractions where needed.)

$$f'(x) = 8x \cdot \cos x + (-\sin x) \cdot 4x^2$$

$$\boxed{= 8x\cos x - 4x^2 \sin x}$$