3.7 The Chain Rule

The Chain Rule is used to differentiate composite functions such as $y = \cos\left(x^3\right)$ and $y = \sqrt{x^4 + 1}$.

Recall that a *composite function* is obtained by evaluating one function at the output of another. The composite of f with g, denoted $f \circ g$, is defined by

$$(f \circ g)(x) = f(g(x))$$

For convenience, we call f the *outside* function and g the *inside* function. Often, we write the composite function as $f(u)$, where $u = g(x)$. For example, $y = \cos\left(x^3\right)$ is the function $y = \cos\ u$, where $u = x^3$.

$$f \quad \& \quad g$$

$$(f+g)' = f' + g'$$

$$(f-g)' = f' - g'$$

$$(f \cdot g)' = f'g + g'f$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

$$(f \circ g)'_{(x)} = \Big(f(g(x))\Big)'$$

$$= \boxed{f'(g(x)) \cdot g'(x)}$$

Chain Rule

THEOREM 1

Chain Rule

If f and g are differentiable, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable and

$$\boxed{(f(g(x)))' = f'(g(x))\, g'(x)}$$

We will prove the Chain Rule at the end of the section.

The Chain Rule says

$$(f(g(x)))' = outside'(inside) \cdot inside'$$

Verbally, it is " the derivative of the outside function at the inside function times the derivative of the inside function."

EXAMPLE 1

$y = \cos x^3$

$= -\sin x^3 \cdot 3x^2$ ✓

Calculate the derivative of $y = \cos\left(x^3\right)$.

u

$y = \cos u \rightarrow y' = -\sin u \cdot u'$

$u = x^3$

$3x^2$

$y' = -\sin x^3 \cdot 3x^2$ ✓

$\boxed{y' = -3x^2 \sin x^3}$ ✓

Solution

As noted above, $y = \cos\left(x^3\right)$ is a composite $f\left(g\left(x\right)\right)$, where

$$\begin{aligned} f\left(u\right) &= \cos\ u, & u = g\left(x\right) &= x^3 \\ f'\left(u\right) &= -\sin\ u, & g'\left(x\right) &= 3x^2 \end{aligned}$$

Since $u = x^3$, $f'\left(g\left(x\right)\right) = f'\left(u\right) = f'\left(x^3\right) = -\sin\left(x^3\right)$. So, by the Chain Rule,

$$\frac{d}{dx}\cos\left(x^3\right) = \underbrace{-\sin\left(x^3\right)}_{f'(g(x))}\ \underbrace{\left(3x^2\right)}_{g'(x)} = -3x^2\ \sin\left(x^3\right).$$

EXAMPLE 2

$y = f'(u) \cdot u'$

Calculate the derivative of $y = \sqrt{x^4 + 1}$.

u

Solution

$y = \sqrt{u}$

$u = x^4 + 1$

$\rightarrow y' = \frac{1}{2\sqrt{u}} \cdot u'$

$y' = \frac{1}{2\sqrt{x^4+1}} \cdot 4x^3 = \frac{\cancel{4}^{2} x^3}{{}_1\cancel{2}\sqrt{x^4+1}} = \boxed{\frac{2x^3}{\sqrt{x^4+1}}}$

The function $y = \sqrt{x^4 + 1}$ is a composite $f(g(x))$, where

$$\begin{aligned} f(u) &= u^{1/2}, & u = g(x) &= x^4 + 1 \\ f'(u) &= \frac{1}{2} u^{-1/2}, & g'(x) &= 4x^3 \end{aligned}$$

Note that $f'(g(x)) = \frac{1}{2}(x^4 + 1)^{-1/2}$, so by the Chain Rule,

$$\frac{d}{dx}\sqrt{x^4 + 1} = \underbrace{\frac{1}{2}(x^4 + 1)^{-1/2}}_{f'(g(x))} \underbrace{(4x^3)}_{g'(x)} = \frac{4x^3}{2\sqrt{x^4 + 1}}$$

$y = e^u$

$y' = e^u \cdot u' \Rightarrow y' = e^{1+\sin x} \cdot (\cos x)$

EXAMPLE 3

Calculate the derivative of $y = e^{\overset{u}{1+\sin\ x}}$.

Solution

The function $y = e^{1+\sin\ x}$ is a composite $f(g(x))$, where

$$\begin{aligned} f(u) &= e^u, & u = g(x) &= 1 + \sin\ x \\ f'(u) &= e^u, & g'(x) &= \cos\ x \end{aligned}$$

Note that $f'(g(x)) = e^{1+\sin\ x}$, so by the Chain Rule,

$$\frac{d}{dx} e^{1+\sin\ x} = (\cos\ x) e^{1+\sin\ x}$$

It is instructive to write the Chain Rule in Leibniz notation. Let

$$y = f(u) = f(g(x))$$

Then, by the Chain Rule, $\dfrac{dy}{dx} = f'(u)\, g'(x) = \dfrac{df}{du}\,\dfrac{du}{dx}$, or

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}}$$

EXAMPLE 5

$$V = \frac{4}{3}\pi r^3 \longrightarrow \frac{dV}{dr} = \frac{4}{\cancel{3}}\pi \cdot \cancel{3} r^2$$

$$= 4\pi r^2$$

$$\frac{dr}{dt} = 3$$

A spherical balloon has a radius r that is increasing at a rate of 3 cm/s. At what rate is the volume V of the balloon increasing when $r = 10$ cm?

$$\frac{dV}{dt} = ? \qquad r = 10 \quad , \quad \frac{dr}{dt} = 3$$

Solution :

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

$$= 4\pi \underset{10}{r^2} \cdot \cancel{\frac{dr}{dt}}\ 3$$

$$= 4\pi (10)^2 \cdot 3 = 1200\pi \approx 3770 \ \text{cm}^3/\text{s}$$

Solution

Because we are asked to determine the rate at which V is increasing, we must find dV/dt. We are given that $dr/dt = 3$ cm/s. The Chain Rule allows us to express dV/dt in terms of dV/dr and dr/dt:

$$\underbrace{\frac{dV}{dt}}_{\substack{\text{Rate of change of volume}\\\text{with respect to time}}} = \underbrace{\frac{dV}{dr}}_{\substack{\text{Rate of change of volume}\\\text{with respect to radius}}} \times \underbrace{\frac{dr}{dt}}_{\substack{\text{Rate of change of radius}\\\text{with respect to time}}}$$

To compute dV/dr, we use the formula for the volume of a sphere, $V = \frac{4}{3}\pi r^3$:

$$\frac{dV}{dr} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$$

Because $dr/dt = 3$, we obtain

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 (3) = 12\pi r^2$$

For $r = 10$,

$$\left.\frac{dV}{dt}\right|_{r=10} = (12\pi)\,10^2 = 1200\pi \approx 3770$$

The volume of the balloon is increasing at a rate of approximately $3770\ \text{cm}^3/\text{s}$.

THEOREM 2

General Power and Exponential Rules

If g is differentiable, then

- $\dfrac{d}{dx}(g(x))^n = n(g(x))^{n-1}g'(x) \quad \text{(for any number } n\text{)}$
- $\dfrac{d}{dx}e^{g(x)} = e^{g(x)}g'(x)$

EXAMPLE 6

General Power and Exponential Rules

$y = u^{-\frac{1}{3}} \implies y' = -\frac{1}{3} u^{-\frac{4}{3}} \cdot u'$

Find the derivatives of

$u = x^2 + 7x + 2$

$$\boxed{y = -\frac{1}{3} (x^2+7x+2)^{-\frac{4}{3}} \cdot (2x+7)}$$

a. $y = (\underbrace{x^2 + 7x + 2}_{u})^{-1/3}$ and

b. $y = e^{\overset{u}{\cos t}}$.

$y = e^{u} \implies y' = e^{u} \cdot u'$

$u = \cos t$

$$\boxed{y' = e^{\cos t} \cdot (-\sin t)}$$

EXAMPLE 7

$y = \sqrt{u}$

$u = 1 + \sqrt{x^2+1}$

$u' = \frac{1}{\cancel{2}\sqrt{x^2+1}} \cdot \cancel{2}x = \frac{x}{\sqrt{x^2+1}}$

$y' = \frac{1}{2\sqrt{u}} \cdot u'$

$= \frac{1}{2\sqrt{1+\sqrt{x^2+1}}} \cdot \frac{x}{\sqrt{x^2+1}}$

Using the Chain Rule Twice

Calculate $\frac{d}{dx}\sqrt{\underbrace{1+\sqrt{x^2+1}}_{u}}.$

Solution

In the computation that follows, we apply the Chain Rule, first to the square root of the inside function $u = 1 + \sqrt{x^2+1}$ and then to the derivative of the inside function:

$$\begin{aligned}
\left(1+(x^2+1)^{1/2}\right)^{1/2} &= \frac{1}{2}\left(1+(x^2+1)^{1/2}\right)^{-1/2}\frac{d}{dx}\left(1+(x^2+1)^{1/2}\right) \\
&= \frac{1}{2}\left(1+(x^2+1)^{1/2}\right)^{-1/2}\left(\frac{1}{2}(x^2+1)^{-1/2}(2x)\right) \\
&= \frac{1}{2}x(x^2+1)^{-1/2}\left(1+(x^2+1)^{1/2}\right)^{-1/2}
\end{aligned}$$

$y = \sin(x^3 + 3x)$

$y' = \cos(x^3 + 3x) \cdot (3x^2 + 3)$

3.7 SUMMARY

- The Chain Rule expresses $(f \circ g)'$ in terms of f' and g':

$$(f(g(x)))' = f'(g(x))\ g'(x)$$

- In Leibniz notation: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$, where $y = f(u)$ and $u = g(x)$
- General Power Rule: $\frac{d}{dx}(g(x))^n = n(g(x))^{n-1} g'(x)$
- General Exponential Rule: $\frac{d}{dx}e^{g(x)} = e^{g(x)} g'(x)$

Consider the functions.

$$f(u) = u^9 + u$$
$$g(x) = \cos(x)$$

Find the following function and derivatives.

(Express numbers in exact form. Use symbolic notation and fractions where needed.)

$f(g(x)) =$

$f'(u) =$

$f'(g(x)) =$

$g'(x) =$

$(f \circ g)' =$

$f(g(x)) = \cos^9(x) + \cos(x)$

$f'(u) = 9u^8 + 1$

$f'(g(x)) = 9\cos^8(x) + 1$

$g'(x) = -\sin(x)$

Use the table of values to calculate the derivative of the function at the given point.

x	1	4	6
$f(x)$	4	0	10
$f'(x)$	5	7	8
$g(x)$	4	1	6
$g'(x)$	5	1	7

(Give your answer as a whole number.)

$$\frac{d}{dx} f(g(x))\Big|_{x=6} = \boxed{56}$$

$f(g(x)) =$

$f'(g(x)) \cdot g'(x)$

$f'(\cancel{g(6)}) \cdot g'(6)$

$f'(6)$

$= 8 \cdot 7$

$= 56$

Note:

$$\frac{d \sin X}{dX} = \boxed{\cos X}$$

$$(\underbrace{\sin x}_{u})^2 = 2 \sin x \cdot \cos x$$

$$\frac{d \sin x^2}{dX} = \cos x^2 \cdot (2x) = \boxed{2x \cdot \cos x^2}$$

$$\frac{d \sin^2 x}{dX} = \boxed{2 \sin x \cdot \cos x}$$

The power P in a circuit is $\boxed{P = Ri^2}$, where R is resistance and i is the current. Find $\frac{dP}{dt}$ at $t = \frac{1}{12}$ if $\boxed{R = 2000\ \Omega}$ and i varies according to $\boxed{i = \sin(4\pi t)}$ (time in seconds).

(Use symbolic notation and fractions where needed.)

$$P = Ri^2$$

$$P = 2000 \cdot \sin^2(4\pi t)$$

$$\left.\frac{dP}{dt}\right|_{t=\frac{1}{12}} = \boxed{}$$

$$\frac{dP}{dt} = 2000 \cdot 2 \sin(4\pi t) \cdot 4\pi$$

$$= 16000 \sin\left(\cancel{4}\pi \cdot \frac{1}{\cancel{12}_{3}}\right) \cdot \pi$$

$$= 16000\pi \sin\frac{\pi}{3} = 16000\pi \cdot \frac{\sqrt{3}}{2}$$

$$\boxed{= 8000\pi\sqrt{3}}$$