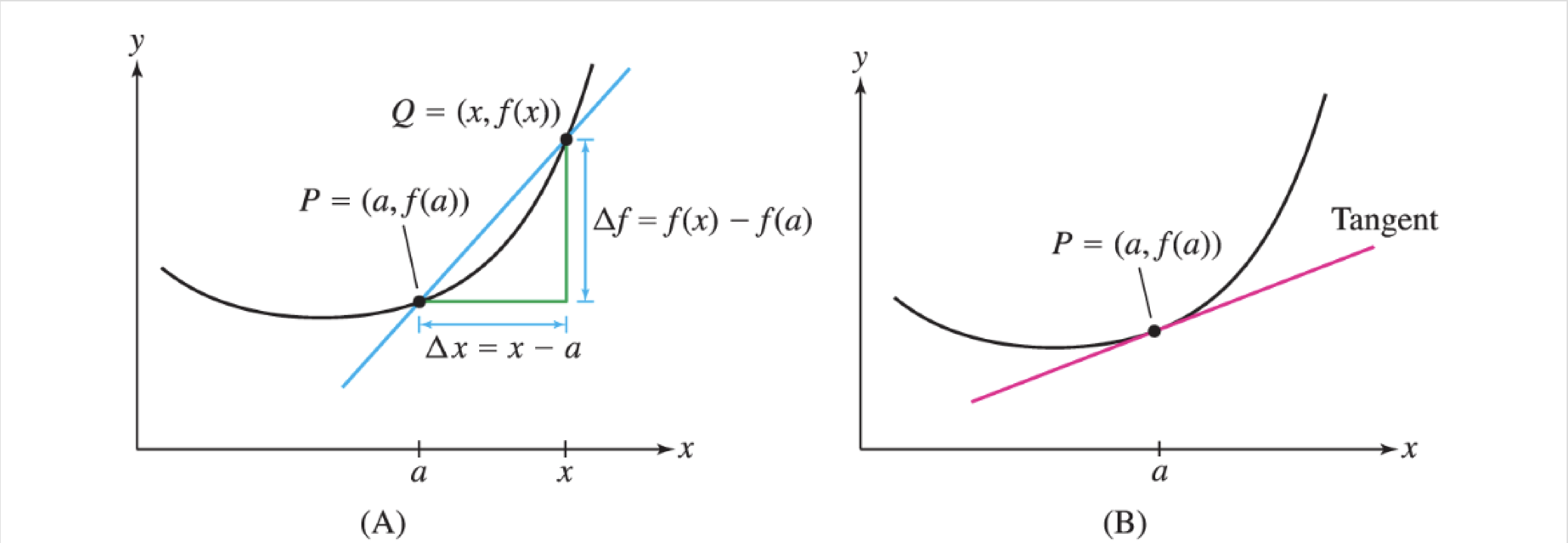
3.1 Definition of the Derivative

We begin with two questions: What is the precise definition of a tangent line? And how can we compute its slope? To answer these questions, let's return to the relationship between tangent and secant lines first mentioned in Section 2.1.

The secant line through distinct points $P = (a,\ f(a))$ and $Q = (x,\ f(x))$ on the graph of a function f has slope [Figure 1(A)]

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(a)}{x - a}$$

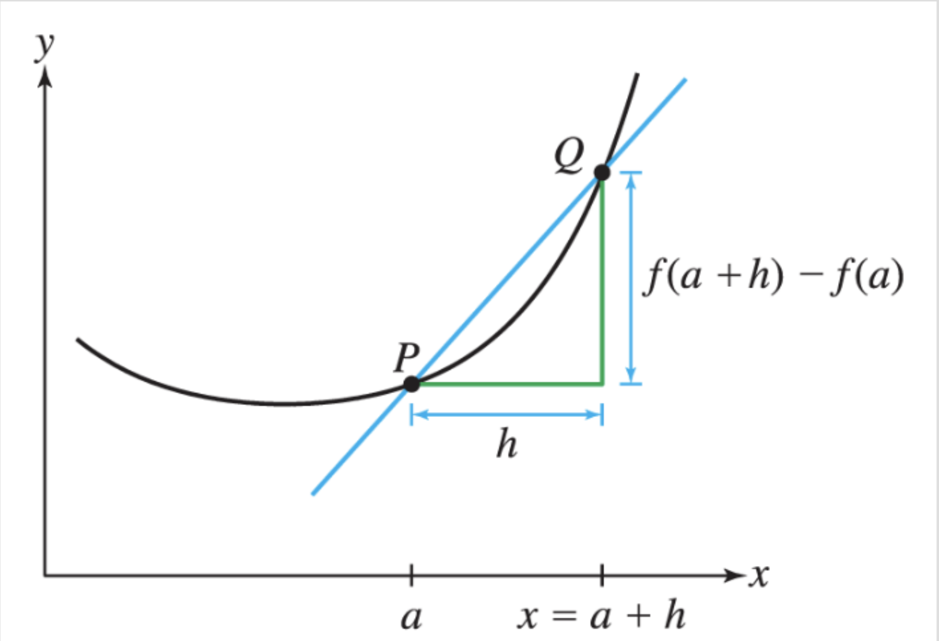
The expression $\frac{f(x)-f(a)}{x-a}$ is called the **difference quotient**. We can think of the secant line through P and Q as a rough approximation to the tangent line at P [Figure 1(B)].



$$\underbrace{f'(a)}_{\substack{\text{Slope of the}\\ \text{tangent line}}} = \underbrace{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}_{\text{Limit of slopes of secant lines}}$$

Another way of writing the difference quotient is to use the variable $h = x - a$ (Figure 3). We have $x = a + h$ and, for $x \neq a$,

$$\frac{f(x)-f(a)}{x-a} = \frac{f(a+h)-f(a)}{h}$$



DEFINITION

The Derivative

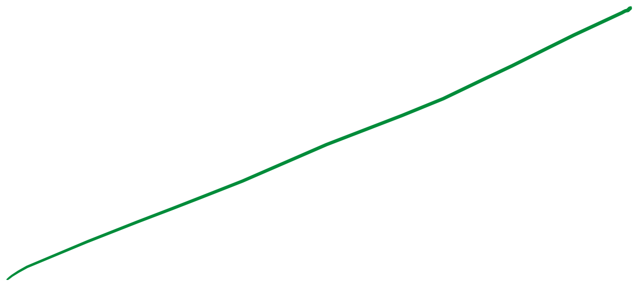
The derivative of f at a point a is the limit of the difference quotients (if it exists):

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad (1)$$

or equivalently:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad (2)$$

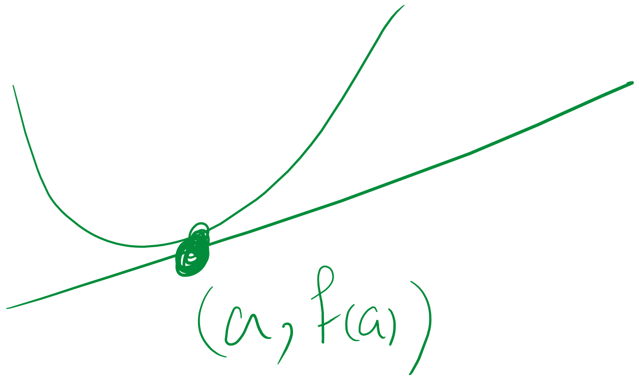
When the limit exists, we say that f is **differentiable** at a.



1 point (x_1, y_1)

Slope = m

$$\boxed{y - y_1 = m(x - x_1)}$$



Point $(a, f(a))$

Slope $f'(a)$

$$\boxed{y - f(a) = f'(a)(x - a)}$$

100% in Exams

We can now define the tangent line in a precise way, as the line of slope $f'(a)$ through $P = (a,\ f(a))$.

DEFINITION

Tangent Line

Important

Assume that f is differentiable at a. The **tangent line** to the graph of $y = f(x)$ at $P = (a,\ f(a))$ is the line through P of slope $f'(a)$. The equation of the tangent line in point-slope form is

$$y - f(a) = f'(a)(x - a) \tag{3}$$

REMINDER

The equation of the line through $P = (a, b)$ of slope m in point-slope form:

$$y - b = m(x - a)$$

Point : $(5, 25)$

Slope ?

$$\lim_{x\to 5} \frac{f(x) - \cancel{f(5)}^{25}}{x-5} = \lim_{x\to 5} \frac{x^2 - 25}{x-5} = \frac{0}{0}$$

EXAMPLE 1

$$\lim_{x\to 5} \frac{\cancel{(x-5)}(x+5)}{\cancel{x-5}} = 5+5 = \boxed{10}$$ slope

Equation of a Tangent Line

Find an equation of the tangent line to the graph of $f(x) = x^2$ at $x = 5$.

$y = 5^2 = 25$

$y - y_1 = m(x - x_1)$

$y - 25 = 10(x - 5)$

$y - 25 = 10x - 50$

$+25 \qquad +25$

$\longrightarrow \boxed{y = 10x - 25}$ Tangent line ✓

Solution

First, we must compute $f'(5)$. We are free to use either Eq. (1) or Eq. (2). Using Eq. (2), we have

$$f'(5) = \lim_{x \to 5} \frac{f(x) - f(5)}{x - 5} = \lim_{x \to 5} \frac{x^2 - 25}{x - 5}$$

This limit is in the indeterminate form $0/0$. We can simplify and then evaluate by substitution:

$$f'(5) = \lim_{x \to 5} \frac{(x-5)(x+5)}{x - 5} = \lim_{x \to 5} (x+5) = 10$$

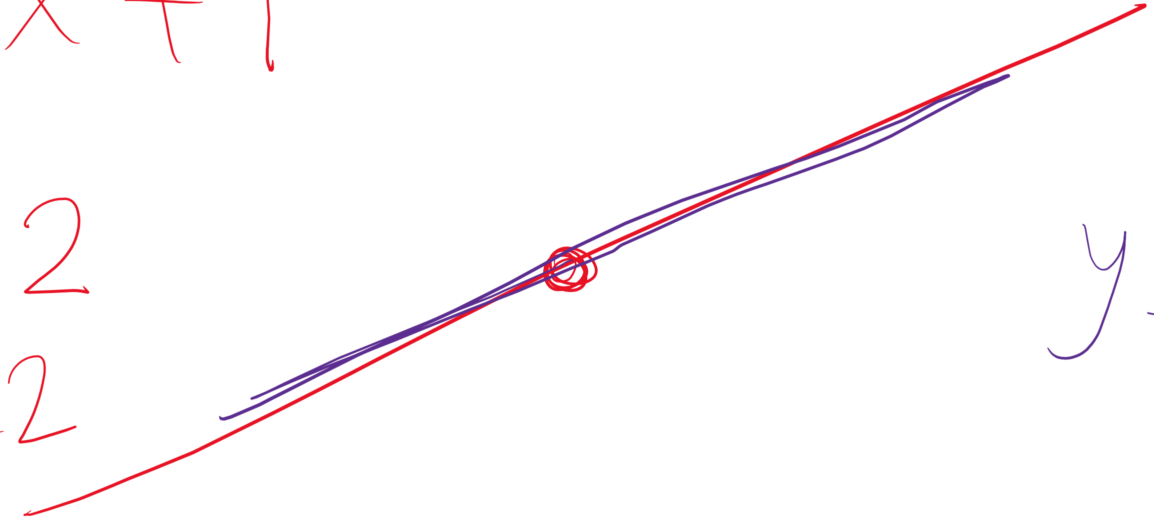
Next, we apply Eq. (3) with $a = 5$. Because $f(5) = 25$, an equation of the tangent line is $y - 25 = 10(x - 5)$, or in slope-intercept form, $y = 10x - 25$ (Figure 4).

$f(x) = x^2 - 8x$, find $f'(3) = ?$

$y = 2x + 1$

$f'(1) = 2$

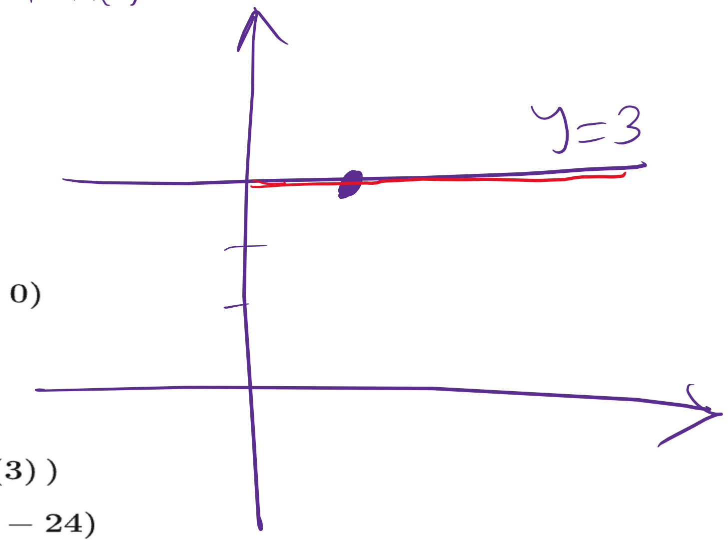
$f'(3) = 2$



$y = mx + b$

$f'(-10) = 2$

$f(x) = 3$



Zero

EXAMPLE 2

Compute $f'(3)$, where $f(x) = x^2 - 8x$.

Solution

Using Eq. (1), we write the difference quotient at $a = 3$ as

$$\frac{f(a+h) - f(a)}{h} = \frac{f(3+h) - f(3)}{h} \qquad (h \neq 0)$$

***Step 1*. Write out the numerator of the difference quotient.**

$$\begin{aligned} f(3+h) - f(3) &= \left((3+h)^2 - 8(3+h)\right) - (3^2 - 8(3)) \\ &= ((9 + 6h + h^2) - (24 + 8h)) - (9 - 24) \\ &= h^2 - 2h \end{aligned}$$

Undefine

***Step 2*. Divide by h and simplify.**

$$\frac{f(3+h) - f(3)}{h} = \frac{h^2 - 2h}{h} = \underbrace{\frac{h(h-2)}{h} = h - 2}_{\text{Cancel } h}$$

$f(x) = 3$

$f'(x) = 0$

***Step 3*. Compute the limit.**

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} (h - 2) = -2$$

$f'(1) = 0$

$f'(0) = 0$

$f'(-5) = 0$

$\frac{y_2 - y_1}{x_2 - x_1}$

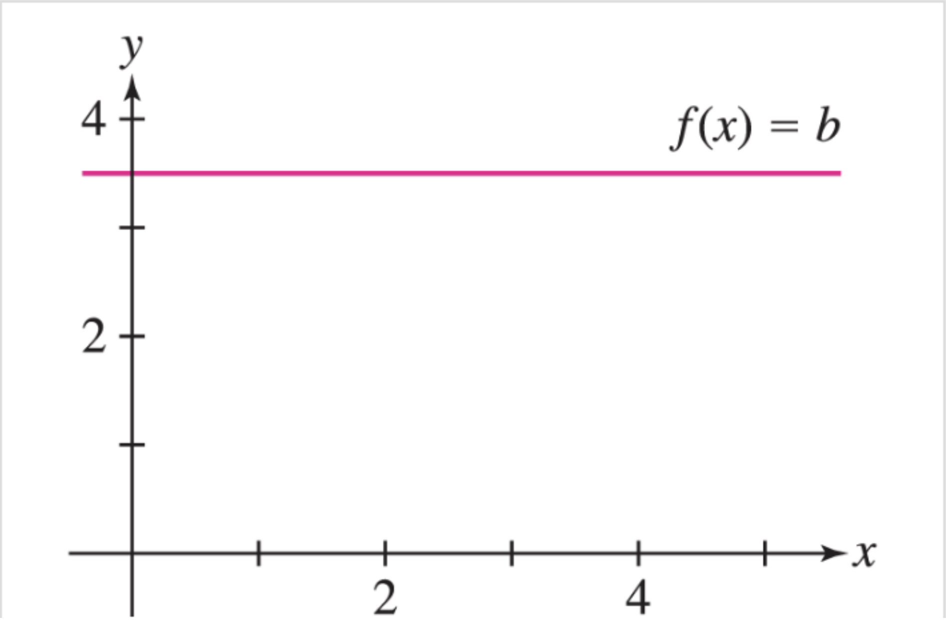
THEOREM 1

$f(x) = 9x - 5$ $\quad x = 2$

$x = 5$

Derivative of Linear and Constant Functions

- If $f(x) = mx + b$ is a linear function, then $f'(a) = m$ for all a.
- If $f(x) = b$ is a constant function, then $f'(a) = 0$ for all a.



EXAMPLE 4

Find the derivative of $f(x) = 9x - 5$ at $x = 2$ and x

Solution

We have $f'(a) = 9$ for all a. Hence, $f'(2) = f'(5) = 9$

EXAMPLE 5

Failure to be Differentiable

Show that the functions $f(x) = |x|$ and $g(x) = x^{1/3}$ are not differentiable at $x = 0$.

Solution

First, note that

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

Since

$$\frac{|h|}{h} = \begin{cases} 1 & \text{if } h > 0 \\ -1 & \text{if } h < 0 \end{cases}$$

we have the one-sided limits

$$\lim_{h \to 0^+} \frac{|h|}{h} = 1 \quad \text{and} \quad \lim_{h \to 0^-} \frac{|h|}{h} = -1.$$

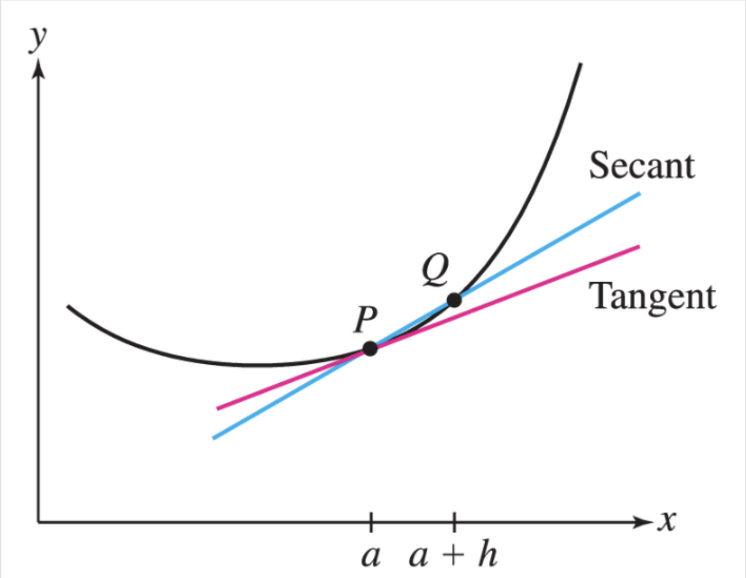
These one-sided limits are not equal; therefore, $\lim_{h \to 0} \frac{|h|}{h}$ does not exist. Thus, f is not differentiable at $x = 0$.

Estimating the Derivative

Approximations to the derivative are useful in situations where we cannot evaluate $f'(a)$ exactly. Since the derivative is the limit of difference quotients, the difference quotient should give a good numerical approximation when h is sufficiently small:

$$f'(a) \approx \frac{f(a+h) - f(a)}{h} \qquad \text{if } h \text{ is small}$$

We refer to this estimate as the **difference quotient approximation**. Graphically, thi[s] says that for small h, the slope of the secant line is nearly equal to the slope of the tangent line (Figure 10).



3.1 SUMMARY

- The *difference quotient* is the slope of the secant line through the points P and Q on the graph of f and equals

$$\frac{f(a+h)-f(a)}{h} \text{ with } P=(a,\ f(a)) \text{ and } Q=(a+h,\ f(a+h))$$

$$\frac{f(x)-f(a)}{x-a} \text{ with } P=(a,\ f(a)) \text{ and } Q=(x,\ f(x))$$

- The *derivative* $f'(a)$ is defined by the following equivalent limits:

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

If the limit exists, we say that f is *differentiable* at $x=a$.

- By definition, the tangent line at $P=(a,\ f(a))$ is the line through P with slope $f'(a)$ [assuming that $f'(a)$ exists].
- Equation of the tangent line in point-slope form:

$$y-f(a)=f'(a)(x-a)$$

- To calculate $f'(a)$ using the definition $f'(a)=\lim\limits_{h\to 0}\frac{f(a+h)-f(a)}{h}$:

  Step 1. Write out the numerator of the difference quotient.

  Step 2. Divide by h and simplify.

Step 2. Divide by h and simplify.

Step 3. Compute the derivative by taking the limit.

- For small h, we have the *difference quotient approximation*: $f'(x) \approx \frac{f(x+h)-f(x)}{h}$.