Discrete Mathematics

Eighth Edition

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Counting Methods and the Pigeonhole Principle

In many discrete problems, we are confronted with the problem of counting. In this chapter we develop several tools for counting. These techniques can be used to derive the binomial theorem. The chapter concludes with a discussion of the Pigeonhole Principle, which often allows us to prove the existence of an object with certain properties.

6.1 Basic Principles

The menu for Kay's Quick Lunch is shown in Figure 6.1.1. As you can see, it features two appetizers, three main courses, and four beverages. How many different dinners consist of one main course and one beverage?

Solution:

MAIN COURSES = { H, C, F}

Hamburger   *Cheeseburger*   *Fish Filet*

APPETIZERS = {N, S}

Nacho   *Salad*

BEVERAGES = {T, M, C, R}

Tea   *Milk*   *Cola*   *Root Beer*

If we list all possible dinners consisting of one main course and one beverage, HT, HM, HC, HR, CT, CM, CC, CR, FT, FM, FC, FR,

Notice that there are three main courses and four beverages and $12 = 3 \cdot 4$.

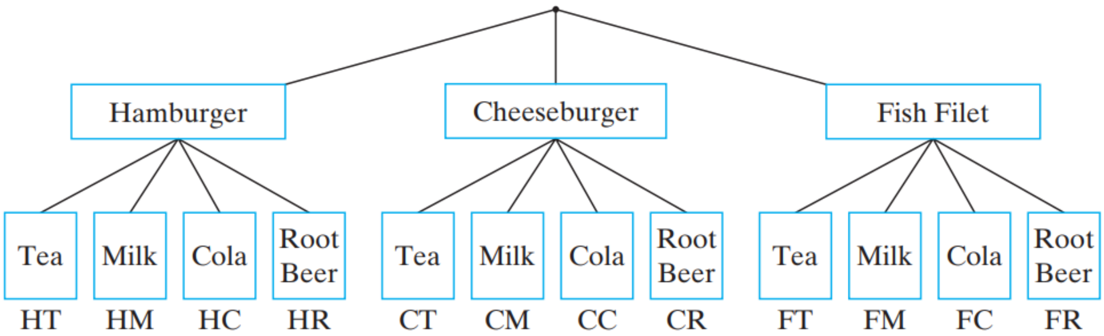

Figure 6.1.2 An illustration of the Multiplication Principle.

There are 24 possible dinners consisting of one appetizer, one main course, and one beverage:

NHT, NHM, NHC, NHR, NCT, NCM, NCC, NCR, NFT, NFM, NFC, NFR, SHT, SHM, SHC, SHR,

SCT, SCM, SCC, SCR, SFT, SFM, SFC, SFR.

Notice that there are two appetizers, three main courses, and four beverages and

$$24 = 2 \cdot 3 \cdot 4.$$

Multiplication Principle

If an activity can be constructed in t successive steps and step 1 can be done in n_1 ways, step 2 can then be done in n_2 ways, . . . , and step t can then be done in n_t ways, then the number of different possible activities is $n_1 \cdot n_2 \cdot \cdots \cdot n_t$.

Example 6.1.3

(a) How many strings of length 4 can be formed using the letters *ABCDE* if repetitions are not allowed?

(b) How many strings of part (a) begin with the letter *B*?

(c) How many strings of part (a) do not begin with the letter *B*?

Solution:

(a) By the Multiplication Principle, there are

$$5 \cdot 4 \cdot 3 \cdot 2 = 120 \text{ strings.}$$

(b) By the Multiplication Principle, there are

$$1 \cdot 4 \cdot 3 \cdot 2 = 24$$

Strings that start with the letter *B*.

(c) there are

$$120 - 24 = 96$$

strings that do not begin with the letter *B*.

Example 6.1.4 In a digital picture, we wish to encode the amount of light at each point as an eight-bit string. How many values are possible at one point?

Solution:

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^8 = 256$$

Example 6.1.5

Use the Multiplication Principle to prove that a set $\{x_1,\ldots, x_n\}$ containing n elements has 2^n subsets; that is, the cardinality of the power set of an n-element set is 2^n.

Solution:

A subset can be constructed in n successive steps: Pick or do not pick x_1; pick or do not pick x_2;…; pick or do not pick x_n. Each step can be done in two ways. Thus the number of possible subsets is

$$\mathbf{2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2=2^n}$$

Example 6.1.6

- Let *X* be an *n*-element set. How many ordered pairs (*A*, *B*) satisfy $A \subseteq B \subseteq X$?
- Solution:
- $3 \cdot 3 \cdot 3 \ldots 3 = 3^n$
- We can make such assignments by the following n-step process:
- Assign the first element of X to one of A, B−A, X−B; assign the second element of X to one of A, B−A, X−B;...; assign the nth

element of X to one of A, B − A, X − B.

- **Since each step can be done in three ways, the number of ordered pairs (A, B) satisfying**
- **$A \subseteq B \subseteq X$.**
- **is**
- **$3 \cdot 3 \cdot 3 \ldots 3 = 3^n$**
- **n factors**

Example 6.1.7

How many reflexive relations are there on an n-element set?

Solution:

We count the number of n × n matrices that represent reflexive relations on an n-element set X. Since (x, x) is in the relation for all x ∈ X, the main diagonal of the matrix must consist of 1's. There is no restriction on the remaining entries; each can be 0 or 1. An n × n matrix has n^2 entries and the diagonal contains n entries. Thus there are $n^2 - n$ off-diagonal entries. Since each can be assigned values in two ways, by the Multiplication Principle there are

$$2 \cdot 2 \ \dots \ 2 = 2^{n^2-n}$$

matrices that represent reflexive relations on an n-element set.

Therefore there are 2^{n^2-n} reflexive relations on an n-element set.

Example 6.1.8 Internet Addresses

In the IPv4 (Internet Protocol, Version 4) addressing scheme, the addresses are divided into five classes—Class A through Class E.

In this example, we count the number of Class A addresses. Exercises 80–82 deal with Class B and C addresses. A Class A address is a bit string of length 32. The first bit is 0 (to identify it as a Class A address). The next 7 bits, called the netid, identify the network. The remaining 24 bits, called the hostid, identify the computer interface. The netid must not consist of all 1's. The hostid must not consist of all 0's or all 1's.

Solution:

Arguing as in Example 6.1.4, we find that there are 2^7 7-bit strings. Since 1111111 is not allowed as a netid, there are $2^7 - 1$ netids. Again, arguing as in Example 6.1.4, we find that there are 2^{24} 24-bit strings. Since the two strings consisting of all 0's or all 1's are not allowed as a hostid, there are $2^{24} - 2$ hostids. By the Multiplication Principle, there are

$(2^7 - 1)(2^{24} - 2) = 127 \cdot 16{,}777{,}214 = 2{,}130{,}706{,}178$ Class A internet addresses.

Example 6.1.9 How many eight-bit strings begin either 101 or 111?

Solution: By the Multiplication Principle, there are

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32$$

eight-bit strings that begin 101.

The same argument can be used to show that there are 32 eight-bit strings that begin 111.

Since there are 32 eight-bit strings that begin 101 and 32 eight-bit strings that begin 111, there are **32 + 32 = 64** eight-bit strings that begin either 101 or 111.

Addition Principle

Suppose that $X_1, \ldots, X_t$ are sets and that the ith set X_i has n_i elements. If

$$\{X_1, \ldots, X_t\}$$

is a pairwise disjoint family (i.e., if $i \neq j$, $X_i \cap X_j = \emptyset$), the number of possible

elements that can be selected from

$$X_1 \text{ or } X_2 \text{ or } \ldots \text{ or } X_t$$

is

$$n_1 + n_2 + \cdots + n_t$$

(Equivalently, the union

$X_1 \cup X_2 \cup \cdots \cup X_t$ contains

$n_1 + n_2 + \cdot \quad \cdot \quad \cdot + n_t$ elements.)

If we are counting objects that are constructed in successive steps, we use the **Multiplication Principle.**

If we have **disjoint sets of objects** and we want to know the total number of objects, we use the **Addition Principle.**

Example 6.1.10 In how many ways can we select two books from different subjects among five distinct computer science books, three distinct mathematics books, and two distinct art books?

Solution: Using the Multiplication Principle we find that we can select two books, one from computer science and one from mathematics, in **$5 \cdot 3 = 15$ ways.**

Similarly, we can select two books, one from computer science and one from art, in **$5 \cdot 2 = 10$ ways**, and we can select two books, one from mathematics and one from art, in **$3 \cdot 2 = 6$ ways**.

we may use the Addition Principle to conclude that there are **$15 + 10 + 6 = 31$**

ways of selecting two books from different subjects among the computer science, mathematics, and art books.

Example 6.1.12

A six-person committee composed of Alice, Ben, Connie, Dolph, Egbert, and Francisco is to select a chairperson, secretary, and treasurer.

(a) In how many ways can this be done?

(b) In how many ways can this be done if either Alice or Ben must be chairperson?

(c) In how many ways can this be done if Egbert must hold one of the offices?

(d) In how many ways can this be done if both Dolph and Francisco must hold office?

Solution:

a) $6 \cdot 5 \cdot 4 = 120$,

b) $5.4 = 20$ & $20 + 20 = 40$,

c) $20 + 20 + 20 = 60$, c) $3 \cdot 5 \cdot 4 = 60$,

d) $3 \cdot 2 \cdot 4 = 24$

Example 6.1.12 We revisit Example 6.1.3(c), which asked how many strings of length 4 that do not begin with the letter B can be formed using the letters $ABCDE$ (**repetitions not allowed**)?

Solution:

$$24 + 24 + 24 + 24 = 96.$$

However, the solution given in Example 6.1.3(c), which subtracted the number of strings that *do* start with B from the total number of strings, was easier.

A variation of the direct approach might also have been used:

By the Multiplication Principle,

there are

$4 \cdot 4 \cdot 3 \cdot 2 = 96$

strings that do not begin with the letter *B*.

Inclusion-Exclusion Principle

we want to count the number of eight-bit strings

start **10** or end **011**

X denote the set of eight-bit strings that start **10**

Y denote the set of eight-bit strings that end **011**

The goal then is to compute

$$|X \cup Y|$$

We *cannot* use the Addition Principle and add $|X|$ and $|Y|$ to compute $|X \cup Y|$

because the Addition Principle requires X and Y to be disjoint. Here X and Y are not disjoint; for example, $10111011 \in X \cap Y$.

suppose that we compute

$$|X| + |Y|$$

Theorem 6.1.13 Inclusion-Exclusion Principle for Two Sets

If X and Y are finite sets, then

$$|\boldsymbol{X} \cup \boldsymbol{Y}| = |\boldsymbol{X}| + |\boldsymbol{Y}| - |\boldsymbol{X} \cap \boldsymbol{Y}|.$$

Proof:

Since $X = (X-Y) \cup (X \cap Y)$ and $X-Y$ and $X \cap Y$ are disjoint, by the Addition Principle

$$|X| = |X - Y| + |X \cap Y| \qquad (1)$$

Similarly,

$$|Y| = |Y - X| + |X \cap Y| \qquad (2)$$

Since

$$X \cup Y = (X - Y) \cup (X \cap Y) \cup (Y - X)$$

and

$$X - Y,\ X \cap Y, \text{ and } Y - X$$

are pairwise disjoint,

by the Addition Principle

$$|X \cup Y| = |X - Y| + |X \cap Y| + Y - X| \quad (3)$$

Combining equations (1)–(3), we obtain

$$|X| + |Y| = |X - Y| + |X \cap Y| + |Y - X| + |X \cap Y| = |X \cup Y| + |X \cap Y|.$$

Subtracting $|X \cap Y|$ from both sides of the preceding equation gives the desired result. ■

Example 6.1.14 A committee composed of Alice, Ben, Connie, Dolph, Egbert, and Francisco is to select a chairperson, secretary, and treasurer. How many selections are there in which either

Alice or Dolph or both are officers?

SOLUTION:

Let X denote the set of selections in which Alice is an officer and let Y denote the set of selections in which Dolph is an officer.

We must compute $|X \cup Y|$.

X and Y

We first count the number of selections in which Alice is an officer.

$$3 \cdot 5 \cdot 4 = 60$$

$$|X| = 60$$

$$|Y| = 60$$

$$3 \cdot 2 \cdot 4 = 24$$

$$|X \cap Y| = 24$$

The Inclusion-Exclusion Principle tells us that

$|X \cup Y| = |X| + |Y| - |X \cap Y| = 60 + 60 - 24 = 96.$

Thus there are 96 selections in which either Alice or Dolph or both are officers. ■

The name "inclusion-exclusion" in Theorem 6.1.13 results from *including*

$$|X \cap Y|$$

twice when computing $|X \cup Y|$ as $|X| + |Y|$ and then *excluding* it by subtracting

$$|X \cap Y|$$

from $|X| + |Y|$.