Divide and Conquer — Merge Sort

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Uppsala University

(Based on previous material by Mohamed Faouzi Atig and Parosh Aziz Abdulla)

Sequential Search

- Problem: Check whether the value of x appears in an array A

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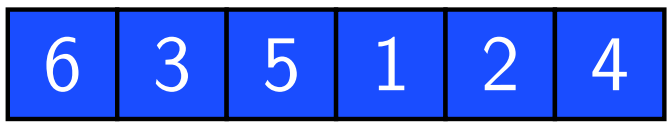

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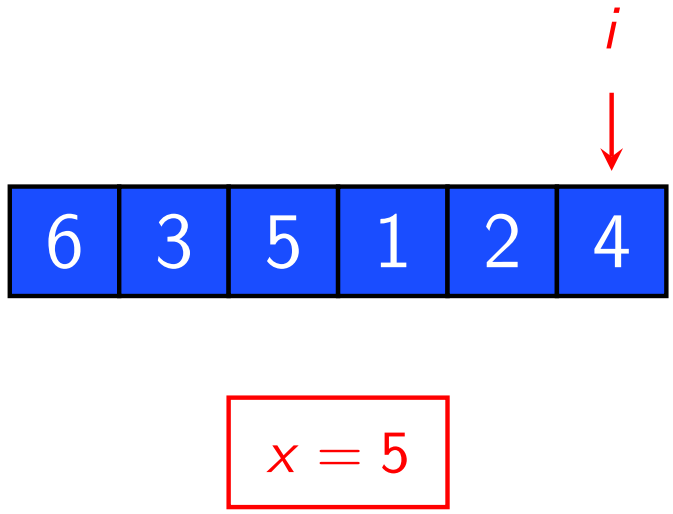

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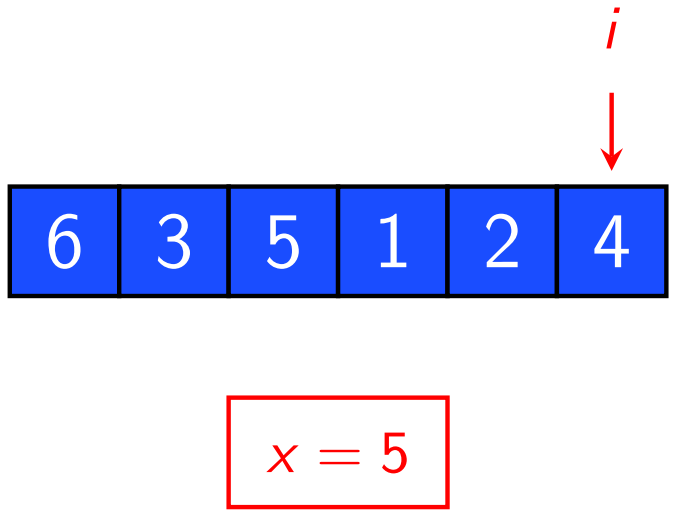

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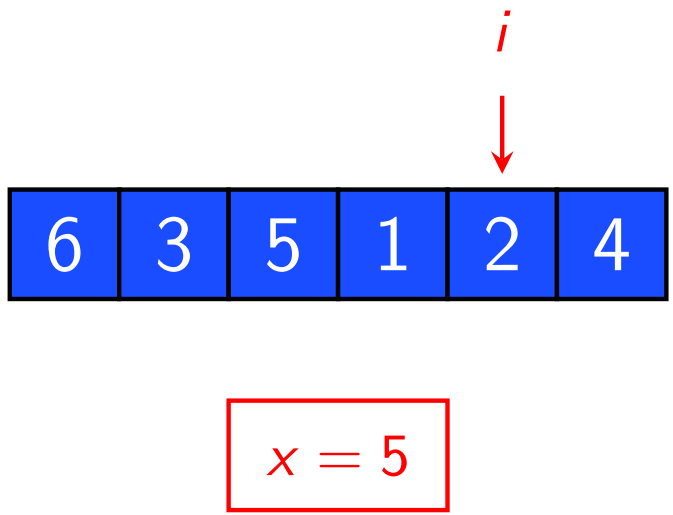

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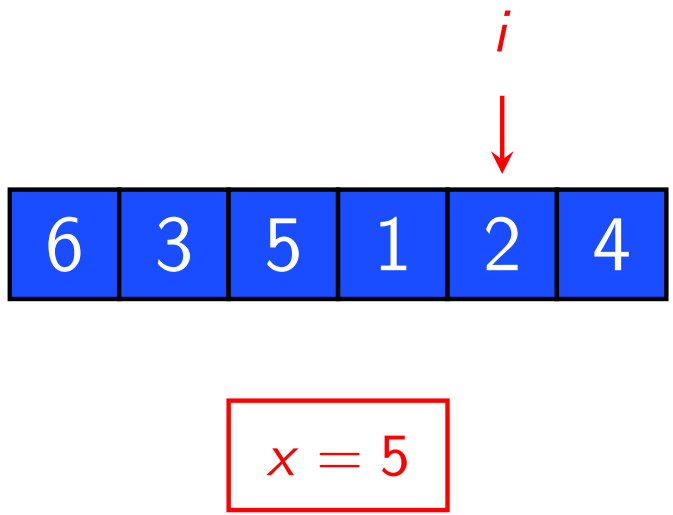

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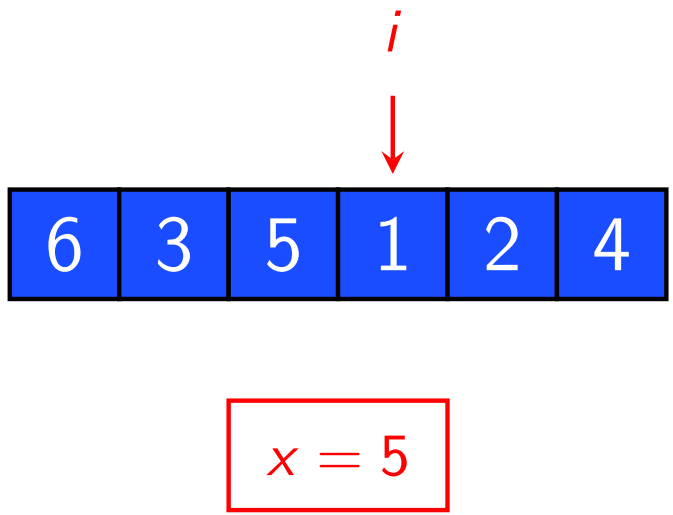

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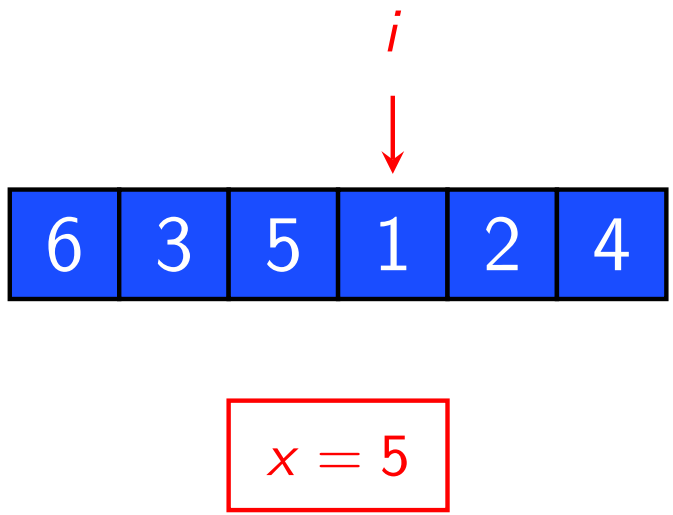

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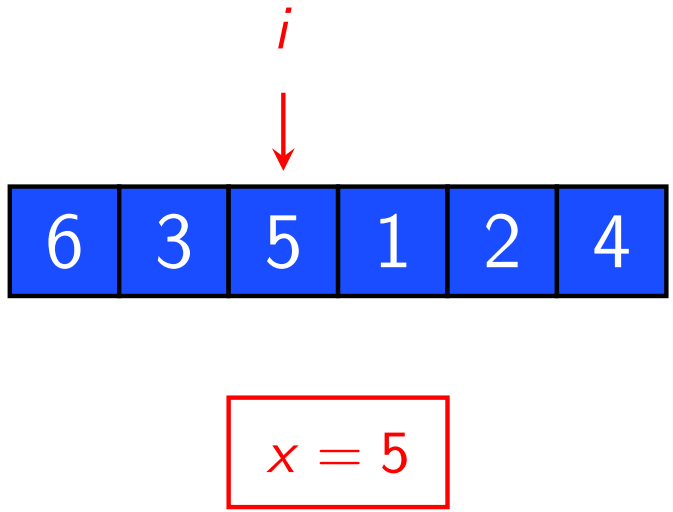

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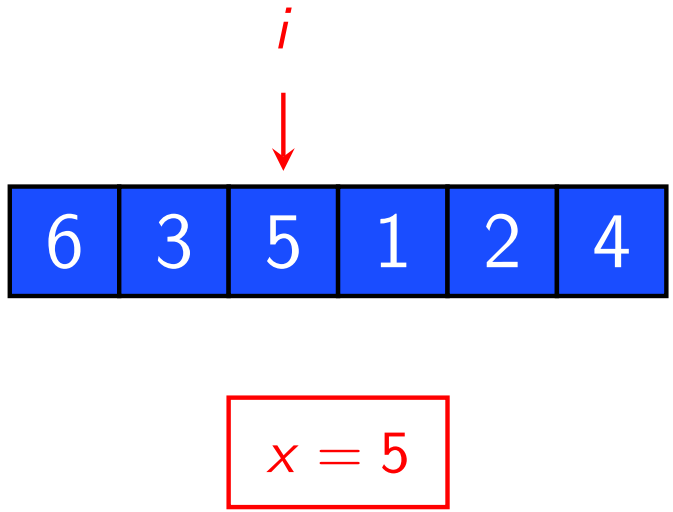

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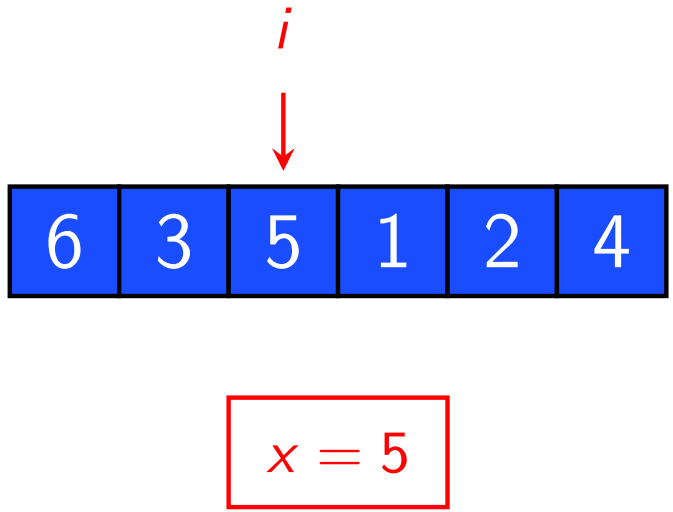

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Insertion Sort

- Problem: Sort an array A of n elements in non-decreasing order

```
INSERTION-SORT(A)
1  for j ← 2 to A.length
2       do key ← A[j]
3          ▷ Insert A[j] into A[1..j − 1]
4          i ← j − 1
5          while i > 0 and A[i] > key
6               do A[i + 1] ← A[i]
7                  i ← i − 1
8          A[i + 1] ← key
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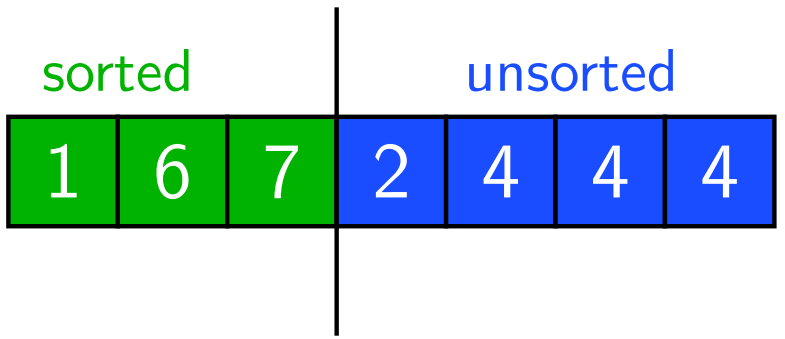

- Incremental approach: Having sorted the subarray $A[1..j-1]$, we inserted $A[j]$ into its proper place, yielding the sorted array $A[1..j]$.

Incremental approach

- Advantages:
  - Simple and applicable to many problems
  - A good starting point to find better algorithms
- Disadvantages:
  - Will often produce less efficient algorithms
  - Less elegant and creative approach

Another Approach

Divide et impera

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Divide and rule / Divide and conquer

Another Approach: The Divide-and-Conquer Methodology

- Divide the problem into a number of smaller subproblems
- Conquer the subproblems individually by solving them respectively
- Combine the solutions to the subproblems into a solution of the main problem
- Base Cases: If the size of the problem does not exceed some given threshold n_0, then a solution can be provided in a straightforward manner

Example: Divide-and-Conquer Search Algorithm

- Problem: Check whether the value of x appears in an array A
- Divide: Divide the input array into two halves of length $n/2$ each (or as close as possible).
- Conquer: Search recursively in each of the two subarrays.
- Combine: Check if the value of x appears in any of the sub-arrays
- Base Cases: The size of A is one. Checking whether the value of x appears in A is trivial.

Divide-and-Conquer Search Algorithm

SEARCH(A, p, r, x)

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2      then q ← ⌊(p+r)/2⌋
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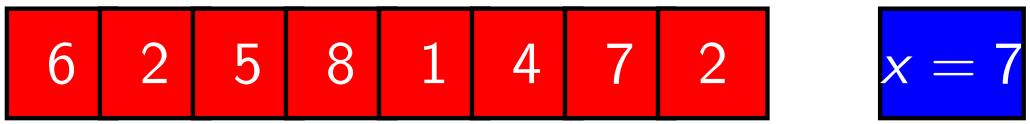

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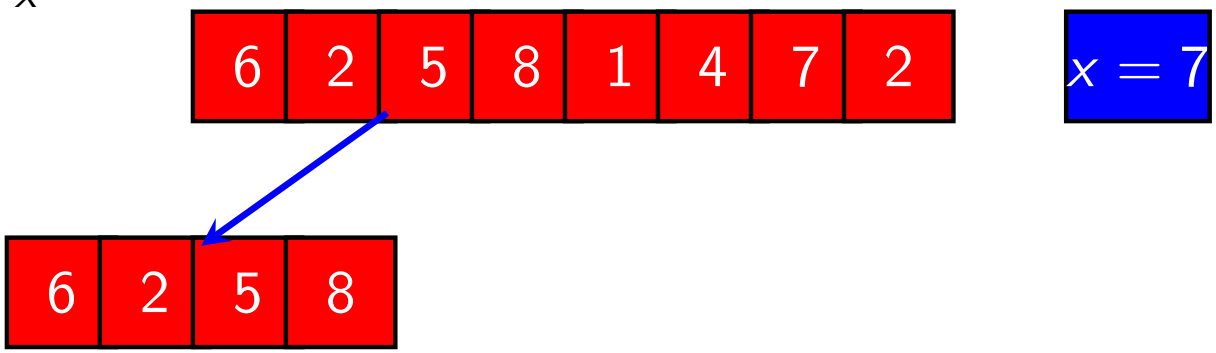

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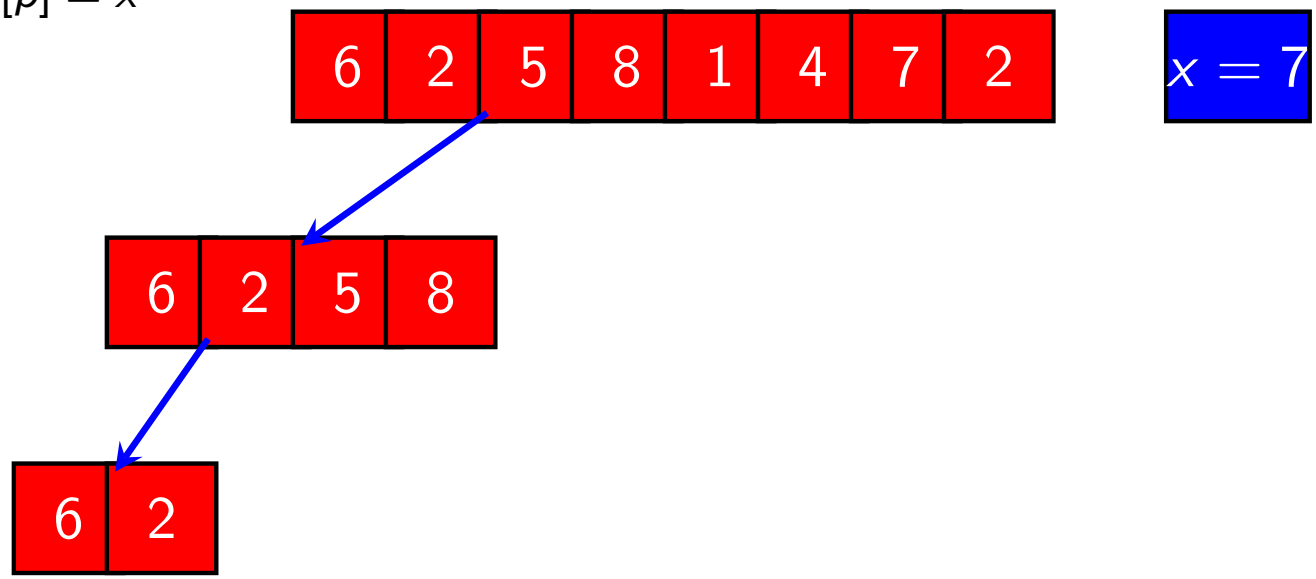

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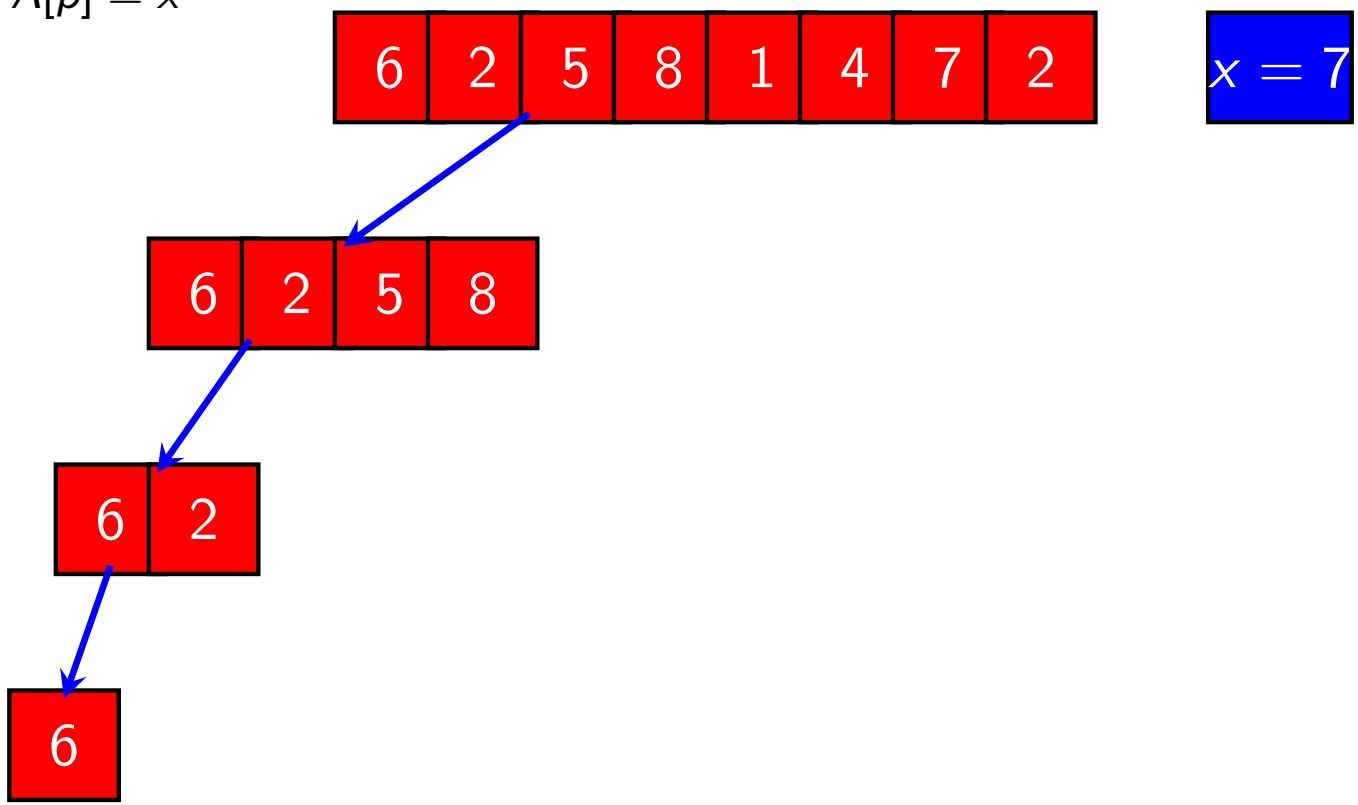

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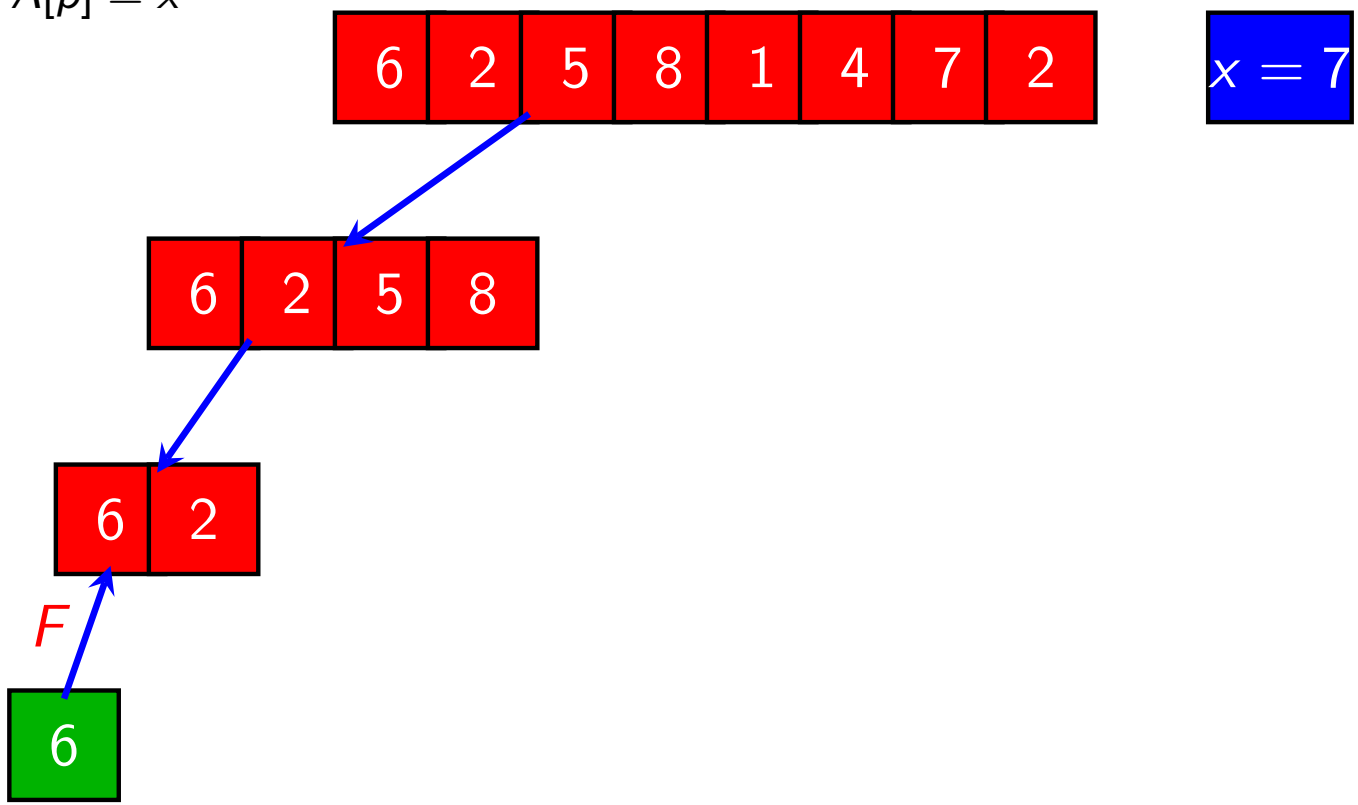

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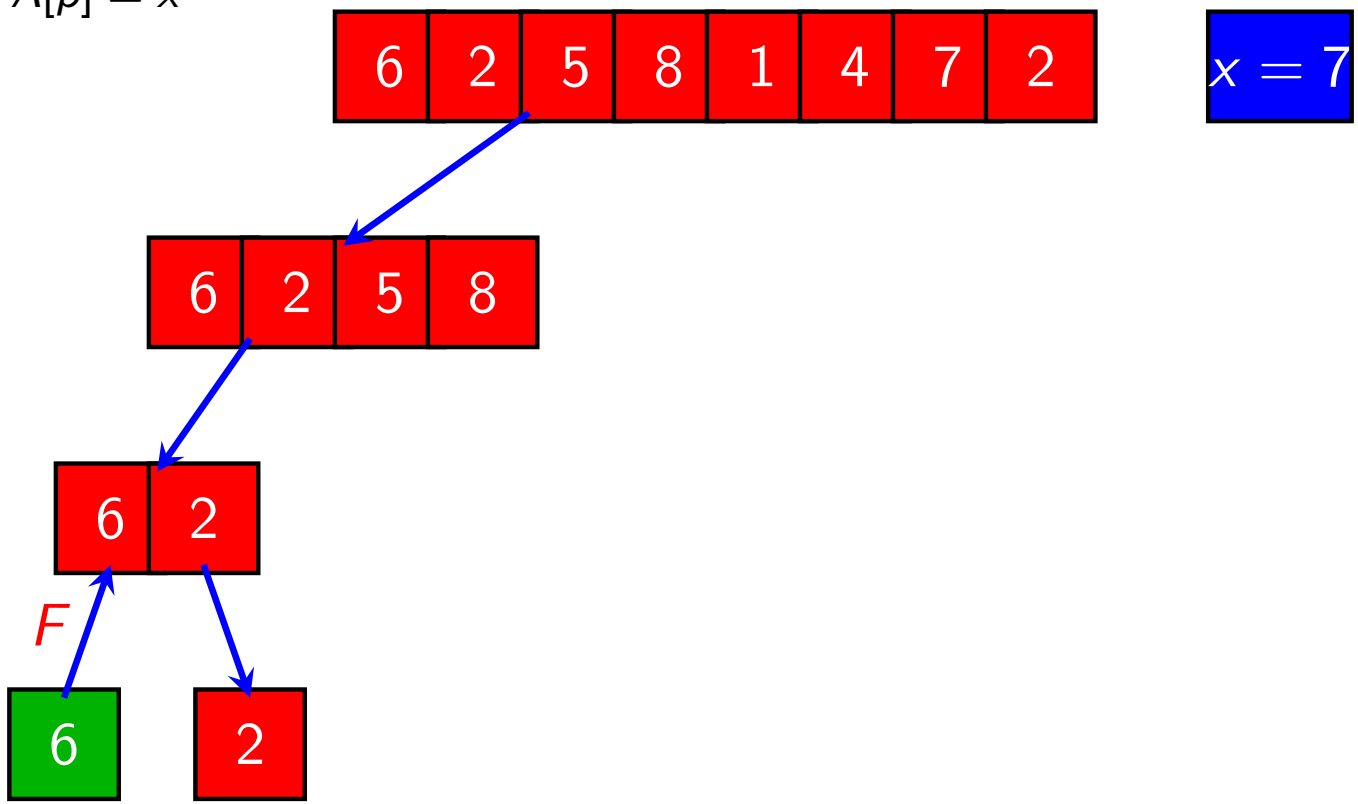

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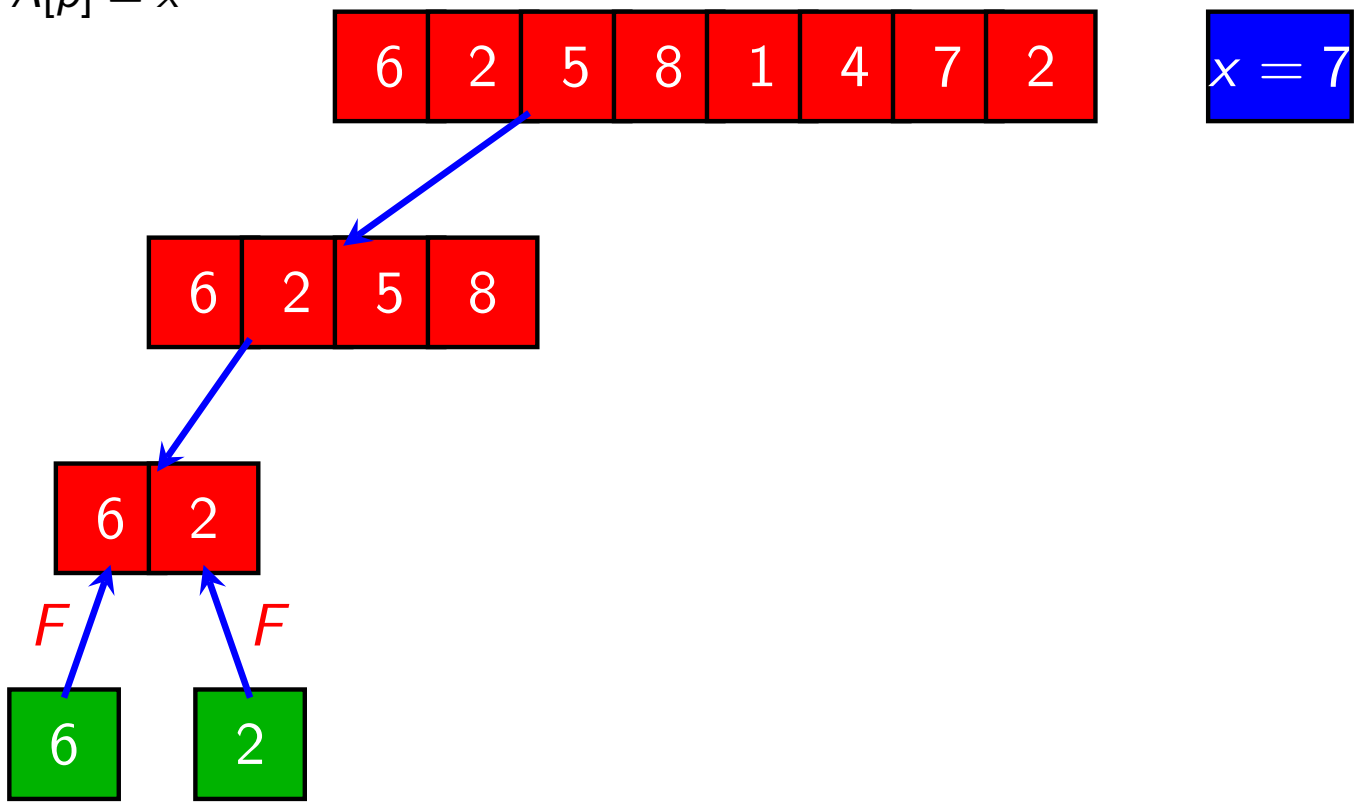

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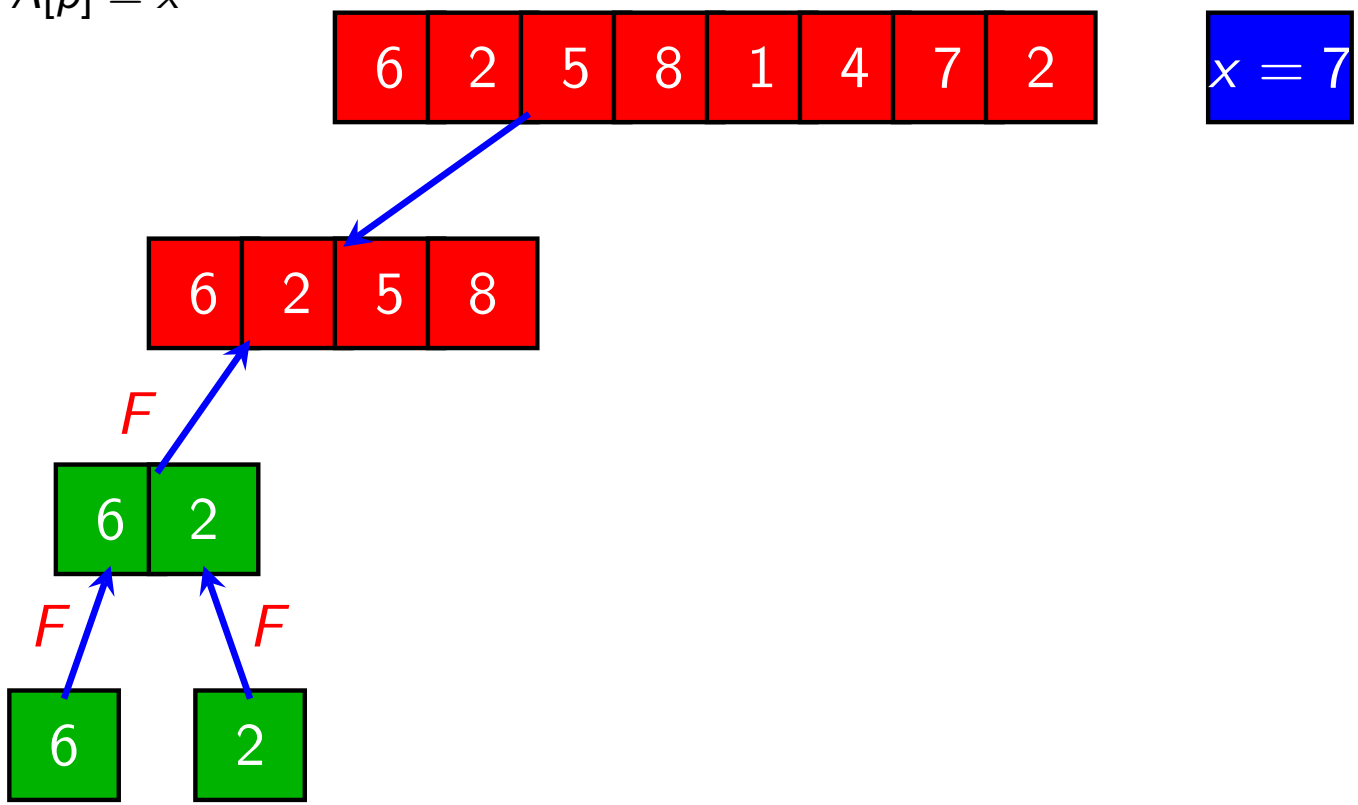

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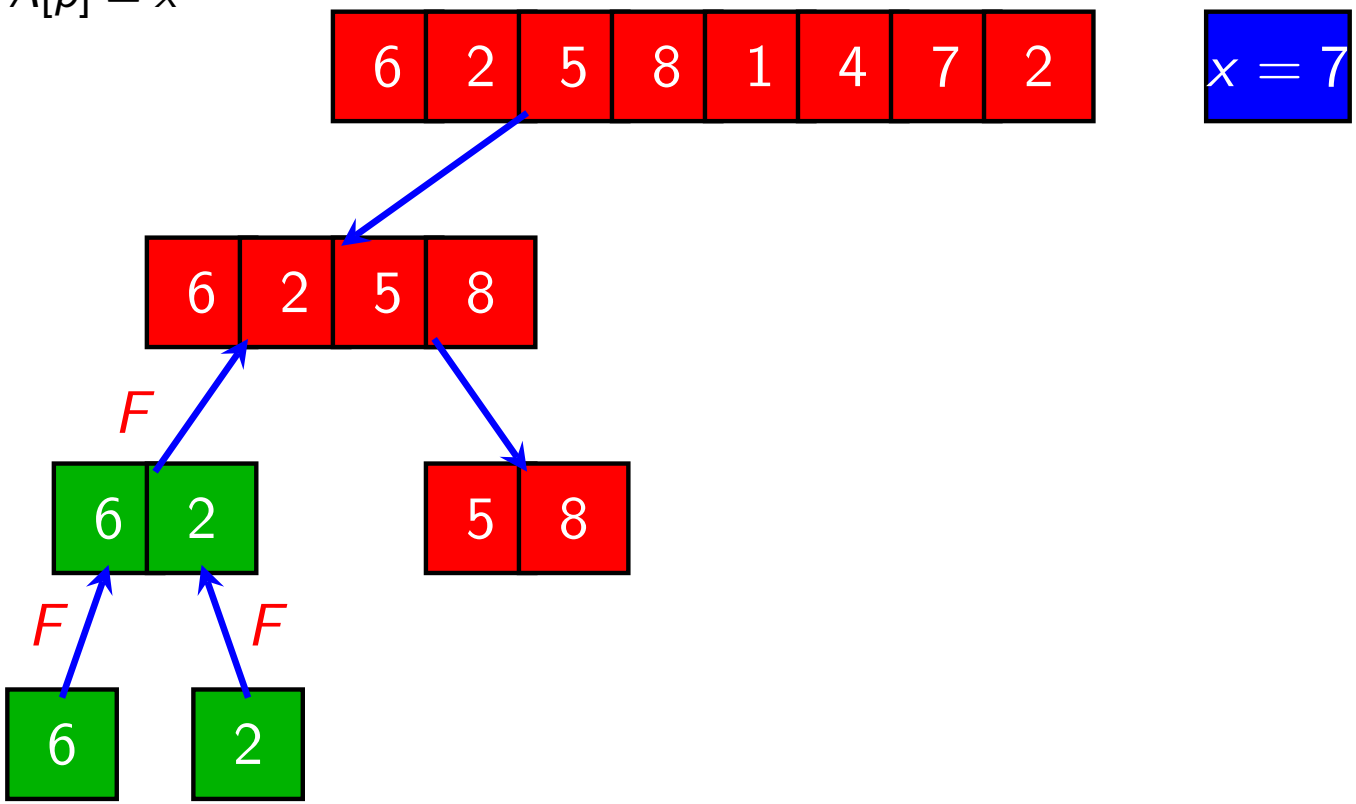

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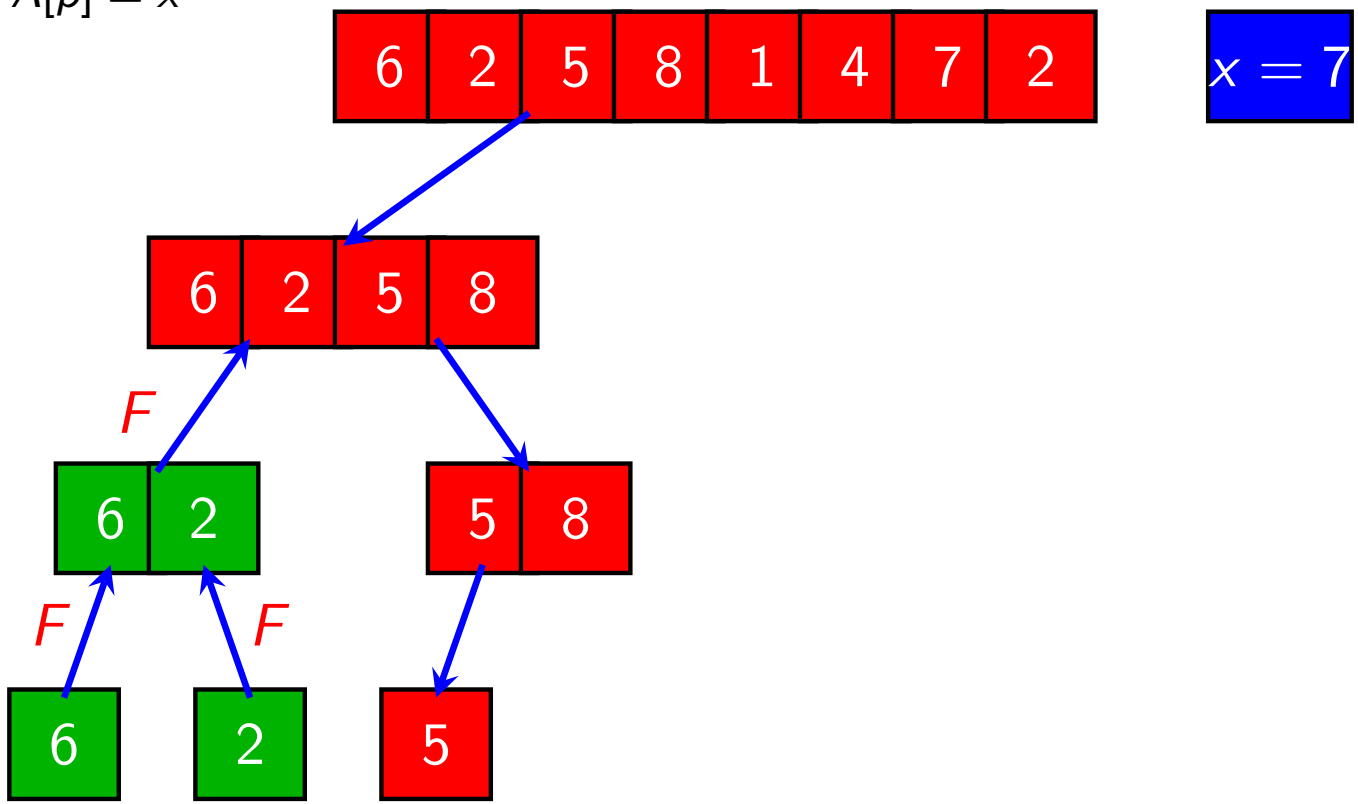

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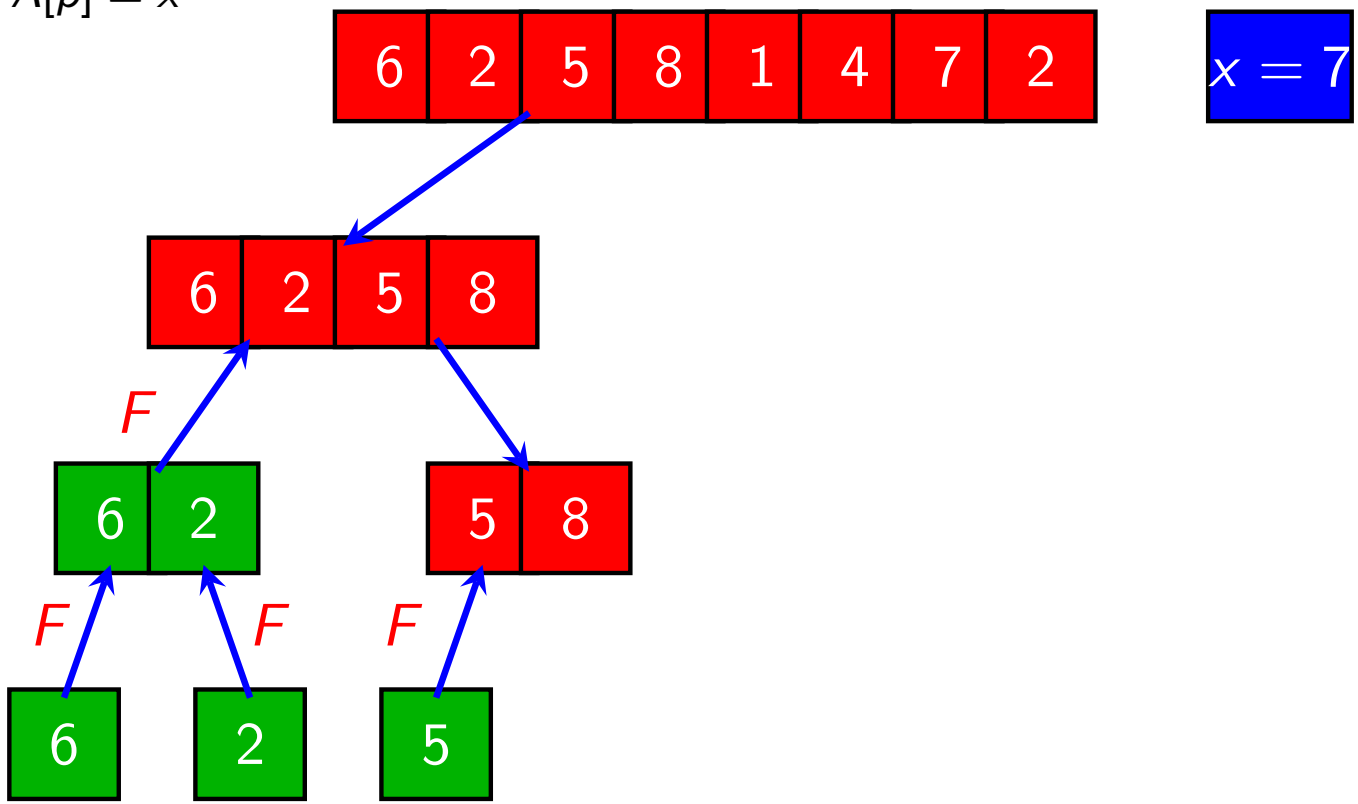

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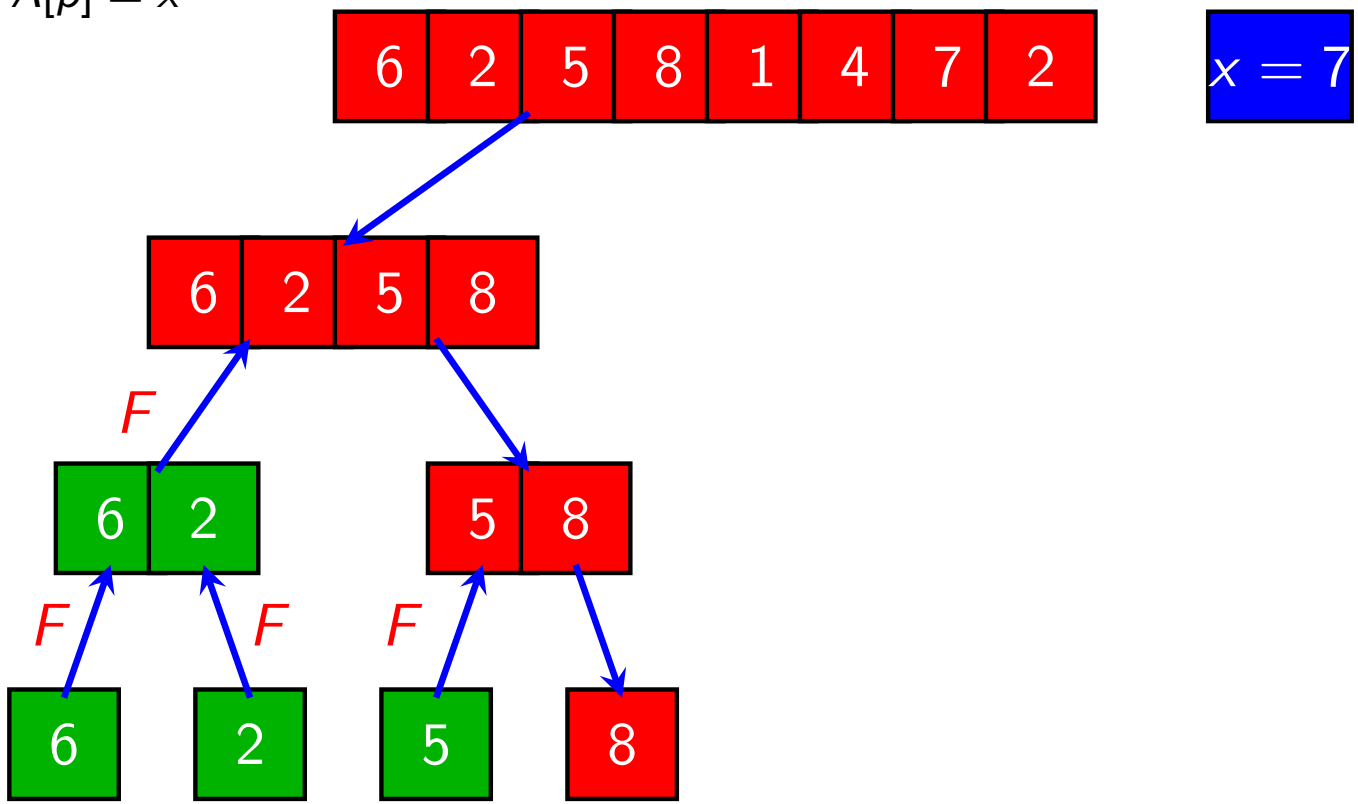

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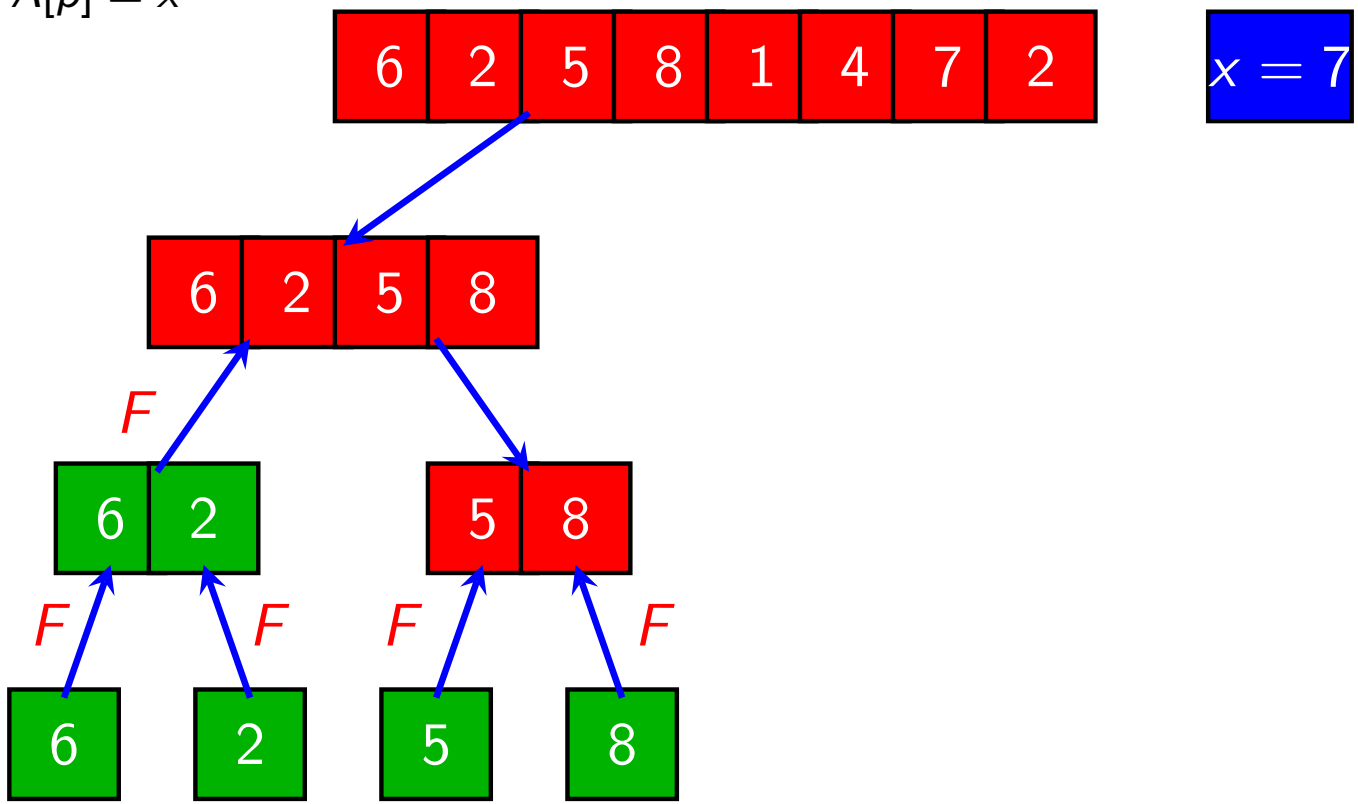

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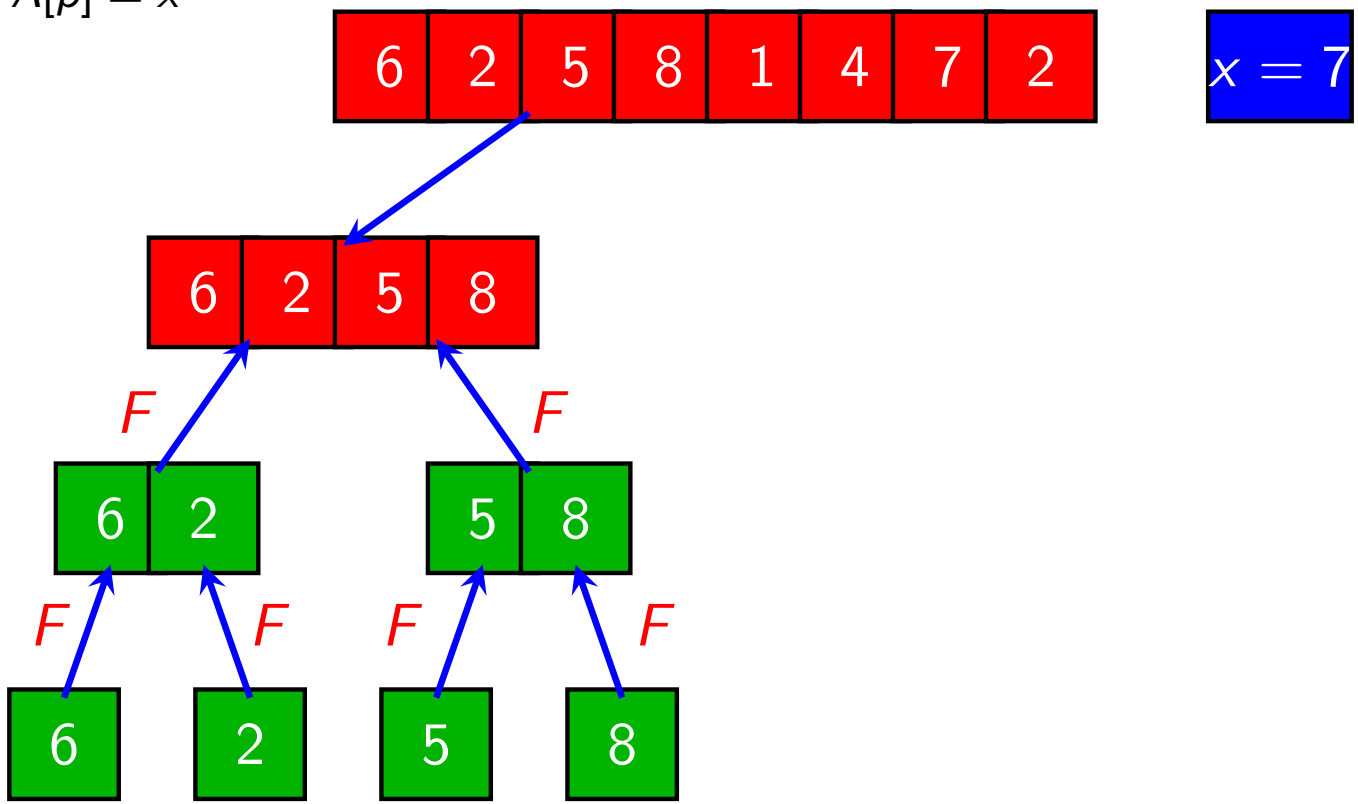

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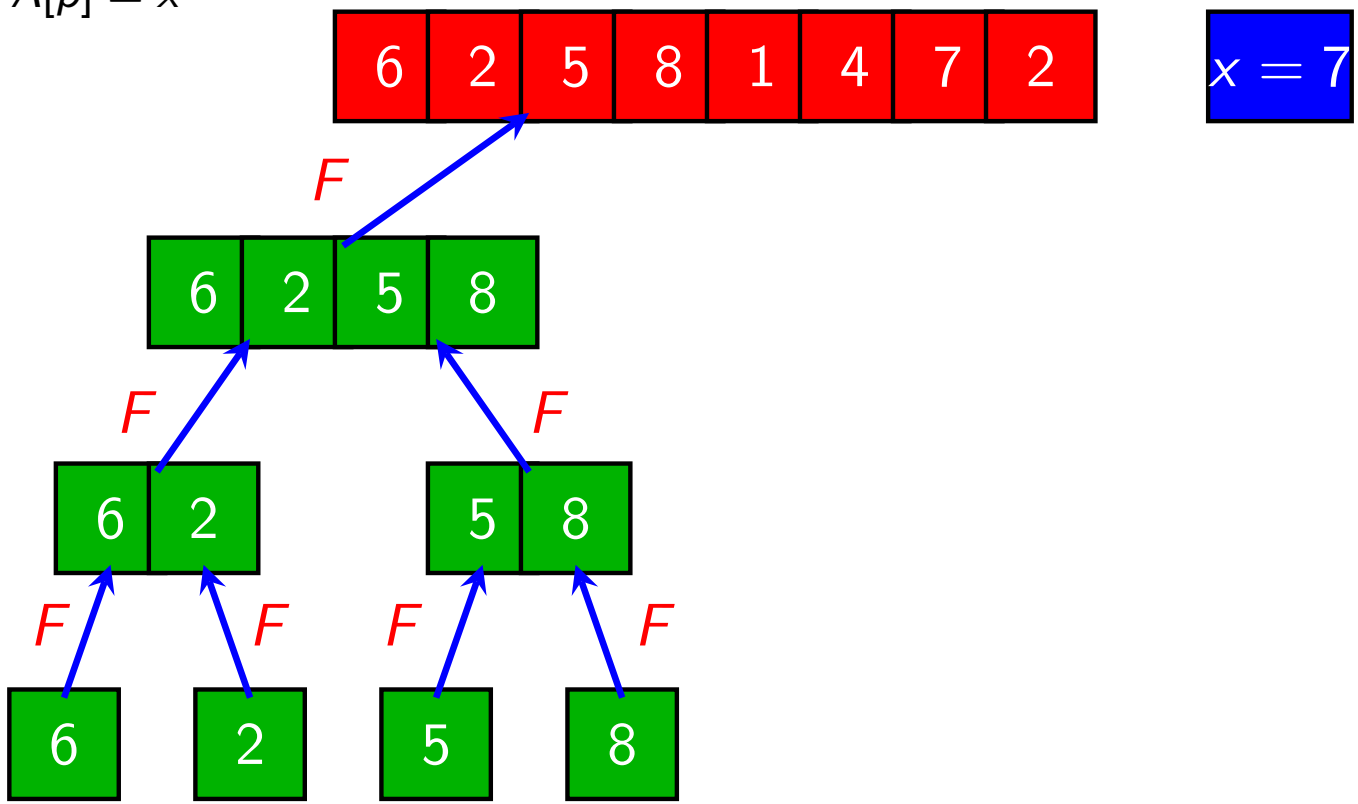

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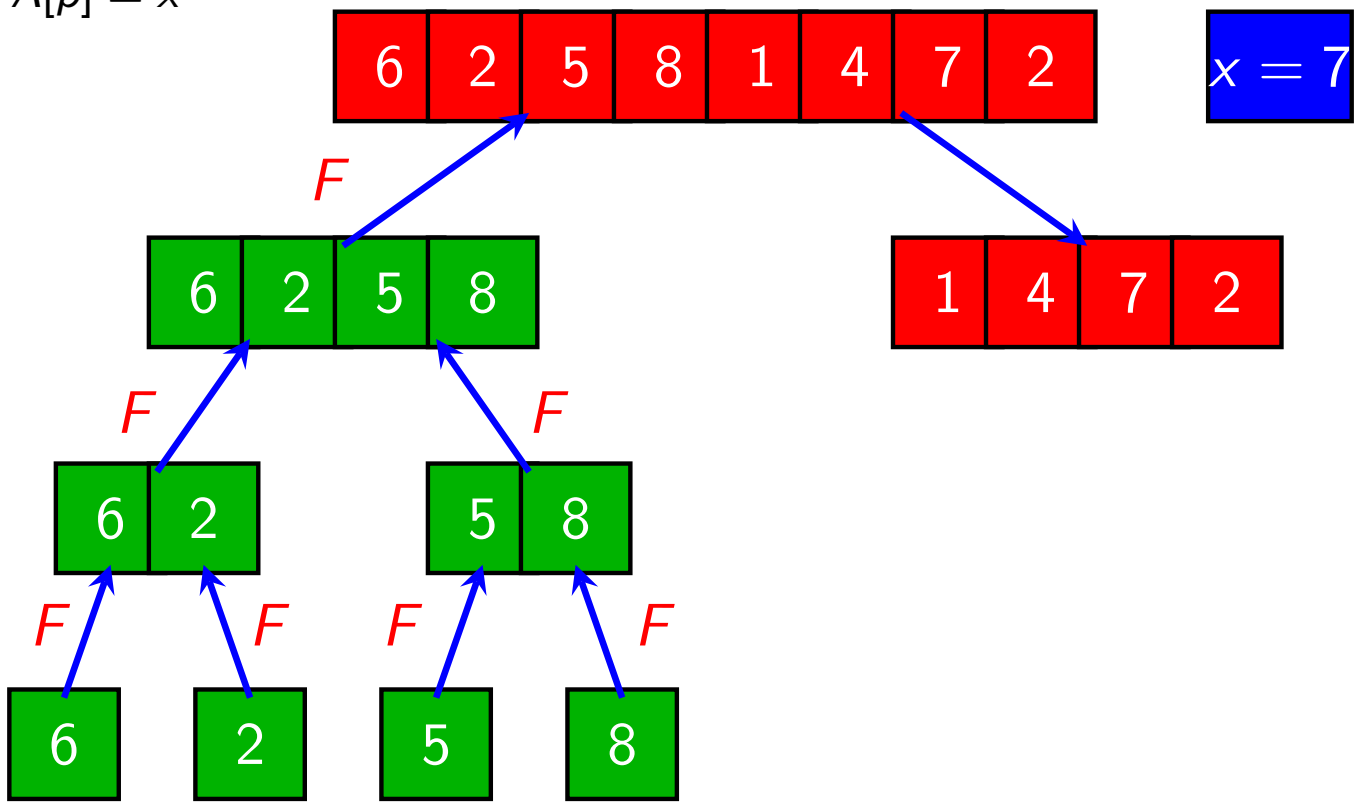

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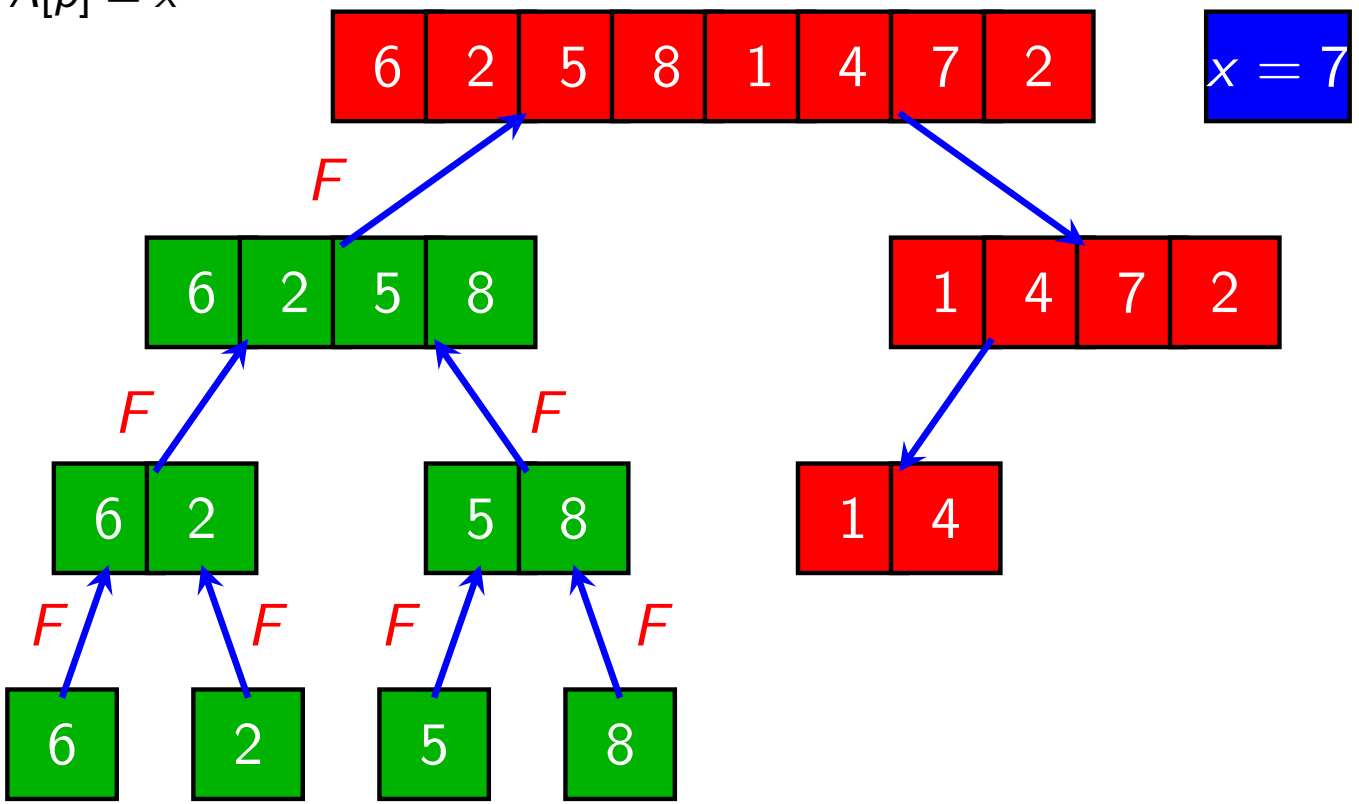

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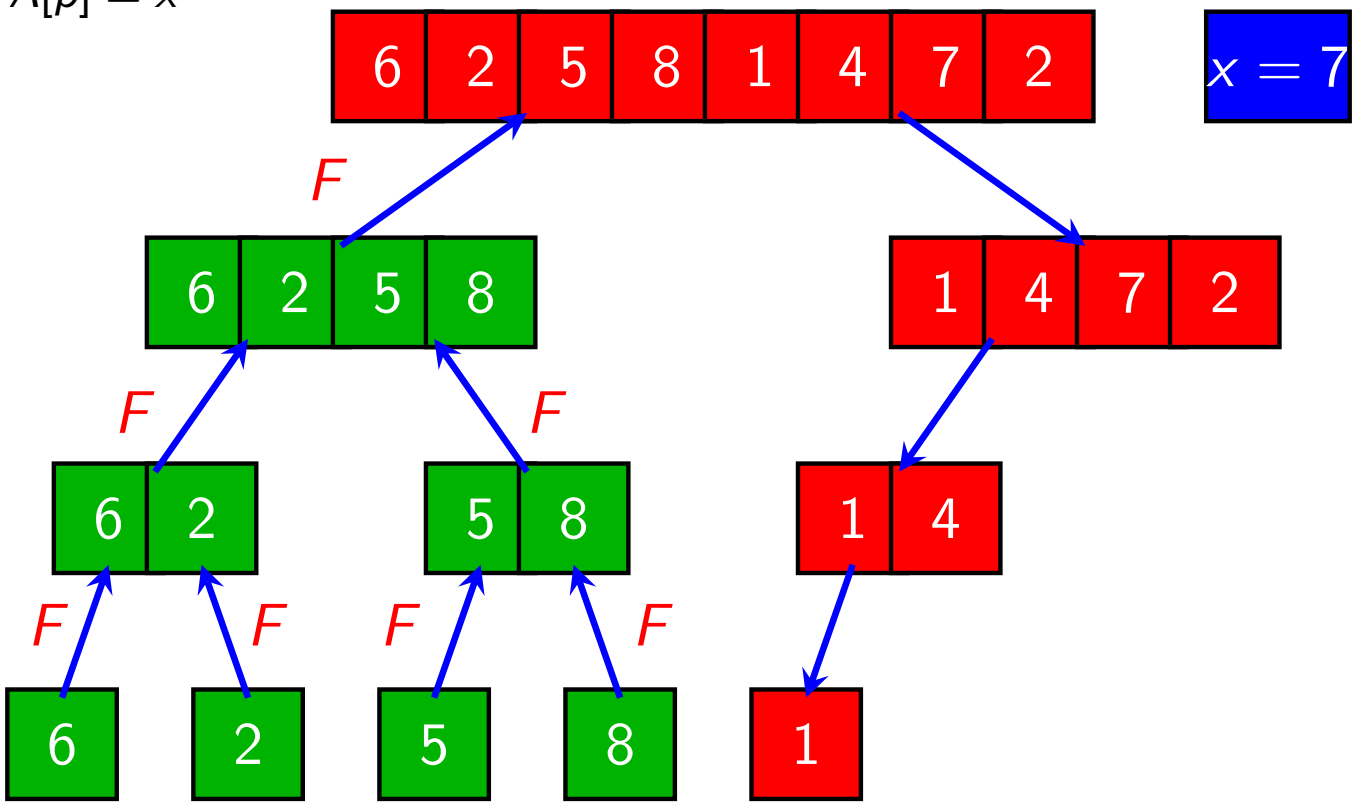

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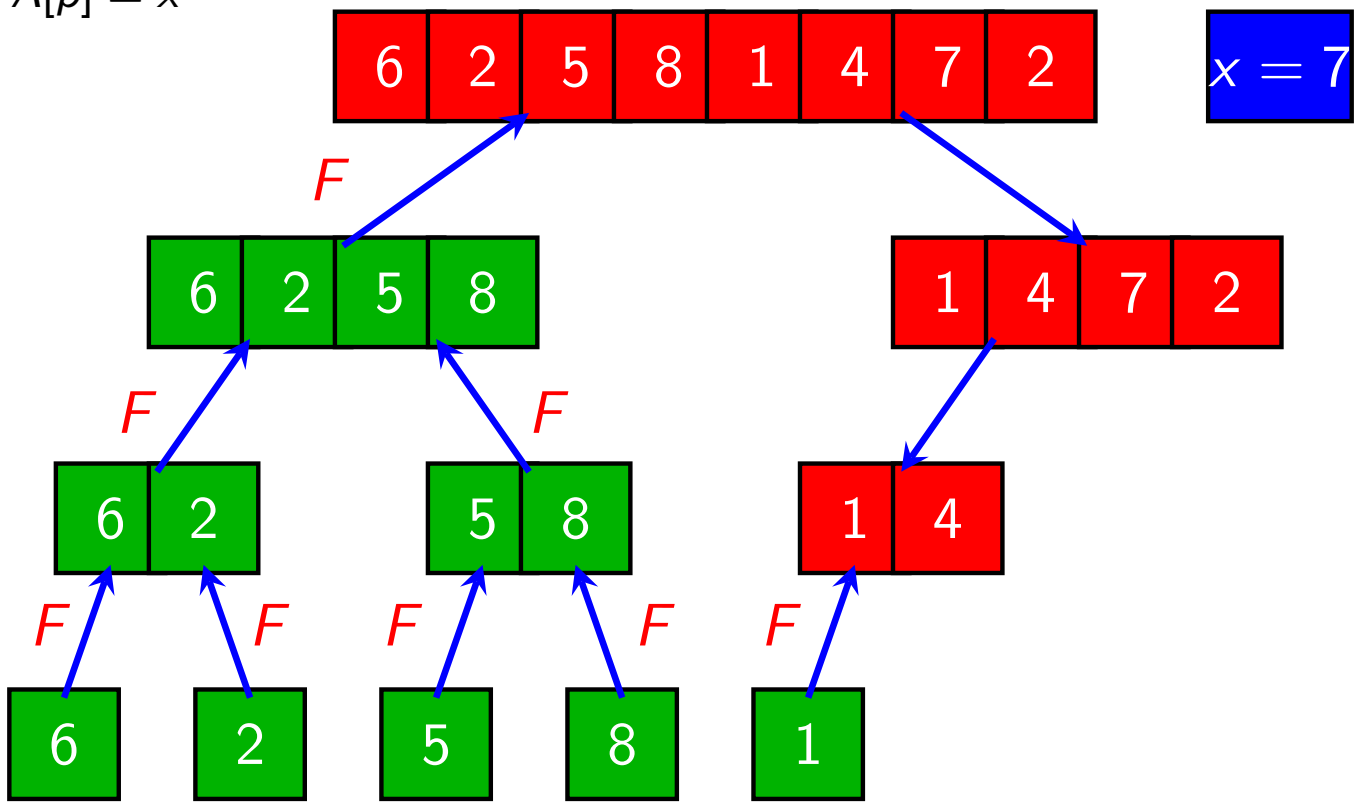

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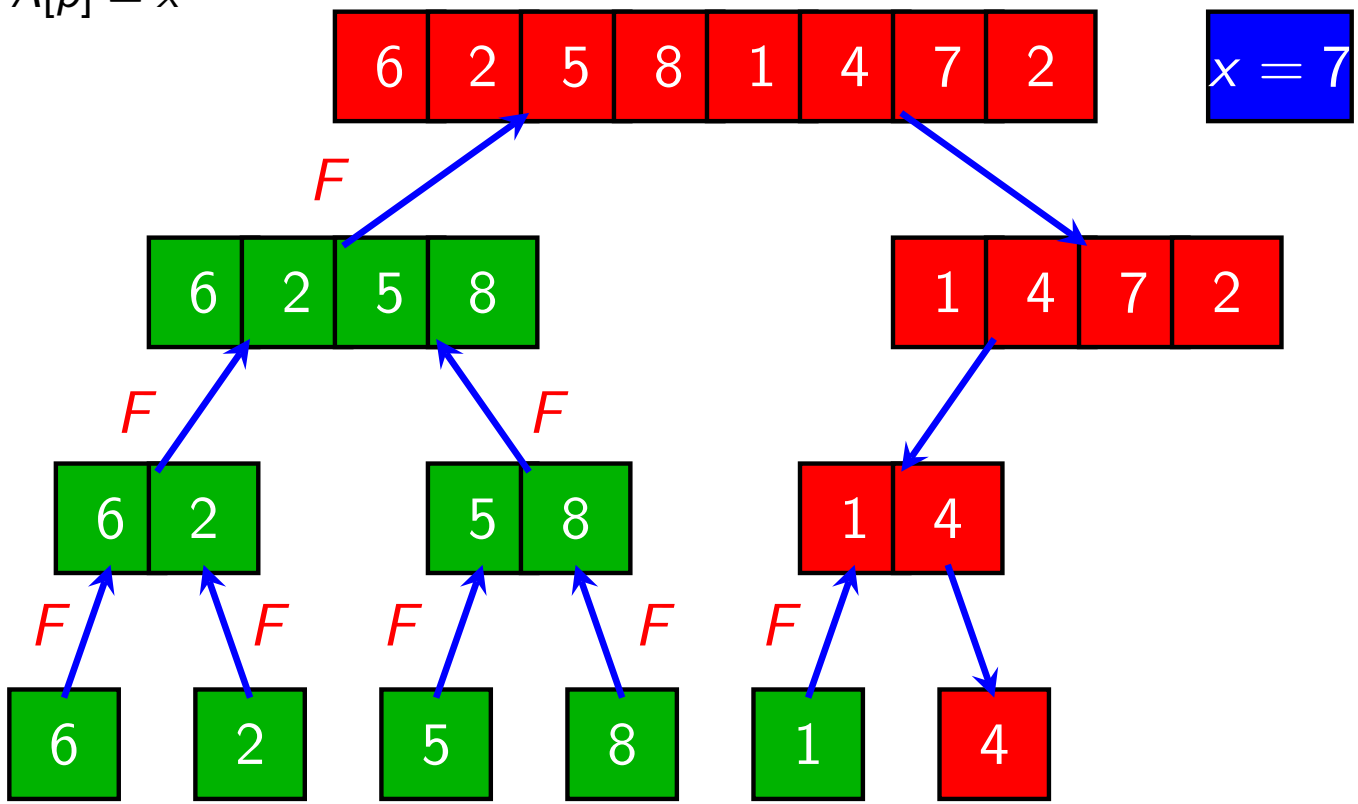

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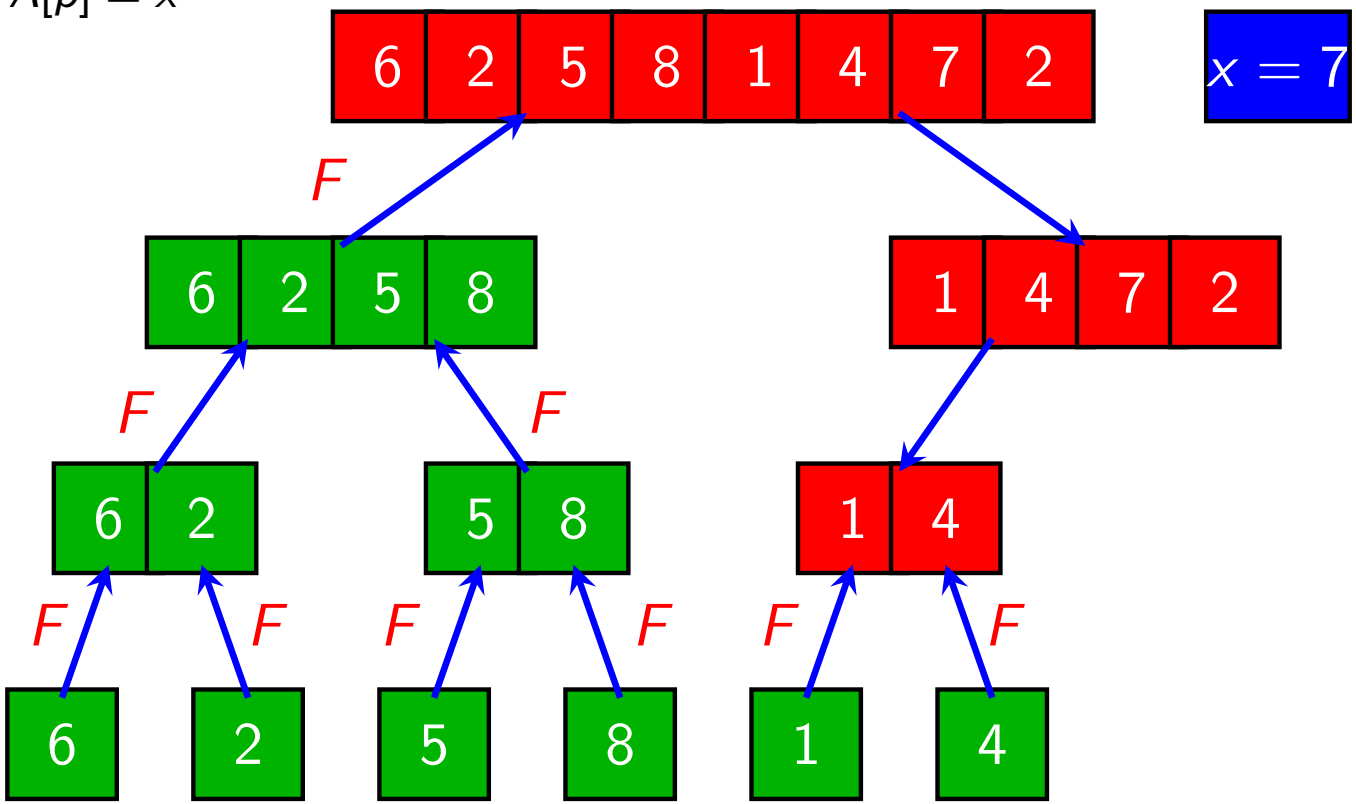

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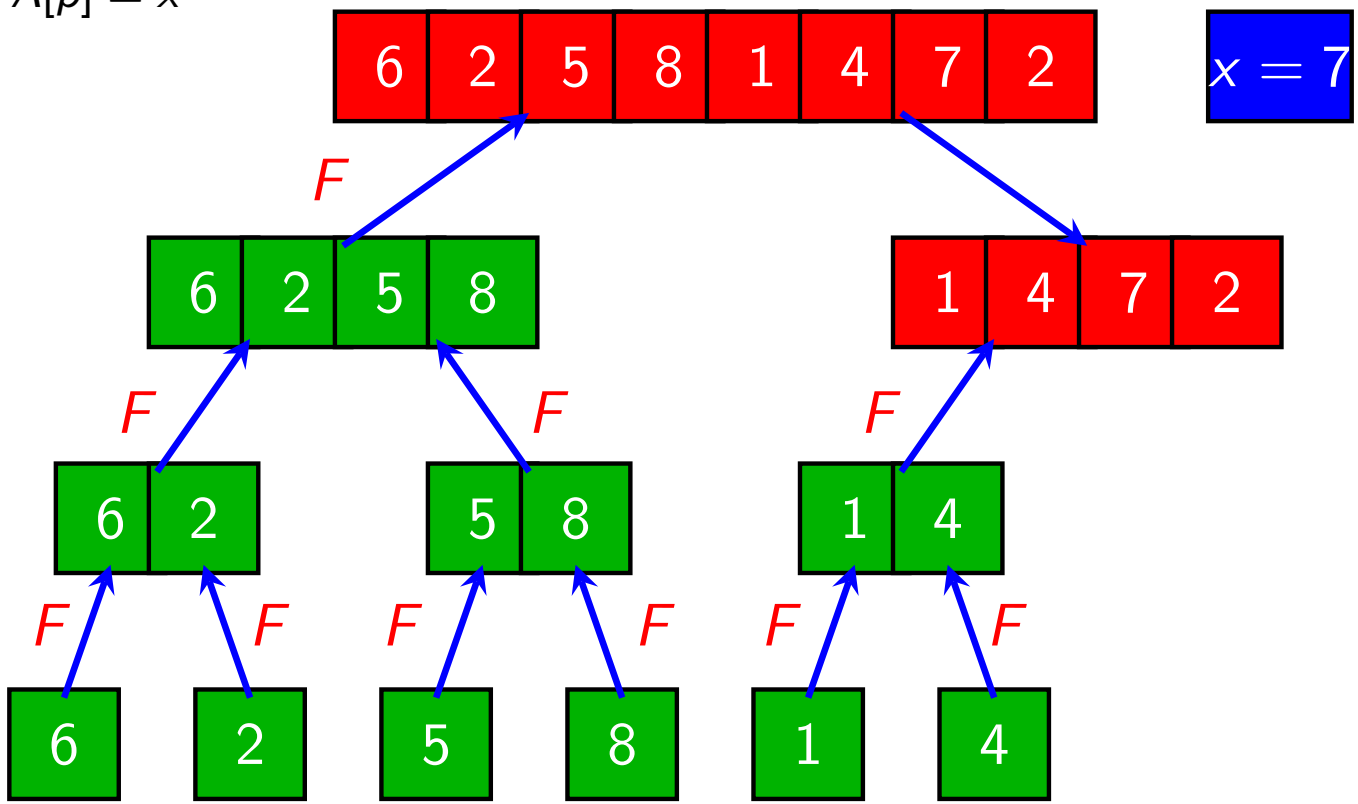

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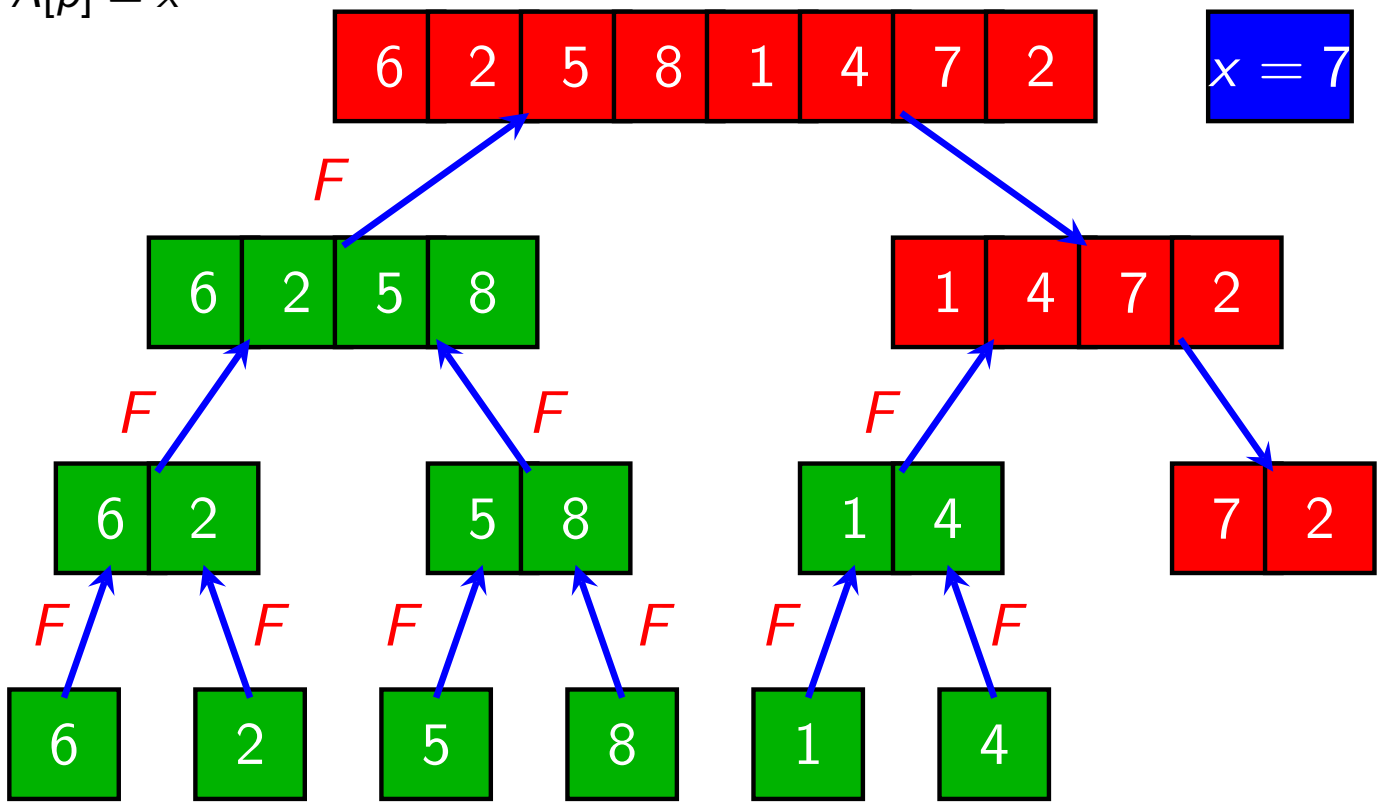

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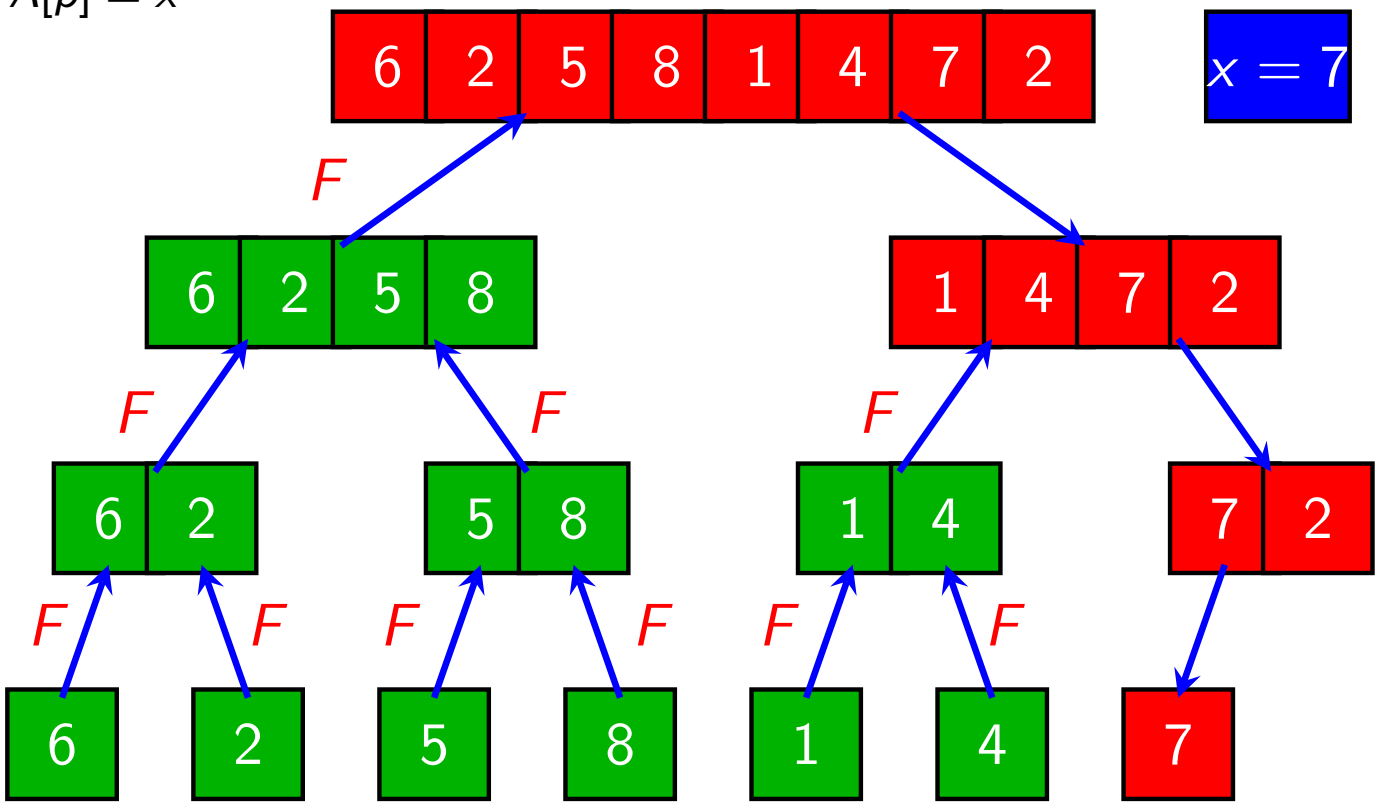

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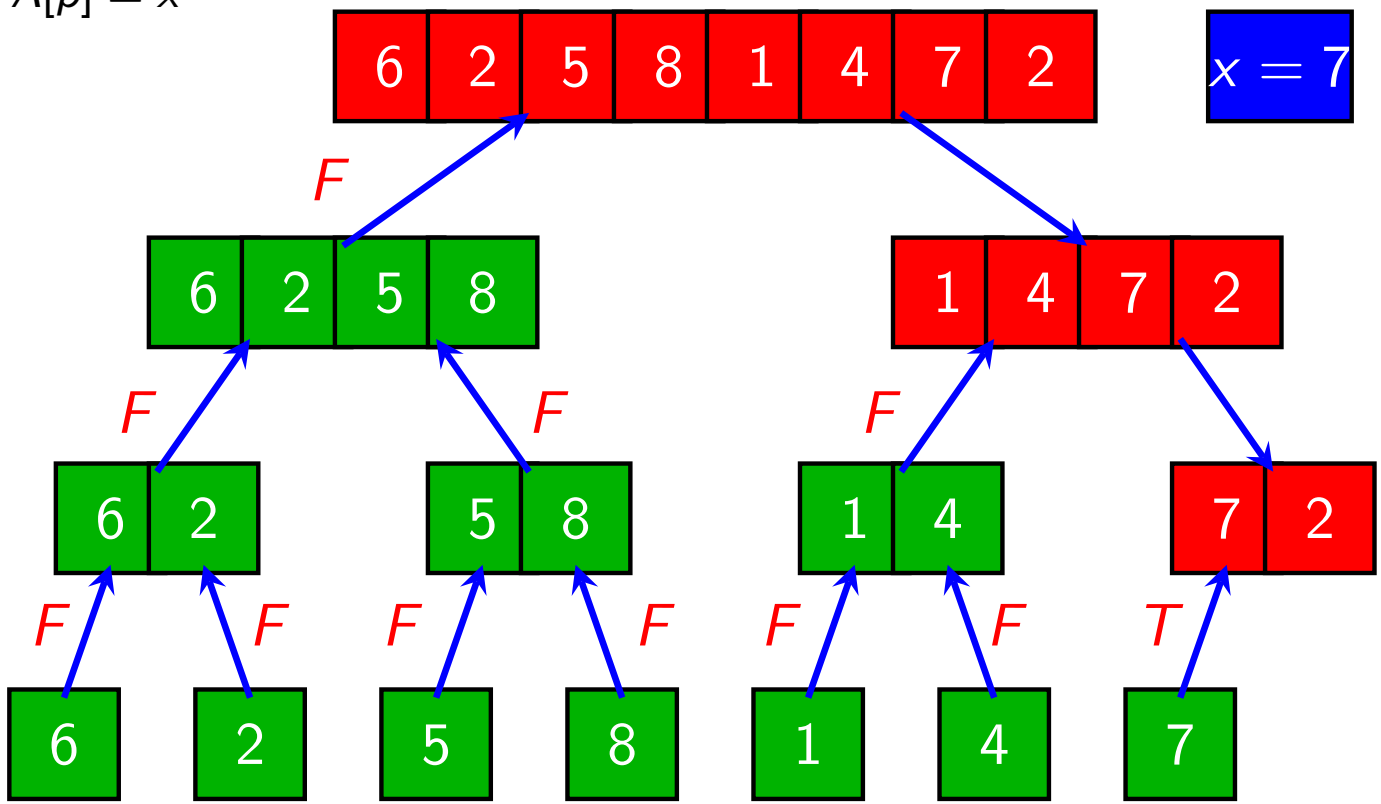

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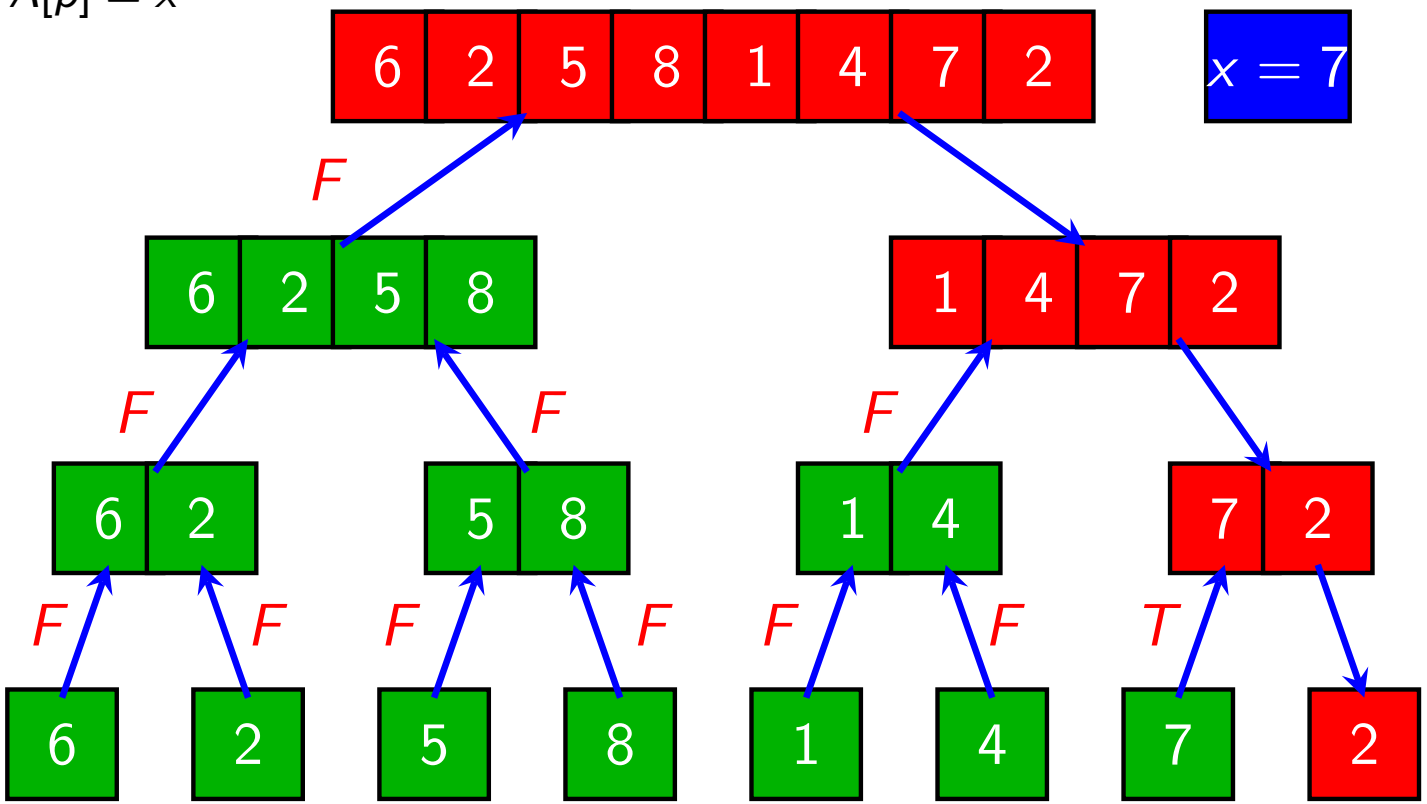

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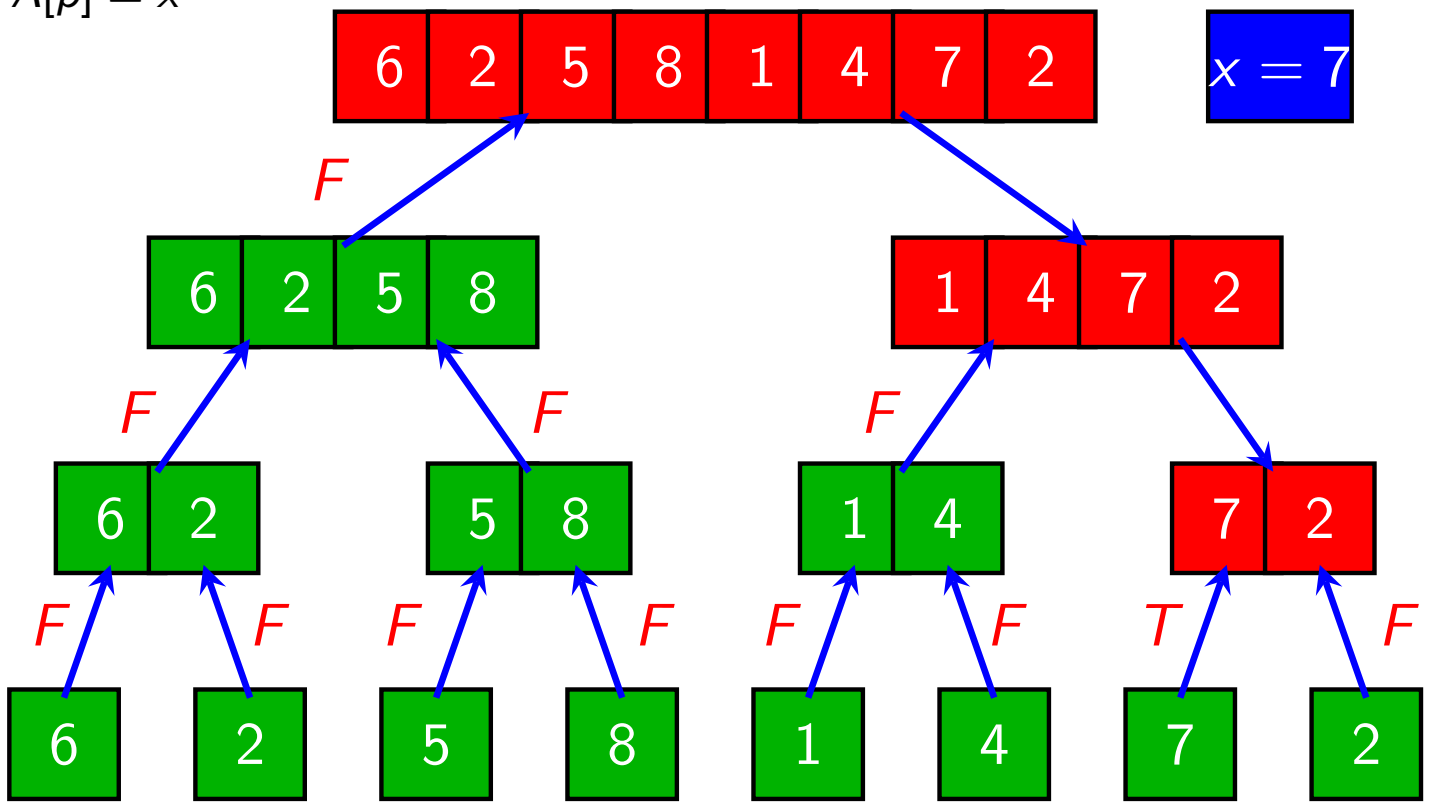

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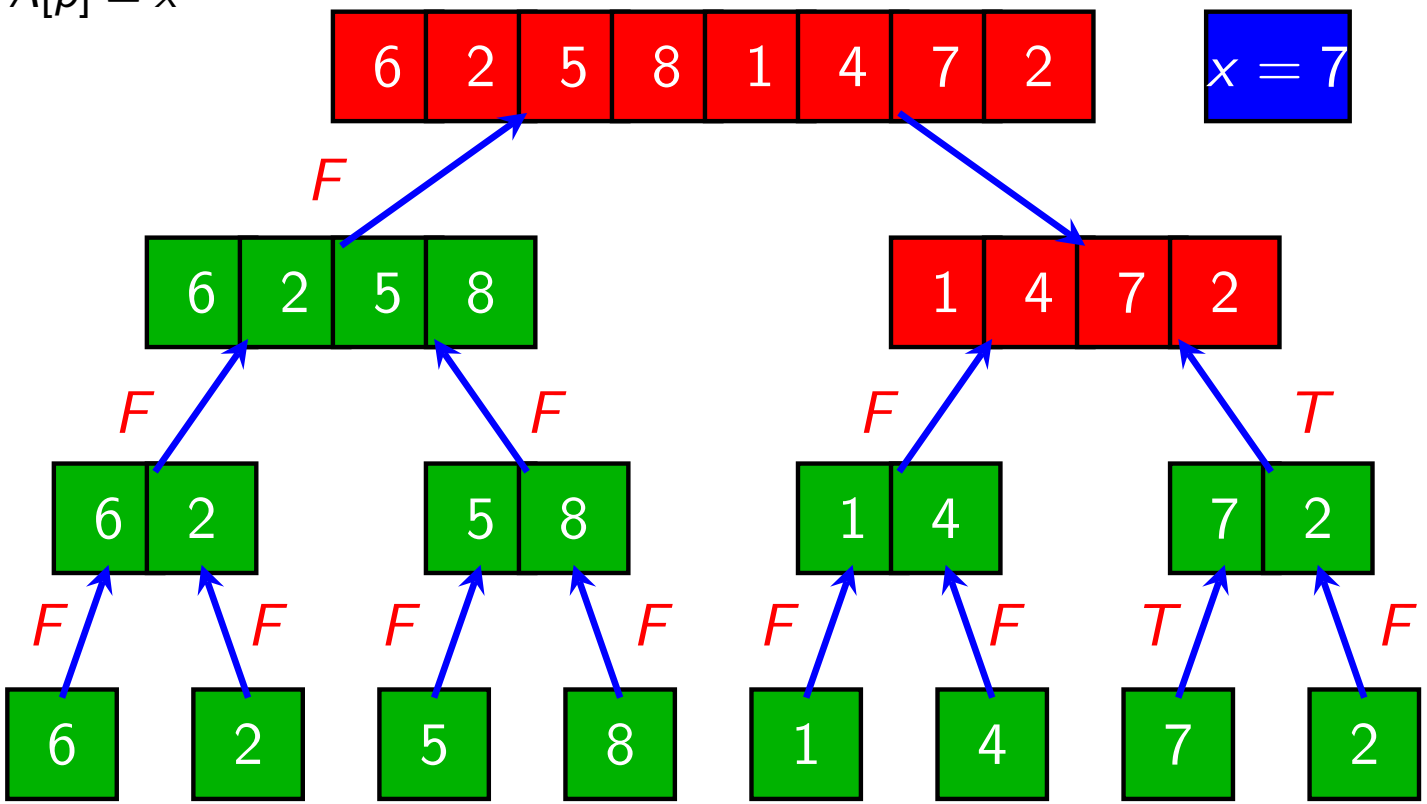

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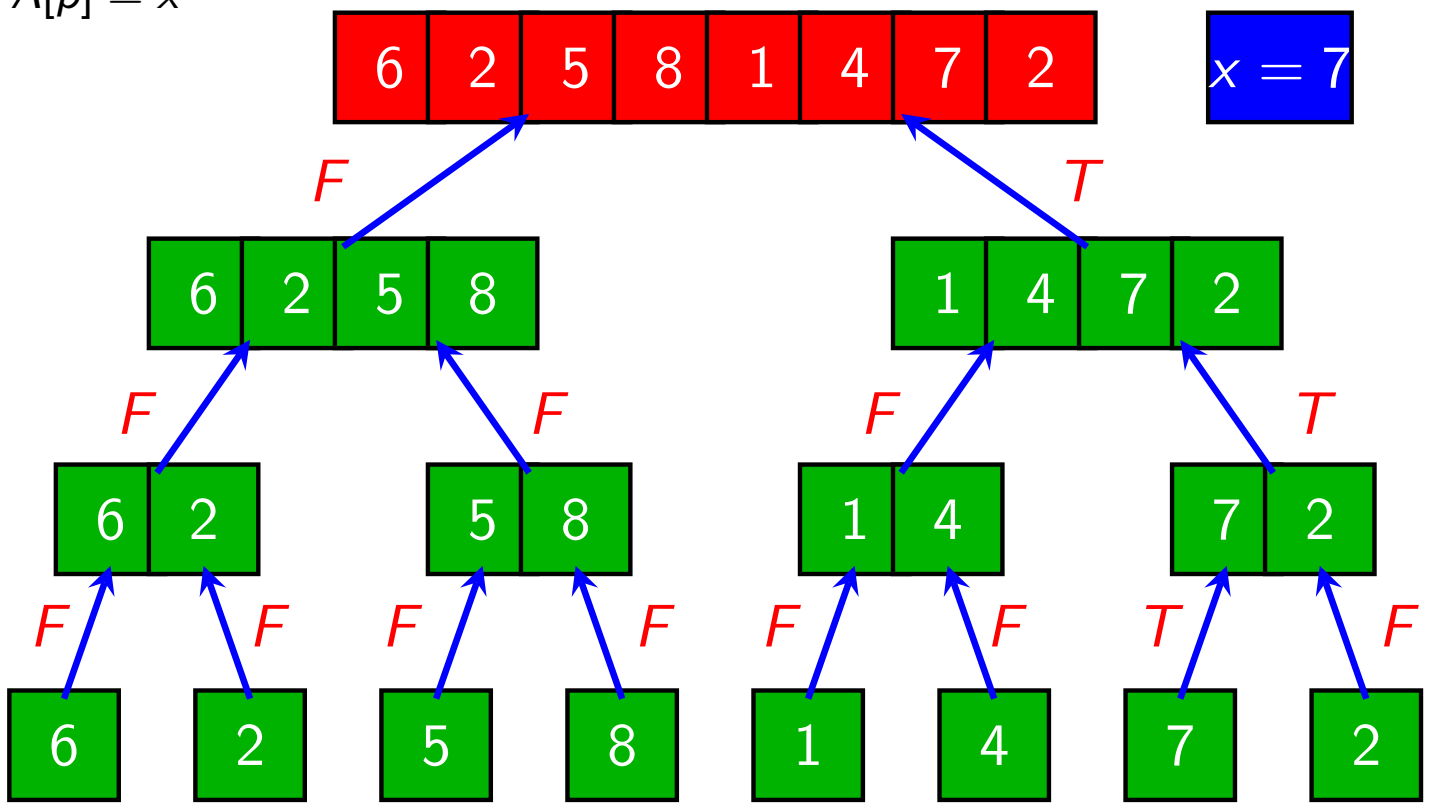

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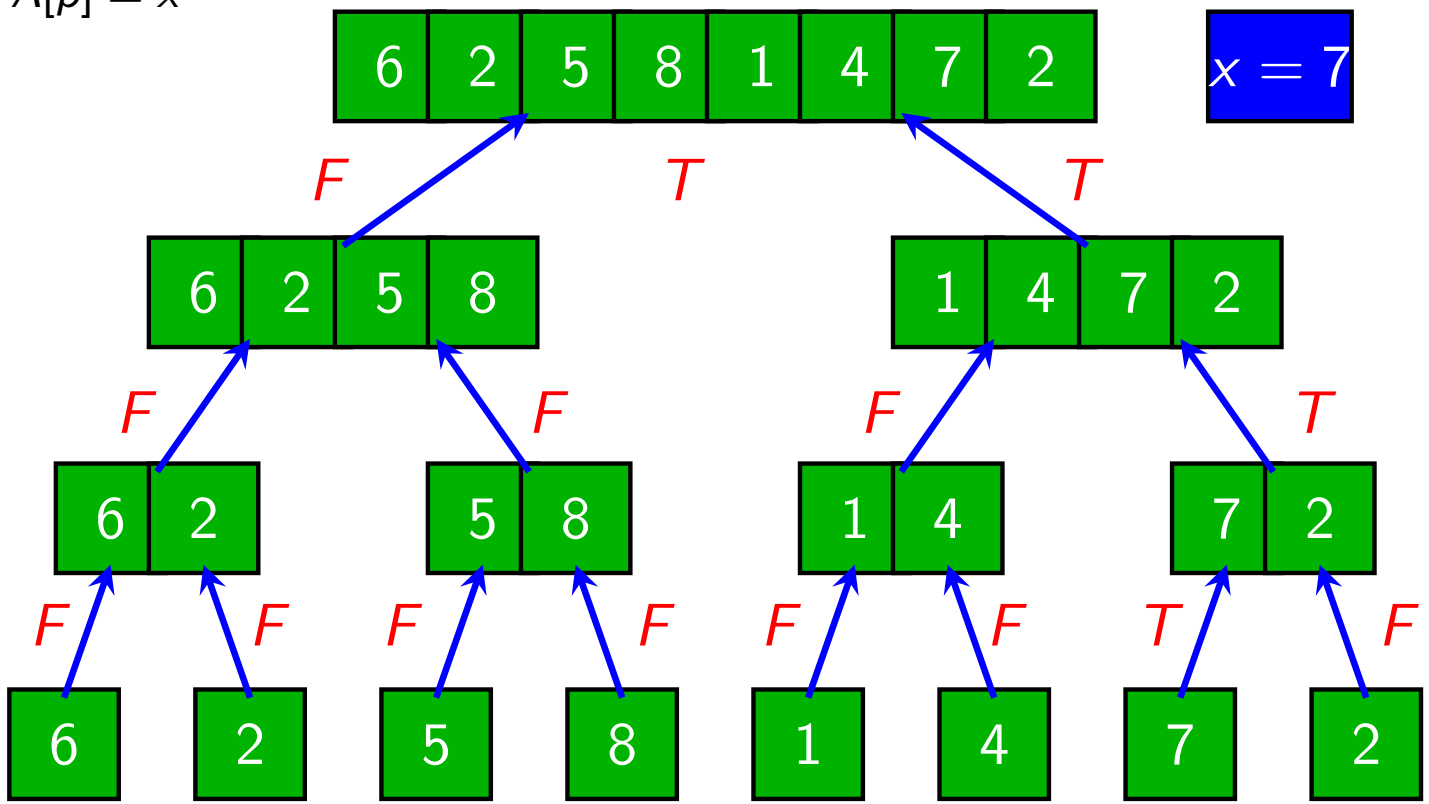

Merge Sort

- Problem: Sort an array A of n elements in non-decreasing order
- Divide: Divide the input array into two halves of length $n/2$ each (or as close as possible).
- Conquer: Sort each of the two subarrays recursively using merge sort.
- Combine: Merge the two sorted subarrays into a single array which is a sorted permutation of the original array.
- Base Cases: The size of A is one. Then, A is trivially sorted.

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The Merge Procedure

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- Input:
  - $A[p..r]$: subarray.
  - $p \leq q < r$.
  - The two subarrays $A[p..q]$ and $A[q+1..r]$ are individually sorted.
- Task:
  - Merge the two subarrays into a single sorted array, which replaces the input subarray $A[p..q]$.

Merge Sort

MERGE(A, p, q, r)

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 1  n1 ← q − p + 1
 2  n2 ← r − q
 3  create arrays L[1 . . n1 + 1]
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 4  for i ← 1 to n1
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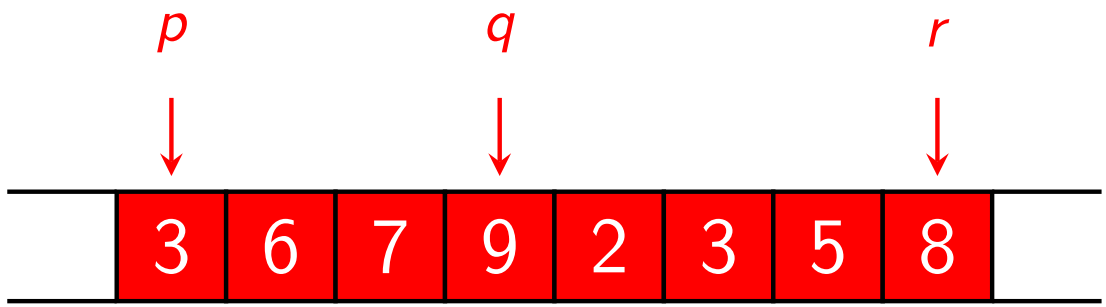

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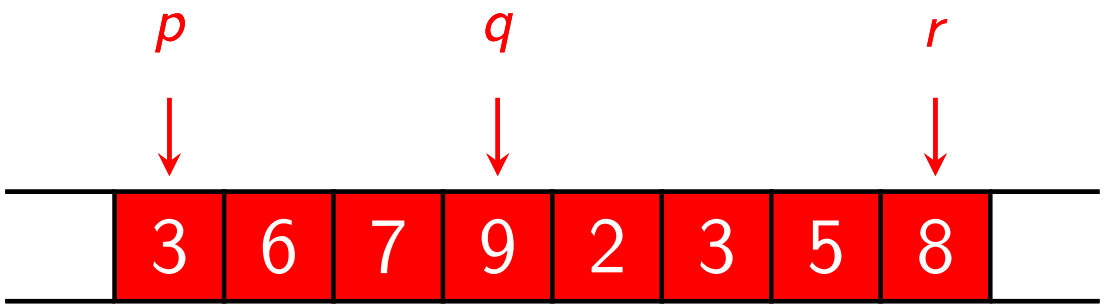

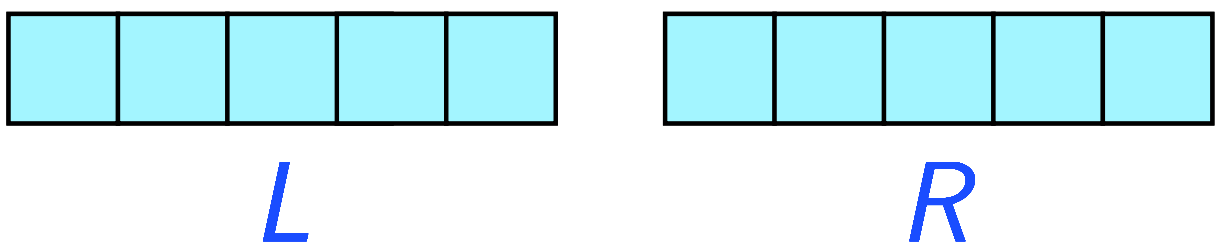

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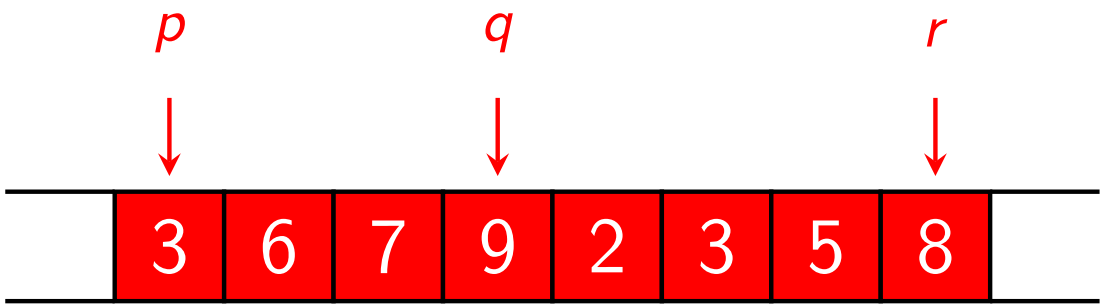

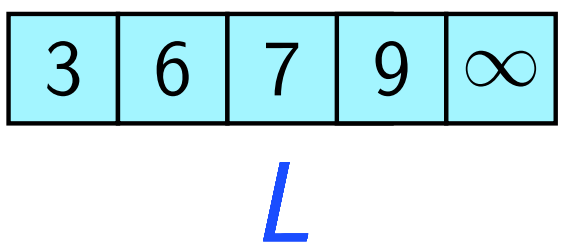

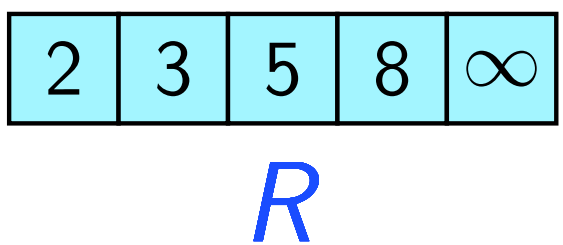

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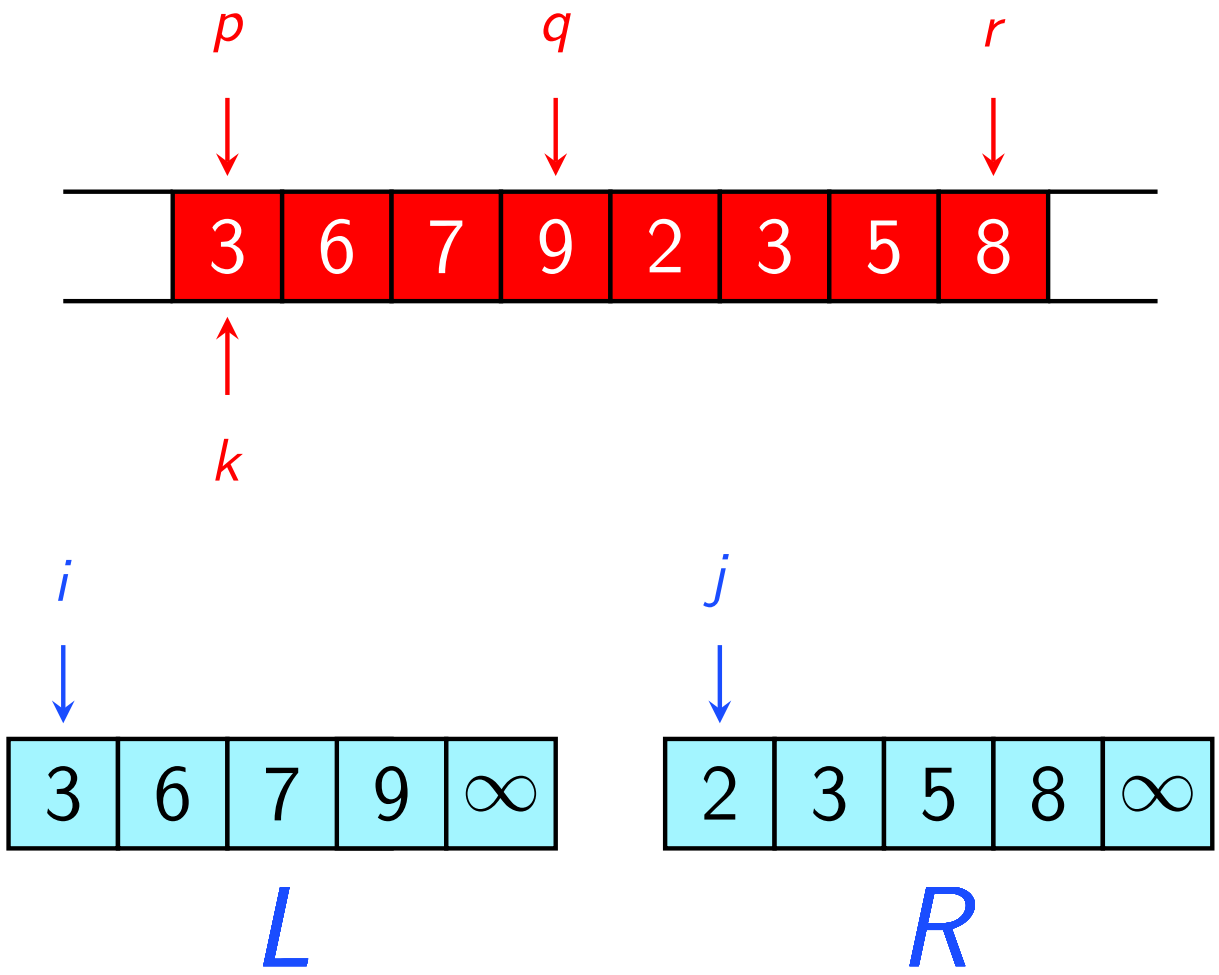

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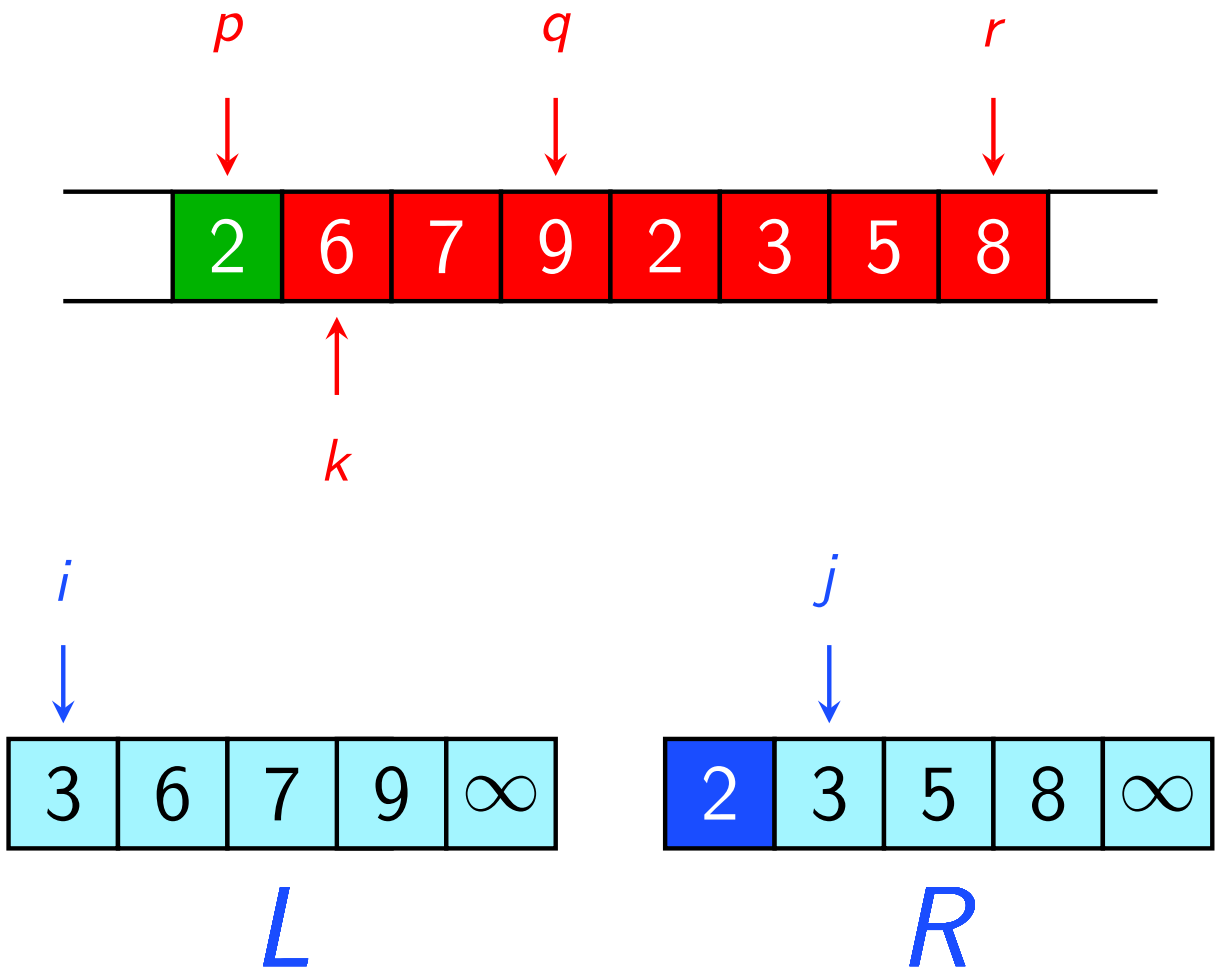

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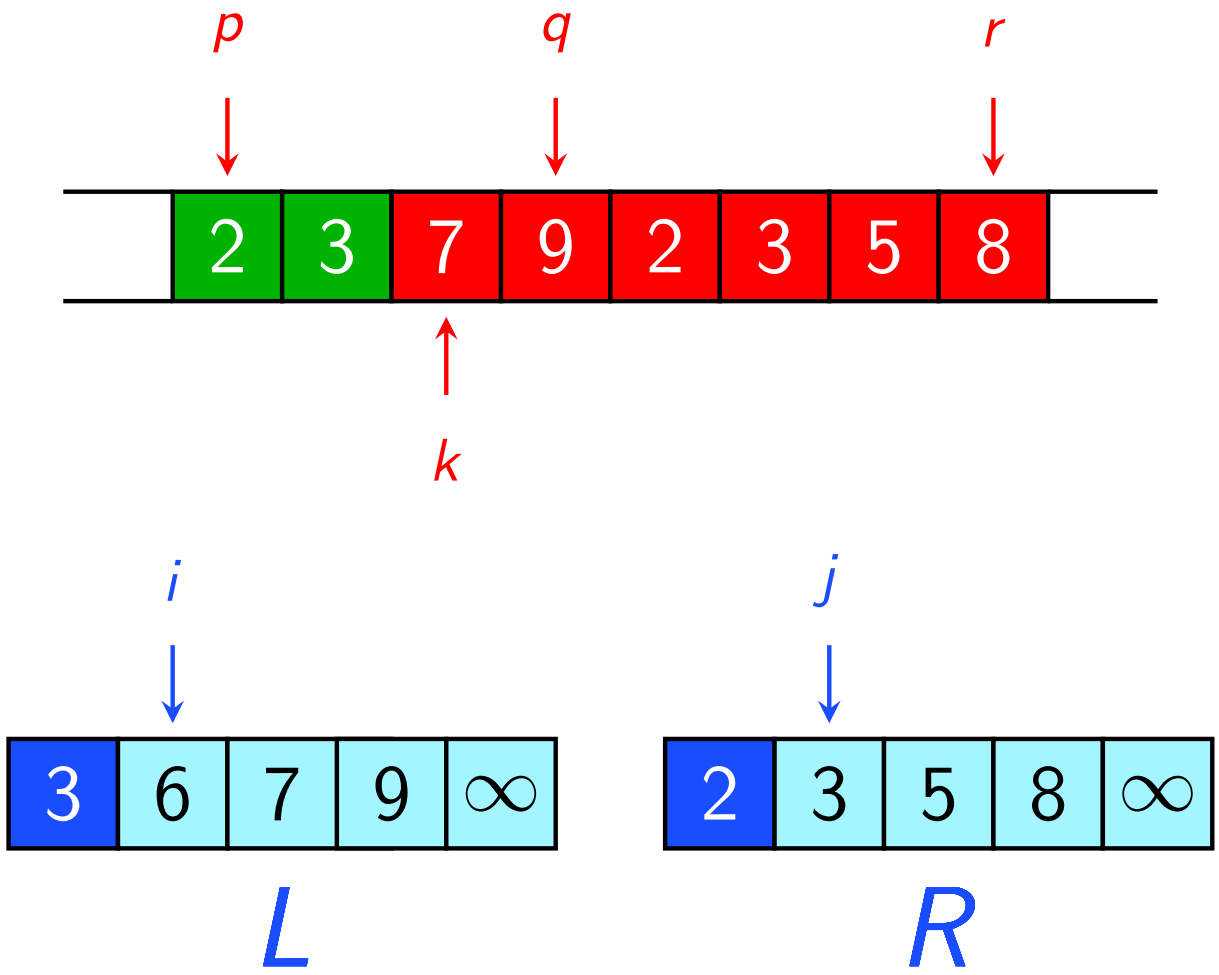

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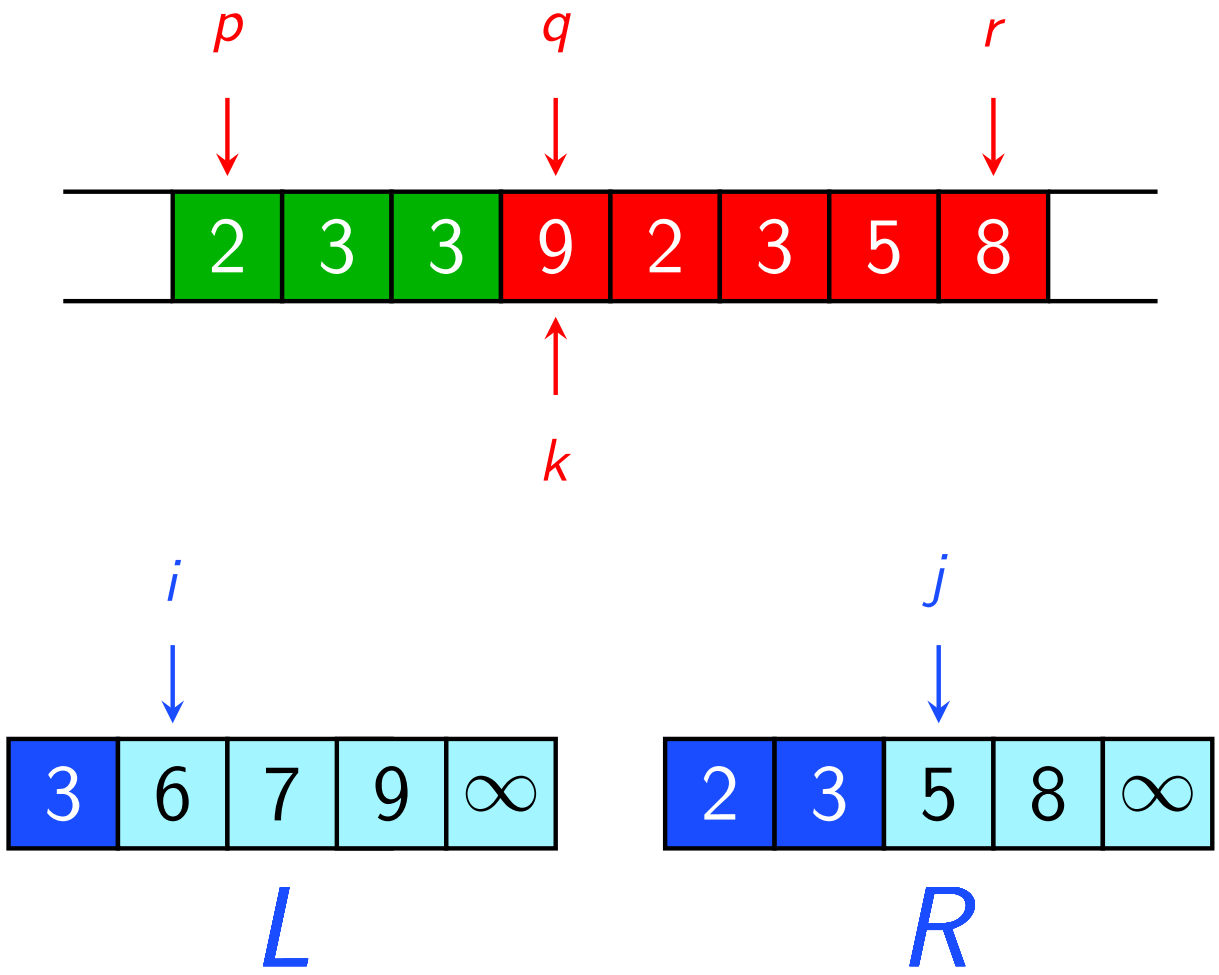

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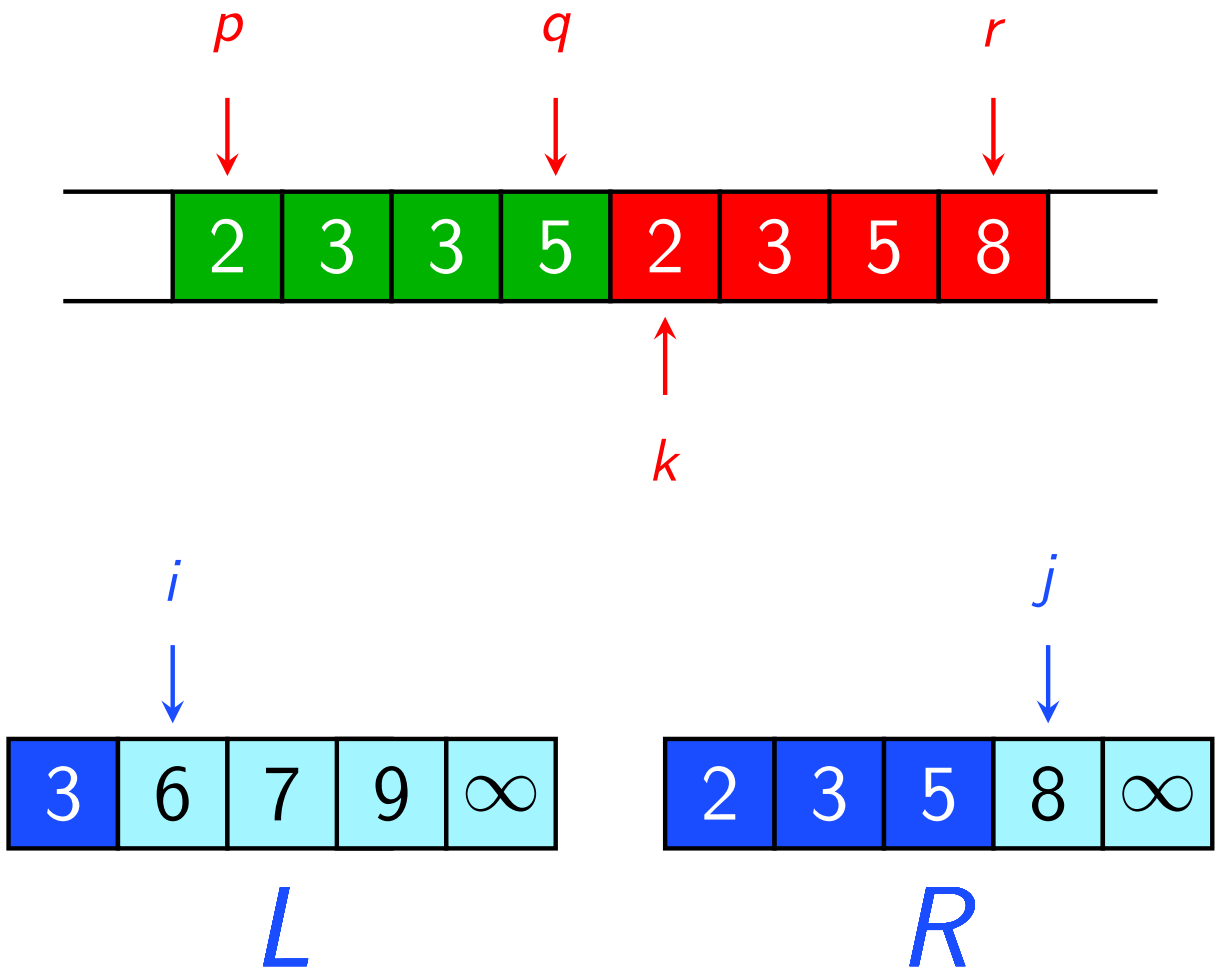

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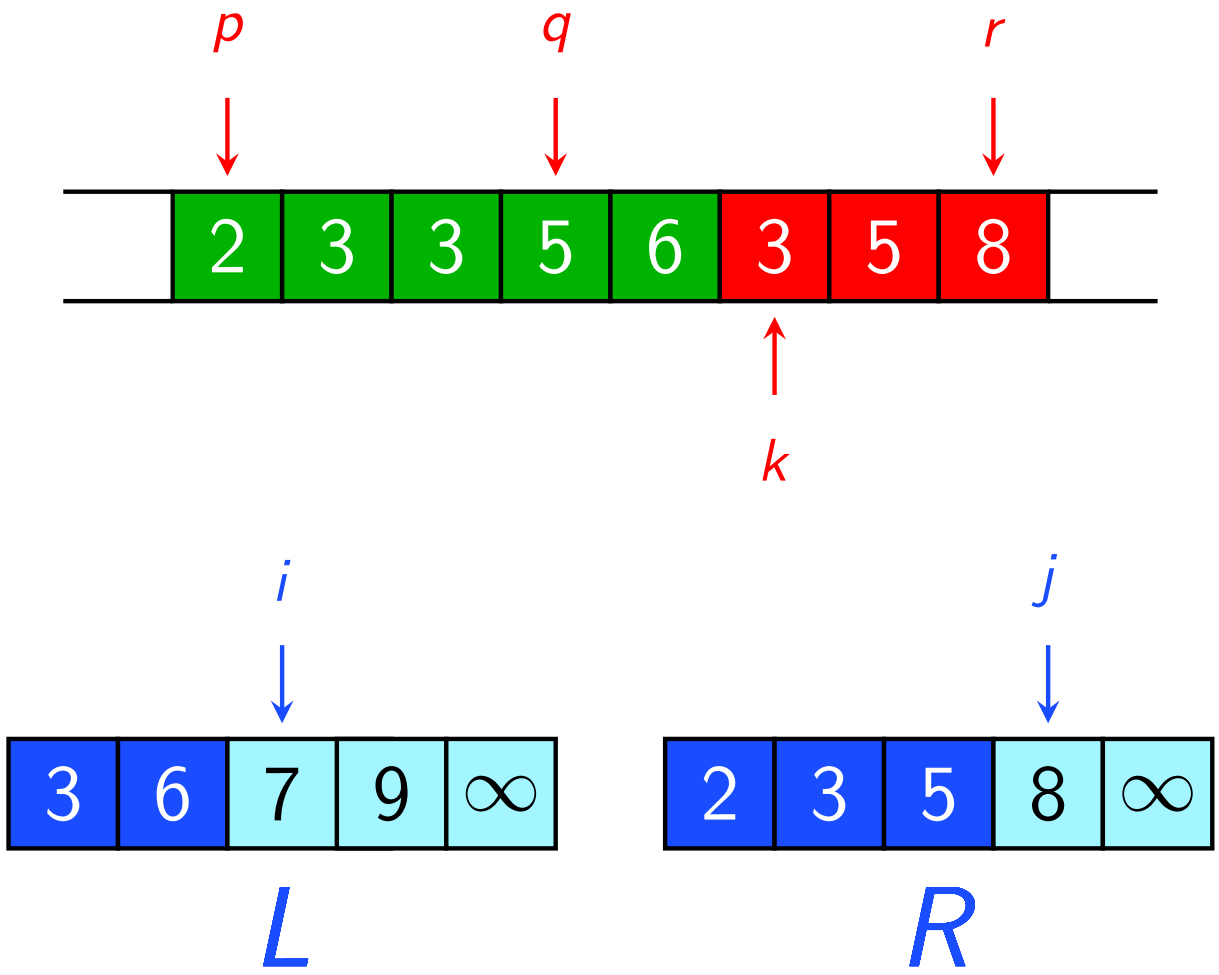

Merge Sort

MERGE(A, p, q, r)

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 1  n1 ← q − p + 1
 2  n2 ← r − q
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 4  for i ← 1 to n1
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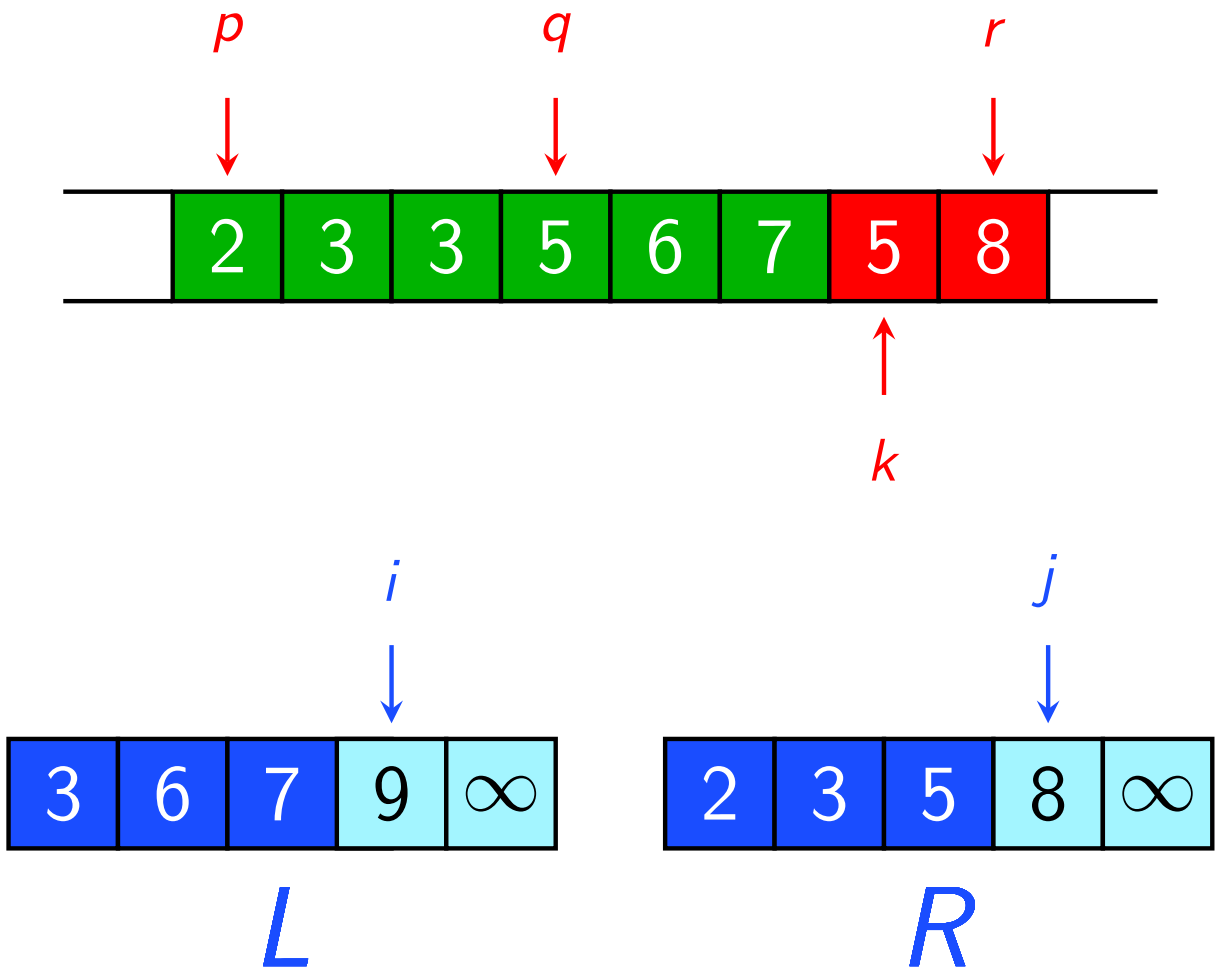

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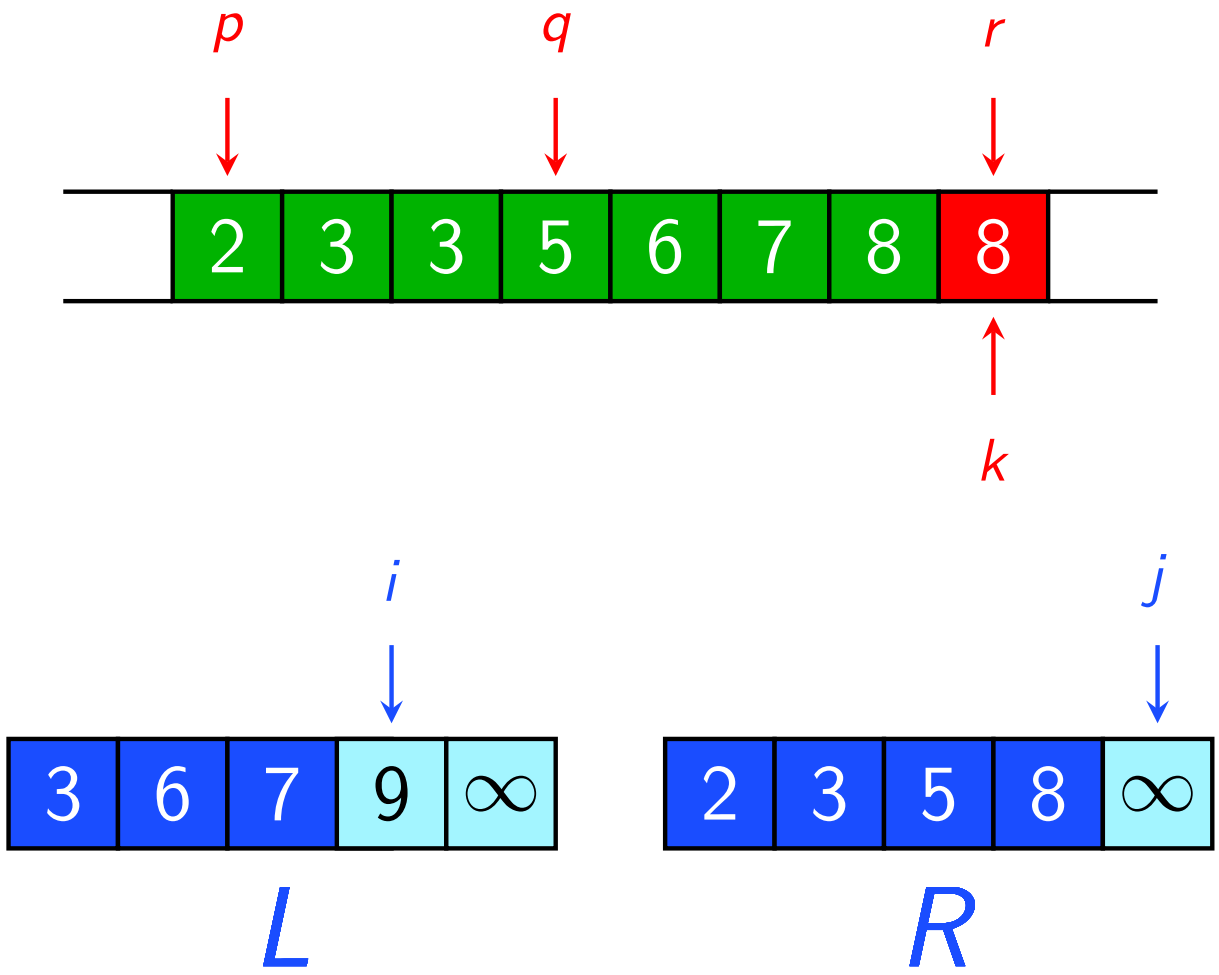

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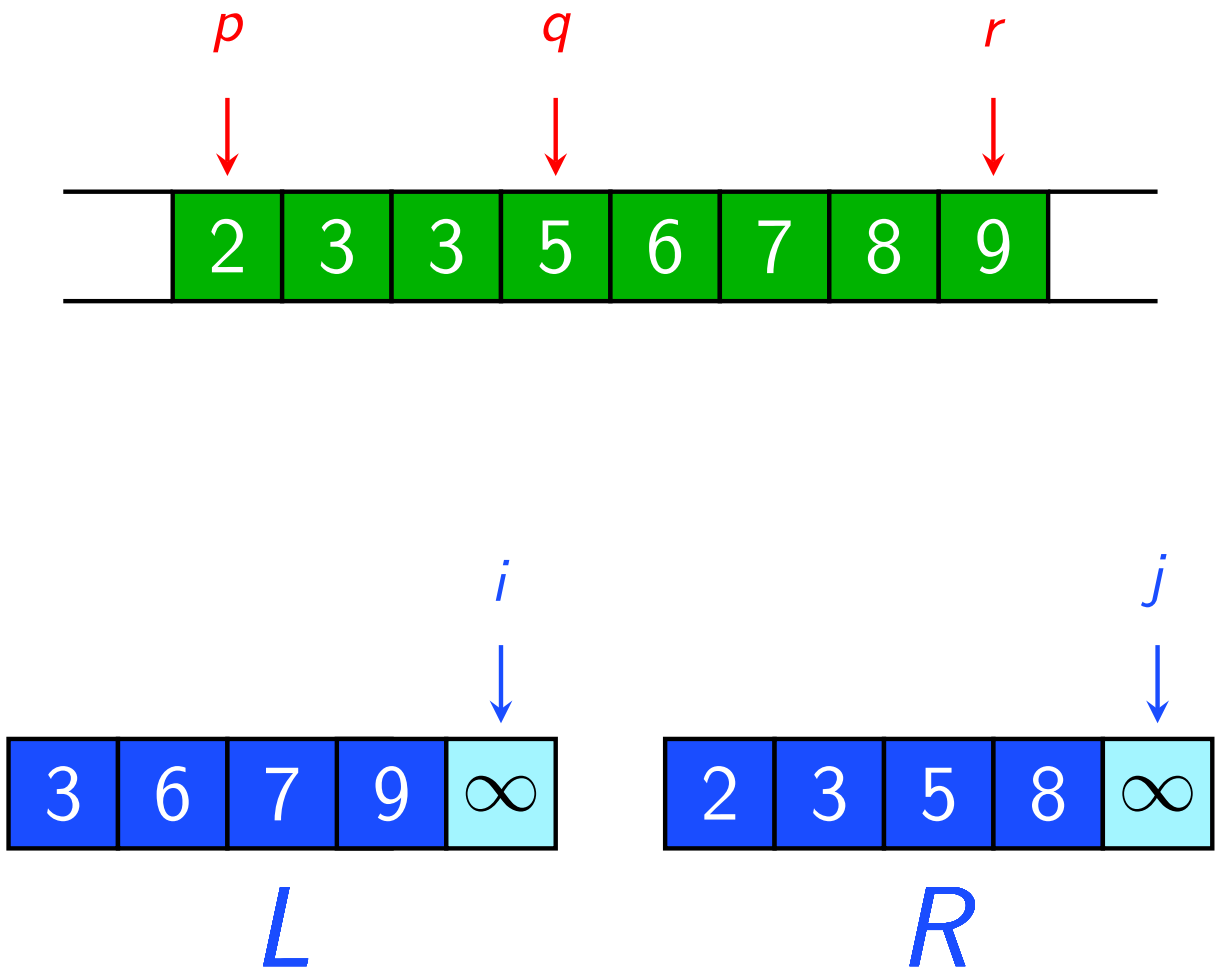

Worst-Case Complexity Analysis of the Merge Procedure

- Let $n = r - p + 1$

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- Each of lines 1-3 and 8-11 takes constant time.

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The Merge Procedure runs in $\Theta(n)$

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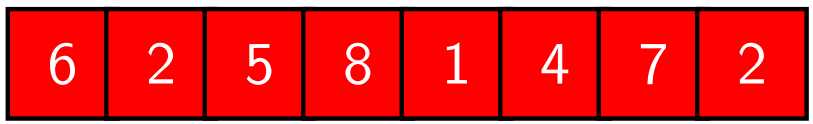

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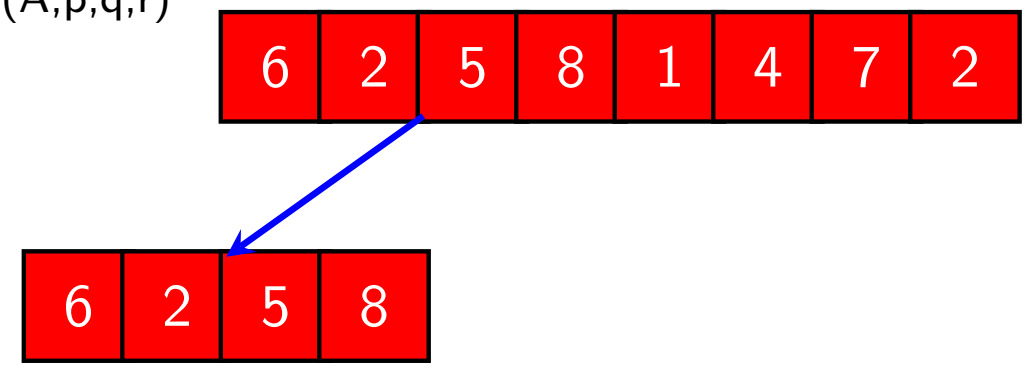

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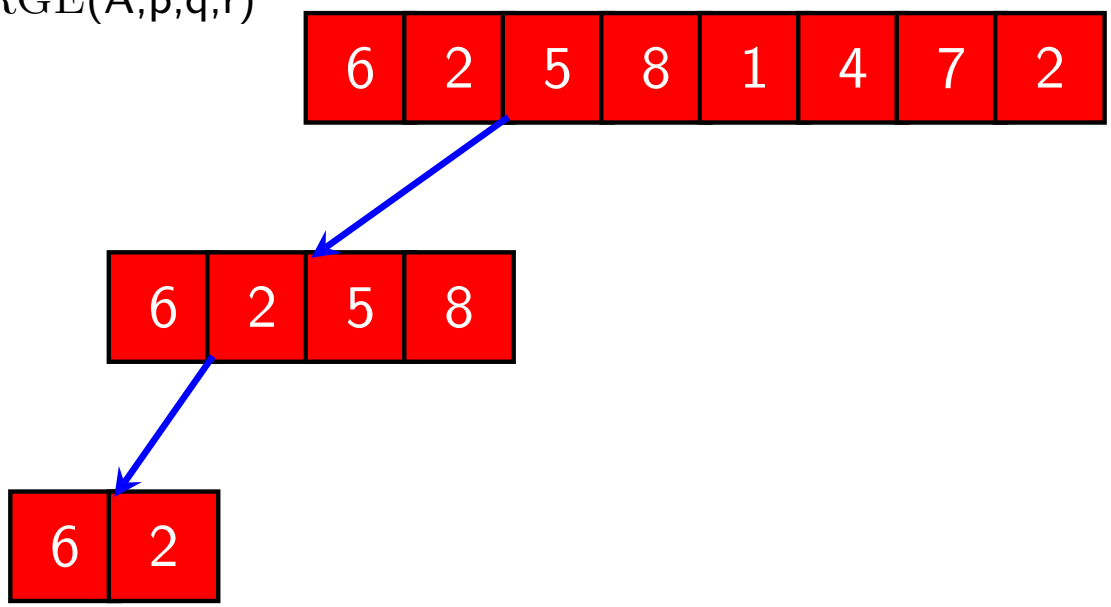

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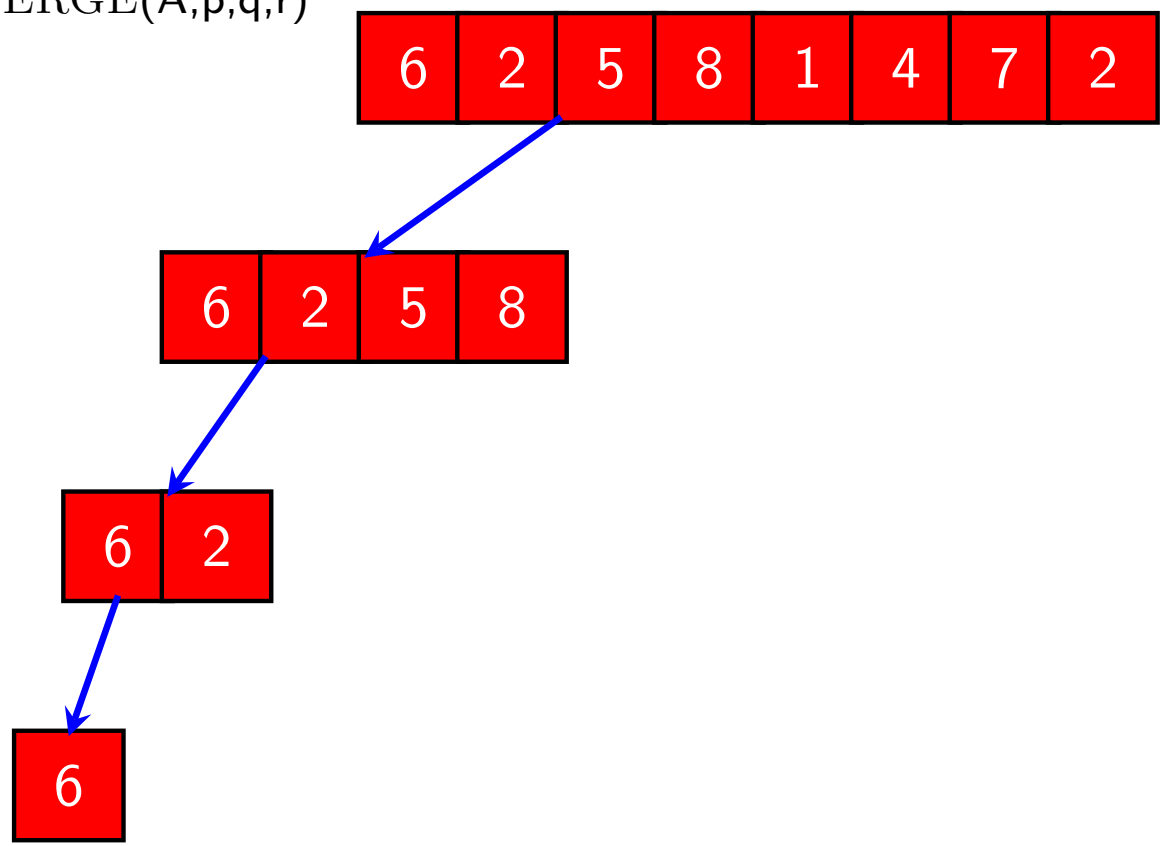

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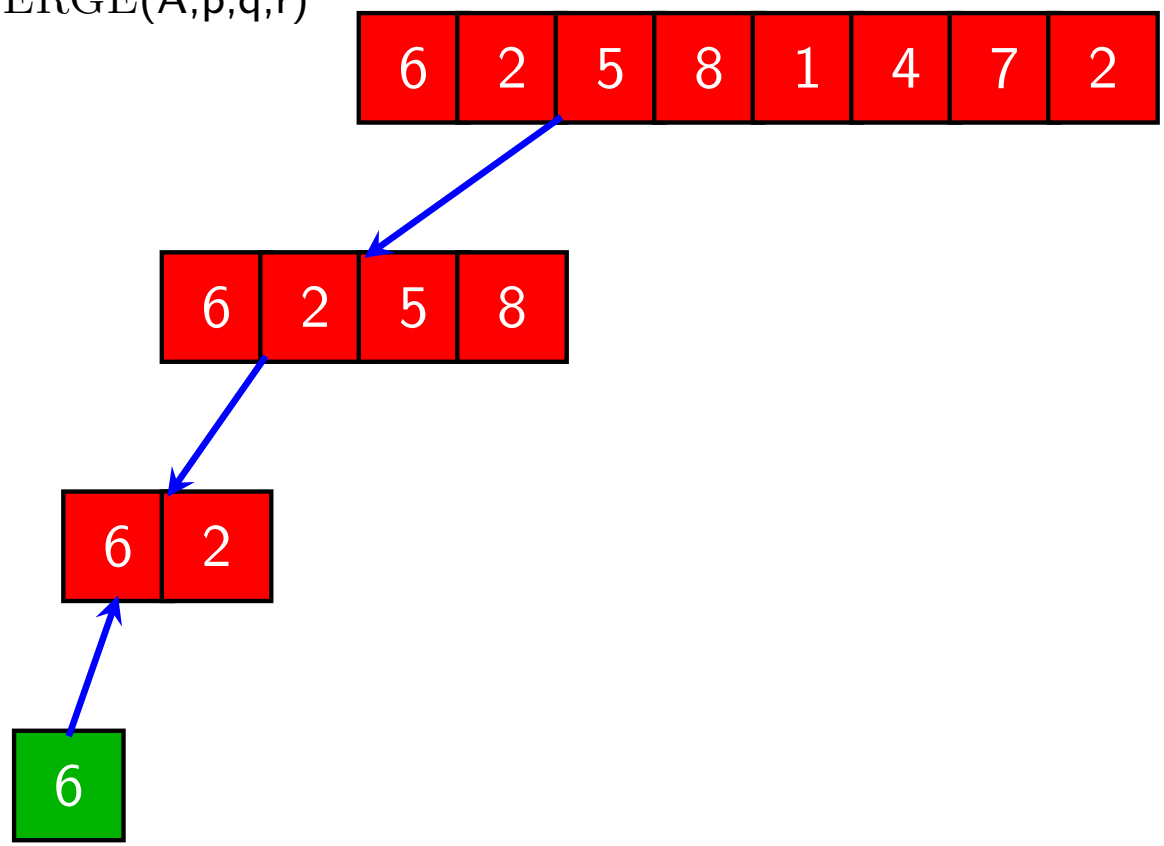

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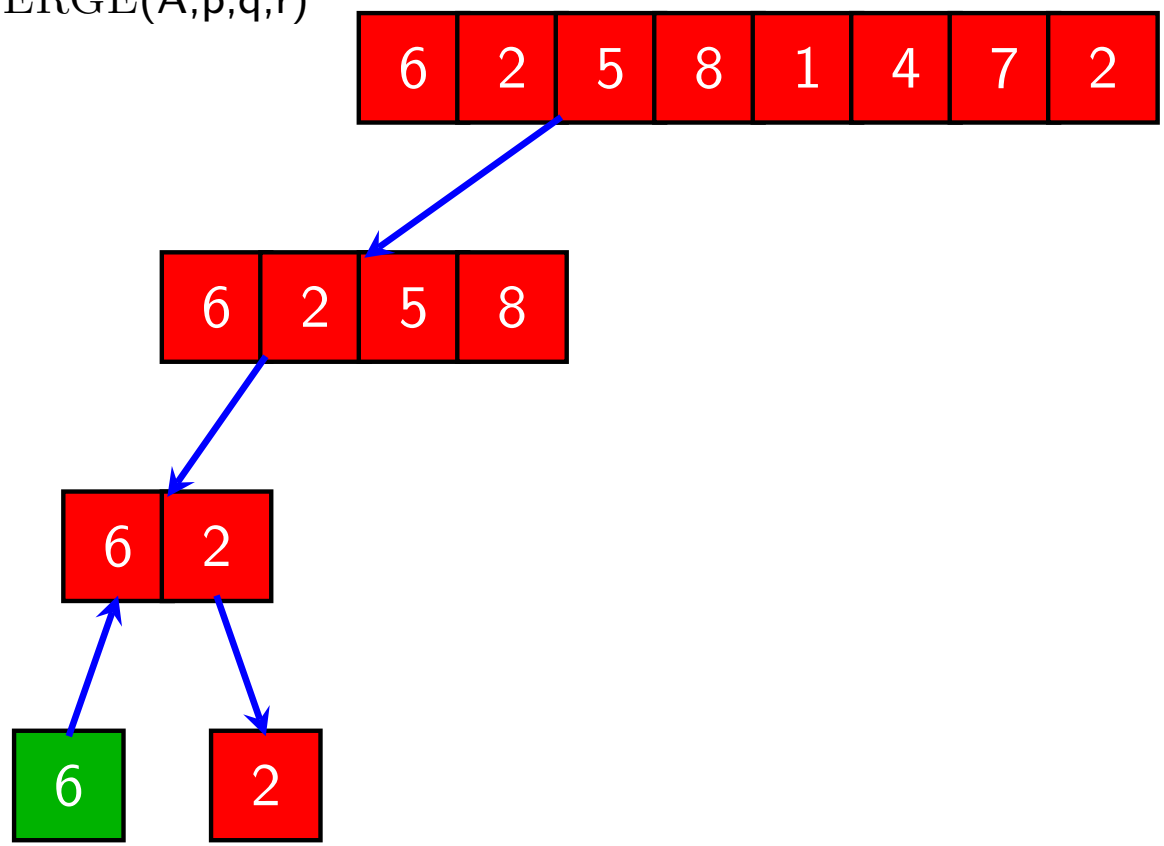

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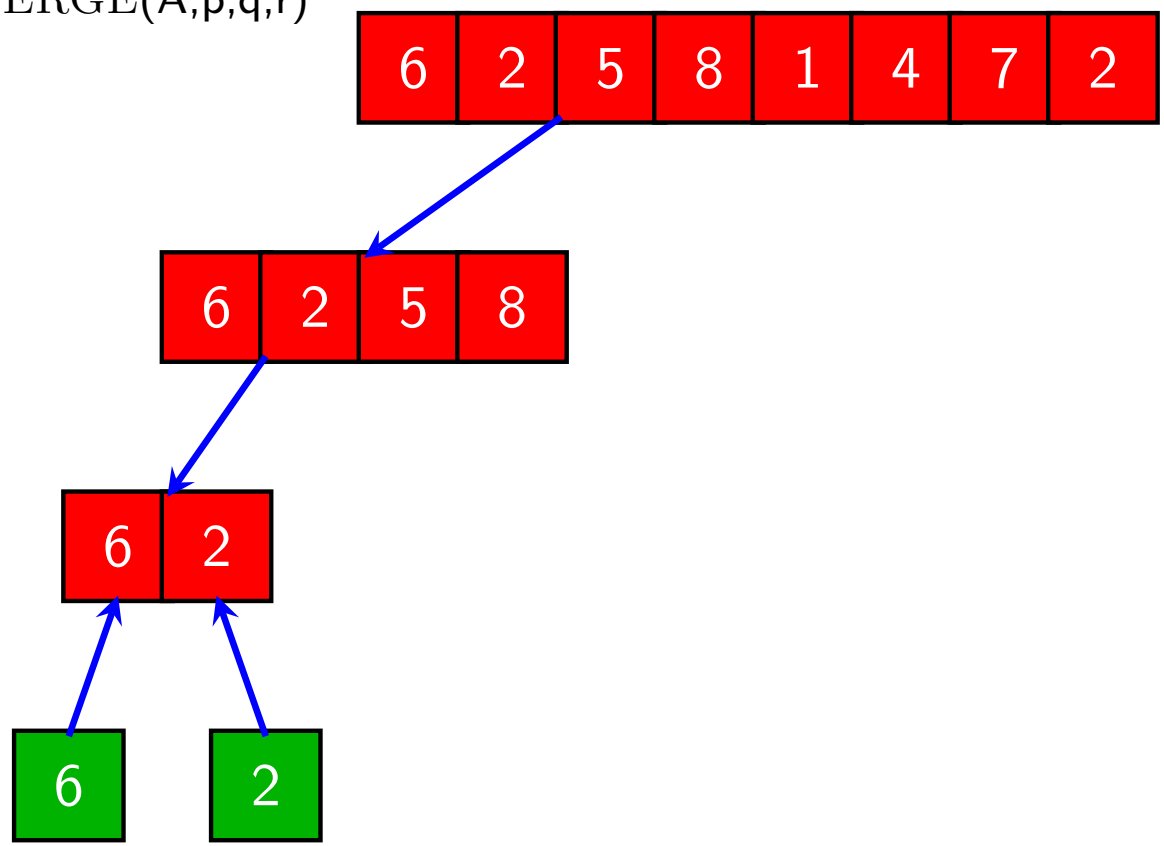

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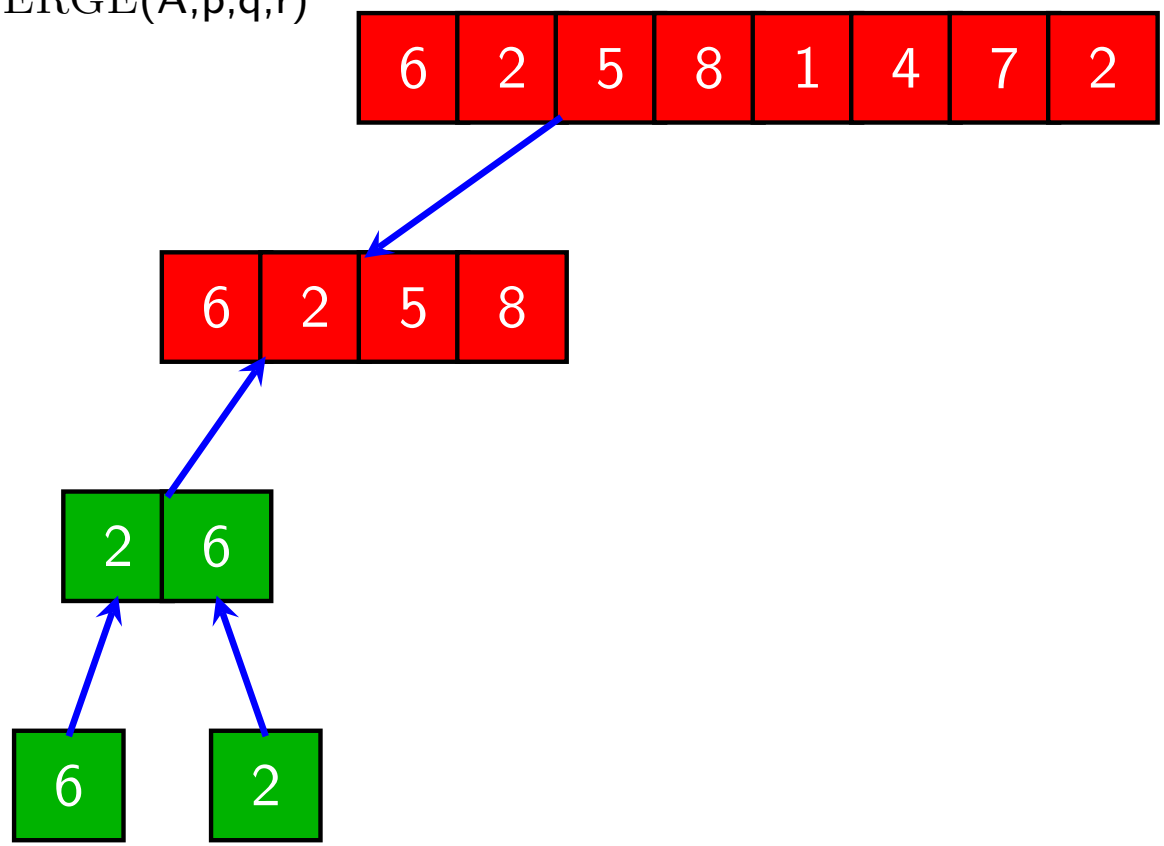

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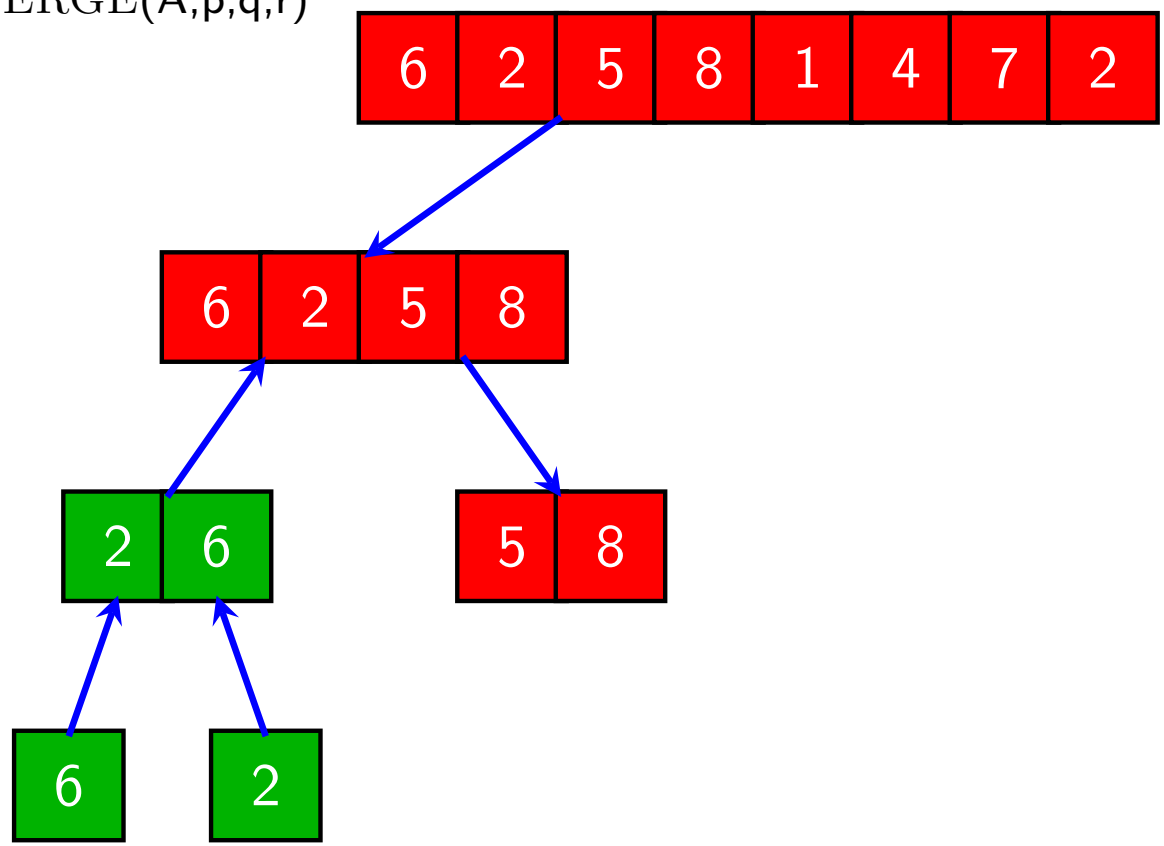

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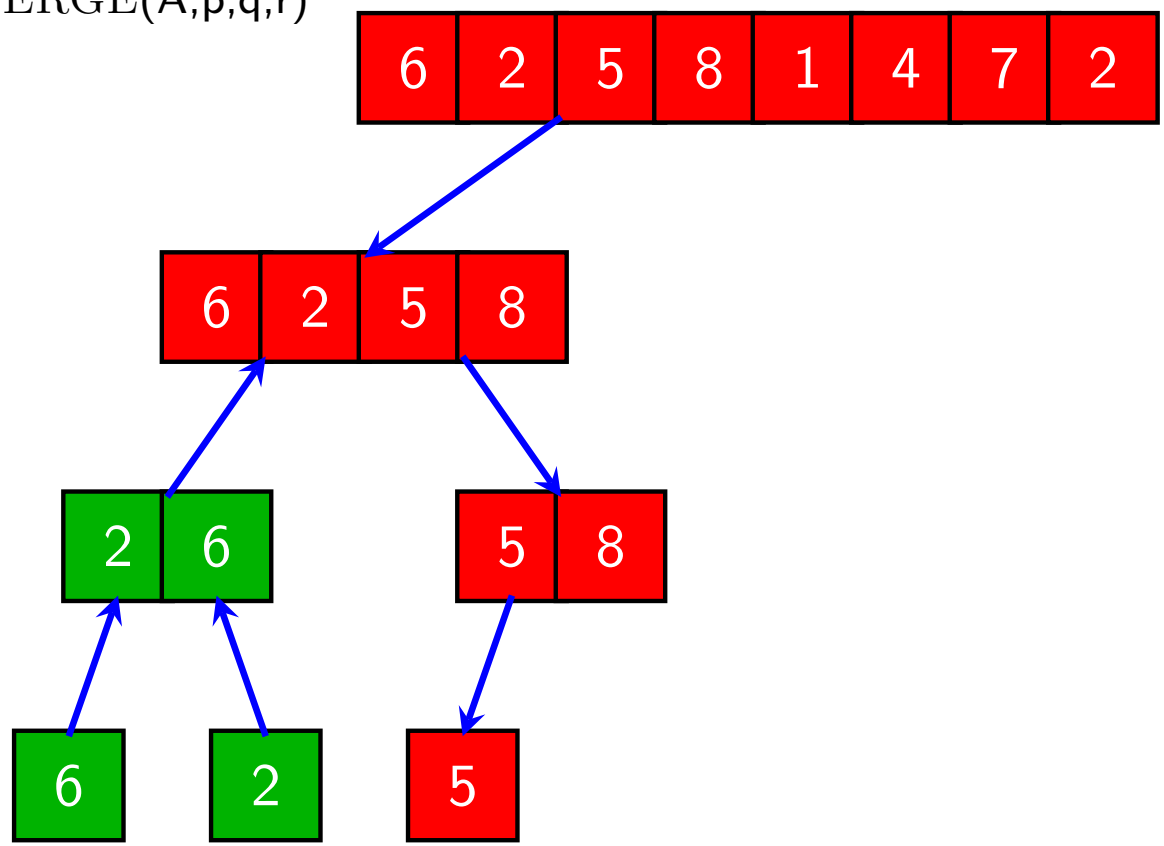

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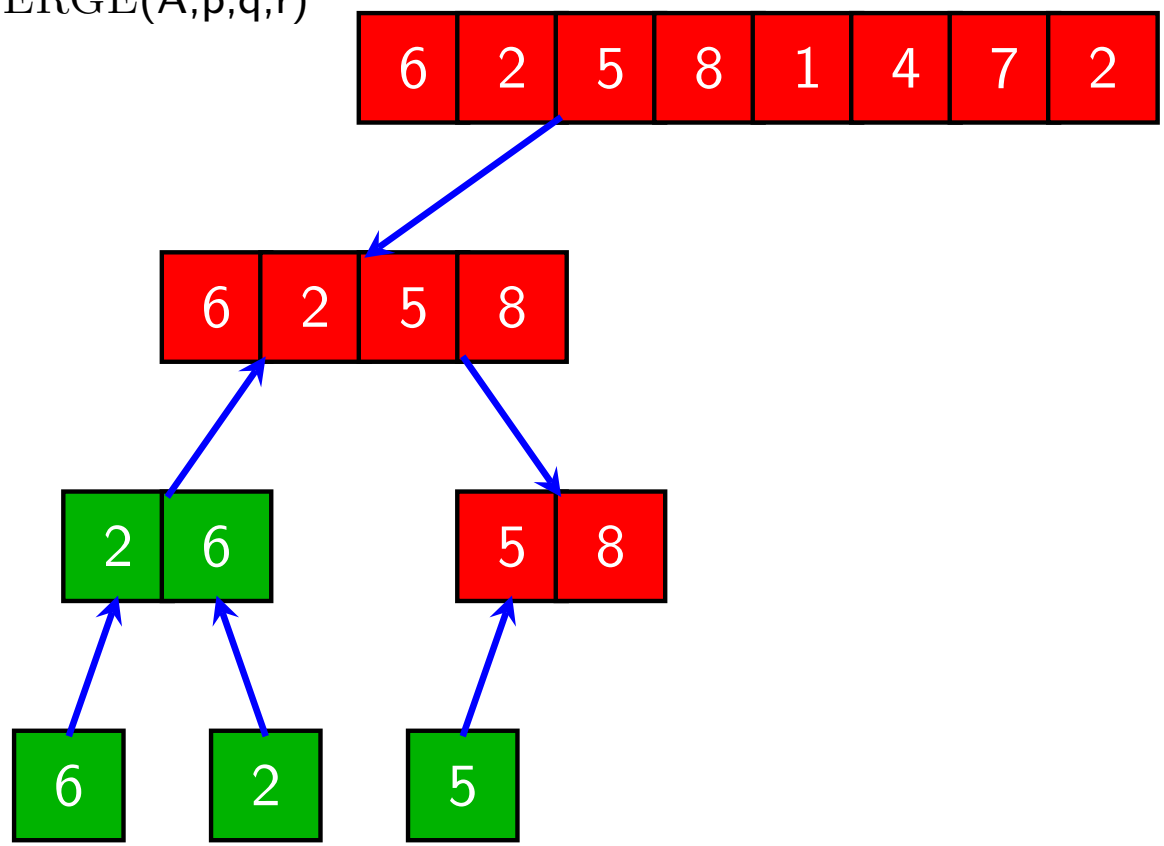

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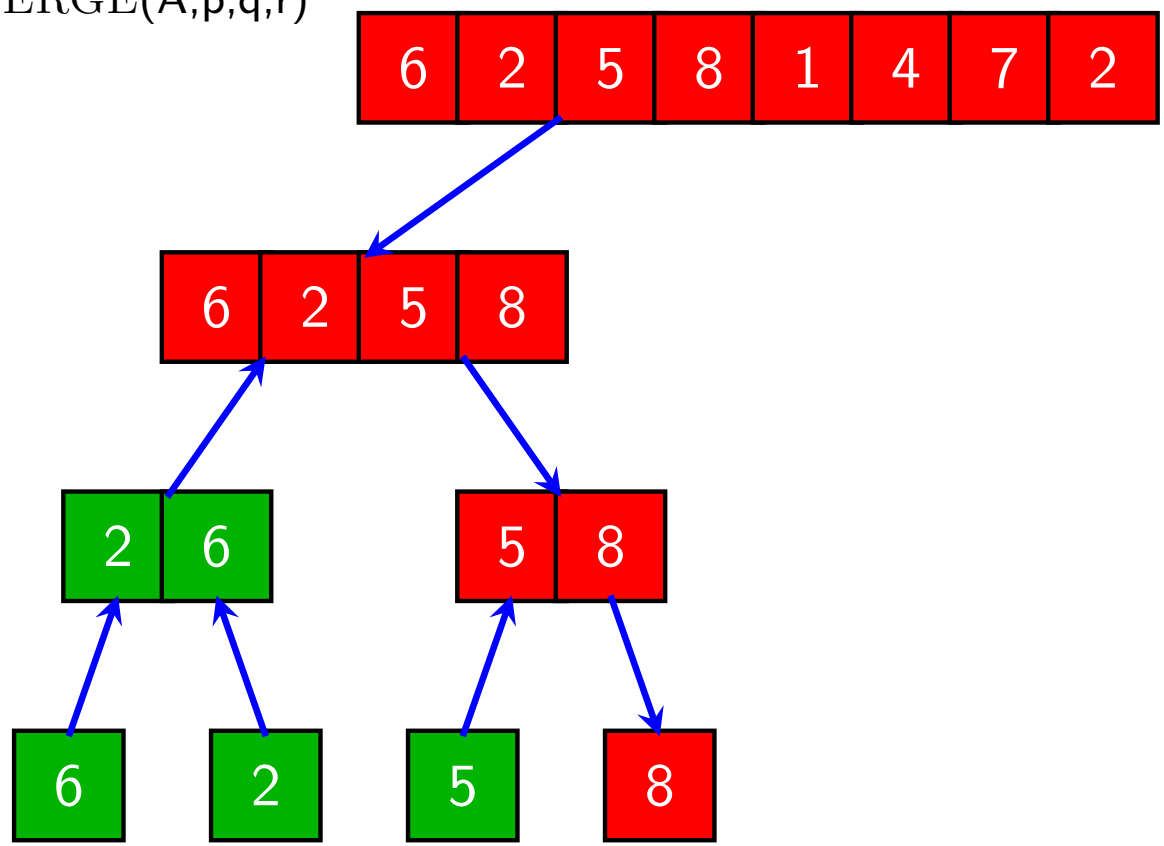

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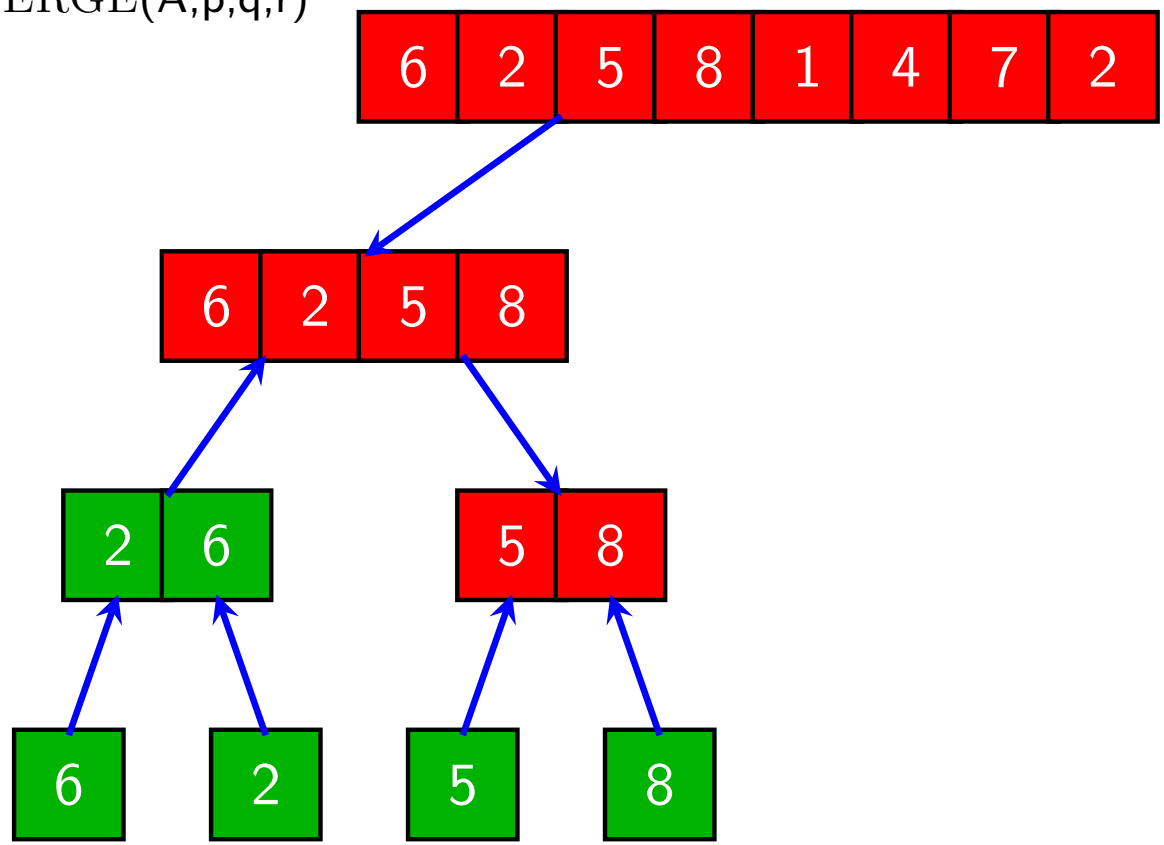

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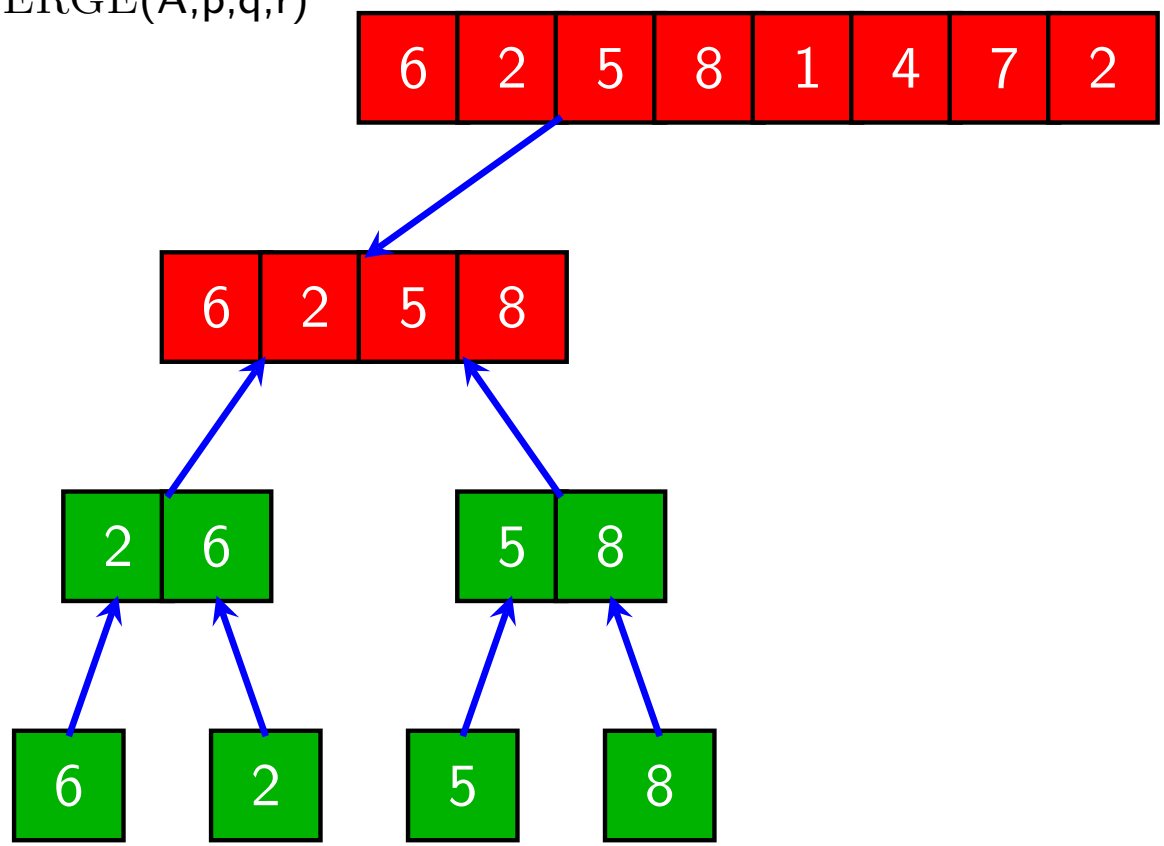

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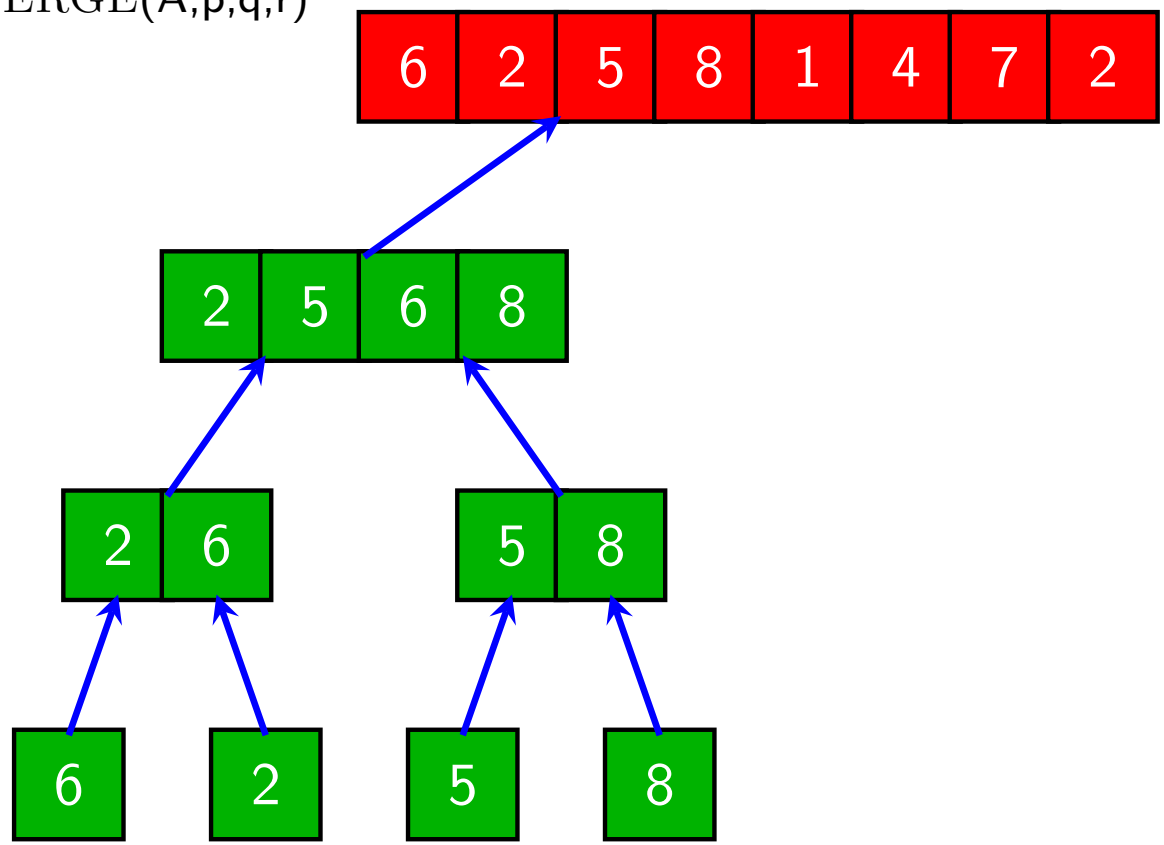

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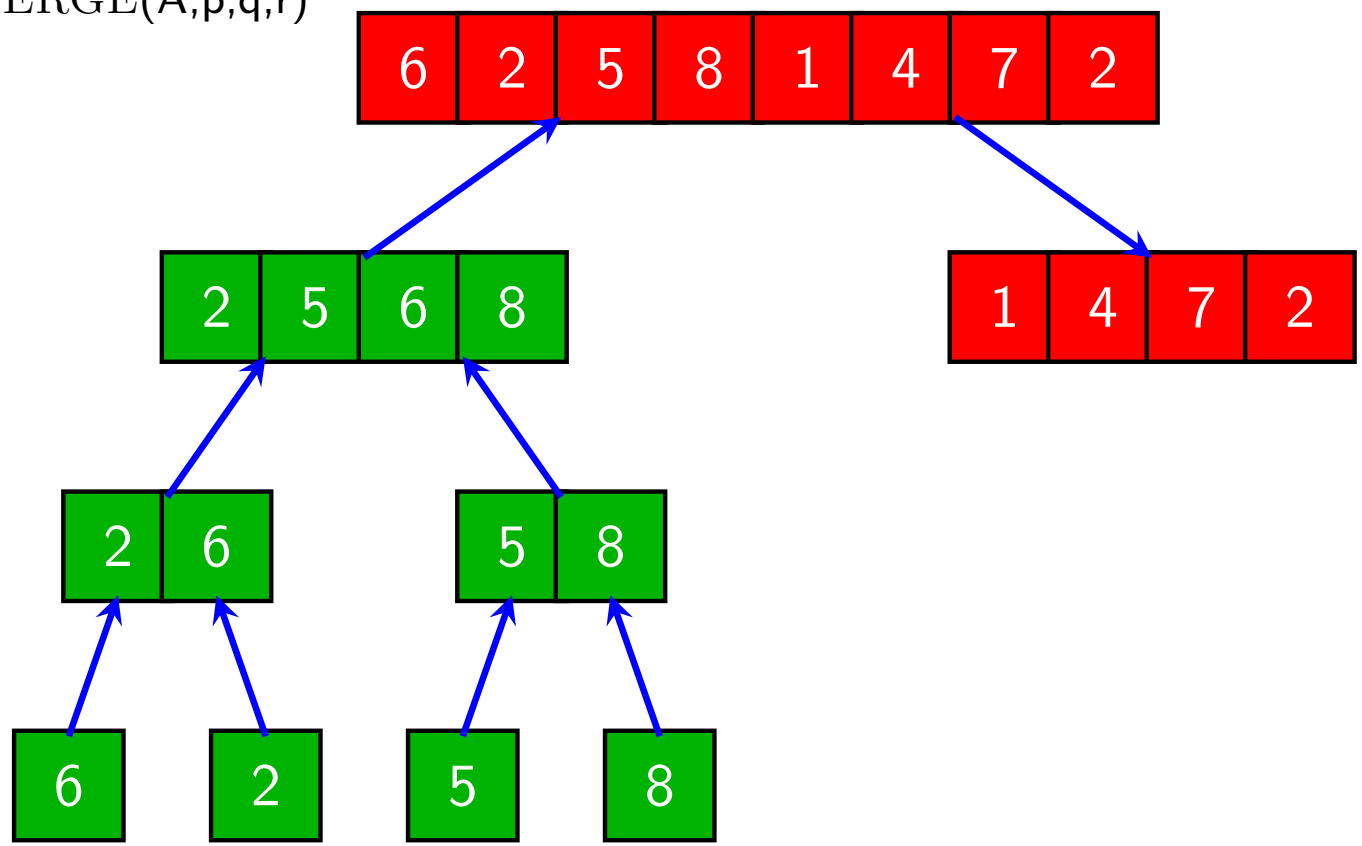

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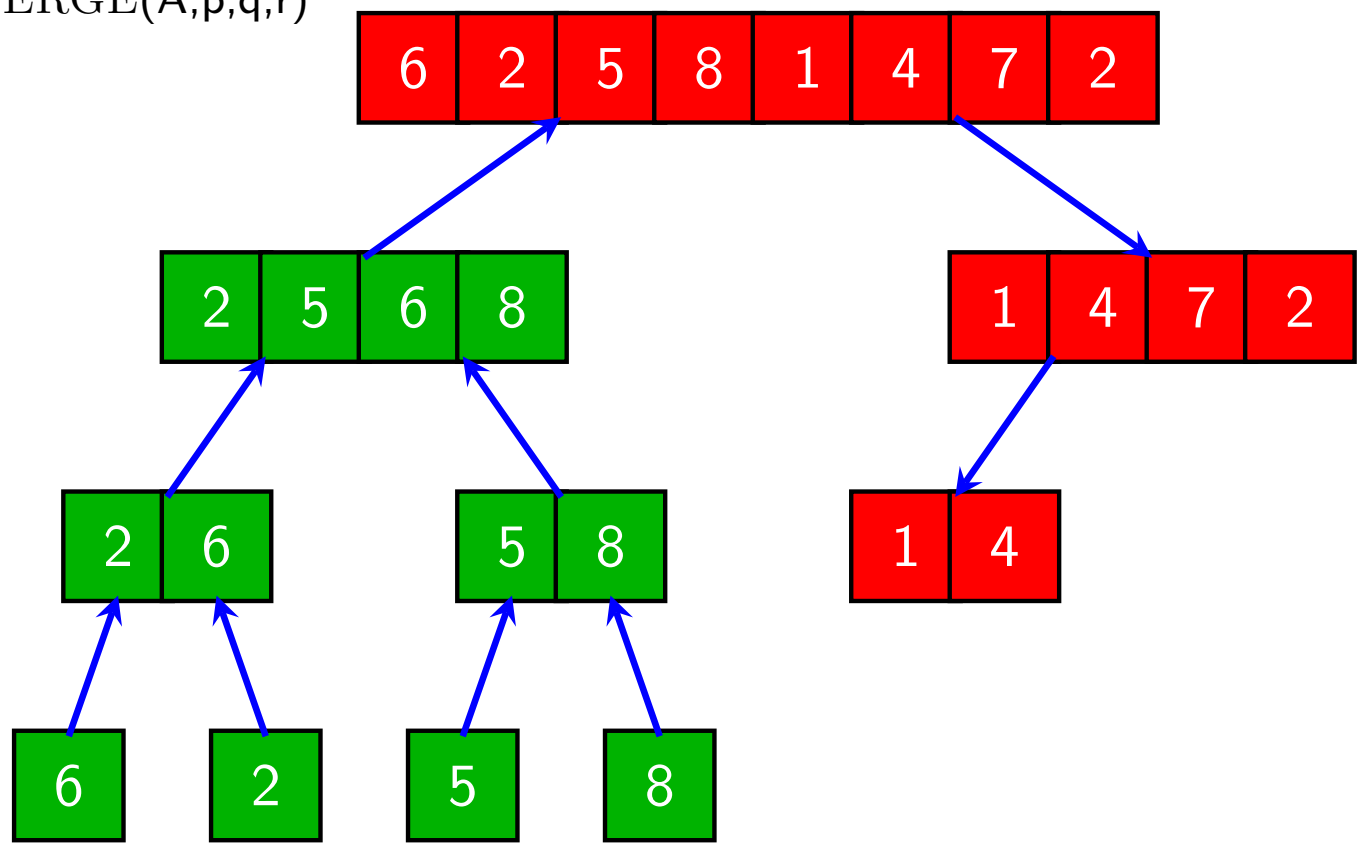

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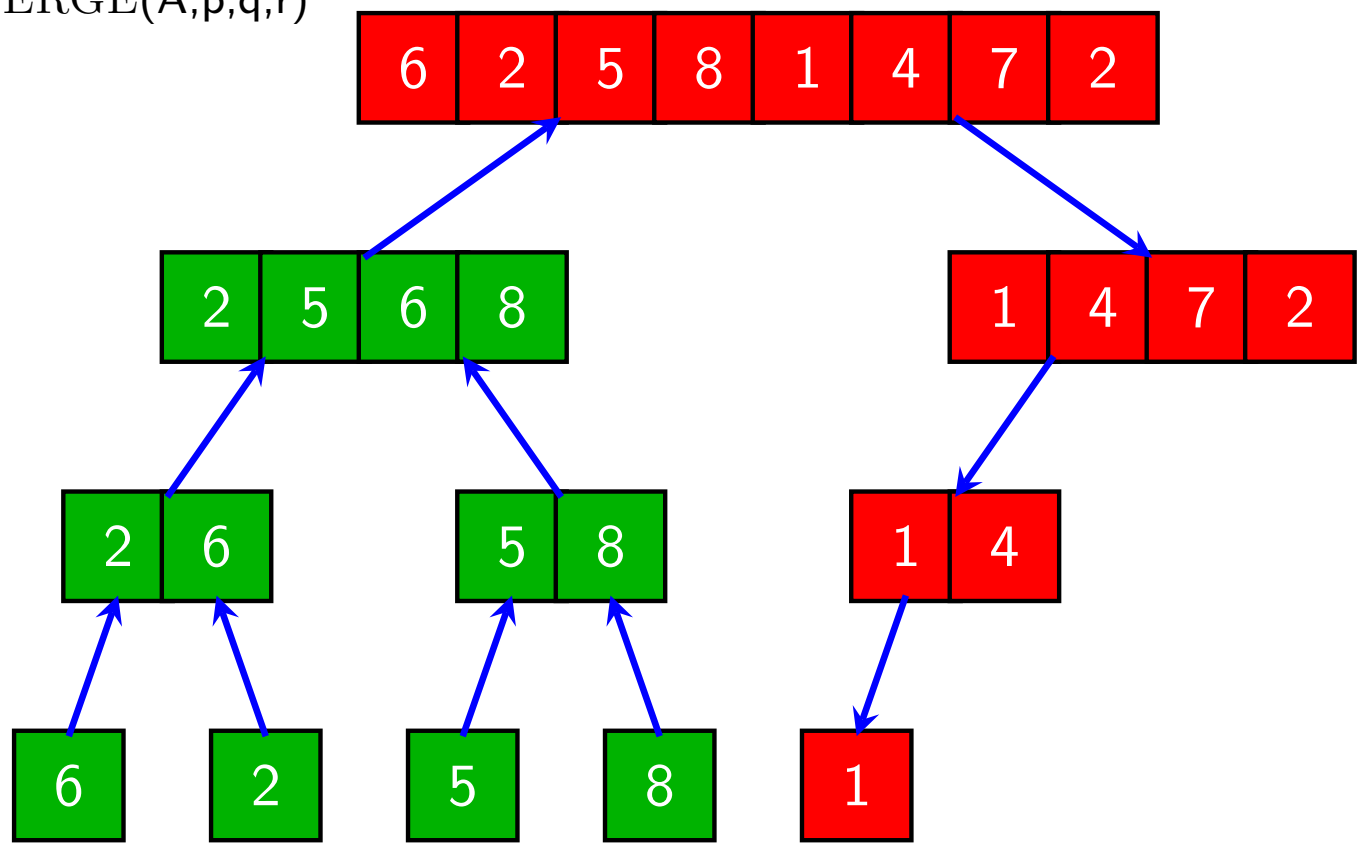

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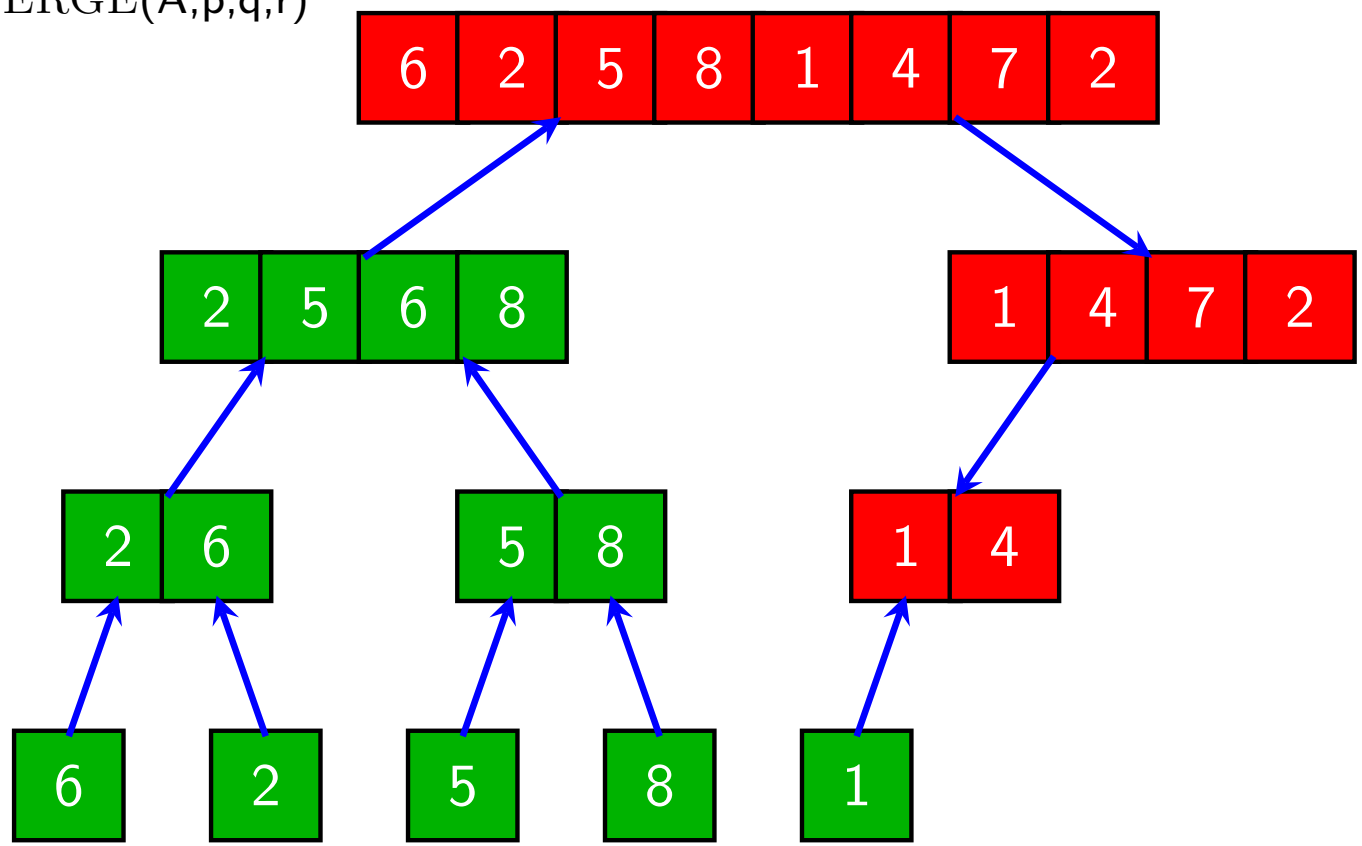

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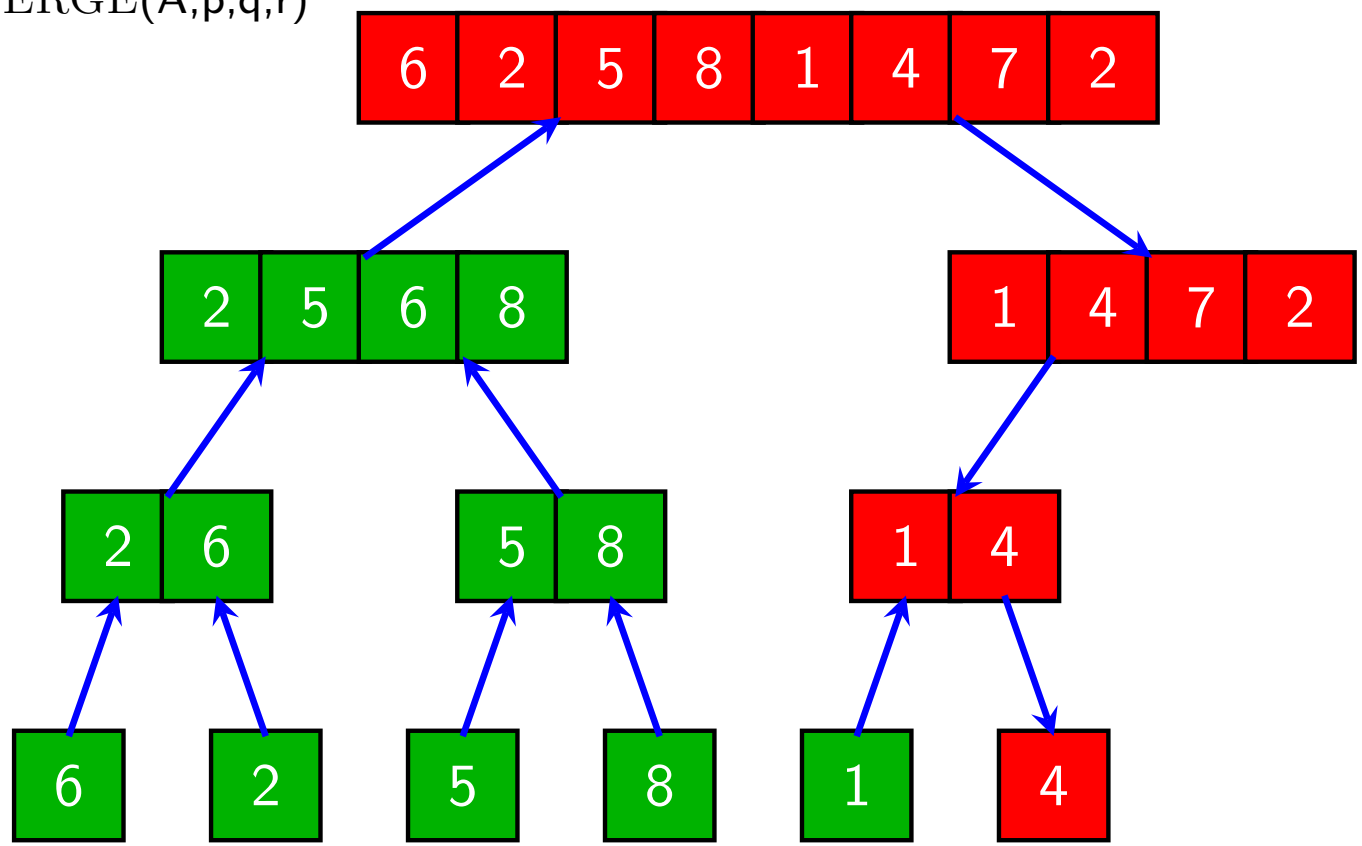

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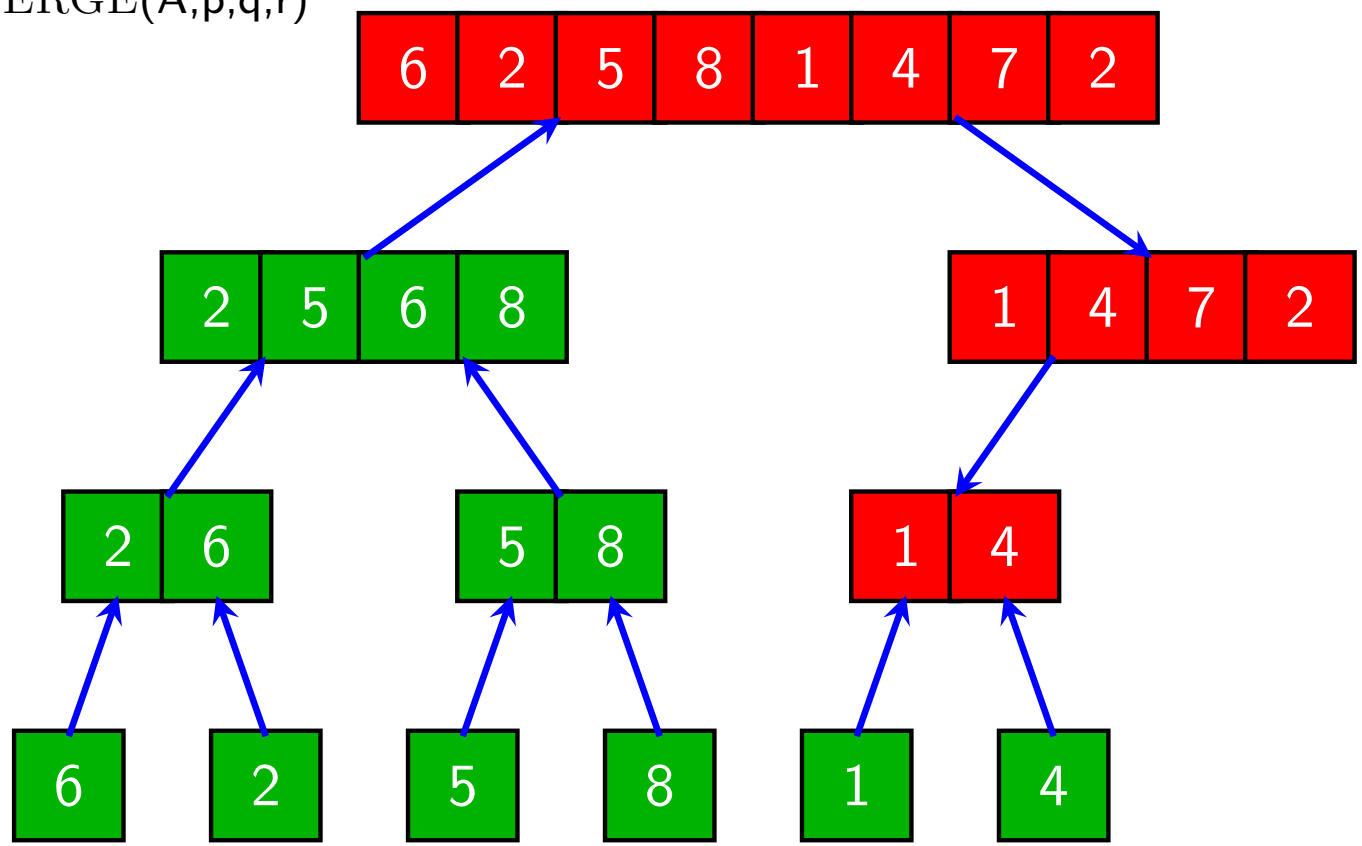

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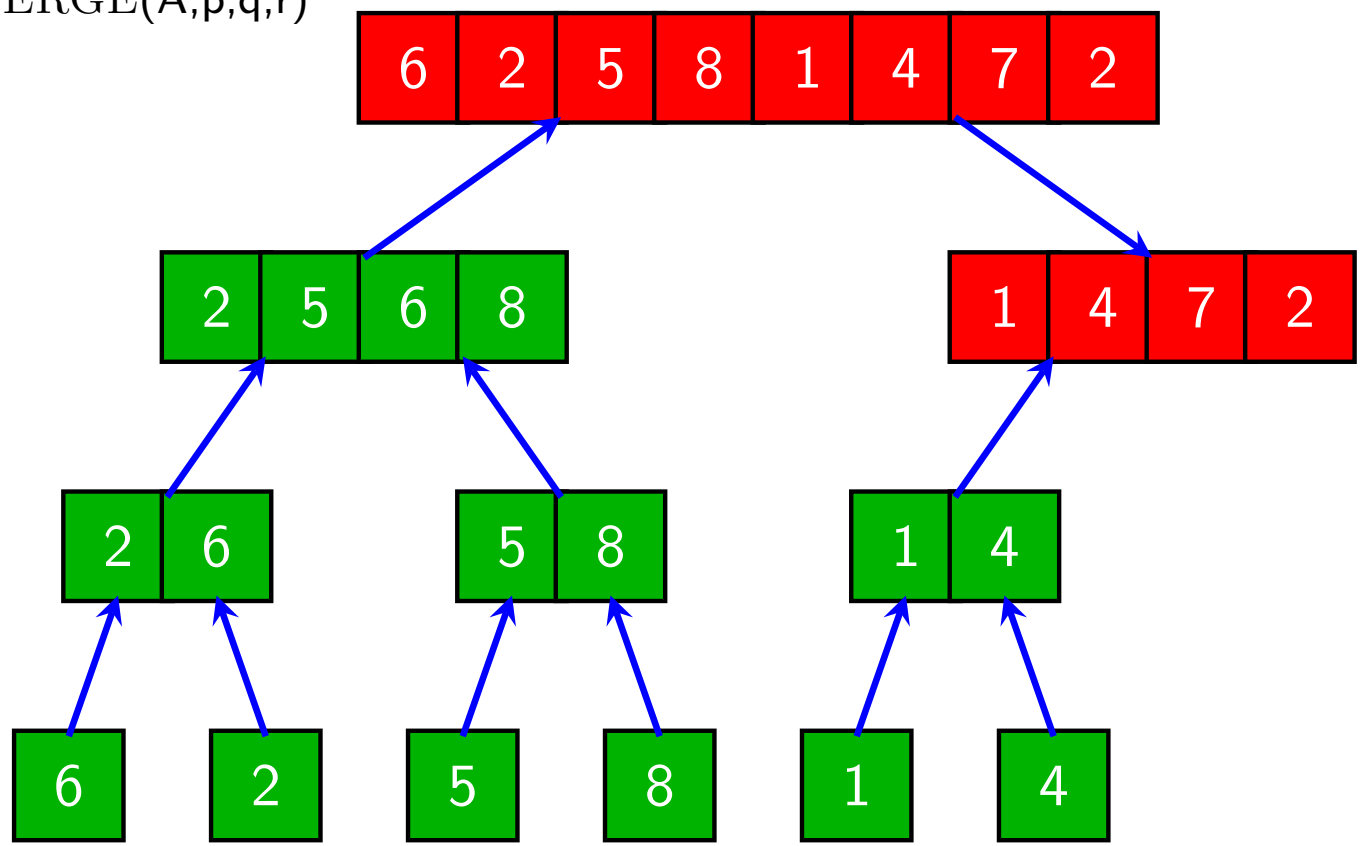

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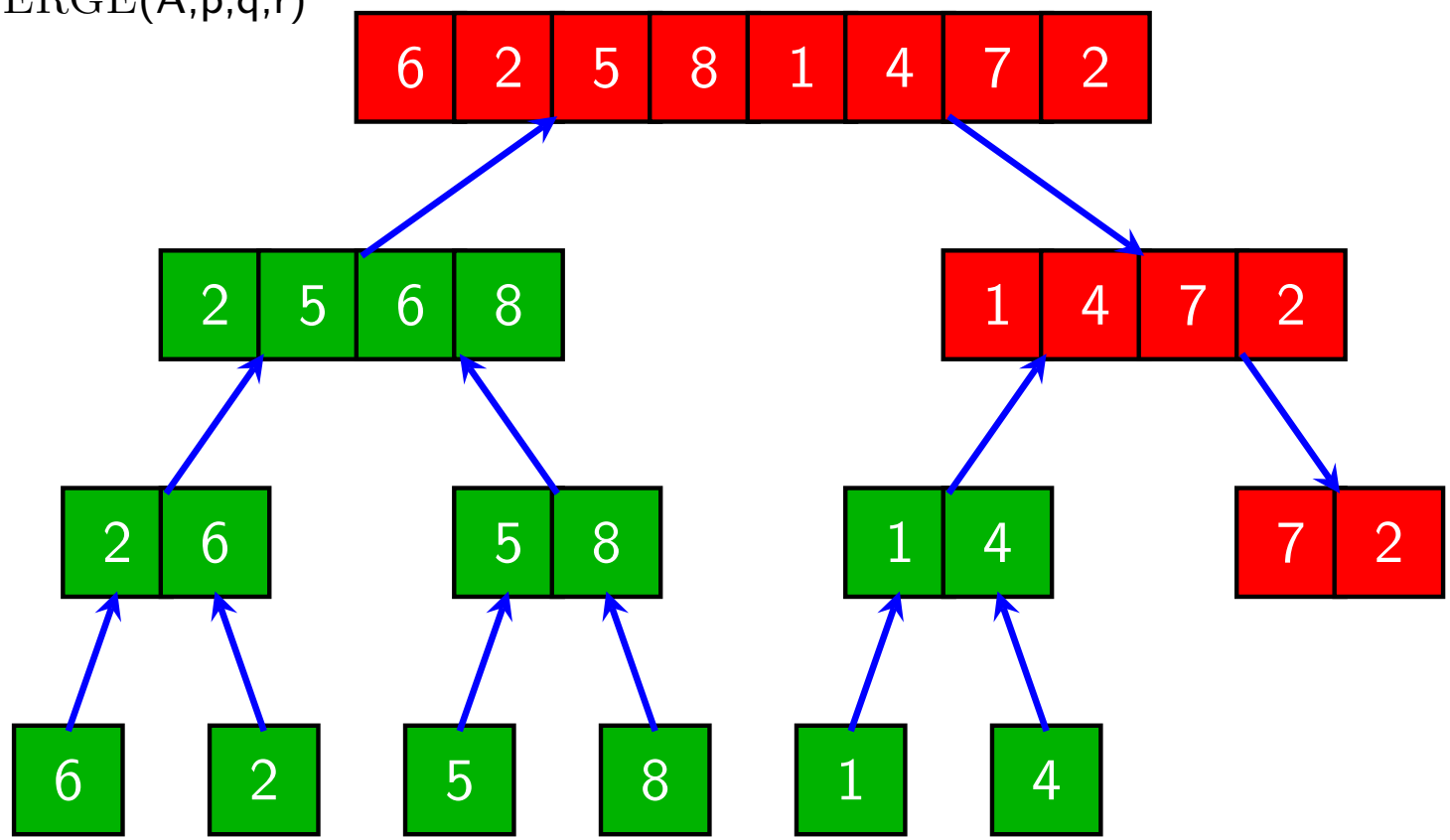

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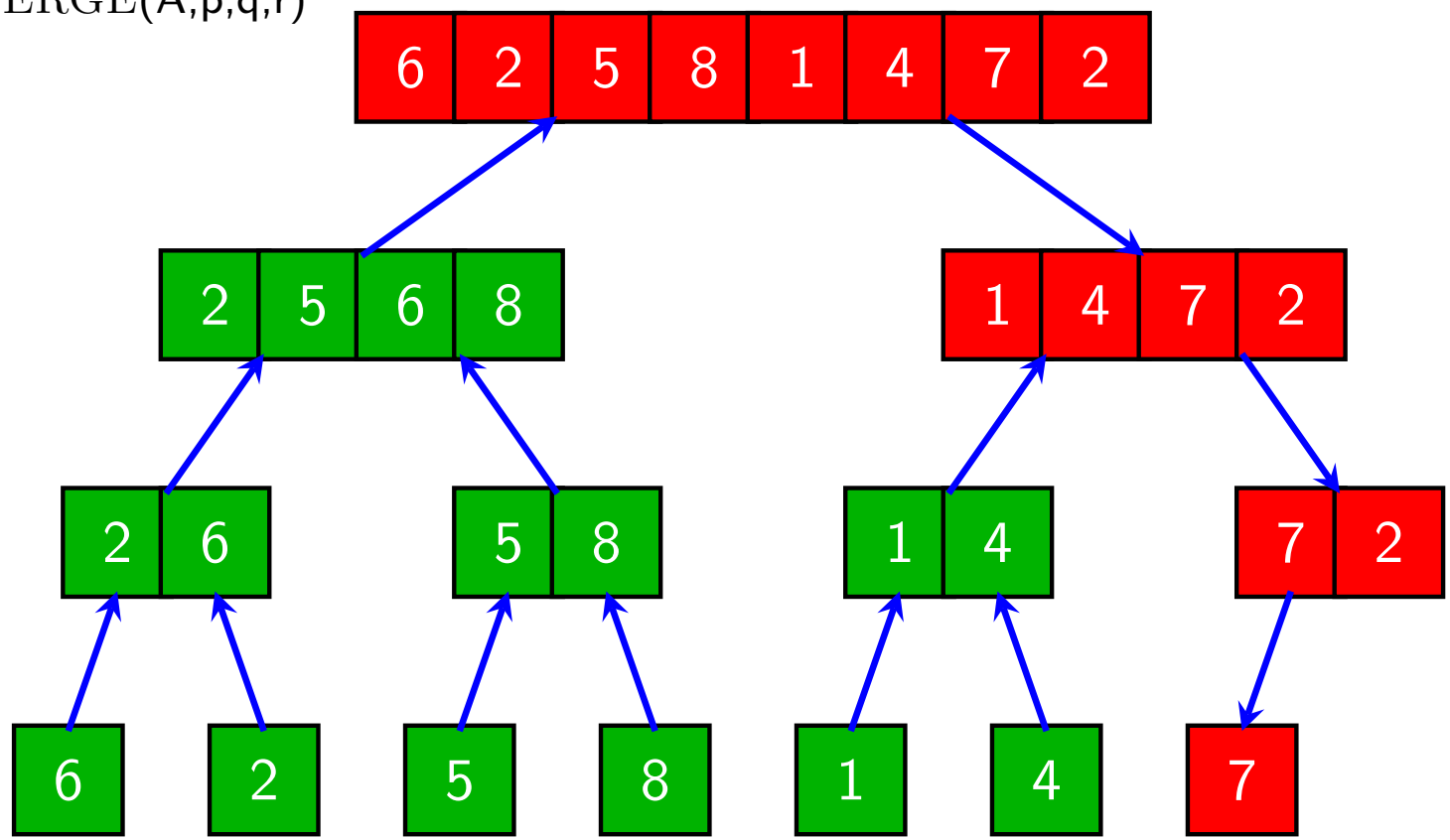

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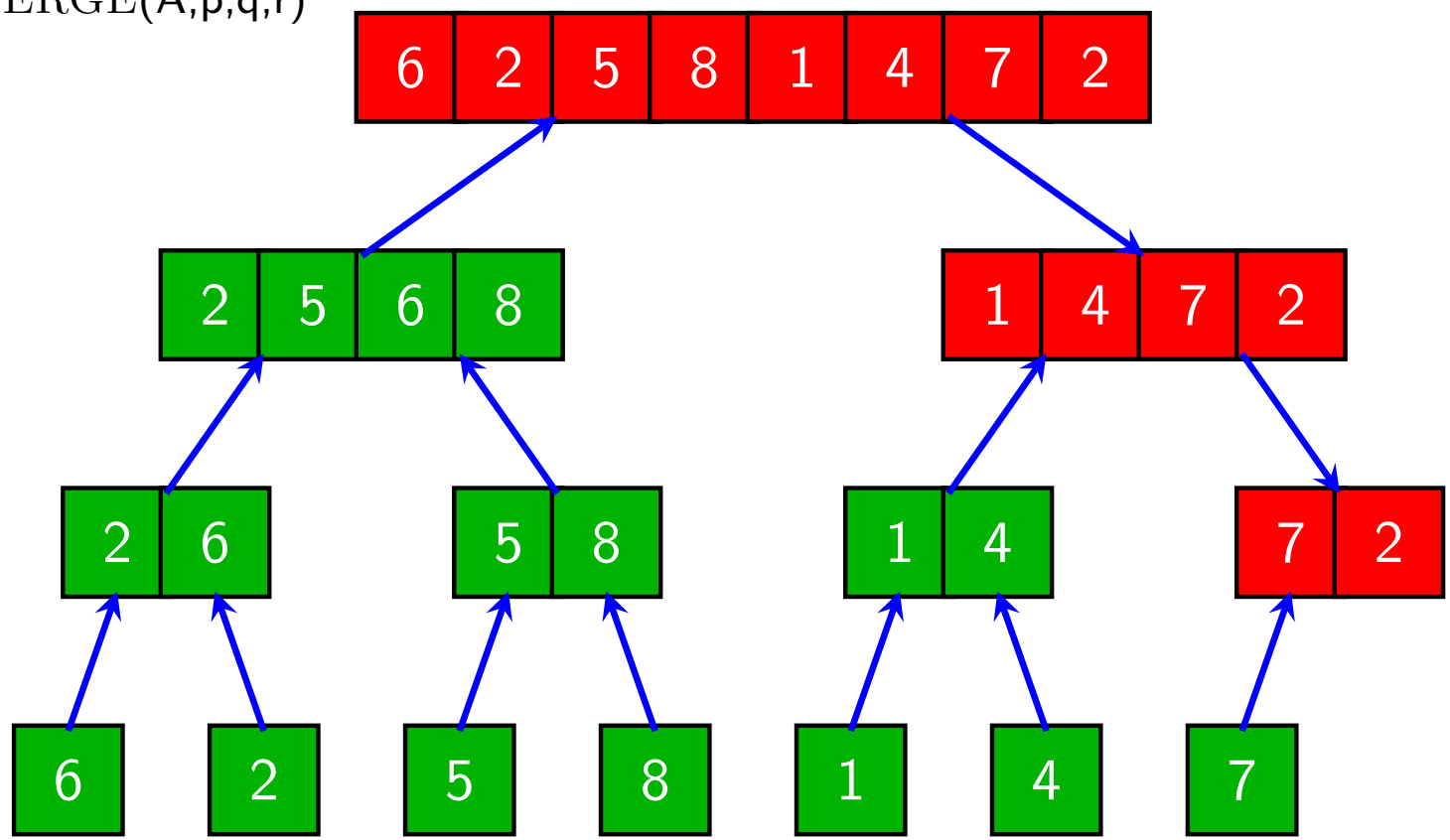

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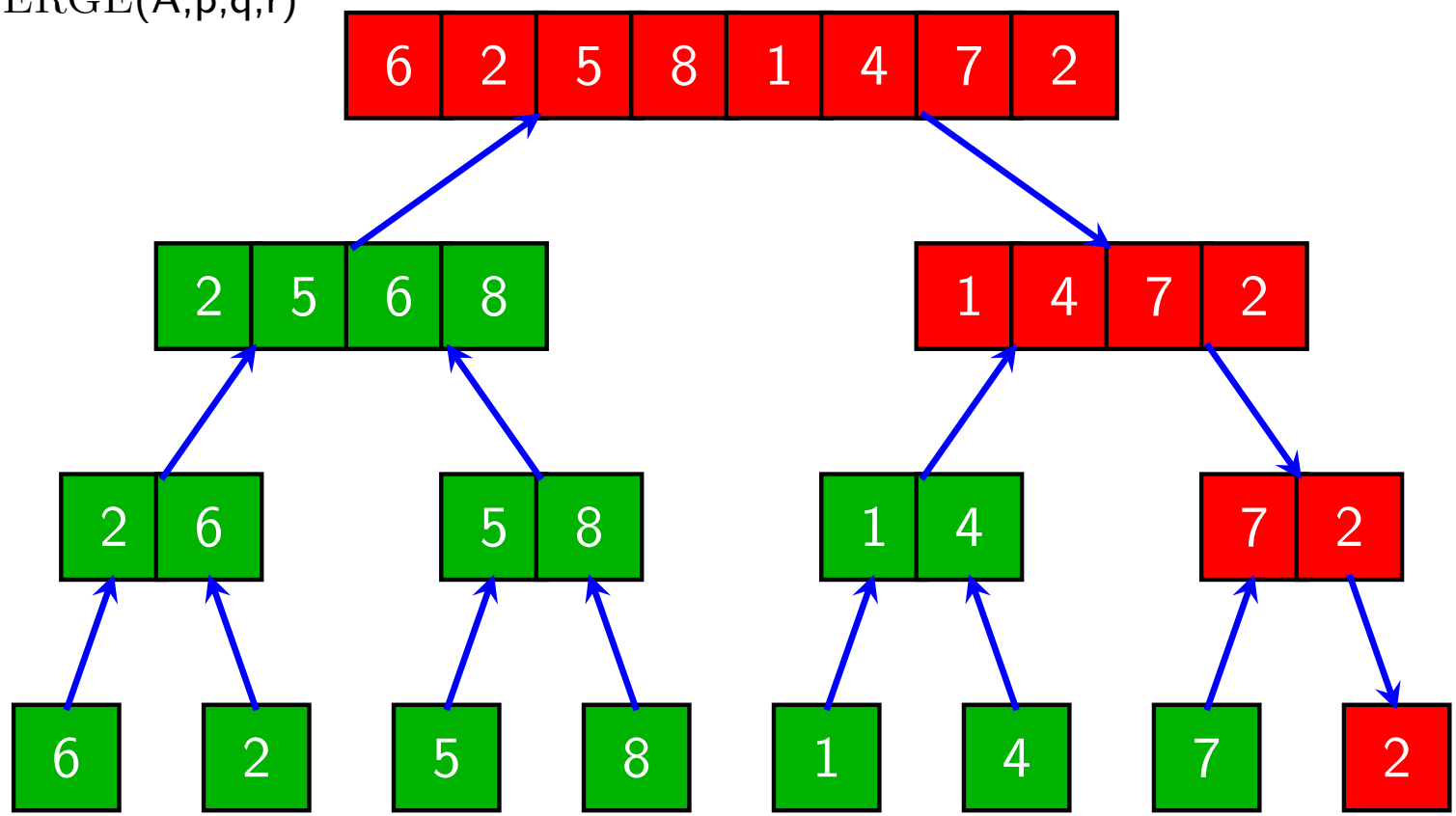

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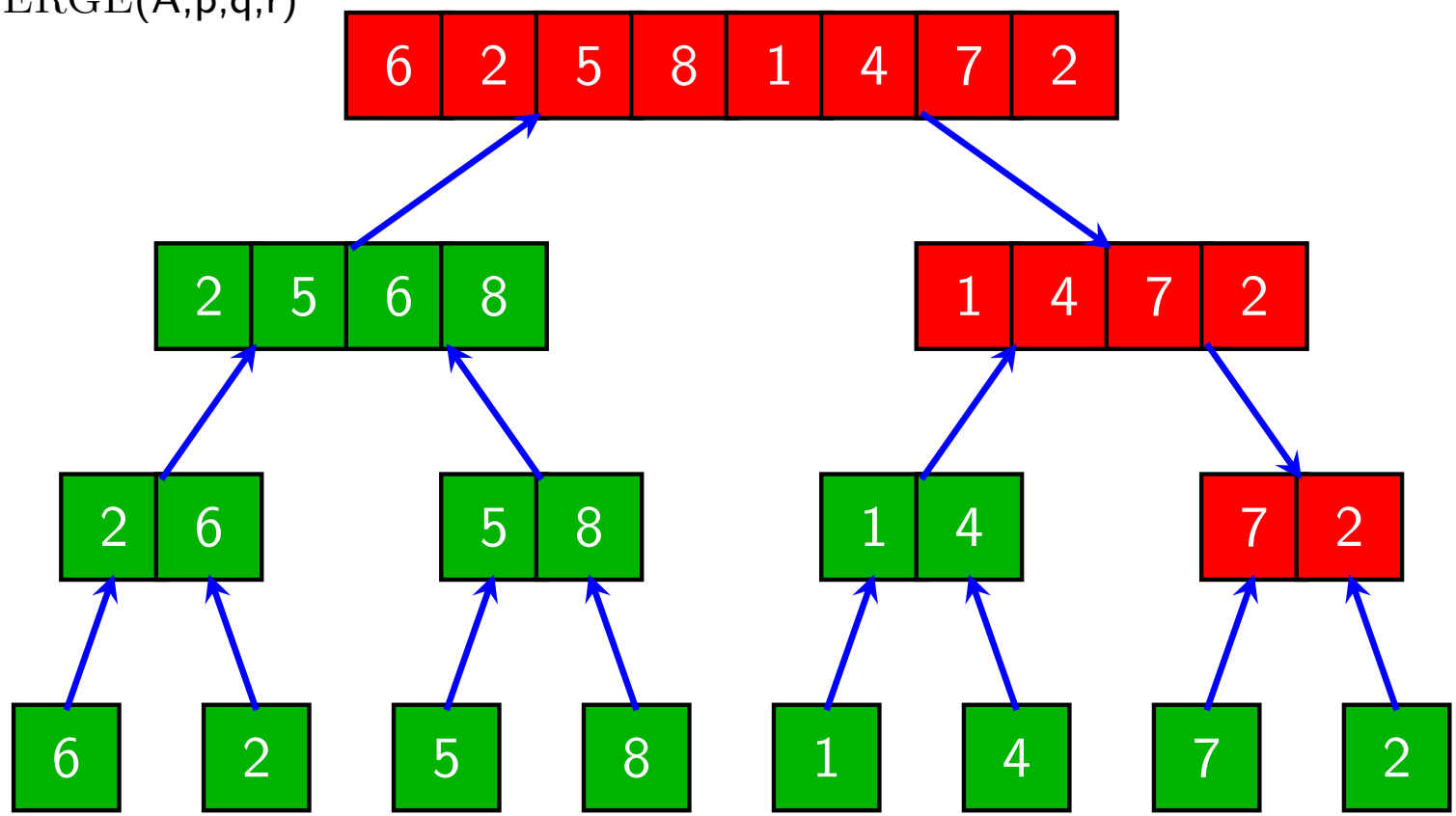

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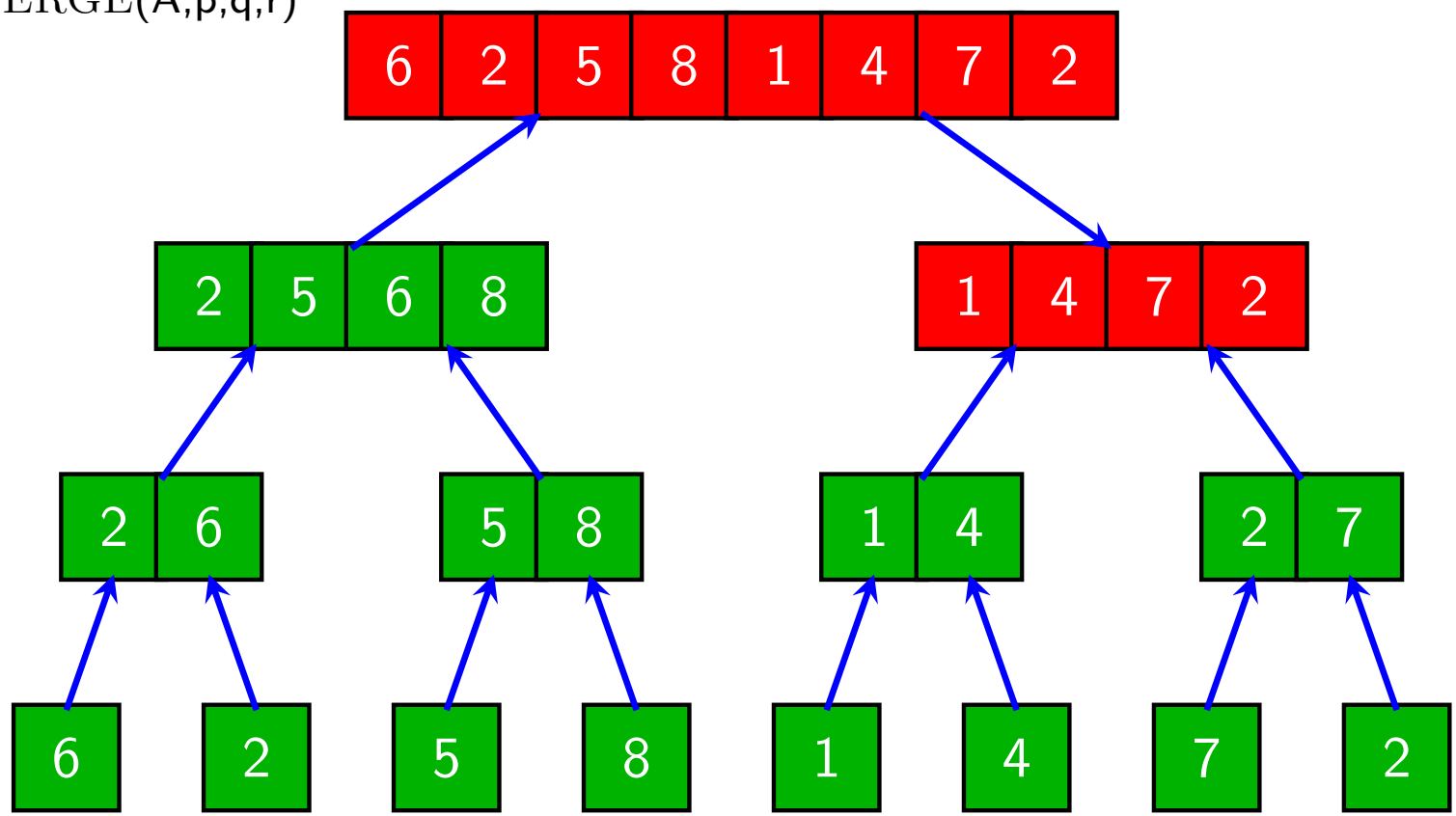

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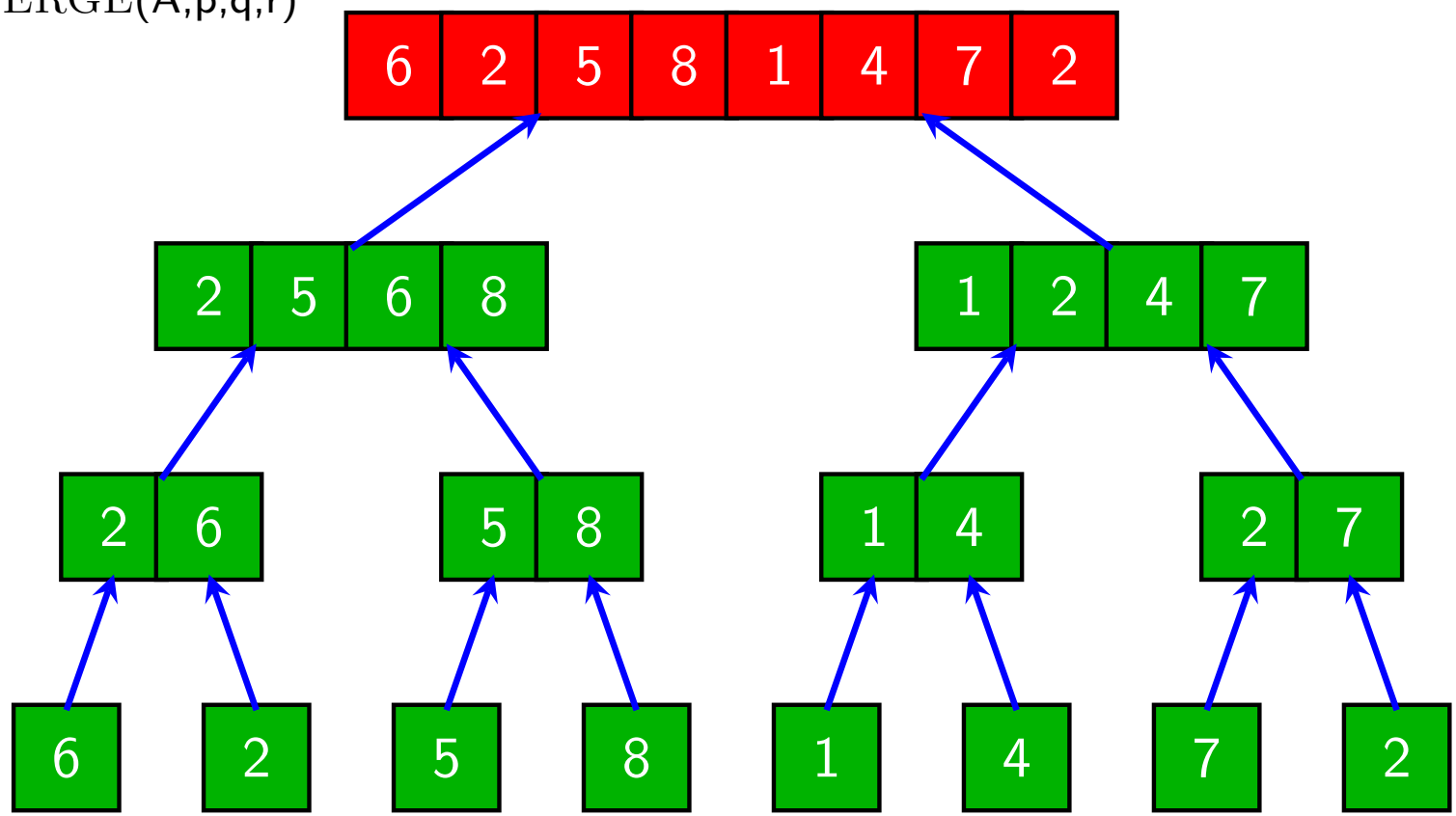

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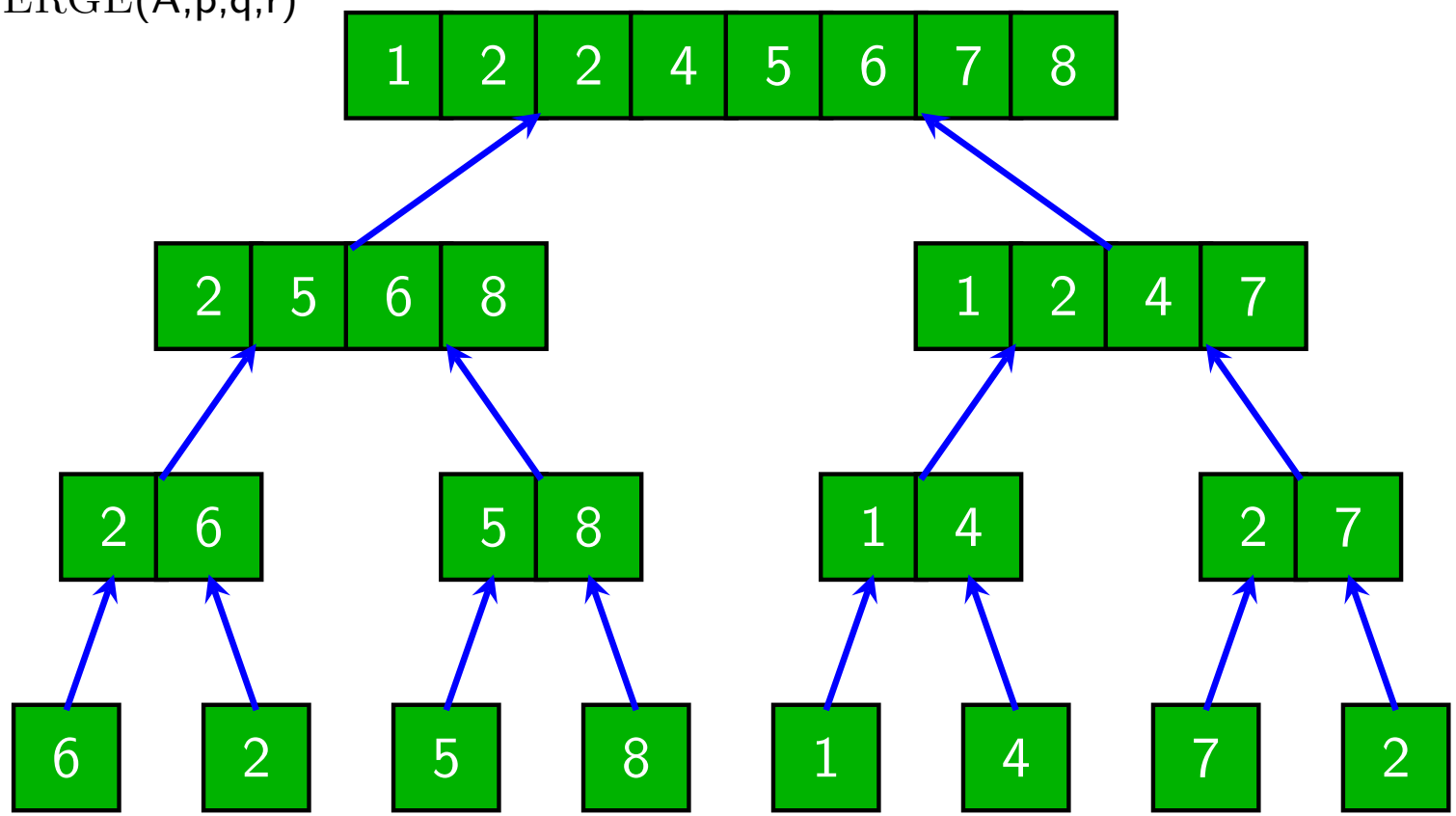

Complexity Analysis of Divide-and-Conquer Algorithms

- Divide the problem into a number of smaller subproblems
- Conquer the subproblems individually by solving them respectively
- Combine the solutions to the subproblems into a solution of the main problem
- Base Cases: If the size of the problem does not exceed some given threshold n_0, then a solution can be provided in a straightforward manner

Complexity Analysis of Divide-and-Conquer Algorithms

- $T(n)$: running time on a problem instance of size n.
- a: number of subproblems.
- $\frac{n}{b}$: size of each subproblem.
- $D(n)$: cost of dividing the problem into subproblems.
- $C(n)$: cost of combining the solutions to the subproblems into the solution of the original problem.
- The cost of each base case ($n \leq n_0$) is constant.
- $$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq n_0 \\ a \cdot T\left(\frac{n}{b}\right) + C(n) + D(n) & \text{if } n > n_0 \end{cases}$$

  This is a recurrence equation.

Example: Divide-and-Conquer Search Algorithm

- $a = 2$: number of subproblems.
- $\frac{n}{2}$ $(b = 2)$: size of each subproblem.
- $D(n) = \Theta(1)$: cost of dividing the problem into subproblems.
- $C(n) = \Theta(1)$: cost of combining the sub-solutions
- Base case $n_0 = 1$
- $T(n) =$
$$\begin{cases} \Theta(1) & \text{if } n \leq n_0 \\ a \cdot T\left(\frac{n}{b}\right) + C(n) + D(n) & \text{if } n > n_0 \end{cases}$$

SEARCH(A, p, r, x)

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Solving More Recurrences

Consider the runtime of Divide-and-Conquer Search Algorithm:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + \Theta(1) & \text{if } n > 1 \end{cases}$$

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How can we construct an tight/upper/lower bound closed form of $T(n)$?

We consider three different methods:

1. The substitution method
2. The recursion-tree method
3. The master method

Method 1: Substitution

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + \Theta(1) & \text{if } n > 1 \end{cases}$$

The substitution method consists in:

- Guess a solution
- Verify the correctness of the solution using mathematical induction

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$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + \Theta(1) & \text{if } n > 1 \end{cases}$$

We rewrite $T(n)$ as follows:

$$T(n) \leq \begin{cases} c & \text{if } n \leq 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + c & \text{if } n > 1 \end{cases}$$

Method 1: Substitution

We guess that $T(n) = O(n)$, and try to show that for all $n \geq n_0 = 1$, $T(n) \leq an - c$ with $a = 2c$.

- Base Case: $(n = n_0 = 1)$:
  $T(n) = T(1) \leq c = a \cdot 1 - c$

- Induction Step: Let $n > n_0$.

  Assume: $T(i) \leq ai - c$ for all $n_0 \leq i < n$

  Show $T(n) \leq an - c$

$$\begin{aligned} T(n) &\leq 2T\left(\tfrac{n}{2}\right) + c \\ &\leq 2(a\tfrac{n}{2} - c) + c \qquad \text{(by the I.H.)} \\ &= an - c \end{aligned}$$

Method 2: Recursive Trees

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + \Theta(1) & \text{if } n > 1 \end{cases}$$

This method consists in visualizing the recursion as a tree where:

- Each node represents the cost of one sub-problem.
- The sum of all node costs within a level gives the total cost of that level.
- The sum of all per-level costs gives the total cost.

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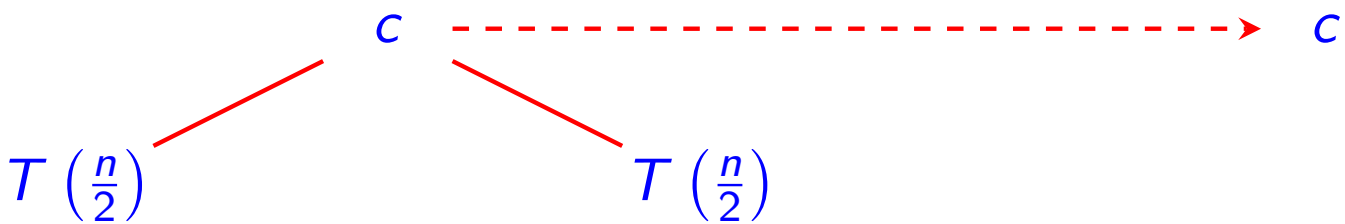

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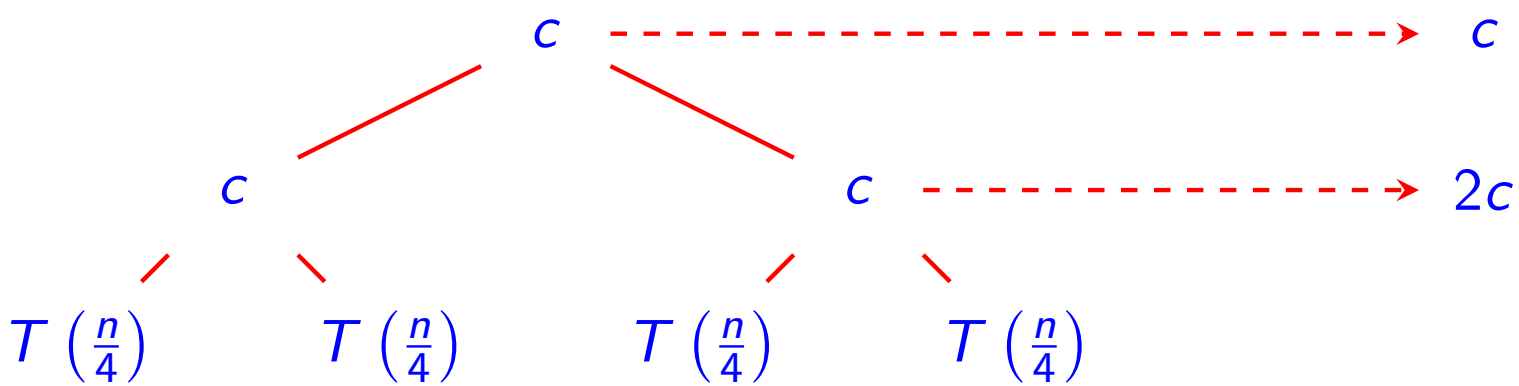

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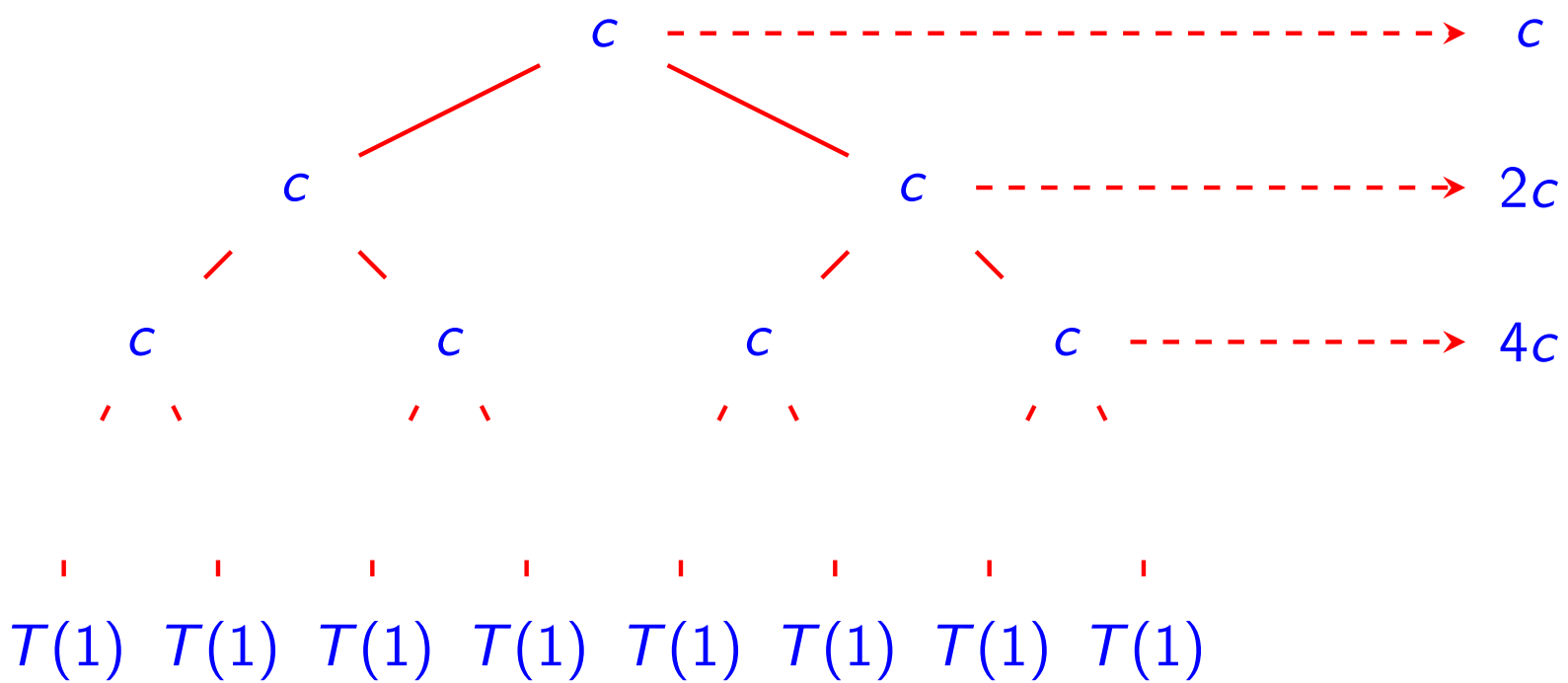

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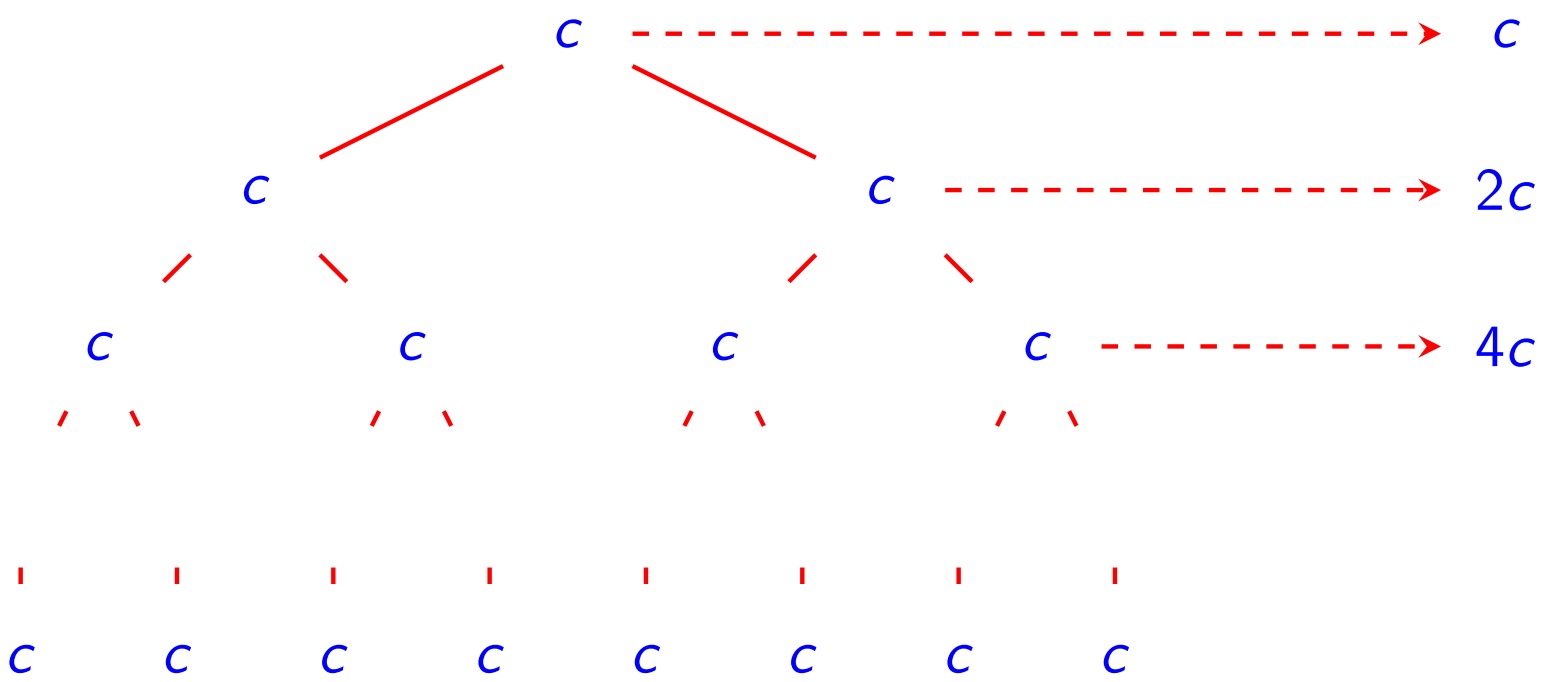

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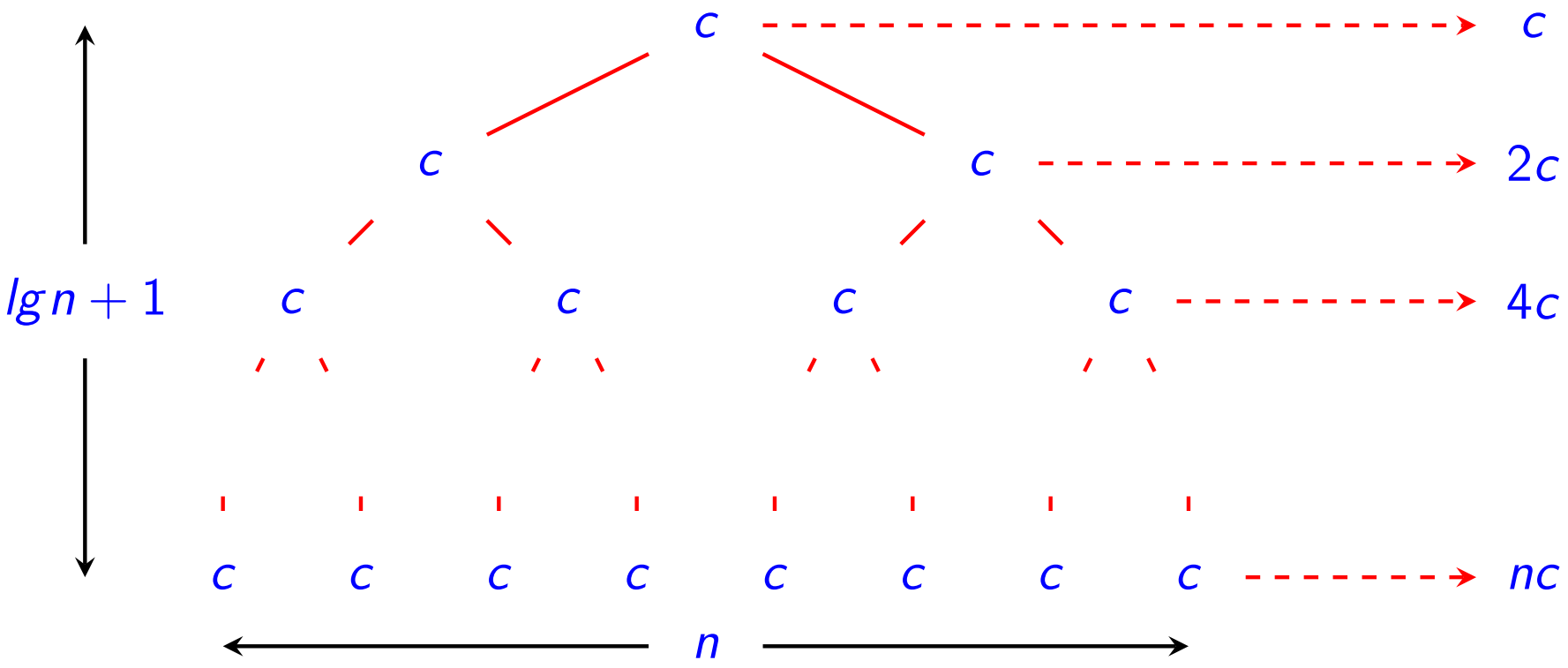

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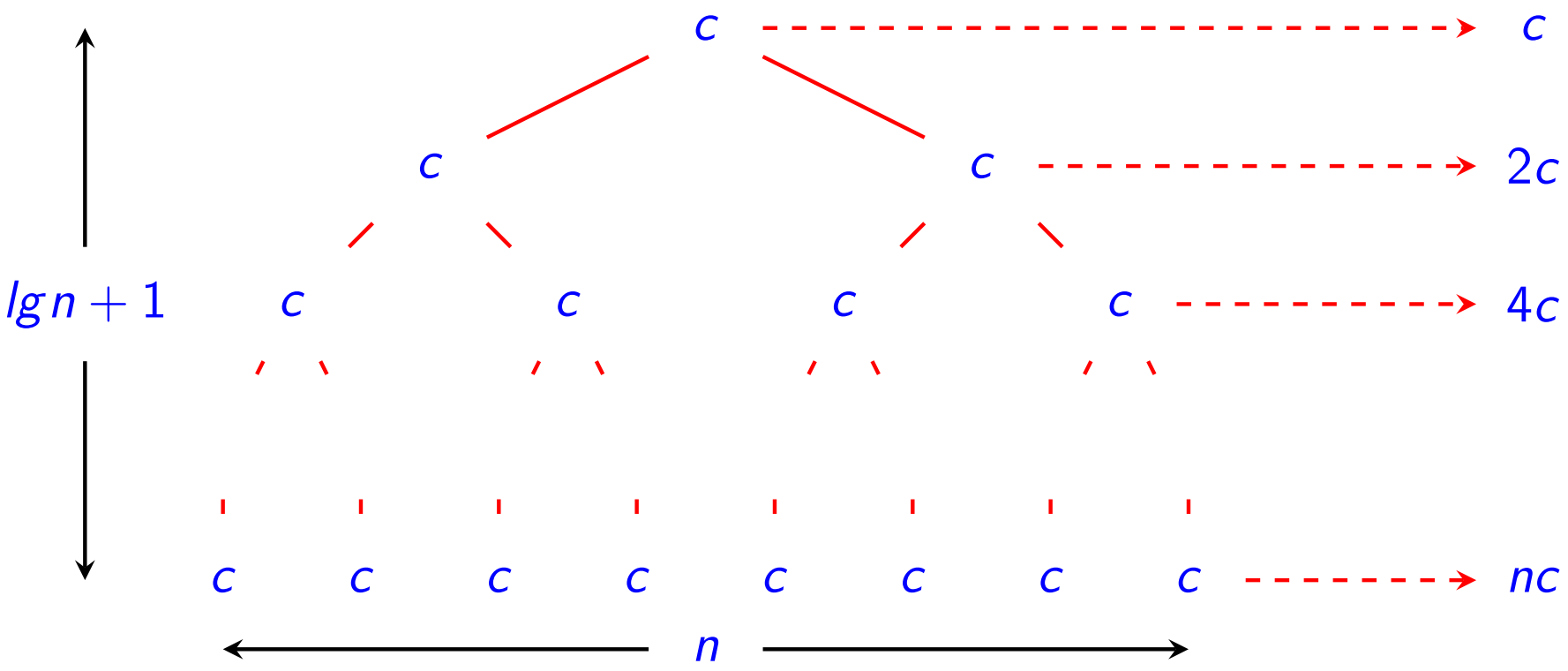

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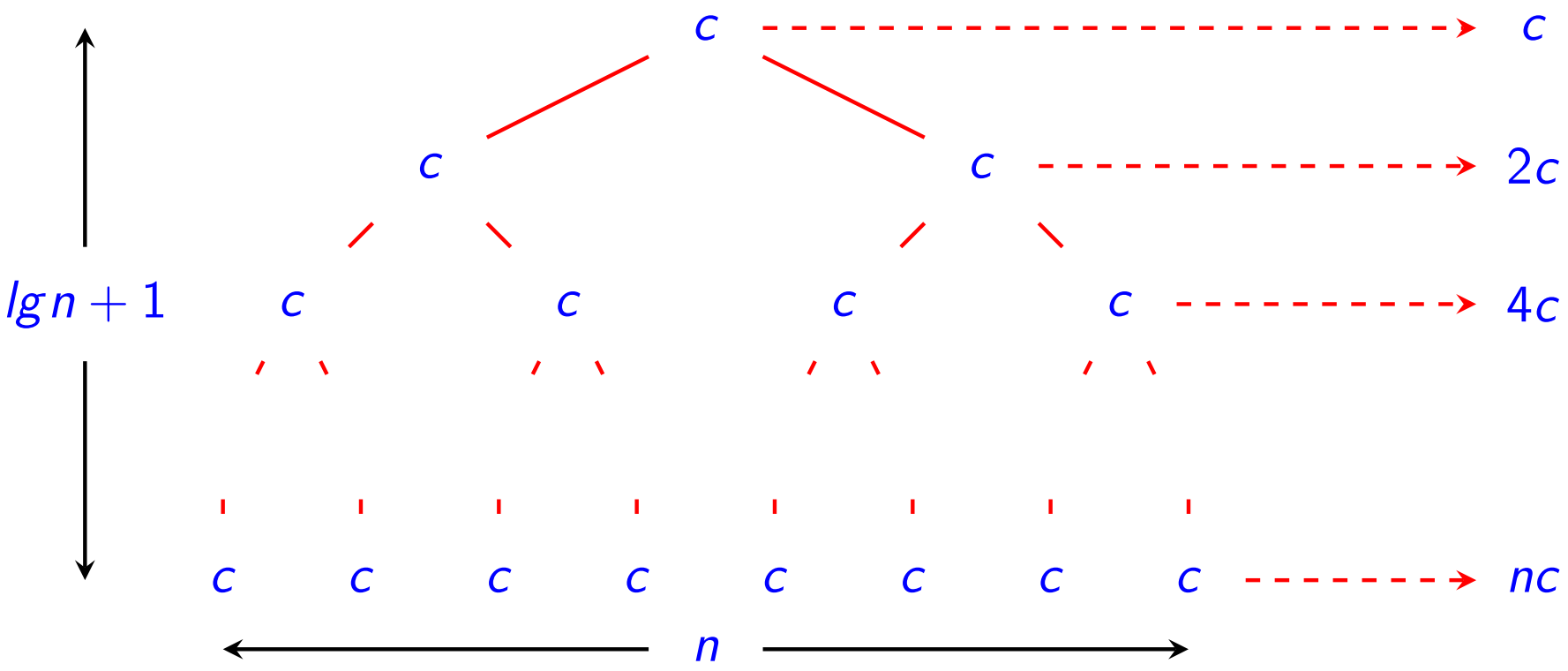

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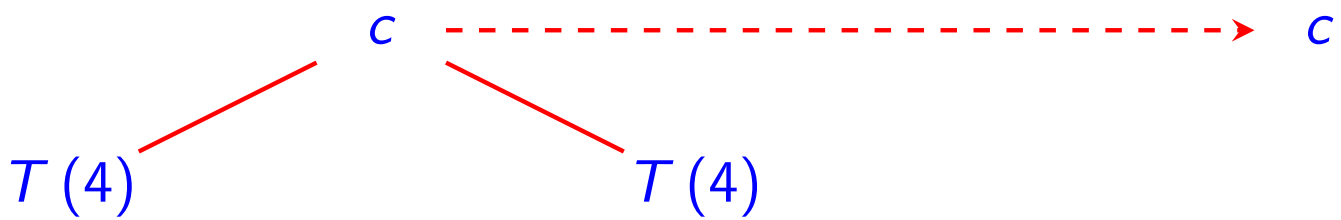

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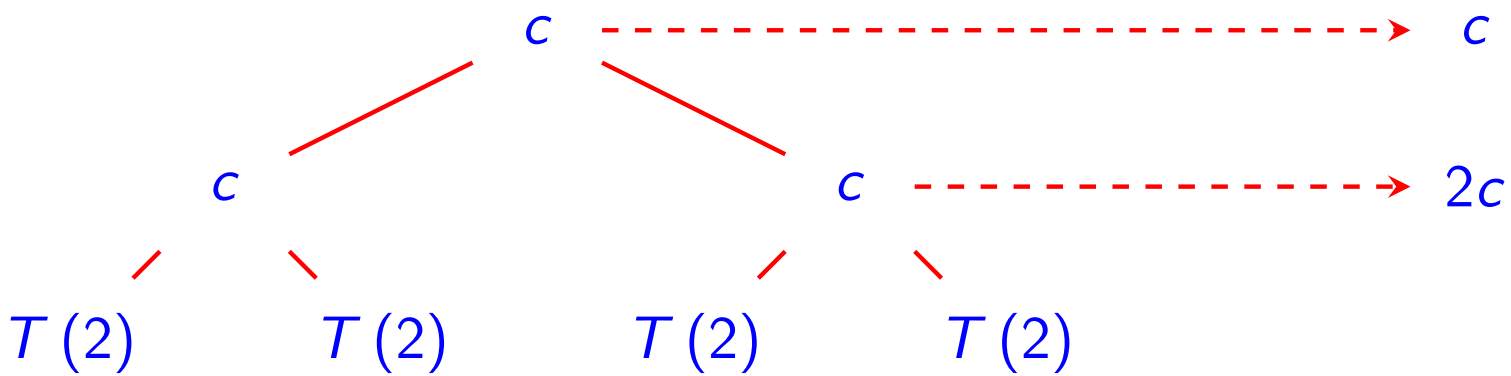

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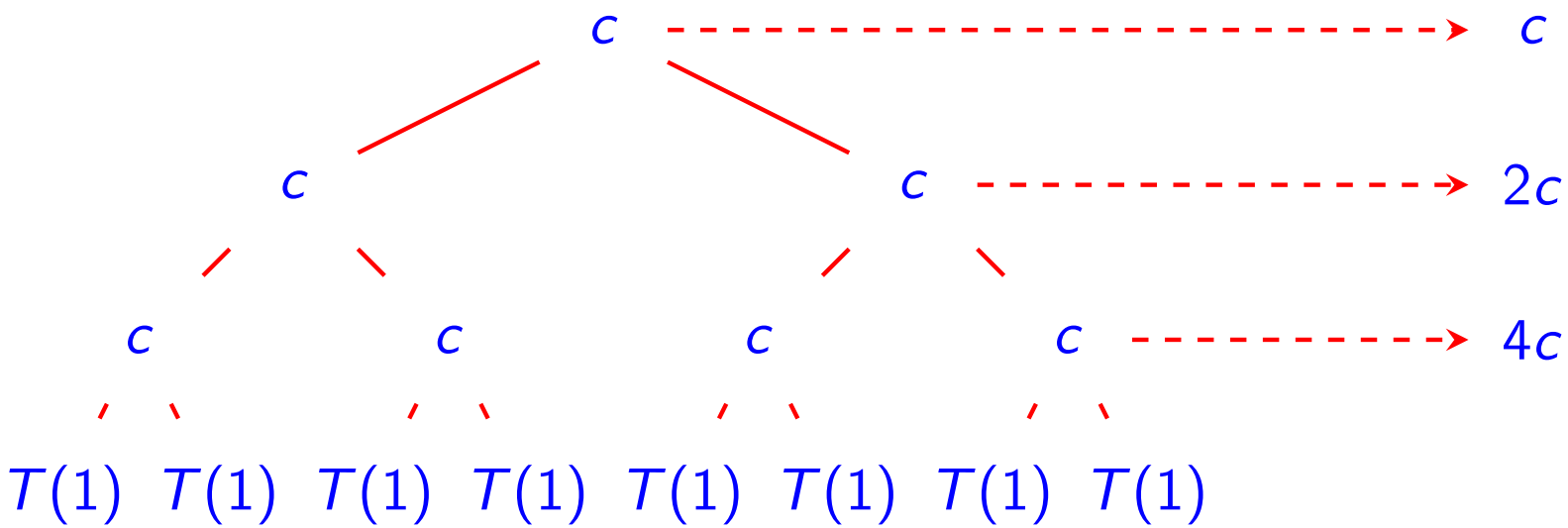

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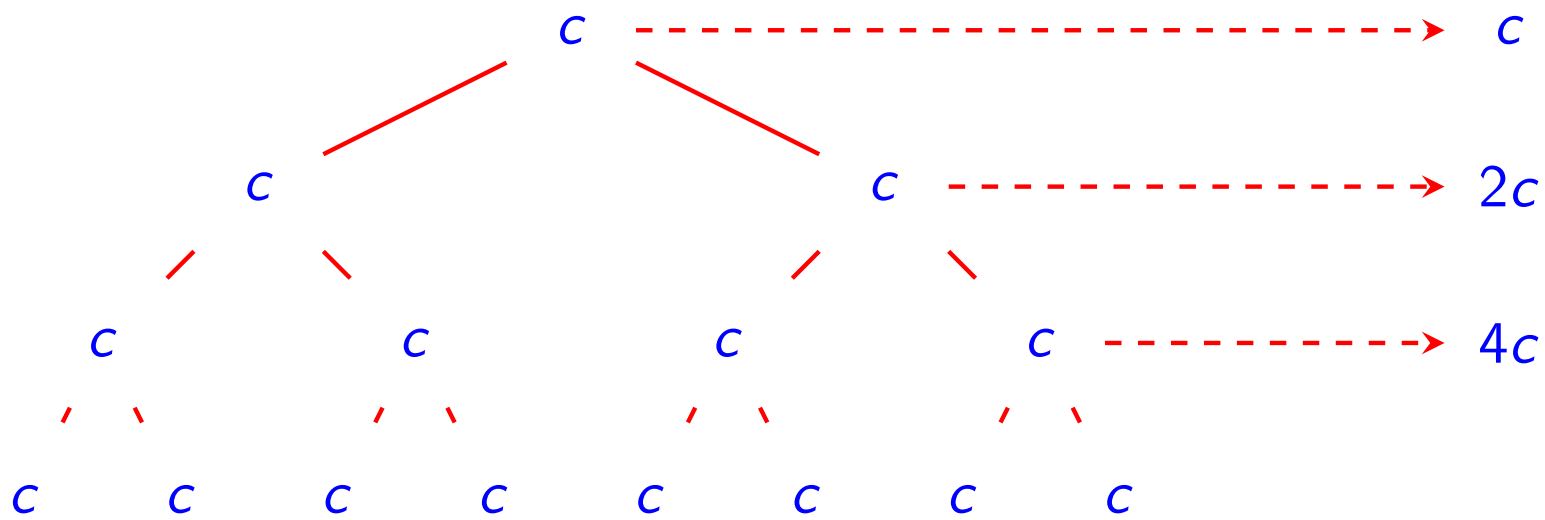

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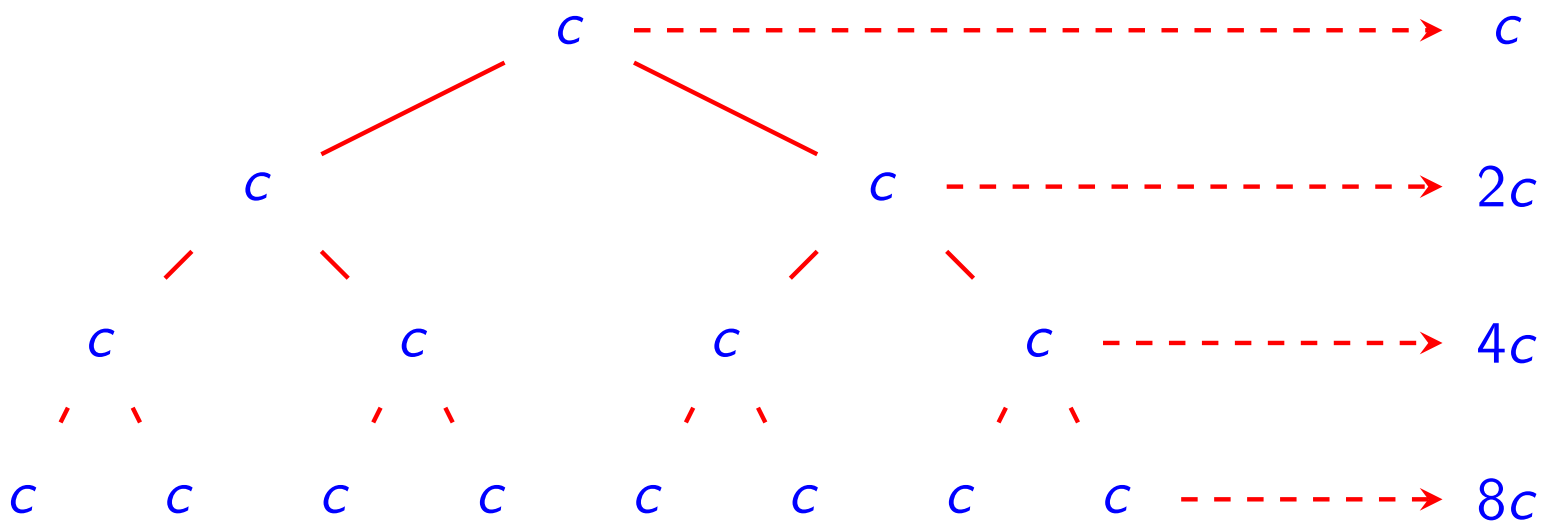

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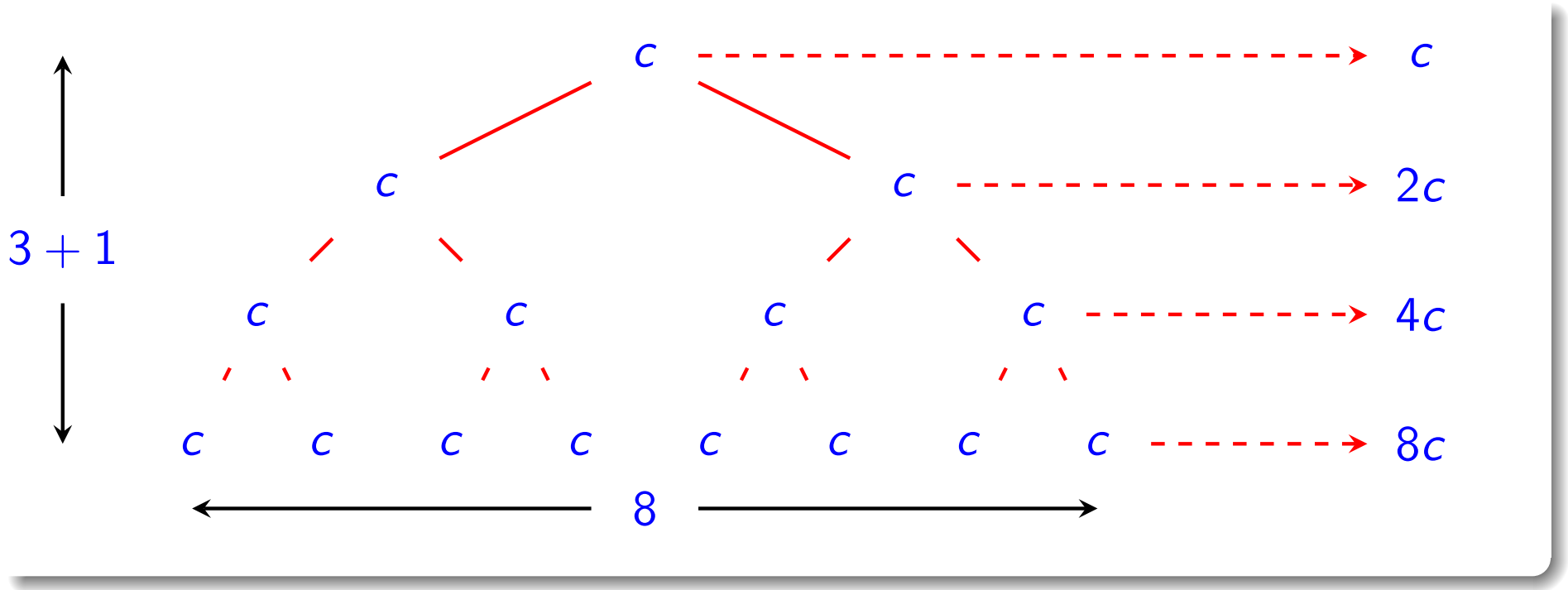

The Master Method and Master Theorem

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$$T(n) = aT(\frac{n}{b}) + f(n)$$

Then, $T(n)$ has the following asymptotic bounds:

- If $f(n) = O(n^{\log_b(a)-\epsilon})$ for some $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b(a)})$
- If $f(n) = \Theta(n^{\log_b(a)})$, then $T(n) = \Theta(n^{\log_b(a)} \log(n))$
- If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ for some $\epsilon > 0$, and if $a \cdot f(\frac{n}{b}) \leq c \cdot f(n)$ for some constant $c < 1$ and sufficiently large n, then $T(n) = \Theta(f(n))$.

Method 3: Master Theorem

Consider the recurrence equation of the search algorithm:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + \Theta(1) & \text{if } n > 1 \end{cases}$$

Then, we have $a = 2$, $b = 2$, $f(n) = \Theta(1)$. Case 1 of the Master Theorem applies since $f(n) = O(n^{\log_2(2)-\epsilon}) = O(1)$ where $\epsilon = 1$. Thus we have:

$$T(n) = \Theta(n^{\log_a(b)}) = \Theta(n)$$

.

Now let's use the three methods to find the complexity of Merge Sort, starting with the recursion tree method.

Complexity Analysis of Divide-and-Conquer Algorithms

- $T(n)$: running time on a problem instance of size n.
- a: number of subproblems.
- $\frac{n}{b}$: size of each subproblem.
- $D(n)$: cost of dividing the problem into subproblems.
- $C(n)$: cost of combining the solutions to the subproblems into the solution of the original problem.
- The cost of each base case ($n \leq n_0$) is constant.
- $$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq n_0 \\ a \cdot T\left(\frac{n}{b}\right) + C(n) + D(n) & \text{if } n > n_0 \end{cases}$$

Example: Merge Sort

- $a = 2$: number of subproblems.
- $\frac{n}{2}$ $(b = 2)$: size of each subproblem.
- $D(n) = \Theta(1)$: cost of dividing the problem into subproblems.
- $C(n) = \Theta(n)$: cost of the procedure Merge
- Base case $n_0 = 1$
- $T(n) =$
$$\begin{cases} \Theta(1) & \text{if } n \leq n_0 \\ a \cdot T\left(\frac{n}{b}\right) + C(n) + D(n) & \text{if } n > n_0 \end{cases}$$

MERGE-SORT(A, p, r)

```
1  if p < r
2     then q ← ⌊(p+r)/2⌋
3          MERGE-SORT(A,p,q)
4          MERGE-SORT(A,q+1,r)
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The Recursion Tree method for Merge Sort

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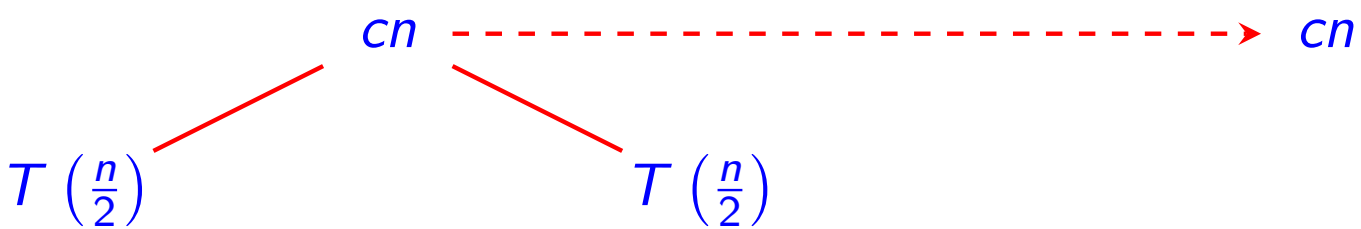

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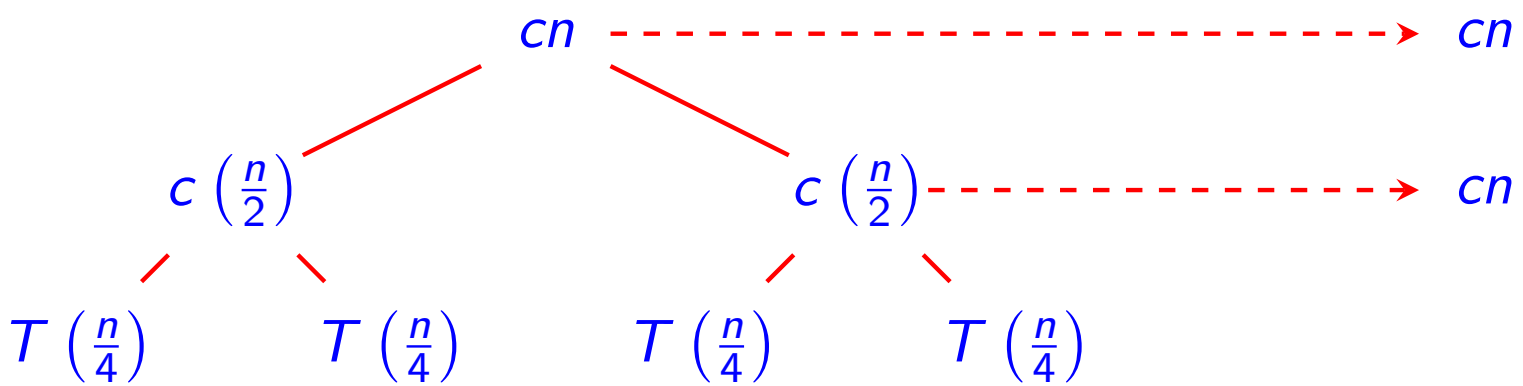

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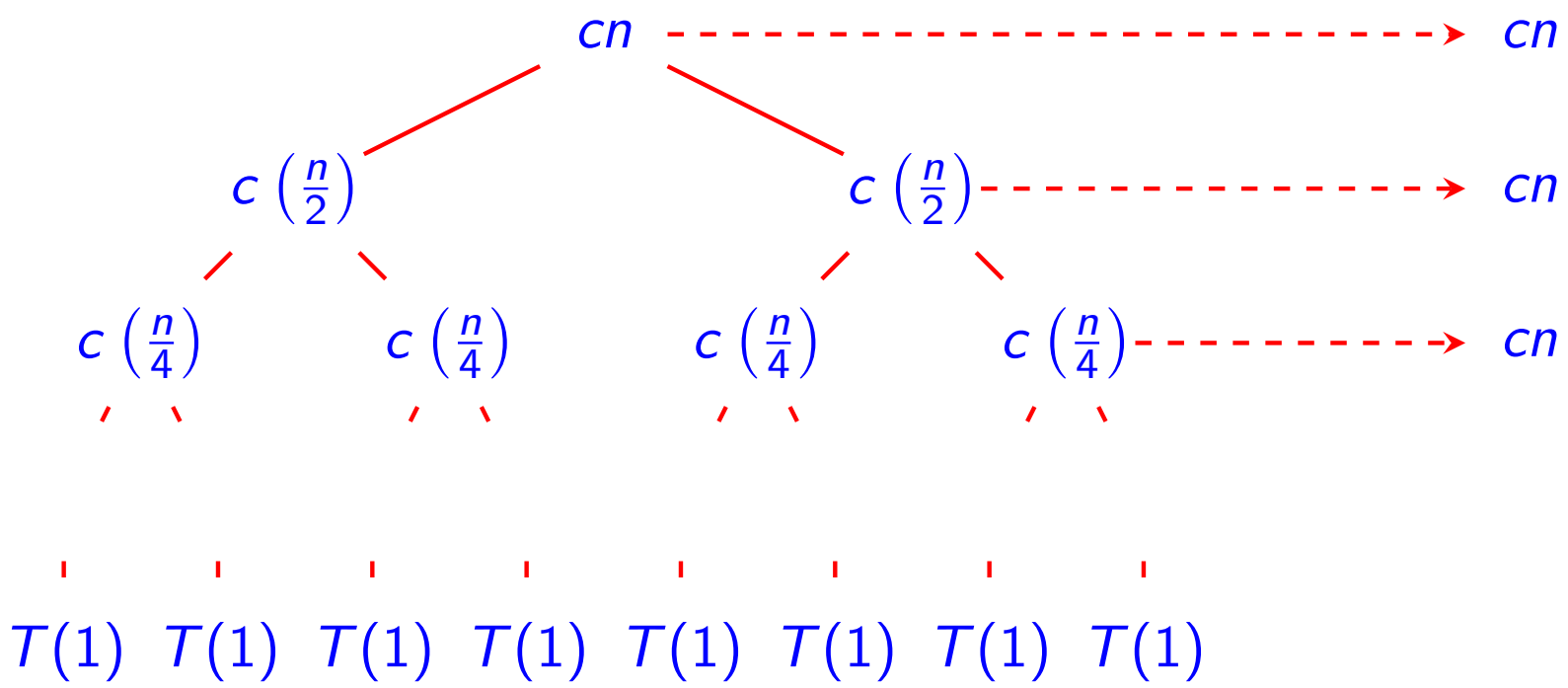

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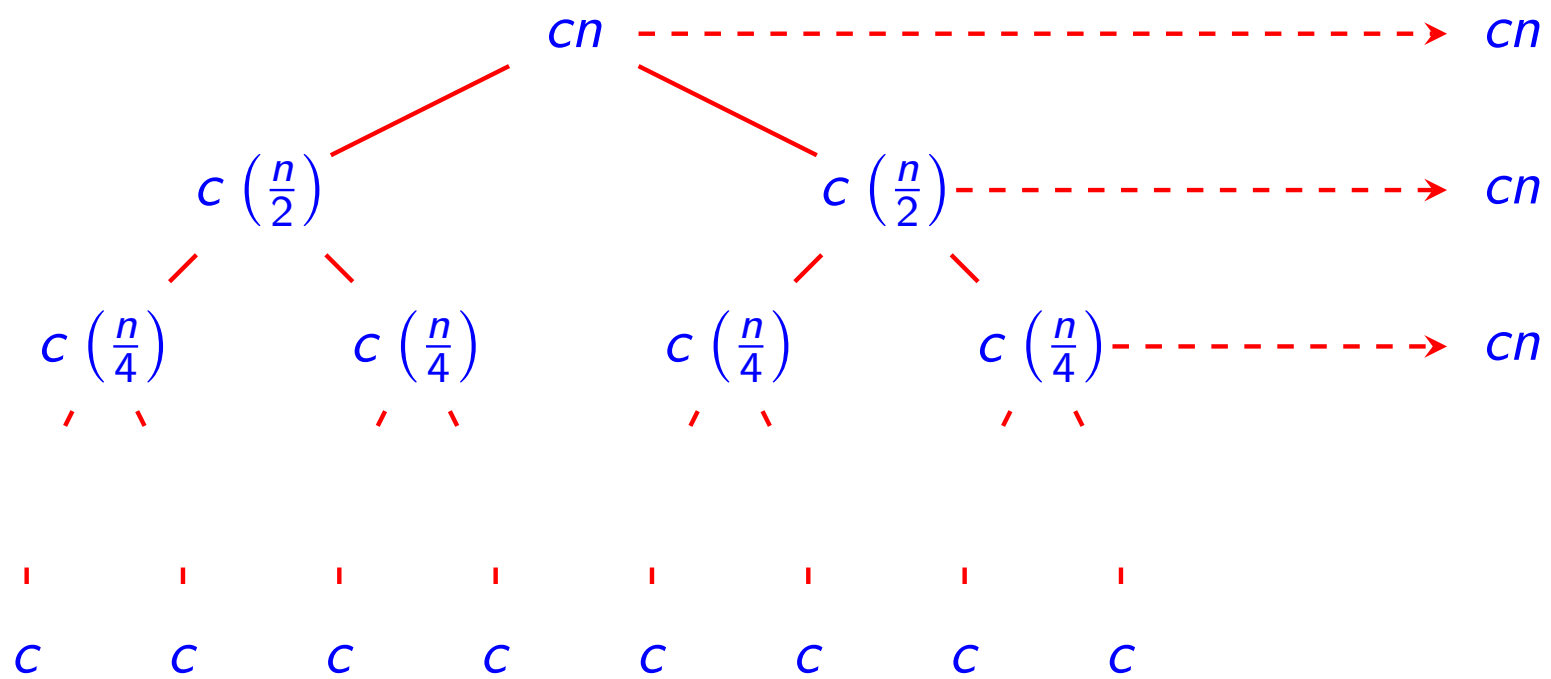

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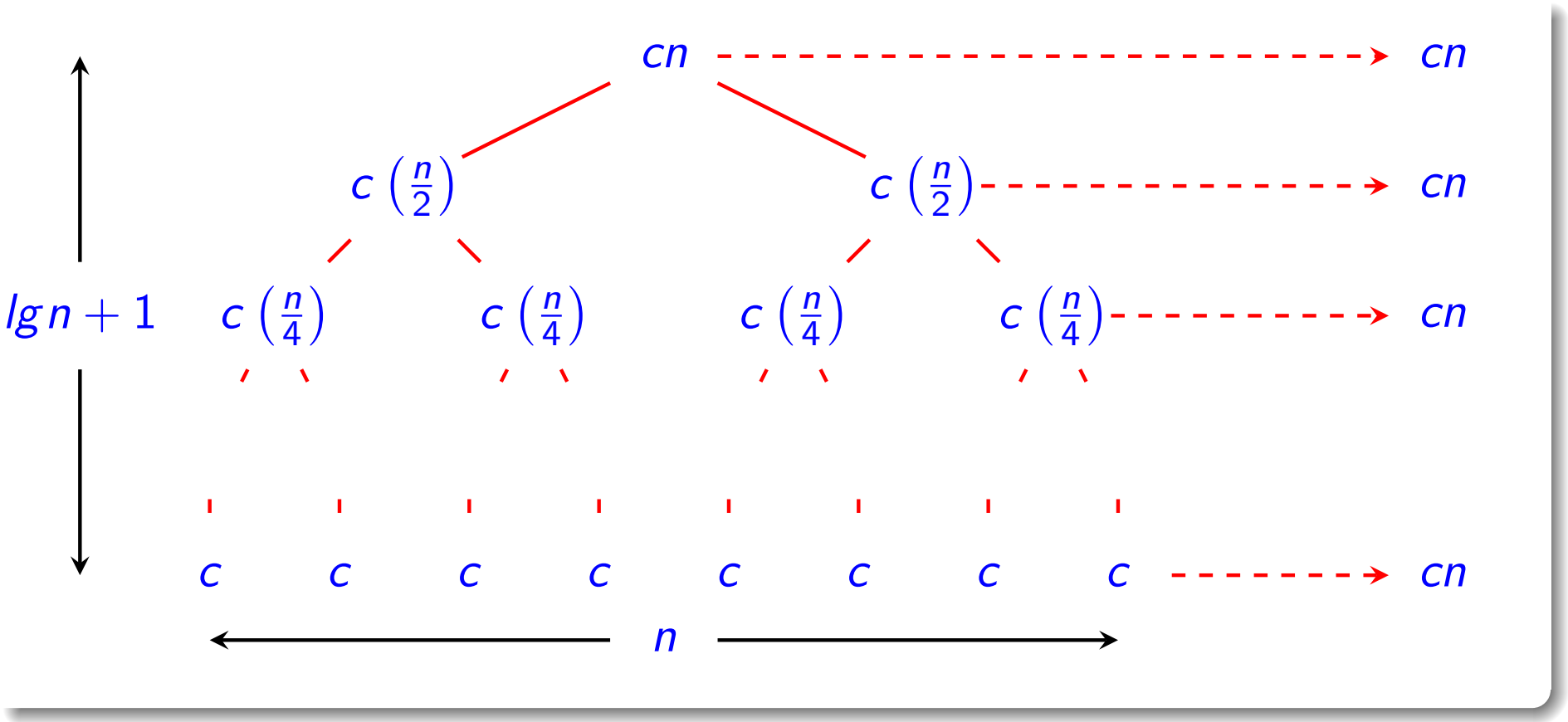

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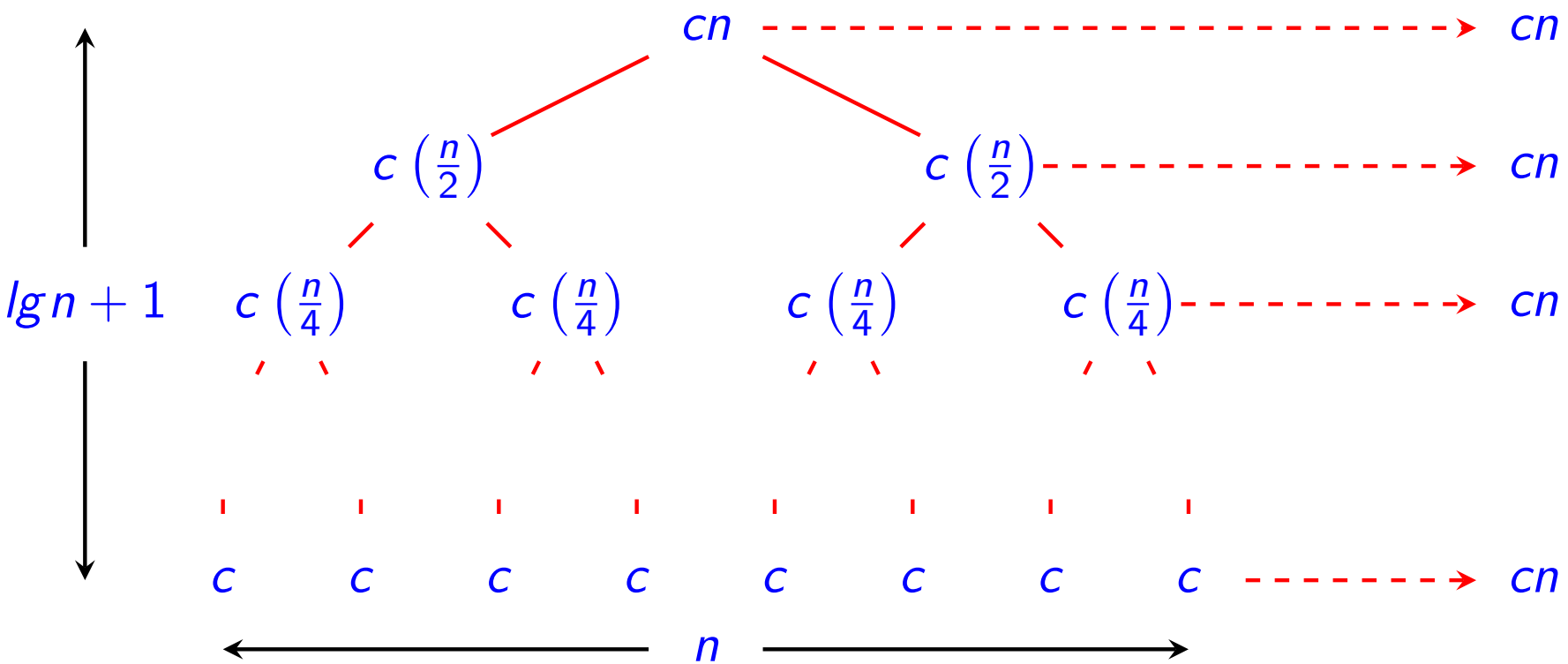

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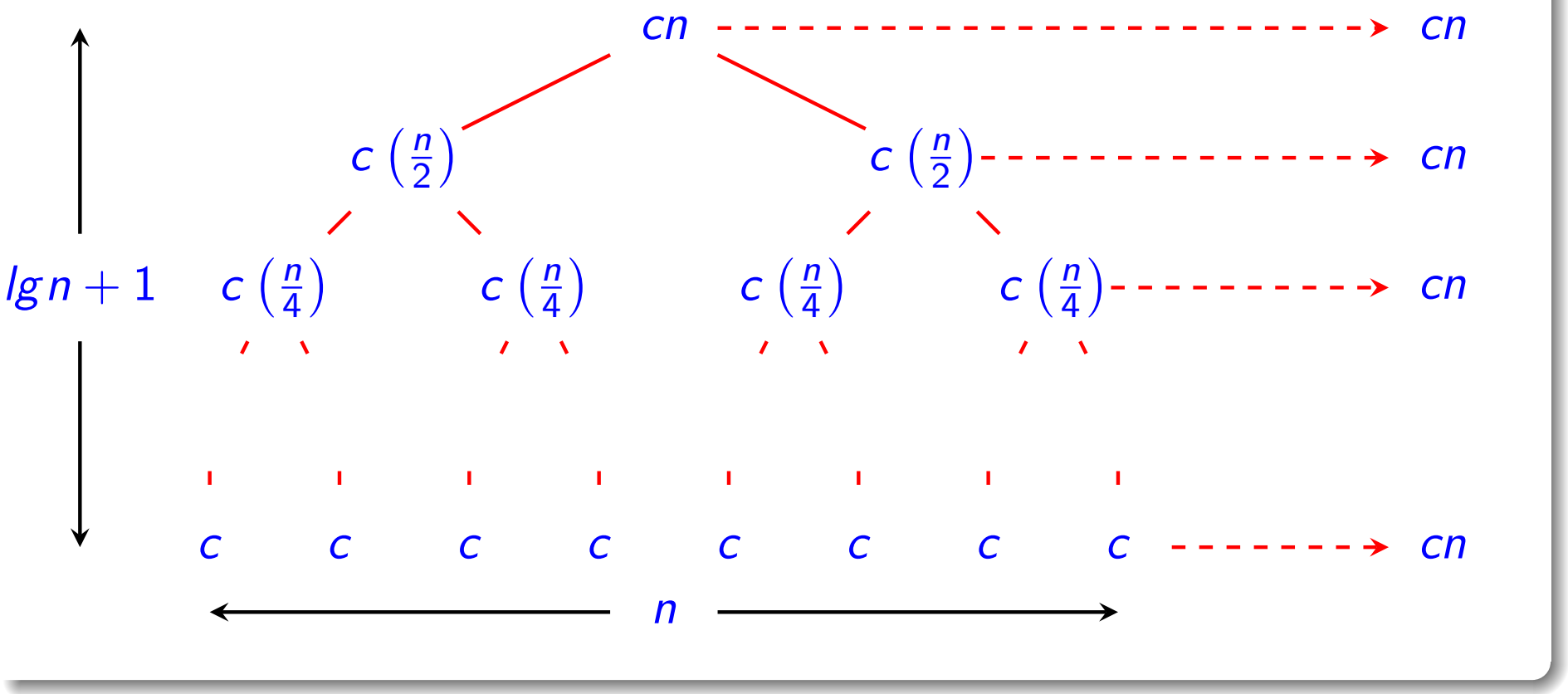

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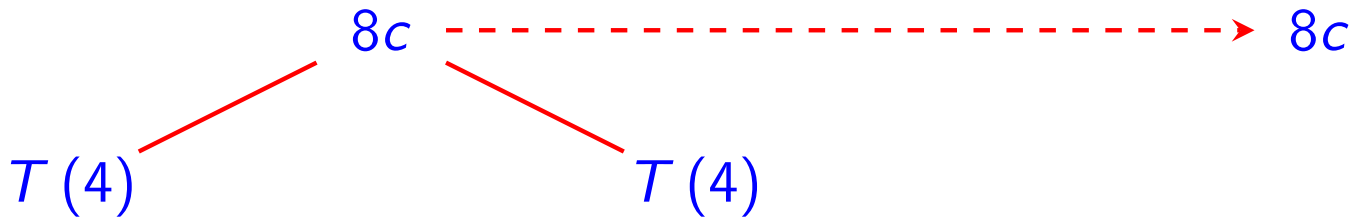

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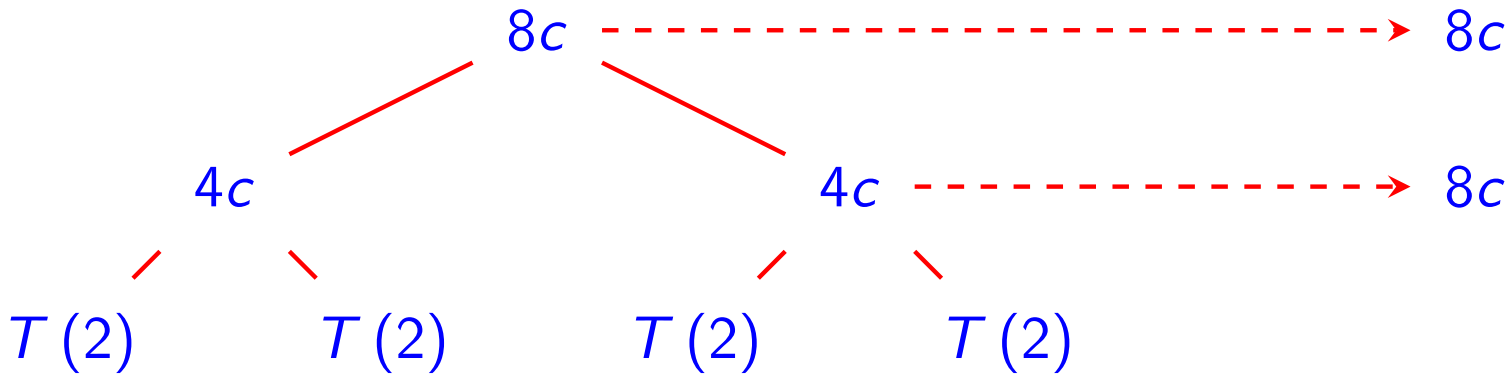

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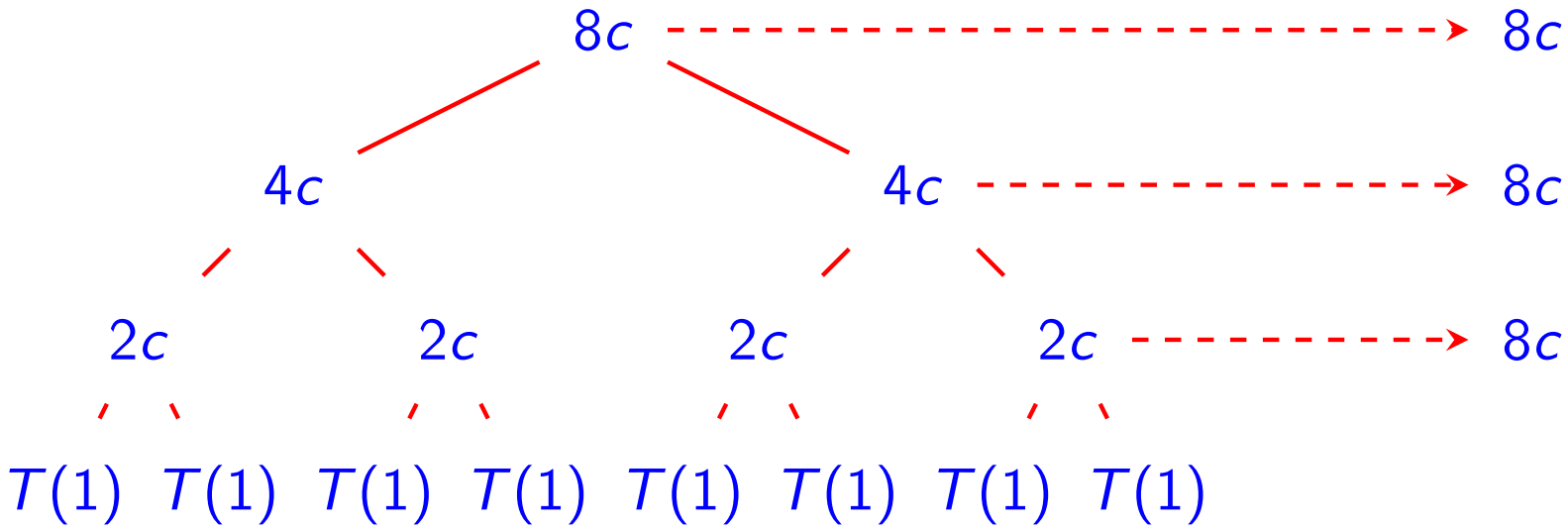

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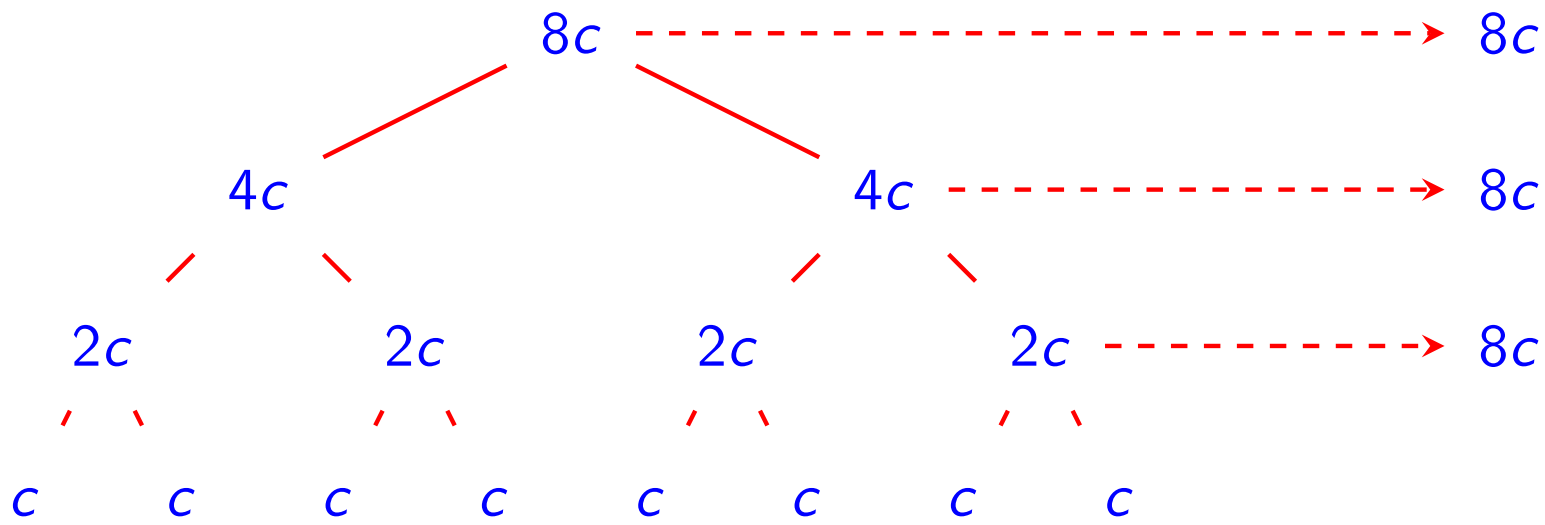

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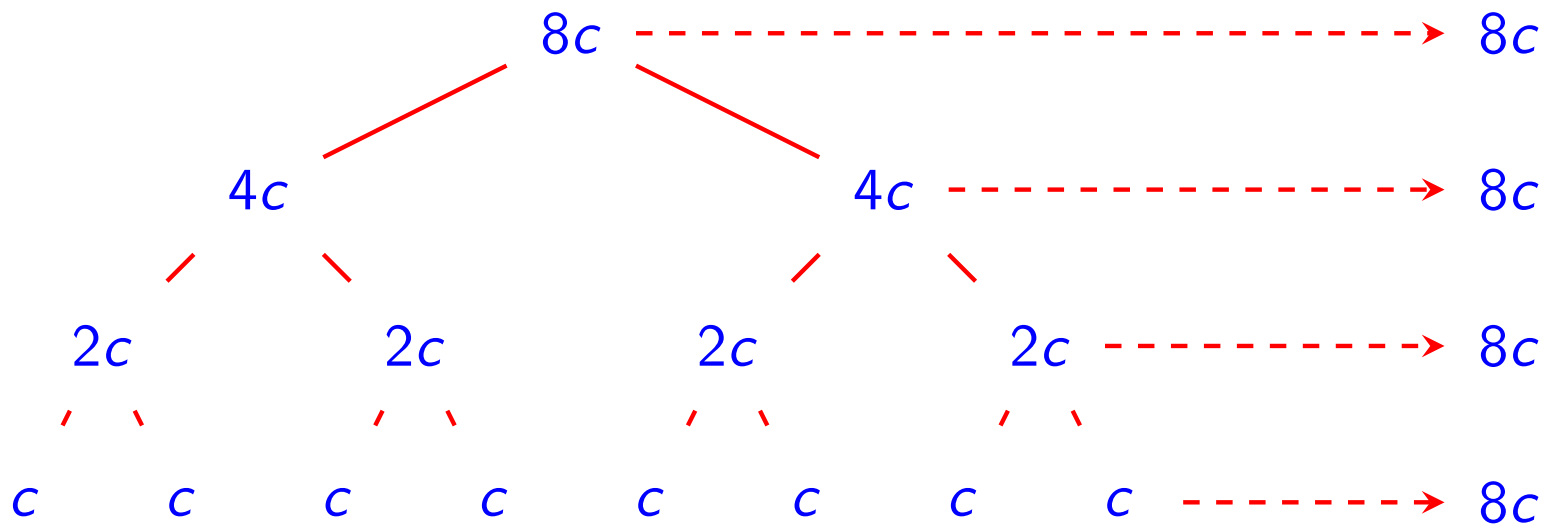

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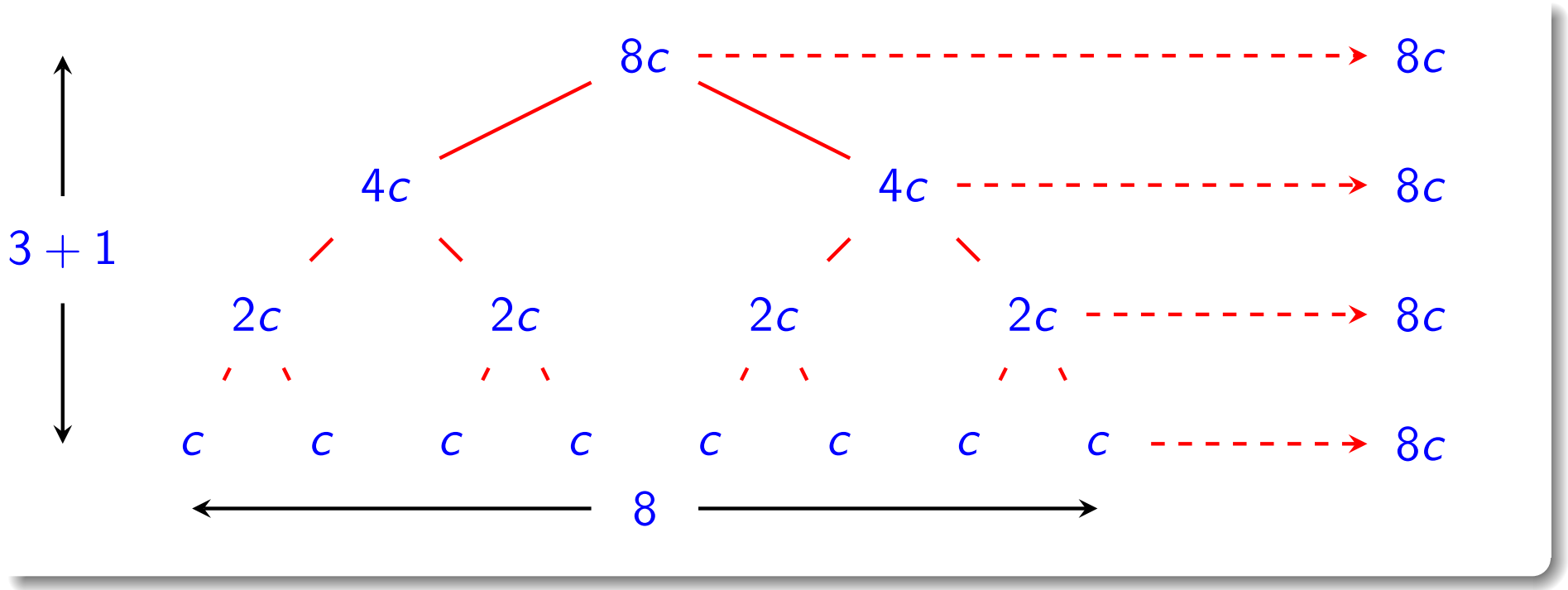

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The Master Method for Merge Sort

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Then, we have $a = 2$, $b = 2$, $f(n) = \Theta(n)$. Case 2 of the Master Theorem applies since $f(n) = \Theta(n^{\log_2(2)}) = \Theta(n)$. Thus we have:

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The Substitution Method for Merge Sort

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + \Theta(n) & \text{if } n > 1 \end{cases}$$

The substitution method consists in:

- Guess a solution
- Verify the correctness of the solution using mathematical induction

The Substitution Method for Merge Sort

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We rewrite $T(n)$ as follows:

$$T(n) = \begin{cases} c & \text{if } n \leq 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + cn & \text{if } n > 1 \end{cases}$$

and try to find an asymptotic upper bound on $T(n)$.

The Substitution Method for Merge Sort

We guess that $T(n) = O(n \cdot \log(n))$, and try to show that for all $n \geq n_0 = 2$, $0 \leq T(n) \leq a \cdot n \cdot \log(n)$ with $a = 2c$.

- Base Case ($n = n_0$):
  $T(n_0) = T(2) = 2T(1) + 2c = 4c \leq a \cdot 2 \cdot \log(2)$.

- Induction Step: Let $n > n_0$.

  Assume (the I.H.): $T(i) \leq a \cdot i \cdot \log(i)$ for all $i < n$

  Show: $T(n) \leq a \cdot n \cdot \log(n)$

$$
\begin{array}{rcll}
T(n) & = & 2T\left(\frac{n}{2}\right) + cn & \text{(by definition of } T(n)\text{)} \\
& \leq & 2(a \cdot \frac{n}{2} \cdot \log(\frac{n}{2})) + cn & \text{(by the I.H.)} \\
& \leq & a \cdot n \cdot \log(n) - an + cn & \\
& \leq & a \cdot n \cdot \log(n) &
\end{array}
$$

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(But for sorting certain things we can do better by not using plain comparison operations.)