Binary Search Trees

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(Based on previous material by Mohamed Faouzi Atig and Parosh Aziz Abdulla)

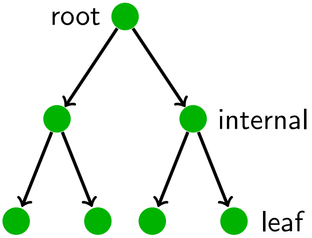
Introduction

Binary Search Trees

- Data structures that support many dynamic-set operations, including SEARCH, MINIMUM, MAXIMUM, PREDECESSOR, SUCCESSOR, INSERT, and DELETE.
- Can be used as both dictionary and as priority queue
- Basic operations take time proportional to the height of the tree:
  - For complete binary tree with n nodes: worst case $\Theta(log_2(n))$
  - For linear chain with n nodes: worst case $\Theta(n))$

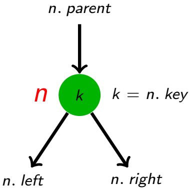
Binary Trees

- A binary tree is a tree such that:
  - Each node has at most two child nodes, distinguished by left and right.
  - The trees we consider here need not be complete binary trees.



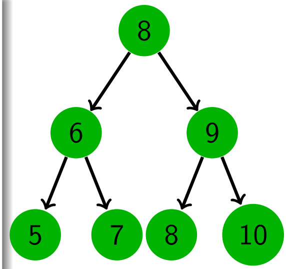
Implementation of Binary Trees

- A binary tree T can be represented as a linked data structure
- Each node n is represented by an object having the following attributes:
  - $n.key$ is the key stored at the node n.
  - ($n.value$ is the value stored at the node n.)
  - $n.left$ points to the left child of n.
  - $n.right$ points to the right child of n.
  - $n.parent$ points to the parent of n.
- If a child or the parent is missing, the appropriate attribute contains the value NIL
- The root node of the tree T is pointed to by the attribute $T.root$.

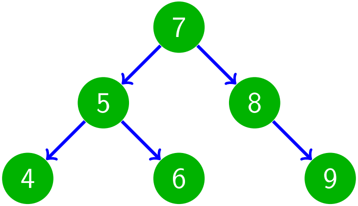


Binary Search Tree

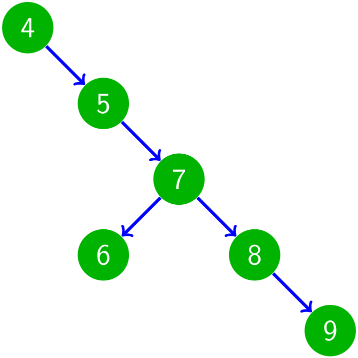
- A binary search tree T is a binary tree such that:
  - For every x and y two nodes of T:
    - If y is in the left subtree of x then $y.key \leq x.\,key$.
    - If y is in the right subtree of x then $x.key \leq y.\,key$.



Binary Search Trees: Example (1)



Binary Search Trees: Example (2)



Binary Search Trees: Operations:

Let T be a tree, x be a node in T, and a key k:

- $\textsc{Inorder-Tree-Walk}(x)$: Print out all the keys of the subtree rooted at x in a sorted order.
- $\textsc{Search}(x, k)$: Return a pointer to a node with key k in the subtree of x if one exists; otherwise, return NIL
- $\textsc{Tree-Minimum}(x)$: Return a pointer to the node with smallest key in the subtree of x.
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TREE-WALK

INORDER-TREE-WALK(x): Print out all the keys of the tree rooted at x in a sorted order.

INORDER-TREE-WALK(x)

if $x \neq NIL$
    then INORDER-TREE-WALK($x.\ left$)
        print $x.\ key$
        INORDER-TREE-WALK($x.\ right$)

INORDER-TREEE-WALK: Principle

- Check whether the node x is not *NIL*
- Recursively, print the keys of the nodes in the left subtree of x
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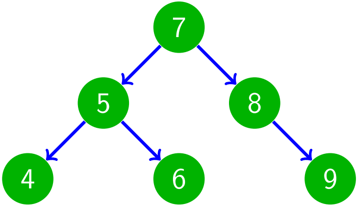

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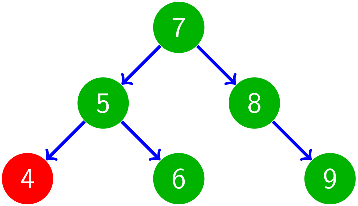

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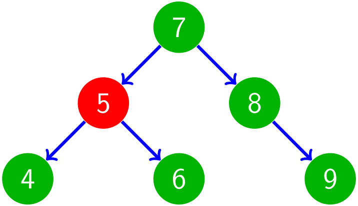

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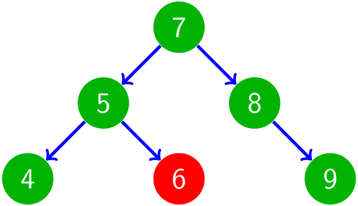

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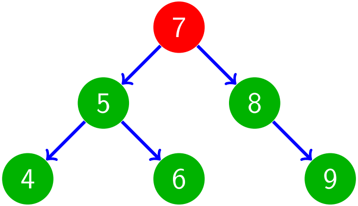

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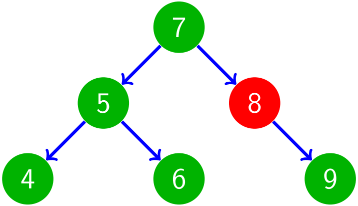

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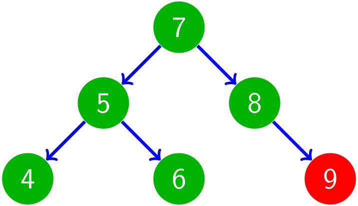

4 5 6 7 8 9

TREE-WALK

- Let n be the number of nodes in the subtree rooted at x
- Let $T(n)$ be the time taken by $\text{INORDER-TREE-WALK}(x)$
- Each of lines 1 and 3 takes constant time.
- Let k be the number of nodes of the left subtree of x then we have:

$$T(n) = T(k) + T(n-k-1) + \Theta(1)$$

INORDER-TREE-WALK(x)

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$$T(n) = \Theta(n)$$

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For each node x it encounters, it compares the key k and $x.key$:

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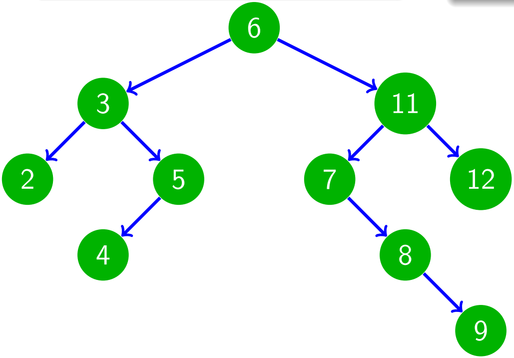

TREE-SEARCH($T.root$, 4)

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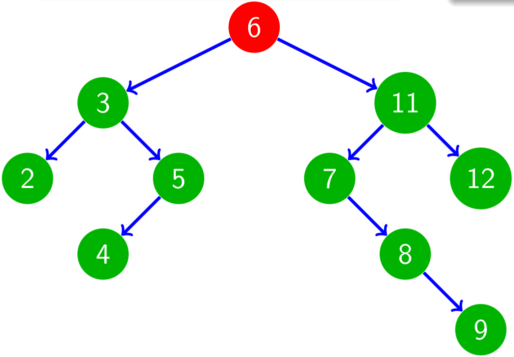

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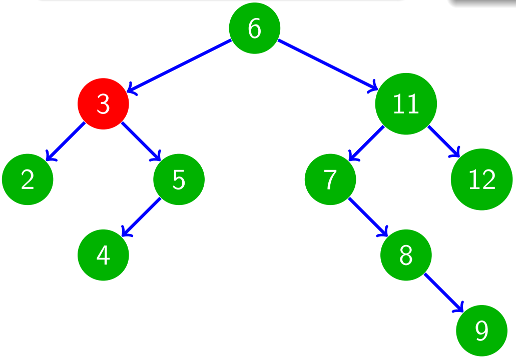

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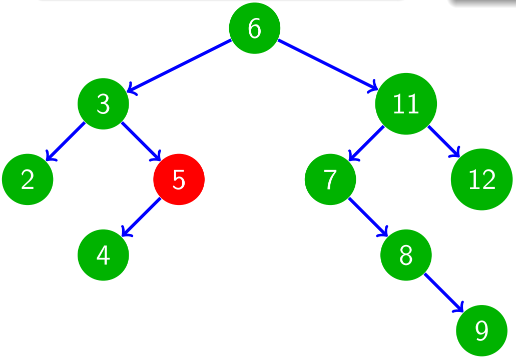

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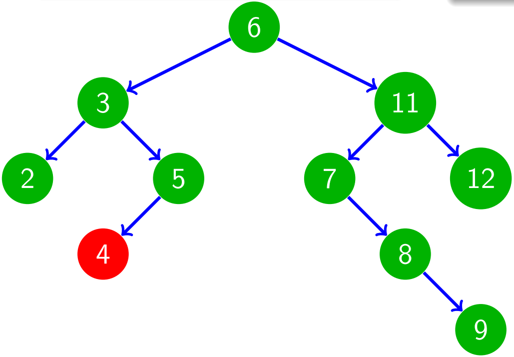

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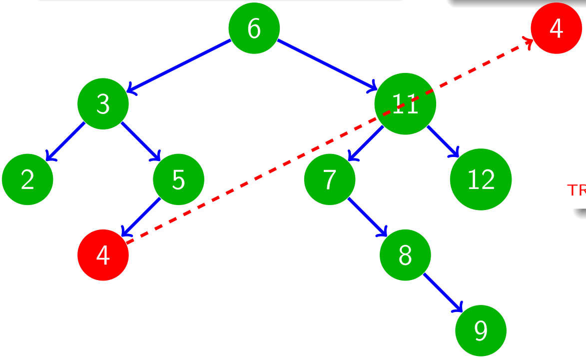

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Binary Search Trees

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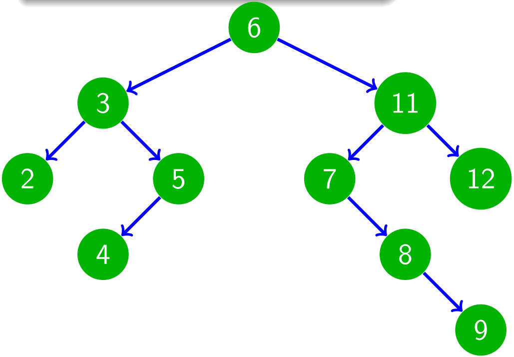
TREE-SEARCH($T.root$, 10)



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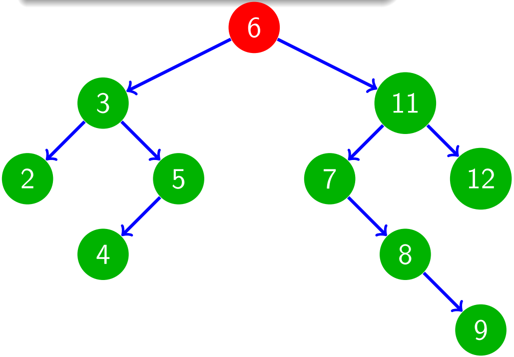

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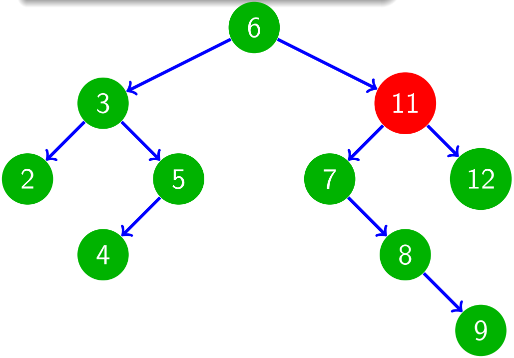
TREE-SEARCH($T.root.right$, 10)



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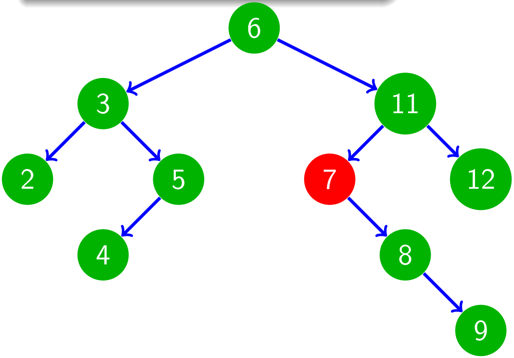
TREE-SEARCH($T.root.right.left$, 10)



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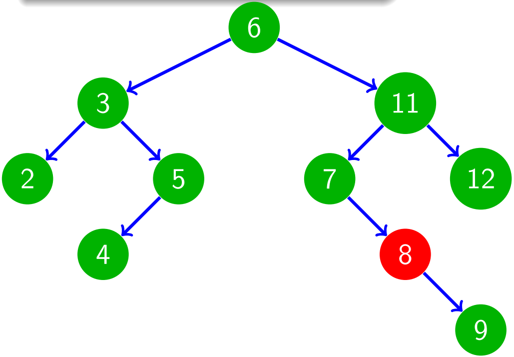

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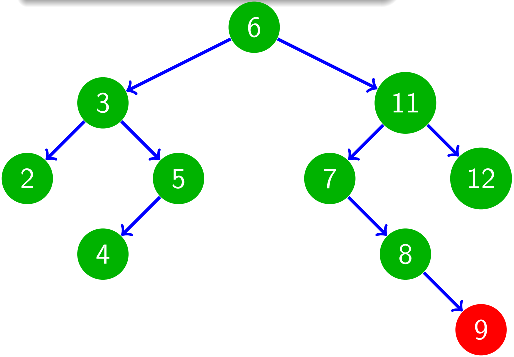

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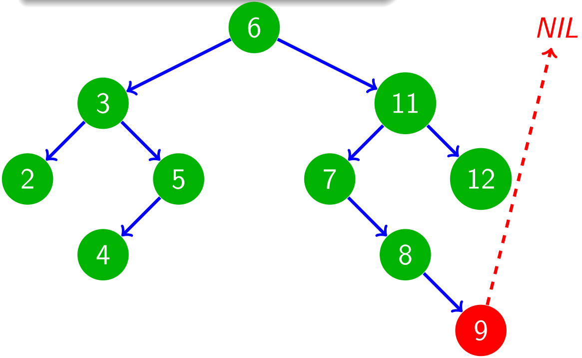

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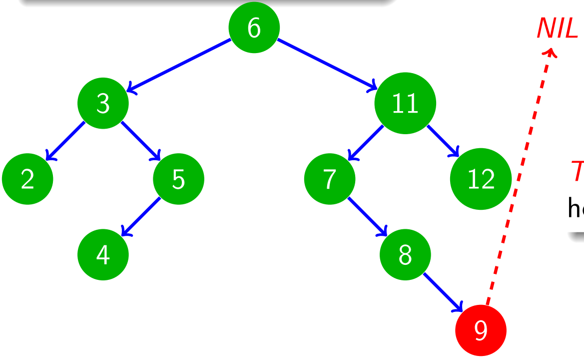

$T(n) = O(h)$ where h is the height of the tree

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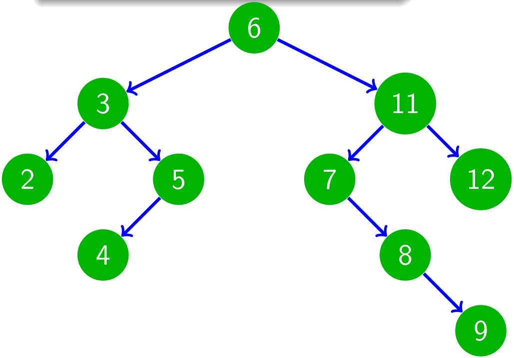
TREE-MINIMUM

TREE-MINIMUM(x)

```
while x. left ≠ NIL
    do x ⟵ x. left
return x
```

TREE-MINIMUM: Principle

The minimum key of a binary search tree is located at the leftmost node



TREE-MINIMUM($T.\ root$)

TREE-MINIMUM

TREE-MINIMUM(x)

while $x.\mathit{left} \neq \mathit{NIL}$
    do $x \longleftarrow x.\mathit{left}$
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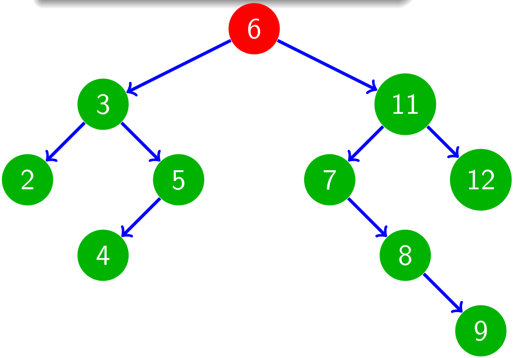

TREE-MINIMUM($T.\mathit{root}$)

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while $x.\,left \neq NIL$
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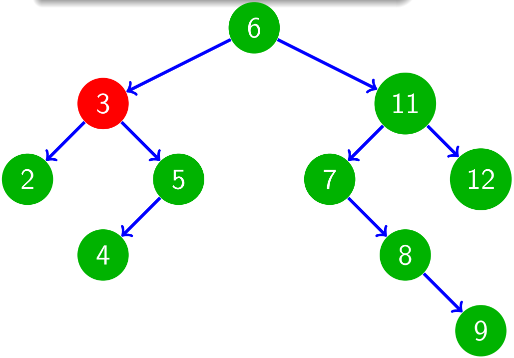

TREE-MINUMUM($T.\,root\,.\,left$)

TREE-MINIMUM

TREE-MINIMUM(x)

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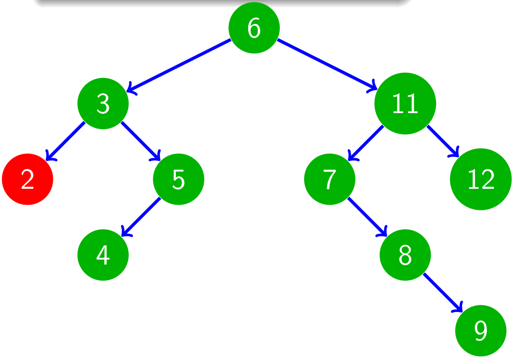

TREE-MINIMUM($T.root.left.left$)

TREE-MINIMUM

TREE-MINIMUM(x)

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    do x ⟵ x.left
return x
```

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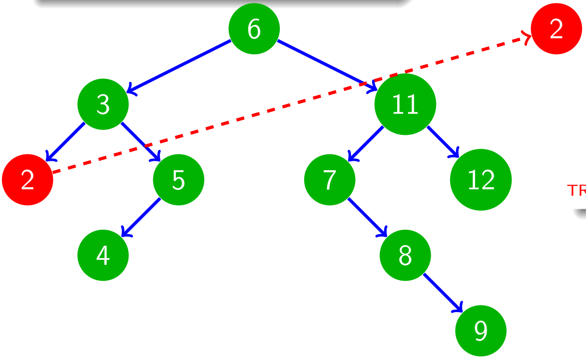

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TREE-MINIMUM

TREE-MINIMUM(x)

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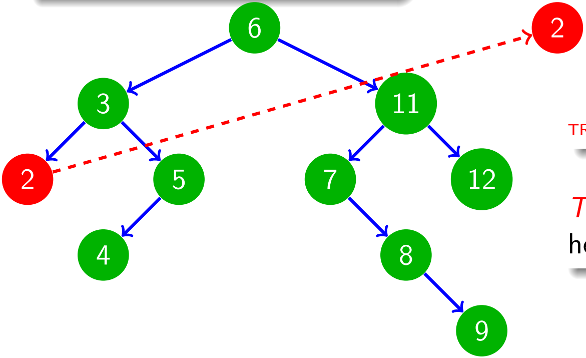

TREE-MINIMUM($T.\,root\,.\,left\,.\,left$)

$T(n) = O(h)$ where h is the height of the tree

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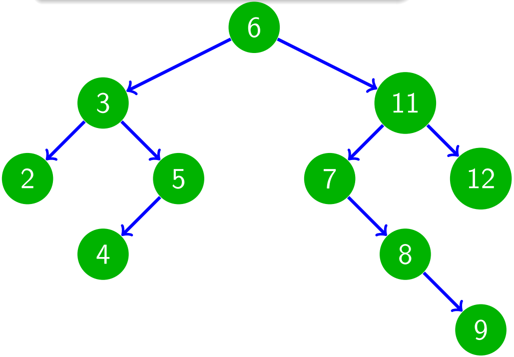
TREE-MAXIMUM

TREE-MAXIMUM(x)

```
while x.right ≠ NIL
    do x ⟵ x.right
return x
```

TREE-MAXIMUM: Principle

The maximum key of a binary search tree is located at the rightmost node



TREE-MINIMUM($T.root$)

TREE-MAXIMUM

TREE-MAXIMUM(x)

while $x.\,right \neq NIL$
    do $x \longleftarrow x.\,right$
return x

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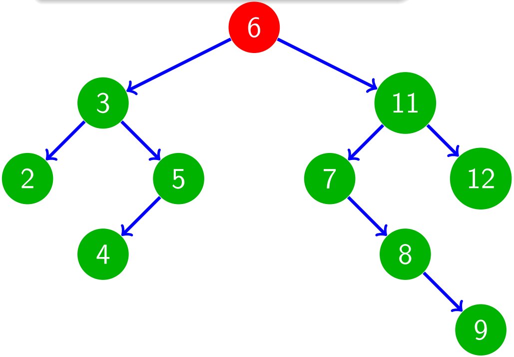

TREE-MINIMUM($T.\,root$)

TREE-MAXIMUM

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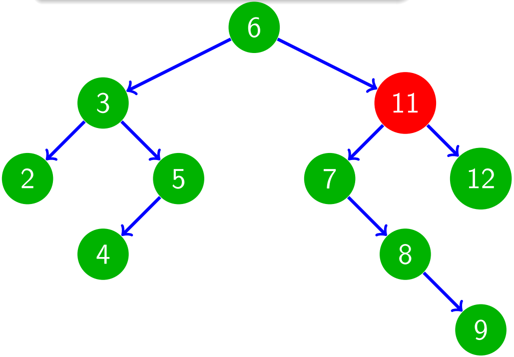

TREE-MINUMUM($T.root.right$)

TREE-MAXIMUM

TREE-MAXIMUM(x)

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return x

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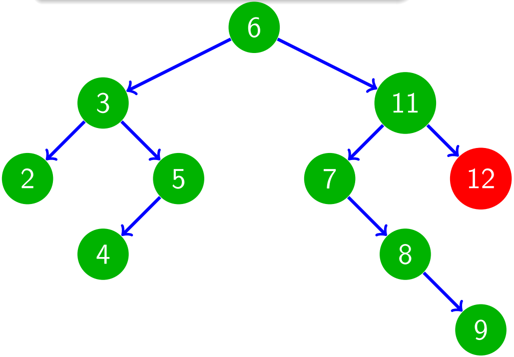

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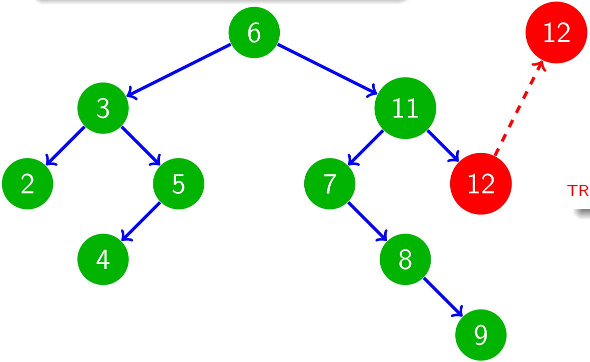

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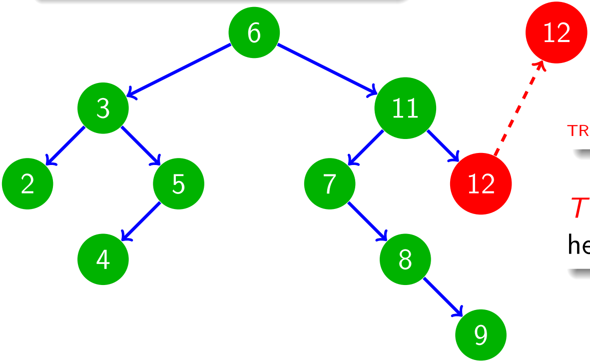

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TREE-SUCCESSOR

TREE-SUCCESSOR(x)

if $x.\,right \neq NIL$
  then return TREE-MINIMUM($x.\,right$)
$y \leftarrow x.\,parent$
while $y \neq NIL$ and $x = y.\,right$
  do $x \leftarrow y$
    $y \leftarrow y.\,parent$
return (y)

TREE-SUCCESSOR: Principle

- Assumption: All keys are different
- To find the node with the smallest key larger than $x.\,key$
  - If the right subtree of x is nonempty, then the successor of x is the minimum in the right subtree of x
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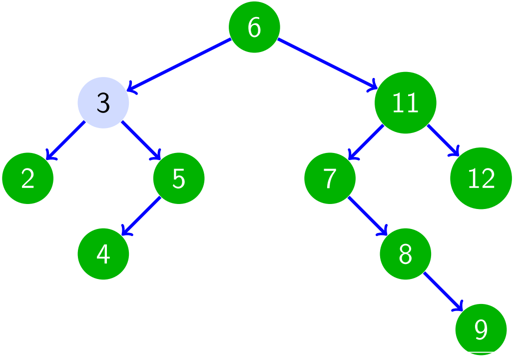

TREE-SUCCESSOR

TREE-SUCCESSOR(x)

```
if x. right ≠ NIL
    then return  TREE-MINIMUM(x. right)
y ← x. parent
while y ≠ NIL and x = y. right
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return (y)
```

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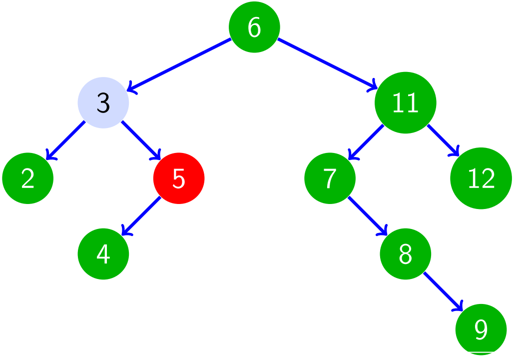

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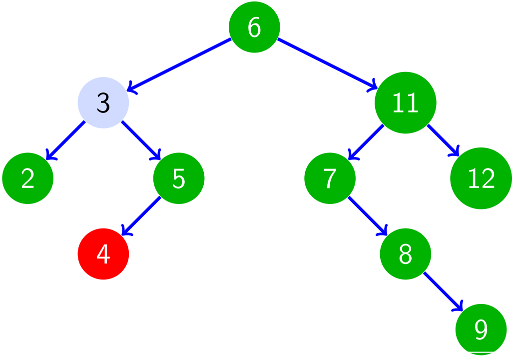

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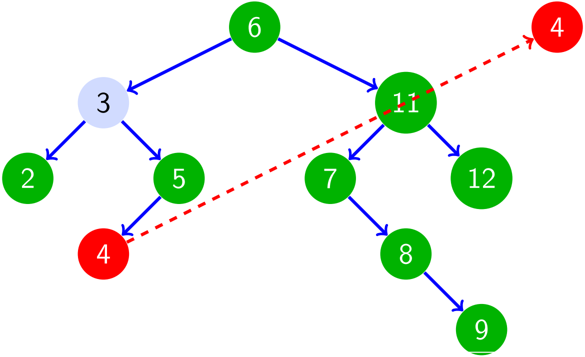

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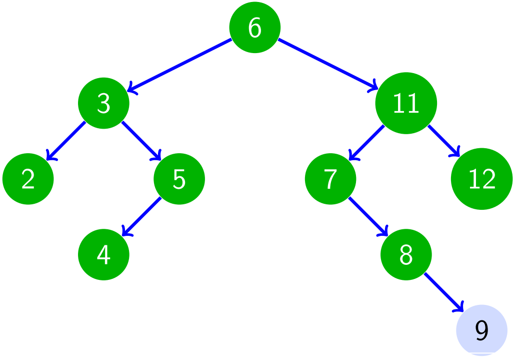

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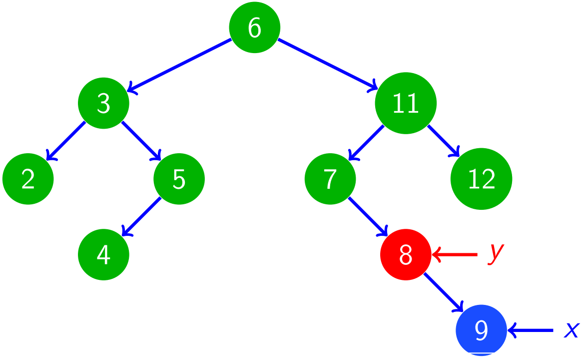

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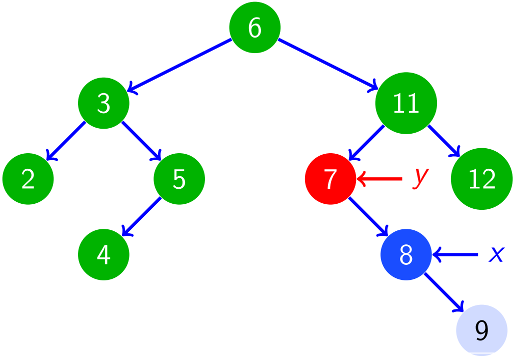

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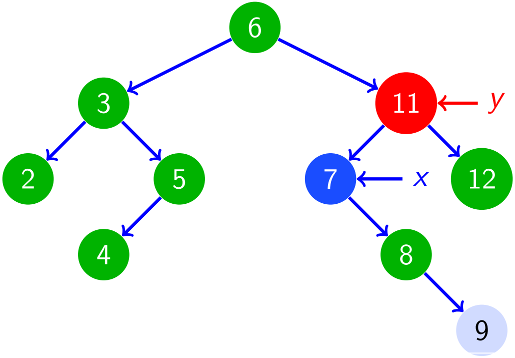

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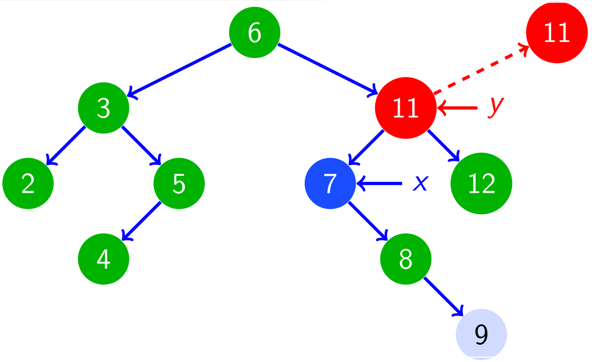

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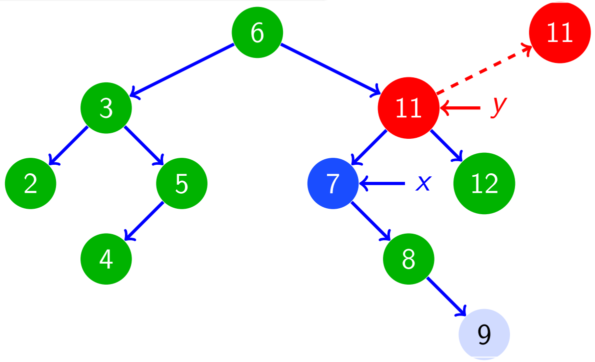

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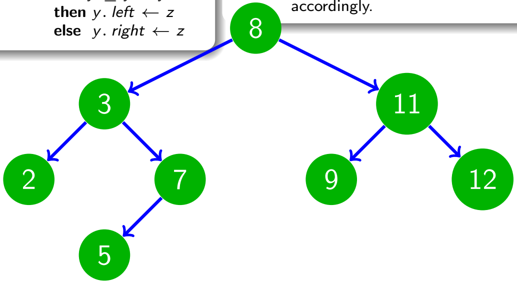
TREE-INSERT

TREE-INSERT(T, z)

```
y ← NIL
x ← T.root
while x ≠ NIL
      do y ← x
         if z.key ≤ x.key
            then x ← x.left
            else  x ← x.right
z.parent ← y
if y = NIL
   then T.root ← z
   else if z.key ≤ y.key
           then y.left ← z
           else  y.right ← z
```

TREE-INSERT: Principle

- Begin at the root and maintain two pointers:
  - Pointer x traces a simple path downward looking for a NIL to replace it with the node z
  - Pointer y: Trailing pointer to keep track of the parent of x
- Traverse the tree downward by comparing $x.key$ to $z.key$, and move to the left or right child accordingly.
- When x is NIL, it is at the right position to insert z
- Compare the key of z to the one of y, and insert z o the left or right of y accordingly.



$z.key = 4$

TREE-INSERT

TREE-INSERT(T, z)

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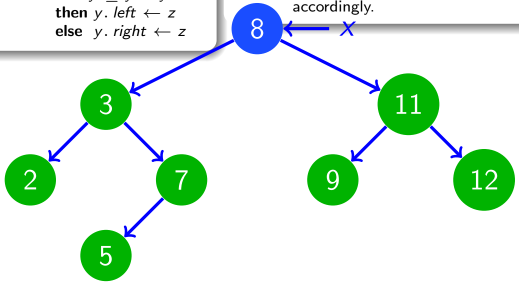

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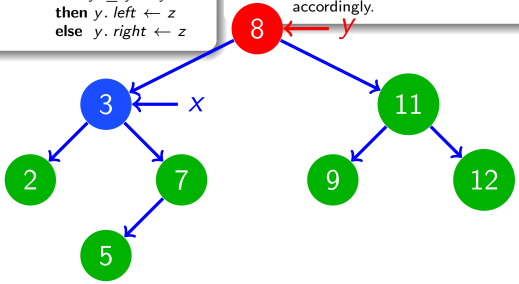

$z.key = 4$

TREE-INSERT

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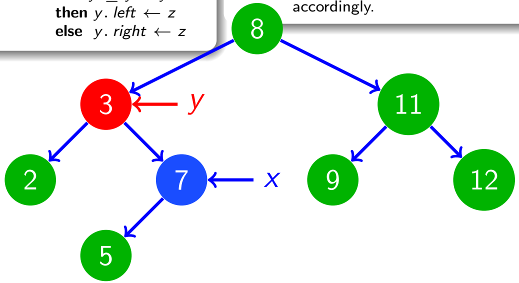

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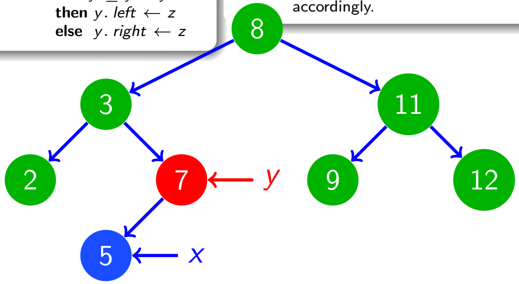

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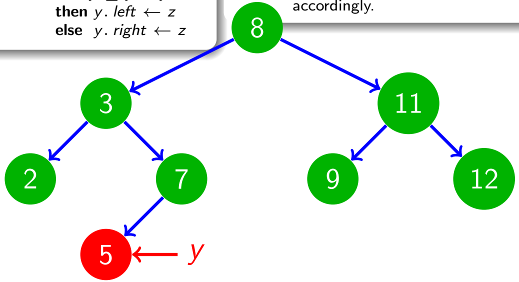

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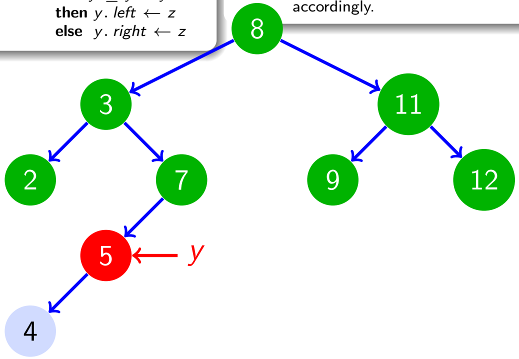

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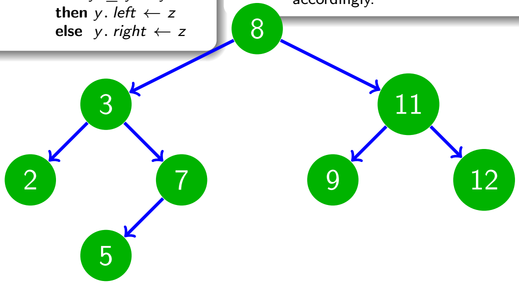

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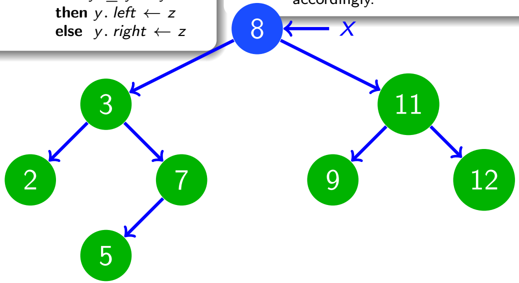

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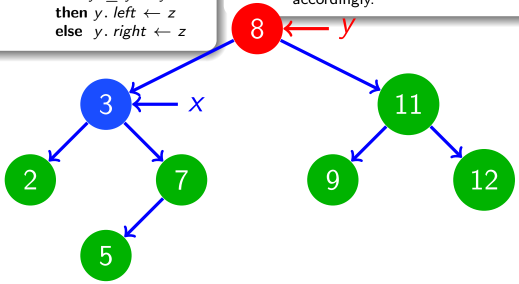

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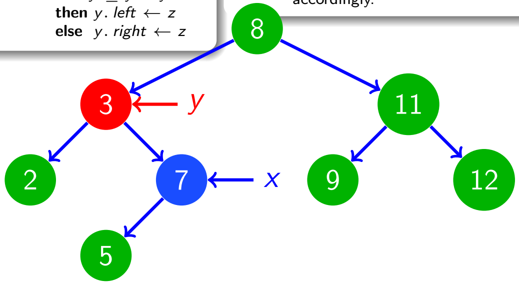

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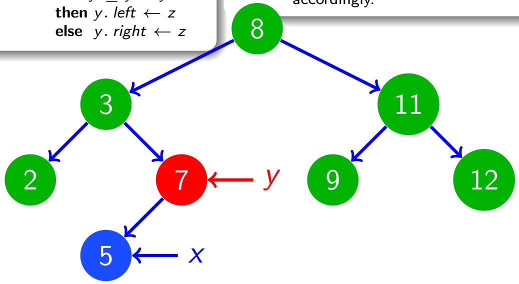

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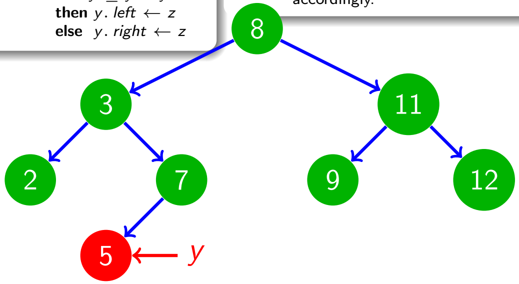

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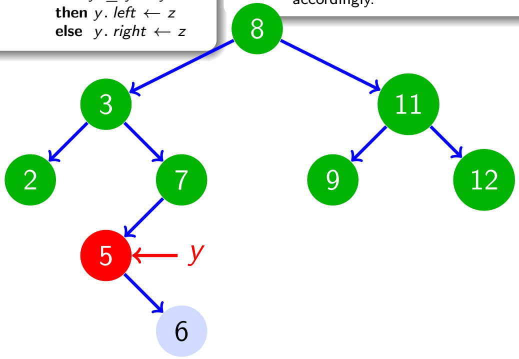

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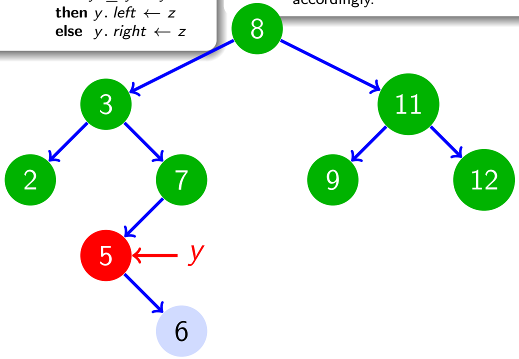

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$T(n) = O(h)$ where h is the height of the tree

Binary Search Trees: Operations:

Let T be a tree, x be a node in T, and a key k:

- $\textsc{Inorder-Tree-Walk}(x)$: Print out all the keys of the subtree rooted at x in a sorted order.
- $\textsc{Search}(x, k)$: Return a pointer to a node with key k in the subtree of x if one exists; otherwise, return NIL
- $\textsc{Tree-Minimum}(x)$: Return a pointer to the node with smallest key in the subtree of x.
- $\textsc{Tree-Maximum}(x)$: Return a pointer to the node with largest key in the subtree of x.
- $\textsc{Tree-Successor}(x)$: Return a pointer to the node with the smallest key larger than $x.key$.
- $\textsc{Tree-Insert}(T, x)$: Insert x in T such that the binary search property is preserved.
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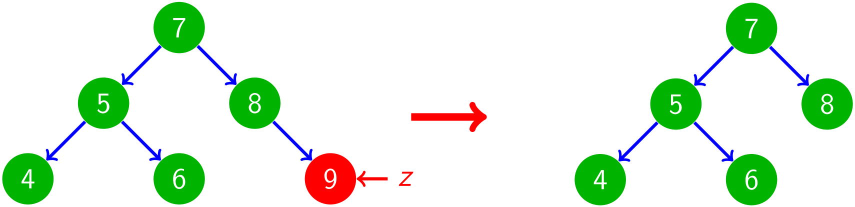
TREE-DELETE: Principle

TREE-DELETE(T, z): Delete z from T such that the binary search property is preserved.

There are three cases to consider depending on the position of the node z:

- Case 1: The node z has no children
    - Delete z by making the parent of z point to *NIL*, instead of to z.



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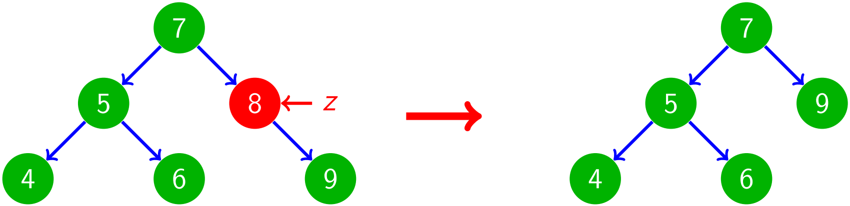

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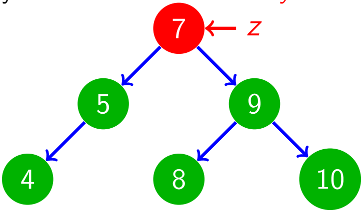
There are three cases to consider depending on the position of the node z:

- Case 3: The node z has two children
  - Compute y the successor of z (y has at most one child).
  - Delete y from the tree (via case 1 or 2)
  - Replace z's key and satellite data with y's



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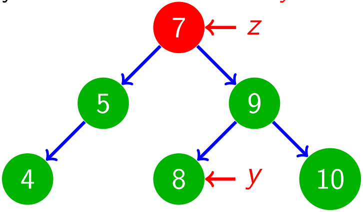

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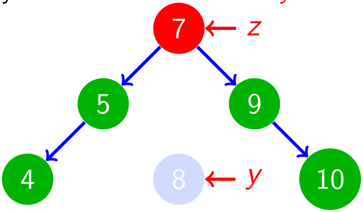

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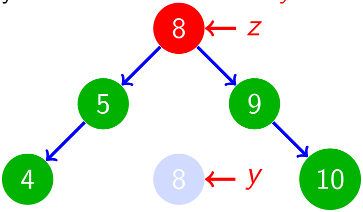

TREE-DELETE

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TREE-DELETE(T, z)
▷ Determine which node y to splice out:
either z or z's successor
if z.left = NIL or z.right = NIL
    then y ← z
    else  y ← TREE-SUCCESSOR(z)
▷ x is set to a non-NIL child of y, or to NIL
if y has no children
if y.left ≠ NIL
    then x ← y.left
    else  x ← y.right
if x ≠ NIL
    then x.parent ← y.parent
▷ y is removed from the tree by manipulating
pointers of y.parent and x
if y.parent = NIL
    then T.root = x
    else  if y = y.parent.left
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▷ If it was z's successor that was spliced out,
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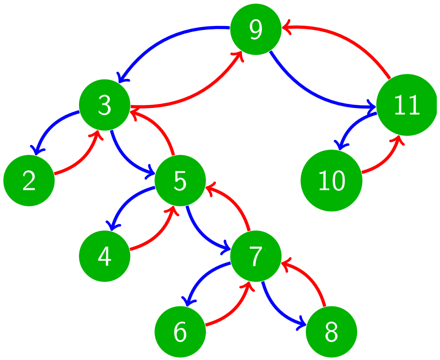

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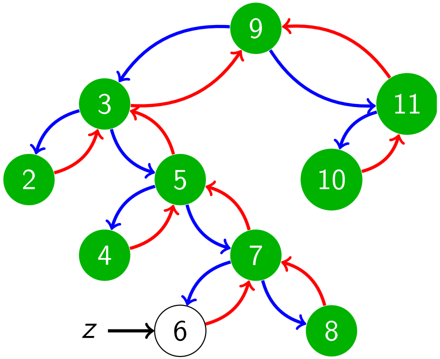

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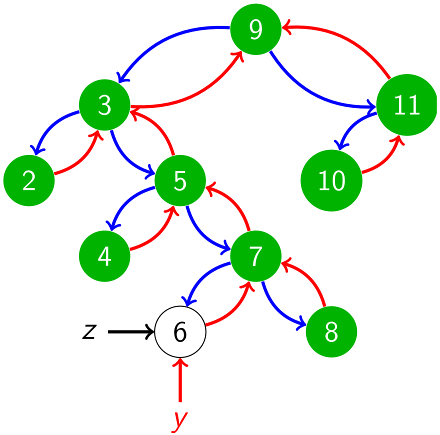

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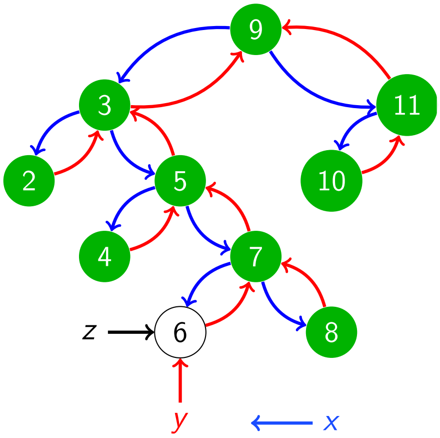

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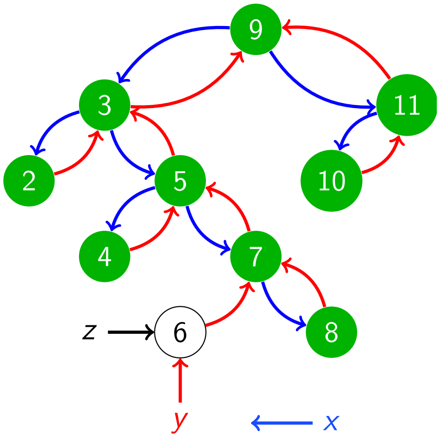

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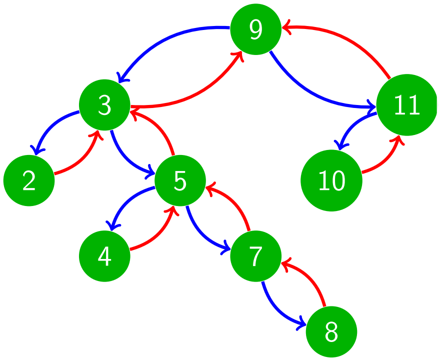

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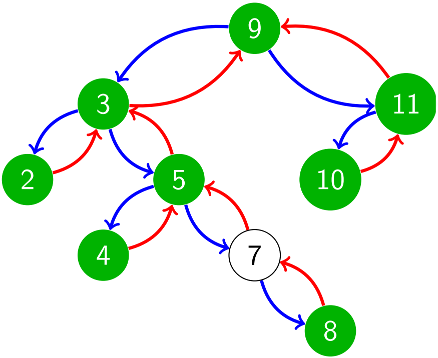

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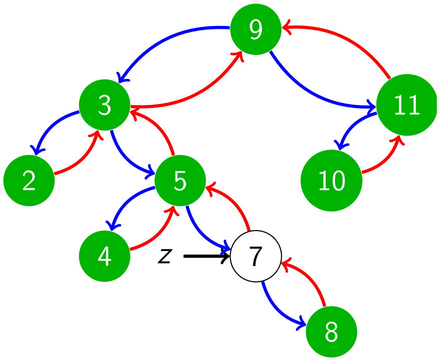

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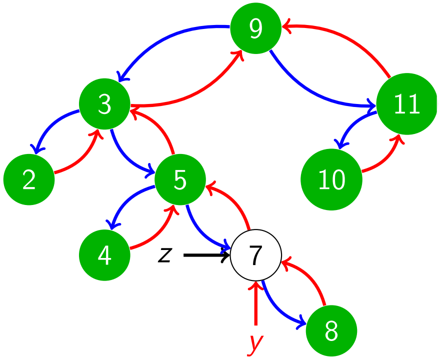

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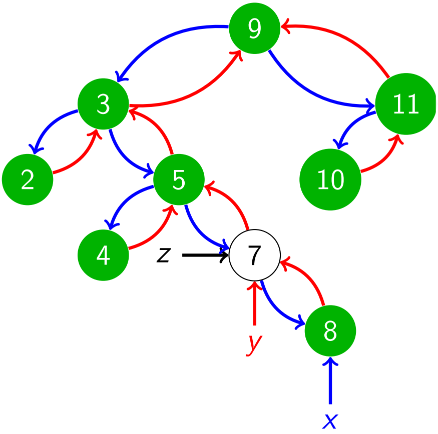

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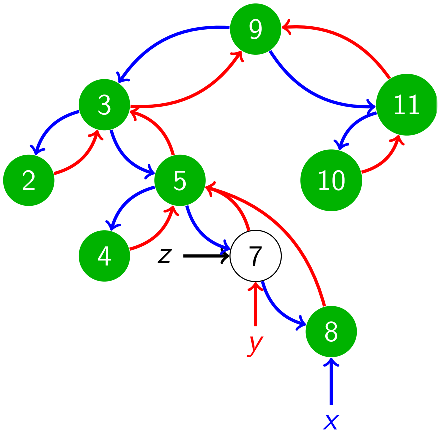

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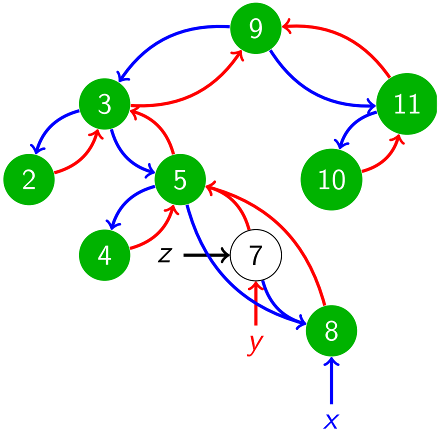

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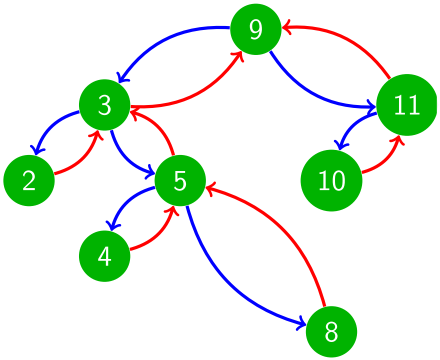

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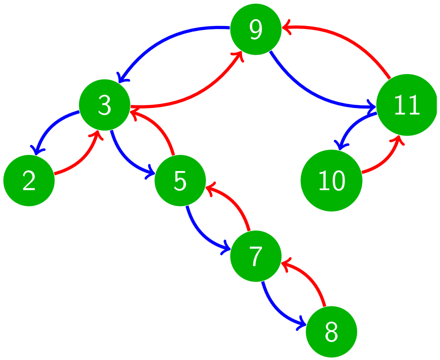

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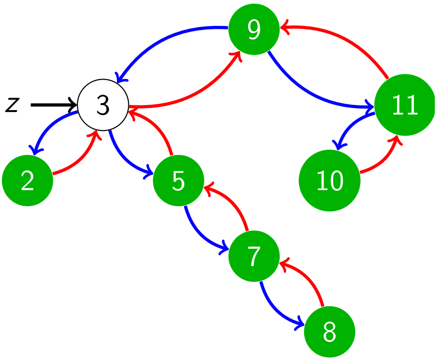

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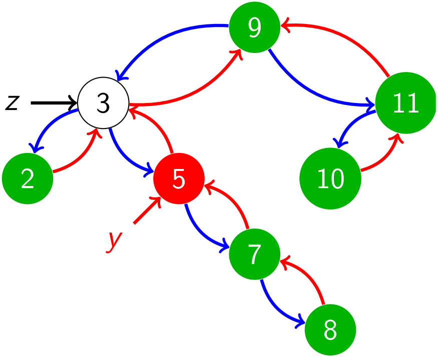

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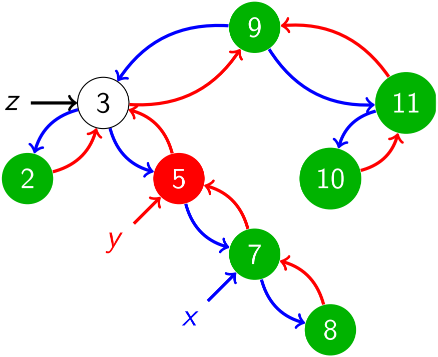

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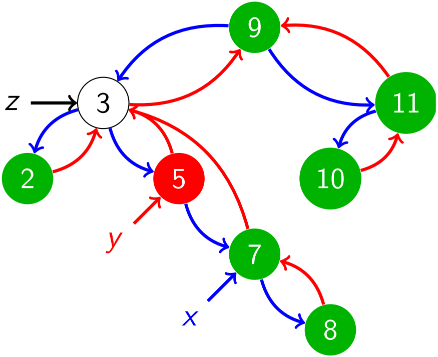

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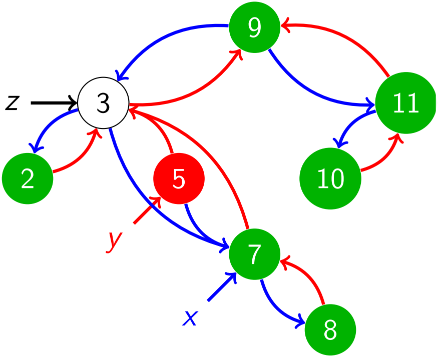

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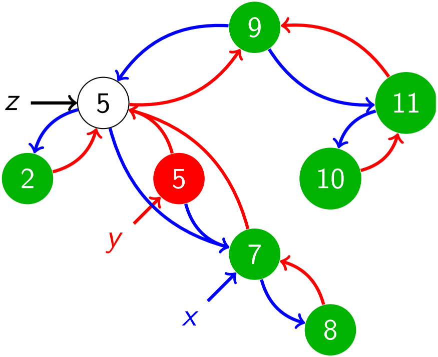

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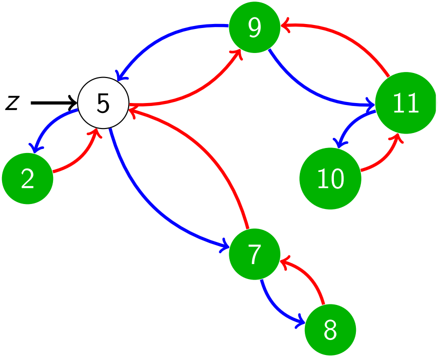

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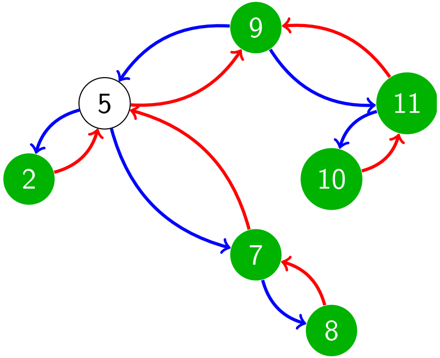

$T(n) = O(h)$ where h is the height of the tree

Binary Search Trees

- Almost all the operations can be performed in time $O(h)$ where h is the height of the tree.
- If the tree contains n nodes then we have:
  - Worst-Case: $h = n - 1 = O(n)$
    - For linear chain of n nodes
  - Best-Case: $h = O(log_2(n))$
    - For a complete binary tree with n nodes
  - Average-Case: $h = O(log_2(n))$