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Instrument for Measuring Moment of Inertia with High Precision

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ABSTRACT

Accurate calculation of the moment of inertia of an irregular body is made difficult by the large number of quantities. A popular method is to use a trifilar suspension system to measure the period of oscillation of the body in the horizontal plane. In this paper, an instrument for measuring the moment of inertia based on trifilar pendulum is designed; some sources of error are discussed; three metal disks with known moments of inertia are used to calibrate the instrument, the other metal disks with known moments of inertia are used to test the accuracy of the instrument. The results are consistent when compared with calculated moment of inertia of the metal disks. In addition, the instrument could be used to measure the moment of inertia of other irregular objects. The period of oscillation is acquired by the capture mode of MSP430 microprocessor, the mass is obtained by the Electronic Balance and the data is transferred to the MSP430 via serial port.

Keywords: trifilar pendulum, moment of inertia, irregular objects

1. INTRODUCTION

In dynamic analyses of solid components it is essential to know the moment of inertia. For simple geometry this can be calculated, but if the geometry is complicated or the exact specification unknown, it is necessary to measure the inertia experimentally. With the development of new sensors and equipment a variety of dedicated machinery has been designed to address this purpose. For example, Witter et al. [1] describe techniques using six degree of freedom load cells and linear accelerometer arrays. With such machinery the majority of measurement error is accounted for as part of a regular calibration schedule, reducing the scope for measurement error in normal use. However, the use of these machines may be less appropriate and difficult to justify in comparison to more traditional methods. Hongwu Wang et al. [2] describe an experiment measuring the moment of inertia of an electric powered wheelchair using four-cable pendulum method. However, it is difficult to suspend the heavy and irregular objects through the center of mass.

The moment of inertia is an inertial property which can be measured only by some sort of dynamic test. The object whose inertial properties are to be measured is subjected to controlled generalized forces, and the resulting motion is introduced, in the present case the motion is a rotational motion and the generalized forces, actually moments, can cause either periodic or non-periodic motion. This allows the test procedures to be divided into two large categories:

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acceleration methods and oscillation methods. The oscillation methods mean that the motion is periodic and the moment of inertia is obtained from the period of oscillation. Genta and Delprete [3] conclude that the results from oscillatory methods are less affected by the presence of damping than those obtained using acceleration-based methods, and that torsional multifilar pendulum are generally considered to be the most accurate. These are reported to be capable of producing results with errors of less than 1%[3,4]. Lyons[5] concurs that for plate rotations of less than 10% the non-linear component of the motion does not cause significant errors in the results.

The use of bifilar torsion pendulum has been employed extensively for the measurement of the moment of inertia of aircraft and is considered to have advantages over simple compound pendulum[6-8]. But an obvious drawback of a bifilar pendulum is its unstable nature; only limited arrangements of an object can be accommodated. In contrast, the trifilar pendulum allow for almost any orientation of the body to be tested and also permit the suspension of a plate from the wires, upon which the test body may be rested. The trifilar pendulum is a widely-used apparatus for the measurement of the moment of inertia[9]. It typically consists of a circular platform suspended by three parallel cables attached equidistantly around a circle near the periphery of the platform.

In Section 2 the trifilar method is summarized and simplified calculations are derived for the determination of moments of inertia. The architecture of the instrument is introduced in Section 3, including the hardware and software design. Section 4 analyzed the error may caused by the system, and a new calibration method is proposed. Several standard regular weights with calculated moment of inertia were used to verify the precision in Section 5. Section 6 gives the summary.

2. METHODOLOGICAL APPROACH

The following is the derivation of equation for calculating the mass moment of inertia of any given object from readily measurable quantities in the trifilar pendulum system. A summary of the variables is given in table 1.

Table 1 Definition of variables

Symbol	Quantity
r	upper plate radius
R	bottom plate radius
L	length of cable
H	distance between upper plate and bottom plate
m_0	bottom plate weight

m_x	object to measured weight
I_0	moment inertia of bottom plate with respect to OO'
θ_m	maximum angle displacement of bottom plate from equilibrium position
h_m	maximum height of bottom plate from equilibrium position
ω_m	maximum rotary angular velocity
$v = \frac{dh}{dt}$	velocity of bottom plate's up-and-down movement
T_0	period time when bottom plate is empty
T_1	period time when bottom plate is with object
g	gravity acceleration

For a group of objects, the combined moment of inertia is equal to the sum of the individual moments about a given axis. Given this, the difference between the moment of inertia of the bottom plate along with the object to be measured, I_1, and bottom plate, I_0, would yield moment of inertia of any object, I_x.

$$I_x = I_1 - I_0 \tag{1}$$

The equations used to calculate I_0 and I_1 were derived from the law of conservation of energy.

$$\frac{1}{2}I_0\omega^2 + \frac{1}{2}m_0v^2 + m_0gh = constant \tag{2}$$

The translational energy $\frac{1}{2}m_0v^2$ is much less than rotational kinetic energy $\frac{1}{2}I_0\omega^2$, the translational energy can be ignored. During the rotation motion, the kinetic energy E_k and potential energy E_p of the plate are kept constant. At the highest point of the bottom plate:

$$E_p = m_0gh_m - E_k = 0 \tag{3}$$

When the plate returns to equilibrium position:

$$E_k = \frac{1}{2} I_0 \omega_m^2 - E_p = 0 \tag{4}$$

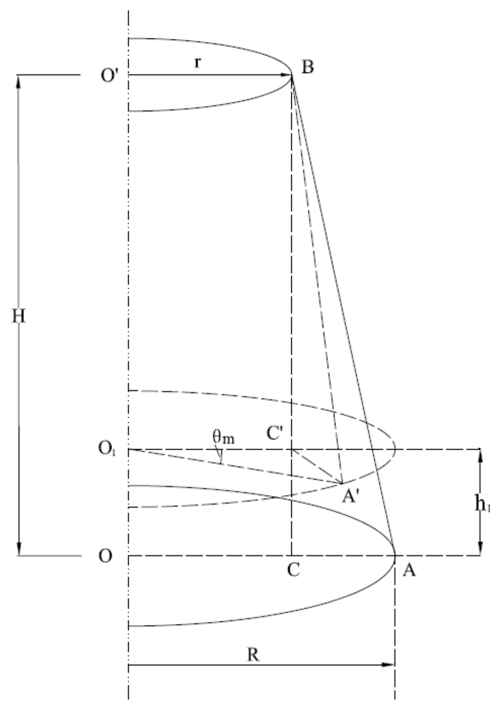

Figure 1 The principle of trifilar pendulum

As shown in figure 1, by ignoring the resistance of the air and applying the conservation of mechanical energy:

$$\frac{1}{2} I_0 \omega_m^2 = m_0 g h_m \tag{5}$$

The parameter m_0 is readily measured with a balance. When the initial rotation angle of the bottom plate is very small, the movement of the plate is consider a harmonic vibration. Hence, the angle displacement is:

$$\theta = \theta_m \sin(2\pi / T_0)t \tag{6}$$

With the rotary angular velocity of:

$$\omega = d\theta / dt = 2\pi\theta_m / T_0 \cos(2\pi / T_0)t \tag{7}$$

The maximum rotary angular velocity is: $\omega_m = 2\pi / T_0 \theta_m$

(8)

From the geometric relations given in figure 1:

$$h_m = \overline{OO_1} = \overline{BC} - \overline{BC'} = \frac{\overline{BC}^2 - \overline{BC'}^2}{\overline{BC} + \overline{BC'}} \tag{9}$$

$$\overline{BC}^2 = \overline{AB}^2 - \overline{AC}^2 = L^2 - (R-r)^2 \tag{10}$$

$$\overline{BC'}^2 = \overline{A'B}^2 - \overline{A'C'}^2 = L^2 - (R^2 + r^2 - 2Rr\cos\theta_m)$$

$$\overline{BC} + \overline{BC'} = 2H_0 - h_m \tag{11}$$

By substituting (9-11) into (8),

$$h_m = 2Rr(1-\cos\theta_m)/(2H_0 - h_m) = 4Rr\sin^2(\frac{\theta_m}{2})/(2H_0 - h_m) \tag{12}$$

When θ_m is small ($\theta_m < 5^\circ$),

$$\sin\theta_m / 2 \approx \theta_m / 2rad, 2H_0 - h_m \approx 2H_0 \tag{13}$$

Then,

$$h_m = Rr{\theta_m}^2 / (2H_0) \tag{14}$$

Substitute (8) and (14) into (5),

$$\frac{1}{2} I_0 (2\pi / T_0 \cdot \theta_m)^2 = m_0 gRr / (2H_0){\theta_m}^2 \tag{15}$$

Solving for I_0,

$$I_0 = \frac{m_0 gRr}{4\pi^2 H_0} T_0^2 \tag{16}$$

From (1), the mathematical model for measuring the moment of inertia about axis OO' to any object with mass m_x is obtained.

$$I_1 = \frac{(m_x + m_0)gRr}{4\pi^2 H_1} T_1^2 \tag{17}$$

Where H_1 is the distance between upper plate and bottom plate after the measuring object m_x has been put, normally $H_1 \approx H_0$, thus the moment of inertia of m_x is:

$$I_x = I_1 - I_0 = \frac{gRr}{4\pi^2 H_0}(m_x + m_0)T_1^2 - \frac{gRr}{4\pi^2 H_0} m_0 T_0^2 \tag{18}$$

3. SYSTEM DESIGN

In order to implement the trifilar pendulum method, a circular steel bottom plate of uniform mass distribution is created, as shown in Figure 2. It is hollowed out in order to alleviate its weight. The air bubble in the middle of the plate is used to adjust the plate kept in the horizontal level. The bottom plate is connected to a small, circular top plate with three inelastic pieces of rope. The top plate is structured to keep the bottom plate rotate within an angle of 5°. The specifications for the trifilar pendulum are given in table 2.

Table 2 Parameters of the trifilar pendulum

Efficiency radius (cm)		Distance between up and bottom (cm)	Weight(kg)
Upper plate	Bottom plate	Empty plate	Bottom plate
12.5	18.0	80.0	2.13

A photoelectric sensor is used to measure the rotating period of the bottom plate. One side of the photoelectric sensor is transmitting side, while the other side is receiving side, when there is no obstacle between them, the output is 3.3V high voltage, and otherwise the output is 0V low voltage. A shade rod is stretched out at the bottom plate, when the bottom plate rotates, the shade rod will rotate with same frequency, and the output of photoelectric sensor is a continuous square wave, the period of the square wave is obtained by the capture mode of MSP430 MCU. MSP430 records the time of each falling edge or rising edge, and the difference of the time is the period of square wave, also the period of the rotating period of bottom plate. In the practical measurement, several periods are recorded; the average of them is used for the final period value to improve the precision.

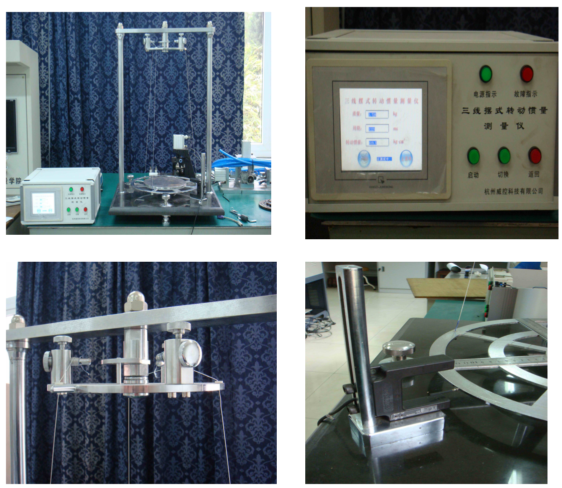

Figure 2 photographs of instrument, display, top plate and photoelectric sensor

The weight of unknown mass m_x is measured by an Electronic Balance with RS-232, the data is transferred into MSP430 by serial port.

A 5.7", 640×480 dots, 65K colours TFT LCM Display with touch screen is used as the man-machine interface, the data of LCM Display module and touch screen are communicated with MSP430 MCU by serial port.

The software flowchart of MCU is designed as Figure 3. The interrupt includes key interrupt when users press the touch-screen, serial port interrupt, which receive the weight signal from Electronic Balance, as well as the capture interrupt, which measures the rotating period time of bottom plate.

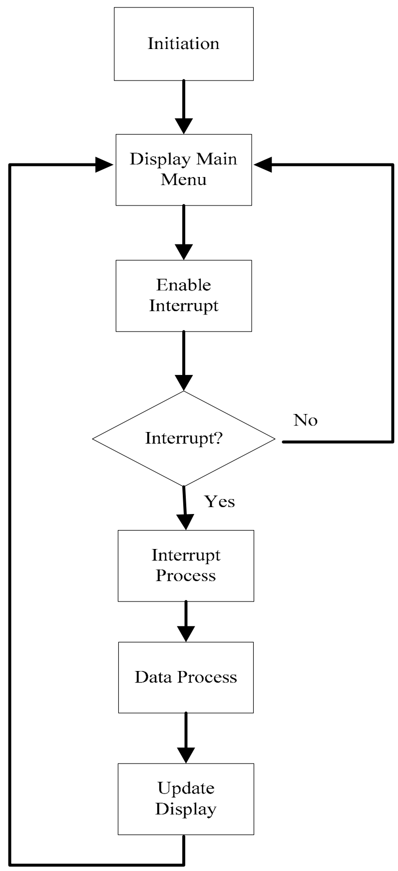

Figure 3 Flowchart of MCU

4. CALIBRATION AND ERROR ANALYSIS

Assume $\alpha = \frac{gRr}{4\pi^2 H_0}, \beta = I_0$, then Eq.(18) can be written as:

$$I_x = \alpha(m_x + m_0)T^2 - \beta \tag{19}$$

It is an equation with two unknown parameters. The instrument should firstly be tested by two metal disks whose theoretical mass moment of inertia could be readily calculated, and α and β could be calculated automatically by the MCU. The parameters α and β are stored in the system. The experiment procedure is as follows: the measuring object was firstly put on the electronic balance, the weight m_x was measured and transferred into the MCU via serial port, then the measuring object was put on the bottom plate, and the bottom plate was rotated to an angle of less than 5°, the rotation period were obtained and recorded, the value of I_x was calculated according to Eq.(18) with known α and β, and then displayed. Some experiments were carried out; Table 3 shows the weights and inertia of three standard metal disks, No.1 and No.2 were taken to calculate α and β, the results were: $\alpha = 92.1$, $\beta = 441.1$. The measured inertial of the No.3 was 61.2 $kg\cdot cm^2$ by applying Eq.(19). The error between measured value and theoretical value is un-acceptable.

Table 3 Parameters of three standard metal disks

No.	T (ms)	m_x (kg)	Theoretical Inertial($kg\cdot cm^2$)
1	1.211	2.728	215.2
2	1.258	3.920	440.9
3	1.031	3.000	52.0

The error analysis is undertaken here to assess the influence to the result by different variables.

1) Error caused by ignoring translational energy

When the bottom plate is under torsion motion at certain equilibrium position, the center of mass is also under up-and-down motion, and its speed $v = \frac{dh}{dt}$, substituting equation (12) and simplifying gives:

$$v = \frac{d(4Rr\sin^2(\frac{\theta_m}{2})/(2H_0))}{dt} = \frac{Rr\sin\theta}{H_0}\omega \tag{20}$$

The translational energy is:

$$E_t = \frac{1}{2}m_0 v^2 = \frac{m_0 R^2 r^2}{2H_0^2}\omega^2 \sin^2\theta \tag{21}$$

$\omega = 0, E_t = 0$ when the bottom plate is at the maximum angular displacement; ω reaches maximum when it is at the equilibrium position, but $\theta = 0, E_t = 0$. So the maximum of E_t occurs at the some position between maximum angular displacement and equilibrium position.

The ratio of translational energy and rotational kinetic energy is:

$$\frac{E_t}{E_k} = \frac{1/2m_0v^2}{1/2I_0\omega^2} = \frac{2r^2}{H_0^2}\sin^2\theta \tag{22}$$

2)Error caused by taking $2H_0 - h_m \approx 2H_0$

The error caused by taking $2H_0 - h_m \approx 2H_0$ is:

$$E_{r2} = \frac{h}{H} \approx \frac{2Rr\sin^2\frac{\theta}{2}}{H^2} \tag{23}$$

3) Error caused by taking $\sin\theta/2 \approx \theta/2$

The error caused by taking $\sin\theta/2 \approx \theta/2$ is:

$$E_{r3} = \frac{(\frac{\theta}{2})^2 - \sin^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} \tag{24}$$

4) Error caused by other factors

The error caused by other factors could be the misalignment of centre, accompanying with swing motion during the torsion motion, measurement error of parameters R, r, m_0, T_0 etc.

The period time obtained by the system is not consistent with the ideal value, influenced by the errors listed above. Thus the result utilizing Eq.(19) seems un-acceptable. A new calibration method is proposed by modifying Eq.(18),

$$I_x = I_1 - I_0 = \frac{gRr}{4\pi^2H_0}(m_x + m_0)(AT_1 + B)^2 - \frac{gRr}{4\pi^2H_0}m_0(AT_0 + B)^2 \tag{25}$$

For the convenient of programming by MCU, Eq.(25) is abbreviated as the following equations:

$$
\begin{aligned}
I_0 &= Am_0T_0^2 + Bm_0T_0 + Cm_0 \\
I_1 &= Am_1T_1^2 + Bm_1T_1 + Cm_1 - I_0 \\
I_2 &= Am_2T_2^2 + Bm_2T_2 + Cm_2 - I_0 \\
I_3 &= Am_3T_3^2 + Bm_3T_3 + Cm_3 - I_0
\end{aligned} \tag{26}
$$

where I_0, I_1, I_2, I_3 are the moment of inertia of bottom plate with empty, No.1, No.2 and No.3 metal disks, m_0, m_1, m_2, m_3 are the weight of bottom plate with empty, No.1, No.2 and No.3 metal disk, T_0, T_1, T_2, T_3 are the period time of bottom plate with empty, No.1, No.2 and No.3 metal disk, A, B, C and I_0 are unknown parameters. The system was calibrated using three metal disks, whose theoretical mass moment of inertia could be readily calculated from the equation given in Eq.(27).

$$
I = \frac{1}{2} mR^2 \tag{27}
$$

where R is the radius of the metal disks.

The physical parameters of three metal disks are given in table 4. The calibrating procedure was complete in similar methods as the exclusively bottom plate trial, only now the object place on the bottom plate with its center of mass place inline with the center of mass of the bottom plate. The parameters of A, B, C and I_0 were then calculated by MCU according to Eq.(26) and recorded by the system.

Each time when the measuring environment were changed, such as the three cable were adjusted, the same calibration procedure could be carried out to obtain the new A, B, C and I_0.

After calibration, the system is ready to be used to test other objects.

Table 4 Parameters of metal disk

Weight (kg)	Radius(cm)	Theoretical Inertial($kg\cdot cm^2$)
2.7	12.5	213.1
3.0	59	52.1
3.9	150	441.0

5. RESULT

Seven metal disks with known moments of inertia were used to test the accuracy of the instrument. Table 5 lists the measurement results and theoretical results of moment of inertial, all had measurement errors of less than 2%.

Table 5 Measurement and theoretical results of inertial

Weight (kg)	Radius(cm)	Theoretical Inertial ($kg\cdot cm^2$)	Measured Inertial ($kg\cdot cm^2$)	Errors in percent (%)
1.0	10.0	54.2	55.1	1.6
2.0	10.0	68.2	67.0	1.7
3.7	12.5	225.1	224.1	0.4
4.9	12.5	453.0	451.3	0.3
4.9	7.4	135.9	136.9	0.6
5.7	12.5	265.0	262.3	1.0
6.6	15.0	650.9	654.1	0.5

6. SUMMARY

In this paper, an instrument for measuring the moment of inertia based on trifilar pendulum is presented. From the measurement results, the instrument could measure the moment of inertia with errors less than 2%. It has been used by some companies which produce automotive components. The advantage of the instrument is that it is convenient and can be performed with a high degree of accuracy, improves the measuring efficiency and precision. It integrates calibration function, keeps the high precision after long time operation or movement.

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