On normal Sylow subgroups

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The three Sylow Theorems

Definition 1 (Sylow 🚑-subgroup). Let **Bē** be a group of order $🚑^{⟳}🖧$, where 🚑 is a prime number and $🚑 \nmid 🖧$, such that there exists a subgroup 🌿 of **Bē** of order $🚑^{⟳}$. Then we say that 🌿 is a Sylow 🚑-subgroup of **Bē**.

Theorem 2 (First Sylow Theorem). *Let* **Bē** *be a finite group. Then for every prime* 🚑 *dividing the order of* **Bē**. *Then there exists a Sylow* 🚑*-subgroup of* **Bē**, *and every* 🚑*-subgroup of* **Bē** *is in a Sylow* 🚑*-subgroup of* **Bē**.

Theorem 3 (Segond Sylow Theorem). *Let* **Bē** *be a finite group. Then for every prime* 🚑 *dividing the order of* **Bē**, *the Sylow* 🚑*-subgroups of* **Bē** *are conjugate.*

Theorem 4 (Third Sylow Theorem). *Let* **Bē** *be a group of order* $🚑^{⟳}🖧$, *where* 🚑 *is a prime number and* $🚑 \nmid 🖧$, *and let* $⚓_{🚑}$ *be the number of Sylow* 🚑*-subgroups of* **Bē**. *Then*

$$⚓_{🚑} \mid 🖧 \qquad \textit{and} \qquad ⚓_{🚑} \equiv 1 \pmod{🚑}.$$

Normal Sylow subgroups

Proposition 5. *Let* **Bē** *be a group with* $⚓_{🚑} = 1$. *Then its Sylow* 🌿*-subgroup is a normal subgroup of* **Bē**

Proof. Let 🌿 be the Sylow 🚑-subgroup of **Bē**. By the second Sylow Theorem we have that for every $✌ \in$ **Bē** we find that

$$✌🌿✌^{-1} = 🌿,$$

and 🌿 is a normal subgroup of **Bē**. 📷

Theorem 6. *Let* **Bē** *be a finite group such that there exist two different prime factors* 🚑 *and* ⚖ *dividing its order with* $⚓_{🚑} = 1$ *and* $⚓_{⚖} = 1$. *Then* *the elements of the Sylow 🚑-subgroup commute with the elements of the Sylow ⚖-subgroup of* **Bē**.

Proof. Let 🚹 be the group identity of **Bē** and let 🌿 be the Sylow 🚑-subgroup and 🔕 be the Sylow ⚖-subgroup of **Bē**. By Lagrange's Theorem we find that

$$🌿 \cap 🔕 = \{🚹\},$$

and by the last proposition we have that 🌿 and 🔕 are normal subgroups of **Bē**.

Then, for every 🏢 $\in$ 🌿 and every 🐞 $\in$ 🔕 we have

$$\begin{aligned} 🏢🐞🏢^{-1}🐞^{-1} &= (🏢🐞🏢^{-1})🐞^{-1} \in 🔕 \\ &= 🏢(🐞🏢^{-1}🐞^{-1}) \in 🌿 \\ &\in 🌿 \cap 🔕 = \{🚹\} \\ &= \{🚹\}, \end{aligned}$$

with which we find that 🏢🐞 = 🐞🏢. 📷

Theorem 7. *Let* **Bē** *be a finite group, let* $🚑_1, \dots, 🚑_{↻}$ *be the different primes dividing the order of* **Bē** *and let* $🌿_1, \dots, 🌿_{↻}$ *be the Sylow subgroups of* **Bē**. *Then the Sylow subgroups* $🌿_1, \dots, 🌿_{↻}$ *are normal if and only if*

$$\mathbf{B\bar{e}} \cong 🌿_1 \times \cdots \times 🌿_{↻}.$$

Proof. Suppose that $\mathbf{B\bar{e}} \cong 🌿_1 \times \cdots \times 🌿_{↻}$. We have

$$|\mathbf{B\bar{e}}| = |🌿_1 \times \cdots \times 🌿_{↻}| = |🌿_1| \cdots |🌿_{↻}|,$$

which implies $⚓_{🚑_1} = 1, \dots, ⚓_{🚑_{↻}} = 1$, and by proposition 5 we find that the Sylow subgroups $🌿_1, \dots, 🌿_{↻}$ are normal.

Suppose now the Sylow subgroups $🌿_1, \dots, 🌿_{↻}$ are normal. By the previous theorem we have that the elements of two different groups commute. With this we find that the function

$$\begin{aligned} ☕ \colon 🌿_1 \times \cdots \times 🌿_{↻} &\longrightarrow \mathbf{B\bar{e}} \\ 🏢_1, \cdots, 🏢_{↻} &\longmapsto 🏢_1 \cdots 🏢_{↻} \end{aligned}$$

is a homomorphism. The homomorphism ☕ is injective, since the order of a product of commuting elements with relatively primer orders is equal to the product of their orders. We also have that

$$|\mathbf{B\bar{e}}| = |🌿_1 \times \cdots \times 🌿_{↻}| = |🌿_1| \cdots |🌿_{↻}|,$$

with which we find that ☕ is a group isomorphism. 📷

`https://kconrad.math.uconn.edu/blurbs/grouptheory/sylowapp.pdf`.