

Artificial Intelligence

Lecture 6: Constraint Satisfaction Problems

Credit: Ansaf Salleb-Aouissi, and “Artificial Intelligence: A Modern Approach”, Stuart Russell and Peter Norvig, and “The Elements of Statistical Learning”, Trevor Hastie, Robert Tibshirani, and Jerome Friedman, and “Machine Learning”, Tom Mitchell.

Recall

- **Search problems:**

- Find the sequence of actions that leads to the goal.
- Sequence of actions means a path in the search space.
- Paths come with different costs and depths.
- We use heuristics to guide the search efficiently.

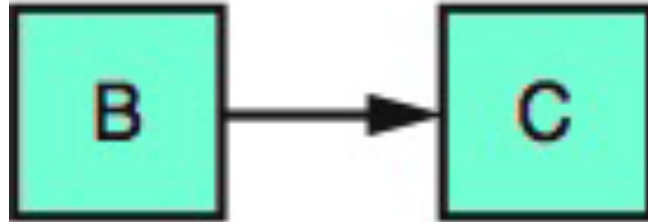
- **Constraint satisfaction problems:**

- A search problem too!
- We care about the **goal itself**.

CSPs definition

- Search problems:

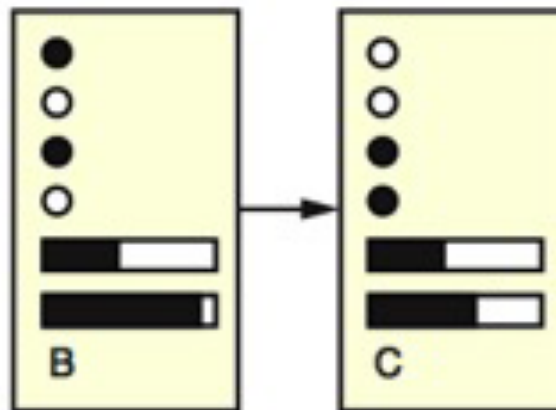
- A state is a **black box**, implemented as some data structure.
Recall **atomic representation**.
- A goal test is a function over the states.



CSPs definition

- CSPs problems:

- A state: defined by variables X_i with values from domain D_i . Recall **factored representation**.
- A goal test is a **set of constraints** specifying **allowable combinations** of values for subsets of variables.



CSPs definition



CSPs definition

- A constraint satisfaction problem consists of **three elements**:
 - A set of **variables**, $X = \{X_1, X_2, \dots, X_n\}$
 - A set of **domains** for each variable: $D = \{D_1, D_2, \dots, D_n\}$
 - A set of **constraints** C that specify allowable combinations of values.
- Solving the CSP: **finding the assignment(s)** that **satisfy all constraints**.
- Concepts: problem formalization, backtracking search, arc consistency, etc.
- We call a solution, a **consistent assignment**.

Example: Map coloring



Variables: $X = \{WA, NT, Q, NSW, V, SA, T\}$

Domains: $D_i = \{\text{red, green, blue}\}$

Constraints: adjacent regions must have different colors;
e.g., $WA \neq NT$ or $(WA, NT) \in \{(\text{red, green}), (\text{red, blue}), \text{etc.}\}$

Example: Map coloring



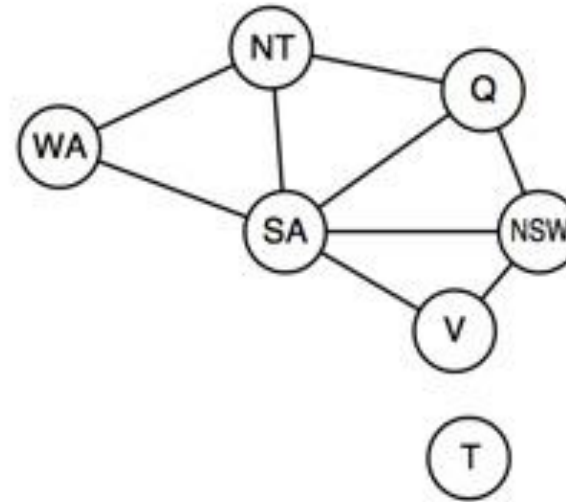
Example:

{WA=red, NT=green, Q=red, NSW=green, V=red, SA=blue, T=green}

Real-world CSPs

- Assignment problems, e.g., who teaches what class?
- Timetabling problems, e.g., which class is offered when and where?
- Hardware configuration
- Transportation scheduling
- Factory scheduling
- Floor planning
- etc.
- Notice that many real-world problems involve real-valued variables

Constraint graph



任何约束都能转化为 binary

- **Binary CSP:** each constraint relates at most two variables
Constraint graph: nodes are variables, arcs show constraints
- **CSP algorithms:** use the graph structure to speed up search. E.g., Tasmania is an independent sub-problem!

Varieties of variables

- **Discrete variables:**

- Finite domains: assume n variables, d values, then the number of complete assignments is $O(d^n)$, e.g., map coloring, 8-queen problem
- Infinite domains (integers, strings, etc.): need to use a constraint language, e.g., job scheduling. $T_1 + d \leq T_2$.

- **Continuous variables:**

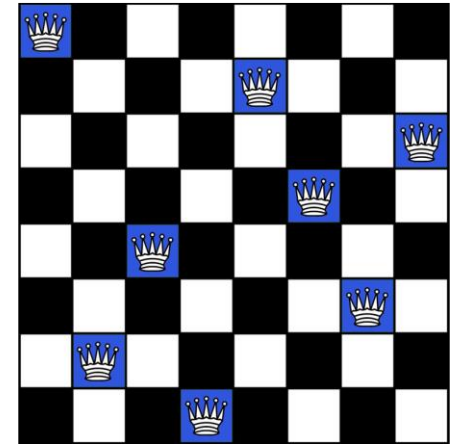
- Common in operation research
- Linear programming problems with linear or non linear equalities

Varieties of constraints

- **Unary constraints:** involve a single variable e.g., $SA \neq \text{green}$
- **Binary constraints:** involve pairs of variables e.g., $SA \neq WA$
- **Global constraints:** involve 3 or more variables e.g., *Alldiff* that specifies that all variables must have different values (e.g., Cryptarithmic puzzles, Sudoku)
- **Preferences (soft constraints):**
 - Example: red is better than green
 - Often represented by a cost for each variable assignment
 - constrained optimization problems

Example: 8-queen

8-Queen: Place 8 queens on an 8x8 chess board so no queen can attack another one.

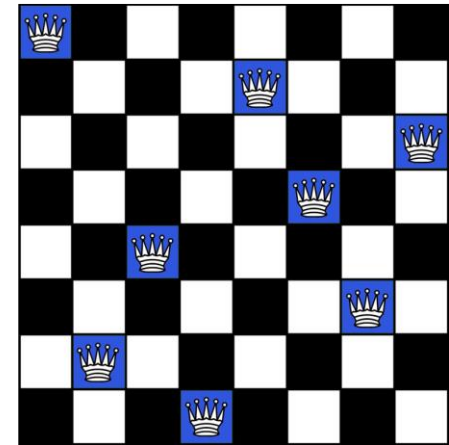


Problem formalization 1:

- One variable per queen, Q_1, Q_2, \dots, Q_8 .
- Each variable could have a value between 1 and 64. 64^8
- Solution: $Q_1 = 1, Q_2 = 13, Q_3 = 24, \dots, Q_8 = 60$.

Example: 8-queen

8-Queen: Place 8 queens on an 8x8 chess board so no queen can attack another one.

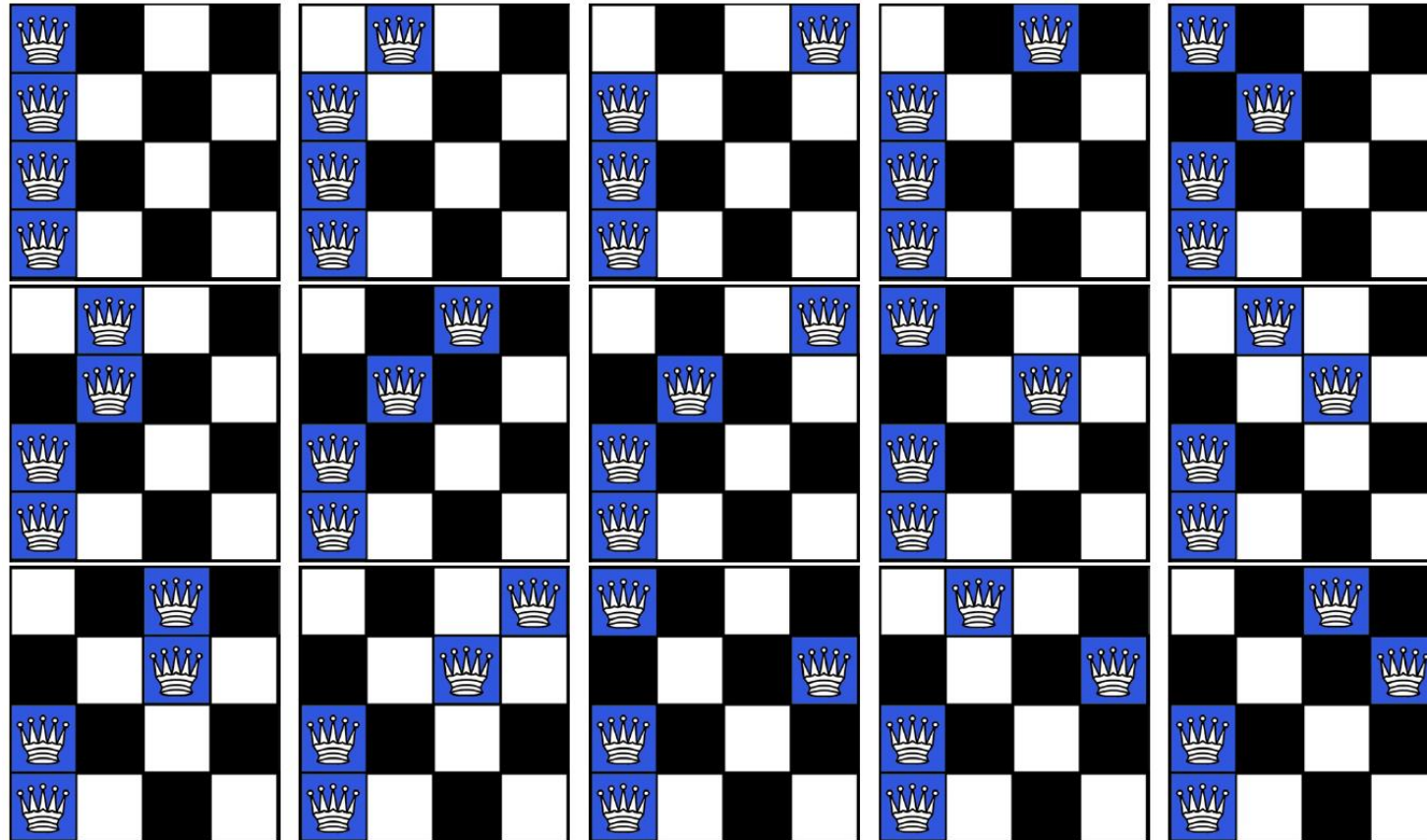


Problem formalization 2:

- One variable per queen, Q_1, Q_2, \dots, Q_8 . ²¹
- Each variable could have a value between 1 and 8 (rows). ^{8⁸}
- Solution: $Q_1 = 1, Q_2 = 7, Q_3 = 5, \dots, Q_8 = 3$.

Brute force?

Should we simply generate and test all configurations?



Example Cryptarithmic

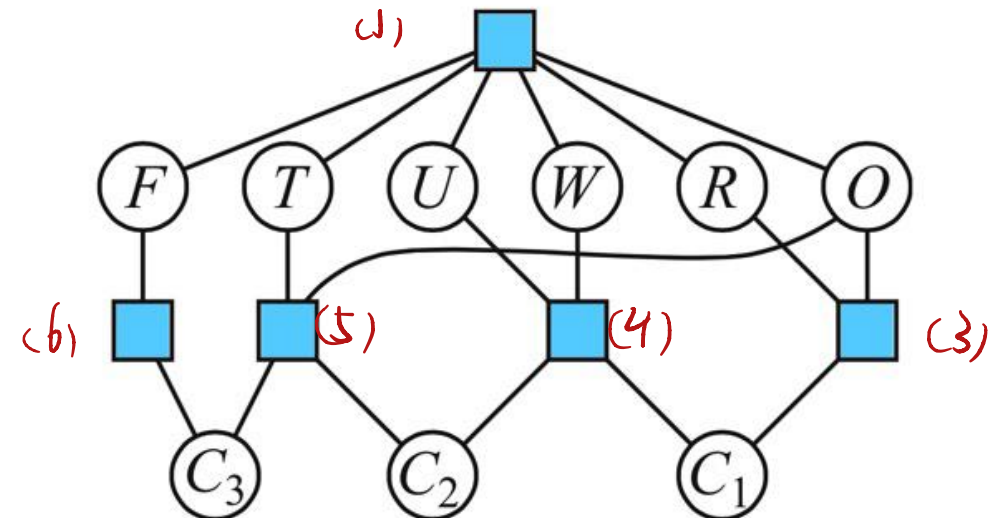
Variables: $X = \{F, T, U, W, R, O, C_1, C_2, C_3\}$

Domain: $D = \{0, 1, 2, \dots, 9\}$

Constraints:

- (1) • $\text{Alldiff}(F, T, U, W, R, O)$
- (2) • $T \neq 0, F \neq 0$
- (3) • $O + O = R + 10 * C_1$
- (4) • $C_1 + W + W = U + 10 * C_2$
- (5) • $C_2 + T + T = O + 10 * C_3$
- (6) • $C_3 = F$

$$\begin{array}{r}
 C_3 \ C_2 \ C_1 \ \text{进位} \\
 7T \ 3W \ O \ 4 \\
 + 7T \ 3W \ O \ 4 \\
 \hline
 F \ O \ U \ R \ 8 \\
 1 \ 4 \ 6
 \end{array}$$



Solving CSPs

IMPORTANT

- **State-space search algorithms:** search!
- **CSP Algorithms:** Algorithm can do two things:
 - **Search:** choose a new variable assignment from many possibilities
 - **Inference:** ^{约束传播} constraint propagation, use the constraints to spread the word: reduce the number of values for a variable which will reduce the legal values of other variables etc.
- As a preprocessing step, constraint propagation can sometimes solve the problem entirely without search.
- Constraint propagation can be intertwined with search.

Solving CSPs

- **BFS:** Develop the complete tree
- **DFS:** Fine but time consuming
- **BTS: Backtracking search** is the basic uninformed search for CSPs. It's a DFS s.t.
 1. Assign one variable at a time (**expansion**): assignments are commutative, e.g., (WA = red, NT = green) is same as (NT = green, WA = red)
 2. Check constraints on the go: consider values that do not conflict with previous assignments.

Backtrack + DFS

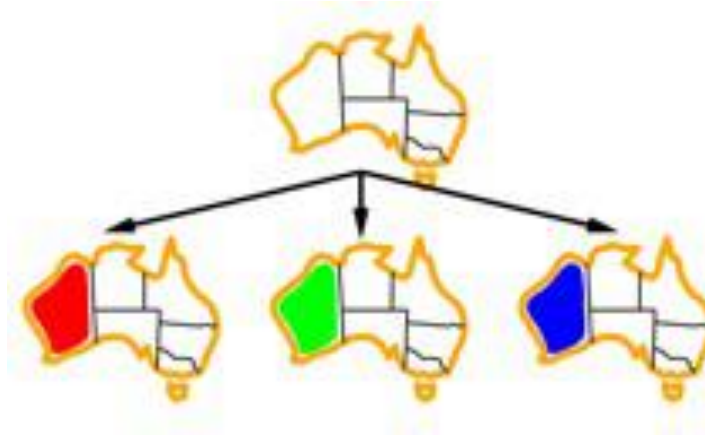
Solving CSPs

- **Initial state:** empty assignment $\{ \}$
- **States:** are partial assignments
- **Successor function:** assign a value to an unassigned variable
- **Goal test:** the current assignment is complete and satisfies all constraints

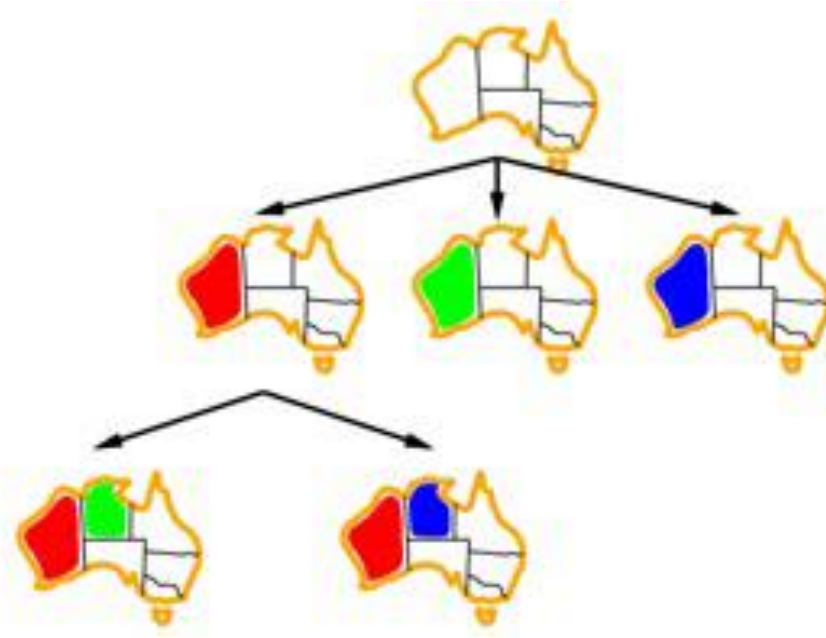
Backtracking search



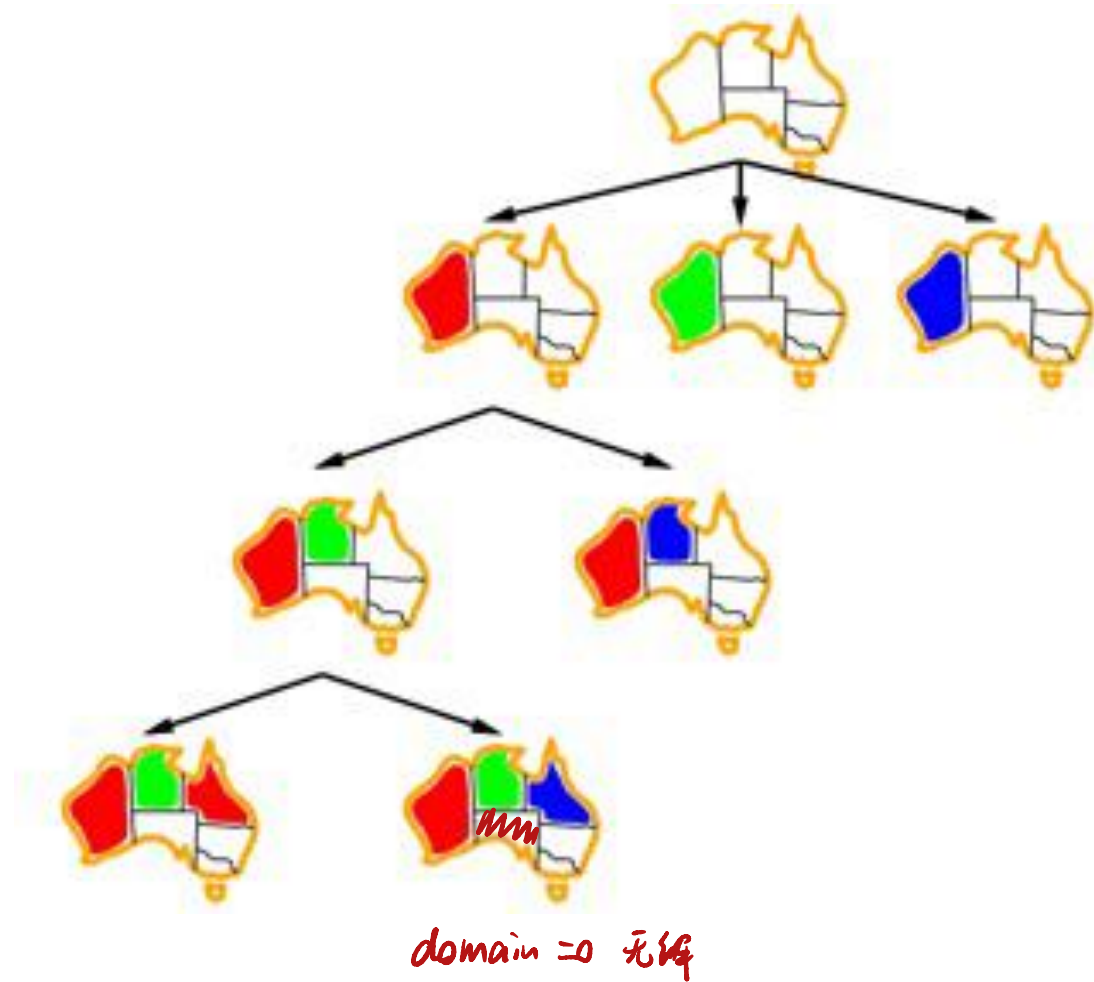
Backtracking search



Backtracking search



Backtracking search



Improving BTS

DFS 框架下提升效率

Heuristics are back!

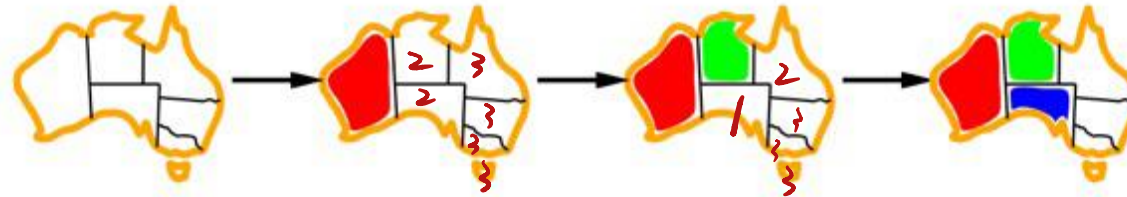
1. Which variable should be assigned next?
2. In what order should its values be tried?
3. Can we detect inevitable failure early?

Minimum Remaining Value

1. Which variable should be assigned next?



- **MRV:** Choose the variable with the fewest legal values in its domain



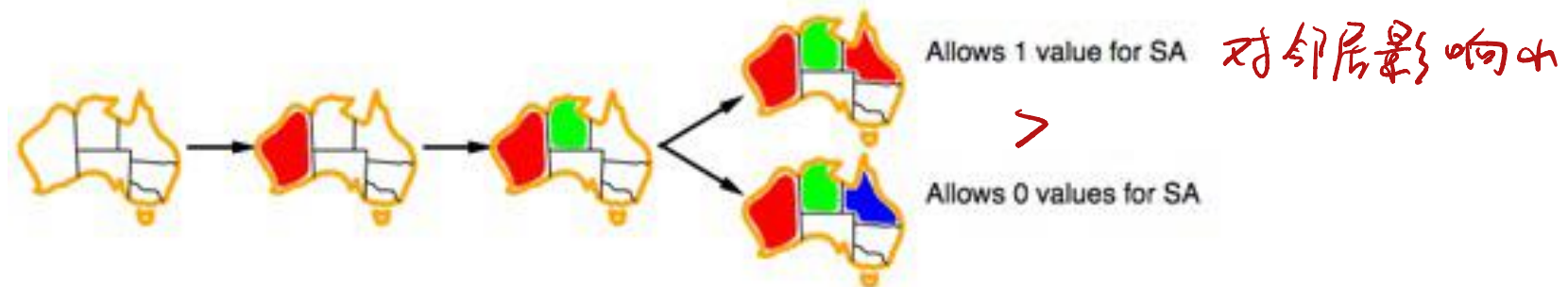
Pick the hardest!

Least constraining value

2. In what order should its values be tried?



- **LCV:** Given a variable, choose the least constraining value: the one that rules out the fewest values in the remaining variables



Pick the ones that are likely to work!

Forward checking

3. Can we detect inevitable failure early?



- **FC:** Keep track of remaining legal values for the unassigned variables. Terminate when any variable has no legal values.



↓
unassigned

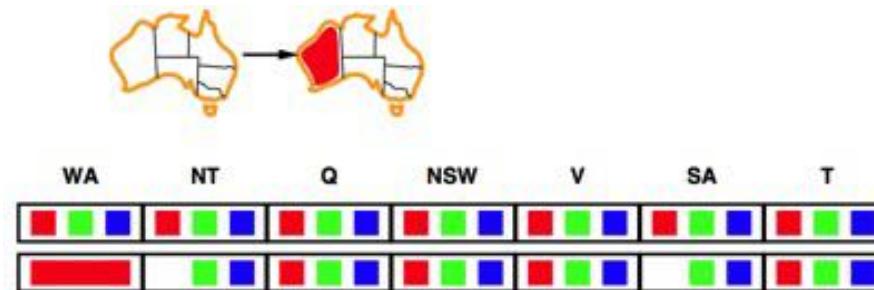
WA	NT	Q	NSW	V	SA	T
<div><div></div><div></div><div></div></div>	<div><div></div><div></div><div></div></div>	<div><div></div><div></div><div></div></div>	<div><div></div><div></div><div></div></div>	<div><div></div><div></div><div></div></div>	<div><div></div><div></div><div></div></div>	<div><div></div><div></div><div></div></div>

Forward checking

3. Can we detect inevitable failure early?



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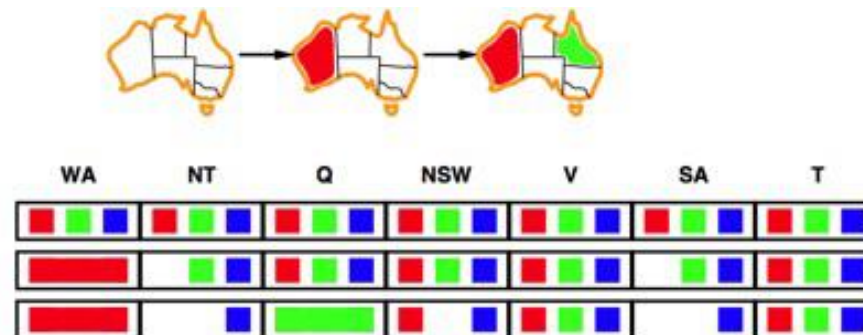


Forward checking

3. Can we detect inevitable failure early?



- **FC:** Keep track of remaining legal values for the unassigned variables. Terminate when any variable has no legal values.

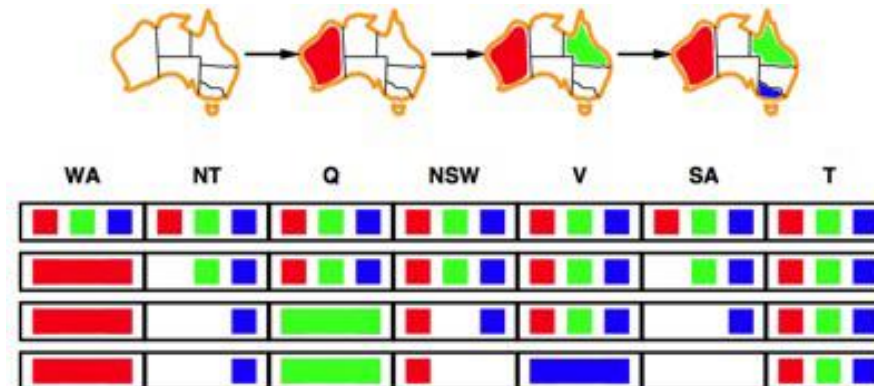


Forward checking

3. Can we detect inevitable failure early?



- **FC:** Keep track of remaining legal values for the unassigned variables. Terminate when any variable has no legal values.



Backtracking search

function BACKTRACKING_SEARCH(*csp*) returns a solution, or failure
return BACKTRACK({}, *csp*)

function BACKTRACK(*assignment*, *csp*) returns a solution, or failure
 if *assignment* is complete **then return** *assignment*
 var = SELECT_UNASSIGNED_VARIABLE(*csp*) *MRV*
 for each *value* in ORDER_DOMAIN_VALUES (*var*, *assignment*, *csp*) *LCV*
 if *value* is consistent with *assignment* **then**
 add {*var* = *value*} to *assignment* *FC*
 result = BACKTRACK(*assignment*, *csp*)
 if *result* ≠ failure **then return** *result*
 remove {*var* = *value*} from *assignment*
 return failure

fail 则跳转到

Solving CSPs: Sudoku

All 3x3 boxes, rows, columns, **must contain all digits 1...9.**

Variables: $V = \{A_1, \dots, A_9, B_1, \dots, B_9, \dots, I_1, \dots, I_9\}$, $|V| = 81$.

Domain: $D = \{1, 2, \dots, 9\}$, the filled squares have a single value.

Constraints: 27 constraints

- $\text{Alldiff}(A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9)$
...
- $\text{Alldiff}(A_1, B_1, C_1, D_1, E_1, F_1, G_1, H_1, I_1)$
...
- $\text{Alldiff}(A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3)$

8	4	9	5	2	1	7	3	6
2	5	7		6	3	9		
1	6	3				2		
3				9		4		7
4	9		3		7	6	2	
7		6		8		3		
9						1	6	3
6			9	3		5		2
5	3	2	6		4	8		9

Solving CSPs: Sudoku

All 3x3 boxes, rows, columns, **must contain all digits 1...9.**

8		9	5		1	7	3	6
2		7		6	3			
1	6							
				9		4		7
	9		3		7		2	
7		6		8				
							6	3
			9	3		5		2
5	3	2	6		4	8		9

- Naked doubles (triples): find two (three) cells in a 3x3 grid that have only the same candidates left, eliminate these two (three) values from all possible assignments in that box.
- Locked pair, Locked triples, etc.

Solving CSPs: Sudoku

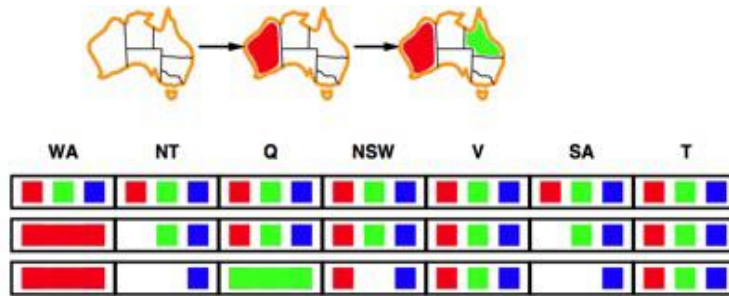
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8		9	5		1	7	3	6
2		7		6	3			
1	6							
				9		4		7
	9		3		7		2	
7		6		8				
							6	3
			9	3		5		2
5	3	2	6		4	8		9

8	4	9	5	2	1	7	3	6
2	5	7	8	6	3	9	1	4
1	6	3	7	4	9	2	5	8
3	2	5	1	9	6	4	8	7
4	9	8	3	5	7	6	2	1
7	1	6	4	8	2	3	9	5
9	8	4	2	7	5	1	6	3
6	7	1	9	3	8	5	4	2
5	3	2	6	1	4	8	7	9

Constraint propagation

- Forward checking propagates information from assigned to unassigned variables.
- Observe:



Forward checking does not check interaction between unassigned variables! Here SA and NT! (They both must be blue but can't be blue!).

- Forward checking improves backtracking search but does not look very far in the future, hence does not detect all failures.
- We use constraint propagation, reasoning from constraint to constraint. e.g., arc consistency test. 弧一致性检验

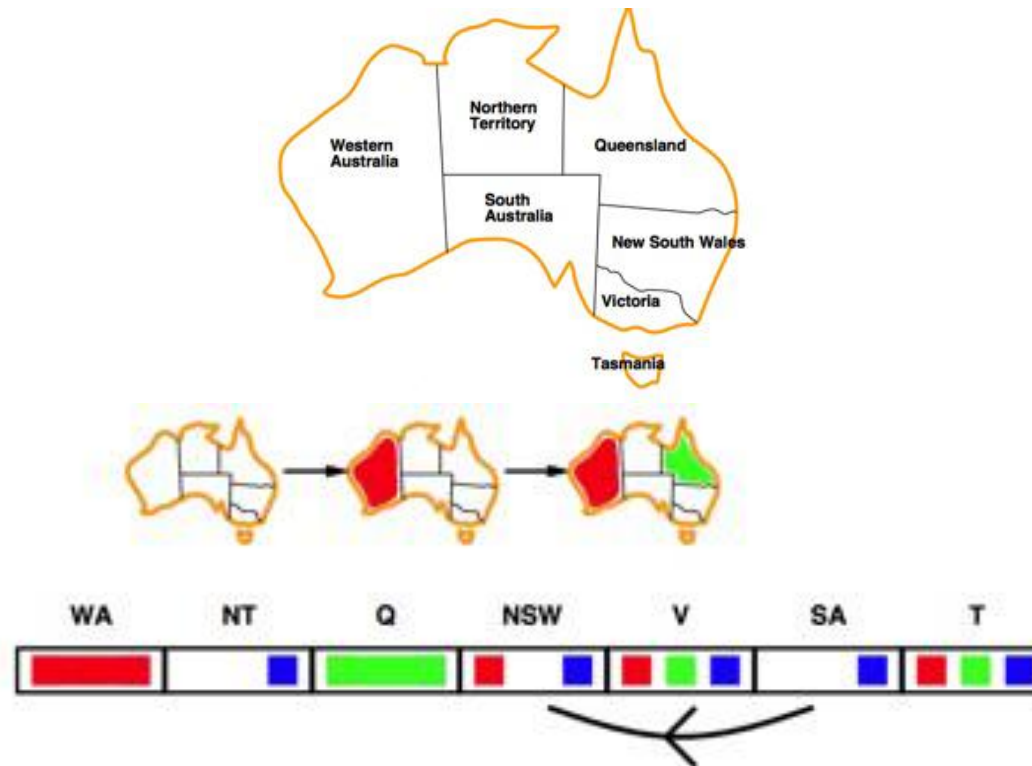
Types of Consistency

- **Node-consistency** (unary constraints): A variable X_i is node-consistent if all the values of $Domain(X_i)$ satisfy all unary constraints.
- **Arc-consistency** (binary constraints): $X \rightarrow Y$ is arc-consistent if and only if every value x of X is consistent with some value y of Y .
 - **E.g.** $Domain(A) = \{3, 4, 5\}$, $Domain(B) = \{3, 4, 5, 6\}$, $A < B$.
 - $A \rightarrow B$ 弧一致 (satisfies)
 - $B \rightarrow A$ 不一致 (fails)
- **Path-consistency and k -consistency** (n-ary constraints): generalizes arc-consistency from binary to multiple constraints.
- **Note:** It is always possible to transform all n-ary constraints into binary constraints. Often, CSPs solvers are designed to work with binary constraints.

Arc consistency

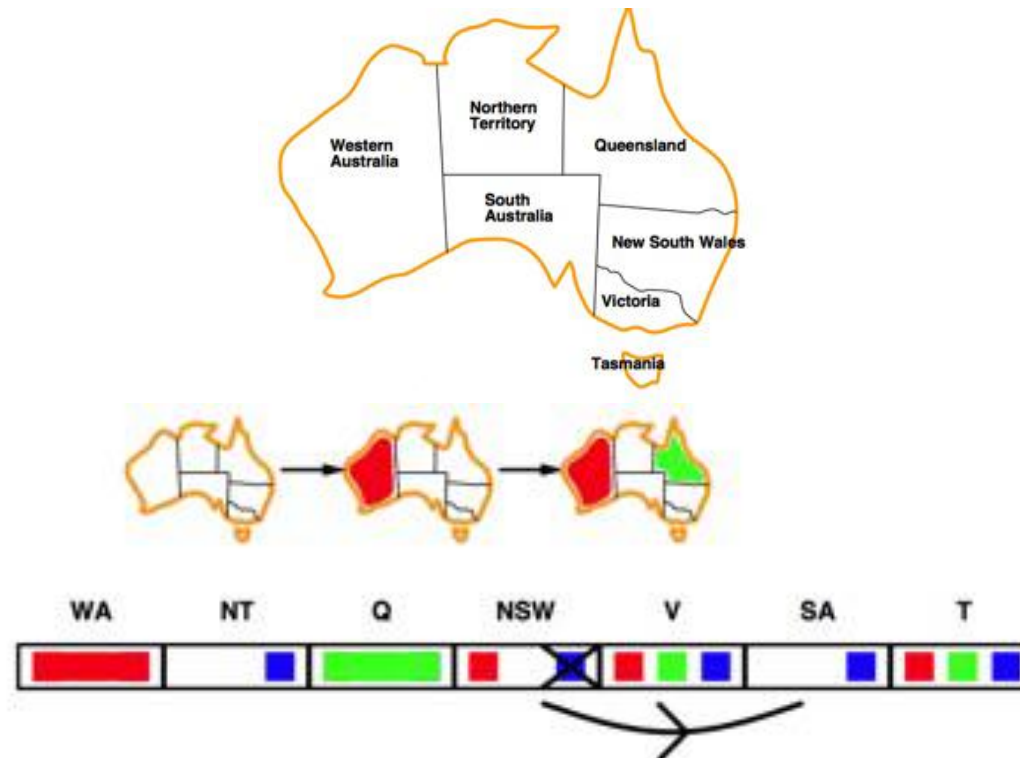
通过修改 domain 满足 AC.

- **AC:** Simplest form of propagation makes each arc consistent.
- $X \rightarrow Y$ is consistent IFF for every value x of X , there is some allowed y .



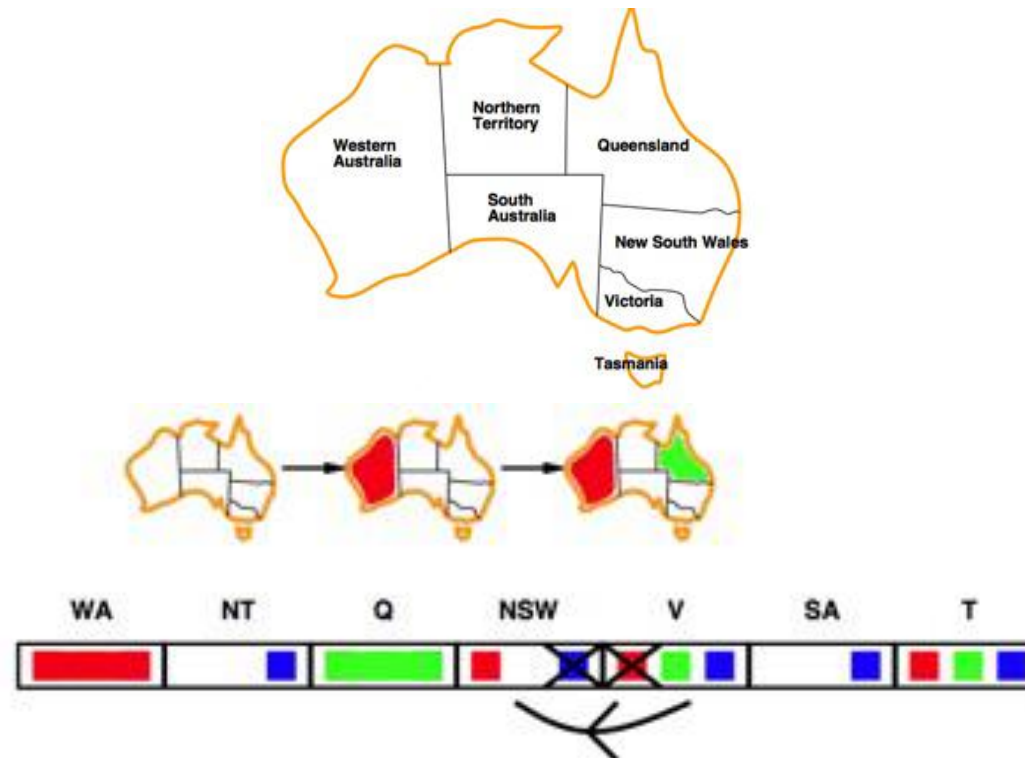
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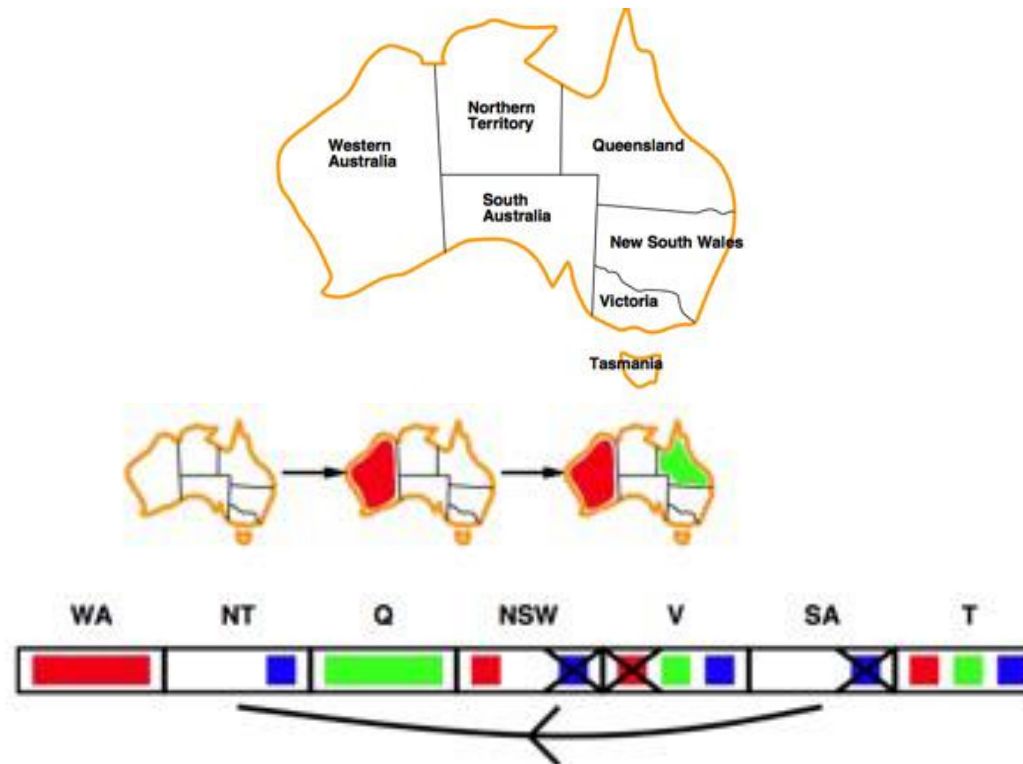
Arc consistency

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Arc consistency

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Arc consistency

Algorithm that makes a CSP arc-consistent!

```
function AC-3( csp)
returns False if an inconsistency is found, True otherwise
inputs: csp, a binary CSP with components (X, D, C)
local variables: queue, a queue of arcs, initially all the arcs in csp
while queue is not empty do
    (Xi, Xj) = REMOVE-FIRST(queue)
    if REVISE(csp, Xi, Xj) then Revise 修改-改
        if size of Di = 0 then return False
        for each Xk in Xi.NEIGHBORS - {Xj} do
            add (Xk, Xi) to queue enqueue
return true
```

```
function REVISE( csp, Xi, Xj)
returns True iff we revise the domain of Xi
revised = False
for each x in Di do
    if no value y in Dj allows (x, y) to satisfy the constraint between Xi and Xj then
        delete x from Di
        revised = True
return revised 标记
```

Complexity of AC-3

- Let n be the number of variables, and d be the domain size.
- If every node (variable) is connected to the rest of the variables, then we have $n * (n - 1)$ arcs (constraints) $\rightarrow O(n^2)$
- Each arc can be inserted in the queue d times $\rightarrow O(d)$ *每个 arc 最多被 revise d 次.*
- Checking the consistency of an arc costs $\rightarrow O(d^2)$.
- Overall complexity is $O(n^2 d^3)$.

Backtracking w/inference

AC3 预处理

function BACKTRACKING-SEARCH(*csp*) **returns** a solution or *failure*

return BACKTRACK(*csp*, { })

function BACKTRACK(*csp*, *assignment*) **returns** a solution or *failure*

if *assignment* is complete **then return** *assignment*

var \leftarrow SELECT-UNASSIGNED-VARIABLE(*csp*, *assignment*)

for each *value* **in** ORDER-DOMAIN-VALUES(*csp*, *var*, *assignment*) **do**

if *value* is consistent with *assignment* **then**

add {*var* = *value*} to *assignment*

inferences \leftarrow INFERENCE(*csp*, *var*, *assignment*)

if *inferences* \neq *failure* **then**

add *inferences* to *csp* 新 csp 问题

result \leftarrow BACKTRACK(*csp*, *assignment*)

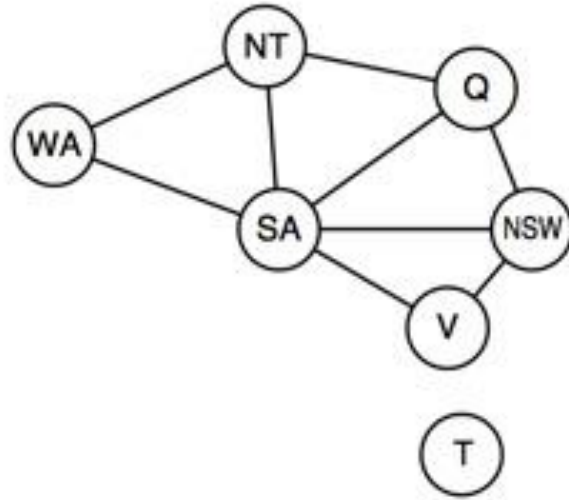
if *result* \neq *failure* **then return** *result*

remove *inferences* from *csp*

remove {*var* = *value*} from *assignment*

return *failure*

Problem structure



- Idea: Leverage the problem structure to make the search more efficient.
- Example: Tasmania is an independent problem.
- Identify the connected component of a constraint graph.
- Work on independent sub-problems.

Problem structure

Complexity:

- Let d be the size of the domain and n be the number of variables.
- Time complexity for BTS is $O(d^n)$.
- Suppose we decompose into sub-problems, with c variables per sub-problem.
- Then we have n/c sub-problems.
- c variables per sub-problem takes $O(d^c)$.
- The total for all sub-problems takes $O(n/c \cdot d^c)$ in the worst case.

Problem structure

Example:

- Assume $n = 80$, $d = 2$.
- Assume we can decompose into 4 sub-problems with $c = 20$.
- Assume processing at 10 million nodes per second.
- Without decomposition of the problem we need:

$$2^{80} = 1.2 \times 10^{24}$$

3.83 million years!

- With decomposition of the problem we need:

$$4 \times 2^{20} = 4.2 \times 10^6$$

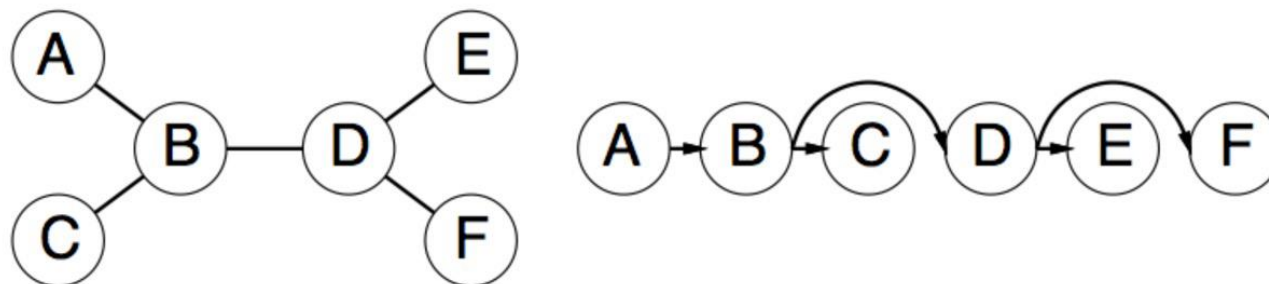
0.4 seconds!

Problem structure

- Turning a problem into independent sub-problems is not always possible.
- Can we leverage other graph structures?
- Yes, if the graph is tree-structured or nearly tree-structured.
- A graph is a **tree** if any two variables are connected by **only one path**.
- Idea: use DAC, Directed Arc Consistency 直接弧相容
- A CSP is said to be **directed arc-consistent** under an ordering X_1, X_2, \dots, X_n IFF every X_i is arc-consistent with each X_j for $j > i$.

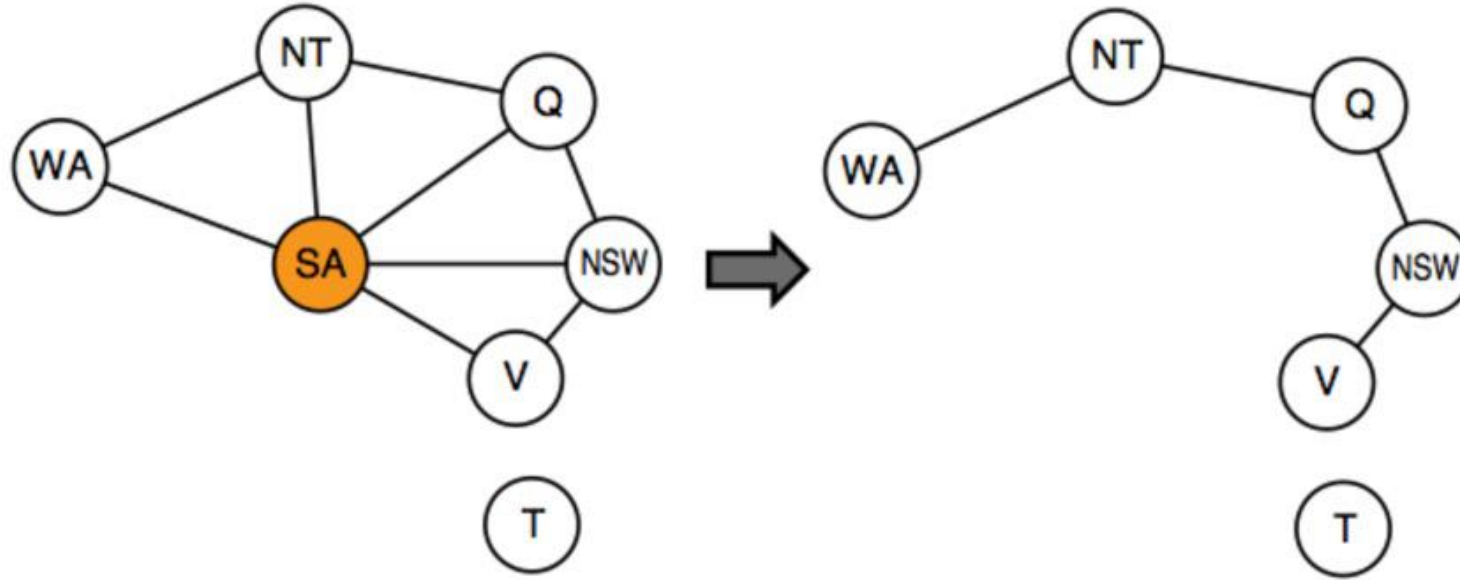
Problem structure

变成树后复杂度大幅降低



- First pick a variable to be the root.
- Do a **topological sorting**: choose an ordering of the variables s.t. each variable appears after its parent in the tree.
- For n nodes, we have $n - 1$ edges.
- Make the tree directed arc-consistent takes $O(n)$
- Each consistency check takes up to $O(d^2)$ (compare d possible values for 2 variables).
多少条边, 多少次 revise
图 (树 n 条边)
- The CSP can be solved in $O(nd^2)$
不变
不需要 insert.

Nearly tree-structured CSPs



- Assign a variable or a set of variables and prune all the neighbors domains.
- This will turn the constraint graph into a tree :)
- There are other tricks to explore, have fun!

Summary

- CSPs are a special kind of search problems:
 - states defined by values of a fixed set of variables
 - goal test defined by constraints on variable values
- Backtracking = depth-first search with one variable assigned per node
- Variable selection and value ordering heuristics help
- Forward checking prevents assignments that guarantee later failure

Summary

- Constraint propagation (e.g., arc consistency) is an important mechanism in CSPs.
- It does additional work to constrain values and detect inconsistencies.
- Tree-structured CSPs can be solved in linear time
- Further exploration: How can local search be used for CSPs?
- **The power of CSPs: domain-independent, that is you only need to define the problem and then use a solver that implements CSPs mechanisms.**

To be continued