CS215: Discrete Math (H)
2022 Fall Semester Written Assignment # 4
Due: Dec. 7th, 2022, please submit at the beginning of class

Q.1 Prove by induction that, for any sets $A_1, A_2, \cdots, A_n$, De Morgan's law can be generalized to

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}.$$

Solution: The **base case** here is $n = 2$, that is De Morgan's law. (It is clear that $\overline{A_1} = \overline{A_1}$, so we could also use $n = 1$ as the base case.) It remains to show the inductive step. Suppose the statement holds for $n = k$, we now show it holds for $n = k + 1$. We have

$$\begin{array}{rcll} \overline{A_1 \cup A_2 \cup \cdots \cup A_{k+1}} & = & \overline{(A_1 \cup A_2 \cup \cdots \cup A_k) \cup A_{k+1}} & \\ & = & \overline{A_1 \cup A_2 \cup \cdots \cup A_k} \cap \overline{A_{k+1}} & \text{De Morgan} \\ & = & \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k} \cap \overline{A_{k+1}} & \text{by i.h.} \end{array}$$

Then by mathematical induction, we have proved the conclusion.

□

Q.2 Use induction to prove that 3 divides $n^3 + 2n$ whenever n is a positive integer.

Solution: Base case: $n = 1$, $n^3 + 2n = 3$, which is divisible by 3.

Inductive hypothesis: Suppose that 3 divides $n^3 + 2n$.

Inductive step: We now prove that 3 divides $(n + 1)^3 + 2(n + 1)$. We have

$$\begin{array}{rcl} (n+1)^3 + 2(n+1) & = & (n^3 + 2n) + (3n^2 + 3n + 3) \\ & = & (n^3 + 2n) + 3(n^2 + n + 1). \end{array}$$

Since $n^3 + 2n$ is divisible by 3 by i.h., and also $3(n^2 + n + 1)$ is divisible by 3. it follows that $(n + 1)^3 + 2(n + 1)$ is divisible by 3.

Conclusion: By mathematical induction, we prove the result.

□

Q.3 Suppose that a and b are real numbers with $0 < b < a$. Prove that if n is a positive integer, then $a^n - b^n \leq na^{n-1}(a-b)$.

Solution: It turns out to be easier to think about the given statement as $na^{n-1}(a-b) \geq a^n - b^n$. The basic step ($n = 1$) is true since $a - b \geq a - b$. Assume that the inductive hypothesis, that $ka^{k-1}(a-b) \geq a^k - b^k$; we must show that $(k+1)a^k(a-b) \geq a^{k+1} - b^{k+1}$. We have

$$\begin{aligned}(k+1)a^k(a-b) &= k \cdot a \cdot a^{k-1}(a-b) + a^k(a-b) \\ &\geq a(a^k - b^k) + a^k(a-b) \\ &= a^{k+1} - ab^k + a^{k+1} - ba^k.\end{aligned}$$

To complete the proof we want to show that $a^{k+1} - ab^k + a^{k+1} - ba^k \geq a^{k+1} - b^{k+1}$. This inequality is equivalent to $a^{k+1} - ab^k - ba^k + b^{k+1} \geq 0$, which factors into $(a^k - b^k)(a-b) \geq 0$, and this is true, because we are given that $a > b$.

□

Q.4 A store gives out gift certificates in the amounts of \$10 and \$25. What amounts of money can you make using gift certificates from the store? Prove your answer using strong induction.

Solution:

By checking the first few values $10, 20, 25, 30, 35, 40, 45, 50, \ldots$, we guess that we can make $\$n$ in amount of money, where

$$n \in \{10\} \cup \{5m : \ m \geq 4 \text{ and } m \in \mathbb{Z}^+\}.$$

Let $P(n)$ be the statement "we can make $\$5m$ in gift certificate in amount of \$10 and \$25."

Base case: $m = 4, 5$, we can make \$20 and \$25 in gift certificate.

Inductive hypothesis: We can make $\$5k$ for $4 \leq k < m$.

Inductive step: We now prove $P(m)$ for $m \geq 6$. Note that $5m = 10 + 5(m-2)$. Since $4 \leq m - 2 < m$, $P(m-2)$ is true. So we can make $\$5(m-2)$ in gift certificate. It then follows that we can $\$5m$ in gift certificate by adding an extra \$10 certificate.

□

Q.5 Show that the principle of mathematical induction and strong induction are equivalent; that is, each can be shown to be valid from the other.

Solution: The strong induction principle clearly implies ordinary induction, for if one has shown that $P(k) \to P(k+1)$, then it automatically follows that $[P(1) \wedge \cdots \wedge P(k)] \to P(k+1)$; in other words, strong induction can always be invoked whenever ordinary induction is used.

Conversely, suppose that $P(n)$ is a statement that one can prove using strong induction. Let $Q(n)$ be $P(1) \wedge \cdots \wedge P(n)$. Clearly $\forall n P(n)$ is logically equivalent to $\forall n Q(n)$. We show how $\forall n Q(n)$ can be proved using ordinary induction. First, $Q(1)$ is true because $Q(1) = P(1)$ and $P(1)$ is true by the basis step for the proof of $\forall n P(n)$ by strong induction. Now suppose that $Q(k)$ is true, i.e., $P(1) \wedge \cdots \wedge P(k)$ is true. By the proof of $\forall n P(n)$ by strong induction, it follows that $P(k+1)$ is true. But $Q(k) \wedge P(k+1)$ is just $Q(k+1)$. Thus, we have proved $\forall n Q(n)$ by ordinary induction.

$\square$

Q.6 Suppose that the function f satisfies the recurrence relation $f(n) = 2f(\sqrt{n}) + \log n$ whenever n is a perfect square greater than 1 and $f(2) = 1$.

(a) Find $f(16)$

(b) Find a big-O estimate for $f(n)$. [Hint: make the substitution $m = \log n$.]

Solution:

(a) $f(16) = 2f(4) + 4 = 2(2f(2) + 2) + 4 = 2(2 \cdot 1 + 2) + 4 = 12$.

(b) Let $m = \log n$, so that $n = 2^m$. Also, let $g(m) = f(2^m)$. Then our recurrence becomes $f(2^m) = 2f(2^{m/2}) + m$, since $\sqrt{2^m} = (2^m)^{1/2} = 2^{m/2}$. Rewriting this in terms of g we have $g(m) = 2g(m/2) + m$. Theorem 2 (with $a = 2, b = 2, c = 1$, and $d = 1$ now tells us that $g(m)$ is $O(m \log m)$. Since $m = \log n$, this means that our function is $O(\log n \cdot \log \log n)$.

$\square$

Q.7 Find $f(n)$ when $n = 4^k$, where f satisfies the recurrence relation $f(n) = 5f(n/4) + 6n$, with $f(1) = 1$.
Solution: $f(n) = 25n^{\log_4 5} - 24n$.

$\square$

Q.8 Suppose that $n \geq 1$ is an integer.

(a) How many functions are there from the set $\{1, 2, \ldots, n\}$ to the set $\{1, 2, 3\}$?

(b) How many of the functions in part (a) are one-to-one functions?

(c) How many of the functions in part (a) are onto functions?

Solution:

(a) There are 3^n functions.

(b) If $n \leq 3$, there are $P(3, n)$ one-to-one functions. Hence, there are 3 when $n = 1$, 6 when $n = 2$, and 6 when $n = 3$. If $n > 3$, then there are 0 injective functions; there cannot be a one-to-one function from A to B if $|A| > |B|$.

(c) By the Inclusion-Exclusion Principle, we have

$$\begin{aligned}
\# &= \#\{f : f(A) \subseteq \{1,2,3\}\} - \#\{f : f(A) \subseteq \{1,2\}\} - \#\{f : f(A) \subseteq \{1,3\}\} \\
&\quad -\#\{f : f(A) \subseteq \{2,3\}\} + \{f : f(A) \subseteq \{1\}\} + \{f : f(A) \subseteq \{2\}\} \\
&\quad \#\{f : f(A) \subseteq \{3\}\} \\
&= 3^n - 2^n - 2^n - 2^n + 1 + 1 + 1 \\
&= 3^n - 3 \cdot 2^n + 3.
\end{aligned}$$

$\square$

Q.9 How many 6-card poker hands consist of exactly 2 pairs? That is two of one rank of card, two of another rank of card, one of a third rank, and one of a fourth rank of card? Recall that a deck of cards consists of 4 suits each with one card of each of the 13 ranks. <u>You should leave your answer as an equation</u>.

Solution: First, we choose the ranks of the 2 pairs, noting that the order we pick these two ranks does not matter, so there are $\binom{13}{2}$ options here. Next we pick the 2 suits for the first pair, $\binom{4}{2}$ and the suits for the second pair $\binom{4}{2}$. Then we decide which 2 ranks of the remaining 11 to use for the other cards, $\binom{11}{2}$, and finally choose each of their suits $\binom{4}{1}\binom{4}{1}$. Altogether, by the product rule, this gives $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{2}\binom{4}{1}\binom{4}{1}$ hands.

$\square$

Q.10 Prove that the binomial coefficient

$$\binom{240}{120}$$

is divisible by $242 = 2 \cdot 121$.

Solution:

Since $\gcd(2, 121) = 1$, it suffices to prove that $2 \big| \binom{240}{120}$ and $121 \big| \binom{240}{120}$. We prove these two divisibilities in general, i.e.,

$$2 \bigg| \binom{2n}{n}, \text{ and } (n+1) \bigg| \binom{2n}{n}.$$

Since

$$\begin{aligned}
\binom{2n}{n} &= \frac{(2n)!}{n!n!} \\
&= \frac{2n \cdot (2n-1)!}{n!n!} \\
&= \frac{2 \cdot (2n-1)!}{(n-1)!n!} \\
&= 2 \cdot \binom{2n-1}{n},
\end{aligned}$$

we have 2 divides $\binom{2n}{n}$. Since

$$\begin{aligned}
\binom{2n}{n} - \binom{2n}{n-1} &= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} \\
&= \frac{(2n)!}{(n+1)!n!} \\
&= \frac{1}{n+1}\binom{2n}{n},
\end{aligned}$$

which is an integer, we have $n+1$ divides $\binom{2n}{n}$. This completes the proof.

$\square$

Q.11 How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that $a_1 \bmod 5 = a_2 \bmod 5$ and $b_1 \bmod 5 = b_2 \bmod 5$.

Solution:

Working modulo 5 there are 25 pairs: $(0, 0), (0, 1), \ldots, (4, 4)$. Thus, we could have 25 ordered pairs of integers (a, b) such that no two of them were equal when reduced modulo 5. The pigeonhole principle, however, guarantees that if we have 26 such pairs, then at least two of them will have the same coordinates, modulo 5.

$\square$

Q.12 Let (x_i, y_i), $i = 1, 2, 3, 4, 5$, be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integers coordinates.

Solution:

The midpoint of the segment whose endpoints are (a, b) and (c, d) is $((a+c)/2, (b+d)/2)$. We are concerned only with integer values of the original coordinates. Clearly the coordinates of these fractions will be integers as well if and only if a and c have the same parity (both odd or both even) and b and d have the same parity. There are four possible pairs of parities: (odd, odd), $(odd, even)$, $(even, odd)$, $(even, even)$. Since we are given five points, the pigeonhole principle guarantees that at least two of them will have the same pair of parities. The midpoint of the segment joining these two points will therefore have integer coordinates.

$\square$

Q.13 Show that if p is a prime and k is an integer such that $1 \le k \le p-1$, then p divides $\binom{p}{k}$.

Solution:

We know that

$$\binom{p}{k} = \frac{p!}{k!(p-k)!}.$$

Clearly p divides the numerator. On the other hand, p cannot divide the denominator, since the prime factorizations of these factorials contains only numbers less than p. Therefore the factor p does not cancel when this fraction is reduced to lowest terms (i.e., to a whole number), so p divides $\binom{p}{k}$.

$\square$

Q.14 Prove the hockeystick identity

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

whenever n and r are positive integers,

(a) using a combinatorial argument

(b) using Pascal's identity.

Solution:

(a) $\binom{n+r+1}{r}$ counts the number of ways to choose a sequence of r 0s and $n+1$ 1s by choosing the positions of the 0s. Alternatively, suppose that the $(j+1)$st term is the last term equal to 1, so that $n \leq j \leq n+r$. Once we have determined where the last 1 is, we decide where the 0s are to be placed in the j spaces before the last 1. There are n 1s and $j-n$ 0s in this range. By the sum rule it follows that there are $\sum_{j=n}^{n+r} \binom{j}{j-n} = \sum_{k=0}^{r} \binom{n+k}{k}$ ways to this.

(b) Let $P(r)$ be the statement to be proved. The basis step is the equation

$\binom{n}{0} = \binom{n+1}{0}$, which is just $1 = 1$. Assume that $P(r)$ is true. Then

$$\begin{aligned}
&\sum_{k=0}^{r+1} \binom{n+k}{k} \\
&= \sum_{k=0}^{r} \binom{n+k}{k} + \binom{n+r+1}{r+1} \\
&= \binom{n+r+1}{r} + \binom{n+r+1}{r+1} \\
&= \binom{n+r+2}{r+1},
\end{aligned}$$

using the inductive hypothesis and Pascal's identity.

□

Q.15 Find the solution to $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n = 3, 4, 5, \ldots$, with $a_0 = 3$, $a_1 = 6$, and $a_2 = 0$.

Solution:

The characteristic equation is $r^3 - 2r^2 - r + 2 = 0$. This factors as $(r-1)(r+1)(r-2) = 0$, so the roots are $1, -1$, and 2. Therefore the general solution is $a_n = \alpha_1 + \alpha_2(-1)^n + \alpha_3 2^n$. Plugging in initial conditions gives $3 = \alpha_1 + \alpha_2 + \alpha_3$, $6 = \alpha_1 - \alpha_2 + 2\alpha_3$, and $0 = \alpha_1 + \alpha_2 + 4\alpha_3$. The solution to this system of equations is $\alpha_1 = 6$, $\alpha_2 = -1$ and $\alpha_3 = -1$. Therefore, the answer is $a_n = 6 - 2(-1)^n - 2^n$.

□

Q.16 Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$ with initial conditions $a_0 = 1$ and $a_1 = 0$.

Solution: The *characteristic equation* is $r^2 - 5r + 6 = 0$, and the solutions are $r = 2, 3$. So the closed-form is $a_n = \alpha_1 \cdot 2^n + \alpha_2 \cdot 3^n$. In particular, we have $1 = \alpha_1 \cdot 2^0 + \alpha_2 \cdot 3^0 = \alpha_1 + \alpha_2$, and $0 = 2\alpha_1 + 3\alpha_2$. Solving for 2 unknowns with 2 equations is $\alpha_1 = 3$ and $\alpha_2 = -2$. Therefore, we have

$$a_n = 3 \cdot 2^n - 2 \cdot 3^n.$$

$\square$

Q.17 A computer system considers a string of decimal digits $(0, 1, \ldots, 9)$ to be a **valid** code word if and only if it contains an **odd number of zero digits**. For example, 12030 and 11111 are **not** valid, but 29046 is. Let $V(n)$ denote the number of valid n-digit code words. Find a recurrence relation for $V(n)$ with initial cases, and give a closed-form solution to this recurrence relation. Please explain how you find the recurrence relation. (Hint: notice that the number of non-valid code words is equal to $10^n - V(n)$.)

Solution: There are two ways to construct a valid code of length n from a string of $n-1$ digits:

(a) take a valid code of length $n-1$, append a number between 1 and 9: there are $9V(n-1)$ ways;

(b) take a non-valid code of length $n-1$, append a 0: there are $10^{n-1} - V(n-1)$ ways.

In total, we have

$$V(n) = 9V(n-1) + 10^{n-1} - V(n-1) = 10^{n-1} + 8V(n-1),$$

with initial cases $V(1) = 1$.

By iterating this recurrence, we have

$$\begin{aligned}
V(n) &= 8V(n-1) + 10^{n-1} \\
&= 8(8V(n-2) + 10^{n-2}) + 10^{n-1} \\
&= 8^2V(n-2) + 8 \cdot 10^{n-2} + 10^{n-1} \\
&= 8^2(8V(n-3) + 10^{n-3}) + 8 \cdot 10^{n-2} + 10^{n-1} \\
&= 8^3V(n-3) + 8^2 \cdot 10^{n-3} + 8 \cdot 10^{n-2} + 10^{n-1} \\
&= \vdots \\
&= 8^{n-1}V(1) + 8^{n-2}10^1 + 8^{n-3}10^2 + \cdots + 8 \cdot 10^{n-2} + 10^{n-1} \\
&= 8^{n-1}\left(1 + \frac{5}{4} + (\frac{5}{4})^2 + \cdots + (\frac{5}{4})^{n-1}\right) \\
&= 8^{n-1} \cdot \frac{1 - (\frac{5}{4})^n}{1 - \frac{5}{4}} \\
&= 5 \cdot 10^{n-1} - 4 \cdot 8^{n-1}.
\end{aligned}$$

$\square$

Q.18 Let $S_n = \{1, 2, \ldots, n\}$ and let a_n denote the number of *non-empty* subsets of S_n that contain **no** two consecutive integers. Find a recurrence relation for a_n. Note that $a_0 = 0$ and $a_1 = 1$.

Solution: We may split S_n into 3 cases :

> Case (1): item 1 is not in the subset. We must now choose a non-empty subset of $\{2, \ldots, n\}$. There are a_{n-1} ways to do this.
>
> Case (2): item 1 is in the subset, and there are more elements. We must now choose a non-empty subset of $\{3, \ldots, n\}$. There are a_{n-2} ways to do this.
>
> Case (3): item 1 is in the subset, and no other elements are. There is 1 way to do this.

Thus, we have the recurrence relation as: $a_n = a_{n-1} + a_{n-2} + 1$.

$\square$

Q.19 Let $\mathbf{A}_n$ be the $n \times n$ matrix with 2's on its main diagonal, 1's in all positions next to a diagonal element, and 0's everywhere else. Find a recurrence relation for d_n, the determinant of $\mathbf{A}_n$. Solve this recurrence relation to find a formula for d_n.

Solution:

We can compute the first few terms by hand. For $n = 1$, the matrix is just the number 2, so $d_1 = 2$. For $n = 2$, the matrix is $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, and its determinant is clearly $d_2 = 4 - 1 = 3$. For $n = 3$, the matrix is

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix},$$

and we get $d_3 = 4$. For the general case, our matrix is

$$\mathbf{A}_n = \begin{bmatrix} 2 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 2 & 1 & 0 & \ldots & 0 \\ 0 & 1 & 2 & 1 & \ldots & 0 \\ 0 & 0 & 1 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 2 \end{bmatrix}.$$

To compute the determinant, we expand along the top row. This gives us a value of 2 times the determinant of the matrix obtained by deleting the first row and first column minus the determinant of the matrix obtained by deleting the first row and second column. The first of these smaller matrices is just $\mathbf{A}_{n-1}$, with determinant d_{n-1}. The second of these smaller matrices has just one nonzero entry in its first column, so we expand its determinant along the first column and see that it equals d_{n-2}. Therefore our recurrence relation is $d_n = 2d_{n-1} - d_{n-2}$, with initial conditions as computed at the start of this solution. If we compute a few more terms we are led to the conjecture that $d_n = n + 1$. If we show that this satisfies the recurrence, then we have proved that it is indeed the solution. And sure enough, $n+1 = 2n - (n-1)$.

□

Q.20 Use generating functions to prove Vandermonde's identity:

$$C(m+n, r) = \sum_{k=0}^{r} C(m, r-k)C(n, k),$$

whenever m, n, and r are nonnegative integers with r not exceeding either m or n. [Hint: Look at the coefficient of x^r in both sides of $(1+x)^{m+n} = (1+x)^m(1+x)^n$.]
Solution: Applying the binomial theorem to the equality $(1+x)^{m+n} = (1+x)^m(1+x)^n$, shows that $\sum_{r=0}^{m+n} C(m+n, r)x^r = \sum_{r=1}^{m} C(m, r)x^r \cdot \sum_{r=0}^{n} C(n, r)x^r = \sum_{r=0}^{m+n} [\sum_{k=0}^{r} C(m, r-k)C(n, k)]x^r$. Comparing coefficients gives the desired identity.

□

Q.21 Generating functions are very useful, for example, provide an approach to solving linear recurrence relations. Read pp. 537-548 of the textbook. [You do not need to write anything for this problem on your submitted assignment paper.]