南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Algorithm Design and Analysis (H)

CS216

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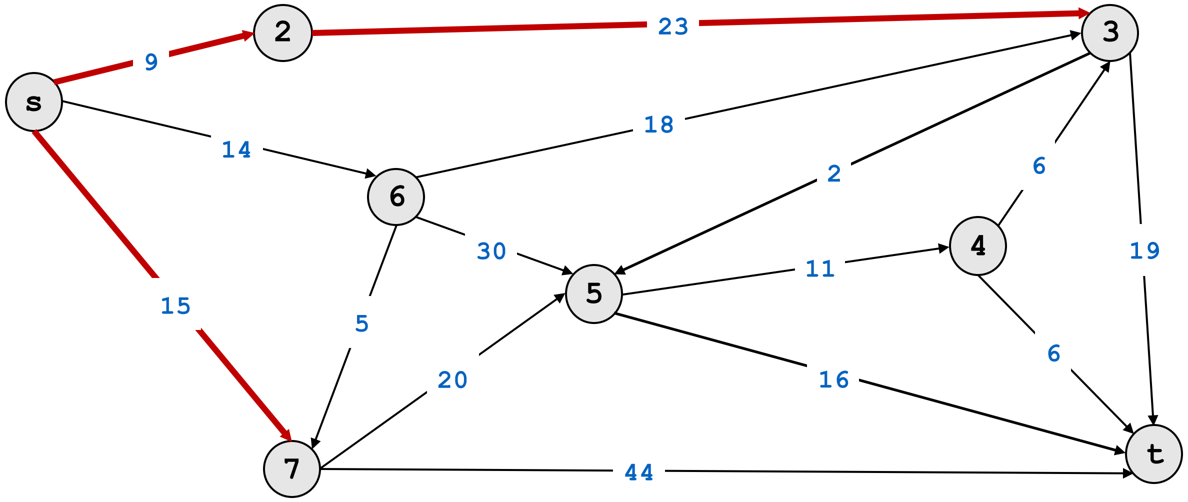
(slides edited from Prof. Shiqi Yu)

Greedy Algorithms

4. Shortest Paths in a Graph

Single-Source Shortest Path Problem

- **Single-source shortest path problem.**
  - Directed graph G = (V, E) with non-negative edge costs.
  - Source s. undirected graph is usually easier
  - ℓ_e = length of edge e. path length = sum of edge lengths on path
  - Goal: find a shortest directed path from s to every node.



shortest parth from s to 3: 9 + 23 = 32
shortest parth from s to 7: 15

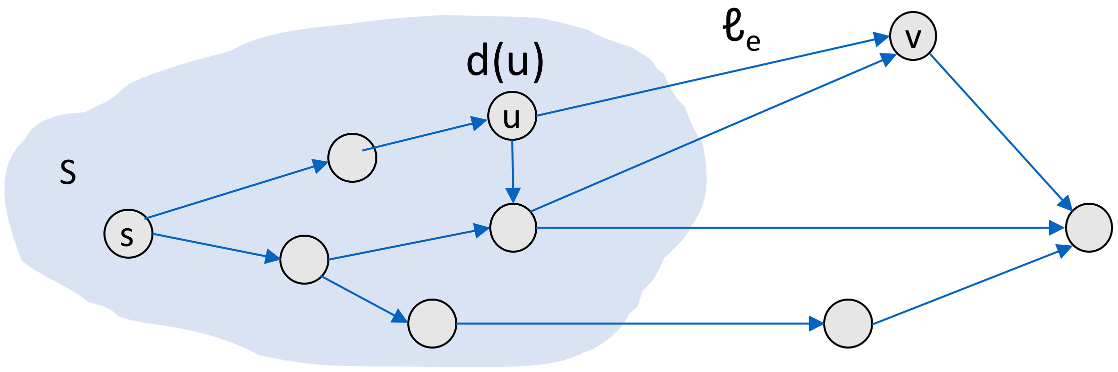
Dijkstra's Algorithm

- **Greedy approach.**
  - Maintain a set S of explored nodes for which we have determined the shortest path distance d(u) from s to u.
  - Initialize S = { s }, d(s) = 0.
  - Repeatedly choose unexplored node v which minimizes

$$\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e$$

shortest path to some u in explored part, followed by a single edge (u, v)

  - Add v to S and set d(v) = π(v).



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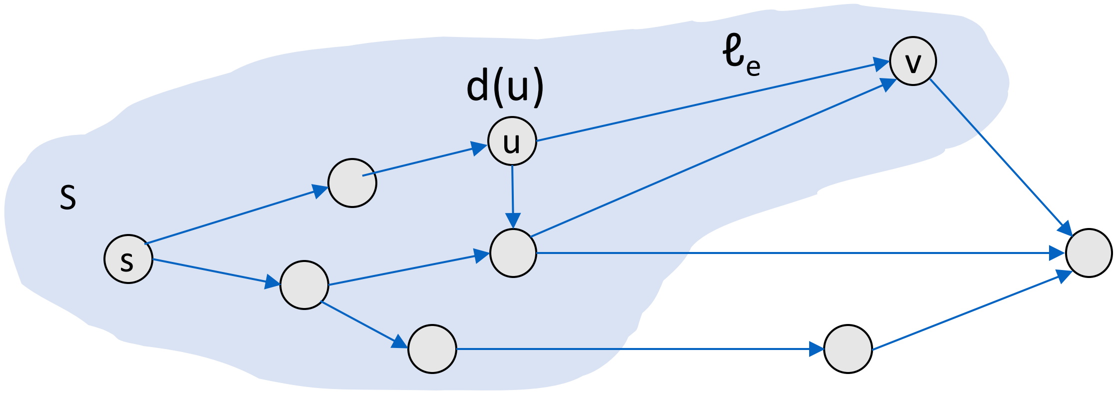
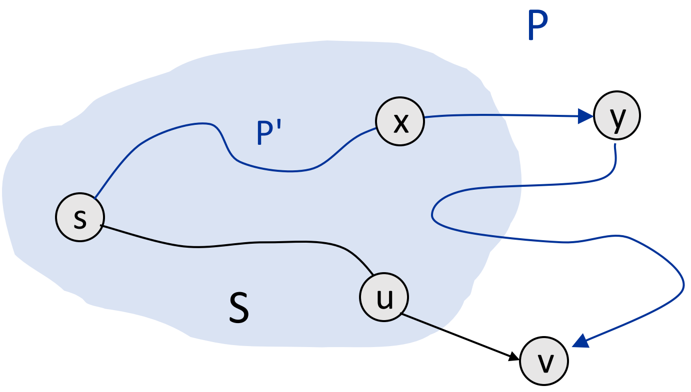


Dijkstra's Algorithm: Proof of Correctness

- **Invariant.** For each $u \in S$, $d(u)$ is the length of the shortest s-u path.
- Pf. (by induction on $|S|$)
  - ➢ Base case: $|S| = 1$ is trivial.
  - ➢ Inductive hypothesis: Assume true for $|S| \geq 1$.
  - ➢ Let v be next node added to S and let u-v be the chosen edge.
  - ➢ The shortest s-u path plus (u, v) is an s-v path of length $\pi(v)$.
  - ➢ Consider any s-v path P. It is no shorter than $\pi(v)$.
    - ✓ Let x-y be the first edge in P that leaves S, and let P' be the subpath to x.
    - ✓ P is already too long as soon as it leaves S.

$$\ell(P) \;\underset{\text{nonnegative weights}}{\geq}\; \ell(P') + \ell(x,y) \;\underset{\text{inductive hypothesis}}{\geq}\; d(x) + \ell(x, y) \;\underset{\text{definition of } \pi(y)}{\geq}\; \pi(y) \;\underset{\text{definition of } \pi(v)}{\geq}\; \pi(v)$$



Dijkstra's Algorithm: Implementation

- For each unexplored node, explicitly maintain $\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e$.
  - ➢ Next node to explore = node with minimum π(v).
  - ➢ When exploring v, for each incident edge e = (v, w), update

$$\pi(w) = \min \{ \pi(w), \pi(v) + \ell_e \}.$$

  - ➢ Maintain a priority queue of unexplored nodes, prioritized by π(v).

PQ Operation	Dijkstra	Array	Binary heap	d-way Heap	Fibonacci heap †
Insert	n	n	log n	$d \log_d n$	1
ExtractMin	n	n	log n	$d \log_d n$	log n
ChangeKey	m	1	log n	$\log_d n$	1
IsEmpty	n	1	1	1	1
Total		n^2	m log n	$m \log_{m/n} n$	m + n log n

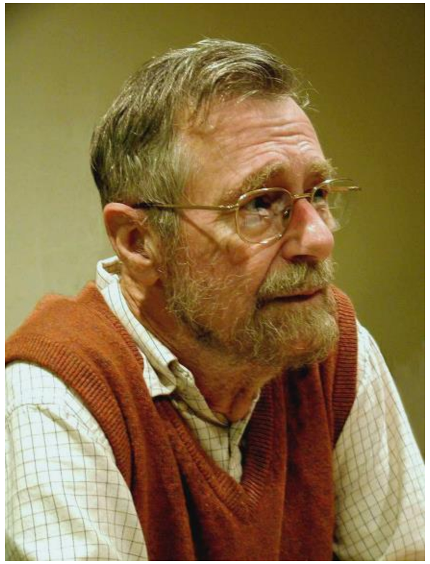
large constant

† Individual ops are amortized bounds

Dijkstra's Algorithm: History

" What's the shortest way to travel from Rotterdam to Groningen? It is the algorithm for the shortest path, which I designed in about 20 minutes. One morning I was shopping in Amsterdam with my young fiancée, and tired, we sat down on the café terrace to drink a cup of coffee and I was just thinking about whether I could do this, and I then designed the algorithm for the shortest path. "

--- Edsger W. Dijkstra

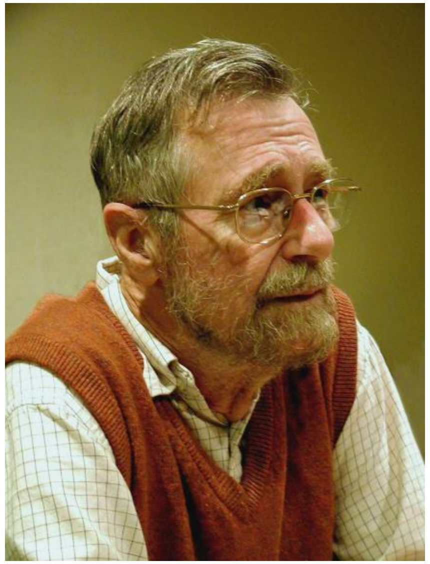


More about Edsger W. Dijkstra

Dijkstra was immensely influential in many fields of computing: compilers, operating systems, concurrent programming, software engineering, programming languages, algorithm design, and teaching (among others!)

It would be hard to pin down what he is most famous for because he has influenced so much CS.

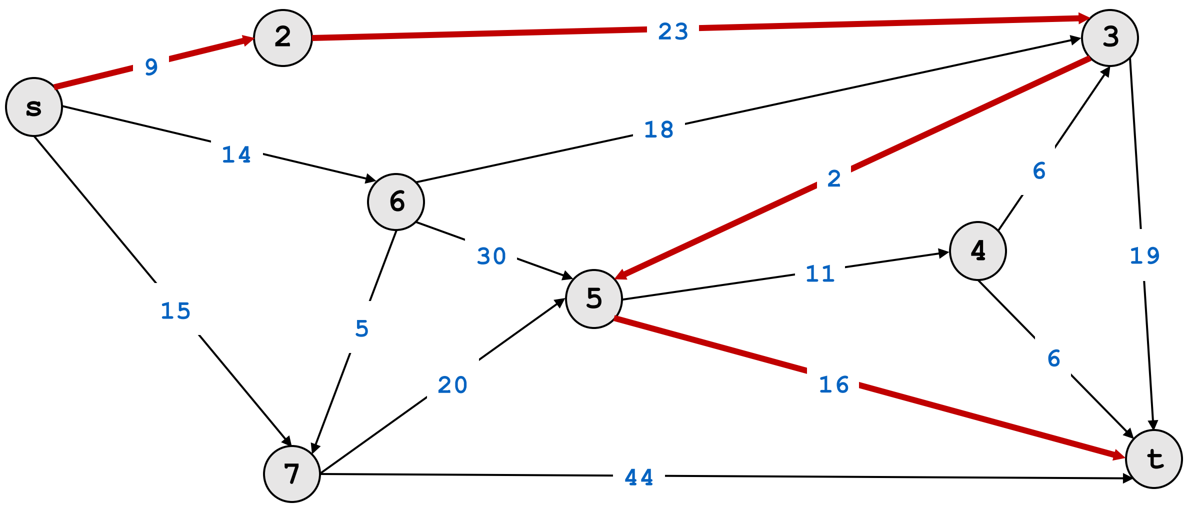
Dijkstra was also influential in making programming more structured -- he wrote a seminal paper titled, "Goto Considered Harmful" where he lambasted the idea of the "goto" statement (which exists in C++ -- you will rarely, if ever, use it!)



Single-Pair Shortest Path Problem

- **Single-pair shortest path problem.**
  - Directed graph G = (V, E) with non-negative edge costs.
  - Source s, destination t.
  - ℓ_e = length of edge e.
  - Goal: find a shortest directed path from s to t.

only one destination is considered



shortest path from s to t:
9 + 23 + 2 + 1 = 50

Single-Pair Shortest Path Problem

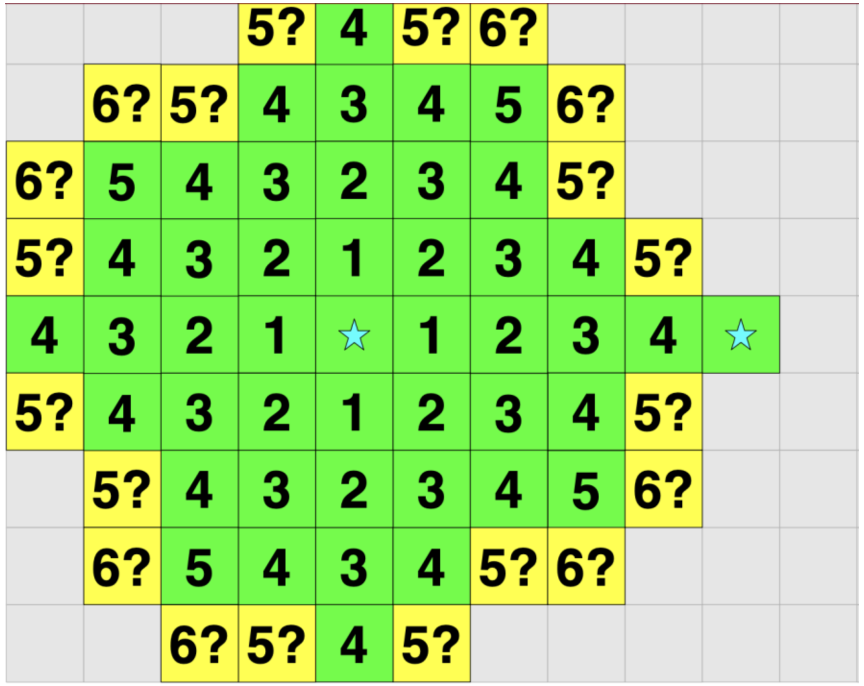
- **Single-pair shortest path problem.**
  - Directed graph G = (V, E) with non-negative edge costs.
  - Source s, destination t.
  - ℓ_e = length of edge e.
  - Goal: find a shortest directed path from s to t.

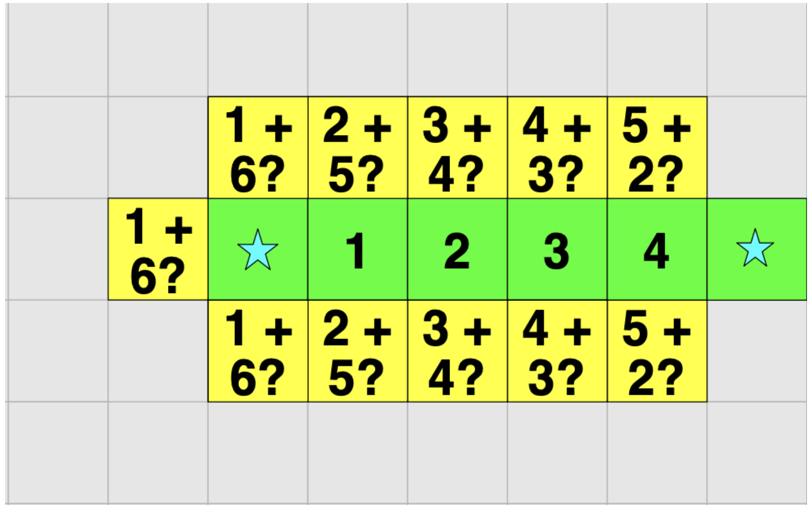
目标确定 only one destination is considered

- **Q.** Can we beat Dijkstra when we have only one destination?
  - E.g., if traveling from Beijing to Shanghai, we will go south.
  - Solution: add some heuristic information to guide the search, which could be direction in the case of a street map. 增加启发式信息

Single-Pair Shortest Path Problem

- **Dijkstra's priority:**
  - ➢ s-v distance $\pi(v)$
- **Ideal priority:**
  - ➢ $\pi(v)$ + v-t distance $d(v, t)$





we should prioritize search to the right

A* Search Algorithm

- **A* priority:**
  - ➢ s-v distance π(v) (same as Dijkstra) + heuristic v-t distance h(v, t)
- **Q.** How do we estimate the true v-t distance d(v, t)?
  - ➢ E.g., hamming distance, Euclidean distance, etc.
- **Choice of h(v, t):**
  - ➢ h(v, t) = 0: same as Dijkstra
  - ➢ h(v, t) < d(v, t): same or faster than Dijkstra (and shortest path guaranteed)
  - ➢ h(v, t) = d(v, t): fastest, but requires perfect knowledge (the shown example)
  - ➢ h(v, t) > d(v, t): won't necessarily find the shortest path (but might run faster)
- **Takeaway:** Always underestimate the true future cost d(v, t) for A*.

A* Search Algorithm

- Example:

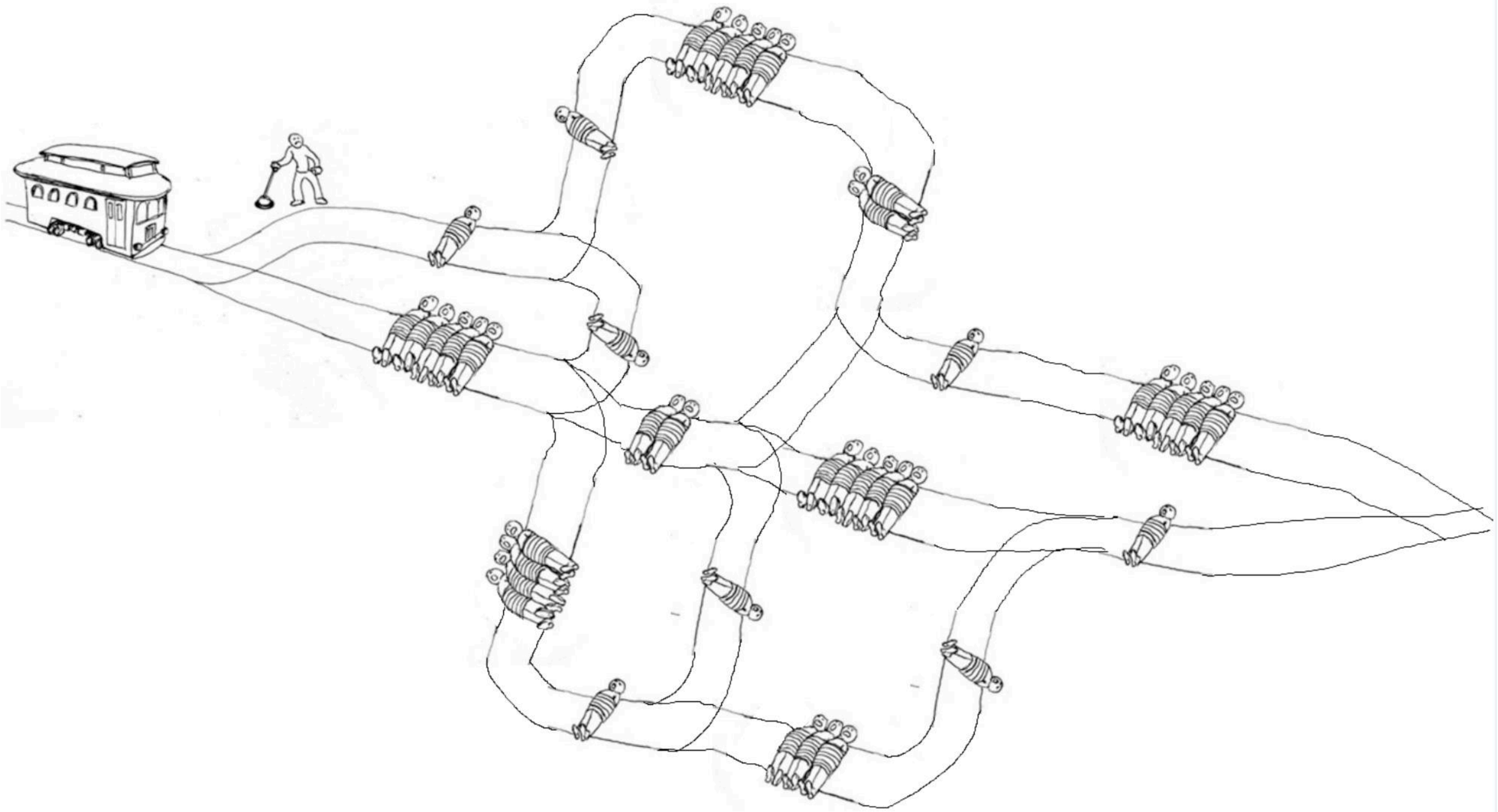
Dijkstra:

8	7	6	5	4	5	6	7	8	9?				
7	6	5	4	3	4	5	6	7	8	9?			
6	5	4	3	2	3	4	5	6	7	8	9?		
5	4	3	2	1	2	3		7	8	9?			
4	3	2	1	☆	1	2		8	☆				
5	4	3	2	1	2	3		7	8	9?			
6	5	4	3	2	3	4	5	6	7	8	9?		
7	6	5	4	3	4	5	6	7	8	9?			
8	7	6	5	4	5	6	7	8	9?				

A*: (h(v, t) = Hamming distance)

				3 + 8?	4 + 7?	5 + 6?	6 + 5?	7 + 4?					
			3 + 8?	2	3	4	5	6	7 + 2?				
		3 + 8?	2	1	2	3		7 + 2?					
	3 + 8?	2	1	☆	1	2		8	☆				
		3 + 8?	2	1	2	3		7	8 + 1?				
			3 + 8?	2	3	4	5	6	7	8 + 3?			
				3 + 8?	4 + 7?	5 + 6?	6 + 5?	7 + 4?	8 + 3?				

Shortest Path Problem: Moral Implications



5. Minimum Spanning Trees

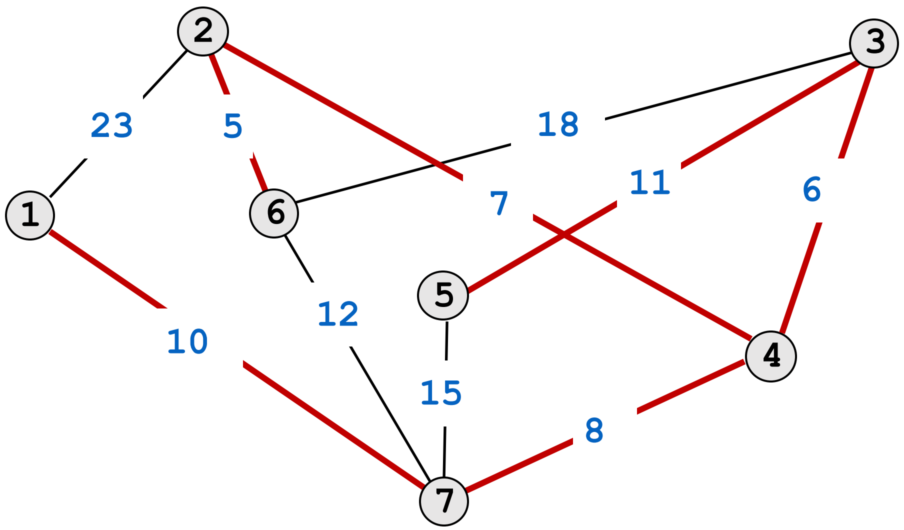
Spanning Trees

- **Def.** Let $H = (V, T)$ be a subgraph of an undirected graph $G = (V, E)$. H is a spanning tree of G if H is both acyclic and connected.
- **Property.** All the following are equivalent:
  - H is a spanning tree of G.
  - H is acyclic and connected.
  - H is connected and has $|V| - 1$ edges.
  - H is acyclic and has $|V| - 1$ edges.
  - H is minimally connected: removal of any edge disconnects it.
  - H is maximally acyclic: addition of any edge creates a cycle.

Minimum Spanning Trees (MSTs)

- **Def.** Given a connected, undirected graph $G = (V, E)$ with edge costs, a minimum spanning tree (MST) (V, T) is a spanning tree of G such that the sum of the edge costs in T is minimized.
- **Cayley's theorem:** K_n has n^{n-2} spanning trees. ($|V| = n$, $|E| = m$)

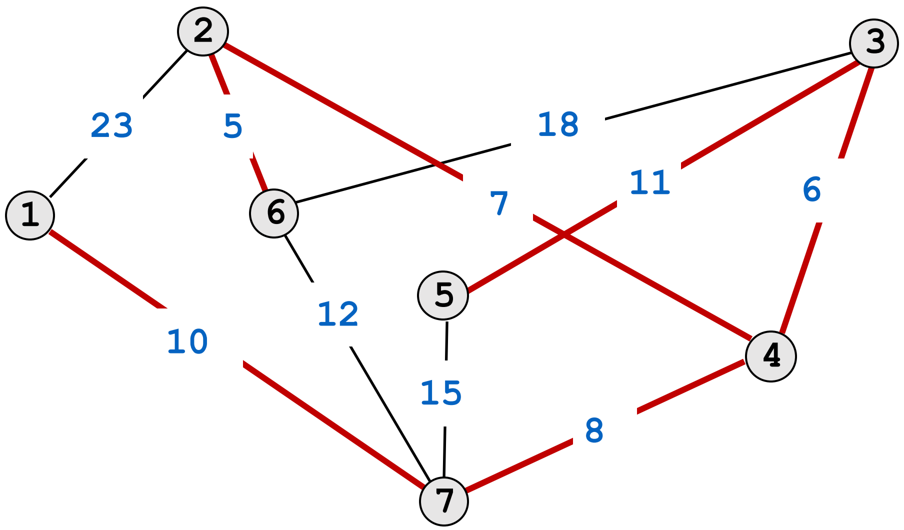
cannot solve by brute-force



MST cost: 5 + 6 + 7 + 8 + 10 + 11 = 47

Prim's Algorithm

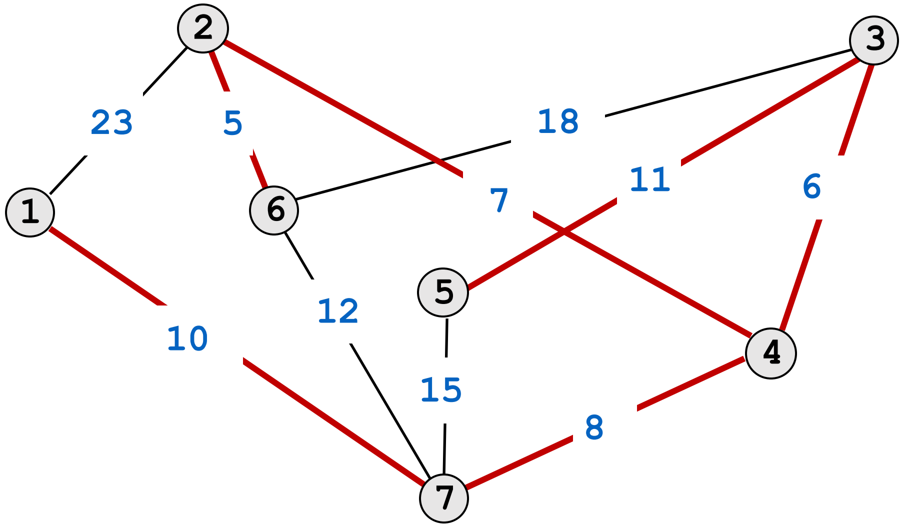
- **Greedy approach.**
  - Initialize S = { s } for any node s and initialize edge set $T = \emptyset$.
  - Repeat $n - 1$ times:
    extract-min
    ✓Add to T a min-cost edge with exactly one endpoint in S.
    ✓Add the other endpoint to S. ← change-key



time complexity: O(m log n)

Kruskal's Algorithm

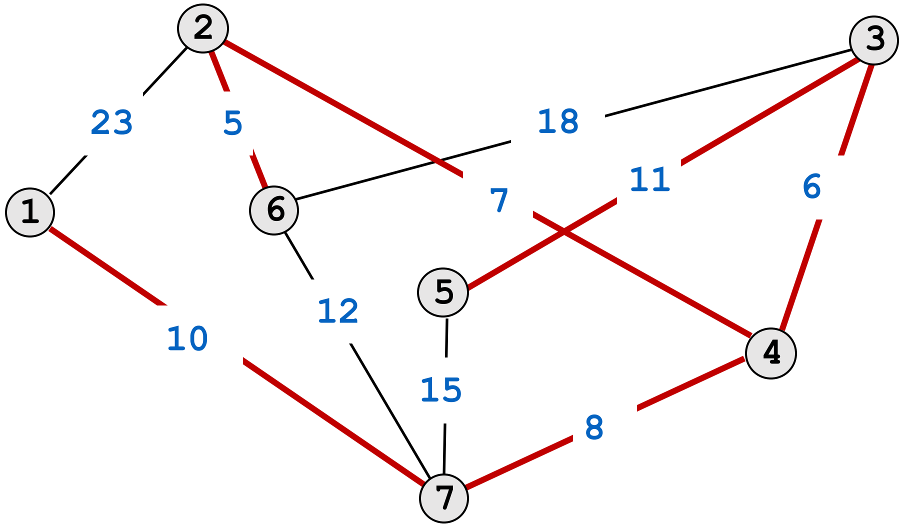
- **Greedy approach.**
  - Initialize edge set $T = \emptyset$.
  - Sort edges in ascending order of cost.
  - Repeat m times:
    - ✓Add to T the considered edge unless it would create a cycle. ← union-find



time complexity: O(m log m)

Reverse-Delete Algorithm

- **Greedy approach.**
  - Initialize edge set $T = E$.
  - Sort edges in descending order of cost.
  - Repeat m times:
    - ✓Delete from T the considered edge unless it would disconnect T.

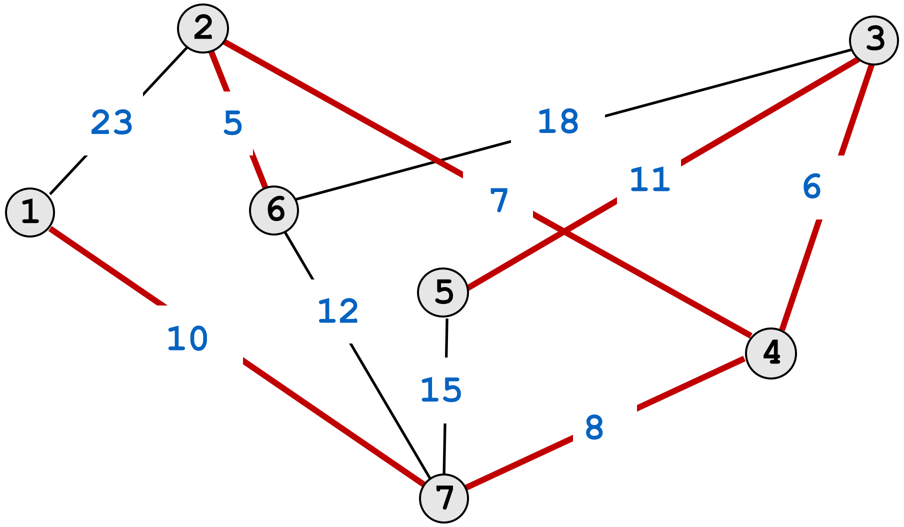


time complexity: $O(m \log n (\log \log n)^3)$

[Thorup 2000]

Borůvka's Algorithm

- **Greedy approach.**
  - Initialize edge set $T = \emptyset$ (T has n connected components, one for each node).
  - Repeat until only one connected component is left: ← O(log n) rounds, no need to sort
    - ✓For each edge (u, v), if u, v are in different components, use the cost of (u, v) to update the min-cost edge for both components.
    - ✓Add to T the min-cost edge for each component.



time complexity: O(m log n)

Minimum Spanning Trees: Summary

- The learned MST greedy algorithms follow similar ideas and share roughly the same time complexity *O(m log n)*.
    - ➢ Prim: extend a single connected component with a min-cost edge
    - ➢ Kruskal: extend connected components with a min-cost edge
    - ➢ Reverse-delete: remove a max-cost edge and maintain connected
    - ➢ Borůvka: extend connected components with a min-cost edge (without sorting)

- **Remark.** The above greedy algorithms can be extended to find minimum spanning forests.
- **Q.** Are there linear MST algorithms?

Minimum Spanning Trees: Linear Algorithms

- Linear randomized MST algorithms do exist! [Karger-Klein-Tarjan 1995]
- It is still open for deterministic compare-based MST algorithms.

year	worst case	discovered by
1975	$O(m \log \log n)$	Yao
1976	$O(m \log \log n)$	Cheriton–Tarjan
1984	$O(m \log^* n)$, $O(m + n \log n)$	Fredman–Tarjan
1986	$O(m \log (\log^* n))$	Gabow–Galil–Spencer–Tarjan
1997	$O(m\, \alpha(n) \log \alpha(n))$	Chazelle
2000	$O(m\, \alpha(n))$	Chazelle
2002	*asymptotically optimal*	Pettie–Ramachandran
20xx	$O(m)$	???

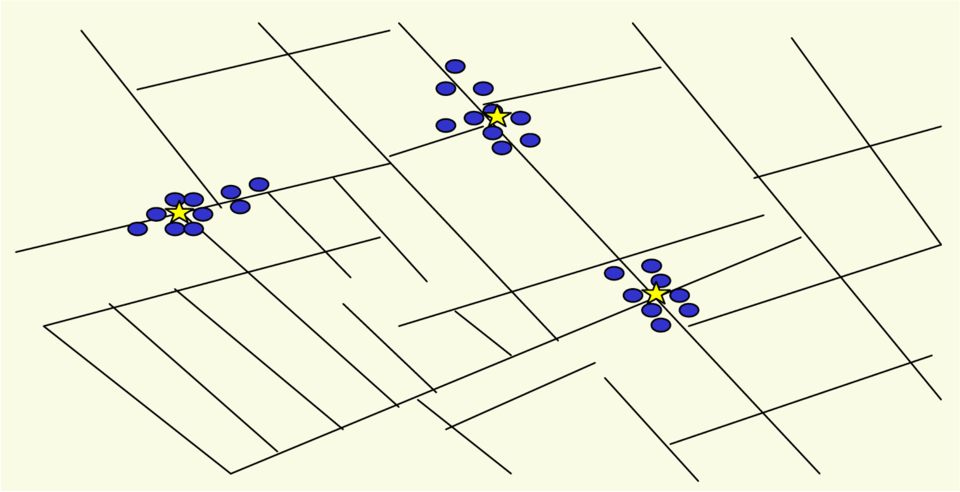
6. Clustering

Clustering

- **Goal.** Given a set U of n objects labeled $p_1, \ldots, p_n$, partition into clusters so that objects in different clusters are far apart.

e.g., photos, documents, etc.

w.r.t. some distance, e.g., number of pixels that differ by some threshold

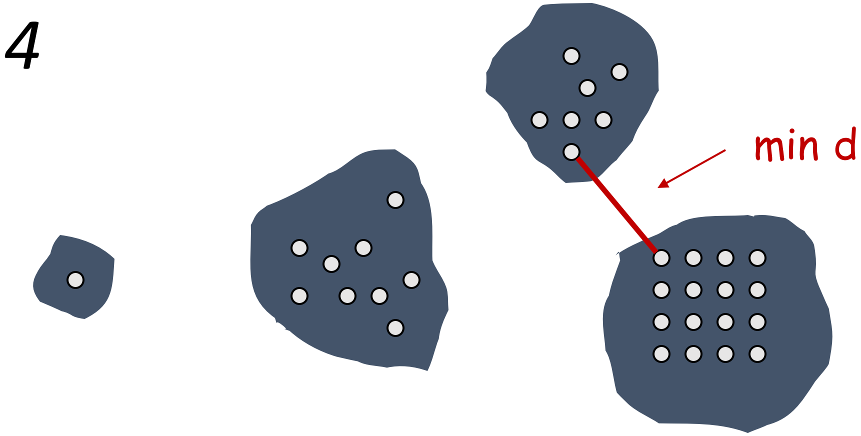


Outbreak of cholera deaths in London in 1850s. Reference: Nina Mishra, HP Labs

- **Applications.** Routing, categorization, similarity searching, etc.

Clustering of Maximum Spacing

- **k-clustering.** Divide objects into k non-empty groups.
- **Distance function.** Numeric value specifying "closeness" of two objects.
  - $d(p_i, p_j) = 0$ iff $p_i = p_j$ (identity of indiscernibles)
  - $d(p_i, p_j) \geq 0$ (nonnegativity)
  - $d(p_i, p_j) = d(p_j, p_i)$ (symmetry)
- **Spacing.** Min distance between any pair of points in different clusters.
- **Goal.** Given an integer k, find a k-clustering of maximum spacing.
- Ex. $k = 4$



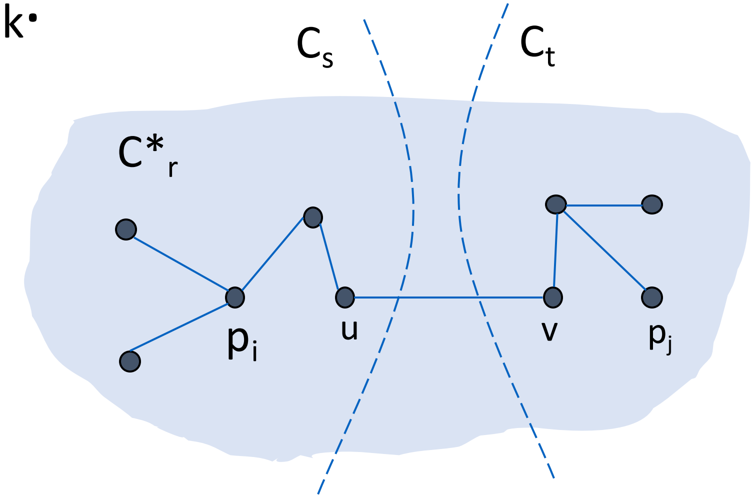
single-linkage: distance between two clusters is determined by a single pair of objects

Single-Link k-Clustering: Greedy Algorithm

- **Greedy approach.**
  - Form a graph on the object set U, with n clusters in the beginning.
  - Find the closest pair of objects such that each object is in a different cluster, and add an edge between them.
  - Repeat $n - k$ times until there are exactly k clusters.
- **Key observation.** This procedure is precisely Kruskal's algorithm, with a complete graph K_n where edge costs are distances (except we stop when there are k connected components).
- **Remark.** Equivalent to finding an MST and deleting the $k - 1$ most expensive edges.

Single-Link k-Clustering: Analysis

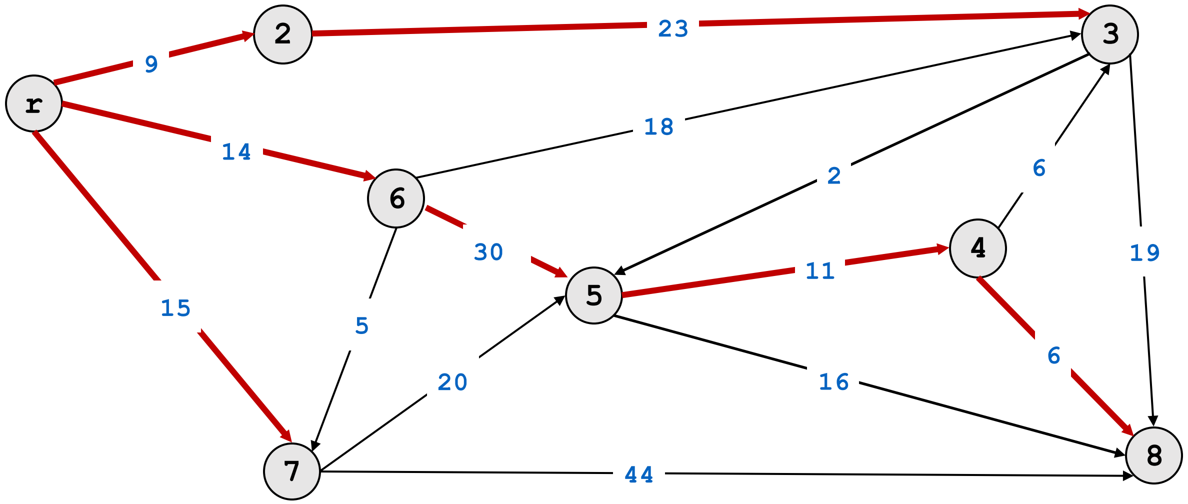
- **Theorem.** Let C* be the clustering $C_1^*, \ldots, C_k^*$ formed by deleting the $k-1$ most expensive edges of an MST. C* is a k-clustering of max spacing.
- Pf. Let C denote any other k-clustering $C_1, \ldots, C_k$.
  - Let p_i, p_j be in the same cluster in C*, say C_r^*, but in different clusters in C, say C_s and C_t.
  - Some edge (u, v) on p_i–p_j path in C_r^* spans two different clusters in C.
  - Spacing of C* = length d* of the ($k-1$)-st longest edge in the corresponding MST.
  - All edges on p_i–p_j path have length ≤ d* since Kruskal already added them.
  - Spacing of C is ≤ d* since u and v are in different clusters in C and d(u,v) ≤ d*. ▪



7. Min-Cost Arborescences

Arborescences

- **Def.** Given a directed graph $G = (V, E)$ and a root $r \in V$, an arborescence (rooted at r) is a subgraph $T = (V, F)$ such that
  - ➢ T is a spanning tree of G if we ignore the direction of edges.
  - ➢ There is a (unique) directed path in T from r to each other node $v \in V$.
- **Observation.** Arborescences are essentially directed spanning trees.

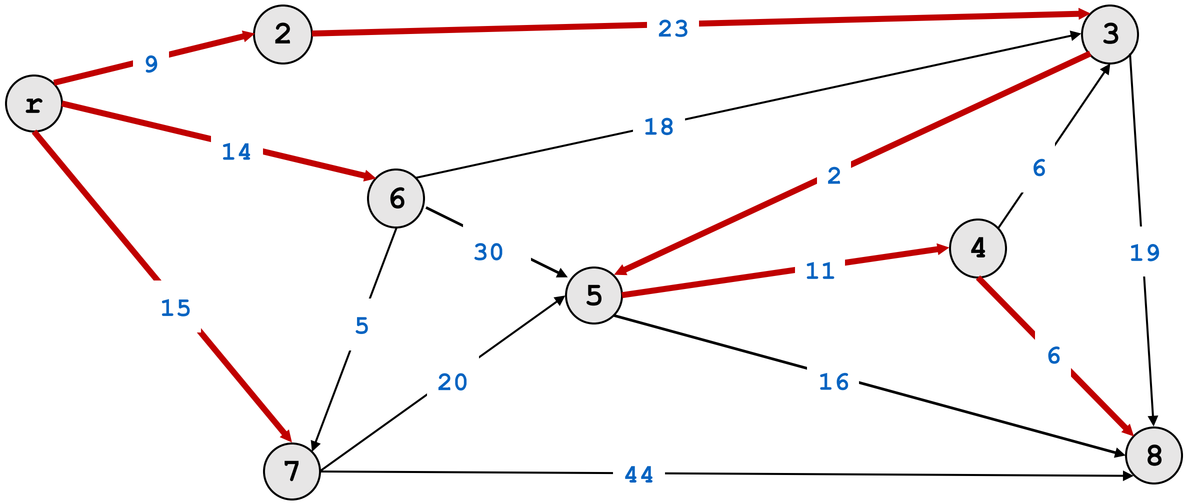


Arborescences

- **Claim.** A subgraph T of G is an arborescence rooted at r iff T has no directed cycles and each node $v \neq r$ has exactly one entering edge.
- Pf. We need prove both the "if" part and the "only if" part.
  - "only if" part $\Rightarrow$:
    - ✓A spanning tree has no cycles.
    - ✓The last edge on the unique r–v path is the only entering edge.
  - "if" part $\Leftarrow$:
    - ✓Suppose T has no cycles and each node $v \neq r$ has exactly one entering edge.
    - ✓To construct an r–v path, start at v and follow edges in the backward direction. Since T has no directed cycles, the process must terminate.
    - ✓It must terminate at r since r is the only node with no entering edge. ▪

Min-Cost Arborescences

- **Goal.** Given a directed graph *G* with a root node *r* and nonnegative edge costs, find an arborescence rooted at *r* of minimum cost.
- **Observation.** Min-cost arborescences are essentially directed MSTs.
- **Assumptions** (w.l.o.g.): All nodes are reachable from *r*. No edge enters *r*.



min-cost arborescence cost:
9 + 14 + 15 + 23 + 2 + 11 + 6 = 80

Min-Cost Arborescences

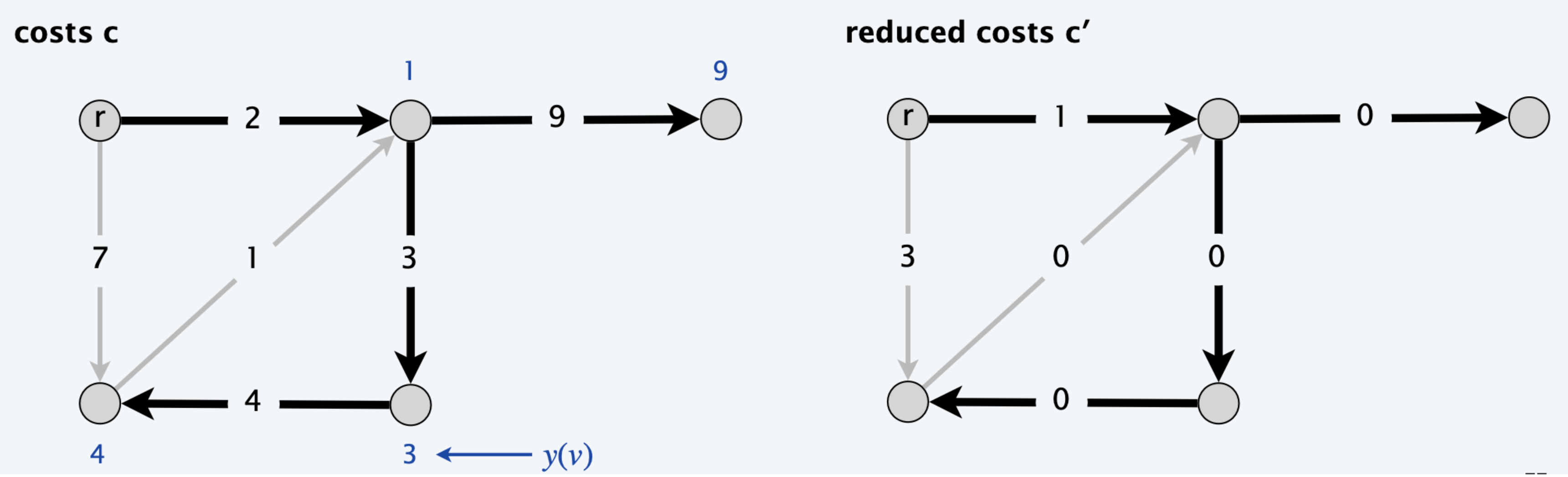
- **Goal.** Given a directed graph G with a root node r and nonnegative edge costs, find an arborescence rooted at r of minimum cost.
- **Observation.** Min-cost arborescences are essentially directed MSTs.
- **Assumptions** (w.l.o.g.): All nodes are reachable from r. No edge enters r.

- **Property.** For each node $v \neq r$, choose a cheapest edge entering v. If such $n - 1$ edges form an arborescence, then it is a min-cost arborescence.
  - What would happen when it is not an arborescence? There are directed cycles.
- **Q.** How can we remove such directed cycles?

Min-Cost Arborescences

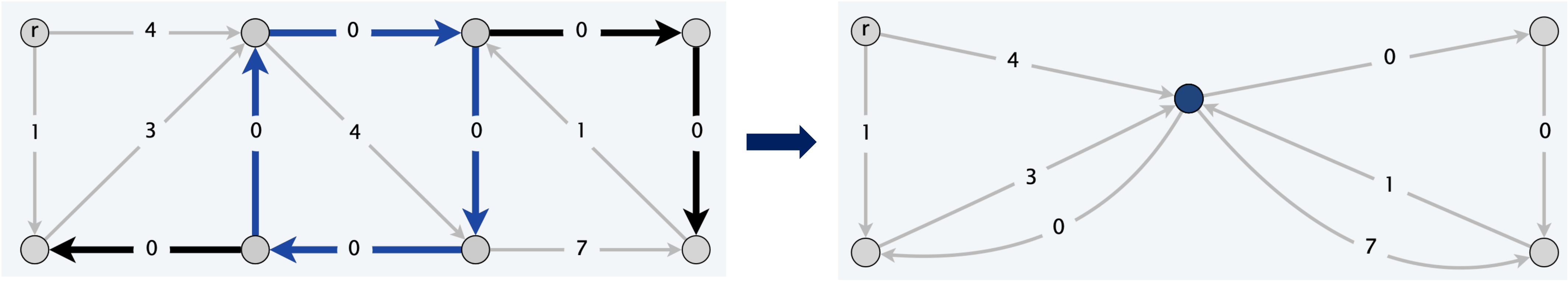
- **Def.** For each $v \neq r$, let $y(v)$ denote the min cost of any edge entering v. The reduced cost of an edge (u, v) is $c'(u, v) = c(u, v) - y(v) \geq 0$.
- **Claim.** T is a min-cost arborescence in G using costs c iff T is a min-cost arborescence in G using reduced costs c'.
- Pf. Recall that any arborescence T has exactly one edge entering $v \neq r$, so the cost difference in c and c' is independent of the chosen T. ▪

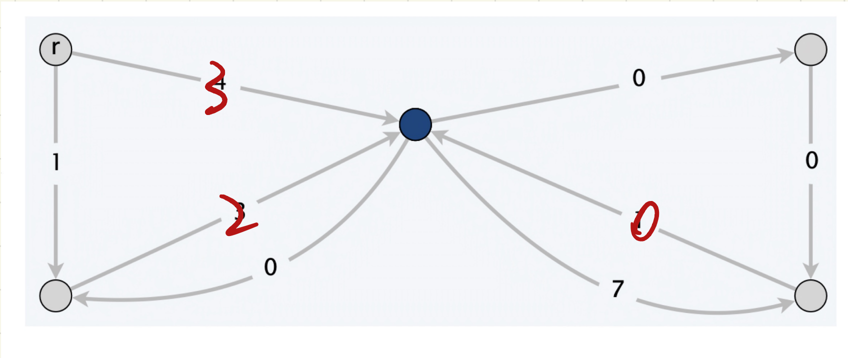
减去最小的
入权重

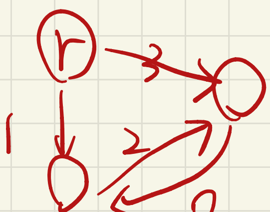


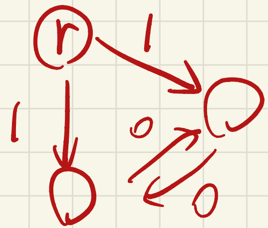
Chu-Liu's Algorithm (aka. Edmonds' Algorithm)

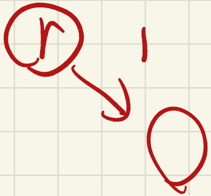
- **Greedy approach.**
  - For each $v \neq r$, choose a cheapest edge entering v and form an edge set E^*.
  - Then, all edges in E^* have 0 cost with respect to reduced costs $c'(u, v)$.
  - If E^* does not contain a cycle, then we find a min-cost arborescence.
  - If E^* contains a cycle C, can afford to use as many such 0-cost edges in C.
  - Therefore, we can contract C to a supernode (and remove any self-loops).
  - Recursively solve problem in contracted graph G' with costs $c'(u, v)$.







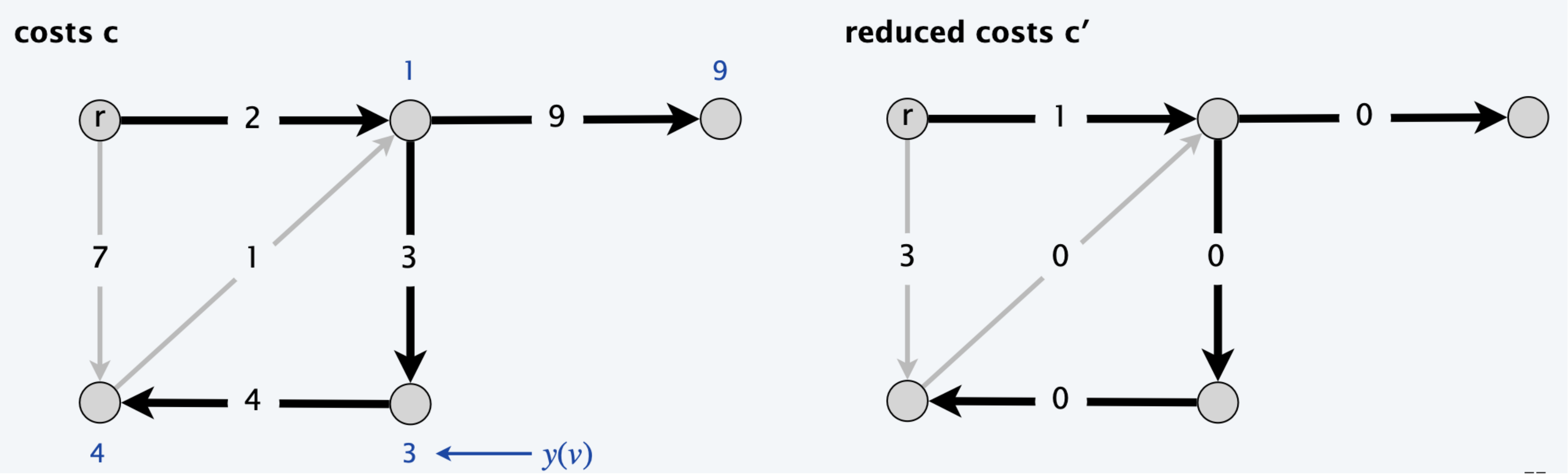




expand时确保 r 到每个节点有 unique path.

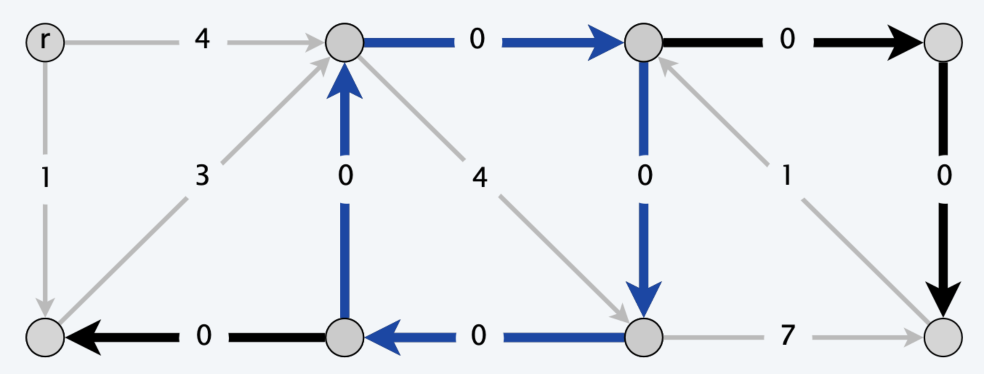
Min-Cost Arborescences

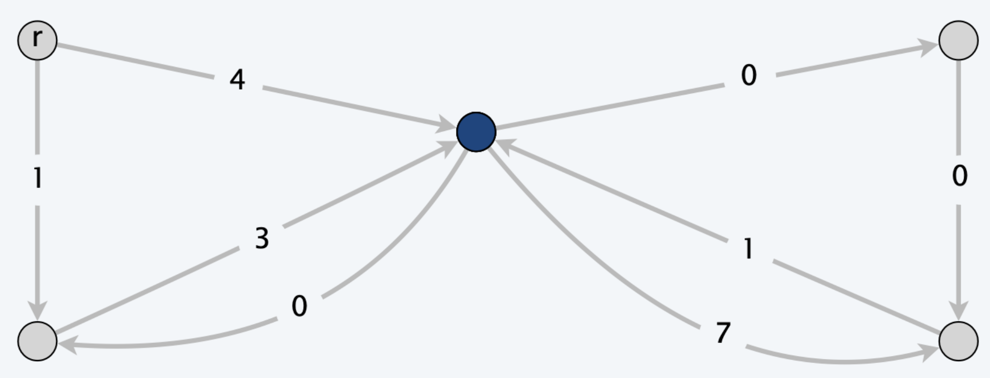
- **Def.** For each $v \neq r$, let $y(v)$ denote the min cost of any edge entering v. The reduced cost of an edge (u, v) is $c'(u, v) = c(u, v) - y(v) \geq 0$.
- **Claim.** T is a min-cost arborescence in G using costs c if and only if T is a min-cost arborescence in G using reduced costs c'.
- Pf. Recall that any arborescence T has exactly one edge entering $v \neq r$, so the cost difference in c and c' is independent of the chosen T. ▪



Chu-Liu's Algorithm (aka. Edmonds' Algorithm)

- **Greedy approach.**
  - For each $v \neq r$, choose a cheapest edge entering v and form an edge set E^*.
  - Then, all edges in E^* have 0 cost with respect to reduced costs $c'(u, v)$.
  - If E^* does not contain a cycle, then we find a min-cost arborescence.
  - If E^* contains a cycle C, can afford to use as many such 0-cost edges in C.
  - Therefore, we can contract C to a supernode (and remove any self-loops).
  - Recursively solve problem in contracted graph G' with costs $c'(u, v)$.





Chu-Liu's Algorithm: Analysis

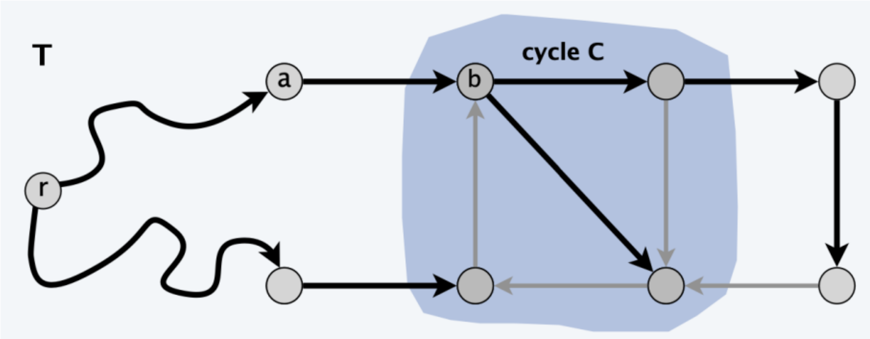
- **Theorem.** [Chu–Liu 1965, Edmonds 1967] The greedy algorithm finds a min-cost arborescence.
- Pf. (by strong induction on number of nodes |V|)
  - If the edges of E^* form an arborescence, then it is a min-cost arborescence.
  - Otherwise, we use reduced costs, which is equivalent.
  - After contracting a 0-cost cycle C to obtain a smaller graph G', the algorithm finds a min-cost arborescence T' in G' (by induction).
  - There exists a min-cost arborescence T in G that corresponds to T'. ▪
- **Q.** What can go wrong for the last step?
  - A min-cost arborescence in G' has exactly one edge entering a node in C (since C is contracted to a single supernode), but a min-cost arborescence in G might have several edges entering C.

sufficient to show the existence of a min-cost arborescence in G with only one edge entering C

Chu-Liu's Algorithm: Analysis

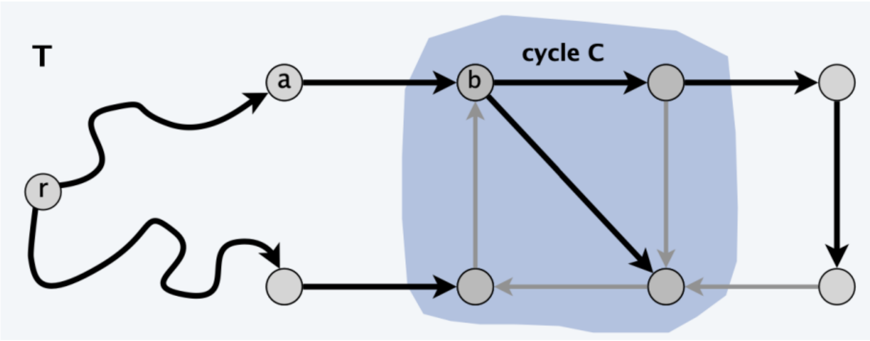
- **Claim.** Let C be a cycle in G containing only 0-cost edges. There exists a min-cost arborescence T rooted at r with exactly one edge entering C.
- Pf. (by cases)
  - Case 0. T has no edges entering C. Impossible!

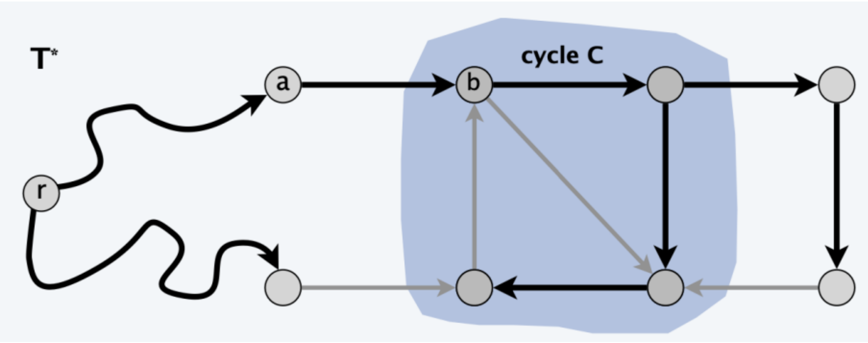
    ✓Arborescence T has an r–v path for each node $v \Rightarrow$ at least one edge enters C.
  - Case 1. T has exactly one edge entering C. Nothing to prove.
  - Case 2. T has two (or more) edges entering C. We construct another min-cost arborescence T^* that has exactly one edge entering C.



Chu-Liu's Algorithm: Analysis

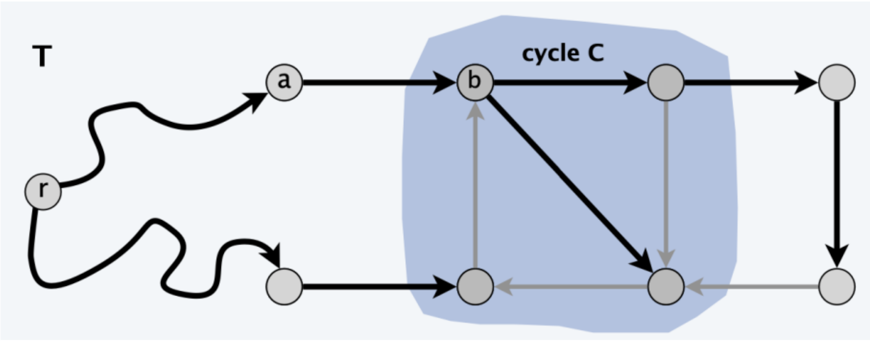
- **Claim.** Let C be a cycle in G containing only 0-cost edges. There exists a min-cost arborescence T rooted at r with exactly one edge entering C.
- Pf. (by cases)
  - Case 2. T has two (or more) edges entering C. We construct another min-cost arborescence T^* that has exactly one edge entering C.
    - ✓Let (a, b) be an edge in T entering C that lies on a shortest path from r.
    - ✓We delete all edges of T that enter a node in C except (a, b). only one node in C
    - ✓We add in all edges of C except the one that enters b.

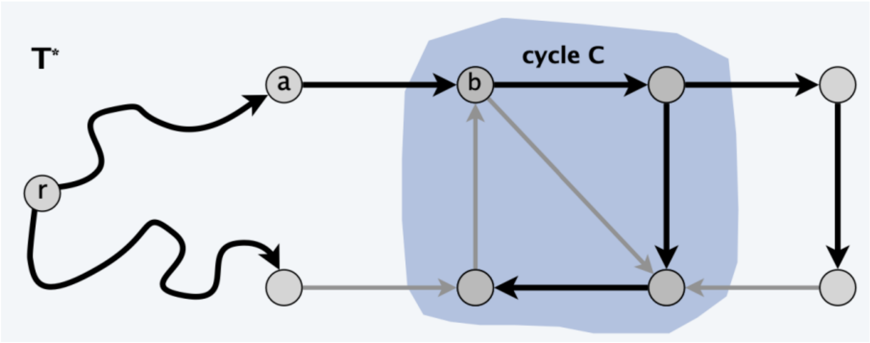




Chu-Liu's Algorithm: Analysis

- **Claim.** Let *C* be a cycle in *G* containing only 0-cost edges. There exists a min-cost arborescence *T* rooted at *r* with exactly one edge entering *C*.
- Pf. (by cases)
  - ➢ Case 2. *T** is a min-cost arborescence.
    - ✓The cost of *T** is no more than that of *T* since we add only 0-cost edges.
    - ✓*T** is an arborescence, i.e., it has exactly one edge entering each node *v* ≠ *r* and has no directed cycles.
      - *T* had no cycles before, now only (*a*, *b*) enters *C* and no cycles within *C*.





Chu-Liu's Algorithm: Analysis

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- Pf. (by strong induction on number of nodes |V|)
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  - Otherwise, we use reduced costs, which is equivalent.
  - After contracting a 0-cost cycle C to obtain a smaller graph G', the algorithm finds a min-cost arborescence T' in G' (by induction).
  - **Proved:** There exists a min-cost arborescence T in G that corresponds to T'. ▪
- **Time complexity.** $O(mn)$.
  - At most n contractions (since each reduces the number of nodes).
    - ✓Finding and contracting cycle C (with reduced costs) takes $O(m)$ time.
    - ✓Transforming T' into T takes $O(m)$ time. ▪

More on Min-Cost Arborescences

- **Remark.** Chu-Liu's algorithm can be extended to find directed minimum spanning forests and can be implemented in $O(m \log n)$ time.

- [Gabow–Galil–Spencer–Tarjan 1986] There exists an $O(m + n \log n)$ time algorithm to compute a min-cost arborescence.