Algorithm Design and Analysis (H)

CS216

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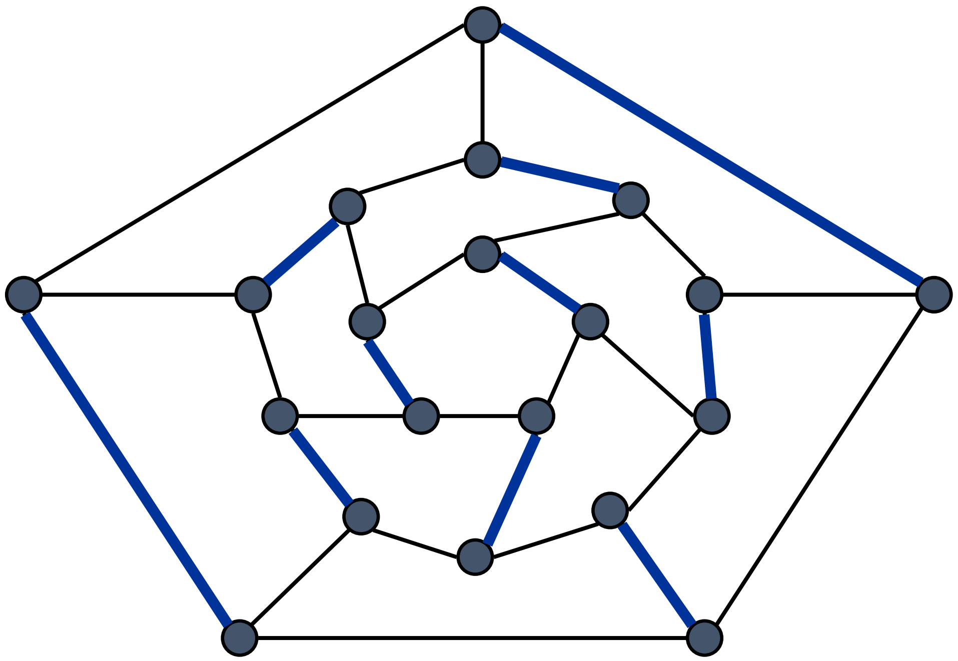
(slides edited from Prof. Shiqi Yu)

Network Flow

7. Bipartite Matching

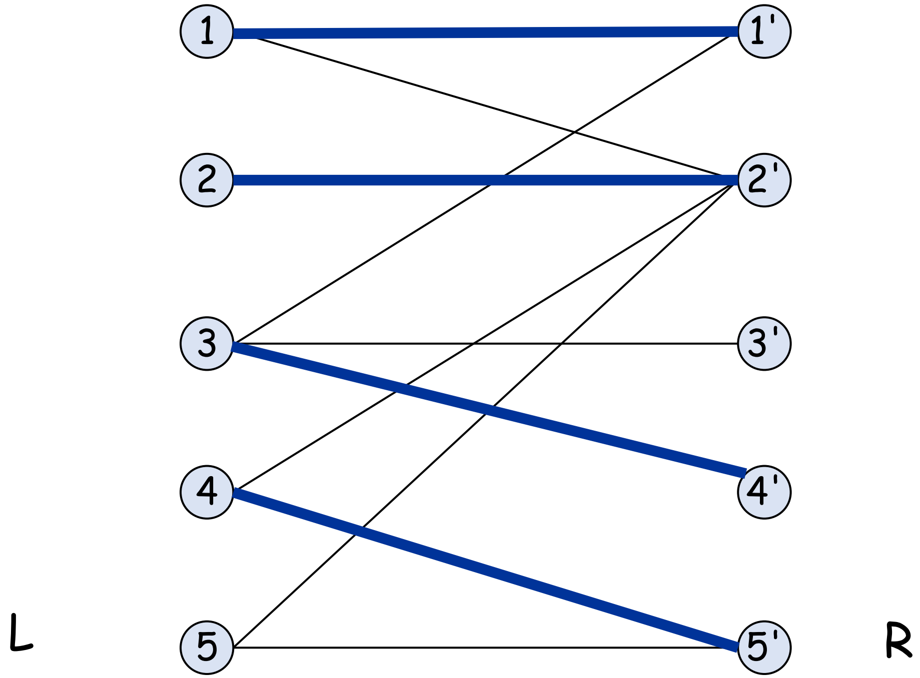
Matching

- **Matching.**
  - Input: undirected graph G = (V, E).
  - $M \subseteq E$ is a matching if each node appears in at most one edge in M.
  - **Max matching:** find a max-cardinality matching.



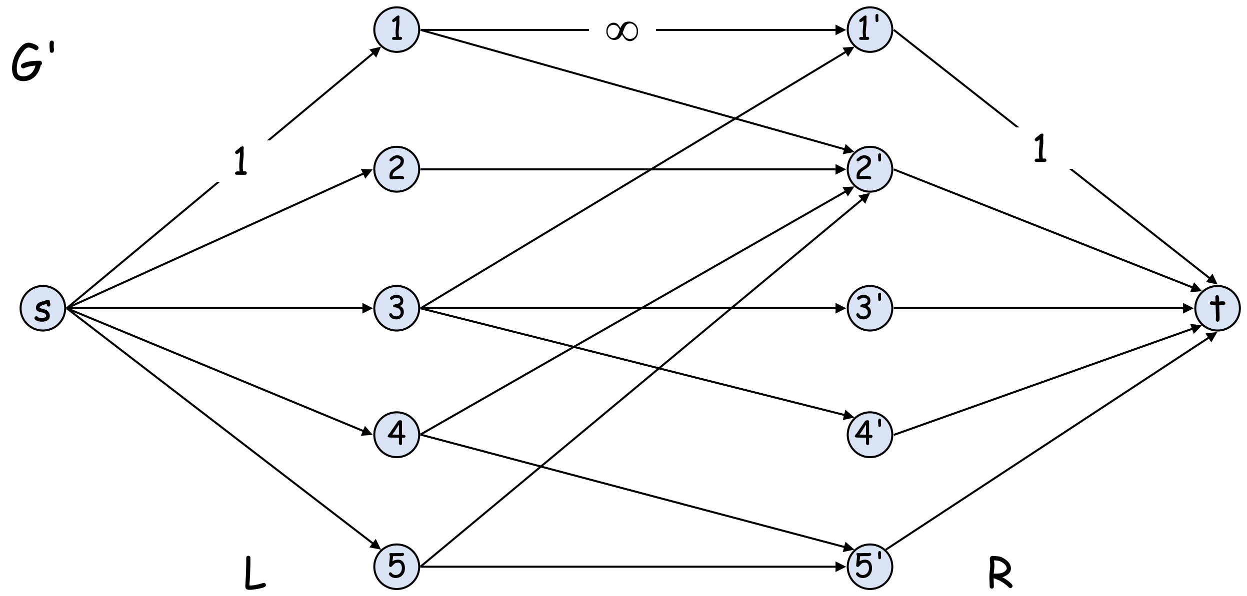
Bipartite Matching

- **Bipartite matching.**
  - Input: undirected, bipartite graph $G = (L \cup R, E)$.
  - $M \subseteq E$ is a matching if each node appears in at most one edge in M.
  - **Max matching:** find a max-cardinality matching.



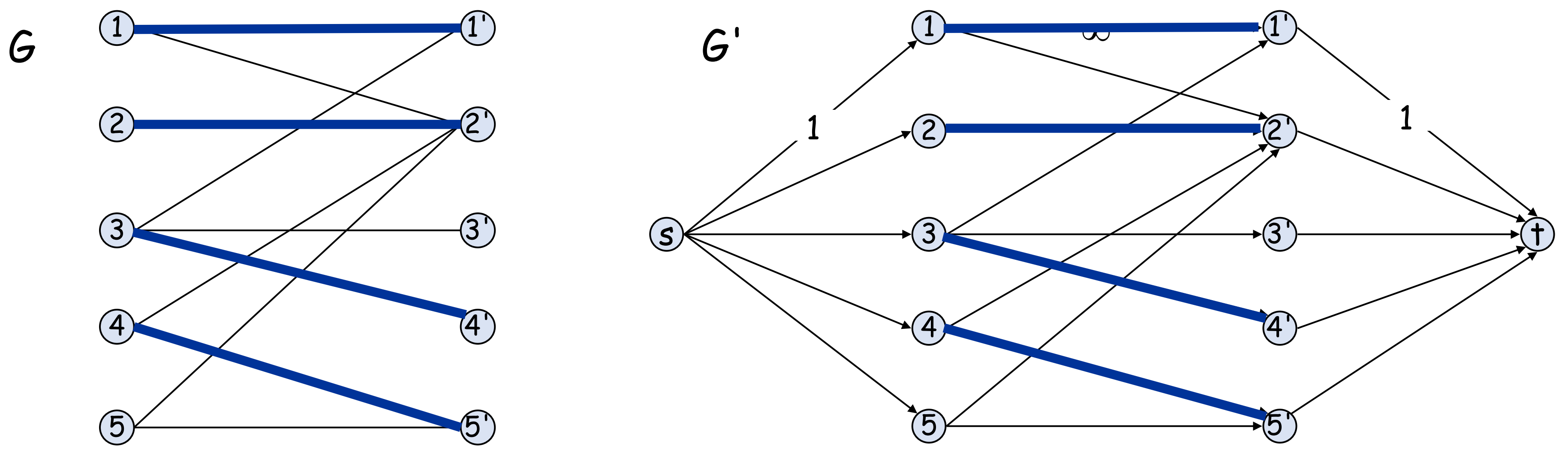
Bipartite Matching: Max-Flow Formulation

- **Max-flow formulation.**
  - ➢ Create digraph G' = (L ∪ R ∪ {s, t}, E').
  - ➢ Direct all edges from L to R, and assign infinite (or unit) capacity.
  - ➢ Add source s, and unit capacity edges from s to each node in L.
  - ➢ Add sink t, and unit capacity edges from each node in R to t.



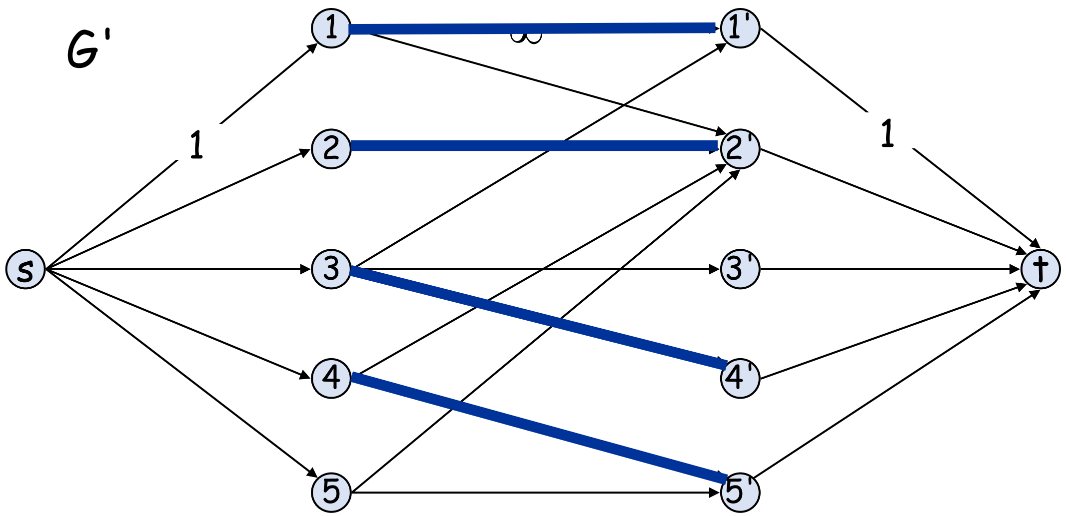
Max-Flow Formulation: Correctness

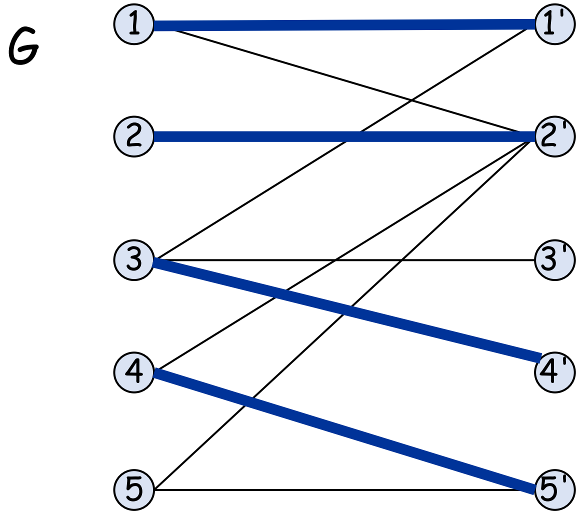
- **Theorem.** 1-1 correspondence between matchings of cardinality k in G and integral flows of value k in G'.
- **Pf.** $\Rightarrow$:
  - Let M be a matching in G of cardinality k.
  - Consider flow f that sends 1 unit on each of the k corresponding paths.
  - f is an integral flow of value k. ▪



Max-Flow Formulation: Correctness

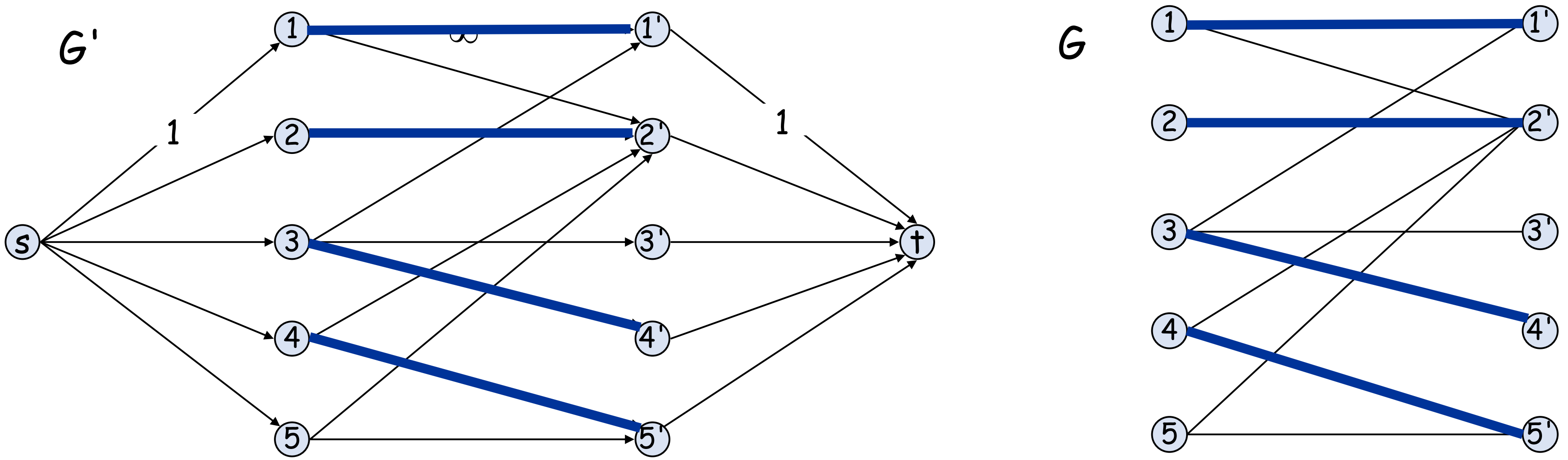
- **Theorem.** 1-1 correspondence between matchings of cardinality k in G and integral flows of value k in G'.
- **Pf.** $\Leftarrow$:
  - Let f be an integral flow in G' of value k.
  - Consider M = set of edges from L to R with $f(e) = 1$.
    - ✓ each node in L and R participates in at most one edge in M
    - ✓ $|M| = k$: apply flow-value lemma to cut $(L \cup \{s\}, R \cup \{t\})$ ▪





Max-Flow Formulation: Correctness

- **Theorem.** 1-1 correspondence between matchings of cardinality *k* in *G* and integral flows of value *k* in *G*′.
- **Corollary.** Can solve bipartite matching via max-flow formulation.
  Pf.
  - Integrality theorem ⇒ there exists a max flow *f** in *G* ′ that is integral.
  - Theorem ⇒ *f** corresponds to max-cardinality matching. ▪



Perfect Matchings in Bipartite Graphs

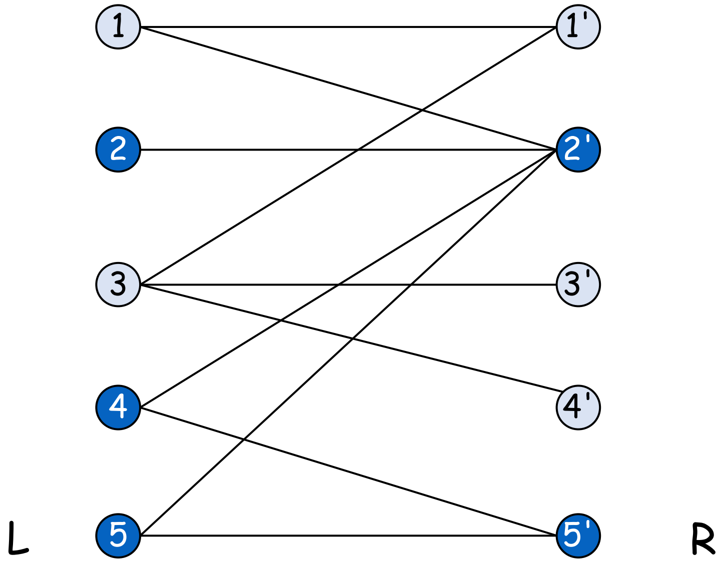
- **Def.** Given a graph $G = (V, E)$, a subset of edges $M \subseteq E$ is a perfect matching if each node appears in exactly one edge in M.

- **Q.** When does a bipartite graph have a perfect matching?

- **Structure of bipartite graphs with perfect matchings.**
  - Clearly we must have $|L| = |R|$.
  - What other conditions are necessary?
  - What other conditions are sufficient?

Perfect Matchings in Bipartite Graphs

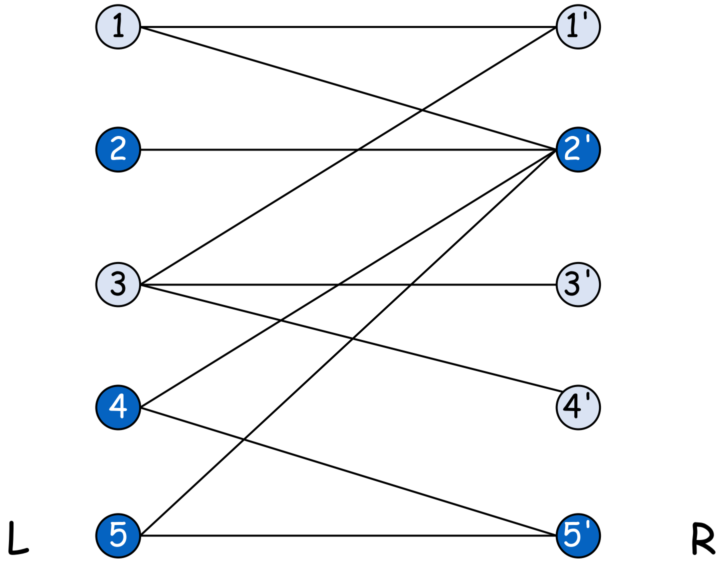
- **Notation.** Let S be a subset of nodes, and let N(S) be the set of nodes adjacent to nodes in S.
- **Observation.** If a bipartite graph $G = (L \cup R, E)$ has a perfect matching, then $|N(S)| \geq |S|$ for all subsets $S \subseteq L$.
  Pf. Each node in S has to be matched to a different node in N(S).



No perfect matching:
S = { 2, 4, 5 }
N(S) = { 2', 5' }.

Hall's Marriage Theorem

- **Theorem.** [Frobenius 1917, Hall 1935] Let $G = (L \cup R, E)$ be a bipartite graph with $|L| = |R|$. Then, G has a perfect matching iff $|N(S)| \geq |S|$ for all subsets $S \subseteq L$.
- **Pf.** $\Rightarrow$: This was the previous observation.

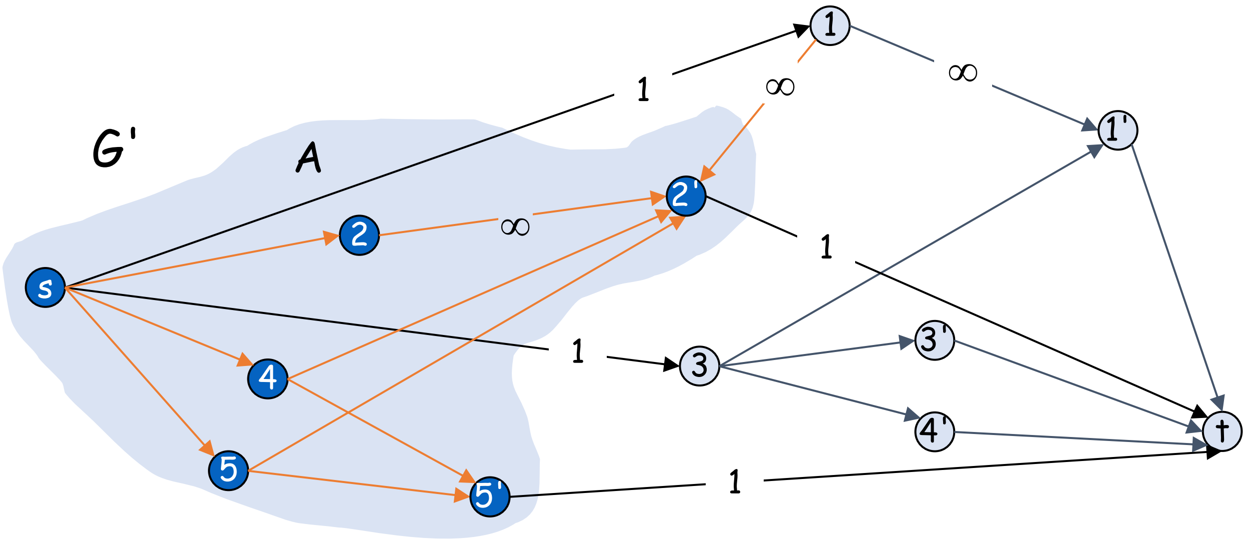


No perfect matching:
$S = \{ 2, 4, 5 \}$
$N(S) = \{ 2', 5' \}$.

Hall's Marriage Theorem

- **Pf.** $\Leftarrow$: Suppose G does not have a perfect matching. (contrapositive)
  - Formulate as a max flow problem and let (A, B) be a min cut in G'.
  - By max-flow min-cut theorem, $c(A, B) < |L|$.
  - Define $L_A = L \cap A$, $L_B = L \cap B$, $R_A = R \cap A$.
  - Min cut cannot use ∞ edges: $N(L_A) \subseteq R \cap A = R_A$
  - $c(A, B) = |L_B| + |R_A| < |L| = |L_A| + |L_B| \Rightarrow |R_A| < |L_A|$
  - $|N(L_A)| \leq |R_A| < |L_A|$.
  - Choose $S = L_A$. ▪

$L_A = \{2, 4, 5\}$
$L_B = \{1, 3\}$
$R_A = \{2', 5'\}$
$N(L_A) = \{2', 5'\}$



Algorithms for Matching

- **Which max-flow algorithm to use for bipartite matching?**
  - Generic augmenting path: $O(m \text{ val}(f^*)) = O(mn)$.
  - Capacity scaling: $O(m^2 \log C) = O(m^2)$.
  - Shortest augmenting path: $O(mn^{1/2})$.
  - Fast matrix multiplication: $O(n^{2.378})$. [Mucha-Sankowsi 2003]

- **Non-bipartite matching.**
  - Structure of non-bipartite (undirected) graphs is more complicated.
  - But well-understood. [Tutte-Berge formula, Edmonds-Galai]
  - Blossom algorithm: $O(n^4)$. [Edmonds 1965]
  - Best known: $O(mn^{1/2})$. [Micali-Vazirani 1980, Vazirani 1994]

Historical Significance (Jack Edmonds 1965)

2. Digression. An explanation is due on the use of the words "efficient algorithm." First, what I present is a conceptual description of an algorithm and not a particular formalized algorithm or "code."

For practical purposes computational details are vital. However, my purpose is only to show as attractively as I can that there is an efficient algorithm. According to the dictionary, "efficient" means "adequate in operation or performance." This is roughly the meaning I want—in the sense that it is conceivable for maximum matching to have no efficient algorithm. Perhaps a better word is "good."

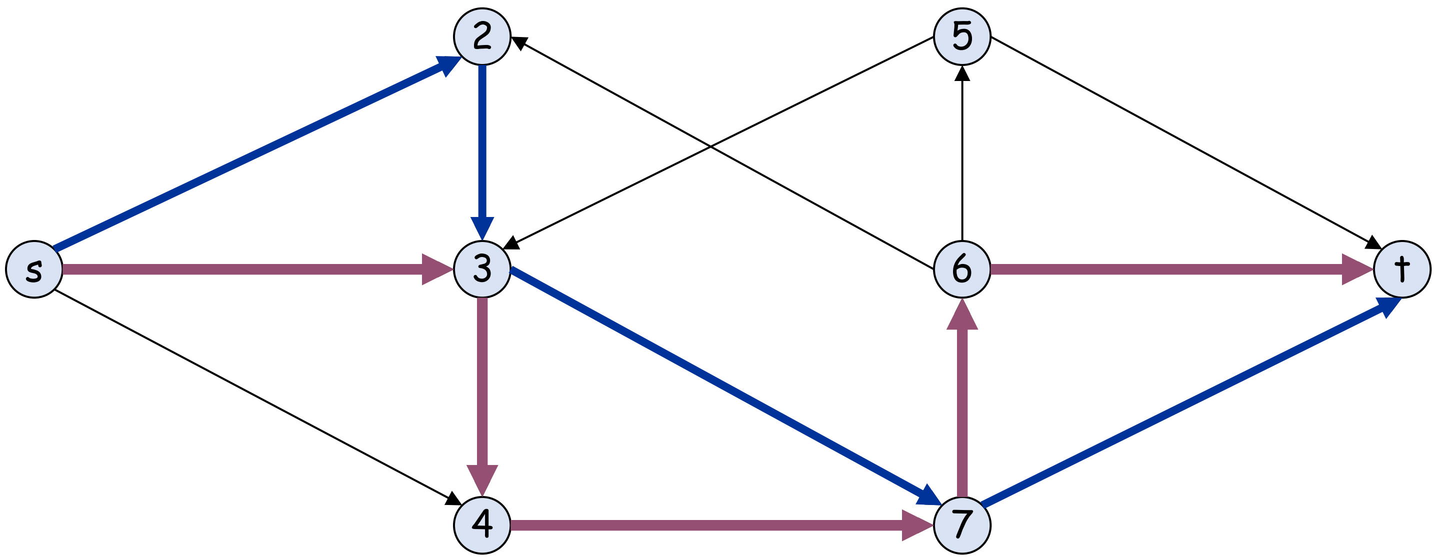
I am claiming, as a mathematical result, the existence of a *good* algorithm for finding a maximum cardinality matching in a graph.

There is an obvious finite algorithm, but that algorithm increases in difficulty exponentially with the size of the graph. It is by no means obvious whether *or not* there exists an algorithm whose difficulty increases only algebraically with the size of the graph.

8. Disjoint Paths

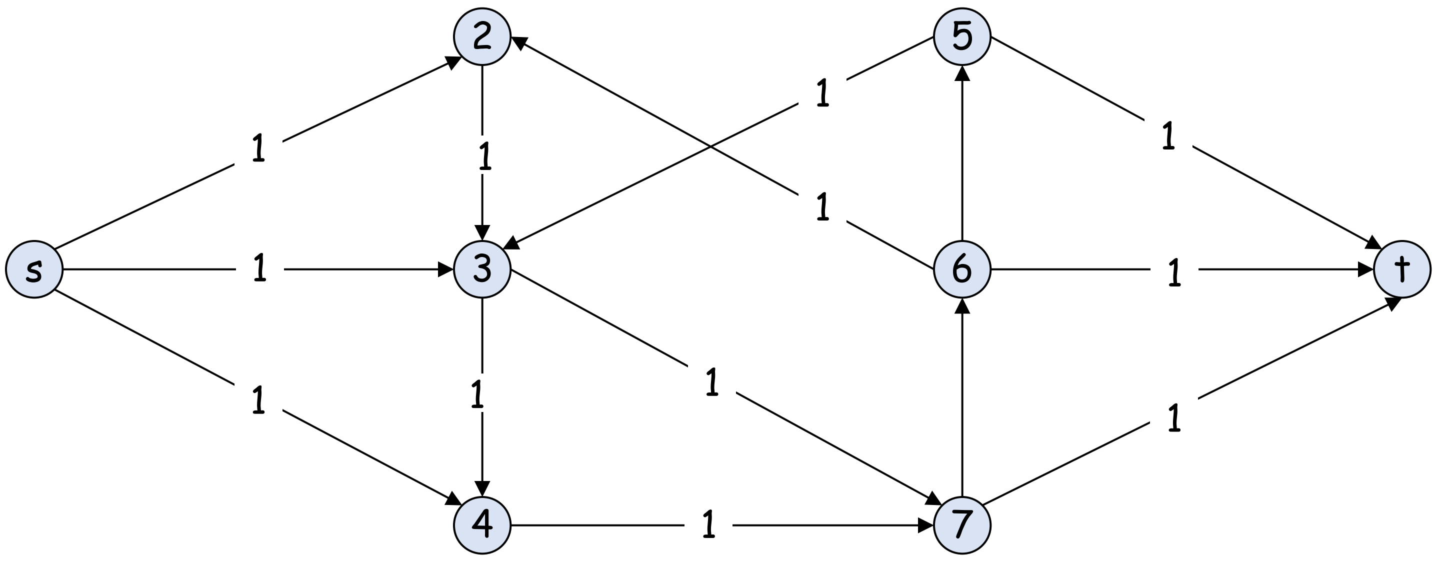
Edge-Disjoint Paths

- **Def.** Two paths are edge-disjoint if they have no edge in common.
- **Edge-disjoint paths problem.** Given a digraph G = (V, E) and two nodes s and t, find the max number of edge-disjoint s-t paths.
- **Ex.** Communication networks.



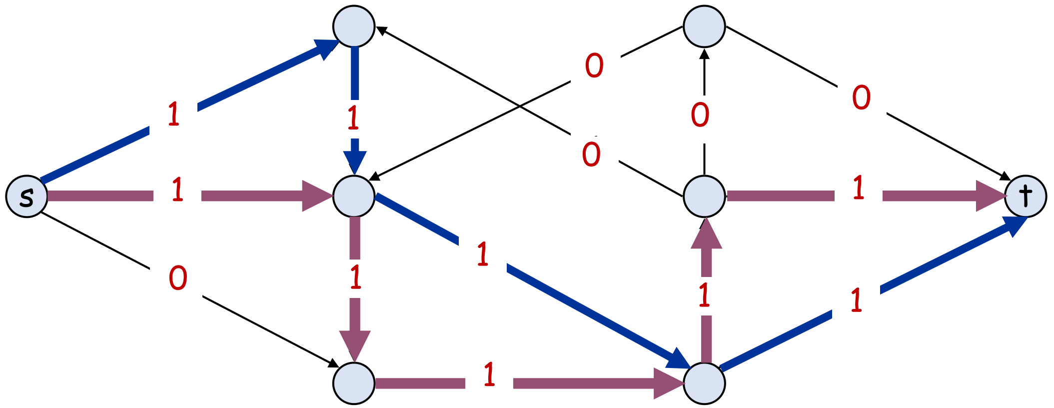
Edge-Disjoint Paths: Max-Flow Formulation

- **Def.** Two paths are edge-disjoint if they have no edge in common.
- **Edge-disjoint paths problem.** Given a digraph G = (V, E) and two nodes s and t, find the max number of edge-disjoint s-t paths.
- **Max-flow formulation.** Assign unit capacity to every edge.



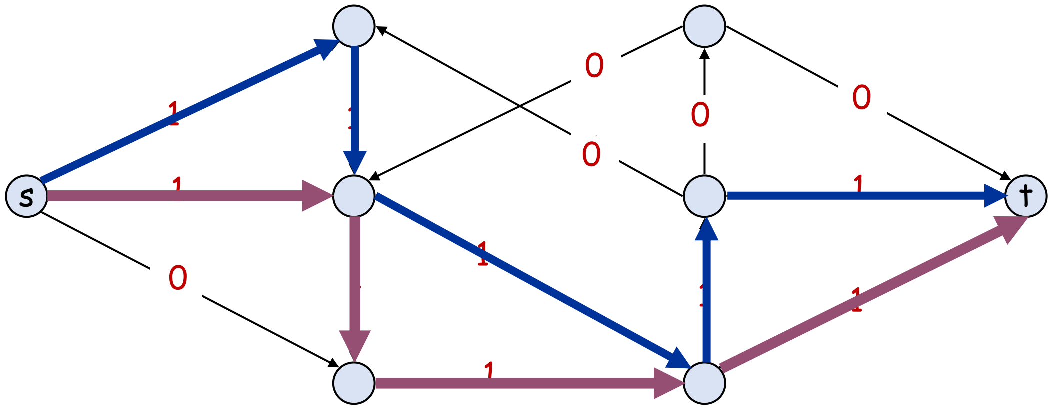
Max-Flow Formulation: Correctness

- **Theorem.** 1-1 correspondence between k edge-disjoint s-t paths in G and integral flows of value k in G'.
- **Pf.** $\Rightarrow$:
  - Let $P_1, \ldots, P_k$ be k edge-disjoint paths in G.
  - Set $f(e) = 1$ if edge e participates in some path P_i; else set $f(e) = 0$.
  - Since paths are edge-disjoint, f is a flow of value k. ▪



Max-Flow Formulation: Correctness

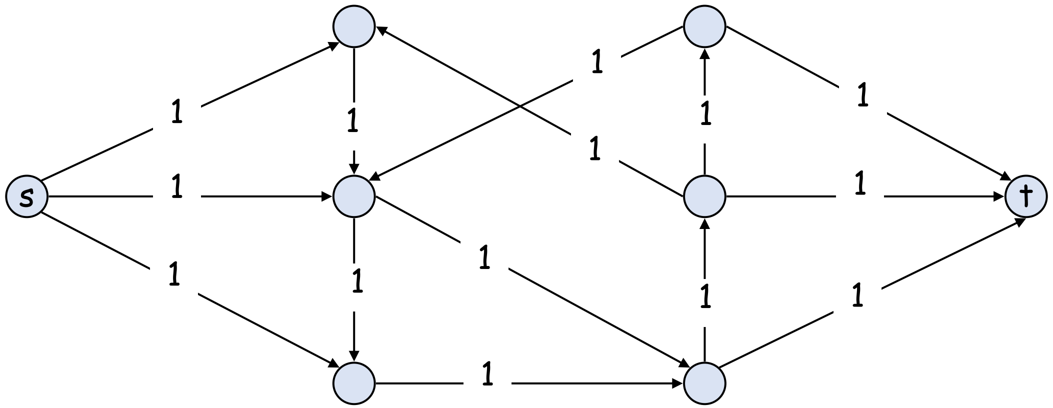
- **Theorem.** 1-1 correspondence between *k* edge-disjoint *s*-*t* paths in *G* and integral flows of value *k* in *G*′.
- **Pf.** ⇐:
  - Let *f* be an integral flow in *G*′ of value *k*.
  - Consider edge (*s*, *u*) with *f*(*s*, *u*) = 1.
    - ✓by flow conservation, there exists an edge (*u*, *v*) with *f*(*u*, *v*) = 1
    - ✓continue until reach *t*, always choosing a new edge.
  - Produces *k* (not necessarily simple) edge-disjoint paths. ▪



can eliminate cycles and get simple paths if desired

Max-Flow Formulation: Correctness

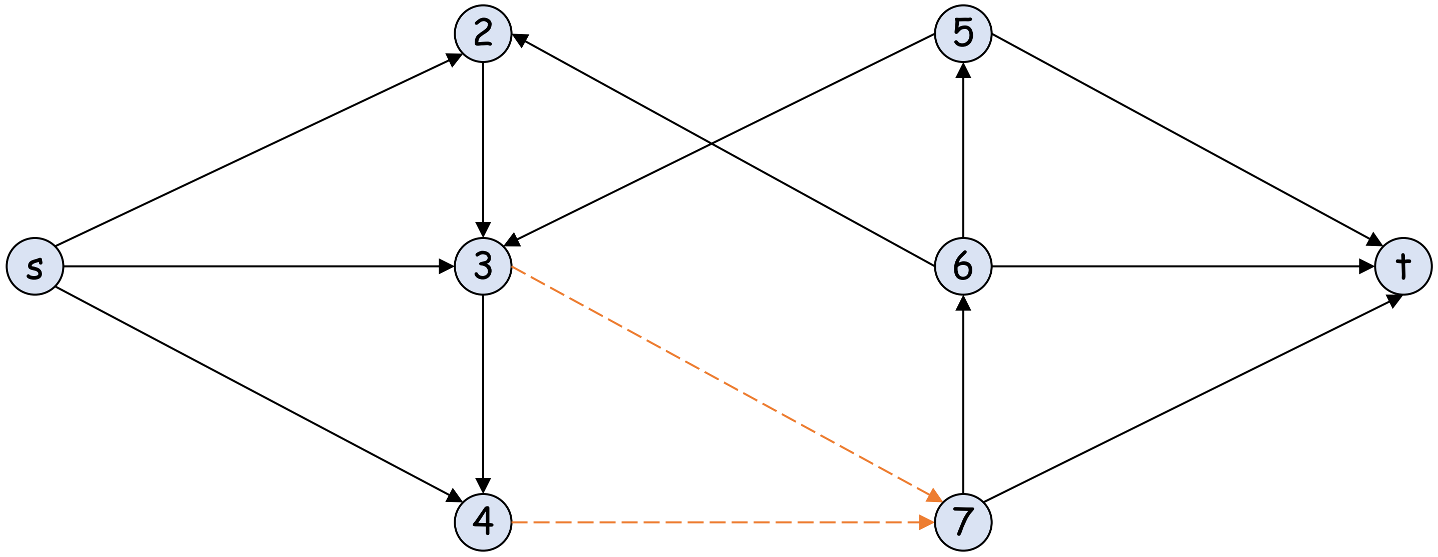
- **Max-flow formulation.** Assign unit capacity to every edge.
- **Theorem.** 1-1 correspondence between k edge-disjoint s-t paths in G and integral flows of value k in G'.
- **Corollary.** Can solve edge-disjoint paths via max-flow formulation.
- **Pf.**
  - Integrality theorem ⇒ there exists a max flow f^* in G' that is integral.
  - Theorem ⇒ f^* corresponds to max number of edge-disjoint s-t paths in G. ▪



Network Connectivity

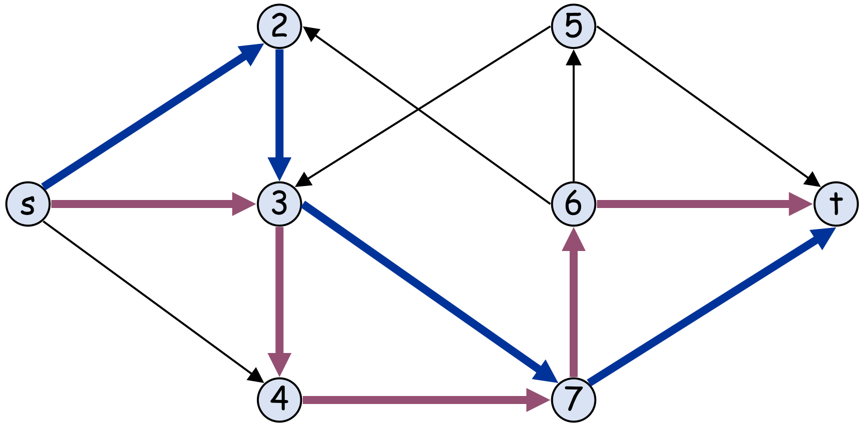
- **Def.** A set of edges $F \subseteq E$ disconnects t from s if every s-t path uses at least one edge in F.

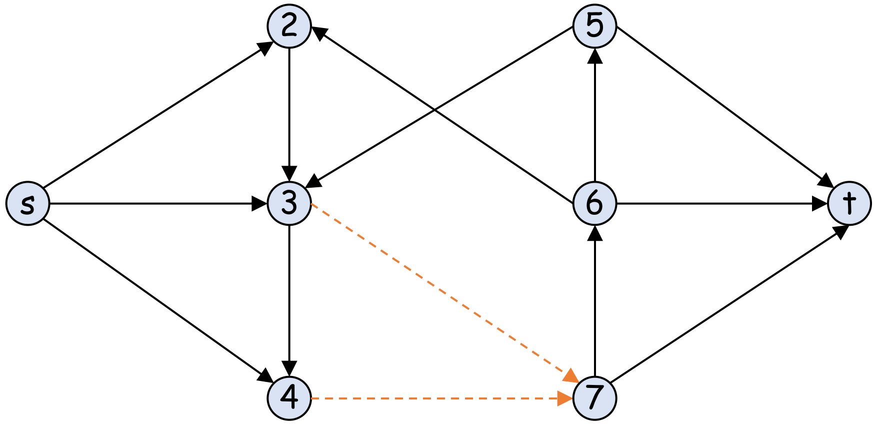
- **Network connectivity.** Given a digraph $G = (V, E)$ and two nodes s and t, find min number of edges whose removal disconnects t from s.



Menger's Theorem

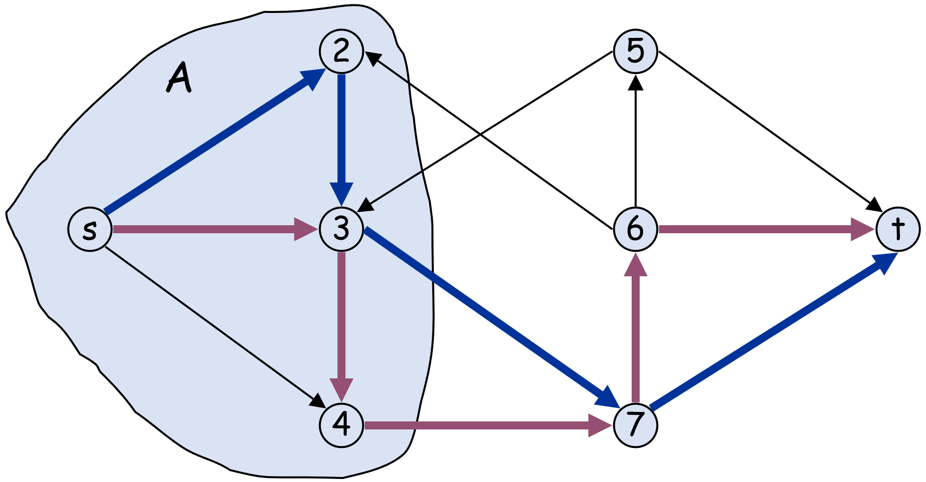
- **Theorem.** [Menger 1927] The max number of edge-disjoint s-t paths equals the min number of edges whose removal disconnects t from s.
- **Pf.** ≤:
  - Suppose the removal of $F \subseteq E$ disconnects t from s, and $|F| = k$.
  - Every s-t path uses at least one edge in F.
  - Hence, the number of edge-disjoint paths is $\leq k$. ▪

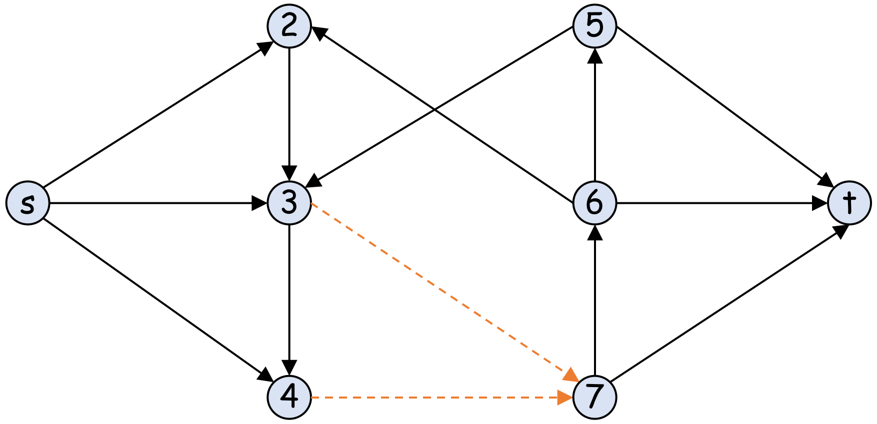




Menger's Theorem

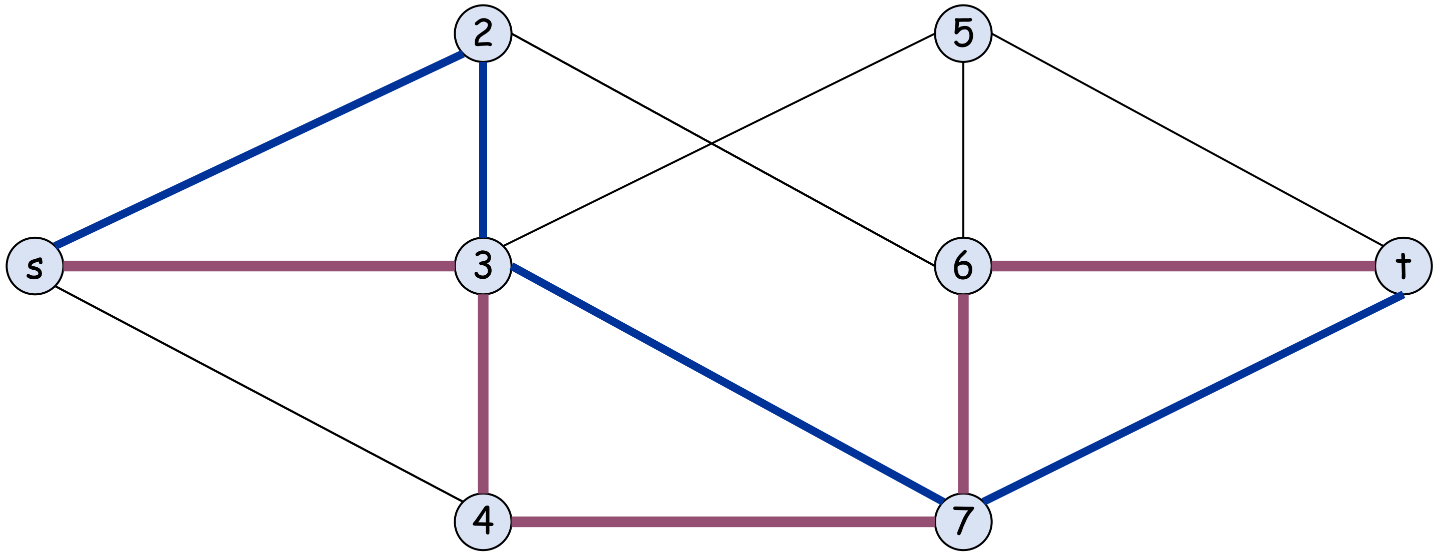
- **Theorem.** [Menger 1927] The max number of edge-disjoint s-t paths equals the min number of edges whose removal disconnects t from s.
- **Pf.** ≥:
  - Suppose max number of edge-disjoint *s*-*t* paths is *k*. Then, value of max flow = *k*.
  - Max-flow min-cut theorem ⇒ there exists a cut (*A*, *B*) of capacity *k*.
  - Let *F* be set of edges going from *A* to *B*.
  - |*F*| = *k* and removing *F* disconnects *t* from *s*. ▪





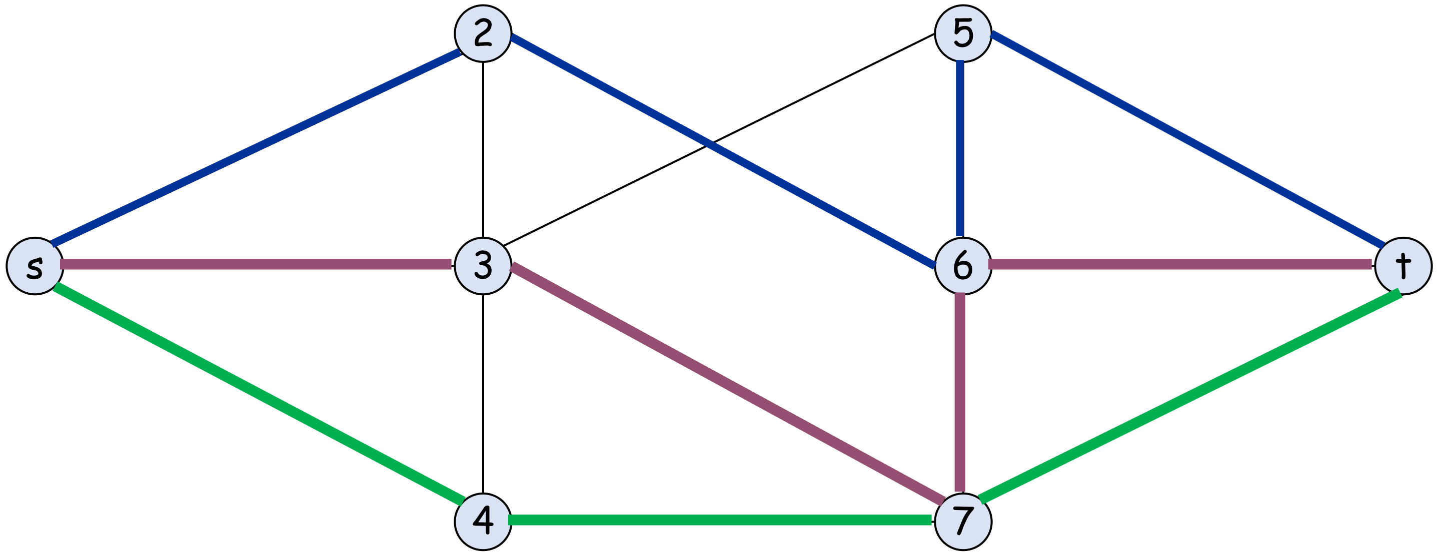
Edge-Disjoint Paths in Undirected Graphs

- **Edge-disjoint paths problem in undirected graphs.** Given an (undirected) graph G = (V, E) and two nodes s and t, find the max number of edge-disjoint s-t paths.
- **Ex.** 2 edge-disjoint paths.



Edge-Disjoint Paths in Undirected Graphs

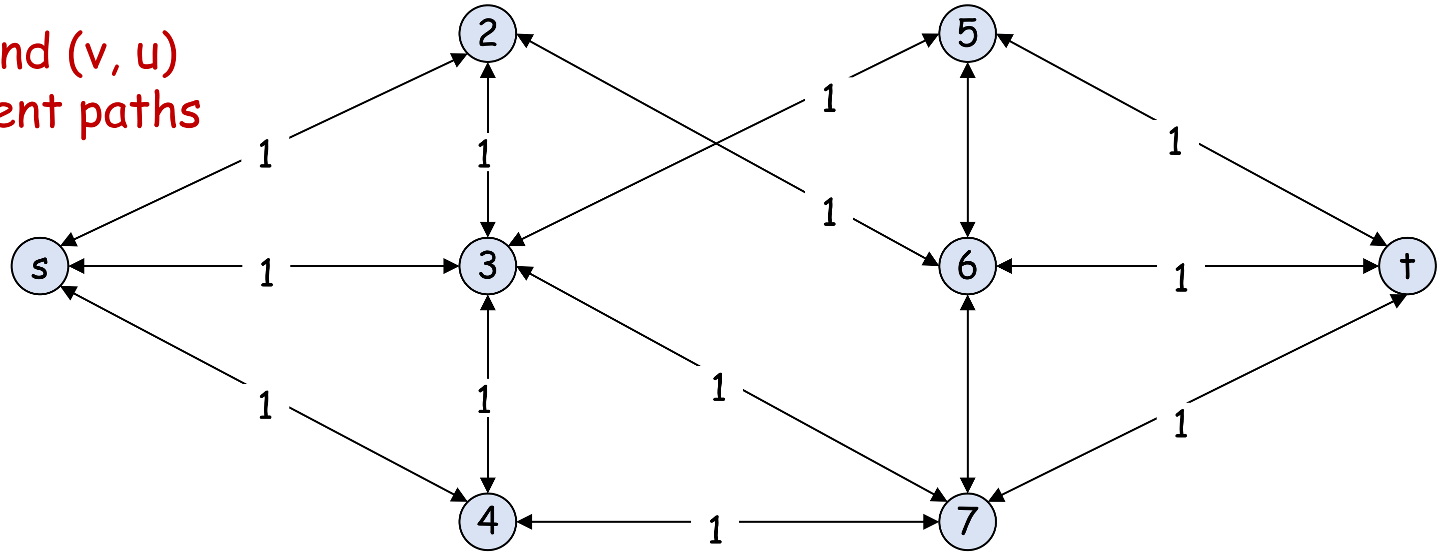
- **Edge-disjoint paths problem in undirected graphs.** Given an (undirected) graph G = (V, E) and two nodes s and t, find the max number of edge-disjoint s-t paths.
- **Ex.** 3 edge-disjoint paths (max number).



Edge-Disjoint Paths: Max-Flow Formulation

- **Max-flow formulation.** Replace each edge with two antiparallel edges and assign unit capacity to every edge.
- **Observation.** Two paths P_1 and P_2 may be edge-disjoint in the digraph but not edge-disjoint in the undirected graph.

edges (u, v) and (v, u)
used in different paths



Max-Flow Formulation: Correctness

- **Max-flow formulation.** Replace each edge with two antiparallel edges and assign unit capacity to every edge.

- **Lemma.** In any flow network, there exists a maximum flow f in which for each pair of antiparallel edges e and e' : either $f(e) = 0$ or $f(e') = 0$ or both. Moreover, integrality theorem still holds.
- **Pf.** (by induction on number of such pairs)
  - Suppose $f(e) > 0$ and $f(e') > 0$ for a pair of antiparallel edges e and e'.
  - Set $f(e) = f(e) - \delta$ and $f(e') = f(e') - \delta$, where $\delta = \min\{ f(e), f(e') \}$.
  - f is still a flow of the same value but has one fewer such pair. ▪

Max-Flow Formulation: Correctness

- **Max-flow formulation.** Replace each edge with two antiparallel edges and assign unit capacity to every edge.

- **Lemma.** In any flow network, there exists a maximum flow f in which for each pair of antiparallel edges e and e' : either $f(e) = 0$ or $f(e') = 0$ or both. Moreover, integrality theorem still holds.

- **Theorem.** Max number of edge-disjoint s-t paths = value of max flow.
- **Pf.** Similar to proof in digraphs; use Lemma.

More Menger Theorems

- **Theorem.** Given an undirected graph and two nodes s and t, the max number of edge-disjoint s–t paths equals the min number of edges whose removal disconnects s and t.

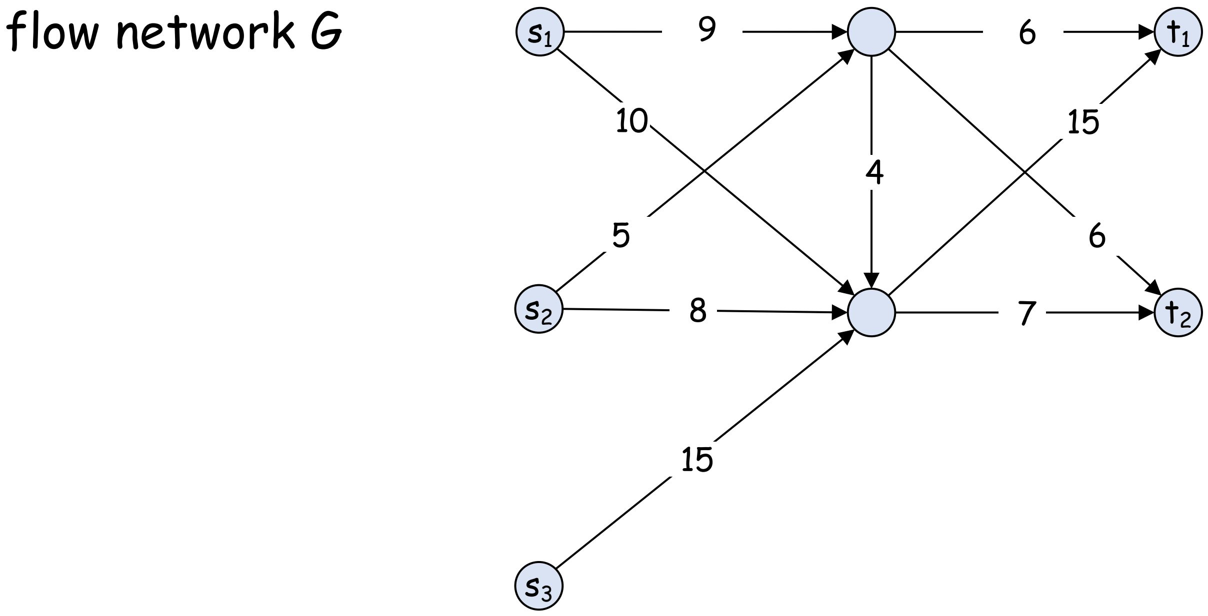
- **Theorem.** Given an undirected graph and two nonadjacent nodes s and t, the max number of internally node-disjoint s–t paths equals the min number of internal nodes whose removal disconnects s and t.

- **Theorem.** Given a directed graph with two nonadjacent nodes s and t, the max number of internally node-disjoint s-t paths equals the min number of internal nodes whose removal disconnects t from s.

9. Extensions to Max Flow

Multiple Sources and Sinks

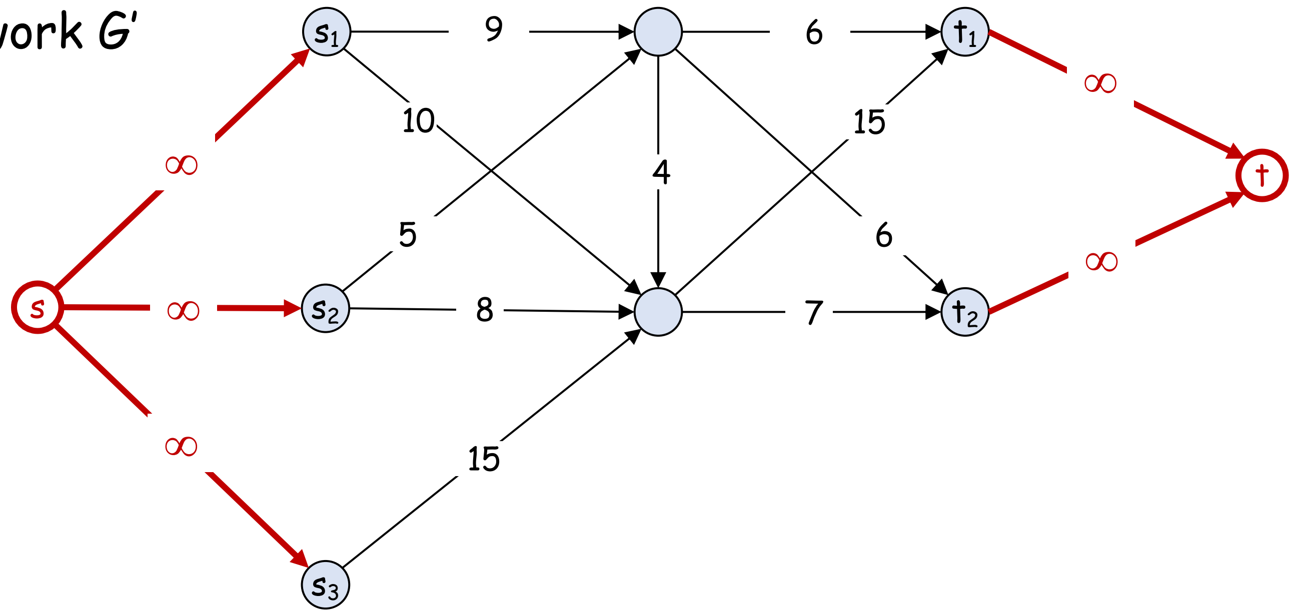
- **Def.** Given a digraph $G = (V, E)$ with edge capacities $c(e) \geq 0$ and multiple source nodes and multiple sink nodes, find max flow that can be sent from the source nodes to the sink nodes.



Multiple Sources and Sinks

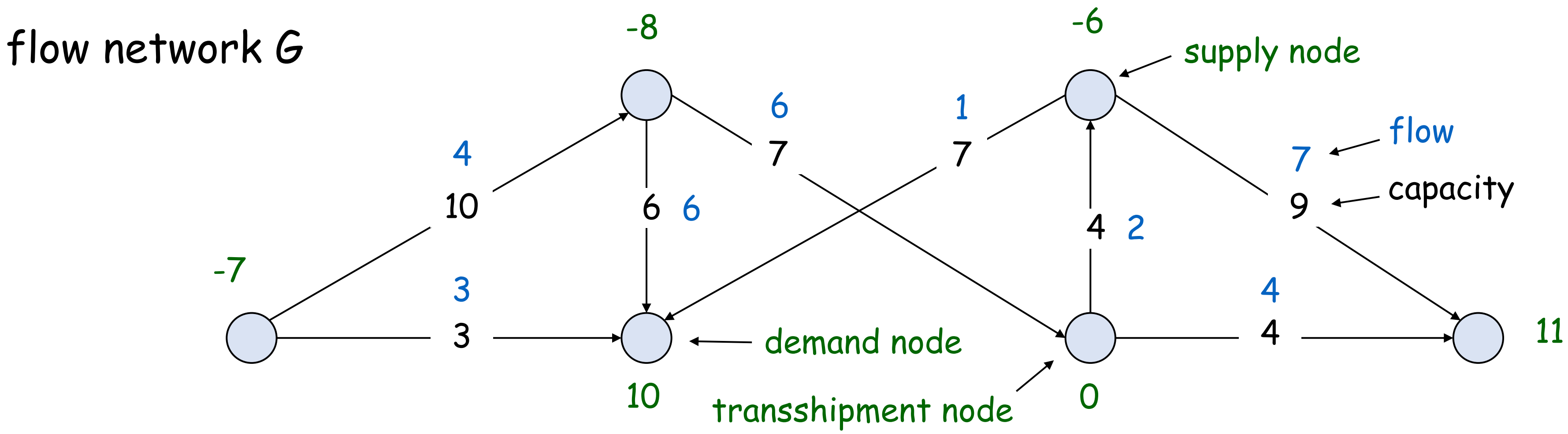
- **Def.** Given a digraph $G = (V, E)$ with edge capacities $c(e) \geq 0$ and multiple source nodes and multiple sink nodes, find max flow that can be sent from the source nodes to the sink nodes.
- **Claim.** 1-1 correspondence between flows in G and G'.

flow network G'



Circulation with Supplies and Demands

- **Def.** Given a digraph $G = (V, E)$ with edge capacities $c(e) \geq 0$ and node demands $d(v)$, a circulation is a function $f(e)$ that satisfies:
  - For each $e \in E$: $0 \leq f(e) \leq c(e)$ [capacity]
  - For each $v \in V$: $\sum_{e \text{ into } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$ [flow conservation]
- **Circulation problem.** Given (V, E, c, d), find a circulation.



Circulation with Supplies and Demands

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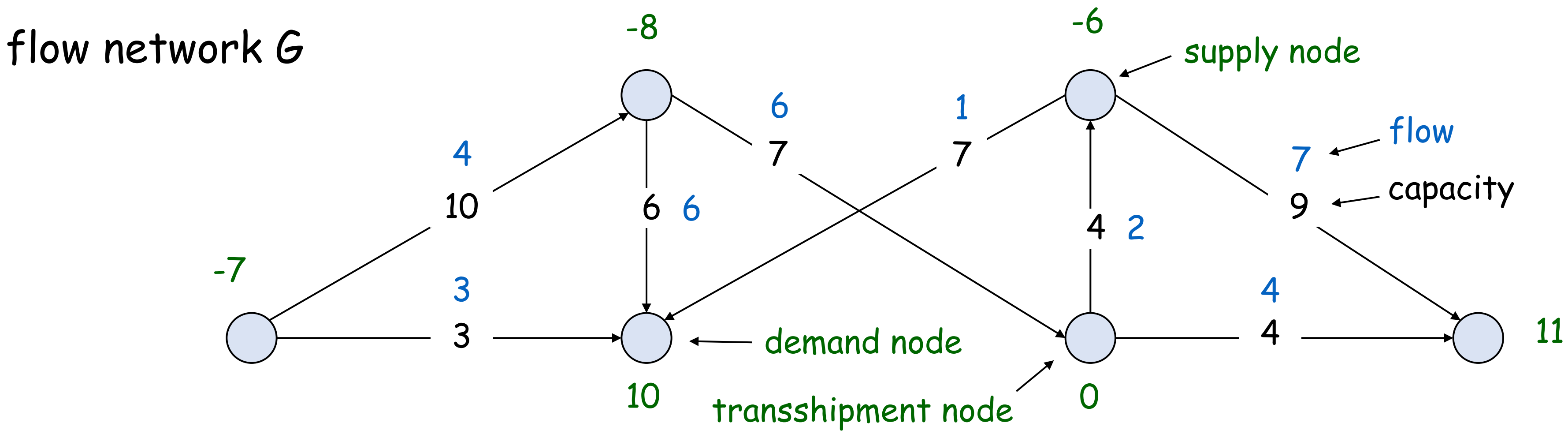
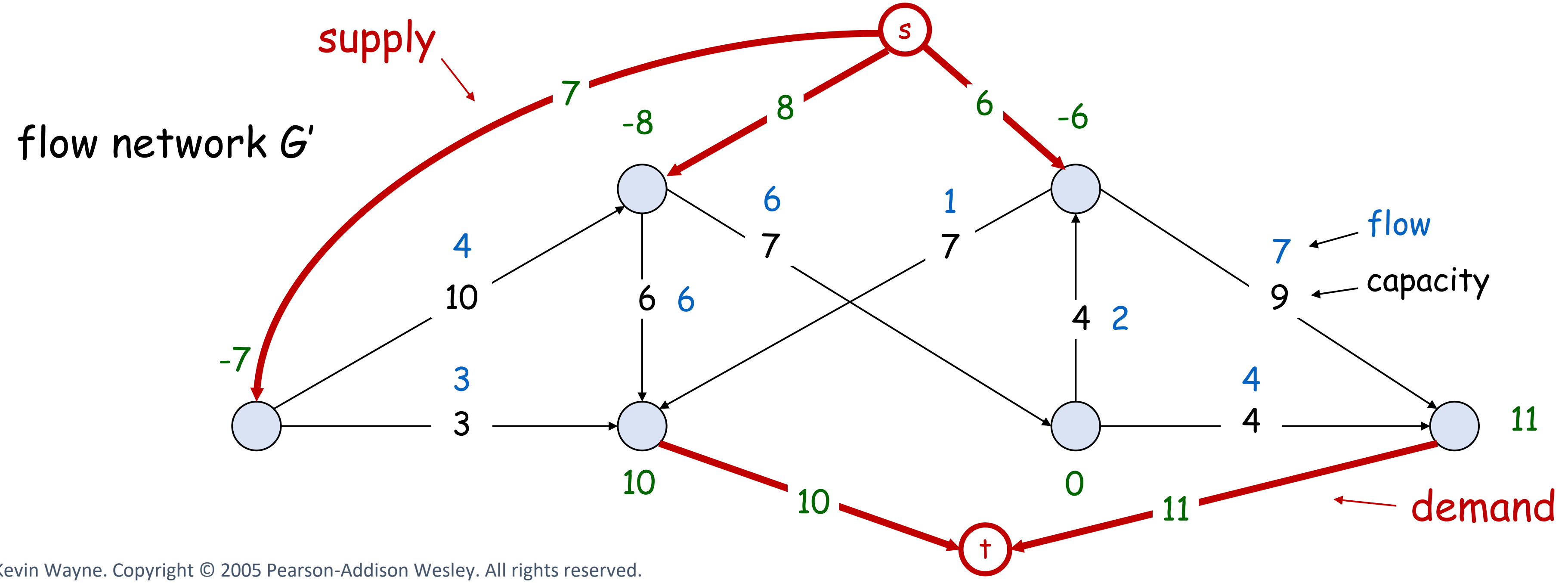
  - For each $e \in E$: $0 \leq f(e) \leq c(e)$ [capacity]
  - For each $v \in V$: $\sum_{e \text{ into } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$ [flow conservation]
- **Observation.** G has a circulation $\Rightarrow \sum_{v:d(v)>0} d(v) = \sum_{v:d(v)<0} -d(v)$.
- **Pf.** Sum of flow conservation for all nodes = 0.

total demand = total supply



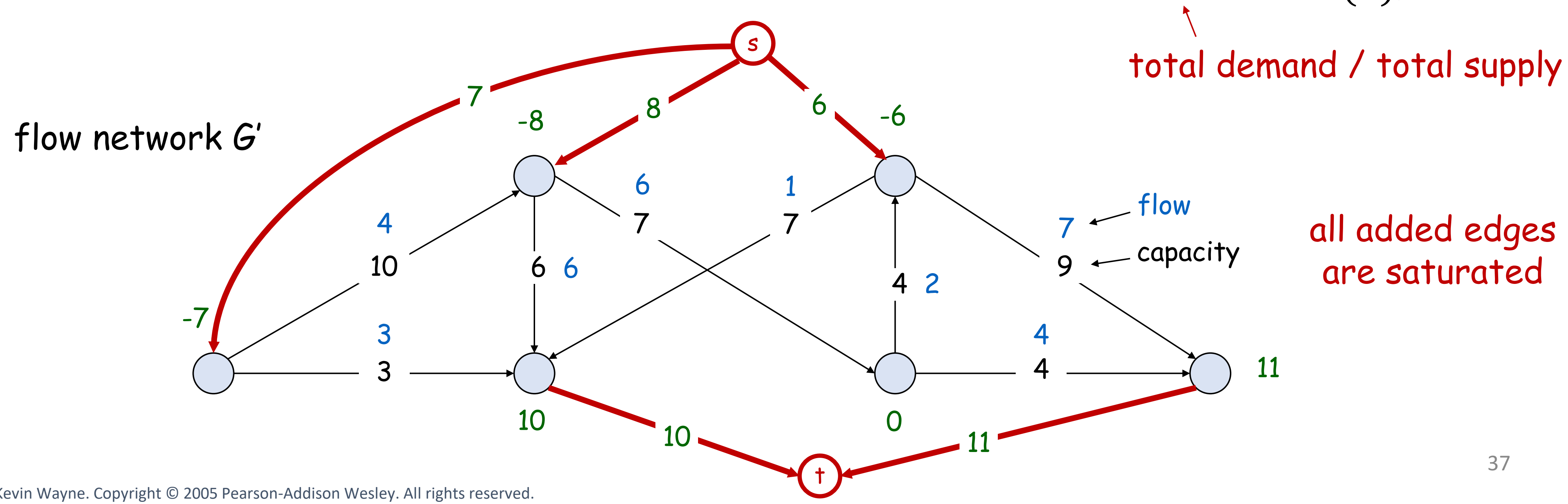
Circulation: Max-Flow Formulation

- **Max-flow formulation.**
  - Add new source *s* and sink *t*.
  - For each *v* with $d(v) < 0$, add edge (s, v) with capacity $-d(v)$.
  - For each *v* with $d(v) > 0$, add edge (v, t) with capacity $d(v)$.



Circulation: Max-Flow Formulation

- **Max-flow formulation.**
  - Add new source *s* and sink *t*.
  - For each *v* with *d*(*v*) < 0, add edge (*s*, *v*) with capacity −*d*(*v*).
  - For each *v* with *d*(*v*) > 0, add edge (*v*, *t*) with capacity *d*(*v*).
- **Claim.** *G* has a circulation iff *G'* has max flow of value *D* = $\sum_{v:d(v)>0} d(v)$.



Circulation with Supplies and Demands

- **Integrality theorem.** If all capacities and demands are integers, and there exists a circulation, then there exists one that is integer-valued.
- **Pf.** Follows from max-flow formulation + integrality theorem for max flow.

- **Theorem.** Given (V, E, c, d), there does not exist a circulation iff there exists a node partition (A, B) such that $\Sigma_{v \in B}\, d(v) > cap(A, B)$.
- **Pf sketch.** Look at min cut in G'.

demand by nodes in B exceeds
supply of nodes in B plus
max capacity of edges going from A to B

exploit the relation between
cut in G' and node partition in G

Circulation with Lower Bounds

- **Def.** Given a digraph $G = (V, E)$ with edge capacities $c(e) \geq 0$, lower bounds $\ell(e) \geq 0$, and node demands $d(v)$, a circulation is a function $f(e)$ that satisfies:
  - For each $e \in E$: $\ell(e) \leq f(e) \leq c(e)$ [capacity]
  - For each $v \in V$: $\sum_{e \text{ into } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$ [flow conservation]

- **Circulation problem with lower bounds.** Given (V, E, ℓ, c, d), does there exist a feasible circulation?

Circulation with Lower Bounds

- **Max-flow formulation.**
  - Model lower bounds as circulation with demands.
  - Send $\ell(e)$ units of flow along edge e. ← this flow can then be abstracted away in G'

- **Theorem.** There exists a circulation in G iff there exists a circulation in G'. Moreover, if all demands, capacities, and lower bounds in G are integers, then there exists a circulation in G that is integer-valued.
- **Pf sketch.** $f(e)$ is a circulation in G iff $f'(e) = f(e) - \ell(e)$ is a circulation in G'.

10. Survey Design

Survey Design

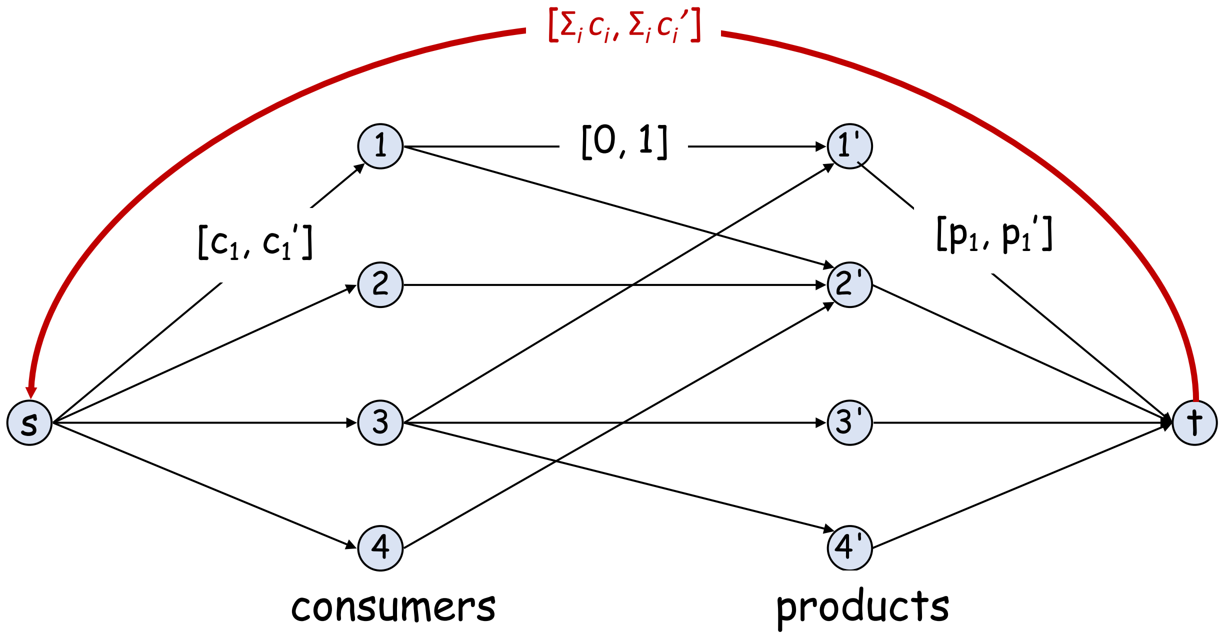
- **Survey design.**
  - Design survey asking n_1 consumers about n_2 products. ← one question per consumer-product
  - Can survey consumer *i* about product *j* only if they own it.
  - Ask consumer *i* between c_i and c_i' questions.
  - Ask between p_j and p_j' consumers about product *j*.

- **Goal.** Design a survey that meets these specs, if possible.

- **Bipartite perfect matching.** Special case when $c_i = c_i' = p_j = p_j' = 1$.

Survey Design: Circulation Formulation

- **Circulation formulation.** A circulation problem with lower bounds.
  - Add edge (i, j) if consumer i owns product j.
  - Add edge from s to consumer i. Add edge from product j to t.
  - Add edge from t to s with capacity $\Sigma_i c_i'$ and lower bound $\Sigma_i c_i$. All demands = 0.
- **Claim.** Integer circulation $\Leftrightarrow$ feasible survey design.



circulation can be solved
by max-flow formulation

11. Airline Scheduling

Airline Scheduling

- **Airline scheduling.**
  - Complex computational problem faced by airline carriers.
  - Must produce schedules that are efficient in terms of equipment usage, crew allocation, and customer satisfaction.
  - One of largest consumers of high-powered algorithmic techniques.

even in presence of unpredictable events, such as weather and breakdowns

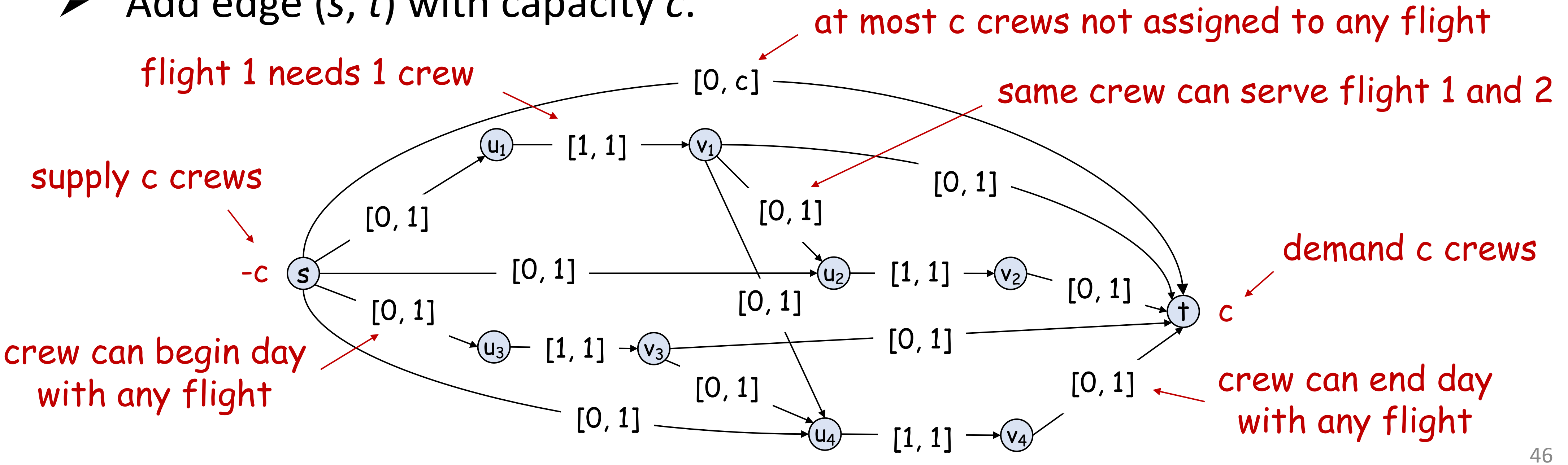
- **"Toy problem."**
  - Manage flight crews by reusing them over multiple flights.
  - Input: set of k flights for a given day.
  - Flight i leaves origin o_i at time s_i and arrives at destination d_i at time f_i.
  - Minimize number of flight crews.

one crew per flight

Airline Scheduling: Circulation Formulation

- **Circulation formulation.** (Check if c crews suffice.)
  - For each i, add nodes u_i, v_i and edge (u_i, v_i) with lower bound and capacity 1.
  - Add source s with demand $-c$, and edges (s, u_i) with capacity 1.
  - Add sink t with demand c, and edges (v_i, t) with capacity 1.
  - If flight j reachable from i, add edge (v_i, u_j) with capacity 1.
  - Add edge (s, t) with capacity c.

u_i = start of flight i
v_i = end of flight i



Airline Scheduling: Running Time

- **Theorem.** Airline scheduling problem can be solved in $O(k^3 \log k)$ time.
- **Pf.** k = number of flights.
  - $O(k)$ nodes, $O(k^2)$ edges.
  - c = number of crews (unknown).
  - At most k crews needed $\Rightarrow$ solve $\log_2 k$ circulation problems.
    binary search for min value c*
  - Any flow value is between 0 and $k \Rightarrow \leq k$ augmentations per circulation problem.
  - Overall time = $O(k^3 \log k)$.

- **Note.** Can solve in $O(k^3)$ time by formulating as minimum-flow problem.

Airline Scheduling: Running Time

- **Remark.** We solved a toy version of a real problem.
- **Real-world problem models countless other factors:**
  - Flight crews can fly only a certain number of hours in a given time window.
  - Need optimal schedule over planning horizon, not just one day.
  - Flights don't always leave or arrive on schedule.
  - Simultaneously optimize both flight schedule and fare structure.
- **Message.**
  - Our solution is a generally useful technique for efficient reuse of limited resources but trivializes real airline scheduling problem.
  - Flow techniques useful for solving airline scheduling problems · (and are widely used in practice).
  - Running an airline efficiently is a very difficult problem.

12. Image Segmentation

Image Segmentation

- **Image segmentation.**
  - ➢ Divide image into coherent regions.
  - ➢ Central problem in image processing.
- **Ex.** Separate human and robot from background scene.

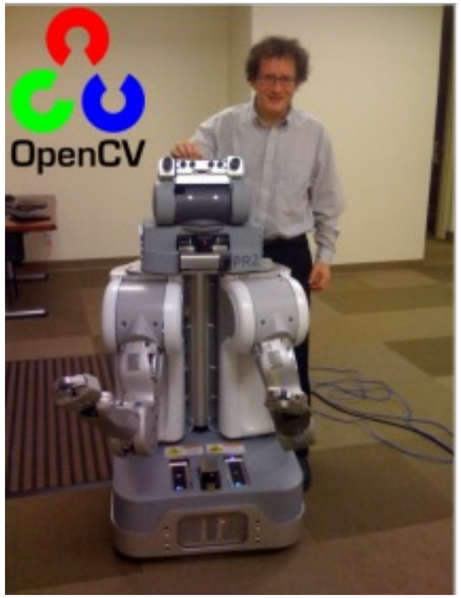
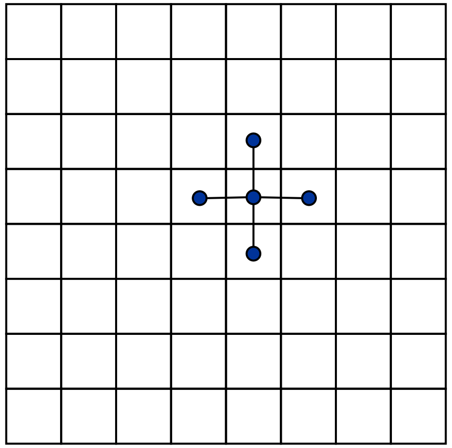



Image Segmentation

- **Foreground / background segmentation.** Label each pixel in picture as belonging to foreground or background.
  - V = set of pixels, E = pairs of neighboring pixels.
  - $a_i \geq 0$ is likelihood pixel i in foreground.
  - $b_i \geq 0$ is likelihood pixel i in background.
  - $p_{ij} \geq 0$ for each $(i, j) \in E$ is separation penalty for labeling one of i and j as foreground and the other as background.

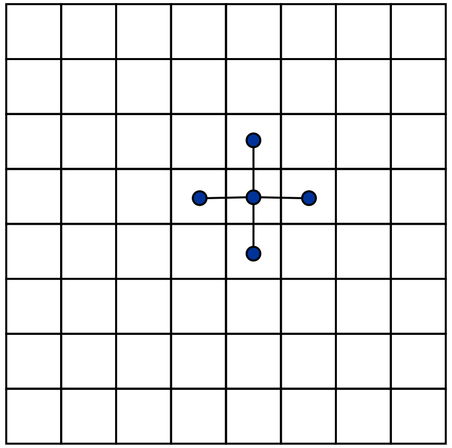


- **Goals.**
  - Accuracy: if $a_i > b_i$ in isolation, prefer to label i in foreground.
  - Smoothness: if many neighbors of i are labeled foreground, we should be inclined to label i as foreground.
  - Find partition (A, B) that maximizes: $q(A,B) = \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$

foreground background

Image Segmentation

- **Formulate as min-cut problem.**
  - Maximization.
  - No source or sink.
  - Undirected graph.
- **Turn into minimization problem.**

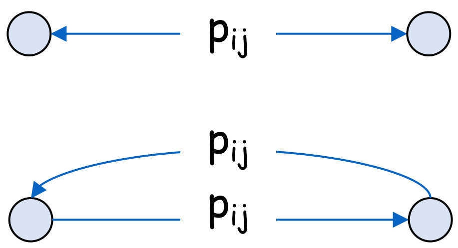


  - Maximizing $q(A,B) = \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$

is equivalent to minimizing $\left(\underbrace{\sum_{i \in A \cup B} a_i + \sum_{j \in A \cup B} b_j}_{\text{constant}} \right) - q(A,B) = \sum_{j \in B} a_j + \sum_{i \in A} b_i + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$

Image Segmentation

- **Formulate as min-cut problem *G'* = (*V'*, *E'*).**
  - Include node for each pixel.
  - Use two anti-parallel edges instead of undirected edge.
  - Add source *s* to correspond to foreground.
  - Add sink *t* to correspond to background.



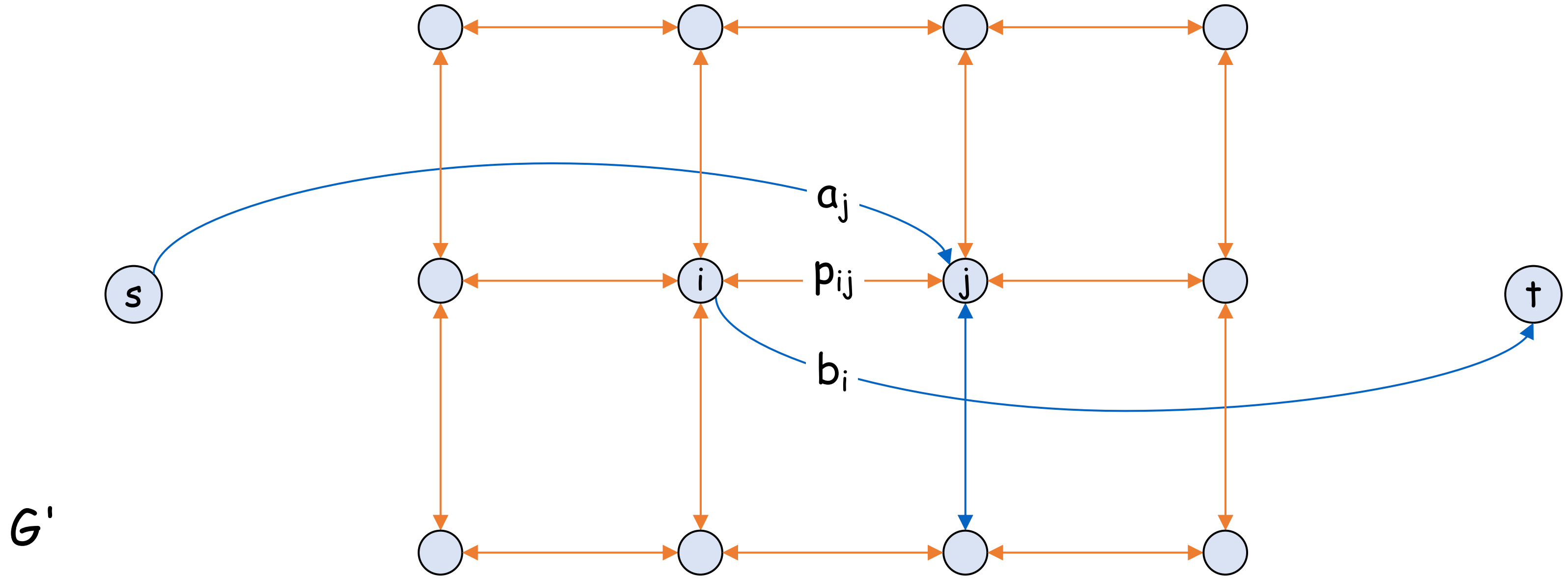
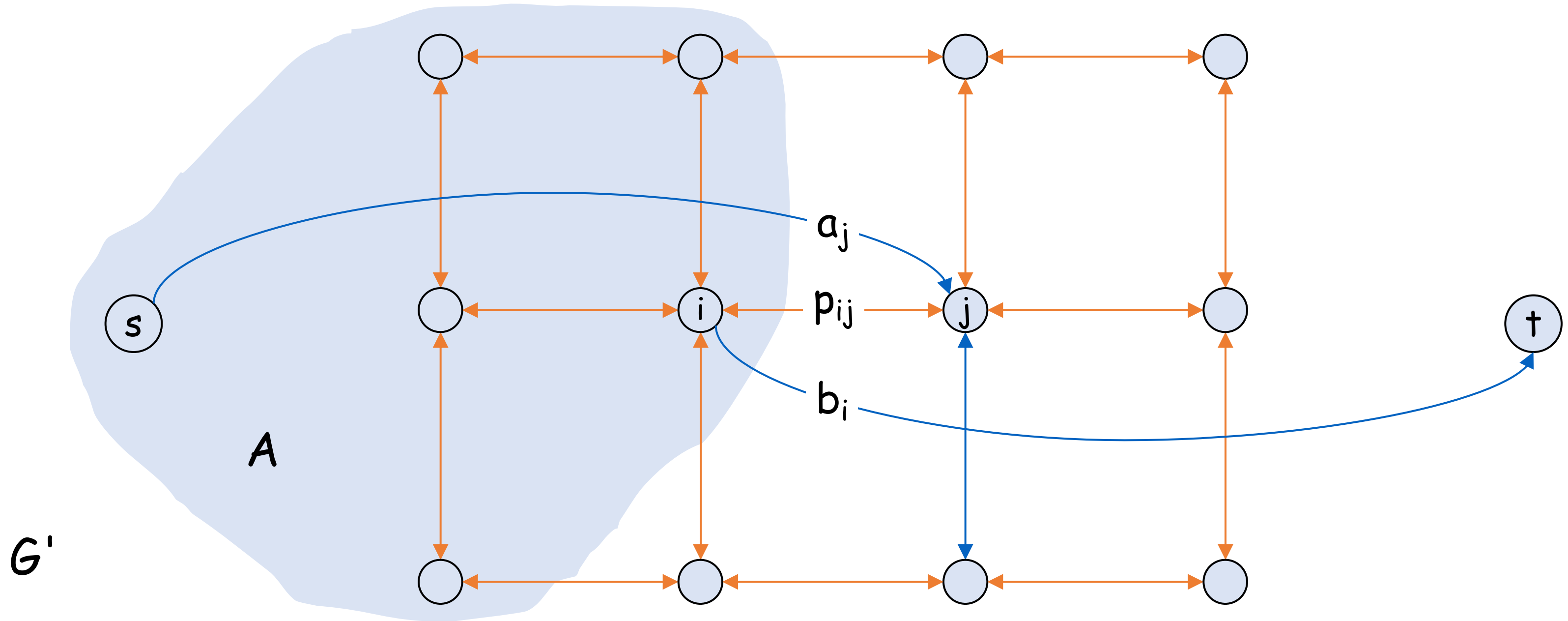


Image Segmentation

- **Consider min cut (A, B) in G'.**
  - A = foreground. $cap(A,B) = \sum_{j \in B} a_j + \sum_{i \in A} b_i + \sum_{\substack{(i,j) \in E' \\ i \in A, j \in B}} p_{ij}$
  - Precisely the quantity we want to minimize.

if i and j on different sides, p_{ij} counted exactly once



More on Network Flow Problems

- **More applications:**
  - Project selection (maximum weight closure problem). [Textbook, Section 7.11]
  - Baseball elimination. [Textbook, Section 7.12]

- **Problems solved by more advanced network flow algorithms:**
  - Minimum-cost perfect matching. [Textbook, Section 7.13]
  - Minimum-cost flow (as a general problem).