CS215: Discrete Math (H)
2022 Fall Semester   Written Assignment # 5
Due: Dec. 21st, 2022, please submit at the beginning of class

Q.1 How many relations are there on a set with n elements that are

(a) symmetric?

(b) antisymmetric?

(c) irreflexive?

(d) both reflexive and symmetric?

(e) neither reflexive nor irreflexive?

(f) both reflexive and antisymmetric?

(g) symmetric, antisymmetric and transitive?

Solution:

(a) $2^{n(n+1)/2}$

(b) $2^n 3^{n(n-1)/2}$

(c) $2^{n(n-1)}$

(d) $2^{n(n-1)/2}$

(e) $2^{n^2} - 2 \cdot 2^{n(n-1)}$

(f) $3^{n(n-1)/2}$

(g) 2^n

$\square$

Q.2 Show that a subset of an *antisymmetric* relation is also *antisymmetric*.
Solution: Suppose that $R_1 \subseteq R_2$ and that R_2 is antisymmetric. We must show that R_1 is also antisymmetric. Let $(a, b) \in R_1$ and $(b, a) \in R_1$. Since these two pairs are also both in R_2, we know that $a = b$, as desired.

$\square$

Q.3 Define a relation R on $\mathbb{R}$, the set of real numbers, as follows: For all x and y in $\mathbb{R}$, $(x, y) \in R$ if and only if $x - y$ is rational. Answer the followings, and explain your answers.

(1) Is R reflexive?

(2) Is R symmetric?

(3) Is R antisymmetric?

(4) Is R transitive?

Solution:

(1) Yes. Note that for all x we have $x - x = 0$, which is rational.

(2) Yes. Suppose that $(x, y) \in R$. Then $x - y = \frac{m}{n}$ for two integers m and n. Hence $y - x = \frac{-m}{n}$, which is again rational.

(3) No. Let $x = \sqrt{2}$ and $y = \sqrt{2} + 2$. Then we have $(x, y) \in R$ and $(y, x) \in R$, but $x \neq y$.

(4) Yes. Let $(x, y) \in R$ and $(y, z) \in R$. Then by definition both $x - y$ and $y - z$ are rational. Consequently, their sum $(x - y) + (y - z) = x - z$ is also rational. By definition, we have $(x, z) \in R$.

Q.4 Prove or give a counterexample to the following: For a set A and a binary relation R on A, if R is reflexive and symmetric, then R must be transitive as well.
Solution: Counterexample: Consider $A = \{1, 2, 3\}$ and

$$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}.$$

Then R is symmetric and reflexive, but not transitive.

$\square$

Q.5 Let R_1 and R_2 be *symmetric* relations. Is $R_1 \cap R_2$ also symmetric? Is $R_1 \cup R_2$ also be symmetric? Explain your answer.

Solution: Yes. Yes. For both R_1 and R_2, the corresponding 0-1 matrices are both symmetric. Thus. the two matrices representing $R_1 \cap R_2$ and $R_1 \cup R_2$ are also symmetric.

$\square$

Q.6 Let R and S both be *transitive* relations on a set A. For each of the relations below, either prove that it is transitive, or give a counterexample, showing that it may not be transitive.

(1) $R \cap S$

(2) $R \cup S$

(3) $R \circ S$

Solution:

(1) $R \cap S$ is transitive. Consider $(a, b), (b, c) \in R \cap S$, we have $(a, b), (b, c) \in R$ and $(a, b), (b, c) \in S$. Since both R and S are transitive, it follows that $(a, c) \in R$ and $(a, c) \in S$ and thus $(a, c) \in R \cap S$. Hence, $R \cap S$ is transitive.

(2) $R \cup S$ may not be transitive. Let $A = \{1, 2, 3\}$, and $R = \{(1, 3)\}$, $S = \{3, 1\}$. It is easy to check that both R and S are transitive. However, $R \cup S = \{(1, 3), (3, 1)\}$, which is not transitive.

(3) $R \circ S$ may not be transitive. Let $A = \{(2, 3), (4, 1)\}$ and $S = \{(1, 2), (3, 4)\}$. Then we have $R \circ S = \{(1, 3), (3, 1)\}$, which is not transitive.

Q.7 Let R be the relation on $\mathbb{Z}$, the set of integers, as follows: For all m and n in $\mathbb{Z}$, $(m, n) \in R$ if and only if 3 divides $(m^2 - n^2)$.

(1) Prove that R is an equivalence relation.

(2) Describe the equivalence classes of R.

Solution:

(1) Since $3|0$, the relation R is obviously reflexive. If $(m,n) \in R$, then $3|(m^2-n^2)$. Hence $3|(n^2-m^2)$. By definition, $(n,m) \in R$. This proves the symmetry. We now prove transitivity. Suppose that $(m,n) \in R$ and $(n,\ell) \in R$, by definition, we then have

$$3x = m^2 - n^2 \text{ and } 3y = n^2 - \ell^2$$

for some integers x and y. It then follows that

$$3(x+y) = m^2 - \ell^2,$$

which means that $3|(m^2 - \ell^2)$. By definition, we have $(m,\ell) \in R$. Hence, R is an equivalence relation on $\mathbb{Z}$.

(2) Every integer $m \in \mathbb{Z}$ can be expressed as $m = 3x + r$, where x is an integer and r is an integer with $0 \leq r \leq 2$.

Let $m = 3x + r$ and $n = 3y + s$, where $0 \leq r \leq 2$ and $0 \leq s \leq 2$. We then have

$$m^2 - n^2 = 9(x^2 - y^2) + 6(xr - ys) + r^2 - s^2.$$

Hence, there are only the following two equivalence classes:

$$\overline{0} = \{a \in \mathbb{Z} : 3|a\} \text{ and } \overline{1} = \{b \in \mathbb{Z} : 3 \nmid b\}.$$

Q.8 Let S be a finite set and T be a subset of S. We define a binary relation R on the power set $\mathcal{P}(S)$ of set S: for subsets A and B of S, $(A,B) \in R$ if and only if $(A \cup B) \setminus (A \cap B) \subseteq T$. Prove that the relation R is an equivalence relation.

Solution: Since $(A \cup A) \setminus (A \cap A) = \emptyset \subseteq T$, we have $(A,A) \in R$ for all $A \subseteq S$. The relation R is *reflexive*.

If $(A,B) \in R$, then $(A \cup B) \setminus (A \cap B) \subseteq T$, but since $\cup$ and $\cap$ are both symmetric, $A \cup B = B \cup A$ and $A \cap B = B \cap A$. So, $(B \cup A) \setminus (B \cap A) \subseteq T$. We then have the relation R is *symmetric*.

Assume that $(A,B),(B,C) \in R$. Note that e is an element of $S = (A \cup B) \setminus (A \cap B)$ if and only if it is in exactly one of A and B. So, $(A,B) \in R$ implies that every such element is in T. Similarly, $(B,C) \in R$ means that every element in exactly one of B and C is in T. Now consider an element e

in exactly one of A and C. Assume that it is in A, hence not in C. If it is also in B, then it satisfies the condition to be an element of $(B \cup C) \setminus (B \cap C)$ and thus is in T. If e is not in B, then it satisfies the condition to be in $(A \cup B) \setminus (A \cap B)$ and hence is in T. An analogous line of reasoning applies to show that if e is in C but not in A then it is in T. So we have $(A, C) \in R$ and the relation R is *transitive*.

To sum up, the relation R is an equivalence relation.

Q.9 How many different equivalence relations with exactly three different equivalence classes are there on a set with five elements?
Solution: 25. There are two possibilities to form exactly three different equivalence classes with 5 elements. One is 3, 1, 1 elements for each equivalence class, and the other is 2, 2, 1 elements for each equivalence class. By counting techniques, there are $\binom{5}{3} + \binom{5}{1} \cdot \binom{4}{2}/2 = 25$.

□

Q.10 Given functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, f is **dominated** by g if $f(x) \leq g(x)$ for all $x \in \mathbb{R}$. Write $f \preceq g$ if f is dominated by g.

(a) Prove that $\preceq$ is a partial ordering.

(b) Prove or disprove: $\preceq$ is a total ordering.

Solution:

(a) **Reflexive** For all $x \in \mathbb{R}$, $f(x) \leq f(x)$, so $f \preceq f$.

Antisymmetric Let $f \preceq g$ and $g \preceq f$. Then for all $x \in \mathbb{R}$, $f(x) \leq g(x) \leq f(x)$ and thus $f(x) = g(x)$. Since this holds for all x, we have $f = g$.

Transitive Let $f \preceq g \preceq h$. Then for all $x \in \mathbb{R}$, $f(x) \leq g(x) \leq h(x)$, giving $f(x) \leq h(x)$. So, $f \preceq h$.

(b) It is not a total ordering. Let $f(x) = x$ and $g(x) = -x$. Then $f(1) = 1 \not\leq -1 = g(1)$ and $g(-1) = 1 \not\leq -1 = f(-1)$. So it is not the case that for all x, $f(x) \leq g(x)$, and it is not the case that for all x, $g(x) \leq f(x)$. That is, these two functions are incomparable.

□

Q.11 Which of these are posets?

(a) $(\mathbf{Z}, =)$

(b) $(\mathbf{Z}, \neq)$

(c) $(\mathbf{Z}, \geq)$

(d) $(\mathbf{Z}, \nmid)$

Solution:

(a) Yes. The only ordered pairs we will have in this relation is (a, a) for all $a \in \mathbf{Z}$. This would mean that the relation is reflexive, antisymmetric, and transitive.

(b) No. It is not reflexive. The relation is also not antisymmetric, and not transitive.

(c) Yes. For reflexive, we can have the ordered pair (a, a) for all $a \in \mathbf{Z}$. This is also antisymmetric because consider the ordered pair (a, b) and $a \neq b$. This would mean that $a > b$. If this is the case, then $b > a$ is not true and you cannot have (b, a). This is also transitive because if $a > b$, $b > c$, and $a \neq b \neq c$. Then it follows that $a > c$ for all $a, b, c \in \mathbf{Z}$.

(d) No. The relations is not reflexive, not antisymmetric, not transitive.

$\square$

Q.12 Consider a relation $\propto$ on the set of functions from $\mathbb{N}^+$ to $\mathbb{R}$, such that $f \propto g$ if and only if $f = O(g)$.

(a) Is $\propto$ an equivalence relation?

(b) Is $\propto$ a partial ordering?

(c) Is $\propto$ a total ordering?

Solution:

(a) No. $\propto$ is not symmetric. Let $f(n) = n$ and $g(n) = n^2$. Here $f = O(g)$ but $g \neq O(f)$.

(b) No. $\propto$ is not antisymmetric. Let $f(n) = n$ and $g(n) = 2n$. Then $f = O(g)$ and $g = O(f)$, but $f \neq g$.

(c) No. It is not partial ordering, then not a total ordering.

$\square$

Q.13 Answer these questions for the poset $(\{3, 5, 9, 15, 24, 45\}, |)$.

(1) Find the maximal elements.

(2) Find the minimal elements.

(3) Is there a greatest element?

(4) Is there a least element?

(5) Find all upper bounds of $\{3, 5\}$.

(6) Find the least upper bound of $\{3, 5\}$, if it exists.

(7) Find all lower bounds of $\{15, 45\}$.

(8) Find the greatest lower bound of $\{15, 45\}$, if it exists.

Solution:

(1) By drawing the Hasse diagram, our maximal elements are 24 and 45.

(2) The minimal elements are 3 and 5.

(3) There is no greatest element because this element would have to be a number that all other elements divide. Since our maximal elements are 24 and 45, and they do not divide each other, we do not have a greatest element.

(4) There is no least element because this element would be a number that can divide all other elements. Since our minimal elements are 3 and 5, and they do not divide each other, we do not have a least element.

(5) 15 and 45.

(6) 15.

(7) 3, 5, and 15.

(8) 15.

$\square$

Q.14 Let $A = \{(m,n) | m,n \in \mathbb{N} \text{ and } \gcd(m,n) = 1\}$, and define the relation $\preceq$ on A according to

$$(a,b) \preceq (c,d) \Leftrightarrow ad \leq bc.$$

Prove that $(A, \preceq)$ is a totally ordered set.

Solution: We first prove that $\preceq$ on the set A is a partial ordering. Since $ab \leq ab)$, we have $(a,b) \leq (a,b)$ for all $(a,b) \in A$. Thus, $\preceq$ is reflexive; If $(a,b) \preceq (c,d)$ and $(c,d) \preceq (a,b)$ for $(a,b),(c,d) \in A$, we have $ad \leq bc$ and $bc \leq ad$ and thereby $ad = bc$. Since all pairs (a,b) in A satisfy $\gcd(a,b) = 1$, we have $(a,b) = (c,d)$. Then, $\preceq$ is antisymmetric; If $(a,b) \preceq (c,d)$ and $(c,d) \preceq (e,f)$, we have $ad \leq bc$ and $cf \leq de$. Then we have $adf \leq bcf \leq bde$ and $af \leq be$. It then follows that $(a,b) \preceq (e,f)$. The relation $\preceq$ is transitive.

For any two elements $(a,b),(c,d)$ in A, by comparing a/b and c/d, they are comparable. Thus, $(A, \preceq)$ is a totally ordered set.

$\square$

Q.15 Define the relation $\preceq$ on $\mathbb{Z} \times \mathbb{Z}$ according to

$$(a,b) \preceq (c,d) \Leftrightarrow (a,b) = (c,d) \text{ or } a^2 + b^2 < c^2 + d^2.$$

Show that $(\mathbb{Z}\times\mathbb{Z}, \preceq)$ is a poset; Construct the Hasse diagram for the subposet $(B, \preceq)$, where $B = \{0,1,2\} \times \{0,1,2\}$.

Solution: We now prove that $\preceq$ on the set $\mathbb{Z} \times \mathbb{Z}$ is a partial ordering. Obviously, $(a,b) \preceq (a,b)$, and we have $\preceq$ is reflexive; Suppose that $(a,b) \preceq (c,d)$ and $(c,d) \preceq (a,b)$, then the only possibility is that $(a,b) = (c,d)$. Then $\preceq$ is antisymmetric; Suppose that $(a,b) \preceq (c,d)$ and $(c,d) \preceq (e,f)$, then we have four possible cases: $(a,b) = (c,d)$ and $c^2 + d^2 < e^2 + f^2$; $(a,b) = (c,d)$ and $(c,d) = (e,f)$; $a^2 + b^2 < c^2 + d^2$ and $(c,d) = (e,f)$; $a^2 + b^2 < c^2 + d^2$ and $c^2 + d^2 < e^2 + f^2$. For each of the four cases above, we have $(a,b) \preceq (e,f)$ and thereby the relation $\preceq$ is transitive.

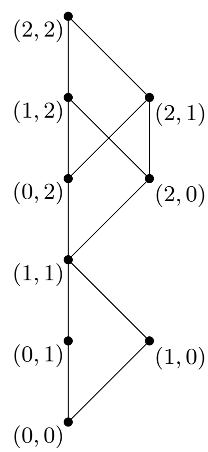

Figure 1: Q.15

□

Q.16 Answer these questions for the partial order represented by this Hasse diagram.

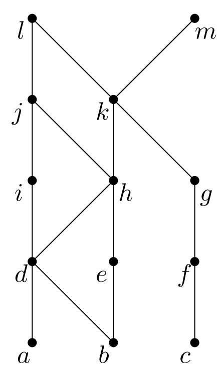

Figure 2: Q.16

(a) Find the maximal elements.

(b) Find the minimal elements.

(c) Is there a greatest element?

(d) Is there a least element?

(e) Find all upper bounds of $\{a, b, c\}$.

(f) Find the least upper bound of $\{a, b, c\}$, if it exists.

(g) Find all lower bounds of $\{f, g, h\}$.

(h) Find the greatest lower bound of $\{f, g, h\}$, if it exists.

Solution:

(a) The maximal elements are the ones with no other elements above them, namely l and m.

(b) The minimal elements are the ones with no other elements below them, namely a, b and c.

(c) There is no greatest element, since neither l nor m is greater than the other.

(d) There is no least elements, since neither a nor b is less than the other.

(e) We need to find elements from which we can find downward paths to all of a, b, and c. It is clear that k, l and m are the elements fitting this description.

(f) Since k is less than both l and m, it is the least upper bound of a, b and c.

(g) No element is less than both f and h, so there are no lower bounds.

(h) Since there is no lower bound, there cannot be greatest lower bound.

$\square$

Q.17 We consider partially ordered sets whose elements are sets of natural numbers, and for which the ordering is given by $\subseteq$. For each such partially ordered set, we can ask if it has a minimal or maximal element. For example, the set $\{\{0\}, \{0, 1\}, \{2\}\}$, has minimal elements $\{0\}, \{2\}$, and maximal elements $\{0, 1\}, \{2\}$.

(a) Prove or disprove: there exists a nonempty $R \subseteq \mathcal{P}(\mathbb{N})$ with no maximal element.

(b) Prove or disprove: there exists a nonempty $R \subseteq \mathcal{P}(\mathbb{N})$ with no minimal element.

(c) Prove or disprove: there exists a nonempty $T \subseteq \mathcal{P}(\mathbb{N})$ that has neither minimal nor maximal elements.

Solution:

(a) There are many choices here. One is to let $R = \{A_0, A_1, A_2, \ldots\}$ where $A_i = \{j \in \mathbb{N} | j < i\}$. Then R has no maximal element, because for any $A_i \in R$, we have $A_i \not\supseteq A_{i+1} \in R$.

(b) For this we will do the same thing as above in reverse. Let $S = \{B_0, B_1, B_2, \ldots\}$ where $B_i = \{j \in \mathbb{N} | j \geq i\}$. Then S has no minimal element, because for any $B_i \in S$, we have $B_i \not\subseteq B_{i+1}$.

(c) Here we can combine the previous two results. Let $T = \{C_{ij} | i \in \mathbb{N},\ j \in \mathbb{N}\}$ where each $x \in \mathbb{N}$ is in C_{ij} if and only if $x = 2k$ and $k < i$, or $x = 2k + 1$ and $K \geq j$. Now T has no minimal or maximal elements, because for any $C_{ij} \in T$, $C_{i,j+1} \not\supseteq C_{ij} \not\supseteq C_{i+1,j}$.

$\square$