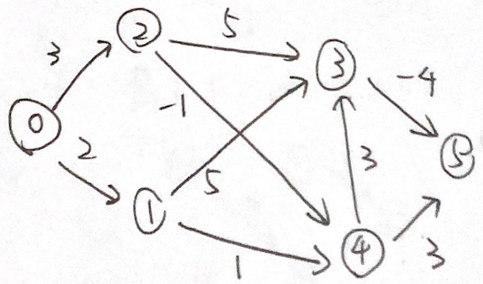
$w_{in}(e) = \{3, 2, 5, -1, 5, 1, -4, 3, 3)$

顶点数 $n=6$

$s_{in} = 0$ 边数 $m=9$



Algorithm 2: Algorithm for SPmain($G_{in} = (V, E, w_{in}), s_{in}$)

```
1  w̄(e) ← w_in(e) · 2n for all e ∈ E, Ḡ ← (V, E, w̄), B ← 2n.      // scale up edge weights
2  Round B up to nearest power of 2                      // still have w̄(e) ≥ −B for all e ∈ E
3  φ_0(v) = 0 for all v ∈ V                              // identity price function
4  for i = 1 to t := log_2(B) do
5  |  ψ_i ← ScaleDown(Ḡ_{φ_{i−1}}, Δ := n, B/2^i)
```

SPmain

$\bar{w}(e) = \{36, 24, 60, -12, 60, 12, -48, 36, 36\}$

$B = 2 \times 6 = 12 \rightarrow B = 16$

for $i = 1$ to 4

$\psi_1 = \text{ScaleDown}(\bar{G}_{\phi_0}, 6, 8)$

Algorithm 1: Algorithm for ScaleDown($G = (V, E, w), \Delta, B$)

```
1  if Δ ≤ 2 then
2  |  Let φ_2 = 0 and jump to Phase 3 (Line 10)
3  Let d = Δ/2. Let G^B_{≥0} := (V, E, w^B_{≥0}) where w^B_{≥0}(e) := max{0, w^B(e)} for all e ∈ E
   // Phase 0: Decompose V to SCCs V_1, V_2, ... with weak diameter dB in G
4  E^rem ← LowDiamDecomposition(G^B_{≥0}, dB) (Lemma 1.2)
```

ScaleDown

$(\bar{G}_{\phi_0}, 6, 8)$ $\Delta = 6$, $B = 8$, $\phi_0 = \{0, \ldots, 0\}$, $\bar{w}$

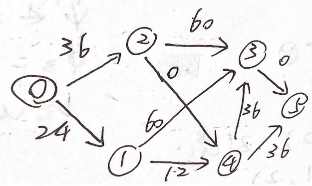
$d = \Delta/2 = 3$

$w^B_{\geq 0} = \{36, 24, 60, 0, 60, 12, 0, 36, 36\}$

$E^{rem} = \text{LowDiamDecomposition}(G^B_{\geq 0}, 24)$

$= E(G^B_{\geq 0})$

$G^B_{\geq 0}$:



Algorithm 3: Algorithm for LDD($G = (V, E), D$)

```
1  Let n be the global variable in Remark 6.3
2  G_0 ← G, E^rem ← ∅
   // Phase 1: mark vertices as light or heavy
3  k ← c ln(n) for large enough constant c
4  S ← {s_1, ..., s_k}, where each s_i is a random node in V   // possible: s_i = s_j for i ≠ j
5  For each s_i ∈ S compute Ball^in_G(s_i, D/4) and Ball^out_G(s_i, D/4)
6  For each v ∈ V compute Ball^in_G(v, D/4) ⋂ S and Ball^out_G(v, D/4) ⋂ S using Line 5
7  foreach v ∈ V do
8  |  If |Ball^in_G(v, D/4) ⋂ S| ≤ .6k, mark v in-light        // whp |Ball^in_G(v, D/4)| ≤ .7|V(G)|
9  |  Else if |Ball^out_G(v, D/4) ⋂ S| ≤ .6k, mark v out-light// whp |Ball^out_G(v, D/4)| ≤ .7|V(G)|
10 |  Else mark v heavy // w.h.p Ball^in_G(v, D/4) > .5|V(G)| and Ball^out_G(v, D/4) > .5|V(G)|
   // Phase 2: carve out balls until no light vertices remain
11 while G contains a node v marked ∗-light for ∗ ∈ {in, out} do
12 |  Sample R_v ∼ Geo(p) for p = min{1, 80 log_2(n)/D}.
13 |  Compute Ball^∗_G(v, R_v).
14 |  E^boundary ← boundary(Ball^∗_G(v, R_v))       // add boundary edges of ball to E^rem.
15 |  If R_v > D/4 or |Ball^∗_G(v, R_v)| > .7|V(G)| then return E^rem ← E(G) and terminate
   |   // Pr[terminate] ≤ 1/n^20
16 |  E^recurse ← LDD(G[Ball^∗_G(v, R_v)], D)                        // recurse on ball
17 |  E^rem ← E^rem ⋃ E^boundary ⋃ E^recurse.
18 |  G ← G \ Ball^∗_G(v, R_v)                                    // remove ball from G
   // Clean Up: check that remaining vertices have small weak diameter in
   initial input graph G_0.
19 Select an arbitrary vertex v in G.
20 If Ball^in_{G_0}(v, D/2) ⊉ V(G) or Ball^out_{G_0}(v, D/2) ⊉ V(G) then return E^rem ← E(G) and
   terminate     // Pr[terminate] ≤ 1/n^20.  If this does not terminate, then all
   remaining vertices in V(G) have weak diameter ≤ D
21 Return E^rem
```

]

$D = 24$

6. $G_0 = G$. $E^{rem} = \phi$ large enough constant $c = 1000$

$c\ln(n) = 1000 \ln 6 \approx 2000$

$\{v_0, v_1, v_2, v_3, v_4, v_5, \ldots, s_k\}$

$\text{ll}^{in}_G(v_0, 6) = \{v_0\}$ $\text{Ball}^{out}_G(v_0, 6) = \{v_0\}$

$\text{ll}^{in}_G(v_1, 6) = \{v_1\}$ $\text{Ball}^{out}_G(v_1, 6) = \{v_1\}$

$\text{ll}^{in}_G(v_2, 6) = \{v_2\}$ $\text{Ball}^{out}_G(v_2, 6) = \{v_4, v_2\}$

${}^{in}_G(v_3, 6) = \{v_3\}$ $\text{Ball}^{out}_G(v_3, 6) = \{v_5, v_3\}$

$\text{l}^{in}_G(v_4, 6) = \{v_2, v_4\}$ $\text{Ball}^{out}_G(v_4, 6) = \{v_4\}$

$\text{l}^{in}_G(v_5, 6) = \{v_3, v_5\}$ $\text{Ball}^{out}_G(v_5, 6) = \{v_5\}$

6 $k = 2000 \times 0.6 = 1200$ V all in-light

$= \min\{1, 80 \times (\log_2 6)/24\} = 1$ $R_v = 1$.

$\text{Ball}^*_G(v, 1)$ 均同 $\text{Ball}^*_G(v, 6)$.

$^{boundary}_0 = \phi$ $E^b_1 = \{e_{01}\}$ $E^b_2 = \{e_{02}\}$

$^b_3 = \{e_{23}, e_{13}, e_{43}\}$ $E^b_4 = \{e_{02}, e_{14}\}$ $E^b_5 = \{e_{23}, e_{13}, e_{43}, e_{45}\}$

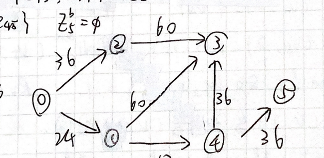
$= \{e_{02}, e_{01}\}$ $E^b_1 = \{e_{13}, e_{14}\}$ $E^b_2 = \{e_{23}, e_{43}, e_{45}\}$

$= \phi$ $E^b_4 = \{e_{43}, e_{45}\}$ $E^b_5 = \phi$

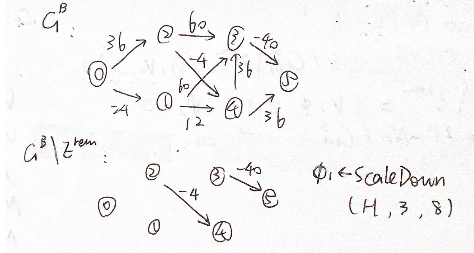
$\text{Ball}^{*_1}_G(v, 1)]$:

$|{}^*_G(v, R_v)| = 7 > 0.7 \times 6$

$\text{rn } E^{rem} \leftarrow E(G)$

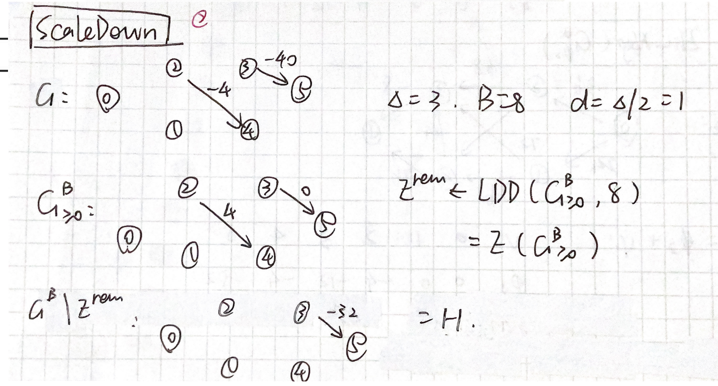


5 Compute Strongly Connected Components (SCCs) of $G^B \setminus E^{rem}$, denoted by $V_1, V_2, \dots$
`// Properties: (Lemma 4.3) For each` $u, v \in V_i$, $\mathrm{dist}_G(u,v) \le dB$.
`// (Lemma 4.4) If` $\eta(G^B) \le \Delta$, `then for every` $v \in V_i$, $E[P_{G^B}(v) \cap E^{rem}] = O(\log^2 n)$
`// Phase 1: Make edges inside the SCCs` $G^B[V_i]$ `non-negative`
6 Let $H = \bigcup_i G[V_i]$, i.e. H only contains edges inside the SCCs.
`// (Lemma 4.5) If` G `has no negative-weight cycle, then` $\eta(H^B) \le d = \Delta/2$.
7 $\phi_1 \leftarrow$ ScaleDown$(H, \Delta/2, B)$ `// (Corollary 4.6)` $w_{H^B_{\phi_1}}(e) \ge 0$ `for all` $e \in H$



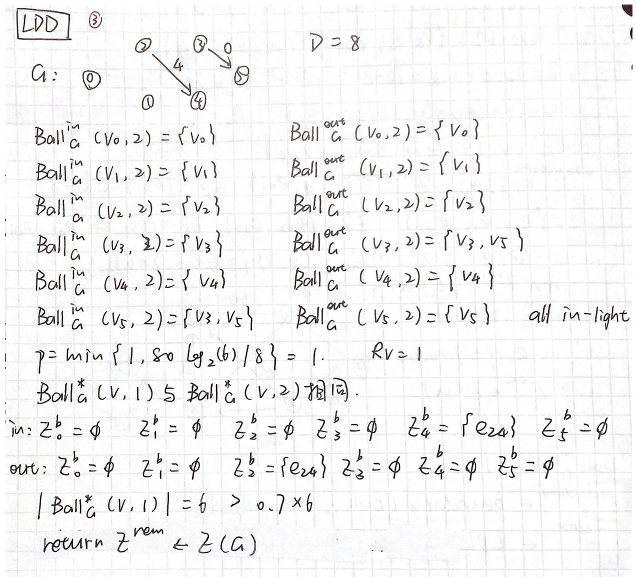
Algorithm 1: Algorithm for ScaleDown$(G = (V, E, w), \Delta, B)$

1 **if** $\Delta \le 2$ **then**
2 Let $\phi_2 = 0$ and jump to Phase 3 (Line 10)
3 Let $d = \Delta/2$. Let $G^B_{\ge 0} := (V, E, w^B_{\ge 0})$ where $w^B_{\ge 0}(e) := \max\{0, w^B(e)\}$ for all $e \in E$
`// Phase 0: Decompose` V `to SCCs` $V_1, V_2, \dots$ `with weak diameter` dB `in` G
4 $E^{rem} \leftarrow$ LowDiamDecomposition$(G^B_{\ge 0}, dB)$ (Lemma 1.2)



Algorithm 3: Algorithm for LDD$(G = (V, E), D)$

1 Let n be the global variable in Remark 6.3
2 $G_0 \leftarrow G$, $E^{rem} \leftarrow \emptyset$
`// Phase 1: mark vertices as light or heavy`
3 $k \leftarrow c \ln(n)$ for large enough constant c
4 $S \leftarrow \{s_1, \dots, s_k\}$, where each s_i is a random node in V `// possible:` $s_i = s_j$ `for` $i \ne j$
5 For each $s_i \in S$ compute $\mathrm{Ball}^{in}_G(s_i, D/4)$ and $\mathrm{Ball}^{out}_G(s_i, D/4)$
6 For each $v \in V$ compute $\mathrm{Ball}^{in}_G(v, D/4) \bigcap S$ and $\mathrm{Ball}^{out}_G(v, D/4) \bigcap S$ using Line 5
7 **foreach** $v \in V$ **do**
8 If $|\mathrm{Ball}^{in}_G(v, D/4) \bigcap S| \le .6k$, mark v *in-light* `// whp` $|\mathrm{Ball}^{in}_G(v, D/4)| \le .7|V(G)|$
9 Else if $|\mathrm{Ball}^{out}_G(v, D/4) \bigcap S| \le .6k$, mark v *out-light* `// whp` $|\mathrm{Ball}^{out}_G(v, D/4)| \le .7|V(G)|$
10 Else mark v *heavy* `// w.h.p` $\mathrm{Ball}^{in}_G(v, D/4) > .5|V(G)|$ `and` $\mathrm{Ball}^{out}_G(v, D/4) > .5|V(G)|$
`// Phase 2: carve out balls until no light vertices remain`
11 **while** G *contains a node* v *marked* $*$*-light for* $* \in \{in, out\}$ **do**
12 Sample $R_v \sim \mathrm{Geo}(p)$ for $p = \min\{1, 80\log_2(n)/D\}$.
13 Compute $\mathrm{Ball}^*_G(v, R_v)$.
14 $E^{boundary} \leftarrow$ boundary$(\mathrm{Ball}^*_G(v, R_v))$ `// add boundary edges of ball to` E^{rem}.
15 If $R_v > D/4$ or $|\mathrm{Ball}^*_G(v, R_v)| > .7|V(G)|$ then return $E^{rem} \leftarrow E(G)$ and terminate
`// Pr[terminate]` $\le 1/n^{20}$
16 $E^{recurse} \leftarrow$ LDD$(G[\mathrm{Ball}^*_G(v, R_v)], D)$ `// recurse on ball`
17 $E^{rem} \leftarrow E^{rem} \bigcup E^{boundary} \bigcup E^{recurse}$.
18 $G \leftarrow G \setminus \mathrm{Ball}^*_G(v, R_v)$ `// remove ball from` G
`// Clean Up: check that remaining vertices have small weak diameter in initial input graph` G_0.
19 Select an arbitrary vertex v in G.
20 If $\mathrm{Ball}^{in}_{G_0}(v, D/2) \not\supseteq V(G)$ or $\mathrm{Ball}^{out}_{G_0}(v, D/2) \not\supseteq V(G)$ then return $E^{rem} \leftarrow E(G)$ and terminate `// Pr[terminate]` $\le 1/n^{20}$. `If this does not terminate, then all remaining vertices in` $V(G)$ `have weak diameter` $\le D$
21 Return E^{rem}

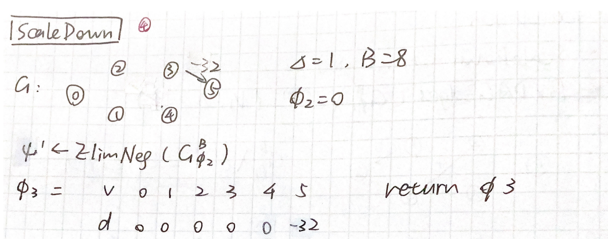


5 Compute Strongly Connected Components (SCCs) of $G^B \setminus E^{rem}$, denoted by $V_1, V_2, \dots$
`// Properties: (Lemma 4.3) For each` $u, v \in V_i$, $\mathrm{dist}_G(u,v) \le dB$.
`// (Lemma 4.4) If` $\eta(G^B) \le \Delta$, `then for every` $v \in V_i$, $E[P_{G^B}(v) \cap E^{rem}] = O(\log^2 n)$
`// Phase 1: Make edges inside the SCCs` $G^B[V_i]$ `non-negative`
6 Let $H = \bigcup_i G[V_i]$, i.e. H only contains edges inside the SCCs.
`// (Lemma 4.5) If` G `has no negative-weight cycle, then` $\eta(H^B) \le d = \Delta/2$.
7 $\phi_1 \leftarrow$ ScaleDown$(H, \Delta/2, B)$ `// (Corollary 4.6)` $w_{H^B_{\phi_1}}(e) \ge 0$ `for all` $e \in H$

$\phi_1 \leftarrow$ ScaleDown$(H, 1, 8)$.

Algorithm 1: Algorithm for ScaleDown$(G = (V, E, w), \Delta, B)$

1 **if** $\Delta \le 2$ **then**
2 Let $\phi_2 = 0$ and jump to Phase 3 (Line 10)
`// Phase 3: Make all edges in` G^B `non-negative`
10 $\psi' \leftarrow$ ElimNeg$(G^B_{\phi_2})$ (Lemma 3.3) `// (Theorem 4.2) expected time` $O(m \log^3 m)$
11 $\phi_3 = \phi_2 + \psi'$ `// (Theorem 4.1) All edges in` $G^B_{\phi_3}$ `are non-negative.`
12 **return** ϕ_3 ; `// Since` $w^B_{\phi_3}(e) \ge 0$, `we have` $w_{\phi_3}(e) \ge -B$



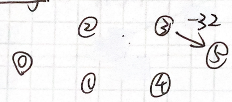
Algorithm 5: Algorithm for ElimNeg(G)

1 Set $d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for $v \neq s$
2 Initialize priority queue Q and add s to Q.
3 Initially, every vertex is unmarked

// Dijkstra Phase
4 **while** *Q is non-empty* **do**
5   Let v be the vertex in Q with minimum $d(v)$
6   Extract v from Q and mark v
7   **foreach** *edge* $(v, x) \in E \setminus E^{neg}(G)$ **do**
8   **if** $d(v) + w(v, x) < d(x)$ **then**
9   add x to Q // x may already be marked or in Q.
10       $d(x) \leftarrow d(v) + w(v, x)$

// Bellman-Ford Phase
11 **foreach** *marked vertex* v **do**
12   **foreach** *edge* $(v, x) \in E^{neg}(G)$ **do**
13   **if** $d(v) + w(v, x) < d(x)$ **then**
14   Add x to Q
15       $d(x) \leftarrow d(v) + w(v, x)$
16   Unmark v
17 If Q is empty: **return** $d(v)$ for each $v \in V$ // labels do not change so we have correct distances.
18 Go to Line 4   // Q is non-empty.

$\boxed{\text{ElimNeg}}$ 6

G : 

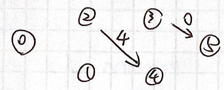
$\phi_2 = 0$.

$d(s) = 0$. $d(v) = \infty$

return

V	0	1	2	3	4	5
d	0	0	0	0	0	-32

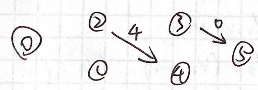
$\psi \leftarrow \text{FixDAGEdges}(G^B_{\phi_1} \setminus E^{rem}, \{V_1, V_2, \ldots\})$

$G^B_{\phi_1}$: 

$G^B_{\phi_1} \setminus E^{rem} = (V, \phi, \phi)$.

// Phase 2: Make all edges in $G^B \setminus E^{rem}$ non-negative
8 $\psi \leftarrow$ FixDAGEdges($G^B_{\phi_1} \setminus E^{rem}, \{V_1, V_2, \ldots\}$) (Lemma 3.2)
9 $\phi_2 \leftarrow \phi_1 + \psi$ // (Lemma 4.7) All edges in $(G^B \setminus E^{rem})_{\phi_2}$ are non-negative

$\psi' \leftarrow \text{ElimNeg}(G^B_{\phi_2})$

$G^B_{\phi_2}$: 

$\phi_3 = \phi_1$.
return ϕ_3.

// Phase 3: Make all edges in G^B non-negative
10 $\psi' \leftarrow$ ElimNeg($G^B_{\phi_2}$) (Lemma 3.3)   // (Theorem 4.2) expected time $O(m \log^3 m)$
11 $\phi_3 = \phi_2 + \psi'$ // (Theorem 4.1) All edges in $G^B_{\phi_3}$ are non-negative.
12 **return** ϕ_3 ;   // Since $w^B_{\phi_3}(e) \geq 0$, we have $w_{\phi_3}(e) \geq -B$

Algorithm 5: Algorithm for ElimNeg(G)

1 Set $d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for $v \neq s$
2 Initialize priority queue Q and add s to Q.
3 Initially, every vertex is unmarked

// Dijkstra Phase
4 **while** *Q is non-empty* **do**
5   Let v be the vertex in Q with minimum $d(v)$
6   Extract v from Q and mark v
7   **foreach** *edge* $(v, x) \in E \setminus E^{neg}(G)$ **do**
8   **if** $d(v) + w(v, x) < d(x)$ **then**
9   add x to Q // x may already be marked or in Q.
10       $d(x) \leftarrow d(v) + w(v, x)$

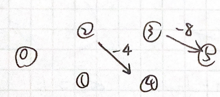
// Bellman-Ford Phase
11 **foreach** *marked vertex* v **do**
12   **foreach** *edge* $(v, x) \in E^{neg}(G)$ **do**
13   **if** $d(v) + w(v, x) < d(x)$ **then**
14   Add x to Q
15       $d(x) \leftarrow d(v) + w(v, x)$
16   Unmark v
17 If Q is empty: **return** $d(v)$ for each $v \in V$ // labels do not change so we have correct distances.
18 Go to Line 4   // Q is non-empty.

$\boxed{\text{ElimNeg}}$ 7

V	0	1	2	3	4	5
d	0	0	0	0	0	-8

// Phase 2: Make all edges in $G^B \setminus E^{rem}$ non-negative
8 $\psi \leftarrow$ FixDAGEdges($G^B_{\phi_1} \setminus E^{rem}, \{V_1, V_2, \ldots\}$) (Lemma 3.2)
9 $\phi_2 \leftarrow \phi_1 + \psi$ // (Lemma 4.7) All edges in $(G^B \setminus E^{rem})_{\phi_2}$ are non-negative

$\psi \leftarrow \text{FixDAGEdges}(G^B_{\phi_1} \setminus E^{rem}, \{V_1, V_2, \ldots\})$

$G^B_{\phi_1} \setminus E^{rem}$: 

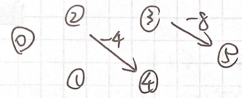
V	0	1	2	3	4	5
ϕ_1	0	0	0	0	0	-32

Algorithm 6: Algorithm for FixDAGEdges($G = (V, E, w), \mathcal{P} = \{V_1, V_2, \ldots\}$)

Relabel the sets $V_1, V_2, \ldots$ so that they are in topological order in G. That is, after relabeling, if $(u, v) \in E$, with $u \in V_i$ and $v \in V_j$, then $i \leq j$.
Define $\mu_j = \min\{w(u, v) \mid (u, v) \in E^{neg}(G), u \notin V_j, v \in V_j\}$; here, let $\min\{\emptyset\} = 0$.
  `// ` μ_j `is min negative edge weight entering` V_j`, or 0 if no such edge exists.`
Define $M_1 \leftarrow \mu_1 = 0$.
for $j = 2$ *to* q **do** `// make edges into each` $V_2, \ldots, V_q$ `non-negative`
  $M_j \leftarrow M_{j-1} + \mu_j$; `// Note:` $M_j = \sum_{k \leq j} \mu_k$
  Define $\phi(v) \leftarrow M_j$ for every $v \in V_j$
return ϕ

FixDAGEdges ⑥

G:



$V_1 = \{v_0\}$
$V_2 = \{v_1\}$
$V_3 = \{v_2, v_4\}$ $V_4 = \{v_3, v_5\}$

$\mu_1 = 0$ $\mu_2 = 0$ $\mu_3 = -4$, $\mu_4 = -8$

$M_1 = \mu_1 = 0$

return

V	0	1	2	3	4	5
ϕ	0	0	-4	-12	-4	-12

$M_2 \leftarrow M_1 + \mu_2 = 0$

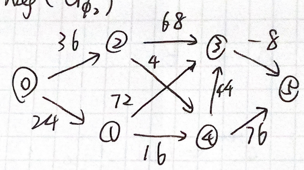
$M_3 \leftarrow M_2 + \mu_3 = -4$

$M_4 \leftarrow M_3 + \mu_4 = -12$

`// Phase 3: Make all edges in` G^B `non-negative`
$\psi' \leftarrow$ ElimNeg($G^B_{\phi_2}$) (Lemma 3.3) `// (Theorem 4.2) expected time` $O(m \log^3 m)$
$\phi_3 = \phi_2 + \psi'$ `// (Theorem 4.1) All edges in` $G^B_{\phi_3}$ `are non-negative.`
return ϕ_3 ; `// Since` $w^B_{\phi_3}(e) \geq 0$`, we have` $w_{\phi_3}(e) \geq -B$

$\psi' \leftarrow$ ElimNeg($G^B_{\phi_2}$)

$G^B_{\phi_2}$:



$\phi_3 = \phi_2 + \psi'$

V	0	1	2	3	4	5
ϕ_3	0	0	-4	-12	-4	-52

return

同理返回SPmain继续计算ScaleDown

SPmain

$\bar{w}(e) = \{36, 24, 60, -12, 60, 12, -48, 36, 36\}$

$B = 2 \times 6 = 12 \rightarrow B = 16$

for $i = 1$ to 4

$\psi_1 = \text{ScaleDown}(\bar{G}_{\phi_0}, 6, 8)$

V	0	1	2	3	4	5
ϕ_1	0	0	-4	-12	-4	-52

$\psi_2 = \text{ScaleDown}(\bar{G}_{\phi_1}, 6, 4)$

$\phi_2 = \phi_1 + \psi_2$

V	0	1	2	3	4	5
ϕ_2	0	0	-8	-20	-16	-64

$\psi_3 \leftarrow \text{ScaleDown}(\bar{G}_{\phi_2}, \Delta = 6, B/2^i = 2)$

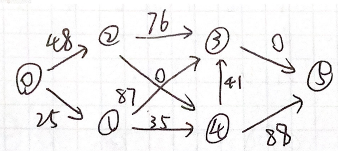
$\phi_3 = \phi_2 + \psi_3$

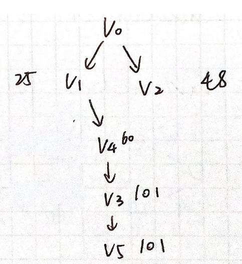
V	0	1	2	3	4	5
ϕ_3	0	0	-10	-24	-20	-70

$\psi_4 \leftarrow \text{ScaleDown}(\bar{G}_{\phi_3}, \Delta = 6, B/2^i = 1)$.

V	0	1	2	3	4	5
ϕ_4	0	0	-11	-26	-22	-73

G^*





原图