CS215: Discrete Math
2022 Fall Semester Written Assignment # 6
Due: Dec. 30th, 2022, please submit online via Sakai

Q.1 The *degree sequence* of a graph is the list of degrees of all the vertices of the graph, usually in nonincreasing order. The *complementary graph* $\overline{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. If the degree sequence of the simple graph G is $4, 3, 3, 2, 2$, what is the degree sequence of $\overline{G}$?
Solution: The degree sequence can be obtained by subtracting each of these numbers from 4 and reversing the order. We obtain $2, 2, 1, 1, 0$.

□

Q.2 A simple graph G is called *self-complementary* if G and $\overline{G}$ are isomorphic. Show that if G is a self-complementary simple graph with v vertices, then $v \equiv 0$ or $1 \pmod 4$.
Solution: If G is self-complementary, then the number of edges of G must equal the number of edges of $\overline{G}$. But the sum of these two numbers is $n(n-1)/2$, where n is the number of vertices of G, since the union of the two graphs is K_n. Therefore, the number of G must be $n(n-1)/4$. Since this number must be an integer, a look at the four cases shows that n may be congruent to either 0 or 1, but not congruent to either 2 or 3, modulo 4.

□

Q.3 Let G be a *simple* graph with n vertices.

(a) What is the *maximum* number of edges G can have?

(b) If G is connected, what is the *minimum* number of edges G can have?

(c) Show that if the minimum degree of any vertex of G is greater than or equal to $(n-1)/2$, then G must be connected.

Solution:

(a) $\binom{n}{2}$

(b) $n-1$

(c) We prove this by contradiction. Suppose that the minimum degree is $(n-1)/2$ and G is not connected. Then G has at least two connected components. In each of the components, the minimum vertex degree is still $(n-1)/2$, and this means that each connected component must have at least $(n-1)/2+1$ vertices. Since there are at least two components, this means that the graph has at least $2(\frac{n-1}{2}+1) = n+1$ vertices, which is a contradiction.

$\square$

Q.4 Let G be a simple graph with n vertices and k connected components.

(a) What is the minimum possible number of edges of G?

(b) What is the maximum possible number of edges of G?

Solution:

(a) Let each component i have c_i vertices. If we put a minimum spanning tree to keep it connected, we get $c_i - 1$ edges. So the total number of edges is

$$\sum_{i=1}^{k}(c_i - 1) = \sum_{i=1}^{k} c_i - k = n - k.$$

Thus, it does not matter how the components are selected, we always get this minimum.

(b) Let each component i have c_i vertices. If we put a complete graph for each connected component, we will maximize edges. So the total number of edges is

$$\sum_{i=1}^{k}\binom{c_i}{2} = \sum_{i=1}^{k}\frac{c_i(c_i-1)}{2} = \frac{1}{2}\left(\sum_{i=1}^{k}(c_i^2 - c_i)\right) = \frac{1}{2}\left(\sum_{i=1}^{k} c_i^2 - n\right).$$

We now need to find some distribution of vertices for each connected component such that we maximize this expression. Consider some sequence $\{c_1, c_2, \ldots, c_k\}$ such that $c_1 \le c_2 \le \cdots \le c_k$. Let's compare the number of edges produced with sequence $\{c_1 - 1, c_2 + 1, c_3, \ldots, c_k\}$.

Notice that this sequence is still in increasing order. The additional edges gained from using this new sequence for number of vertices for each of the components:

$$(c_1 - 1)^2 + (c_2 + 1)^2 - c_1^2 - c_2^2 = 2(c_2 - c_1 + 1).$$

So this is a positive increase as long as $c_1 \leq c_2 + 1$. We have assumed that $c_1 \leq c_2$, so this means that decreasing c_1 by 1 and increasing c_2 by 1 results in creation of additional edges. We can apply this argument to any two consecutive c_i and c_j repeatedly, thus resulting in $c_1 = c_2 = \cdots = c_{k-1} = 1$ and $c_k = n - (k-1)$. Therefore, the maximal number of edges that can be created is

$$\frac{1}{2}(k - 1 + (n - (k-1))^2 - n) = \frac{1}{2}\left((n-k)^2 + (n-k)\right).$$

$\square$

Q.5 Suppose that G is a graph on a finite set of n vertices. Prove the following

(a) If every vertex of G has degree 2, then G contains a circuit.

(b) If G is disconnected, then its complement is connected.

Solution:

(a) Assume for contradiction that G has no circuit, and consider the longest path P in G (one must exist, since the graph is finite). Let v be the final vertex in P – since v has degree 2, it must have two edges e_1 and e_2 incident on it, of which one, say e_1, is the last edge of the path P. Then e_2 cannot be incident on any other vertex of P since that would create a circuit $(v, e_2, [\text{section of } P \text{ ending in } e_1], v)$. So e_2 an its other endpoint are not part of P, and can be appended to P to give a strictly longer path, which contradicts our choice of P. Hence, G must contain a circuit.

(b) Let $\overline{G}$ denote the complement of G. Consider any two vertices u, v in G. If u and v are in different connected components in G, then no edge of G connects them, so there will be an edge $\{u, v\}$ in $\overline{G}$. If u and v are in the same connected component in G, then consider any vertex w in a different connected component (since G is disconnected, there must be at least one other connected component). By our first argument, the edges $\{u, w\}$ and $\{v, w\}$ exist in $\overline{G}$, so u and v are connected by the path (u, w, v). Hence, any two vertices are connected in $\overline{G}$, so the whole graph is connected.

$\square$

Q.6 Let $G = (V, E)$ be an undirected graph and let $A \subseteq V$ and $B \subseteq V$. Show that

(1) $N(A \cup B) = N(A) \cup N(B)$.

(2) $N(A \cap B) \subseteq N(A) \cap N(B)$, and give an example where $N(A \cap B) \neq N(A) \cap N(B)$.

Solution:

(1) If $x \in N(A \cup B)$, then x is adjacent to some vertex $v \in A \cup B$. W.l.o.g., suppose that $v \in A$. Then $x \in N(A)$ and therefore also in $N(A) \cup N(B)$. Conversely, if $x \in N(A) \cup N(B)$, then w.l.o.g. suppose that $x \in N(A)$. Thus, x is adjacent to some vertex $x \in A \subseteq A \cup B$, so $x \in N(A \cup B)$.

(2) If $x \in N(A \cap B)$, then x is adjacent to some vertex $v \in A \cap B$. Since both $v \in A$ and $v \in B$, it follows that $x \in N(A)$ and $x \in N(B)$, whence $x \in N(A) \cap N(B)$. For the counterexample, let $G = (\{u, v, w\}, \{\{u, v\}, \{v, w\}\})$, $A = \{u\}$, and $B = \{w\}$.

$\square$

Q.7 Given a graph $G = (V, E)$, an edge $e \in E$ is said to be a *bridge* if the graph $G' = (V, E \setminus \{e\})$ has more connected components than G. Prove that if all vertex degrees in a graph G are even, then G has no bridge.
Solution: We may assume that G is connected, for otherwise the lemma could be applied to each component separately. For contradiction, suppose that an edge $\{v_1, v_2\} = e$ is a bridge of G. The graph $G' = (V, E \setminus \{e\})$ has exactly 2 components. Let G_1 be the component containing v_1. All vertices of G_1 have an even degree except for v_1 whose degree in G_1 is odd. But this is impossible by the Handshaking Theorem.

$\square$

Q.8 Let G be a connected graph, with the vertex set V. The *distance* between two vertices u and v, denoted by $dist(u, v)$, is defined as the *minimal* length of a path from u to v. Show that $dist(u, v)$ is a metric, i.e., the following properties hold for any $u, v, w \in V$:

(i) $dist(u, v) \geq 0$ and $dist(u, v) = 0$ if and only if $u = v$.

(ii) $dist(u, v) = dist(v, u)$.

(iii) $dist(u, v) \leq dist(u, w) + dist(w, v)$.

Solution:

(i) By definition, the $dist(u, v)$ is the minimal length of a path from u to v, and the length if the number of edges in the path. Thus, $dist(u, v)$ cannot be negative. Furthermore, $dist(u, v) = 0$ if and only if there is a path of length 0 from u to v, which means that $u = v$.

(ii) Suppose that P is path from u to v of the minimal length. We reverse all the edges in the path P, and will get a path P' from v to u. Note that P' must be the minimal path from v to u. Otherwise, we reverse P' and will get a shorter path from u to v, which is a contradiction. Thus, $dist(u, v) = dist(v, u)$.

(iii) By definition, $dist(u, v) = \#$ of edges in the path P, where P is the path from u to v with the minimum length. Suppose that P_1 and P_2 are the paths of minimal length from u to w, and from w to v, respectively. Then $u\tilde{w}\tilde{v}$ is a new path P' from u to v. By the minimality of P, we must have $dist(u, v) \leq dist(u, w) + dist(w, v)$.

$\square$

Q.9 Show that if G is bipartite simple graph with v vertices and e edges, then $e \leq v^2/4$.

Solution: Suppose that the parts are of sizes k and $v-k$, respectively. Then the maximum number of edges of the graph may have is $k(v-k)$. By algebra, we know that the function $f(k) = k(v-k)$ achieves its maximum value when $k = v/2$, giving $f(k) = v^2/4$. Thus there are at most $v^2/4$ edges.

□

Q.10 Use paths either to show that these graphs are not isomorphic or to find an isomorphism between these graphs.

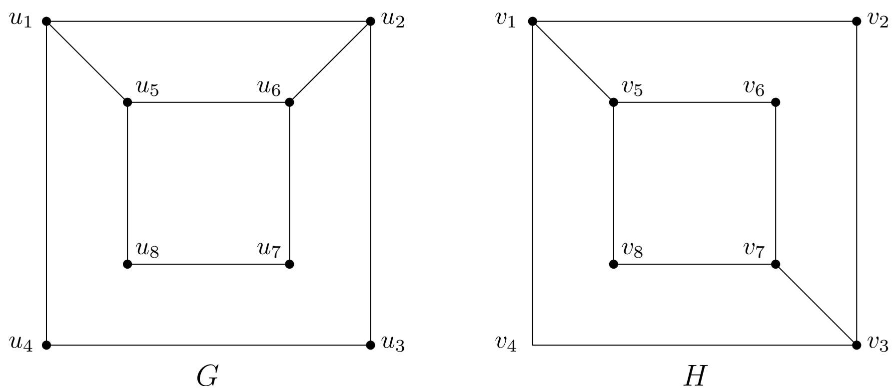

Figure 1: Q.10

Solution: The graph G has a simple closed path containing exactly the vertices of degree 3, namely $u_1u_2u_6u_5u_1$. The graph H has no simple closed path containing exactly the vertices of degree 3. Therefore the two graphs are not isomorphic.

□

Q.11 Let G be a graph in which all vertices have degree at least d. Prove that G contains a path of length d.
Solution: Let the longest path have length p. Consider the last vertex in the path. It has degree at least d. Therefore, they must all be in the path otherwise we can make a longer path by adding any of those. Hence, the longest path must include at least $d + 1$ vertices, meaning the longest path must be at least length d, so a path of length d can be found by taking a subpath of the longest path.

□

Q.12 Show that isomorphism of simple graphs is an equivalence relation.

Solution:

G is isomorphic to itself by the identity function, so isomorphism is reflexive. Suppose that G is isomorphic to H. Then there exists a one-to-one correspondence f from G to H that preserves adjacency and nonadjacency. It follows that f^{-1} is a one-to-one correspondence from H to G that preserves adjacency and nonadjacency. Hence, isomorphism is symmetric. If G is isomorphic to H and H is isomorphic to K, then there are one-to-one correspondences f and g from G to H and from H to K that preserve adjacency and nonadjacency. It follows that $g \circ f$ is a one-to-one correspondence from G to K that preserves adjacency and nonadjacency. Hence, isomorphism is transitive.

$\square$

Q.13 Given a graph G, its *line graph* $L(G)$ is defined as follows: every edge of G corresponds to a unique vertex of $L(G)$; any two vertices of $L(G)$ are adjacent if and only if their corresponding edges of G share a common endpoint. Prove that if G is regular (all vertices have the same degree) and connected, then $L(G)$ has an Euler circuit.
Solution: If the degree of regular graph G is d, then every edge of G has $2(d-1)$ neighbours in $L(G)$. Since this is even, $L(G)$ has an Euler circuit.

$\square$

Q.14 Show that if G is simple graph with at least 11 vertices, then either G or its complement graph $\overline{G}$, the complement of G, is nonplanar.
Solution: If G is planar, then because $e \leq 3v - 6$, G has at most 27 edges. (If G is not connected it has even fewer edges.) Similarly, $\overline{G}$ has at most 27 edges. However, the union of G and $\overline{G}$ is K_{11}, which has 55 edges, and $55 > 27 + 27$.

$\square$

Q.15 The **distance** between two distinct vertices v_1 and v_2 of a connected simple graph is the length (number of edges) of the shortest path between v_1 and v_2. The **radius** of a graph is the *minimum* over all vertices v of the maximum distance from v to another vertex. The **diameter** of a graph is the maximum distance between two distinct vertices. Find the radius and diameter of

(1) K_6

(2) $K_{4,5}$

(3) Q_3

(4) C_6

Solution:

(1) K_6: The diameter is clearly 1, since the maximum distance between two vertices is 1; the radius is also 1, with any vertex serving as the center.

(2) $K_{4,5}$: The diameter is clearly 2, since vertices in the same part are not adjacent, but no pair of vertices are at a distance greater than 2. Similarly, the radius is 2 with any vertex serving as the center.

(3) Q_3: Vertices at diagonally opposite corners of the cube are a distance of 3 from each other, and this is the worst case, so the diameter is 3. By symmetry we can take any vertex as the center, so it is clear that the radius is also 3.

(4) C_6: Vertices at opposite corners of the hexagon are a distance 3 from each other, and this it the worst case, so the diameter is 3. By symmetry we can take any vertex as the center, so it is clear that the radius is also 3. (Note that despite the appearances in this exercise, it is not always the case that the radius equals the diameter; for example, $K_{1,n}$ has radius 1 and diameter 2.)

□

Q.16 Let n be a positive integer. Construct a **connected** graph with $2n$ vertices, such that there are *exactly* **two** vertices of degree i for each $i = 1, 2, \ldots, n$. (You can sketch some pictures, but your graph has to be described by a concise adjacency rule. Remember to prove that your graph is connected.)

Solution:
We draw a bipartite graph in the following way:

The vertex set contains two sets of vertices, V_1 and V_2, with each containing n vertices. For the ith vertex in V_1, it is connected to the first i vertices in the set V_2. In this way, the ith vertex in V_1 has degree i, and the ith vertex in V_2 has degree $n-i+1$, since the previous $i-1$ vertices in $V-1$ are not connected to the ith vertex in V_2 by the adjacency rule. The constructed graph is connected in an obvious way: $(1,1,2,2,3,3,4,4,\ldots,n,n)$, where the first i denotes the ith vertex in V_1 and the second i denotes the ith vertex in V_2 (see the following figure).

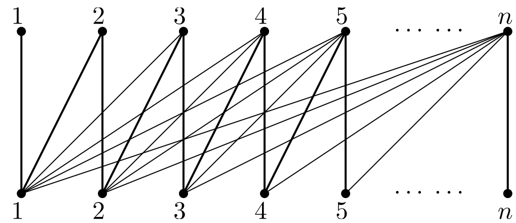

Figure 2: Q.16

$\square$

Q.17 An n-cube is a cube in n dimensions, denoted by Q_n. The 1-cube, 2-cube, 3-cube are a line segment, a square, a normal cube, respectively, as shown below. In general, you can construct the $(n+1)$-cube Q_{n+1} from the n-cube Q_n by making two copies of Q_n, prefacing the labels on the vertices with a 0 in one copy of Q_n and with a 1 in the other copy of Q_n, and adding edges connecting two vertices that have labels differing only in the first bit. Answer the following questions, and explain your answers.

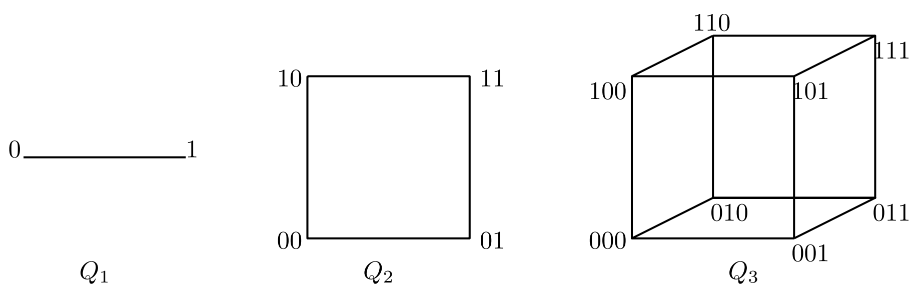

Figure 3: Q.17

(1) How many edges does an n-cube Q_n have?

(2) For what n, the n-cube Q_n has an Euler circuit?

(3) Is an n-cube Q_n bipartite or not?

(4) For what n, the n-cube Q_n is planar?

(5) For what n, the n-cube Q_n has an Hamilton circuit?

Solution:

(1) Fix any vertex v of Q_n. All its neighbors differ from v in exactly one position. There are n positions possible to differ at. Hence, every vertex has degree n. Then by the Handshaking Theorem, we have $2e = \sum_{v \in V} \deg(v) = n \cdot 2^n$. Thus, the number of edges is $n \cdot 2^{n-1}$.

(2) Q_n has an Euler circuit if and only if all its degrees are even. Since for each vertex of Q_n, its degree is n, we have that Q_n has an Euler circuit if and only if n is even.

(3) Q_n is bipartite. Let V_1 be the set of vertices of Q_n with an even number of 0's, and V_2 be the set of vertices of Q_n with an odd number of 0's. Clearly, every vertex must either have an odd or and even number of 0's. Hence, the disjoint union of V_1 and V_2 constitutes the vertex set of Q_n. For two vertices $x, y \in V_1$, there is an edge between x and y if and only if x and y differ in exactly one position. But this would imply that if one of them has an even number of 0's while the other has an odd number of 0's. So these two vertices cannot be both from V_1. This is a contradiction. Similarly we can prove that it is not possible to have an edge with both vertices from V_2.

(4) Q_n is planar only for $n \leq 3$. By (4), we know that Q_n is bipartite, and thereby does not have a circuit of length 3. Then by the necessary condition in Corollary 3, we have $e \leq 2v - 4$. For Q_4, there are $v = 16$ vertices and $e = 4 \cdot 2^{4-1} = 32$. It is easily seen that $32 > 2 \cdot 16 - 4$. Thus, Q_4 cannot be a planar graph. Obviously Q_4 is a minor of Q_n for $n > 4$. Therefore, Q_n is only planar for $n = 1, 2, 3$.

(5) For all n, Q_n has a Hamilton circuit. We prove this by induction. If $n = 1$, we simply need to visit each vertex of a two-vertex graph with an edge connecting them.

Assume that it is true for $n = k$. To build a $(k+1)$-cube, we take two copies of the k-cube and connect the corresponding edges. Take that Hamilton circuit on one cube and reverse it on the other. Then choose an edge on one that is part of the circuit and the corresponding edge on the other and delete them from the circuit. Finally, add to the path connections from the corresponding endpoints on the cubes which will produce a circuit on the $(k+1)$-cube.

Q.18

Let $G = (V, E)$ be a nonempty connected undirected simple graph (no multi-edge and no self-loop), in which every vertex has degree 4. Prove or disprove: For any partition of the vertices V into two nonempty subset S and T, i.e., $S \cup T = V$ and $S \cap T = \emptyset$, there are at least two edges that have one endpoint in S and one in T.

Solution: Because G is connected and every vertex has even degree, there is an Euler circuit of the graph (a circuit that uses every edge exactly once). Fix a particular circuit and consider a partition of V into two sets S and T. There must be at least one edge between S and T. Otherwise, G is not connected; but if there is only one, then the circuit cannot return to S or T once it leaves. It follows that there are at least 2 edges between S and T as claimed.

$\square$

Q.19 There are 17 students who communicates with each other discussing problems in discrete math. They are only 3 possible problems, and each pair of students discuss one of these three 3 problems. Prove that there are at least 3 students who are all pairwise discussing the same problem.

Solution: We use vertices to denote the 17 students and edges to denote the communication among these students. In addition, we use 3 different colors to color the edges to denote the 3 problems they are discussing. For one fixed student A, A communicates with the other 16 students. By the

Pigeonhole Principle, at least 6 edges are of the same color, w.l.o.g., we assume that the edges AB, AC, AD, AE, AF, AG are all of color red.

If among the six students B, C, D, E, F, G there is one edge, e.g., BC whose color is also red, then all the three edges of the triangle ABC are red.

If among the six students B, C, D, E, F, G there is no red edge, we consider the edges BC, BD, BE, BF, BG. There are only two colors for these five edges, so at least there are three of these five edges of the same color. W.l.o.g., assume that the three edges BC, BD, BE are of the same color, yellow. We consider the triangle CDE. If there is one yellow edge of the triangle CDE, say, CD is yellow, then the triangle BCD is a triangle with three edges all yellow. If the triangle CDE does not have yellow edge, which means all edges of CDE are blue, we again have a triangle with three edges of the same color.

□

Q.20 Which complete bipartite graphs $K_{m,n}$, where m and n are positive integers, are trees?

Solution:

If both m and n are at least 2, then clearly there is a simple circuit of length 4 in $K_{m,n}$. On the other hand, $K_{m,1}$ is cleraly a tree (as is $K_{1,n}$). Thus we conclude that $K_{m,n}$ is a tree if and only if $m = 1$ or $n = 1$.

□

Q.21

What is the value of each of these postfix expressions?

(a) $5\ 2\ 1\ -\ -\ 3\ 1\ 4\ +\ +\ *$

(b) $3\ 2\ *\ 2\ \uparrow\ 5\ 3\ -\ 8\ 4\ /\ *\ -$

Solution:

We exhibit the answers by showing with parentheses the operation that is applied next, working from left to right (it always involves the first occurrence of an operator symbol).

(a) $5\ (2\ 1\ -)\ -3\ 1\ 4\ +\ +\ * = (5\ 1\ -)3\ 1\ 4\ +\ +\ * = 4\ 3\ (1\ 4\ +)\ +\ * =$ $4\ (3\ 5\ +) * = (4\ 8\ *) = 32$

(b) $(3\ 2\ *)\ 2\ \uparrow\ 5\ 3\ -\ 8\ 4\ /\ *\ -\ = (6\ 2\uparrow)\ 5\ 3\ -\ 8\ 4\ /\ *\ - =$
$36\ (5\ 3\ -)\ 8\ 4\ /\ *\ - = 36\ 2(8\ 4\ /)\ *\ - = 36(2\ 2\ *) - = (36\ 4\ -) = 32$

□

Q.22

How many different spanning trees does each of these simple graphs have?

a) K_3 b) K_4 c) $K_{2,2}$ d) C_5

Solution:

a) 3 b) 16 c) 4 d) 5

□

Q.23

How many nonisomorphic spanning trees does each of these simple graphs have?

a) K_3 b) K_4 c) K_5

Solution:

a) 1 b) 2 c) 3

□

Q.24

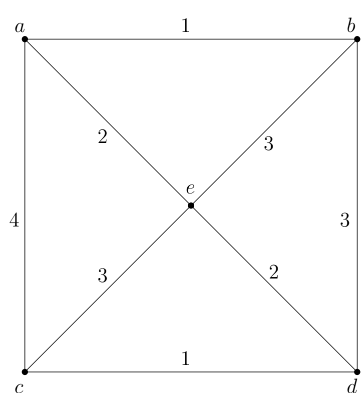

Figure 4: Q.24

(1) Use Prim's algorithm to find a minimum spanning tree for the given weighted graph.

(2) Use Kruskal's algorithm to find a minimum spanning tree for the same weighted graph.

Solution:

(1) We start with the minimum weight edge $\{a, b\}$. The least weight edge incident to the tree constructed so far is edge $\{a, e\}$, with weight 2, so we add it to the tree. Next we add edge $\{d, e\}$, and then edge $\{c, d\}$. This completes the tree, whose total weight is 6.

(2) With Kruskal's algorithm, we add at each step the shortest edge and will not complete a simple circuit. Thus we pick edge $\{a, b\}$ first, and then edge $\{c, d\}$ (alphabetical order breaks ties), followed by $\{a, e\}$ and $\{d, e\}$. The total weight is 6.

$\square$