Quiz 4 周恩呈 12110644

1° For a GMM, $D = \{x_1, \dots, x_N\}$, $\theta = \{\pi_k, \mu_k, \Sigma_k\}_{k=1,\dots K}$

(a) what is the ML solution of θ?

高斯混合概率 $p(x) = \sum_{k=1}^{K} \pi_k N(x | \mu_k, \Sigma_k)$

$$\ln p(X | \pi, \mu, \Sigma) = \sum_{n=1}^{N} \ln \left\{ \sum_{k=1}^{K} \pi_k N(x_n | \mu_k, \Sigma_k) \right\}$$

$$Q(\theta, \theta^{old}) = \sum_{Z} p(Z | X, \theta^{old}) \ln p(X, Z | \theta)$$

$$\frac{\partial Q}{\partial \theta} = 0 \Rightarrow \pi_k = \frac{N_k}{N} = \frac{\sum_{n=1}^{N} r_{nk}}{\sum_{k=1}^{K} \sum_{n=1}^{N} r_{nk}}$$

$$\mu_k = \frac{\sum_{n=1}^{N} r_{nk} x_n}{N_k}$$

$$\Sigma_k = \frac{\sum_{n=1}^{N} r_{nk} (x_n - \mu_k)}{N_k}$$

(b) If $\pi \sim Dir(N_{10}, \dots, N_{K0})$ $\mu_k \sim N(\mu_k | m_{k0}, \Sigma_{k0})$, what is the MAP solution of θ?

$$\sum_{Z} p(Z | X, \theta^{old}) \ln p(X, Z | \theta) + \ln p(\theta) = Q(\theta, \theta^{old}) + \ln p(\theta)$$

$$\frac{\partial Q}{\partial \theta} = 0 \Rightarrow \pi_k = \frac{N_k + N_{k0}}{N + N_0}$$

$$\Sigma_{MAP}^{-1} = \Sigma_{ML}^{-1} + \Sigma_0^{-1}$$

$$\Sigma_{MAP}^{-1} \mu_{MAP} = \Sigma_{ML}^{-1} \mu_{ML} + \Sigma_0^{-1}$$

(c) what is $p(x_{N+1} | \theta_{MAP})$

2° For a HMM, $D=\{x_1,\dots,x_N\}$, $\theta=\{\pi, A, \mu_k, \Sigma_k\}_{k=1,\dots,K}$

(a) what is the ML solution of θ?

$$p(x|z)=\prod_{k=1}^{K} N(x|\mu_k,\Sigma_k)^{z_k}$$

$$\gamma(z_n)=p(z_n|X,\theta^{old})$$

$$\xi(z_{n-1},z_n)=p(z_{n-1},z_n|X,\theta^{old})$$

$$\gamma(z_{nk})=\sum \gamma(z)z_{nk}$$

$$\xi(z_{n-1,j},z_{nk})=\sum_{z}\gamma(z)z_{n-1,j}z_{nk}$$

$$\pi_k=\frac{\gamma(z_{1k})}{\sum_{j=1}^{K}\gamma(z_{1j})} \qquad \mu_k=\frac{\sum_{n=1}^{N}\gamma(z_{nk})x_n}{\sum_{n=1}^{N}\gamma(z_{nk})}$$

$$A_{jk}=\frac{\sum_{n=2}^{N}\xi(z_{n-1,j},z_{nk})}{\sum_{l=1}^{K}\sum_{n=2}^{N}\xi(z_{n-1,j},z_{nl})} \qquad \Sigma_k=\frac{\sum_{n=1}^{N}\gamma(z_{nk})(x_n-\mu_k)(x_n-\mu_k)^T}{\sum_{n=1}^{N}\gamma(z_{nk})}$$

(b) If $\pi\sim Dir(N_{10},\dots,N_{K0})$ $\mu_k\sim N(\mu_k|m_{k0},\Sigma_{k0})$, $A^{(k)}\sim Dir(M_{10}^{(k)},\dots,M_{K0}^{(k)})$
what is the MAP solution of θ?

(C) what is $p(x_{N+1} | \theta_{MAP})$?

$$p(x_{N+1} | X) = \sum_{z_{N+1}} p(x_{N+1}, z_{N+1} | X)$$

$$= \sum_{z_{N+1}} p(x_{N+1} | z_{N+1}) \sum_{z_N} p(z_{N+1} | z_N) \frac{p(z_N, X)}{p(X)}$$

$$= \frac{1}{p(X)} \sum_{z_{N+1}} p(x_{N+1} | z_{N+1}) \sum_{z_N} p(z_{N+1} | z_N) \alpha(z_N)$$

3° For a stock market model, $\pi = [0.5, 0.5]$ $A = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$ $B = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$

$z = \{bull, bear\}$, $X = \{rise, fall\}$

If we have an observation of $D = \{fall, fall, rise\}$

(a) what is $p(D | \pi, A, B)$?

$$\alpha(z_1) = p(z_1, x_1) = p(x_1 | z_1) p(z_1)$$

$$= \begin{bmatrix} 0.2 \\ 0.9 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.45 \end{bmatrix}$$

$$\alpha(z_2) = p(z_2, x_1, x_2) = p(x_2 | z_2) \sum_{z_1} p(z_2 | z_1) \alpha(z_1)$$

$$= \begin{bmatrix} 0.2 \\ 0.9 \end{bmatrix} \begin{bmatrix} 0.6 \times 0.1 + 0.3 \times 0.45 \\ 0.4 \times 0.1 + 0.7 \times 0.45 \end{bmatrix} = \begin{bmatrix} 0.039 \\ 0.3195 \end{bmatrix}$$

$$\alpha(z_3) = p(z_3, x_1, x_2, x_3) = p(x_3 | z_3) \sum_{z_2} p(z_3 | z_2) \alpha(z_2)$$

$$= \begin{bmatrix} 0.8 \\ 0.1 \end{bmatrix} \begin{bmatrix} 0.6 \times 0.039 + 0.3 \times 0.3195 \\ 0.4 \times 0.039 + 0.7 \times 0.3195 \end{bmatrix} = \begin{bmatrix} 0.0954 \\ 0.023925 \end{bmatrix}$$

$$p(x_1, x_2, x_3) = \sum_{z_3} \alpha(z_3) = 0.119325$$

(b) what are $p(z_3 | D, \pi, A, B)$ and $p(z_2, z_3 | D, \pi, A, B)$ respectively?

$$\delta(z_1) = p(z_1, x_1) = p(x_1 | z_1) p(z_1)$$

$$= \begin{bmatrix} 0.2 \\ 0.9 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.45 \end{bmatrix}$$

$$\delta(z_2) = p(z_2, x_1, x_2) = p(x_2 | z_2) \max_{z_1} p(z_2 | z_1) \delta(z_1)$$

$$= \begin{bmatrix} 0.2 \\ 0.9 \end{bmatrix} \begin{bmatrix} 0.45 \times 0.6 \\ 0.45 \times 0.4 \end{bmatrix} = \begin{bmatrix} 0.054 \\ 0.162 \end{bmatrix}$$

$$\delta(z_3) = p(z_3, x_1, x_2, x_3) = p(x_3 | z_3) \max_{z_2} p(z_3 | z_2)\, \delta(z_2)$$

$$= \begin{bmatrix} 0.8 \\ 0.1 \end{bmatrix} \begin{bmatrix} 0.162 \times 0.6 \\ 0.162 \times 0.4 \end{bmatrix} = \begin{bmatrix} 0.07776 \\ 0.00648 \end{bmatrix}$$

(c) optimal $\{z_1, z_2, z_3\}$?

bear → bear → bull

(d) if x_4 = "rise" what is $p(x_4 | D, Z, A, B)$?

$$a(z_4) = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} \begin{bmatrix} 0.6 \times 0.0954 + 0.3 \times 0.023925 \\ 0.4 \times 0.0954 + 0.7 \times 0.023925 \end{bmatrix} = \begin{bmatrix} 0.051534 \\ 0.0109815 \end{bmatrix}$$

$$p(x_4) = \sum_{z_4} a(z_4) = 0.0625155$$