Exponential growth of $H^{4g-6}(\mathcal{M}_g; \mathbf{Q})$, Part II: Estimating $\dim \mathrm{Lie}(V)_n$

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Rephrase the question

Let

$$V = \mathbf{Q}\langle \sigma_{2i+1} \mid i \geq 1 \rangle$$

where σ_{2i+1} lies in degree $2i+1$.

By our previous discussion, we know that there is an injection

$$\mathrm{Lie}(V) \hookrightarrow \mathfrak{grt}_1 \cong H^0(\mathrm{GC}) \cong \Big(\bigoplus_{g\geq 2} H_0(G^{(g)})\Big)^\vee \subset \bigoplus_{g\geq 2} H^{4g-6}(\mathcal{M}_g, \mathbf{Q}).$$

Therefore, to show that

$$\dim H^{4g-6}(\mathcal{M}_g, \mathbf{Q})$$

grows exponentially with g, it is sufficient to show that

$$\dim_{\mathbf{Q}} \mathrm{Lie}(V)_g$$

grows exponentially with g.

Poincaré series

Definition. For a graded vector space $U = \oplus_{n\geq 0} U_n$, we define its Poincaré series to be

$$f_U(t) = \sum_n \dim U_n t^n.$$

For example,

$$f_V(t) = t^3 + t^5 + \cdots = \frac{t^3}{1-t^2}.$$

To phrase our question in terms of Poincaré series, we can write

$$f_{\mathrm{Lie}(V)} = \sum_{n\geq 0} A_n t^n,$$

and estimate the integers A_n.

Poincaré series

Example. Calculate the Poincaré series of the polynomial ring $A = \mathbf{Q}[x_1, \ldots, x_n]$ with the standard grading where each x_i has degree 1.

Poincaré series

Example. Calculate the Poincaré series of the polynomial ring $A = \mathbf{Q}[x_1, \ldots, x_n]$ with the standard grading where each x_i has degree 1.

Solution. The homogeneous degree d part of A has dimension the number of monomials of degree d in n variables. Therefore,

$$f_A(t) = \sum_{d \geq 0} \binom{n+d-1}{d} t^d = \frac{1}{(1-t)^n}.$$

Lemma. For two graded vector spaces V and V', we have

- $f_{V \otimes V'} = f_V \cdot f_{V'}$
- $f_{V \oplus V'} = f_V + f_{V'}$.

Poincaré series

Generalization. Let $(N(1), N(2), \ldots, N(p), \ldots)$ be a sequence of non-negative integers. Let $A = \mathbf{Q}[Z_{p,n}]$ be the polynomial rings with indeterminates $Z_{p,n}$ for $p \geq 1$ and $1 \leq n \leq N(p)$ where $Z_{p,n}$ has degree p. Then

$$f_A(t) = \prod_{p \geq 1} \frac{1}{(1 - t^p)^{N(p)}}.$$

Connection to the universal enveloping algebra

We saw the two descriptions of the universal enveloping algebra of Lie(V).

1. By construction,

$$U(\mathrm{Lie}(V)) \cong \bigoplus_{n \geq 0} V^{\otimes n}.$$

   This means

$$\begin{aligned} f_{U(\mathrm{Lie}(V))} &= 1 + f_V + f_V^2 + \cdots \\ &= \frac{1}{1 - f_V(t)}. \end{aligned}$$

2. By the PBW theorem,

$$U(\mathrm{Lie}(V)) \cong \mathrm{Sym}(\mathrm{Lie}(V)) \text{ as vector spaces.}$$

   This means

$$f_{U(\mathrm{Lie}(V))} = \prod_{n \geq 1} \frac{1}{(1 - t^n)^{A_n}},$$

   because $\mathrm{Sym}(\mathrm{Lie}(V)) \cong \mathbf{Q}[B]$ for some basis B of Lie(V).

Some complex analysis

We have

$$\frac{1}{1 - f_V(t)} = \prod_{n \geq 0} \frac{1}{(1 - t^n)^{A_n}}.$$

Apply $t\frac{d}{dt}\log$ to both, we get

$$p(t) := \frac{t^3(3 - t^2)}{(1 - t^2)(1 - t^2 - t^3)} = \sum_{d \geq 0} dA_d \frac{t^d}{1 - t^d}.$$

Write $p(t) = \sum_{n \geq 0} a_n t^n$. Let's analyze the value of a_n.

Some complex analysis

$$p(t) := \frac{t^3(3-t^2)}{(1-t^2)(1-t^2-t^3)} = \sum_{n\geq 0} a_n t^n.$$

Its smallest pole in terms of magnitude is at $\alpha \approx 0.75488\ldots$. It is a simple pole and has some residue there. The exact value of the residue is not important, so for easy computation we assume $\mathrm{Res}_\alpha p(t) = -\alpha$.

So we can write

$$p(t) = \frac{-\alpha}{t-\alpha} + \sum_{n\geq 0} b_n t^n = \sum_{n\geq 0}(\frac{1}{\alpha^n} + b_n)t^n,$$

where $\sum_{n\geq 0} b_n t^n$ converges on a disk centered at 0 with radius $> \alpha$. Note: this is why we need the smallest pole.

Therefore, $b_n\alpha^n \to 0$ and $a_n\alpha^n = (\frac{1}{\alpha^n} + b_n)\alpha^n \to 1$. Now if we set $\beta_0 = 1/\alpha$, then $a_n \to \beta_0^n$. In particular, a_n grows exponentially with n.

Going back to Lie(V)

Recall that we have

$$p(t) = \sum_{n \geq 0} a_n t^n = \sum_{d \geq 0} dA_d \frac{t^d}{1 - t^d},$$

where $A_d = \dim \mathrm{Lie}(V)_d$. Equating coefficients, we get

$$a_n = \sum_{d|n} dA_d.$$

To obtain an expression for A_d, we appeal to Möbius inversion.

Möbius inversion

Theorem. If g and f are functions from the positive integers to the complex numbers such that

$$g(n) = \sum_{d|n} f(d),$$

then

$$f(n) = \sum_{d|n} \mu(d) g(\frac{n}{d}).$$

Here μ is the Möbius function:

$$\mu(n) = \begin{cases} 1 & \text{if } n \text{ is square-free with an even number of prime factors} \\ -1 & \text{if } n \text{ is square-free with an odd number of prime factors} \\ 0 & \text{otherwise} \end{cases}.$$

Final step

We have

$$a_n = \sum_{d|n} dA_d.$$

So by Möbius inversion,

$$nA_n = \sum_{d|n} \mu(d) a_{\frac{n}{d}}$$

$$A_n = \frac{1}{n} \sum_{d|n} \mu(\frac{n}{d}) a_d$$

Since a_n grows exponentially, the summand when $d = n$ eventually dominates the other terms in the sum. So $A_n \sim a_n/n \to \beta_0^n/n$, hence A_n grows faster than β^n for any $\beta < \beta_0$.

Summary

What have we done?

- We computed the Poincaré series for $U(\mathrm{Lie}(V))$ in two different ways.
- We showed exponential growth of the coefficients in a derivative of one of the Poincaré series by examining its poles.
- We used Möbius inversion to show that $\dim \mathrm{Lie}(V)_n$ grows exponentially with n as well.

Next: everything comes together and we go through the proof of the main theorem.