SCHUR POLYNOMIALS

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1. Question

The irreducible representations of the special unitary group $SU(3)$ are indexed by Young diagrams with exactly two rows, i.e., partitions λ with two parts. Now consider Schur polynomials in three variables x_1, x_2, x_3 obtained from these Young diagrams. We obtain the monomials in s_λ by enumerating semistandard Young tableaux of shape λ.

Example 1.1. We have

$$s_{2,1}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

Let $P = \sum c_\lambda s_\lambda$, $c_\lambda > 0$, be a positive linear combination of Schur polynomials. Given a vector $v = (a, b, c)$ such that $a + b + c = 0$, we may perform the substitution $P_v(z) = P(z^a, z^b, z^c)$.

Example 1.2. Let $P = s_{2,1}$. If $v = (1, 0, -1)$, then $P_v(z) = z^2 + 2z + 2 + 2z^{-1} + z^{-2}$. If $w = (1, 2, -3)$, then $P_w(z) = z^5 + z^4 + z + 2 + z^{-1} + z^{-4} + z^{-5}$.

Question 1.3. Can we find P and vectors $v \neq w \neq 0$ such that $P_v(z) = P_w(z)$?

Observation 1.4. When $P = s_{2,1}$, we have $P_v(z) = P_{-v}(z)$. However, this is not true in general. For example, $s_{2,2}(1, 2, -3)(z) = z^6 + z^2 + z + z^{-2} + z^{-3} + z^{-4}$ but $s_{2,2}(-1, -2, 3)(z) = z^4 + z^3 + z^2 + z^{-1} + z^{-2} + z^{-6}$.

Lemma 1.5. *In $P_{(a,b,c)}(z)$, the sum of the exponents of z is zero.*

Proof. The statement reduces to the case where P is a single Schur polynomial. The sum of the exponents of x_1 in P must be the same as that of x_2, which must be the same as that of x_3, because otherwise P is not symmetric. Let that number be k. In $P_v(z)$, the sum of exponents of z is $(a + b + c)k$, but $a + b + c = 0$. □

References

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