Master in Foundations of Data Science — 2023-2024

NUMERICAL LINEAR ALGEBRA

Exercises from previous exams on *linear equation solving*, the *least square problem* and the *singular value decomposition.*

1. Let A^{-1} be the inverse matrix of A.

(1) Show that $A^{-1} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}$ where i-th column c_i verifies

$$Ac_i = e_i \tag{0.1}$$

with e_i the i-th vector in the standard basis of $\mathbb{R}^n$.

(2) Given a PLU factorization of A, show that solving each equation in (0.1) costs $4n^2/3$ flops.

(3) Show that the time complexity of obtaining A^{-1} by computing a PLU factorization of A and then solving the equations (0.1) is $2n^3 + O(n^2)$.

2. Let $A \in \mathbb{R}^{m\times n}$ be a matrix of rank n and $b \in \mathbb{R}^m$.

(1) Explain how to use the QR factorization of A to solve the least square problem (LSP) that asks to find the vector $x_{\min} \in \mathbb{R}^n$ minimizing the quantity $\|A\,x - b\|_2$ for $x \in \mathbb{R}^n$, and give the expression for the residual error $\|A\,x_{\min} - b\|_2$.

(2) Find the affine function $\ell(x) = \alpha x + \beta$ whose graph fits better the points $(-1, 1)$, $(0, 0)$ and $(1, 1)$, in the sense that the Euclidean norm of the vector

$$\big(\ell(-1) - 1, \ell(0) - 0, \ell(1) - 1\big) \in \mathbb{R}^3$$

is minimal among all possible choices of $\alpha, \beta \in \mathbb{R}$.

3. Consider the matrix

$$A = \begin{pmatrix} 4 & 1 \\ -2 & -1 \end{pmatrix}$$

(1) Compute its QR factorization using Householder reflexions.

(2) Compute the same factorization, but this time using Givens rotations instead of reflexions.

5. Consider the *singular value decomposition (SVD)*

$$A = \begin{pmatrix} 1 & 1 & 0.41 \\ -1 & 0 & 0.41 \\ 0 & 1 & -0.41 \end{pmatrix} = \begin{pmatrix} -0.82 & 0 & -0.58 \\ 0.41 & -0.71 & -0.58 \\ -0.41 & -0.71 & 0.58 \end{pmatrix} \cdot \begin{pmatrix} 1.73 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.71 \end{pmatrix} \cdot \begin{pmatrix} -0.71 & -0.71 & 0 \\ 0.71 & -0.71 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(1) Compute the condition number of the matrix A with respect to the 2-norm.

(2) Compute the best rank 1 and rank 2 approximations of A with respect to the same norm, and determine the distance to A of these approximations.

4. Let $A \in \mathbb{R}^{2\times 2}$ such that the eigenvalues of $A \cdot A^T$ are 9 and $\frac{1}{4}$ with respective eigenvectors

$$\begin{pmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{pmatrix} \approx \begin{pmatrix} 0.71 \\ -0.71 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} \approx \begin{pmatrix} 0.71 \\ 0.71 \end{pmatrix},$$

and the eigenvalues of $A^T \cdot A$ are 9 and $\frac{1}{4}$ with respective eigenvectors

$$\begin{pmatrix} 1/2 \\ -\sqrt{3}/2 \end{pmatrix} \approx \begin{pmatrix} 0.50 \\ -0.87 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \approx \begin{pmatrix} 0.87 \\ 0.50 \end{pmatrix}.$$

(1) Compute a singular value decomposition (SVD) of A.

(2) Using this SVD, compute condition number of A with respect to the 2-norm.

(3) Determine the image of the unit disk of $\mathbb{R}^2$ under the linear map defined by A.