On a bounded remainder set for a digital Kronecker sequence

Mordechay B. Levin

Abstract

Let $\mathbf{x}_0, \mathbf{x}_1, ...$ be a sequence of points in $[0, 1)^s$. A subset S of $[0, 1)^s$ is called a bounded remainder set if there exist two real numbers a and C such that, for every integer N,

$$|\operatorname{card}\{n < N \mid \mathbf{x}_n \in S\} - aN| < C.$$

Let $(\mathbf{x}_n)_{n\geq 0}$ be an s-dimensional digital Kronecker-sequence in base $b\geq 2$, $\boldsymbol{\gamma}=(\gamma_1,...,\gamma_s)$, $\gamma_i\in[0,1)$ with b-adic expansion $\gamma_i=\gamma_{i,1}b^{-1}+\gamma_{i,2}b^{-2}+...,\ i=1,...,s.$ In this paper, we prove that $[0,\gamma_1)\times...\times[0,\gamma_s)$ is the bounded remainder set with respect to the sequence $(\mathbf{x}_n)_{n\geq 0}$ if and only if

$$\max_{1 \le i \le s} \max\{j \ge 1 \mid \gamma_{i,j} \ne 0\} < \infty.$$

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1 Introduction

1.1. Discrepancy. Let $\mathbf{x}_0, \mathbf{x}_1, \dots$ be a sequence of points in $[0,1)^s, S \subseteq [0,1)^s$,

$$\Delta(S, (\mathbf{x}_n)_{n=0}^{N-1}) = \sum_{n=0}^{N-1} (\underline{1}_S(\mathbf{x}_n) - \lambda(S)),$$
 (1.1)

where $1_S(\mathbf{x}) = 1$, if $\mathbf{x} \in S$, and $1_S(\mathbf{x}) = 0$, if $\mathbf{x} \notin S$. Here $\lambda(S)$ denotes the s-dimensional Lebesgue-measure of S. We define the star discrepancy of an N-point set $(\mathbf{x}_n)_{n=0}^{N-1}$ as

$$D^*((\mathbf{x}_n)_{n=0}^{N-1}) = \sup_{0 < y_1, \dots, y_s \le 1} |\Delta([\mathbf{0}, \mathbf{y}), (\mathbf{x}_n)_{n=0}^{N-1})/N|,$$
(1.2)

where $[\mathbf{0}, \mathbf{y}) = [0, y_1) \times \cdots \times [0, y_s)$. The sequence $(\mathbf{x_n})_{\mathbf{n} \geq \mathbf{0}}$ is said to be uniformly distributed in $[0, 1)^s$ if $D_N \to 0$.

In 1954, Roth proved that $\limsup_{N\to\infty} N(\ln N)^{-\frac{s}{2}} D^*((\mathbf{x}_n)_{n=0}^{N-1}) > 0$. According to the well-known conjecture (see, e.g., [BeCh, p.283]), this estimate can be improved to $\limsup_{N\to\infty} N(\ln N)^{-s} D^*((\mathbf{x}_n)_{n=0}^{N-1}) > 0$. See [Bi] and [Le1] for the results on this conjecture.

An s-dimensional sequence $((\mathbf{x}_n)_{n\geq 0})$ is of low discrepancy (abbreviated l.d.s.) if $D^*((\mathbf{x}_n)_{n=0}^{N-1}) = O(N^{-1}(\ln N)^s)$ for $N \to \infty$. For examples of l.d.s. see, e.g., in [BeCh], [DiPi], [Ni].

1.2. Digital Kronecker sequence.

For an arbitrary prime power b, let \mathbb{F}_b be the finite field of order b, $\mathbb{F}_b^* = \mathbb{F}_b \setminus 0$, $\mathbb{Z}_b = \{0, 1, ..., b-1\}$. Let $\mathbb{F}_b[z]$ be the set of all polynomials over \mathbb{F}_b , and let $\mathbb{F}_b((z^{-1}))$ be the field of formal Laurent series. Every element L of $\mathbb{F}_b((z^{-1}))$ has a unique expansion into a formal Laurent series

$$L = \sum_{k=w}^{\infty} u_k z^{-k} \quad \text{with} \quad u_k \in \mathbb{Z}_b, \quad w \in \mathbb{Z} \quad \text{where} \quad u_w \neq 0.$$
 (1.3)

The discrete exponential evaluation ν of L is defined by

$$\nu(L) := w, \quad L \neq 0, \qquad \nu(0) := \infty.$$

Furthermore, we define the "fractional part" of L by

$$\{L\} = \sum_{k=\max(1,w)}^{\infty} u_k z^{-k}.$$
 (1.4)

We choose bijections $\psi_r : \mathbb{Z}_b \to \mathbb{F}_b$ with $\psi_r(0) = 0$, and for i = 1, 2, ..., s and j = 1, 2, ... we choose bijections $\eta_{i,j} : \mathbb{F}_b \to \mathbb{Z}_b$. For n = 0, 1, ..., let

$$n = \sum_{r=0}^{\infty} a_r(n)b^r \tag{1.5}$$

be the digit expansion of n in base b, where $a_r(n) \in \mathbb{Z}_b$ for $r \geq 0$ and $a_r(n) = 0$ for all sufficiently large r.

With every $n = 0, 1, \ldots$, we associate the polynomial

$$n(z) = \sum_{r=0}^{\infty} \psi_r(a_r(n)) z^r \in \mathbb{F}_b[z]$$
 (1.6)

and if $L \in \mathbb{F}_b((z^{-1}))$ is as in (1.3), then we define

$$\eta^{(i)}(L) = \sum_{k=\max(1,w)}^{\infty} \eta_{i,k}(u_k) b^{-k}.$$
 (1.7)

In [Ni], Niederreiter introduced a non-Archimedean analogue of the classical Kronecker sequences. For every s-tuple $\mathbf{L} = (L_1, ..., L_s)$ of elements of $\mathbb{F}_b((z^{-1}))$, we define the sequence $S(\mathbf{L}) = (\mathbf{l}_n)_{n \geq 0}$ by

$$\mathbf{l}_n = (l_n^{(1)}, ..., l_n^{(s)}), \quad l_n^{(i)} = \eta^{(i)}(n(z)L_i(z)), \quad \text{for} \quad 1 \le i \le s, \ n \ge 0.$$
 (1.8)

This sequence is sometimes called a digital Kronecker sequence (see [LaPi, p.4]). In analogy to classical Kronecker sequences, in [LaNi, Theorem 1], the following theorem has been proven

Theorem A. A digital Kronecker sequence S(L) is uniformly distributed in $[0,1)^s$ if and only if $1, L_1, ..., L_s$ are linearly independent over $\mathbb{Z}_b[x]$.

By μ_1 we denote the normalized Haar-measure on $\mathbb{F}_b((z^{-1}))$ and by μ_s the s-fold product measure on $\mathbb{F}_b((z^{-1}))^s$. In [La1], Larcher proved the following metrical upper bound on the star discrepancy of digital Kronecker sequences $D_N(S(L)) = O(N^{-1}(\log N)^s(\log \log N)^{2+\epsilon})$. For μ_s -almost all $L \in \mathbb{F}_b((z^{-1}))^s$, $\epsilon > 0$.

In [LaPi, p.4], Larcher and Pillichshammer were able to give corresponding metrical lower bounds for the discrepancy of digital Kronecker sequences $D_N(S(L)) \geq c(b,s)N^{-1}(\log N)^s \log \log N$ for μ_s -almost all $L \in \mathbb{F}_b((z^{-1}))^s$, for infinitely many $N \geq 1$ with some c(b,s) > 0 not depending on N.

1.3. Bounded remainder set.

Definition 1. Let $\mathbf{x}_0, \mathbf{x}_1, ...$ be a sequence of point in $[0, 1)^s$. A subset S of $[0, 1)^s$ is called a bounded remainder set for $(\mathbf{x}_n)_{n\geq 0}$ if the discrepancy function $\Delta(S, (\mathbf{x}_n)_{n=0}^{N-1})$ is bounded in N.

Let α be an irrational number, let I be an interval in [0,1) with the length |I|, let $\{n\alpha\}$ be the fractional part of $n\alpha$, n=1,2,. Hecke, Ostrowski and Kesten proved that $\Delta(S, (\{n\alpha\})_{n=1}^N)$ is bounded if and only if $|I| = \{k\alpha\}$ for some integer k (see references in [GrLe]).

The sets of bounded remainder for the classical s-dimensional Kronecker sequence were studied by Lev and Grepstad [GrLe]. The case of Halton's se-

quence was studied by Hellekalek [He]. For references to others investigations on bounded remainder set see [GrLe].

Let $\gamma = (\gamma_1, ..., \gamma_s)$, $\gamma_i \in (0, 1)$ with b-adic expansion $\gamma_i = \gamma_{i,1}b^{-1} + \gamma_{i,2}b^{-2} + ..., i = 1, ..., s$. In this paper, we prove

Theorem. Let $(\mathbf{l}_n)_{n\geq 0}$ be a uniformly distributed digital Kronecker sequence. The set $[0, \gamma_1) \times ... \times [0, \gamma_s)$ is of bounded remainder with respect to $(\mathbf{l}_n)_{n\geq 0}$ if and only if

$$\max_{1 \le i \le s} \max\{j \ge 1 \mid \gamma_{i,j} \ne 0\} < \infty. \tag{1.9}$$

In [Le2], we proved similar results for digital sequences described in [DiPi, Sec. 8]. Note that according to Larcher's conjecture [La2, p.215], the assertion of the Theorem is true for all digital (t, s)-sequences in base b.

2 Notations.

A subinterval E of $[0,1)^s$ of the form

$$E = \prod_{i=1}^{s} [a_i b^{-d_i}, (a_i + 1)b^{-d_i}),$$

with $a_i, d_i \in \mathbb{Z}$, $d_i \geq 0$, $0 \leq a_i < b^{d_i}$ for $1 \leq i \leq s$ is called an elementary interval in base $b \geq 2$.

Definition 2. Let $0 \le t \le m$ be integers. A (t, m, s)-net in base b is a point set $\mathbf{x}_0, ..., \mathbf{x}_{b^m-1}$ in $[0, 1)^s$ such that $\#\{n \in [0, b^m - 1] | x_n \in E\} = b^t$ for every elementary interval E in base b with $vol(E) = b^{t-m}$.

Definition 3. ([DiPi, Definition 4.30]) For a given dimension $s \geq 0$, an integer base $b \geq 2$, and a function $\mathbf{T} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\mathbf{T}(m) \leq m$ for all $m \in \mathbb{N}_0$, a sequence $(\mathbf{x}_0, \mathbf{x}_1, ...)$ of points in $[0, 1)^s$ is called a (\mathbf{T}, s) -sequence in base b if for all integers $m \geq 1$ and $k \geq 0$, the point set consisting of the points $x_{kb^m}, ..., x_{kb^m+b^m-1}$ forms a $(\mathbf{T}(m), m, s)$ -net in base b.

A (\mathbf{T}, s) -sequence in base b is called a strict (\mathbf{T}, s) -sequence in base b if for all functions $\mathbf{U} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\mathbf{U}(m) \leq m$ for all $m \in \mathbb{N}_0$ and with $\mathbf{U}(m) < \mathbf{T}(m)$ for at least one $m \in \mathbb{N}_0$, it is not a (\mathbf{U}, s) -sequence in base b.

Definition 4. ([DiNi, Definition 1]) Let $m, s \ge 1$ be integers. Let $C^{(1,m)}, ..., C^{(s,m)}$ be $m \times m$ matrices over \mathbb{F}_b . Now we construct b^m points in $[0,1)^s$. For $n=0,1,...,b^m-1$, let $n=\sum_{j=0}^{m-1}a_j(n)b^j$ be the b-adic expansion of n. For r=0,1,... we choose bijections $\psi_r: \mathbb{Z}_b \to \mathbb{F}_b$ with $\psi_r(0)=0$, and for i=1,2,...,s and j=1,2,... we choose bijections $\eta_{i,j}: \mathbb{F}_b \to \mathbb{Z}_b$. We map the vectors

$$y_n^{(i,m)} = (y_{n,1}^{(i,m)}, ..., y_{n,m}^{(i,m)}), \quad y_{n,j}^{(i,m)} = \sum_{r=0}^{m-1} \psi_r(a_r(n)) c_{j,r}^{(i,m)} \in \mathbb{F}_b$$
 (2.1)

to the real numbers

$$x_n^{(i)} = \sum_{j=1}^m \eta_{i,j}(y_{n,j}^{(i,m)})/b^j$$
(2.2)

to obtain the point

$$\mathbf{x}_n = (x_n^{(1)}, ..., x_n^{(s)}) \in [0, 1)^s.$$

The point set $\{\mathbf{x}_0, ..., \mathbf{x}_{b^m-1}\}$ is called a digital net (over \mathbb{F}_b) (with generating matrices $(C^{(1,m)}, ..., C^{(s,m)})$).

For $m = \infty$, we obtain a sequence $\mathbf{x}_0, \mathbf{x}_1, ...$ of points in $[0, 1)^s$ which is called a digital sequence $(over \mathbb{F}_b)$ (with generating matrices $(C^{(1,\infty)}, ..., C^{(s,\infty)})$). We abbreviate $C^{(i,m)}$ as $C^{(i)}$ for $m \in \mathbb{N}$ and for $m = \infty$.

Lemma A ([LaNi, ref. 1-8]). A digital Kronecker sequence in base b can be expressed as some digital (\mathbf{T}, s) -sequence in base b.

Lemma B ([DiPi, Theorem 4.86]). Let b be a prime power. A strict digital (\mathbf{T}, s) -sequence over \mathbb{F}_b is uniformly distributed modulo one, if and only if $\lim \inf_{m\to\infty} (m-\mathbf{T}(m)) = \infty$.

For m > n, we put $\sum_{j=m}^{n} c_j = 0$ and $\prod_{j=m}^{n} c_j = 1$. For $x = \sum_{j \ge 1} x_j b^{-j}$, where $x_i \in \mathbb{Z}_b = \{0, ..., b-1\}$, we define the truncation

$$[x]_m = \sum_{1 \le j \le m} x_j b^{-j}$$
 with $m \ge 1$.

If $\mathbf{x} = (x^{(1)}, ..., x^{(s)}) \in [0, 1)^s$, then the truncation $[\mathbf{x}]_m$ is defined coordinatewise, that is, $[\mathbf{x}]_m = ([x^{(1)}]_m, ..., [x^{(s)}]_m)$.

For $x = \sum_{j\geq 1} x_j b^{-j}$ and $y = \sum_{j\geq 1} y_j b^{-j}$ where $x_j, y_j \in \mathbb{Z}_b$, we define the (b-adic) digital shifted point v by $v = x \oplus y := \sum_{j\geq 1} v_j b^{-j}$, where $v_j \equiv x_j + y_j \pmod{b}$ and $v_j \in \mathbb{Z}_b$. For $\mathbf{x} = (x^{(1)}, ..., x^{(s)}) \in [0, 1)^s$ and $\mathbf{y} = (y^{(1)}, ..., y^{(s)}) \in [0, 1)^s$, we define the (b-adic) digital shifted point \mathbf{v} by $\mathbf{v} = \mathbf{x} \oplus \mathbf{y} = (x^{(1)} \oplus y^{(1)}, ..., x^{(s)} \oplus y^{(s)})$. For $n_1, n_2 \in [0, b^m)$, we define $n_1 \oplus n_2 := (n_1/b^m \oplus n_2)b^m)b^m$.

For $x = \sum_{j \geq 1} x_j b^{-j}$, where $x_j \in \mathbb{Z}_b$, $x_j = 0$ (j = 1, ..., k) and $x_{k+1} \neq 0$, we define the absolute valuation $\|.\|_b$ of x by $\|x\|_b = b^{-k-1}$. Let $\|n\|_b = b^k$ for $n \in [b^k, b^{k+1})$.

Definition 5. A sequence $(\mathbf{x}_n)_{n\geq 0}$ in $[0,1)^s$ is weakly admissible in base b if

$$\mathbf{x}_{m} := \min_{0 \le k < n < b^{m}} \|\mathbf{x}_{n} \ominus \mathbf{x}_{k}\|_{b} > 0 \quad \forall m \ge 1 \text{ where } \|\mathbf{x}\|_{b} := \prod_{i=1}^{s} \|x^{(i)}\|_{b}. (2.3)$$

Let p be a prime, $b = p^{\kappa}$,

$$E(\alpha) := exp(2\pi i \operatorname{Tr}(\alpha)/p), \qquad \alpha \in \mathbb{F}_b$$

where $\operatorname{Tr}: \mathbb{F}_b \to \mathbb{F}_p$ denotes the usual trace of an element of \mathbb{F}_b in \mathbb{F}_p . Let

$$\delta(\mathfrak{T}) = \begin{cases} 1, & \text{if } \mathfrak{T} \text{ is true,} \\ 0, & \text{otherwise.} \end{cases}$$
 (2.4)

By [LiNi, ref. 5.6 and ref. 5.8], we get

$$\frac{1}{q} \sum_{\beta \in \mathbb{F}_b} E(\alpha \beta) = \delta(\alpha = 0), \quad \text{where} \quad \alpha \in \mathbb{F}_b.$$
 (2.5)

3 Proof

Lemma 1. Let $(\mathbf{x}_n)_{n\geq 0}$ be a weakly admissible digital sequence in base b, $m\geq 1$, $\tau_m=[\log_b(\kappa_m)]+m$. Then we have for all integers $A\geq 1$

$$|\Delta([\mathbf{0}, \gamma), (\mathbf{x}_n)_{n=b^m A}^{b^m A+N-1}) - \Delta([\mathbf{0}, [\gamma]_{\tau_m}), (\mathbf{x}_n)_{n=b^m A}^{b^m A+N-1})| \le s, \quad \forall N \in [1, b^m].$$

Proof. Let

$$B = [\mathbf{0}, \gamma), \quad B_i = \prod_{1 \le j \le s, j \ne i} [0, \gamma^{(j)}) \times [0, [\gamma^{(i)}]_{\tau_m}) \quad \text{and} \quad B_0 = \bigcup_{i=1}^s (B \setminus B_i).$$

It is easy to see that $B = [0, [\gamma]_{\tau_m}) \cup B_0$. By (1.1), we get

$$\Delta([\mathbf{0}, \boldsymbol{\gamma}), (\mathbf{x}_n)_{n=b^m A}^{b^m A+N-1})$$

$$= \Delta([\mathbf{0}, [\boldsymbol{\gamma}]_{\tau_m}), (\mathbf{x}_n)_{n=b^m A}^{b^m A+N-1}) + \Delta(B_0, (\mathbf{x}_n)_{n=b^m A}^{b^m A+N-1}).$$

Hence

$$|\Delta([\mathbf{0}, \boldsymbol{\gamma}), (\mathbf{x}_n)_{n=b^m A}^{b^m A+N-1}) - \Delta([\mathbf{0}, [\boldsymbol{\gamma}]_{\tau_m}), (\mathbf{x}_n)_{n=b^m A}^{b^m A+N-1})|$$

$$\leq \sum_{i=1}^{s} |\Delta(B \setminus B_i, (\mathbf{x}_n)_{n=b^m A}^{b^m A+N-1})|. \tag{3.1}$$

Suppose that there exist $i \in [1, s]$, $k, n \in [0, b^m)$, $k \neq n$ and $A \geq 1$ such that $x_{n+b^m A}, x_{k+b^m A} \in B \setminus B_i$. Therefore

$$x_{n+b^m A,j}^{(i)} = x_{k+b^m A,j}^{(i)}$$
 for $j = 1, ..., \tau_m$.

From (1.5), (2.1) and (2.2), we have

$$y_{n+b^m A,j}^{(i)} = y_{k+b^m A,j}^{(i)}$$
 for $j = 1, ..., \tau_m$,

$$y_{n+b^mA,j}^{(i)} = y_{n,j}^{(i)} + y_{b^mA,j}^{(i)}$$
, and $y_{k+b^mA,j}^{(i)} = y_{k,j}^{(i)} + y_{b^mA,j}^{(i)}$ for $j = 1, ..., \tau_m$.

Hence

$$y_{n,j}^{(i)} = y_{k,j}^{(i)}, \quad j = 1, ..., \tau_m \quad \text{and} \quad x_{n,j}^{(i)} = x_{k,j}^{(i)}, \quad j = 1, ..., \tau_m.$$

Therefore

$$\left\|x_n^{(i)} \ominus x_k^{(i)}\right\|_b < b^{-\tau_m} \le \kappa_m \quad \text{and} \quad \left\|\mathbf{x}_n \ominus \mathbf{x}_k\right\|_b \ge \varkappa_m.$$

By (2.3) we have a contradiction. Thus

$$\operatorname{card}\{n \in [0, b^m) \mid \mathbf{x}_{n+b^m A} \in B \setminus B_i\} \le 1, \text{ and } |\Delta(B \setminus B_i, (\mathbf{x}_n)_{n=b^m A}^{b^m A+N-1})| \le 1.$$

Using (3.1), we get the assertion of Lemma 1.

Let $\beta_1, ..., \beta_{\kappa}$ be a F_p basis of \mathbb{F}_b , and let Tr be a standard trace function. Let

$$\omega(\alpha) = \sum_{j=1}^{\kappa} p^{j-1} \operatorname{Tr}(\alpha \beta_j), \qquad b = p^{\kappa}. \tag{3.2}$$

We use notations (1.5), (2.1) and (2.2). Let $n = \sum_{r\geq 0} a_r(n)b^r$ be the b-adic expansion of n, and let

$$\tilde{n} = \sum_{r \ge 0} \omega(\psi_r(a_r(n)))b^r. \tag{3.3}$$

Therefore

$$\{\tilde{n} \mid 0 \le n < b^m\} = \{0, 1, ..., b^m - 1\}.$$
 (3.4)

Hence

$$\psi_r(a_r(n)) = \omega^{-1}(a_r(\tilde{n})) \tag{3.5}$$

and

$$u_{\tilde{n},j}^{(i)} := \sum_{r>0} \omega^{-1}(a_r(\tilde{n}))c_{j,r}^{(i)} = \sum_{r>0} \psi_r(a_r(n))c_{j,r}^{(i)} = y_{n,j}^{(i)}, \quad 1 \le i \le s.$$
 (3.6)

Let

$$x_n^{(s+1)} = \{n/b^m\}, \ x_{n,j}^{(s+1)} = a_{m-j}(n), \ y_{n,j}^{(s+1)} = \psi_{m-j}(x_{n,j}^{(s+1)}).$$
 (3.7)

Bearing in mind that $\psi_{m-j}(a_{m-j}(n)) = \omega^{-1}(a_{m-j}(\tilde{n}))$, we put

$$u_{\tilde{n},j}^{(s+1)} := \omega^{-1}(a_{m-j}(\tilde{n})) = \psi_{m-j}(a_{m-j}(n)) = y_{n,j}^{(s+1)}, \quad j \in [1, m].$$
 (3.8)

Let

$$u_n^{(i)} = (u_{n,1}^{(i)},...,u_{n,\tau_m}^{(i)}) \in \mathbb{F}_b^{\tau_m} \quad \text{and} \quad u_n^{(s+1)} = (u_{n,1}^{(s+1)},...,u_{n,m}^{(s+1)}).$$

We abbreviate s+1-dimensional vectors $(u_n^{(1)},...,u_n^{(s+1)}), (k^{(1)},...,k^{(s+1)})$ and $(r^{(1)},...,r^{(s+1)})$ by symbols \mathbf{u}_n , \mathbf{k} and \mathbf{r} , and s-dimensional vectors $(u_n^{(1)},...,u_n^{(s)}), (k^{(1)},...,k^{(s)})$ by symbols \mathbf{u}_n and \mathbf{k} .

By (3.2) - (3.8), we get
$$u_n^{(s+1)} = u_{n+b^mA}^{(s+1)}, A = 1, 2, ...,$$

$$u_{n_1 \oplus n_2, j}^{(i)} = u_{n_1, j}^{(i)} + u_{n_2, j}^{(i)}, \ j \ge 1, i \in [1, s+1], \ \mathbf{u}_{n_1 \oplus n_2} = \mathbf{u}_{n_1} + \mathbf{u}_{n_2}.$$
 (3.9)

Let
$$N \in [1, b^m]$$
, $\gamma^{(s+1)} = N/b^m$, $k = \sum_{j=1}^{\tau_m} k_j b^{-j} > 0$, with $k_j \in \mathbb{Z}_b$,

$$v(k) := \max\{j \in [1, \tau_m] \mid k_j \neq 0\}, \quad v(0) = 0. \tag{3.10}$$

Similarly to [Ni, Theorem 3.10] (see also [DiPi, Lemma 14.8]), we consider the following Fourier series decomposition of the discrepancy function:

Lemma 2. Let $A \ge 1$ be an integer, $N \in [1, b^m]$, $\gamma^{(s+1)} = N/b^m$, and let $(\mathbf{x}_n)_{n>0}$ be a digital sequence in base b. Then

$$\Delta([\mathbf{0}, [\boldsymbol{\gamma}]_{\tau_m}), (\mathbf{x}_n)_{n=b^m A}^{b^m A+N-1})$$

$$= \sum_{n=0}^{b^m-1} \sum_{(k^{(1)},\dots,k^{(s)}) \in \mathbb{F}_b^{s\tau_m}} \sum_{k^{(s+1)} \in \mathbb{F}_b^m} E(\mathbf{k} \cdot \mathbf{u}_{n+b^m A}) \hat{1}(\mathbf{k}) - b^m \prod_{i=1}^{s+1} [\gamma^{(i)}]_{\tau_m},$$

where

$$\mathbf{k} \cdot \mathbf{u}_{n} = \sum_{i=1}^{s} \sum_{j=1}^{\tau_{m}} k_{j}^{(i)} u_{n,j}^{(i)} + \sum_{j=1}^{m} k_{j}^{(s+1)} u_{n,j}^{(s+1)}, \quad \hat{\mathbf{1}}(\mathbf{k}) = \prod_{i=0}^{s+1} \hat{\mathbf{1}}^{(i)}(k^{(i)}), \quad (3.11)$$

$$\hat{\mathbf{1}}^{(i)}(0) = [\gamma^{(1)}]_{\tau_m} \ (1 \le i \le s), \ \hat{\mathbf{1}}^{(s+1)}(0) = \gamma^{(s+1)} \quad \text{and}$$

$$\hat{1}^{(i)}(k) = b^{-v(k)} E\left(-\sum_{j=1}^{v(k)-1} k_j \eta_{i,j}^{-1}(\gamma_j^{(i)})\right) \left(\sum_{b=0}^{\gamma_{v(k)}^{(i)}-1} E(-k_{v(k)} \eta_{i,v(k)}^{-1}(b))\right)
+ E(-k_{v(k)} \eta_{i,v(k)}^{-1}(\gamma_{v(k)}^{(i)})) \{b^{v(k)} [\gamma^{(i)}]_{\tau_m}\}\right), \qquad i \in [1, s],
\hat{1}^{(s+1)}(k) = b^{-v(k)} E\left(-\sum_{j=1}^{v(k)-1} k_j \psi_j(\gamma_j^{(s+1)})\right) \left(\sum_{b=0}^{\gamma_{v(k)}^{(s+1)}-1} E(-k_{v(k)} \psi_{v(k)}(b))\right)
+ E(-k_{v(k)} \psi_{v(k)}(\gamma_{v(k)}^{(s+1)})) \{b^{v(k)} \gamma^{(s+1)}\}\right).$$
(3.12)

Proof. Let $\gamma = \sum_{j=1}^{m} \gamma_j b^{-j} > 0$, $w = \sum_{j=1}^{m} w_j b^{-j}$, with $\gamma_j, w_j \in \mathbb{Z}_b$. It is easy to verify (see also [Ni, p. 37,38]) that

$$1_{[0,\gamma)}(w) = \sum_{r=1}^{\dot{m}} \sum_{b=0}^{\gamma_r - 1} \prod_{j=1}^{r-1} \delta(w_i = \gamma_i) \delta(w_r = b).$$

By (2.2) and (3.6), we have that

$$x_{j,n}^{(i)} = b \iff y_{j,n}^{(i)} = \eta_{i,j}^{-1}(b) \iff u_{j,\tilde{n}}^{(i)} = \eta_{i,j}^{-1}(b),$$

and

$$\begin{split} \mathbf{1}_{[0,[\gamma^{(i)}]_{\tau_m})}(x_n^{(i)}) &= \sum_{r=1}^{\tau_m} \sum_{b=0}^{\gamma_r^{(i)}-1} \prod_{j=1}^{r-1} \delta(x_{j,n}^{(i)} = \gamma_j^{(i)}) \delta(x_{r,n}^{(i)} = b) \\ &= \sum_{r=1}^{\tau_m} \sum_{b=0}^{\gamma_r^{(i)}-1} \prod_{j=1}^{r-1} \delta(y_{j,n}^{(i)} = \eta_{i,j}^{-1}(\gamma_j^{(i)})) \delta(y_{r,n}^{(i)} = \eta_{i,r}^{-1}(b)) \\ &= \sum_{r=1}^{\tau_m} \sum_{b=0}^{\gamma_r^{(i)}-1} \prod_{j=1}^{r-1} \delta(u_{j,\tilde{n}}^{(i)} = \eta_{i,j}^{-1}(\gamma_j^{(i)})) \delta(u_{r,\tilde{n}}^{(i)} = \eta_{i,r}^{-1}(b)), \quad i = 1, \dots, s. \end{split}$$

Similarly, we derive

$$1_{[0,\gamma^{(s+1)}]}(x_n^{(s+1)}) = \sum_{r=1}^m \sum_{b=0}^{\gamma_r^{(s+1)}-1} \prod_{j=1}^{r-1} \delta(u_{j,\tilde{n}}^{(s+1)} = \psi_j^{-1}(\gamma_j^{(s+1)})) \delta(u_{r,\tilde{n}}^{(s+1)} = \psi_r^{-1}(b)).$$
(3.13)

Let $k \cdot u_{\tilde{n}}^{(i)} = \sum_{j=1}^{\tau_m} k_j u_{\tilde{n},j}^{(i)}$. By (2.5), we have

$$1_{[0,[\gamma^{(i)}]_{\tau_m})}(x_n^{(i)}) = \sum_{r=1}^{\tau_m} \sum_{b=0}^{\gamma_r^{(i)}-1} b^{-r} \sum_{k_1,\dots,k_r \in \mathbb{F}_b} \dot{1}^{(i)}(k), \quad \text{where}$$

$$\dot{\mathbf{I}}^{(i)}(k) = E\Big(\sum_{i=1}^{r-1} k_j (u_{j,\tilde{n}}^{(i)} - \eta_{i,j}^{-1}(\gamma_j^{(i)})) + k_r (u_{r_i,\tilde{n}}^{(i)} - \eta_{i,r}^{-1}(b))\Big) = E(k \cdot u_{\tilde{n}}^{(i)}) \tilde{\mathbf{I}}^{(i)}(k)$$

with
$$\tilde{\mathbf{I}}^{(i)}(k) = E\left(-\sum_{j=1}^{r-1} k_j \eta_{i,j}^{-1}(\gamma_j^{(i)}) - k_r \eta_{i,r}^{-1}(b)\right).$$
 (3.14)

Hence

$$\begin{split} &\mathbf{1}_{[0,[\gamma^{(i)}]_{\tau_m})}(x_n^{(i)}) = \sum_{r=1}^{\tau_m} \sum_{b=0}^{\gamma_r^{(i)}-1} b^{-r} \sum_{k_1,\dots,k_{\tau_m} \in \mathbb{F}_b} \delta(v(k) \leq r) E(k \cdot u_{\tilde{n}}^{(i)}) \tilde{\mathbf{1}}^{(i)}(k) \\ &= \sum_{k_1,\dots,k_{\tau_m} \in \mathbb{F}_b} \sum_{r=1}^{\tau_m} \sum_{b=0}^{\gamma_r^{(i)}-1} b^{-r} \delta(v(k) \leq r) E(k \cdot u_{\tilde{n}}^{(i)}) \tilde{\mathbf{1}}^{(i)}(k) \\ &= \sum_{k_1,\dots,k_{\tau_m} \in \mathbb{F}_b} E(k \cdot u_{\tilde{n}}^{(i)}) \ddot{\mathbf{1}}^{(i)}(k), \quad \text{where} \quad \ddot{\mathbf{1}}^{(i)}(k) = \sum_{r=v(k)} \sum_{b=0}^{\tau_m} \sum_{b=0}^{\gamma_r^{(i)}-1} b^{-r} \tilde{\mathbf{1}}^{(i)}(k). \end{split}$$

Applying (3.12) and (3.14), we derive

$$\ddot{\mathbf{I}}^{(i)}(k) = \sum_{r=v(k)}^{\tau_m} \sum_{b=0}^{\gamma_r^{(i)}-1} b^{-r} E\left(-\sum_{j=1}^{r-1} k_j \eta_{i,j}^{-1}(\gamma_j^{(i)}) - k_r \eta_{i,r}^{-1}(b)\right)$$

$$= \sum_{b=0}^{\gamma_{v(k)}^{(i)}-1} b^{-v(k)} E\left(-\sum_{j=1}^{v(k)-1} k_j \eta_{i,j}^{-1}(\gamma_j^{(i)}) - k_{v(k)} \eta_{i,v(k)}^{-1}(b)\right) + E\left(-\sum_{j=1}^{v(k)-1} k_j \eta_{i,j}^{-1}(\gamma_j^{(i)})\right) \sum_{r=v(k)+1}^{\tau_m} \sum_{b=0}^{\gamma_r^{(i)}-1} b^{-r}$$

$$= b^{-v(k)} E\left(-\sum_{j=1}^{v(k)-1} k_j \eta_{i,j}^{-1}(\gamma_j^{(i)})\right) \left(\sum_{b=0}^{\gamma_{v(k)}^{(i)}-1} E\left(-k_{v(k)} \eta_{i,v(k)}^{-1}(b)\right) + E\left(-k_{v(k)} (\eta_{i,v(k)}^{-1}(\gamma_{v(k)}^{(i)}))\right) \left\{b^{v(k)}[\gamma]_{\tau_m}^{(i)}\right\} = \hat{\mathbf{I}}^{(i)}(k).$$

Hence

$$1_{[0,[\gamma^{(i)}]_{\tau_m})}(x_n^{(i)}) = \sum_{k_1,\dots,k_{\tau_m} \in \mathbb{F}_h} E(k \cdot u_{\tilde{n}}^{(i)}) \hat{1}^{(i)}(k).$$

Similarly, we obtain from (3.12) and (3.13) that

$$1_{[0,\gamma^{(s+1)}]}(x_n^{(s+1)}) = \sum_{k_1,\dots,k_m \in \mathbb{F}_h} E(k \cdot u_{\tilde{n}}^{(s+1)}) \hat{1}^{(s+1)}(k).$$

Using (3.11), we obtain

$$\prod_{i=1}^{s+1} 1_{[0,[\gamma^{(i)}]_{\tau_m})}(x_n^{(i)}) = \sum_{(k^{(1)},\dots,k^{(s)})\in\mathbb{F}_b^{\tau_m}} \sum_{k^{(s+1)}\in\mathbb{F}_b^m} E(\mathbf{k}\cdot\mathbf{u}_{\tilde{n}})\hat{1}(\mathbf{k}).$$
(3.15)

Bearing in mind that $x_n^{(s+1)} = \{n/b^m\}$ and $\gamma^{(s+1)} = N/b^m$, we have

$$\mathbf{x}_{n+b^mA} \in [\mathbf{0}, [\boldsymbol{\gamma}]_{\tau_m}), n \in [0, N) \iff (\mathbf{x}_{n+b^mA}, x_{n+b^mA}^{(s+1)}) \in [\mathbf{0}, [\boldsymbol{\gamma}]_{\tau_m}) \times [0, \gamma^{(s+1)}).$$

From (3.15) and (1.1), we derive

$$\Delta([\mathbf{0}, [\boldsymbol{\gamma}]_{\tau_m}), (\mathbf{x}_n)_{n=b^m A}^{b^m A+N-1}) = \sum_{n=0}^{b^m-1} \prod_{i=1}^{s+1} 1_{[\mathbf{0}, [\boldsymbol{\gamma}^{(i)}]_{\tau_m})} (x_{n+b^m A}^{(i)}) - b^m \prod_{i=1}^{s+1} [\boldsymbol{\gamma}^{(i)}]_{\tau_m}$$

$$= \sum_{n=0}^{b^m-1} \sum_{(k^{(1)},\dots k^{(s)}) \in \mathbb{F}^{\tau_m}} \sum_{k^{(s+1)} \in \mathbb{F}^m} E(\mathbf{k} \cdot \mathbf{u}_{\widetilde{n+b^m}A}) \prod_{i=1}^{s+1} \widehat{\mathbf{1}}^{(i)}(k^{(i)}) - b^m \prod_{i=1}^{s+1} [\gamma^{(i)}]_{\tau_m}.$$

Hence Lemma 2 is proved. ■

Let

$$\mathbf{k} = (k^{(1)}, ..., k^{(s+1)}), \ k^{(i)} = (k_1^{(i)}, ..., k_{\tau_m}^{(i)}), i \in [1, s], \ k^{(s+1)} = (k_1^{(s+1)}, ..., k_m^{(s+1)}), k^{(s+1)} = (k_1$$

$$G_m = \{ \mathbf{k} \mid k_j^{(i)} \in \mathbb{F}_b \text{ with } j \in [1, \tau_m], i \in [1, s], \text{ and } j \in [1, m] \text{ for } i = s + 1 \},$$

$$G_m^* = G_m \setminus \{\mathbf{0}\}, \text{ and let}$$

$$D_m = \{ \mathbf{k} \in G_m \mid \mathbf{k} \cdot \mathbf{u}_n = 0 \ \forall \ n \in [0, b^m - 1] \}, \quad D_m^* = D_m \setminus \{ \mathbf{0} \}.$$
 (3.16)

It is easy to see that

$$\mu \mathbf{k} \in D_m^* \quad \text{for all} \quad \mu \in \mathbb{F}_b^*, \ \mathbf{k} \in D_m^*.$$
(3.17)

Lemma 3. Let $(\mathbf{x}_n)_{n\geq 0}$ be a digital sequence in base b. Then

$$\Delta([\mathbf{0}, [\boldsymbol{\gamma}]_{\tau_m}), (\mathbf{x}_n)_{n=b^m A}^{b^m A+N-1}) = \sum_{\mathbf{k} \in G_m^*} \hat{\mathbf{1}}(\mathbf{k}) \sum_{n=0}^{b^m -1} E(\mathbf{k} \cdot \mathbf{u}_n + \mathbf{k} \cdot \mathbf{u}_{\widetilde{b^m A}}).$$
(3.18)

Proof. By (3.12) we have $\hat{1}(\mathbf{0}) = \prod_{i=1}^{s+1} [\gamma^{(i)}]_{\tau_m}$. Applying Lemma 2, we get

$$\Delta([\mathbf{0},[\boldsymbol{\gamma}]_{\tau_m}),(\mathbf{x}_n)_{n=b^mA}^{b^mA+N-1}) = \sum_{\mathbf{k}\in G_m^*} \hat{\mathbf{1}}(\mathbf{k}) \sum_{n=0}^{b^m-1} E(\mathbf{k}\cdot \mathbf{u}_{\widetilde{n+b^m}A}).$$

Using (3.3), (3.6) and (3.9), we obtain

$$\widetilde{n+b^mA}=\widetilde{n}+\widetilde{b^mA}=\widetilde{n}\oplus\widetilde{b^mA}\quad\text{and}\quad \mathbf{u}_{\widetilde{n+b^mA}}=\mathbf{u}_{\widetilde{n}}+\mathbf{u}_{\widetilde{b^mA}}$$

Now from (3.4), we get (3.18). Hence Lemma 3 is proved.

Lemma 4. Let $(\mathbf{x}_n)_{n\geq 0}$ be a digital sequence in base b. Then

$$\sigma := \sum_{n=0}^{b^m - 1} E(\mathbf{k} \cdot \mathbf{u}_n) = b^m \delta(\mathbf{k} \in D_m). \tag{3.19}$$

Proof. Using (3.6) and (3.8) and (3.11), we have

$$\mathbf{k} \cdot \mathbf{u}_{\tilde{n}} = \sum_{i=1}^{s} \sum_{j=1}^{\tau_{m}} \sum_{r=0}^{m-1} k_{j}^{(i)} \psi_{r}(a_{r}(n)) c_{j,r}^{(i)} + \sum_{j=1}^{m} k_{j}^{(s+1)} \psi_{m-j}(a_{m-j}(n))$$

$$= \sum_{i=1}^{m-1} \psi_{r}(a_{r}(n)) \left(\sum_{i=1}^{s} \sum_{j=1}^{\tau_{m}} k_{j}^{(i)} c_{j,r}^{(i)} + k_{m-r}^{(s+1)} \right) = \sum_{j=1}^{m-1} f_{r} \xi_{r},$$

where

$$f_r = \psi_r(a_r(n)) \in \mathbb{F}_b$$
 and $\xi_r = \sum_{i=1}^s \sum_{j=1}^{\tau_m} k_j^{(i)} c_{j,r}^{(i)} + k_{m-r}^{(s+1)}$. (3.20)

By (3.4), (1.5) and (2.5), we obtain

$$\sigma = \sum_{\tilde{n}=0}^{b^m-1} E(\mathbf{k} \cdot \mathbf{u}_{\tilde{n}}) = \sum_{f_0, \dots, f_{m-1} \in \mathbb{F}_b} E(\sum_{r=0}^{m-1} f_r \xi_r) = b^m \prod_{r=0}^{m-1} \delta(\xi_r = 0).$$

Now from (3.16), we get that $\mathbf{k} \in D_m$ and Lemma 4 follows.

Let

$$\Lambda_{m} = \{ \mathbf{k} = (k^{(1)}, ..., k^{(s+1)}) \in G_{m} \mid k^{(s+1)} = \mathbf{0} \},
g_{\mathbf{w}} = \{ A \ge 1 \mid y_{b^{m}A, j}^{(i)} = w_{j}^{(i)}, i \in [1, s], j \in [1, \tau_{m}] \}, \quad \rho_{\mathbf{w}} = 0 \text{ for } g_{\mathbf{w}} = \emptyset,
\rho_{\mathbf{w}} = \min_{A \in g_{\mathbf{w}}} A \text{ for } g_{\mathbf{w}} \ne \emptyset, \qquad M_{m} = \{ \rho_{\mathbf{w}} \mid \mathbf{w} \in \Lambda_{m} \}.$$
(3.21)

We consider the following conditions:

$$g_{\mathbf{w}} \neq \emptyset \quad \text{for all} \quad \mathbf{w} \in \Lambda_m$$
 (3.22)

and

$$\sigma_{1} := \frac{1}{\operatorname{card}(R_{m})} \sum_{A \in R_{m}} |\Delta([\mathbf{0}, [\boldsymbol{\gamma}]_{\tau_{m}}), (\mathbf{x}_{n})_{n=b^{m}A}^{b^{m}A+N-1})|^{2} = \sum_{\mathbf{k} \in D_{m}^{*}} b^{2m} |\hat{\mathbf{1}}(\mathbf{k})|^{2}$$
(3.23)

for some finite set R_m .

Bearing in mind (3.6), we get

$$g_{\mathbf{w}} = \{ A \ge 1 \mid u_{\widetilde{b^m A}}^{(i)} = w^{(i)}, \ i \in [1, s] \}, \text{ where } w^{(i)} = (w_1^{(i)}, ..., w_{\tau_m}^{(i)}).$$
 (3.24)

Lemma 5. Let $(\mathbf{x}_n)_{n\geq 0}$ be a weakly admissible uniformly distributed digital (\mathbf{T}, s) -sequence in base b, satisfying to (3.22) for all $m \geq m_0$ with some $m_0 \geq 1$. Then (3.23) is true for $R_m = M_m$.

Proof. By (3.21) and (2.5), we obtain

$$\frac{1}{b^{s\tau_m}} \sum_{\mathbf{w} \in \Lambda_m} E(\mathbf{k} \cdot \mathbf{w}) = \frac{1}{b^{s\tau_m}} \sum_{w_j^{(i)} \in \mathbb{F}_b, \ i \in [1, s], \ j \in [1, \tau_m]} E\left(\sum_{i=1}^s \sum_{j=1}^{\tau_m} k_j^{(i)} w_j^{(i)}\right)$$

$$= \prod_{i=1}^{s} \prod_{j=1}^{\tau_m} \delta(k_j^{(i)} = 0) = \prod_{i=1}^{s} \delta(k^{(i)} = 0), \quad \text{where} \quad \mathbf{k} \in G_m.$$
 (3.25)

Using (3.18), we derive

$$|\Delta([\mathbf{0}, [\boldsymbol{\gamma}]_{\tau_m}), (\mathbf{x}_n)_{n=b^m A}^{b^m A+N-1})|^2$$
(3.26)

$$= \sum_{\dot{\mathbf{k}},\ddot{\mathbf{k}} \in G_m^*} \widehat{1}(\dot{\mathbf{k}}) \overline{\widehat{1}(\ddot{\mathbf{k}})} \sum_{\dot{n},\ddot{n}=0}^{b^m-1} E(\dot{\mathbf{k}} \cdot \mathbf{u}_{\dot{n}} + \dot{\mathbf{k}} \cdot \mathbf{u}_{\widetilde{b^m A}} - \ddot{\mathbf{k}} \cdot \mathbf{u}_{\ddot{n}} - \ddot{\mathbf{k}} \cdot \mathbf{u}_{\widetilde{b^m A}}).$$

It is easy to see that if condition (3.22) is true, than $\operatorname{card}(M_m) = b^{s\tau_m}$ and $\{\mathbf{u}_{\widetilde{bm_A}} \mid A \in M_m\} = \Lambda_m$.

Applying (3.21), (3.25), (3.26) and (3.23) with $R_m = M_m$, we have

$$\begin{split} \sigma_1 &= \sum_{\dot{\mathbf{k}}, \ddot{\mathbf{k}} \in G_m^*} \hat{\mathbf{l}}(\dot{\mathbf{k}}) \overline{\hat{\mathbf{l}}(\ddot{\mathbf{k}})} \sum_{\dot{n}, \ddot{n} = 0}^{b^m - 1} b^{-s\tau_m} \sum_{A \in M_m} E(\dot{\mathbf{k}} \cdot \mathbf{u}_{\dot{n}} - \ddot{\mathbf{k}} \cdot \mathbf{u}_{\ddot{n}} + (\dot{\mathbf{k}} - \ddot{\mathbf{k}}) \cdot \mathbf{u}_{\widetilde{b^m A}}) \\ &= \sum_{\dot{\mathbf{k}}, \ddot{\mathbf{k}} \in G_m^*} \hat{\mathbf{l}}(\dot{\mathbf{k}}) \overline{\hat{\mathbf{l}}(\ddot{\mathbf{k}})} \sum_{\dot{n}, \ddot{n} = 0}^{b^m - 1} E(\dot{\mathbf{k}} \cdot \mathbf{u}_{\dot{n}} - \ddot{\mathbf{k}} \cdot \mathbf{u}_{\ddot{n}}) b^{-s\tau_m} \sum_{\mathbf{w} \in \Lambda_m} E(\dot{\mathbf{k}} - \ddot{\mathbf{k}}) \cdot \mathbf{w}) \\ &= \sum_{\dot{\mathbf{k}}, \ddot{\mathbf{k}} \in G_m^*} \hat{\mathbf{l}}(\dot{\mathbf{k}}) \overline{\hat{\mathbf{l}}(\ddot{\mathbf{k}})} \sum_{\dot{n}, \ddot{n} = 0}^{b^m - 1} E(\dot{\mathbf{k}} \cdot \mathbf{u}_{\dot{n}} - \ddot{\mathbf{k}} \cdot \mathbf{u}_{\ddot{n}}) \prod_{i = 1}^{s} \delta(\dot{k}^{(i)} = \ddot{k}^{(i)}). \end{split}$$

Let $\ddot{n} = \ddot{n} \ominus \dot{n}$. From (1.5), we obtain $\{\ddot{n} \mid 0 \le \ddot{n} < b^m\} = \{0, 1, ..., b^m - 1\}$. By (3.3) - (3.9), we get $\mathbf{u}_{\ddot{n}} = \mathbf{u}_{\ddot{n}} - \mathbf{u}_{\dot{n}}$. Hence

$$\sigma_1 = \sum_{\dot{\mathbf{k}}, \ddot{\mathbf{k}} \in G_{\infty}^*} \hat{1}(\dot{\mathbf{k}}) \overline{\hat{1}(\ddot{\mathbf{k}})} \sum_{\dot{n}, \ddot{n} = 0}^{b^m - 1} E((\dot{\mathbf{k}} - \ddot{\mathbf{k}}) \cdot \mathbf{u}_{\dot{n}} - \ddot{\mathbf{k}} \cdot \mathbf{u}_{\ddot{n}}) \prod_{i=1}^{s} \delta(\dot{k}^{(i)} - \ddot{k}^{(i)}).$$

We get $\dot{\mathbf{k}} - \ddot{\mathbf{k}} = (0, ..., 0, \dot{k}^{(s+1)} - \ddot{k}^{(s+1)}).$ From (3.8), we have $u_{\tilde{n},j}^{(s+1)} = \omega^{-1}(a_{m-j+1}(n))$ and

$$(\dot{\mathbf{k}} - \ddot{\mathbf{k}}) \cdot \mathbf{u}_{\tilde{n}} = (\dot{k}^{(s+1)} - \ddot{k}^{(s+1)}) \cdot u_{\tilde{n}}^{(s+1)} = \sum_{j=1}^{m} (\dot{k}_{j}^{(s+1)} - \ddot{k}_{j}^{(s+1)}) \omega^{-1}(a_{m-j}(\tilde{n})).$$

Taking into account (2.5), we get

$$\sum_{\dot{n}=0}^{b^m-1} E((\dot{\mathbf{k}} - \ddot{\mathbf{k}}) \cdot \mathbf{u}_{\dot{n}}) = \sum_{\tilde{n}=0}^{b^m-1} E((\dot{\mathbf{k}} - \ddot{\mathbf{k}}) \cdot \mathbf{u}_{\tilde{n}}) = b^m \prod_{j=1}^m \delta(\dot{k}_j^{(s+1)} = \ddot{k}_j^{(s+1)}).$$

Hence $\dot{\mathbf{k}} = \ddot{\mathbf{k}}$. Using Lemma 4, we obtain

$$\sigma_1 = b^m \sum_{\dot{\mathbf{k}} \in G_m^*} |\hat{\mathbf{l}}(\dot{\mathbf{k}})|^2 \sum_{\ddot{n} = 0}^{b^m - 1} E(-\dot{\mathbf{k}} \cdot \mathbf{u}_{\ddot{n}}) = \sum_{\mathbf{k} \in D_m^*} b^{2m} |\hat{\mathbf{l}}(\mathbf{k})|^2.$$

Therefore Lemma 5 is proved.

Let Ψ be a set of all bijections $\dot{\psi}: \mathbb{Z}_b \to \mathbb{F}_b$, $\psi \in \Psi$, $k \in \mathbb{F}_b$, $c \in \mathbb{Z}_b$, $A_{k,c,\psi} = E(-k\psi(c)) \sum_{b=0}^{c-1} E(k\psi(b))$, $\langle x \rangle = \min(\{x\}, 1 - \{x\})$, $x \in [0,1]$ and let

$$B_{k,c,\psi}(x) = \sum_{b=0}^{c-1} E(k\psi(b)) + E(k\psi(c))x = E(k\psi(c)/q)(A_{k,c,\psi} + x). \quad (3.27)$$

By (3.12) and (3.27), we get

$$b^{v(\dot{k})}|\hat{1}^{(s+1)}(\mu\dot{k})| = |B_{\tilde{k},\tilde{c},\psi_{1}}(x_{1})| \quad \text{and} \quad b^{v(\ddot{k})}|\hat{1}^{(s)}(\mu\ddot{k})| = |B_{\check{k},\check{c},\psi_{2}}(x_{2})|.$$
with $\tilde{k} = -\mu\dot{k}_{v(\dot{k})}$, $\check{k} = -\mu\ddot{k}_{v(\ddot{k})}$, $\tilde{c} = \gamma_{v(\dot{k})}^{(s+1)}$, $\check{c} = \gamma_{v(\ddot{k})}^{(s)}$, $\psi_{1} = \psi_{v(\dot{k})}^{-1}$, $\psi_{2} = \eta_{s,v(\ddot{k})}^{-1}$, $x_{1} = \{b^{v(\dot{k})}\gamma^{(s+1)}\}$, $x_{2} = \{b^{v(\ddot{k})}[\gamma^{(s)}]_{\tau_{m}}\}$.

Lemma 6. With the notations as above, there exist $a_1, ..., a_{b+7} \in \mathbb{Z}_b$, $a_1^2 + ... + a_{b+5}^2 > 0$, $a_{b+6} = a_{b+7} = 0$ such that

$$\left| B_{k,c,\psi} \left(\sum_{j=1}^{b+7} \frac{\mathsf{a}_j}{b^j} + \frac{y}{b^{b+7}} \right) \right| \ge b^{-b-7}, \quad \forall \ k \in \mathbb{F}_b, c \in \mathbb{Z}_b, y \in [0,1], \psi \in \Psi \quad (3.29)$$

and

$$\sum_{k \in \mathbb{F}_b^*} |B_{k,c,\psi}(x)|^2 \ge b^{-2r} \quad \forall \ c \in \mathbb{Z}_b, \quad \text{where} \quad \langle x \rangle \ge b^{-r}, \ r \ge 1.$$
 (3.30)

Proof. Let

$$\dot{A} = \{\theta_{k,c,\psi} := \operatorname{Re}(A_{k,c,\psi}) \mid k \in \mathbb{F}_b, \ c \in \mathbb{Z}_b, \ \psi \in \Psi\}.$$

Taking into account that $\operatorname{card}(\Psi) = b!$, we get $\operatorname{card}(\dot{A}) \leq b!b^2 + 2 < b^{b+4}$. Let

$$\ddot{A} = \{\mathbf{a} = (\mathbf{a}_1, ..., \mathbf{a}_{b+7}) \in \mathbb{Z}_b^{b+7} \mid \mathbf{a}_1^2 + ... + \mathbf{a}_{b+5}^2 > 0, \quad \mathbf{a}_{b+6} = \mathbf{a}_{b+7} = 0\}$$

and let $z_{\bf a} = {\sf a}_1/b + \cdots + {\sf a}_{b+7}/b^{b+7}$. By (3.27), we derive

$$|B_{k,c,\psi}(x)| = |A_{k,c,\psi} + x| \ge |\text{Re}(A_{k,c,\psi}) + x|.$$

Suppose that (3.29) is not true. Then for all $\mathbf{a} \in \ddot{A}$ there exist $k(\mathbf{a}), c(\mathbf{a}), \psi(\mathbf{a})$ and $y(\mathbf{a})$ such that

$$b^{-b-7} > \left| B_{k(\mathbf{a}),c(\mathbf{a}),\psi(\mathbf{a})} \left(\sum_{j=1}^{b+7} \frac{\mathsf{a}_j}{b^j} + \frac{y(\mathbf{a})}{b^{b+7}} \right) \right| \ge \left| \theta_{k(\mathbf{a}),c(\mathbf{a}),\psi(\mathbf{a})} + z_{\mathbf{a}} + \frac{y(\mathbf{a})}{b^{b+7}} \right|.$$

Hence $|\theta_{k(\mathbf{a}),c(\mathbf{a}),\psi(\mathbf{a})} + z_{\mathbf{a}}| < b^{-b-6}$. Suppose that $\theta_{k(\mathbf{a}_1),c(\mathbf{a}_1),\psi(\mathbf{a}_1)} = \theta_{k(\mathbf{a}_2),c(\mathbf{a}_2),\psi(\mathbf{a}_2)}$ for some $\mathbf{a}_1, \mathbf{a}_2 \in \ddot{A}, \ \mathbf{a}_1 \neq \mathbf{a}_2$. Hence $|z_{\mathbf{a}_1} - z_{\mathbf{a}_2}| < b^{-b-5}$. Bearing in mind that $|z_{\mathbf{a}_1} - z_{\mathbf{a}_2}| \geq b^{-b-5}$ for all $\mathbf{a}_1 \neq \mathbf{a}_2$, we get a contradiction. Therefore $\theta_{k(\mathbf{a}_1),c(\mathbf{a}_1),\psi(\mathbf{a}_1)} \neq \theta_{k(\mathbf{a}_2),c(\mathbf{a}_2),\psi(\mathbf{a}_2)}$ for all $\mathbf{a}_1, \mathbf{a}_2 \in \ddot{A}, \ \mathbf{a}_1 \neq \mathbf{a}_2$. Thus $\operatorname{card}(\dot{A}) > \operatorname{card}(\ddot{A})$. Hence

$$b^{b+4} > \operatorname{card}(\dot{A}) \ge \operatorname{card}(\ddot{A}) = b^{b+5} - 1 > b^{b+4}$$

We have a contradiction. Therefore (3.29) is true.

Now we consider assertion (3.30). If c = 0, then $|B_{k,c,\psi}(x)| = |x|$ and (3.30) follows.

Now let $c \in \{1, ..., b-1\}$. By (3.27), we have

$$|B_{k,c,\psi}(x)|^2 = |A_{k,c,\psi}|^2 + x(A_{k,c,\psi} + \overline{A_{k,c,\psi}}) + x^2.$$
(3.31)

Using (2.5), we get

$$\sum_{k \in \mathbb{F}_b^*} |A_{k,c,\psi}|^2 = -c^2 + \sum_{b_1,b_2=0}^{c-1} \sum_{k \in \mathbb{F}_b} E(k(\psi(b_1) - \psi(b_2)))$$

$$= -c^2 + b \sum_{b_1, b_2=0}^{c-1} \delta(\psi(b_1) = \psi(b_2)) = -c^2 + bc.$$

Taking into account that $\psi(0) = 0$, we obtain

$$\sum_{k \in \mathbb{F}_b^*} A_{k,c,\psi} = -c + \sum_{b=0}^{c-1} \sum_{k \in \mathbb{F}_b} E(k\psi(b)) = -c + b \sum_{b=0}^{c-1} \delta(\psi(b) = 0) \ge b - c.$$

Now from (3.31), we derive

$$\sum_{k \in \mathbb{F}_b^*} |B_{k,c,\psi}(x)|^2 \ge c(b-c) + x^2 + 2x(b-c) \ge x^2$$

and (3.30) follows. Thus Lemma 6 is proved.

Applying (3.28) - (3.30), we have

Corollary. Let $a_1, ..., a_{b+7}$ be integers chosen in Lemma 6 and let $\gamma_{v(k)+j}^{(s+1)} = a_j$, j = 1, ..., b+7, with some \dot{k} . Then

$$|\hat{\mathbf{l}}^{(s+1)}(\mu \dot{k})| \ge b^{-v(\dot{k})-b-7} \quad \forall \ \mu \in \mathbb{F}_b^*$$
 (3.32)

and

$$\sum_{\mu \in \mathbb{F}_b^*} |\hat{\mathbf{1}}^{(i)}(\mu \ddot{k})|^2 \ge b^{-2v(\ddot{k}) - 2\dot{r}}, \text{ where } \langle b^{v(\ddot{k})} \gamma^{(i)} \rangle \ge b^{-\dot{r}}.$$
 (3.33)

Lemma 7. Let $(\mathbf{x}_n)_{n\geq 0}$ be a digital sequence in base b and let $\rho \in [2, m-2]$ be an integer. Then there exists $\mathbf{k} \in D_m^*$ such that $k^{(1)} = ... = k^{(s-1)} = 0$, $k_{v(k^{(s)})}^{(s)} = 1$, $1 \leq v(k^{(s)}) \leq \rho - 1$ and $v(k^{(s+1)}) \leq m - \rho + 2$.

Proof. From (3.3)-(3.8), (3.16) and (3.20), we get that $\mathbf{k} \in D_m^*$ if and only if

$$\sum_{i=1}^{s} \sum_{j=1}^{\tau_m} k_j^{(i)} c_{j,r}^{(i)} + k_{m-r}^{(s+1)} = 0, \quad \text{for all} \quad r = 0, 1, ..., m - 1.$$
 (3.34)

We put $k^{(1)} = \dots = k^{(s-1)} = 0$, $k_j^{(s)} = 0$, for $j \ge \rho$ and $k_j^{(s+1)} = 0$, for $j > m - \rho + 2$. Hence (3.34) is true if and only if

$$k_{m-r}^{(s+1)} = -\sum_{j=1}^{\rho-1} k_j^{(s)} c_{j,r}^{(s)} \text{ for } r = 0, 1, ..., m-1, \quad k_{m-r}^{(s+1)} = 0 \text{ for } m-r > m-\rho+2.$$

$$(3.35)$$

Therefore, in order to obtain the statement of the lemma, it is sufficient to show that there exists a nontrivial solution of the following system of linear equations

$$\sum_{j=1}^{\rho-1} k_j^{(s)} c_{j,r}^{(i)} + k_{m-r}^{(s+1)} \delta(m-r \le m-\rho+2) = 0, \quad r = 0, ..., m-1.$$
 (3.36)

In this system, we have m+1 unknowns $k_1^{(s)},...,k_{\rho-1}^{(s)},k_1^{(s+1)},...,k_{m-\rho+2}^{(s+1)}$ and m linear equations. Hence there exists a nontrivial solution of (3.36). By (3.36),

we get that if $k^{(s)} = 0$, then $k^{(s+1)} = 0$. Hence $k^{(s)} \neq 0$ and $1 \leq v(k^{(s)}) \leq \rho - 1$. Taking into account that if $\mathbf{k} \in D_m$ then $\mu \mathbf{k} \in D_m$ for all $\mu \in \mathbb{F}_b^*$. Therefore there exists $\mathbf{k} \in D_m^*$ such that $k_{v(k^{(s)})}^{(s)} = 1$ and $1 \leq v(k^{(s)}) \leq \rho - 1$. Thus Lemma 7 is proved.

Proposition. Let $(\mathbf{x}_n)_{n\geq 0}$ be a weakly admissible uniformly distributed digital (\mathbf{T}, s) -sequence in base b, satisfying to (3.22) for all $m \geq m_0 \geq 1$. Then $[0, \gamma_1) \times ... \times [0, \gamma_s)$ is of bounded remainder with respect to $(\mathbf{x}_n)_{n\geq 0}$ if and only if (1.9) is true.

Proof. The sufficient part of the Theorem and of the Proposition follows directly from the definition of (\mathbf{T}, s) sequence and Lemma B. We will consider only the necessary part of the Theorem and of the Proposition.

Suppose that (1.9) does not true. Then

$$\max_{1 \le i \le s} \operatorname{card} \{ j \ge 1 \mid \gamma_j^{(i)} \ne 0 \} = \infty.$$

Let, e.g.,

$$\operatorname{card}\{j \ge 1 \mid \gamma_j^{(s)} \ne 0\} = \infty.$$

Let

$$W = \{ j \ge 1 \mid \gamma_i^{(s)} \in \{1, ..., b - 2\} \text{ or } \gamma_i^{(s)} = b - 1, \ \gamma_{i+1}^{(s)} = 0 \}.$$
 (3.37)

Bearing in mind that $\{j \geq 1 \mid \gamma_l^{(s)} = b - 1 \,\forall \, l > j\} = \emptyset$, we obtain that $\operatorname{card}(W) = \infty$.

Suppose that there exists H>0 such that $b^{2H}c_1>4H^2,\,c_1=\gamma_0^2b^{-4b-36},$

$$|\Delta([\mathbf{0}, \boldsymbol{\gamma}), (\mathbf{x}_n)_{n=M}^{M+N-1})| \le H - s \quad \text{for all} \quad M \ge 0, \ N \ge 1,$$
 (3.38)

with
$$[\mathbf{0}, \boldsymbol{\gamma}) = [0, \gamma_1) \times \cdots \times [0, \gamma_s), \ \gamma_0 = \gamma_1 \gamma_2 \cdots \gamma_{s-1}.$$

Let $W = \{\dot{w}_j \mid \dot{w}_i < \dot{w}_j \text{ for } i < j, \ j = 1, 2, ...\}$ and let

$$r(1) = \dot{w}_1, \ r(j+1) = \min(\dot{w}_k \in W \mid \dot{w}_k \ge r(j) + H^2), \ j = 1, 2, \dots$$
 (3.39)

We choose m and J from the following conditions

$$m = r(J) + b + 10, \ 2 \prod_{i=1}^{s-1} [\gamma_i]_{\tau_m} \ge \prod_{i=1}^{s-1} \gamma_i = \gamma_0, \ J \ge H^2 b^{2b+30} \gamma_0^{-2}, \ m \ge m_0.$$
(3.40)

Applying Lemma 1 and (3.38), we have

$$|\Delta([\mathbf{0}, [\boldsymbol{\gamma}]_{\tau_m}), (\mathbf{x}_n)_{n=b^m A}^{b^m A+N-1})| \le H \qquad \forall \ A \ge 0, \ N \in [1, b^m].$$
 (3.41)

By Lemma 7, we get that there exists a sequence $(\mathbf{k}(j))_{i=1}^{J}$ such that

$$\mathbf{k}(j) \in D_m^*, \qquad k^{(1)}(j) = \dots = k^{(s-1)}(j) = 0, \quad k_{v(k^{(s)}(j))}^{(s)}(j) = 1,$$

$$v(k^{(s)}(j)) \le r(j) - 1, \qquad v(k^{(s+1)}(j)) \le m - r(j) + 2, \quad j \in [1, J].$$
 (3.42)

We see that the sequence $(\mathbf{k}(j))_{j=1}^J$ does not depend on $\gamma^{(s+1)}$.

Using (3.37) and (3.39), we obtain $\gamma_{r(j)}^{(s)} \neq 0$. Hence

$$\langle b^{v(k^{(s)}(j))} \gamma^{(s)} \rangle = \gamma_{v(k^{(s)}(j))+1}^{(s)} \cdots \gamma_{r(j)}^{(s)} \cdots \ge b^{v(k^{(s)}(j))-r(j)-2}, \tag{3.43}$$

j = 1, ..., J. Let $H_1 = \{1, 2, ..., J\}$ if

$$|v(k^{(s+1)}(j)) - v(k^{(s+1)}(j_1))| \ge b + 8 \tag{3.44}$$

for all $1 \le j < j_1 \le J$, and let $H_1 = \{j\}$ if there exist $1 \le j < j_1 \le J$ such that (3.44) is false. Let $a_1, ..., a_{b+7}$ be integers chosen in Lemma 6 and let

$$N = b^m \gamma^{(s+1)} \quad \text{with} \quad \gamma^{(s+1)} = \sum_{j \in H_1} \sum_{\nu=1}^{b+7} \mathsf{a}_{\nu} b^{\nu + v(k^{(s+1)}(j))}. \tag{3.45}$$

From Lemma 5, (3.21), (3.23), (3.41) and conditions of the Proposition, we have

$$H^2 \ge \sigma_1 = \sum_{\mathbf{k} \in D_m^*} b^{2m} |\hat{\mathbf{l}}(\mathbf{k})|^2.$$
 (3.46)

Taking into account (3.17), (3.42) and (3.44), we get that if $\mathbf{k}(j) \in D_m$ then $\mu \mathbf{k}(j) \in D_m$ for $\mu \in \mathbb{F}_b^*$, and if $j_1, j_2 \in H_1$, $j_1 \neq j_2$, then $\mu_1 \mathbf{k}(j_1) \neq \mu_2 \mathbf{k}(j_2)$ for all $\mu_1, \mu_2 \in \mathbb{F}_b^*$.

According to (3.11), (3.12) and (3.46), we have

$$\sigma_1 \ge \sum_{\mu \in \mathbb{F}_b^*} \sum_{j \in H_1} b^{2m} |\hat{1}(\mu \mathbf{k}(j))|^2$$

$$= ([\gamma_1]_{\tau_m} \cdots [\gamma_{s-1}]_{\tau_m})^2 \sum_{\mu \in \mathbb{F}_b^*} \sum_{j \in H_1} b^{2m} |\hat{\mathbf{1}}^{(s)}(\mu k^{(s)}(j))|^2 |\hat{\mathbf{1}}^{(s+1)}(\mu k^{(s+1)}(j))|^2.$$

From Corollary and (3.45), we obtain

$$|\hat{1}^{(s+1)}(\mu k^{(s+1)}(j))|^2 \ge b^{-2v(k^{(s+1)}(j))-2b-14}$$
 for all $\mu \in \mathbb{F}_b^*$, $j \in H_1$.

By (3.43), we can apply Corollary with $\dot{r} = r(j) - v(k^{(s)}(j)) + 2$. Hence

$$\sum_{\mu \in \mathbb{F}_b^*} |\hat{\mathbf{I}}^{(s)}(\mu k^{(s)}(j))|^2 \ge b^{-2v(k^{(s)}(j)) - 2(r(j) - v(k^{(s)}(j)) + 2)} = b^{-2r(j) - 4}, \quad j \in H_1.$$
(3.47)

Using (3.46)-(3.47) and (3.40), we obtain

$$4H^2 \ge 4\sigma_1 \ge \sigma_1 \gamma_0^2 ([\gamma_1]_{\tau_m} \cdots [\gamma_{s-1}]_{\tau_m})^{-2} \ge \gamma_0^2 \sum_{j \in H_1} \sum_{\mu \in \mathbb{F}_b^*} |\hat{1}^{(s+1)}(\mu k^{(s)}(j))|^2$$

$$\times b^{2m-2v(k^{(s+1)}(j))-2b-14} \ge \gamma_0^2 \sum_{j \in H_1} b^{2m-2r(j)-2v(k^{(s+1)}(j))-2b-18}. \tag{3.48}$$

Suppose that $card(H_1) = J$. From (3.40) and (3.42), we get

$$4H^2 \ge 4\sigma_1 \ge \gamma_0^2 \sum_{j=1}^J b^{2m-2r(j)-2v(k^{(s+1)}(j))-2b-18} \ge \gamma_0^2 J b^{-2b-22} > 4H^2. \quad (3.49)$$

We have a contradiction. Now let $card(H_1) = 1$.

By (3.44), we obtain that there exist $j, j_1 \in [1, J]$ such that $j \in H_1, j < j_1$ and $|v(k^{(s+1)}(j)) - v(k^{(s+1)}(j_1))| \le b + 7$.

According to (3.39) and (3.42), we have

$$r(j) + v(k^{(s+1)}(j)) \le r(j_1) - H^2 + v(k^{(s+1)}(j_1)) + b + 7 \text{ and } m - r(j)$$
$$-v(k^{(s+1)}(j)) \ge m - r(j_1) - v(k^{(s+1)}(j_1)) + H^2 - b - 7 \ge H^2 - b - 9.$$

Applying (3.38), (3.41) and (3.48), we get

$$4H^2 \geq 4\sigma_1 \geq \gamma_0^2 b^{2m-2r(j)-2v(k^{(s+1)}(j))-2b-18} \geq \gamma_0^2 b^{2H^2-4b-36} = b^{2H^2} c_1 > 4H^2,$$

with $c_1 = \gamma_0^2 b^{-4b-36}$. We have a contradiction. By (3.49), the Proposition is proved.

Completion of the proof of the Theorem. By Lemma A, $S(\mathbf{L}^{(m)})$ is a uniformly distributed digital (T, s)-sequence in base b.

By Theorem A, we get that $1, L_1, ..., L_s$ are linearly independent over $\mathbb{F}_b[z]$. Hence $1, z^m L_1, ..., z^m L_s$ are linearly independent over $\mathbb{F}_b[z]$. Let $\mathbf{L}^{(m)} = (z^m L_1, ..., z^m L_s)$, and let $S(\mathbf{L}^{(m)}) = (\mathbf{l}_n^{(m)})_{n \geq 0}$ (see (1.8)) with

$$\mathbf{l}_{n}^{(m)} = (l_{n}^{(m,1)}, ..., l_{n}^{(m,s)}), \quad l_{n}^{(m,i)} = \eta^{(i)}(n(z)z^{m}L_{i}(z)), \quad \text{for} \quad 1 \le i \le s, \ n \ge 0.$$

Using Theorem A, we obtain that $S(\mathbf{L}^{(m)})$ is a uniformly distributed sequence in $[0,1)^s$. Therefore, for all $\mathbf{w} \in \Lambda_m$ there exists an integer $A \geq 1$ with

$$l_{b^m A, j}^{(m,i)} = \eta_j^{(i)}(w_j^{(i)})$$
 for $1 \le i \le s, \ 1 \le j \le \tau_m$.

Thus $S(\mathbf{L}^{(m)})$ satisfies the condition (3.22).

Bearing in mind that $1, L_1, ..., L_s$ are linearly independent over $\mathbb{F}_b[z]$, we get that $\{n(z)L_i\} \neq 0$ for all $n \geq 1$. Hence $\{l^{(i)}(n)\} \neq 0$ for all $n \geq 1$ (i = 1, ..., s). Therefore the sequence $S(\mathbf{L})$ is weakly admissible.

Applying the Proposition, we get the assertion of the Theorem.

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Address: Department of Mathematics, Bar-Ilan University, Ramat-Gan, 5290002, Israel

E-mail: mlevin@math.biu.ac.il