Función de autocorrelación Parcial (PACF)

→ Objetivo de la PACF:

Encontrar $\mathrm{Corr}(X_t, X_{t+K} \mid \overset{\uparrow f}{\underline{X_{t+1}, X_{t+2}, \dots, X_{t+K-1}}})$

Consideremos un proceso estacionario $\{X_t\}_{t=1}^{-}$, asumir sin pérdida de generalidad que $\mathbb{E}[X_t]=0$

Podemos obtener un estimador lineal de X_{t+K} en función a $X_{t+1}, X_{t+2}, \dots, X_{t+K-1}$

$$\hat{X}_{t+K} = \alpha_1 X_{t+K-1} + \alpha_2 X_{t+K-2} + \alpha_3 X_{t+K-3} + \dots + \alpha_{K-1} X_{t+1}$$

↳ Cómo encontramos $\alpha_1, \alpha_2, \dots, \alpha_{K-1}$

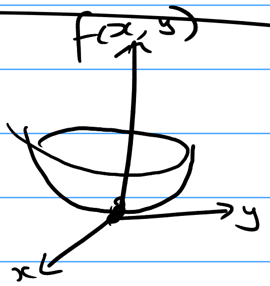
Podemos encontrar: $\min\limits_{\alpha_1, \alpha_2, \dots, \alpha_{K-1}} \mathbb{E}[(X_{t+K} - \hat{X}_{t+K})^2]$

$$\min_{\Theta} f(x, \Theta) \begin{cases} \dfrac{\partial f}{\partial \theta_1} = 0 \\ \dfrac{\partial f}{\partial \theta_2} = 0 \; \dots \; \dfrac{\partial f}{\partial \theta_n} = 0 \end{cases}$$

$\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$

$$\min_{x,y} f(x,y) = x^2 + y^2 \hookrightarrow \frac{\partial f}{\partial x} = 2x = 0 \iff x = 0$$

$$\frac{\partial f}{\partial y} = 2y = 0 \iff y = 0$$



$$f = \mathbb{E}[(X_{t+K} - \hat{X}_{t+K})^2] = \mathbb{E}[(X_{t+K} - \alpha_1 X_{t+K-1} - \alpha_2 X_{t+K-2} \dots - \alpha_{K-1} X_{t+1})^2]$$

$$\Rightarrow \frac{\partial f}{\partial \alpha_1} = \mathbb{E}[2(X_{t+K} - \alpha_1 X_{t+K-1} - \alpha_2 X_{t+K-2} - \dots - \alpha_{K-1} X_{t+1}) \cdot (-X_{t+K-1})$$

$$\frac{\partial f}{\partial \alpha_1} = \mathbb{E}[2(X_{t+K} - \alpha_1 X_{t+K-1} - \alpha_2 X_{t+K-2} - \dots - \alpha_{K-1} X_{t+1})(-X_{t+}]$$

$$= \cancel{2}\, \mathbb{E}[-X_{t+K} X_{t+K-1} + \alpha_1 X_{t+K-1} X_{t+K-1} + \alpha_2 X_{t+K-2} X_{t+K-1} + \dots + \alpha_{K-1} X_{t+1} X_{t+K-1}] = 0$$

$$\Leftrightarrow -\mathbb{E}[X_{t+K} X_{t+K-1}] + \alpha_1 \mathbb{E}[X_{t+K-1} X_{t+K-1}] + \alpha_2 \mathbb{E}[X_{\cancel{t+K}-2} X_{t+K-1}] + \dots + \alpha_{K-1} \mathbb{E}[X_{t+1} X_{t+K-1}] = 0$$

$(t+K-1) - (t+1)$

$\cancel{t}+K-1-\cancel{t}-1 \Rightarrow \underline{K-2}$

$$\left(\text{si} \quad \mathbb{E}[X] = 0,\ \mathbb{E}[Y] = 0 \Rightarrow \text{Cov}(X,Y) = \mathbb{E}[XY]\right)$$

$$-\gamma(1) + \alpha_1 \gamma(0) + \alpha_2 \gamma(1) + \dots + \alpha_{K-1} \gamma(K-2) = 0$$

$$\gamma(1) = \alpha_1 \gamma(0) + \alpha_2 \gamma(1) + \alpha_3 \gamma(2) + \dots , \gamma_{K-1} \gamma(K-2)$$

$$\underset{\overset{\parallel}{0}}{\frac{\partial f}{\partial \alpha_2}}, \underset{\overset{\parallel}{0}}{\frac{\partial f}{\partial \alpha_3}}, \underset{\overset{\parallel}{0}}{\frac{\partial f}{\partial \alpha_4}}, \dots, \underset{\overset{\parallel}{0}}{\frac{\partial f}{\partial \alpha_{K-1}}}$$

Si nosotros hacemos lo mismo para α_i

$$\gamma(i) = \alpha_1 \gamma(i-1) + \alpha_2 \gamma(i-2) + \dots + \alpha_{k-1} \gamma(i-k+1)$$

Podemos encontrar $\rho(i) = \dfrac{\gamma(i)}{\gamma(0)} \quad \forall i$

$$\rho(\underset{2}{i}) = \alpha_1 \rho(\underset{1}{i-1}) + \alpha_2 \rho(\underset{0}{i-2}) + \dots + \alpha_{k-1} \rho(\underset{2-k+1}{i-k+1})$$

$\hookrightarrow 3-k$

$\hookrightarrow \underline{\rho(k-3)}$

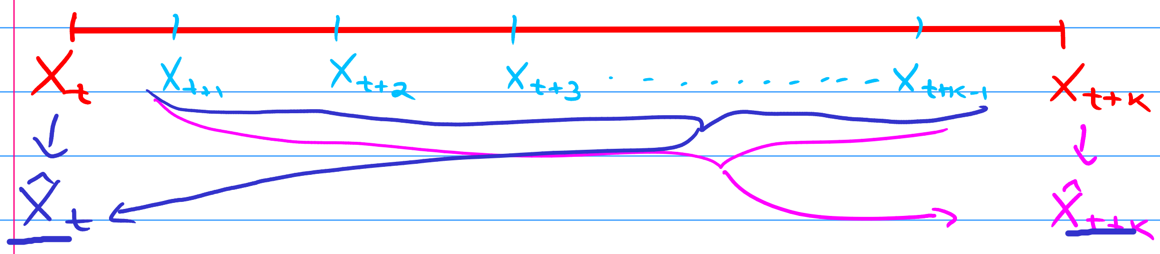
En términos matriciales:

$$\begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \vdots \\ \rho_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \rho_3 \dots & \rho_{k-2} \\ \rho_1 & 1 & \rho_1 & \rho_2 \dots & \rho_{k-3} \\ \rho_2 & \rho_1 & 1 & \rho_1 \dots & \rho_{k-4} \\ \vdots & & & & \\ \rho_{k-2} & \rho_{k-3} & \rho_{k-4} \dots & \dots & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{k-1} \end{bmatrix} \Bigg\}$$

Para llegar a esta expresión, recordar que $\rho(h) = \rho(-h)$

Ya con esto, encontramos $\{\alpha_1, \alpha_2, \dots, \alpha_{k-1}\}$ tales que $\hat{X}_{t+k}$ es el mejor estimador lineal de X_{t+k}

Ahora, vamos a encontrar el mejor estimador lineal de X_t $(\hat{X}_t)$ utilizando $X_{t+1}, X_{t+2}, \dots, X_{t+k-1}$



$$\hat{X}_t = \beta_1 X_{t+1} + \beta_2 X_{t+2} + \beta_3 X_{t+3} + \dots + \beta_{K-1} X_{t+K-1}$$

$$\min_{\beta_1,\dots,\beta_{K-1}} \mathbb{E}[(X_t - \hat{X}_t)^2]$$

$$\swarrow$$

$$\min_{\beta_1,\dots,\beta_n} \mathbb{E}[\overbrace{(X_t - \beta_1 X_{t+1} - \beta_2 X_{t+2} - \dots - \beta_{K-1} X_{t+K-1})^2}^{f}]$$

Encontremos $\dfrac{\partial f}{\partial \beta_i}$, $i \in [1, K-1]$

$$\frac{\partial f}{\partial \beta_i} = \mathbb{E}[2(X_t - \beta_1 X_{t+1} - \beta_2 X_{t+2} - \dots - \beta_{K-1} X_{t+K-1})(-X_{t+i})] = 0$$

$$(\cancel{t}+i) - (\cancel{t}+K-1) = i-K+1$$

$$\mathbb{E}[X_t X_{t+i}] = \beta_1 \mathbb{E}[X_{t+1} X_{t+i}] + \beta_2 \mathbb{E}[X_{t+2} X_{t+i}] + \dots \beta \mathbb{E}[X_{t+K-1} X_{t+i}]$$

$$\gamma(i) = \beta_1 \gamma(i-1) + \beta_2 \gamma(i-2) + \beta_{K-1} \gamma(i-K+1)$$

$$\rho(i) = \beta_1 \rho(i-1) + \beta_2 \rho(i-2) + \dots + \beta_{K-1} \rho(i-K+1)$$

$$\begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{K-1} \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{K-2} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{K-3} \\ \vdots & & & & \\ \rho_{K-2} & \rho_{K-3} & \rho_{K-4} & \cdots & \rho_1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{K-1} \end{bmatrix}$$

Vemos que $\alpha_i = \beta_i$, para $i = 1, 2, \dots, K-1$

Dado lo anterior podemos decir que
la PACF entre X_t y X_{t+K}

↓

$$\text{Corr}\left(X_t, X_{t+K} \mid X_{t+1}, X_{t+2} \cdots X_{t+k}\right)$$

→ Es igual a la correlación entre

$$X_t - \hat{X}_t \quad \text{y} \quad X_{t+K} - \hat{X}_{t+K}$$

$$\text{PACF}(X_t, X_{t+K}) = \text{Corr}\left(X_t - \hat{X}_t, X_{t+K} - \hat{X}_{t+K}\right)$$