Función de autocorrelación Parcial

$\rightarrow \text{Corr}\left(X_t, X_{t+K} \mid X_{t+1}, X_{t+2}, \ldots, X_{t+K-1}\right)$

$$\hat{X}_{t+K} = \alpha_1 X_{t+K-1} + \alpha_2 X_{t+K-2} + \ldots + \alpha_{K-1} X_{t+1}$$

$$\hat{X}_{t} = \beta_1 X_{t+1} + \beta_2 X_{t+2} + \ldots + \beta_{K-1} X_{t+K-1}$$

Resolviendo
$$\left\{ \begin{array}{l} \min\limits_{\alpha_1,\ldots,\alpha_{K-1}} \mathbb{E}\left[(X_{t+K} - \hat{X}_{t+K})^2\right] \checkmark \\ \min\limits_{\beta_1,\ldots,\beta_{K-1}} \mathbb{E}\left[(X_t - \hat{X}_t)^2\right] \checkmark \end{array} \right\}$$
Tenemos que $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$, $\vdots$, $\beta_{K-1} = \alpha_{K-1}$

Lo único que tenemos que hacer para encontrar la PACF es calcular:

$$P_K = \frac{\left.\text{Cov}\left(X_t - \hat{X}_t, X_{t+K} - \hat{X}_{t+K}\right)\right\} C}{\underbrace{\sqrt{\text{Var}(X_t - \hat{X}_t)}}_{A}\ \underbrace{\sqrt{\text{Var}(X_{t+K} - \hat{X}_{t+K})}}_{B}}$$

$$\text{Var}(X_{t+K} - \hat{X}_{t+K}) = \mathbb{E}\left[(X_{t+K} - \alpha_1 X_{t+K-1} - \alpha_2 X_{t+K-2} - \ldots - \alpha_{K-1} X_{t+1})^2\right]$$

$$= \mathbb{E}\left[X_{t+K}(X_{t+K} - \alpha_1 X_{t+K-1} - \alpha_2 X_{t+K-2} - \ldots - \alpha_{K-1} X_{t+1})\right]$$

$$\left.\begin{array}{l} - \alpha_1 \mathbb{E}\left[X_{t+K-1}(X_{t+K} - \hat{X}_{t+K})\right] \\ - \alpha_2 \mathbb{E}\left[X_{t+K-2}(X_{t+K} - \hat{X}_{t+K})\right] \\ \quad \vdots \\ - \alpha_{K-1} \mathbb{E}\left[X_{t+1}(X_{t+K} - \hat{X}_{t+K})\right] \end{array}\right\} (*)$$

Vemos que $\circledast = \alpha_1 \mathbb{E}[X_{t+k-1}(X_{t+k} - \hat{X}_{t+k})]$ $\to 0$

$= \alpha_1 \mathbb{E}[X_{t+k-1} X_{t+k} - [\alpha_1 X_{t+k-1} + \alpha_2 X_{t+k-2} + \dots + \alpha_{k-1} X_{t+1}] X_{t+k-1}]$

$\cancel{t}+k-1-\cancel{t}-1$

$= \alpha_1 [\gamma(1) - \alpha_1 \gamma(0) - \alpha_2 \gamma(1) - \alpha_3 \gamma(2) - \dots - \alpha_{k-1} \gamma(k-2)$

$\searrow$ Peeero!!!!

$\alpha_1, \alpha_2, \dots, \alpha_{k-1}$ son tales que:

$\gamma(1) = \alpha_1 \gamma(0) + \alpha_2 \gamma(1) + \alpha_3 \gamma(2) + \dots + \alpha_{k-1} \gamma(k-2)$

De lo anterior, podemos concluir que el valor de (B) es:

$B = \sqrt{\mathbb{E}[X_{t+k}(X_{t+k} - \hat{X}_{t+k})]}$

Para (A):

$Var(X_t - \hat{X}_t) = \mathbb{E}[(X_t - \hat{X}_t)^2]$

$\mathbb{E}[(X_t - \beta_1 X_{t+1} - \beta_2 X_{t+2} \dots - \beta_{k-1} X_{t+k-1})^2] =$

$= \mathbb{E}[X_t(X_t - \hat{X}_t)] -$
$\underbrace{\beta_1 \mathbb{E}[X_{t+1}(X_t - \hat{X}_t)]}_{\circledast} - \beta_2 \mathbb{E}[X_{t+2}(X_t - \hat{X}_t)] - \dots$
$\beta_{k-1} \mathbb{E}[X_{t+k-1}(X_t - \hat{X}_t)^2]$

$\circledast = \mathbb{E}[X_{t+1} X_t] - \beta_1 \mathbb{E}[X_{t+1} X_{t+1}] - \beta_2 \mathbb{E}[X_{t+1} X_{t+2}] -$
$\dots \beta_{k-1} \mathbb{E}[X_{t+1} X_{t+k-1}]$

$= \gamma(1) - \beta_1 \gamma(0) - \beta_2 \gamma(1) - \dots - \beta_{k-1} \gamma(k-2)$

Entonces $A = \mathbb{E}[X_t(X_t - \hat{X}_t)]$

(La demostración completa, la encuentran en "Time Series Analysis, Univariate & Multivariate Methods", "William, W.S. Wei")

$\Rightarrow$ Resulta que $A = B$

$\Rightarrow$ Para el cálculo de $Cov(X_t - \hat{X}_t, X_{t+k} - \hat{X}_{t+k})$

$$= \gamma_k - \alpha_1 \gamma_{k-1} - \dots - \alpha_{k-1} \gamma_1$$

El denominador $(\sqrt{A} \cdot \sqrt{B})$

$$= \gamma_0 - \alpha_1 \gamma_1 - \dots - \alpha_{k-1} \gamma_{k-1}$$

$$\boxed{P_k = \frac{\rho_k - \alpha_1 \rho_{k-1} - \alpha_2 \rho_{k-2} - \dots - \alpha_{k-1} \rho_1}{1 - \alpha_1 \rho_1 - \alpha_2 \rho_2 - \dots - \alpha_{k-1} \rho_{k-1}}}$$

Lo que tenemos que hacer para encontrar la PACF, es resolver el sistema de ecuaciones para $\alpha_1, \alpha_2, \dots, \alpha_{k-1}$, y sustituirlos en P_k.

$$
\alpha_1 = \frac{\begin{vmatrix} 1 & \rho_1 & \cdots & \rho_1 & \rho_{k-2} \\ \rho_1 & 1 & \cdots & \rho_2 & \rho_{k-3} \\ \vdots & \vdots & & \vdots & \vdots \\ \rho_{k-2} & \rho_{k-3} & & \rho_{k-1} & 1 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_{i-1} & \rho_{k-2} \\ \rho_1 & 1 & \rho_{i-2} \cdots & \rho_{k-3} \\ \vdots & \vdots & \vdots & \vdots \\ \rho_{k-2} & \rho_{k-3} & \rho_{k-i-1} & 1 \end{vmatrix}}
$$

Fórmula simplificada para calcular la PACF (Durbin)

Vamos a definir a $\phi_{k,k} = P_k$

$$
\boxed{\begin{cases} \phi_{1,1} = \rho_1 \\ \phi_{k+1,k+1} = \dfrac{\rho_{k+1} - \sum_{j=1}^{k} \phi_{kj}\, \rho_{k+1-j}}{1 - \sum_{j=1}^{k} \phi_{k,j}\, \rho_j} \\ \phi_{k+1,j} = \phi_{k,j} - \phi_{k+1,k+1}\, \phi_{k,k+1-j} \end{cases}}
$$

Ejemplo: Consideremos un proceso $AR(1)$

$X_t = \phi X_{t-1} + Z_t, \quad Z_t \sim WN(0, \sigma^2)$

Sabemos que $\rho(k) = \phi^k$

Calculemos la PACF $(\phi_{k,k})$

$\phi_{1,1} = \rho(1) = \phi$

$$\phi_{2,2} = \frac{\rho(2) - \sum_{j=1}^{1} \phi_{1,j}\,\rho(2-j)}{1 - \sum_{j=1}^{1} \phi_{1,j}\,\rho(j)}$$

$$= \frac{\phi^2 - \phi_{1,1}\,\rho(1)}{1 - \phi_{1,1}\,\rho(1)} = \frac{\phi^2 - \phi \cdot \phi}{1 - \phi \cdot \phi}$$

$$= \frac{\phi^2 - \phi^2}{1 - \phi^2} = 0$$

$$\phi_{2,1} = \phi_{1,1} - \cancelto{0}{\phi_{2,2}}\,\phi_{1,1}$$

$$\phi_{2,1} = \phi_{1,1} = \rho$$

$$\phi_{3,3} = \frac{\phi^3 - \sum_{j=1}^{2} \phi_{2,j}\,\rho(3-j)}{\underbrace{1 - \sum_{j=1}^{2} \phi_{2,j}\,\rho(j)}_{D}} = \frac{\phi^3 - \phi_{2,1}\,\rho(2) - \cancelto{0}{\phi_{2,2}}\,\rho(1)}{D}$$

$$= \frac{\phi^3 - \phi \cdot \phi^2}{D} = 0$$

Ej:

Demostrar que para un proceso AR(1),

$\phi_{k,k} = 0, \quad k > 1$ (Inducción)