→ Definición de un ruido blanco.

→ Procesos estacionarios

→ Ejemplos de procesos estacionarios

a) $\{X_t\}_{t=1}^{\infty}$, donde $X_t \sim N(0,1)$ para $t=1,2,\dots$

$$\mathbb{E}[X_t]=0,\ Var(X_t)=1 \quad \forall t$$

$$\gamma(t,s)=\mathbb{E}[X_t X_s]=\begin{cases} 1 & si\ t=s \\ 0 & si\ t\neq s \end{cases}$$

$$\rho(t,s)=\frac{\gamma(t,s)}{\gamma(t,t)}=\begin{cases} 1 & si\ t=s \\ 0 & si\ t\neq s \end{cases}$$

b) Sea $\{S_t\}_{t=1}^{\infty}$ una caminata aleatoria, es decir, $S_t=X_1+X_2+X_3+\dots+X_t$, o

$$S_t=\sum_{i=1}^{t} X_i, \quad \text{donde } X_i \sim WN(0,\sigma^2)$$

$$\mathbb{E}[S_t]=0=\mathbb{E}\left[\sum_{i=1}^{t} x_i\right]=\sum_{i=1}^{t}\mathbb{E}[X_i] \xrightarrow{} 0$$

$$Var(S_t)=Var\left(\sum_{i=1}^{t} X_i\right)=\sum_{i=1}^{t} Var(X_i)=\sum_{i=1}^{t}\sigma^2$$

$$\boxed{=t\sigma^2}$$

→ Depende explícitamente de t.

$$S_t + X_{t+1} + X_{t+2} + \dots + X_{t+h}$$

$$\gamma(t+h, t) = Cov(S_{t+h}, S_t)$$

$$= Cov\left(S_t + \sum_{i=1}^{h} X_{t+i}, S_t\right)$$

$$= \underbrace{Cov(S_t, S_t)}_{Var(S_t)} + \underbrace{Cov\left(\sum_{i=1}^{h} X_{t+i}, S_t\right)}_{0}$$

posteriores

$$= t\sigma^2$$

De aquí, concluimos que S_t no es un proceso estacionario.

→ Consideremos un proceso de medias móviles de orden 1, MA(1)

$$X_t = W_t + \theta W_{t-1}, \quad \text{donde } W_t \sim WN(0, \sigma^2)$$

$t = \pm 1, \pm 2, \dots\dots, \quad t \in \mathbb{Z}$

¿Es estacionario? $\theta \in \mathbb{R}$ $\quad X_t = W_{t-1} + \theta W_{t-2}$

$$\mu_t = \mathbb{E}[X_t] = \mathbb{E}[\cancel{W_t}]^{0} + \theta\,\mathbb{E}[\cancel{W_{t-1}}]^{0} = 0$$

$$Var(X_t) = Var(W_t) + \theta^2 Var(W_{t-1})$$

$$= \sigma^2 + \theta^2\sigma^2 = \sigma^2(1+\theta^2)$$

$$\gamma(t, t_{+h}) = Cov(X_t, X_{t+h})$$

* Recordemos que $X_t = W_t + \theta W_{t-1}$

- Cuando $h = 0 \Rightarrow Cov(X_t, X_{t+0}) = Var(X_t)$

$$= \sigma^2(1+\theta^2)$$

· Cuando $h = \pm 1$

$$Cov(X_t, X_{t+1}) = Cov(W_t + \Theta W_{t-1}, W_{t+1} + \Theta W_t)$$

$$= \underbrace{\cancel{Cov(W_t, W_{t+1})}}_{0} + Cov(W_t, \Theta W_t) + \underbrace{\cancel{Cov(\Theta W_{t-1}, W_{t+1})}}_{0} + \underbrace{\cancel{Cov(\Theta W_{t-1}, \Theta W_t)}}_{0}$$

$$= \Theta \, Var(W_t) = \boxed{\Theta \sigma^2}$$

$$Cov(X_t, X_{t-1}) = Cov(X_{s+1}, X_s) = \underline{\Theta \sigma^2}$$

$t = s+1 \rightarrow t-1 = s$ renombrar

Si $|h| > 1 \rightsquigarrow Cov(X_{t+h}, X_t)$

$$= Cov(W_{t+h} + \Theta W_{t+h-1}, W_t + \Theta W_{t-1})$$

$$= \underbrace{\cancel{Cov(W_{t+h}, W_t)}}_{0} + \Theta \underbrace{\cancel{Cov(W_{t+h-1}, W_t)}}_{0} + \Theta \underbrace{\cancel{Cov(W_{t+h}, W_{t-1})}}_{0} + \Theta^2 \underbrace{\cancel{Cov(W_{t+h-1}, W_{t-1})}}_{0} = 0$$

$$\boxed{\gamma_h = \begin{cases} \sigma^2(1+\Theta^2) & \text{si } h = 0 \\ \sigma^2 \Theta & \text{si } h = 1 \text{ o } h = -1 \\ 0 & \text{si } |h| > 1 \end{cases}}$$

$$Cov(X_t, X_s) = \mathbb{E}[(X_t - \mu_t)(X_s - \mu_s)]$$

$$= \mathbb{E}[X_t X_s] - \underbrace{\cancel{\mu_t}}_{0} \cdot \underbrace{\cancel{\mu_s}}_{0}$$

$$= \mathbb{E}[X_t X_s]$$

$$\mathrm{Cov}(\alpha X + \beta Y, Z), \quad X, Y, Z \text{ v.a's}$$
$$\alpha, \beta \in \mathbb{R}$$

$$= \alpha \, \mathrm{Cov}(X, Z) + \beta \, \mathrm{Cov}(Y, Z)$$

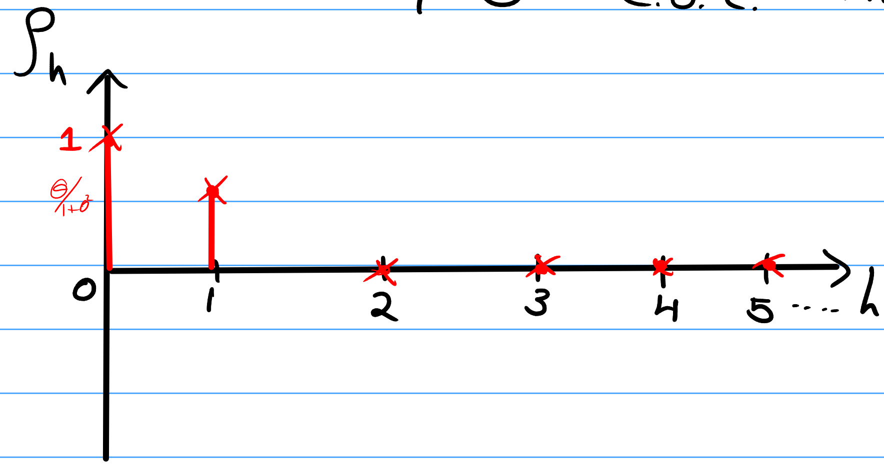
$$\mathbb{E}[(\alpha X + \beta Y) Z] - \mathbb{E}[\alpha X + \beta Y] \, \mathbb{E}[Z]$$

$$\alpha \mathbb{E}[XZ] + \beta \mathbb{E}[YZ] - \alpha \mathbb{E}[X]\mathbb{E}[Z] - \beta \mathbb{E}[Y]\mathbb{E}[Z]$$

$$= \alpha \underbrace{\left[\mathbb{E}[XZ] - \mathbb{E}[X]\mathbb{E}[Z] \right]}_{\alpha \cdot \mathrm{Cov}(X, Z)} + \beta \underbrace{\left[\mathbb{E}[YZ] - \mathbb{E}[Y]\mathbb{E}[Z] \right]}_{+\beta \cdot \mathrm{Cov}(Y, Z)}$$

$\rightarrow$ De aquí, concluimos que X_t es estacionario.

$$\rho_h = \frac{\gamma_h}{\gamma_0} = \begin{cases} 1 & h = 0 \\ \theta/(1+\theta^2) & h = \pm 1 \\ 0 & \text{e.o.c.} \quad |h| > 1 \end{cases}$$



$\rightarrow$ Proceso Autoregresivo de orden 1
(AR(1))

$$X_t = \varphi X_{t-1} + W_t, \quad \varphi \in \mathbb{R}, \quad W_t \sim WN(0, \sigma^2)$$

$$\text{Si } \varphi = 0.5 \Rightarrow X_t = 0.5 X_{t-1} + W_t$$

$$\mathbb{E}[X_t] = 0 \quad xq \quad \mathbb{E}[X_t] = \psi\, \mathbb{E}[X_{t-1}] + 0$$

$$\mathbb{E}[X_t] = \psi\, \mathbb{E}[X_{t-1}] \rightsquigarrow \mathbb{E}[X_{t-1}] = \psi\, \mathbb{E}[X_{t-2}]$$

$$\mathrm{Cov}(X_{t+h}, X_t) = \mathbb{E}[X_{t+h} X_t] - \cancel{\mathbb{E}[X_{t+h}]}_{\to 0}\, \cancel{\mathbb{E}[X_t]}_{\to 0}$$

$$\mathrm{Cov}(\psi X_{t+h-1} + W_t, X_t) = \psi\, \mathrm{Cov}(X_{t+h-1}, X_t) + \mathrm{Cov}\cancel{(W_t, X_t)}_{\to 0}$$

$$= \psi\, \gamma(h-1) = \psi\left[\mathrm{Cov}(X_{t+h-1}, X_t)\right]$$

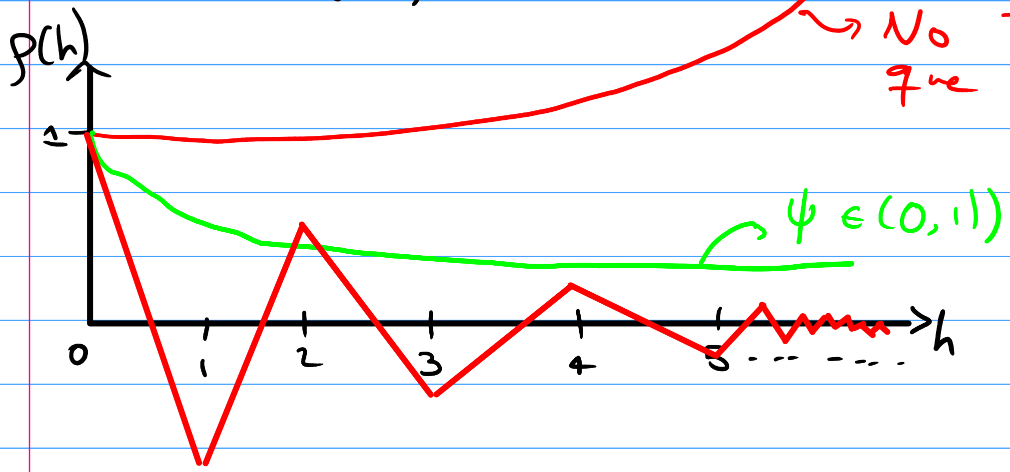
$$= \psi\left[\mathrm{Cov}(\psi X_{t+h-2} + W_t, X_t)\right] =$$

$$= \psi^2\, \mathrm{Cov}(X_{t+h-2}, X_t) + \psi\, \mathrm{Cov}\cancel{(W_t, X_t)}_{\to 0}$$

$$= \psi^2 \gamma(h-2) = \dots = \psi^h \gamma(0)$$

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \psi^h \longrightarrow \begin{cases} |\psi| > 1 \\ \text{o } \underline{|\psi| < 1} \end{cases}$$

No tiene sentido que $|\psi| > 1$



X_t es estacionaria

$$\rho(h) = \psi^h, \quad \text{para} \quad |\psi| < 1$$

¿Cómo calculemos $Var(X_t)$?

$$Var(X_t) = Cov(X_t, X_t) =$$
$$= Cov(\psi X_{t-1} + W_t, \psi X_{t-1} + W_t)$$
$$= Cov(\psi X_{t-1}, \psi X_{t-1}) + Cov(W_t, \psi X_{t-1})$$
$$+ Cov(\psi X_{t-1}, W_t) + Cov(W_t, W_t)$$
$$= \psi^2 Cov(X_{t-1}, X_{t-1}) + 0$$
$$+ 0 + \sigma^2$$

$$\Rightarrow Var(X_t) = \psi^2 Var(X_{t-1}) + \sigma^2$$
$$\gamma(0) = \psi^2 \gamma(0) + \sigma^2 \Leftrightarrow \gamma(0) = \frac{\sigma^2}{1-\psi^2}$$

→ Supongamos que tenemos $x_1, x_2, \ldots, x_n$ n observaciones de una serie de tiempo.

La media muestral se define como:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

La función muestral de autocovarianza se define como:

$$\hat{\gamma}(h) = \frac{1}{n}\sum_{t=1}^{n-h}(x_{t+h} - \bar{x})(x_t - \bar{x})$$

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}$$