AR(1)

$$x_t = \phi x_{t-1} + \omega_t, \quad \omega_t \sim WN(0, \sigma^2)$$

$$x_t = \phi x_{t-1} + \omega_t$$

ACF

$$\hookrightarrow \mathbb{E}[X_t X_{t-k}] = \mathbb{E}\left[\left[\phi^k x_{t-k} + \sum_{i=1}^{k} \phi^{k-1} \omega_{t-k}\right][X_{t-k}]\right] = \phi^k \gamma_k$$

$$\hookrightarrow \rho_k = \phi^k$$

PACF

$$\phi_{11} = \rho_1$$

$$\phi_{22} = \frac{\rho_2 - \sum_{i=1}^{1} \phi_{ij} \rho_{2-j}}{1 - \sum \phi_{ij} \rho_j} = \frac{\rho_2 - \rho_2}{\sim} = 0$$

$$\phi_{2,1} = \phi_{11} - \cancel{\phi_{22}}^{\,0} \phi_{11} = \phi_{11} = \phi^1$$

$$\phi_{3,1} = \frac{\rho_3 - \sum_{i=1}^{2} \phi_{2j} \hat{\rho}_{2+1-j}}{1 - \sum_{i=1}^{2} \phi_{2j} \rho_j} = \frac{\rho_3 - \phi_{21} \rho_1 - \phi_{22} \rho_2}{1 - \sum}$$

$$= \frac{\phi^3 - \phi_{21} \rho_2}{\sim} = 0$$

$$\therefore \phi_{kk} = 0 \quad \forall k > 1$$

$$\phi_{kj} = \phi_{kj}$$

$$\phi_{k+1\,k+1} = \frac{\rho_{k+1} - \sum_{j=1}^{k} \phi_{kj} \rho_{k+1-j}}{C} = \rho_{k+1} - \phi_{11} \rho_k = 0$$

$$\phi_{k1} \overbrace{\rho_k}^{\phi_{11}\rho_k} + \phi_{k2} \rho_{k-1} + \phi_{k3} \rho_{k-2} \dots + \cancel{\phi_{kk}}^{\,0} \rho_1$$

$$x_t = \phi w_{t-1} + w_t$$

$$\rho \quad -\frac{\theta_1}{1+\theta_1^2}$$

$$\phi_{11} = \rho_1$$

$$\phi_{22} = \frac{\rho_2 - \phi_{11}\rho_1}{1 - \phi_{11}\rho_1} = \frac{\rho_2 - \rho_1^2}{1-\rho_1^2} = = \frac{\left(-\frac{\theta_1}{1+\theta_1}\right)^2}{1-\left(\frac{-\theta_1}{1+\theta_1}\right)^2}$$

$$\theta_{kk} = \frac{-\theta_1^k(1-\theta_1^2)}{1-\theta_1^{2(k+1)}}$$

AR(1)

$x_t = \phi x_{t-1} + w_t$; $w_t \sim WN(0, \sigma_w^2)$; $\{w_t\}_{t=1}^{\infty}$; secuencia de va's. no correl.

$(w_t \sim N(0, \sigma^2)$, x_t)

$¿\mu_t, \gamma_0, \gamma_k, \rho_k \rightsquigarrow P_k = \phi_{k,k}$)

$|\phi| < 1 \rightsquigarrow \boxed{x_t = \sum_{j=1}^{\infty} \phi^j w_{t-j}}$ } Principio de causalidad

$$\mu_t = \mathbb{E}[X_t] = \mathbb{E}\left[\sum_{j=1}^{\infty} \phi^j w_{t-j}\right] \overset{|\phi|<1}{=} \sum_{j=1}^{\infty} \phi^j \mathbb{E}[w_{t-j}] = 0$$

Teorema de convergencia acotada

$$\gamma_k = \mathbb{E}[X_t \cdot X_{t-k}] = \mathbb{E}[(\phi X_{t-1} + w_{t-1})(X_{t-k})]$$

$$= \mathbb{E}[(\phi^2 X_{t-2} + \phi w_{t-2} + w_{t-1}) X_{t-k})$$

$\vdots$ k-veces

$$= \mathbb{E}\left[(\phi^k X_{t-k} + \sum_{j=1}^{k-1} \phi^{j-1} w_{t-j}) X_{t-k}\right]$$

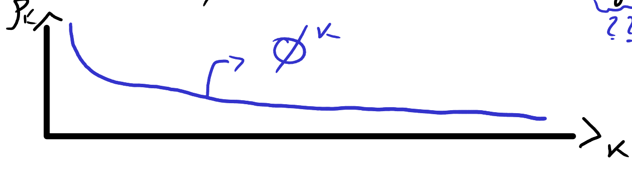
$\{\phi^0 w_{t-1}, \dots, \phi^{k-2} w_{t-(k-1)}\}$

$$= \mathbb{E}[\phi^k \overbrace{X_{t-k} X_{t-k}}^{\gamma_0} + \underbrace{\cancel{X_{t-k} \sum_{j=1}^{k-1} \phi^{j-1} w_{t-j}}}_{\to 0}]$$

$$= \mathbb{E}[\phi^k X_{t-k} X_{t-k}] = \phi^k \underbrace{\mathbb{E}[X_{t-k}^2]}_{\gamma_0} = \phi^k \gamma_0$$

$|\phi| < 1$

$\Rightarrow \gamma_k = \phi^k \gamma_0 \rightsquigarrow \rho_k = \dfrac{\gamma_k}{\underbrace{\gamma_0}_{???}} = \phi^k$ }



$\{z\}_{i=1}^{n}$ v.a. i.i.d $\rightsquigarrow Var(\sum z_i) = \sum Var(z_i)$

$|\phi| < 1$

$$\gamma_0 = \mathbb{E}[X_{t-k}^2] = \mathbb{E}\left[\left(\sum_{j=1}^{\infty} \phi^j w_{t-j}\right)^2\right] = Var\left(\sum_{j=1}^{\infty} \phi^j w_{t-j}\right)$$

$$= \sum_{j=1}^{\infty} \phi^{2j} Var(w_{t-j})$$

$$= \sum_{j=1}^{\infty} \phi^{2j} \sigma_w^2 = \sigma_w^2 \underbrace{\sum_{j=1}^{\infty} \phi^{2j}}$$

$$\sigma_w^2 \cdot \left(\frac{-\phi^2}{1-\phi^2}\right)$$

$$\boxed{= \frac{-\phi^2 \sigma_w^2}{1-\phi^2}}$$

$$\left(\begin{array}{l} S = a + \cancel{a^2} + \cancel{a^3} + \dots a^n \\ aS = \cancel{a^2} + \cancel{a^3} + a^4 + \dots + a^{n+1} \\ aS - S = a - a^{n+1} \\ S(a-1) = a - a^{n+1} \end{array}\right)$$

$$S = \frac{a - a^{n+1}}{a-1} = \frac{a^{n+1} - a}{1-a} = \frac{a(a^n - 1)}{1-a}$$

Si $|a| < 1$ $\lim_{n\to\infty} a^n = 0$

$$\lim_{n\to\infty} \frac{a(\cancel{a^n}^{\to 0} - 1)}{1-a} = \frac{-a}{1-a}$$

$$\rho_k = \phi^k \Rightarrow \rho_1 = \underbrace{\phi}$$

$$Corr(X_t, X_{t-1}) = \hat{\rho} \approx \hat{\phi}$$

$$\hookrightarrow \frac{\sum (X_t - \bar{X})(X_{t-1} - \bar{X})}{\sqrt{\sigma(X_t)} \cdot \sqrt{\sigma(X_{t-1})}}$$

$$\hat{\sigma}_w^2 =$$

Recordemos que

$$\gamma_0 = -\underbrace{\sigma_w^2}\left(\underbrace{\frac{\phi^2}{1-\phi^2}}\right)$$

$$\{\hat{\phi} \,;\, \hat{\sigma}_w^2\}$$

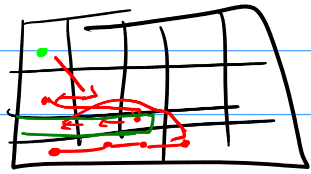
$$\hookrightarrow \frac{1}{n}\sum (X_t - \bar{X})^2$$

Proceso centrado en cero

$$\left(\begin{array}{l} \text{Si } \mu_t = \mu \neq 0 \Rightarrow Y_t = X_t - \mu \\ \mathbb{E}[X_t] = \mathbb{E}[\cancel{Y_t}^{\to 0} + \mu] = \mu \\ \hat{\mu} = \bar{X} \end{array}\right)$$

PACF para AR(1)

$$x_t = \phi x_{t-1} + w_t \quad , \quad w_t \sim WN(0, \sigma_w^2)$$



$$\begin{cases} \hat{\phi}_{k+1,k+1} = \dfrac{\hat{\rho}_{k+1} - \sum_{j=1}^{k} \hat{\phi}_{k,j}\, \hat{\rho}_{k+1-j}}{1 - \sum_{j=1}^{k} \hat{\phi}_{k,j}\, \hat{\rho}_j} \\ \hat{\phi}_{k+1,j} = \hat{\phi}_{k,j} - \hat{\phi}_{k+1,k+1}\, \hat{\phi}_{k,k+1-j} \quad , \; j = 1, \dots, k \end{cases}$$

$$\hat{\phi}_{1,1} = \rho_1$$

$$\rho_k = \phi^k$$

$$\phi_{1,1} = \rho_1$$

$$\phi_{2,2} = \frac{\rho_2 - \sum_{j=1}^{k=1} \hat{\phi}_{1,j}\, \rho_{2-j}}{1 - \sum_{j=1}^{k=1} \hat{\phi}_{1,j}\, \rho_j} = \frac{\hat{\phi}_*^2 - \hat{\phi}_{1,1}\rho_1}{1 - \hat{\phi}_{1,1}\rho_1}$$

$$= \frac{\hat{\phi}_*^2 - \rho_1 \rho_1}{1 - \hat{\phi}_{1,1}\rho_1} = \frac{\cancelto{0}{\hat{\phi}^2 - \hat{\phi}\hat{\phi}}}{1 - \hat{\phi}^2} = 0$$

¿Será que $\phi_{k,k} = 0 \;\; \forall_{k>1,\, k \in \mathbb{N}}$? ⇝ SI!! :)

AR(p) ⇝ $\phi_{k,k} = 0 \;\; \forall k > p$!

$$\hat{\phi}_{2,1} = \hat{\phi}_{1,1} - \cancelto{0}{\hat{\phi}_{2,2}}\, \hat{\phi}_{1,1} \Rightarrow \hat{\phi}_{1,1}$$

$K = 1$ ✓ $\qquad \phi_{1,1} = \rho$

$K = 2 \qquad \phi_{2,2} = 0$

Supongamos que $\phi_{K,K} = 0$ para alguna K. (HI)

$\underbrace{\phi_{K+1,K+1}} \rightsquigarrow$ P.D. $\quad \phi_{K+1,K+1} = 0$

Sea $R = K - 1$

$$\phi_{R+1,j} = \phi_{R,j} - \phi_{R+1,R+1}\,\hat{\phi}_{R,R+1-j}$$

$$= \phi_{K-1,j} - \cancel{\tilde{\phi}_{K,K}}^{\to 0}\,\tilde{\phi}_{K-1,K-j}$$

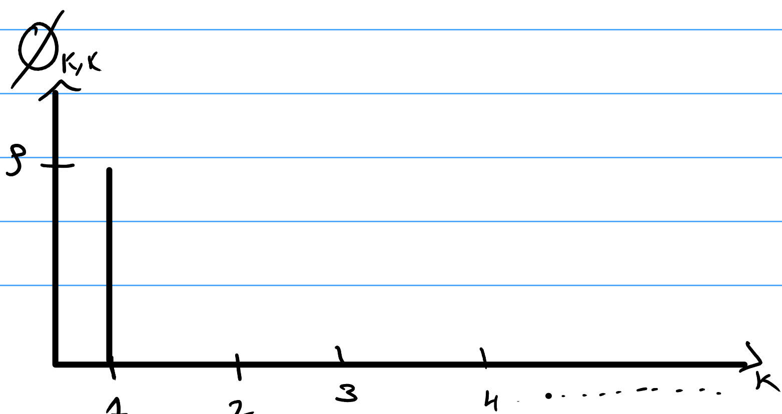
$$\Rightarrow \phi_{K,j} = \phi_{K-1,j} = \phi_{K-2,j} = \phi_{K-3,j} = \dots = \phi_{1,j}$$

$$\phi_{K+1,K+1} = \frac{\hat{\rho}_{K+1} - \sum_{j=1}^{K} \hat{\phi}_{K,j}\,\rho_{K+1-j}}{\underbrace{1 - \sum_{j=1}^{K} \hat{\phi}_{K,j}\,\rho_j}_{A}} = \frac{\hat{\rho}_{K+1} - \hat{\phi}_{K,1}\rho_K - \hat{\phi}_{K,2}\rho_{K-1} - \hat{\phi}_{K,3}\rho_{K-2} \cdots \phi_{K,K}\rho_1}{A}$$

$$= \frac{\hat{\rho}_{K+1} - \hat{\phi}_{1,1}\rho_K - \cancel{\hat{\phi}_{2,2}}^{\to 0}\rho_{K-1} - \cancel{\hat{\phi}_{3,3}}^{\to 0}\rho_{K-3} \cdots \cancel{\phi_{K,K}}^{0}\rho^{K-1}}{A}$$

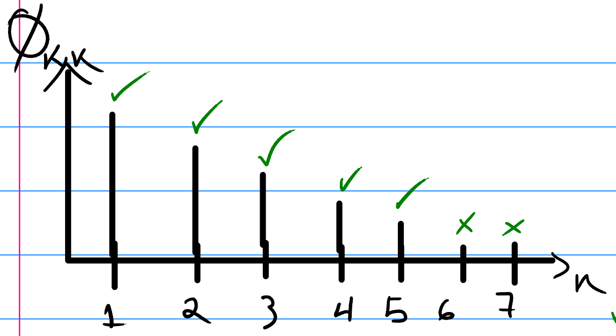
$$= \frac{\phi^{K+1} - \phi\,\phi^{K}}{A} = \frac{\phi^{K+1} - \phi^{K+1}}{A} = 0$$

$$\phi_{K,K} \begin{cases} \phi & K = 1 \\ 0 & \text{e.o.c.} \end{cases}$$



Sea X_t una serie de tiempo tal que $\phi_{K,K} = \hat{\rho}$, para $K=1$, 0 e.o.c. $\longrightarrow$ Un modelo AR(1) es adecuado para X_t.

AR(p)



Un modelo adecuado sería un AR(5)

Modelo de medias móviles de órden 1 ~ MA(1)

$$Z_t = w_t - \theta w_{t-1} = (1-\theta B) w_t \;; \quad w_t \sim WN(0, \sigma_w^2)$$

¿Z_t es estacionario?

$$E[Z_t] = E[w_t] - E[\theta w_{t-1}] = 0$$

$$\gamma_k = E[Z_t Z_{t-k}]$$

- Si $k=0 \Rightarrow \gamma_k = \gamma_0 = E[Z_t^2] = E[w_t^2 - 2\theta \cancel{w_t w_{t-1}} + \theta^2 w_{t-1}^2]$
  $= \sigma_w^2 + \theta^2 \sigma_w^2 = \boxed{(1+\theta^2)\sigma_w^2}$

- Si $k=1 \Rightarrow E[Z_t Z_{t-1}] = E[(w_t - \theta w_{t-1})(w_{t-1} - \theta w_{t-2})]$
  $= E[\cancel{w_t w_{t-1}} - \theta w_{t-1}^2 - \theta \cancel{w_t w_{t-2}} + \theta^2 \cancel{w_{t-1} w_{t-2}}]$
  $\boxed{= -\theta \sigma_w^2}$

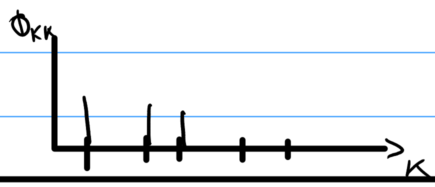
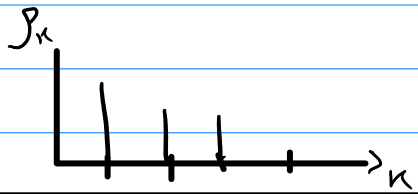
$$\gamma_k = \begin{cases} (1+\theta^2)\sigma_w^2 & k=0 \\ -\theta \sigma_w^2 & k=1 \\ 0 & \text{e.o.c.} \end{cases}$$

- $k>1 \quad E[Z_t Z_{t-k}] = E[\cancel{w_t w_{t-k}} - \theta \cancel{w_t w_{t-k-1}} - \theta \cancel{w_{t-1} w_{t-k}} + \theta^2 \cancel{w_{t-1} w_{t-k-1}}]$
  $= 0$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \frac{\gamma_k}{(1+\theta^2)\sigma_w^2} = \begin{cases} 1 & k=0 \\ \dfrac{-\theta \cancel{\sigma_w^2}}{(1+\theta^2)\cancel{\sigma_w^2}} & k=1 \\ 0 & e.o.c. \end{cases}$$

Si Z_t es un proceso $MA(1)$, su función de autocorrelación, solo dependerá de $k=1$, para $k>1$, $\rho_k = 0$

Si X_t una serie de tiempo tal que $\rho_k = 0$ para $k \geq 1$ => el modelo adecuado será un $MA(1)$

AR(p)	MA(q)
→ PACF	→ ACF
	

PACF de $MA(1)$

$$\phi_{k,k} = \frac{-\theta^k (1-\theta^2)}{1-\theta^{2(k+1)}} \propto -\theta^k$$