Análisis de Supervivencia

→ Función de vida media residual.
"mrl(t)"

$$mrl(t) = \mathbb{E}[T-t \mid T>t] \rightarrow \text{(Expected shortfall)}$$

T: Tiempo de supervivencia
$t \in \mathbb{R}$

$$\mathbb{E}[T-t \mid T>t] = \int_0^{\infty} (u-t)\, f(u \mid u>t)\, du$$

$$\rightarrow \frac{\mathbb{P}[\overbrace{\{u \in (t, t+\Delta t)\}}^{f(u)} \cap \{u>t\}]}{\underbrace{\mathbb{P}[u>t]}_{s(t)}}$$

$$= \int_0^{\infty} (u-t)\, \frac{f(u)}{S(t)}\, du$$

??

Ejercicio:

↝ Teorema de Fubinni

$$mrl(t) = \int_t^{\infty} \frac{S(u)}{S(t)}\, du$$

→ Modelos paramétricos de supervivencia

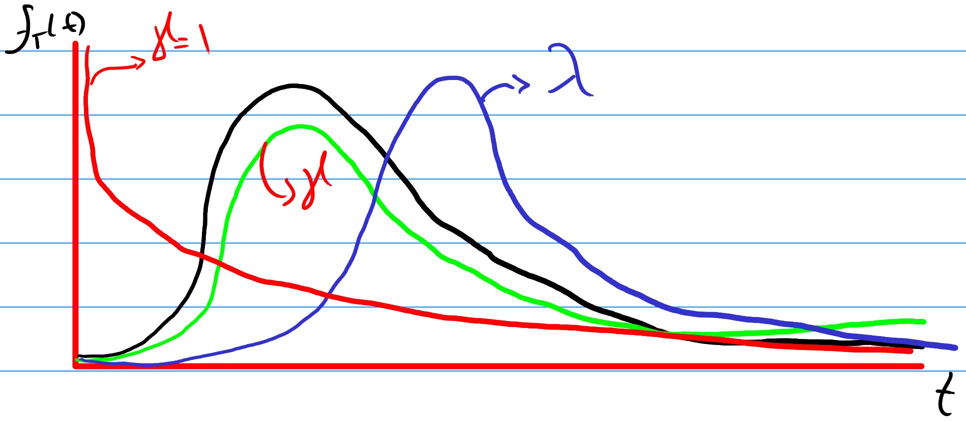
- Distribución exponencial
- Distribución de Weibull

Tenemos varias parametrizaciones para la distribución de Weibull:

$$f_T(t,\gamma,\lambda)\begin{cases} \gamma\lambda t^{\gamma-1}\exp(-\lambda t^{\gamma}) \quad ; \gamma>0, \lambda>0, t\geq 0 \\ \frac{\gamma}{\lambda}\left(\frac{t}{\lambda}\right)^{\gamma-1}\exp\left(-\left(\frac{t}{\lambda}\right)^{\gamma}\right) \\ \gamma\lambda(\lambda t)^{\gamma-1}\exp\left(-(\lambda t)^{\gamma}\right) \end{cases}$$

$\gamma \to$ Parámetro de forma

$\lambda \to$ Parámetro de escala



Función de supervivencia:

$$S_T(t) = \int_t^{\infty} f_T(u)\,du = \boxed{\int_t^{\infty}\gamma\lambda u^{\gamma-1}\exp(\lambda u^{\gamma})\,du} =$$

$\to$ Cambio de variable $u\in(t,\infty)$

$$z = u^{\gamma} \qquad dz = \gamma u^{\gamma-1}du \qquad z\in(t^{\gamma},\infty)$$

$$= \int_{t^{\gamma}}^{\infty}\lambda e(-\lambda z)\,dz = \cancel{\lambda}\left(-\frac{1}{\cancel{\lambda}}e^{-\lambda z}\right)\Bigg|_{t^{\gamma}}^{\infty}$$

$$= \cancel{-e^{-\lambda z}\Big|_{z=\infty}}\ 0 - \left[-e^{-\lambda z}\Big|_{z=t^{\gamma}}\right] = e^{-\lambda t^{\gamma}}$$

→ función de riesgo:

$$h(t) = \frac{f(t)}{S(t)} = \frac{\gamma \lambda t^{\gamma-1} \cancel{\exp(-\lambda t^{\gamma})}}{\cancel{\exp(-\lambda t^{\gamma})}}$$

$$= \gamma \lambda t^{\gamma-1}$$

→ Si $\gamma = 1 \Rightarrow$ $\boxed{h(t) = \lambda}$ Distr. exponencial.

Nota: Si $h_T(t) = \lambda;\ \lambda \in \mathbb{R} \Rightarrow T \sim \exp(\lambda)$?

Dem:

Suponer que T tiene una distr. distinta a la exponencial, pero $h_T(t) = \lambda;\ \lambda \in \mathbb{R}$.

$$\frac{du}{u} = d(\log(u))$$

$$h_T(t) = \frac{f_T(t)}{S_T(t)} = \frac{-\frac{d}{dt} S_T(t)}{S_T(t)} = -d(\log(S_T(t)))$$

$$h_T(t) = \lambda$$

$$-d(\log_e(S_T(t)) = \lambda$$

$$-\int_0^t d(\log(S_T(u))du = \int_0^t \lambda\, du$$

$$-[\log(S_T(t)) - \log(\cancel{S_T(0)})] = \lambda t$$

$$-\log(S_T(t)) = \lambda t \Leftrightarrow S_T(t) = e^{-\lambda t}$$

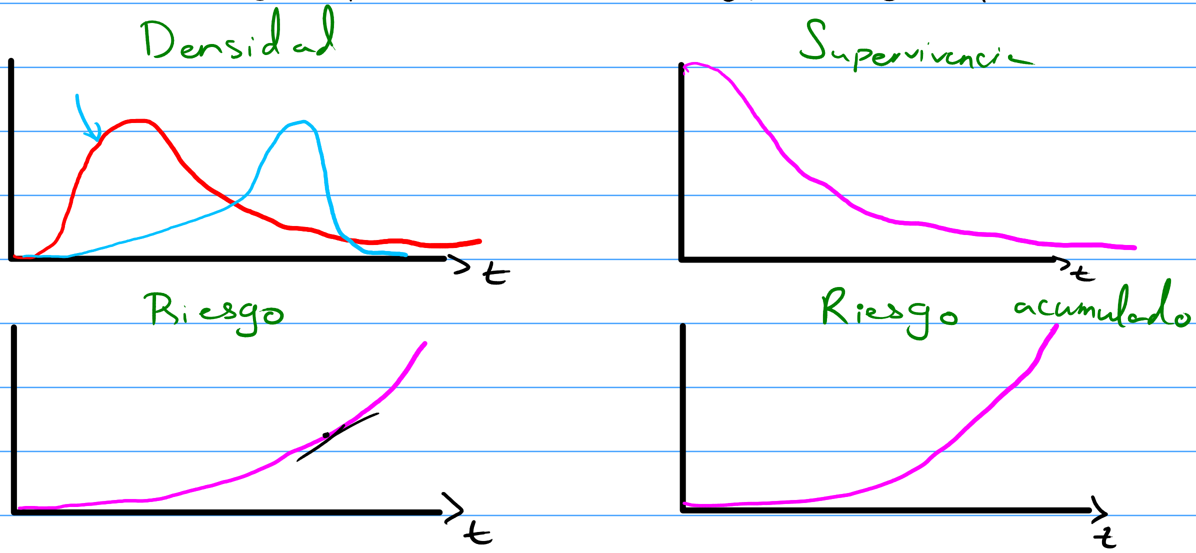
$$F_T(t) = 1 - e^{-\lambda t}$$

$$\Rightarrow f_T(t) = \lambda e^{-\lambda t}$$

Podemos concluir que $T \sim \exp(\lambda)$ $\Leftrightarrow h_T(t) = \lambda;\ \lambda \in \mathbb{R}$

¡

Retomando la distribución Weibull:



¿Qué tipo de riesgos podemos modelar con la distribución Weibull?

$$h'(t) = \frac{d h(t)}{dt} = \frac{d}{dt} \gamma \lambda t^{\gamma-1} = (\gamma-1)\gamma\lambda t^{\gamma-1}$$

$$h(t) = \begin{cases} \text{función de riesgo creciente} & \gamma > 1 \\ \text{decreciente} & \gamma \in (0,1) \\ \text{constante} & \gamma = 1 \end{cases}$$

Media y desv. estándar de una distribución Weibull

$$E[T] = \int_0^\infty u f_T(u)\, du$$

$$= \int_0^\infty u \gamma \lambda u^{\gamma-1} e^{-\lambda u^{\gamma}}\, du$$

$$= \int_0^\infty \gamma \lambda u^{\gamma} e^{-\lambda u^{\gamma}}\, du$$

$$= \int_0^\infty \gamma \lambda u^\gamma \exp(-\lambda u^\gamma)\, du =$$

Sea $z = u^\gamma \quad dz = \gamma u^{\gamma-1}\, du$

$u = z^{-\gamma} \Rightarrow u^{-1} = z^\gamma \qquad \hookrightarrow du = \frac{dz}{\gamma u^{\gamma-1}}$ $\hookrightarrow u^\gamma \cdot u^{-1}$

$$= \int_0^\infty \cancel{z}\, \cancel{\gamma} \lambda e^{-\lambda z} \left(\frac{dz}{\cancel{\gamma z} \cdot z^\gamma} \right)$$

$$= \int_0^\infty z^{-\gamma} \lambda e^{-\lambda z}\, dz \, ;$$

$X \sim \text{Gamma}(r, \lambda)$

$f_X(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} \mathbb{1}_{(0,\infty)}$

$= \Big(\; u = z^{-\gamma} \qquad du = \gamma z^{\gamma-1}$

$v = \frac{1}{\lambda} e^{-\lambda z} \qquad dv = e^{-\lambda z}$

$$\int u\,dv = \Big(\hookrightarrow uv - \int v\,du \Rightarrow z^{-\gamma} \gamma z^{\gamma-1} - \underbrace{\int \frac{1}{\lambda} e^{-\lambda z} \gamma z^{\gamma-1}}_{\frac{\gamma}{\lambda} \int z^{\gamma-1} e^{-\lambda z}\, dz}$$

$\gamma = r - 1$

$$= \int z^{-\gamma} \lambda e^{\lambda z}\, dz = \int_0^\infty z^{-(r-1)} \lambda e^{-\lambda z} =$$

$$= \underbrace{\Gamma(1-\gamma)\lambda}\Big\}\Big\}$$ Esperanza

Varianza:

$$Var(T) = \mathbb{E}[(T-\mu)^2] = \mathbb{E}[T^2] - \mathbb{E}^2[T]$$

$$= \lambda^2 \left[\Gamma(1-2\gamma) - \Gamma^2(1-\gamma)\right]$$

Si $T \sim exp(\lambda)$ $\mathbb{E}[T] = \lambda$, $Var(T) = \lambda^2$

-> Mediana

↳ t' tal que $S_T(t') = 0.5$

$$S_T(t) = e^{-\lambda t^\gamma} = 0.5 \qquad \log(x) = -\log\left(\frac{1}{x}\right)$$

$$\Leftrightarrow \log(0.5) = -\lambda t^\gamma \qquad 0.5 = \frac{1}{2}$$

$$\Leftrightarrow t = \left[\frac{-\log(0.5)}{\lambda}\right]^{1/\gamma} = \boxed{\left[\frac{\log(2)}{\lambda}\right]^{1/\gamma}}$$

-> Distribución Log-normal.

Si $T \sim \log Normal(\mu, \sigma^2) \Rightarrow \log(T) \sim N(\mu, \sigma^2)$

$\Leftrightarrow$ Si $y \sim N(\mu, \sigma^2) \Rightarrow e^y \sim N(\mu, \sigma^2)$

→ Es claro ver que $T \geq 0$

$$f_T(t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \cdot \exp\left\{-\frac{(\log(t)-\mu)^2}{2\sigma^2}\right\}$$

$\Phi(x) = \mathbb{P}[Z < x]$; $Z \sim N(0,1)$

$$S_T(t) = \int_t^\infty f_T(u)\,du = 1 - \Phi\left(\frac{\log(t)-\mu}{\sigma}\right)$$

$\log(T) \sim N(\mu, \sigma)$

$$S_T(t) = \mathbb{P}[T \geq t] = 1 - \mathbb{P}[T \leq t]$$

$$= 1 - \mathbb{P}[\log(T) \leq \log(t)]$$

$$= 1 - \mathbb{P}\Big[\underbrace{\frac{\log(T)-\mu}{\sigma}}_{\frac{\log(T)-\mu}{\sigma} \sim N(0,1)} \leq \frac{\log(t)-\mu}{\sigma}\Big]$$

$$= \boxed{1 - \Phi\left(\frac{\log(t)-\mu}{\sigma}\right)}$$

$$h_T(t) = \frac{f(t)}{S_T(t)} = \frac{\frac{1}{\sqrt{2\pi\sigma^2 t}} \cdot \exp\left(-\frac{(\log(t)-\mu)^2}{\sigma^2}\right)}{1 - \Phi\left(\frac{\log(t)-\mu}{\sigma}\right)}$$

¿Qué pasa con $h_T(t)$ cuando $t \to \infty$?

$$\lim_{t\to\infty} h_T(t) = \frac{\cancel{\frac{1}{\sqrt{2\pi\sigma^2 t}}}^{\to 0} \exp\left(\cancel{-\frac{(\log(t)-\mu)^2}{\sigma^2}}^{\to 0}\right)}{1 - \cancel{\Phi\left(\frac{\log(t)-\mu}{\sigma}\right)}^{\to 0}} = \frac{0}{0} \,!$$

Para solucionar la indeterminación en $\lim_{t\to\infty} h_T(t)$, utilizaremos la regla de L'Hopital.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \dots = \lim_{x\to a}\frac{f^{(n)}(x)}{g^{(n)}(x)}$$

$$\frac{d}{dt}S_T(t) = -f_T(t) \qquad \frac{d}{dt}t^{-1/2} = -\frac{1}{2}\cdot t^{-3/2}$$

$$f'_T(t) = \overbrace{\frac{1}{\sqrt{2\pi\sigma^2 t}}\cdot \exp\left(-\frac{(\log(t)-\mu)^2}{2\sigma}\right)}^{f_T(t)}\cdot\frac{2(\log(t)-\mu)}{2\sigma}\cdot\frac{1}{t}$$

$$+\underbrace{\exp\left(\frac{-(\log(t)-\mu)^2}{2\sigma}\right)\cdot\frac{1}{\sqrt{2\pi\sigma^2 t}}}_{f_T(t)}\cdot\left(-\frac{1}{2}t^{-3/2}\right)t$$

$$\frac{f'(t)}{S'(t)} = \frac{\cancel{f_T(t)}\cdot(\log(t)-\mu) - \cancel{f_T(t)}\cdot\left(-\frac{1}{2}t^{-1/2}\right)}{-\cancel{f_T(t)}}$$

$$\lim_{t\to\infty}\frac{f'(t)}{S'(t)} = \lim_{t\to\infty}\left(\frac{1}{2}\cdot\frac{1}{\sqrt{t}} - (\log(t)-\mu)\right)$$

$$= \frac{1 - 2\sqrt{t}(\log(t)-\mu)}{2\sqrt{t}} = \frac{-\infty}{\infty} \quad !$$

$$\lim_{t\to\infty}\frac{f''(t)}{S''(t)} = \frac{\cancel{2}\left(-\frac{1}{2}t^{-3/2}\right)(\log(t)-\mu) - 2\sqrt{t}\left(\frac{1}{t}\right)}{\cancel{2}\cdot\left(-\frac{1}{2}t^{-3/2}\right)}$$

$$= \frac{t^{-3/2}(\log(t)-\mu) - 2/\sqrt{t}}{-1/2\, t^{-3/2}} = \log(t)-\mu$$

$$\frac{t^{-3/2}(\log(t)-\mu) - 2t^{-1/2}}{-\frac{1}{2}t^{-3/2}} =$$

$$\lim_{t\to\infty} = \cancel{\frac{\log(t)-\mu}{-1/2}}^{\to\infty} - \cancel{\frac{2}{1/2\,t}}^{\to 0} = \underline{\infty}$$

func. de riesgo monótona.