Estimación de la función Media, Autocovarianza y autocorrelación parcial.

$$\hat{\mu},\quad \mu = \mathbb{E}[Z_t] \qquad \hat{\mu} = \bar{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_n$$

Z_t es un proceso estacionario $\Big\}$ No es un proceso de variables indep.

El estimador $\hat{\mu}$ es insesgado

$$\mathbb{E}[\hat{\mu}] = \mathbb{E}[\bar{Z}] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} Z_n\right] = \frac{1}{n}\sum_{i=1}^{n}\mu = \mu$$

$$Var(\hat{\mu}) = Var(\bar{Z}) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} Z_n\right) \sim \cancel{\frac{Var(Z_n)}{n}}$$

$$= \frac{1}{n^2}\sum_{t=1}^{n}\sum_{s=1}^{n} Cov(Z_t, Z_s) = \frac{1}{n^2}\sum_{t=1}^{n}\sum_{s=1}^{n}\gamma_{t-s}$$

$t = s+n$

$t-s = n$

$t = 1, 2, \ldots n$

$s = 1, 2, \ldots, n$

t	s	$t-s=k$
1	1	0
1	2	1
2	2	0

$$\frac{1}{n}\sum_{k=1}^{n}\gamma_k$$

$$= \frac{1}{n^2}\sum_{k=1}^{n}(n\,\gamma_k) = \Longrightarrow$$

multiplicamos $\times\ \gamma_0$

Dividimos $\times\ \gamma_0$

$$= \frac{\gamma_0}{n}\sum_{k=1}^{n}\rho_k \qquad \rho_k = \frac{\gamma_k}{\gamma_0}$$

$\Rightarrow$ Si $\displaystyle\lim_{n\to\infty}\frac{\gamma_0}{n}\sum_{k=1}^{n}\rho_k < \infty \Rightarrow$ Sabremos que $\hat{\mu}$ es un estimador consistente de μ.

$$\left|\frac{1}{n}\sum_{i=1}^{n}\rho_K\right| = \left|\frac{1}{n}\sum_{K=1}^{N}\rho_K + \frac{1}{n}\sum_{K=N+1}^{n}\rho_K\right|$$

$$\leq \left|\frac{1}{n}\sum_{K=1}^{N}\rho_K\right| + \left|\frac{1}{n}\sum_{K=N+1}^{n}\rho_K\right|$$

$$\leq \frac{1}{n}\sum_{K=1}^{N}|\rho_K| + \frac{1}{n}\sum_{K=N+1}^{n}|\rho_K|$$

Vamos a escoger N tal que

$$|\rho_K| \leq \varepsilon/4$$

$$\leq \frac{1}{n}\sum_{K=1}^{N}|\rho_K| + \frac{1}{\cancel{n}}\cdot\frac{\varepsilon}{4}\cdot(\cancel{n}-N+1) < \infty$$

$$n \to \infty$$

Resultado del tipo ergódico

$$n \to \infty \quad X_t^{(n)} \longrightarrow X_t$$

Estimador para la fn. de autocovarianza.

$$\boxed{\hat{\gamma}_K = \frac{1}{n}\sum_{t=1}^{n-K}(Z_t - \bar{Z})(Z_{t+K} - \bar{Z})} \checkmark$$

$$\hat{\hat{\gamma}}_K = \frac{1}{n-K}\sum_{t=1}^{n-K}(Z_t - \bar{Z})(Z_{t+K} - \bar{Z})$$

$$\hat{\rho}_K = \frac{\hat{\gamma}_K}{\hat{\gamma}_0}$$

Datos $\underbrace{\bar{Z}_t}$

t	Z_t	Z_{t+1}	Z_{t-1}	Z_{t+2}
1	13	8	–	15
2	8	15	13	4
3	15	4	8	4
4	4	4	15	12
5	4	12	4	11
6	12	11	4	–
7	11	–	12	–

$$\hat{\rho}_1 = \frac{Cov(Z_{t+1}, Z_t)}{\sqrt{var(Z_{t+1})}\sqrt{Var(Z_t)}} = \frac{\cancel{\frac{1}{n}} \sum (Z_t - \bar{Z})(Z_{t+1} - \bar{Z})}{\sqrt{\cancel{\frac{1}{n}} \sum (Z_t - \bar{Z})^2} \sqrt{\cancel{\frac{1}{n}} \sum (Z_t - \bar{Z})^2}}$$

$\hat{\rho}_1 \simeq -0.188$

$\hat{\rho}_t = \hat{\rho}_{-t}$

Sample PACF (Estimador de la función de autocorrelación parcial)

$$P_K = \hat{\phi}_{KK}$$

$$\hat{\phi}_{1,1} = \hat{\rho}_{1,1}$$

$$\hat{\phi}_{K+1,K+1} = \frac{\hat{\rho}_{K+1} - \sum_{j=1}^{K} \hat{\phi}_{Kj}\, \hat{\rho}_{K+1-j}}{1 - \sum_{j=1}^{K} \hat{\phi}_{Kj}\, \rho_j}$$

Durbin (1960)

$$\hat{\phi}_{K+1,j} = \hat{\phi}_{Kj} - \hat{\phi}_{K+1,K+1}\, \hat{\phi}_{K,K+1-j} \quad ; j = 1, 2, \ldots, K$$

$$\hat{\phi}_{2,2} = \hat{\phi}_{1+1,1+1} = \frac{\hat{\rho}_2 - \sum_{j=1}^{1} \hat{\phi}_{Kj}\, \rho_{K+1-j}}{1 - \sum_{j=1}^{K} \hat{\phi}_{Kj}\, \rho_j}$$

$$= \frac{\hat{\rho}_2 - \hat{\phi}_{1,1}\, \rho_1}{1 - \hat{\phi}_{1,1}\, \rho_1}$$

$$\hat{\phi}_{1+1,1}$$

$$\hat{\phi}_{2,1} = \hat{\phi}_{1,1} - \hat{\phi}_{2,2}\, \hat{\phi}_{1,1}$$

$$\boxed{\hat{\phi}_{3,3}} = \hat{\phi}_{2+1,2+1} = \frac{\hat{\rho}_3 - \sum_{j=1}^{2} \hat{\phi}_{2,j}\, \hat{\rho}_{3-j}}{1 - \sum_{j=1}^{2} \hat{\phi}_{2,j}\, \hat{\rho}_j} = \frac{\hat{\rho}_3 - \boxed{\hat{\phi}_{2,1}} \cdot \rho_2 - \boxed{\hat{\phi}_{2,2}}\, \rho_1}{1 - \phi_{2,1}\hat{\rho}_1 - \phi_{2,2}\, \rho_2}$$

ϕ_{11}				
$\phi_{2,1}$	$\phi_{2,2}$			
$\phi_{3,1}$	$\phi_{3,2}$	$\phi_{3,3}$		
$\phi_{4,1}$	$\phi_{4,2}$	$\phi_{4,3}$	$\phi_{4,4}$	
$\phi_{5,1}$	$\phi_{5,2}$	$\phi_{5,3}$	$\phi_{5,4}$	$\phi_{5,5}$

$$\hat{\phi}_{\overset{k=3+1}{4},\overset{k=3+1}{4}} = \frac{\hat{\rho}_4 - \sum_{j=1}^{3} \phi_{3,j}\hat{\rho}_{4-j}}{1 - \sum_{j=1}^{3} \phi_{3,j}\hat{\rho}_j} = \frac{\hat{\rho}_4 - \phi_{3,1}\rho_3 - \phi_{3,2}\rho_2 - \phi_{3,3}\rho_1}{1 - \phi_{3,1}\rho_1 - \phi_{3,2}\rho_2 - \phi_{3,3}\rho_3}$$

$(\hat{\rho}_1, \hat{\rho}_2, \ldots, \hat{\rho}_n)$

$n=1 \quad \phi_{1,1} = \hat{\rho}_1$

$n=k \quad \phi_{k,k} =$

$\phi_{k,k+j} =$

$\hat{\phi}_{n,n}$

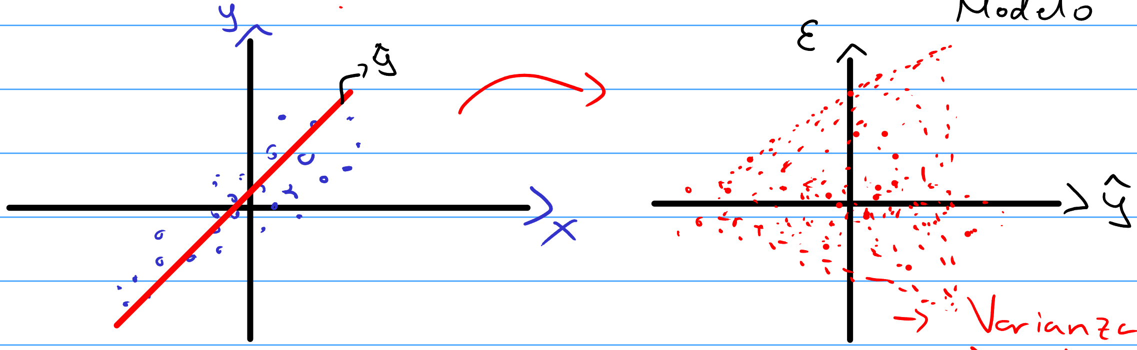
$(0.3,\ 0.4,\ 0.1)$

$\phi_{1,1} = 0.3$

$$\phi_{2,2} = \frac{0.4 - 0.3(0.3)}{1 - 0.3(0.3)} = \frac{0.4 - (0.3)^2}{1 - (0.3)^2}$$

Modelos ARIMA

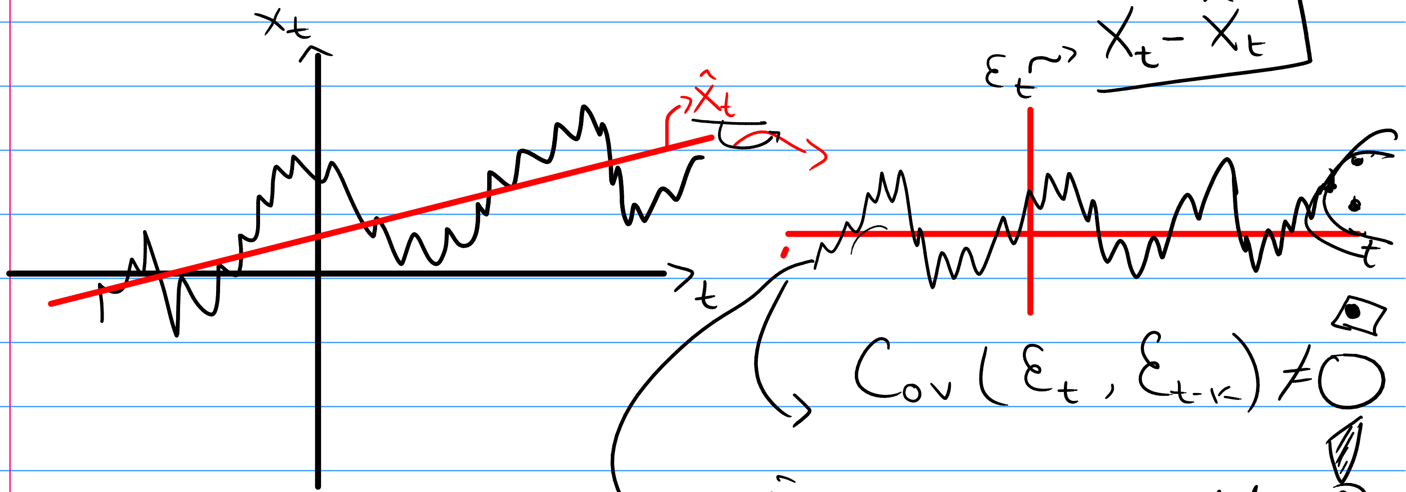
$$Y = \alpha X + \underbrace{\varepsilon}_{?} \quad ; \quad \varepsilon \sim N(0, \sigma^2) \Bigg\}$$ Modelo de regresión lineal "Modelo lineal"



-> Varianza constante
'Heteroskedacidad'

$\varepsilon_t \rightsquigarrow Cov(\varepsilon_t, \varepsilon_{t-k}) = 0 \quad \forall k > 0$ -> No autocorrelación

$\varepsilon_t \rightsquigarrow \underline{X_t - \hat{X}_t}$



$Cov(\varepsilon_t, \varepsilon_{t-k}) \neq 0$!

A ε_t podríamos modelarla con teoría de series de tiempo.

Procesos Autorregresivos (AR) — Procesos de medias móviles (MA)

Procesos Autorregresivos con medias móviles (ARMA)

Procesos autoregresivos Integrados con medias móviles (ARIMA)

Procesos autoregresivos Integrados Estacionales con medias móviles. SARIMA

Introducción a los modelos autoregresivos (AR)

Intuición: La información a tiempo t está influida por los p-últimos valores, es decir $\{X_{t-1}, X_{t-2}, \dots, X_{t-p}\}$

Ejemplo:

$$x_t = x_{t-1} - 0.9X_{t-2} + w_t \qquad w_t \sim WN(0, \sigma^2)$$

$\hookrightarrow AR(2)$

Un modelo autoregresivo de orden p $(AR(p))$ va a definirse de la sig. forma:

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \dots + \phi_p x_{t-p} + w_t \; ; \; w_t \sim WN(0, \sigma^2)$$

$\{\phi_1, \dots, \phi_p\}$ parámetros donde $\phi_i \in \mathbb{R}$, $i = 1, 2, \dots, p$

$\phi_p \neq 0$

Otra forma de nombrar a un proceso $AR(p)$ es la siguiente.

$$\boxed{\begin{aligned} x_t &= \phi_1 B x_t + \phi_2 B^2 x_t + \dots + \phi_p B^p x_t + w_t \\ &= x_t(\phi_1 B + \phi_2 B^2 + \dots + \phi_p B^p) + w_t \end{aligned}}$$

B^n es el operador "lag", donde $B^n x_t = x_{t-n}$

$$\boxed{w_t = x_t(1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p)} \qquad \boxed{w_t = \phi(B) x_t}$$

$$\hookrightarrow \phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p \; \}\text{Operador autoregresivo}$$

Supongamos que tenemos un proceso AR(1)

$x_t = \phi x_{t-1} + w_t \quad \forall\, t \in (-\infty, \dots, 0, \dots \infty) \cap \mathbb{N}$

Iterando hacia atrás.

$x_t = \phi(\phi x_{t-2} + w_{t-1}) + w_t = \phi^2 x_{t-2} + w_t + \phi w_{t-1}$

$= \phi^3 x_{t-3} + w_t + \phi w_{t-1} + \phi^2 w_{t-2} = \dots = \phi^k x_{t-k} + \sum_{j=1}^{k-1} \phi^j w_{t-j}$

→ Si x_t es un proceso estacionario y además $|\phi| < 1$

Podemos representar un proceso AR(1) como un proceso lineal de la sig. forma.

$x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j}$ } Principio de causalidad.

$\mathcal{L}^2$

$\lim_{k \to \infty} \mathbb{E}\left[\left(x_t - \sum_{j=0}^{\infty} \phi^j w_{t-j}\right)^2\right] = \lim_{k \to \infty} \phi^{2k}\, \mathbb{E}[x_{t-k}^2] = 0$

$|\phi| < 1$

$\{x_t\}$ una sucesión de v.a's. $x_t \xrightarrow{\mathcal{L}^2} x$ si

$\lim_{t \to \infty} \mathbb{E}[(x_t - x)^2] = 0$

$\lim_{k \to \infty} \mathbb{E}\left[\left(x_t - \sum_{j=0}^{\infty} \phi^j w_{t-j}\right)^2\right] = \lim_{k \to \infty} \mathbb{E}\left[x_t^2 - \cancel{2x_t \sum_{j=0}^{\infty} \phi^j w_{t-j}}^{\,0} + \left(\sum_{j=0}^{\infty} \phi^j w_{t-j}\right)^2\right]$

$$\lim_{K\to\infty} \overbrace{\mathbb{E}[X_t^2]}^{\gamma_0} + \mathbb{E}\left[\left(\sum_{j=0}^{K}\phi^j w_{t-j}\right)^2\right]$$

$$= \gamma_0 + \lim_{K\to\infty} \mathbb{E}\left[\underbrace{\left(\sum_{j=0}^{\infty}\phi^j w_{t-j}\right)^2}_{Var\left(\sum \phi^j w_{t-j}\right)}\right] = \gamma_0 + \lim_{K\to\infty}\sum_{i=1}^{K} Var(\phi^j w_{t-j})$$

$$= \gamma_0 + \lim_{K\to\infty}\sum_{i=1}^{K}\phi^{2j} Var(w_{t-j})$$

$$\gamma_0 = \phi^0\gamma_0 = \phi^0 Var(w_t)$$

$$= \lim_{K\to\infty}\sum_{i=0}^{K}\phi^{2j}\underline{Var(w_{t-j})} = \lim_{K\to\infty}\sum_{i=1}^{K}\phi^{2j} Var(X_t) \leq$$

la suma sea absolutamente sumable.

$$\sum_{i=1}^{\infty}|a_i| <-> 0$$

$$\leq \lim_{K\to\infty}\sum|\phi^{2j}|\, Var(X_t) \to 0$$

$$\underbrace{X_t = \sum_{k=1}^{\infty}\phi^k w_{t-k}}\quad |\phi| < 1$$

Apéndice A del Shumway