Función de autocorrelación parcial (PACF)

$\rightarrow P_K = Corr(X_{t+K}, X_t \mid X_{t+K+1}, X_{t+K+2}, \dots, X_{t+1})$

$\rightarrow X_t$ un proceso estacionario.

$\hat{X}_t$ lo definimos como un estimador lineal de X_t.

$$\boxed{\hat{X}_{t+K} = \alpha_1 X_{t+K-1} + \alpha_2 X_{t+K-2} + \dots + \alpha_{K-1} X_{t+1}, \quad \alpha_i \in \mathbb{R}, \; i \in [1, K-1]}$$

Encontrar $\{\alpha_1, \alpha_2, \dots, \alpha_{K-1}\}$ tal que $\mathbb{E}\left[\left(X_{t+K} - \sum_{i=1}^{K-1} \alpha_i X_{t+K+i}\right)^2\right]$ sea mínima.

$\rightarrow \frac{\partial \mathbb{E}}{\partial \alpha_i}$

$$\gamma_i = \alpha_1 \gamma_{i-1} + \alpha_2 \gamma_{i-2} + \dots + \alpha_{K-1} \gamma_{i-K+1} \rightarrow (*)$$

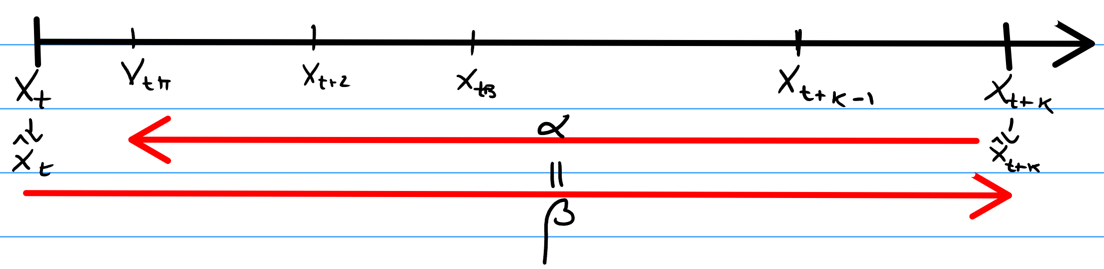
$$\rho_i = \alpha_1 \rho_{i-1} + \alpha_2 \rho_{i-2} + \dots + \alpha_{K-1} \rho_{i-K+1} \qquad i \in [1, K-1]$$

$\rho_{-1} = \rho_1$, ρ_0

$$\begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{K-1} \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \dots & \rho_{K-2} \\ \rho_1 & 1 & \rho_1 & \dots & \rho_{K-3} \\ \vdots & & & & \\ \rho_{K-2} & \rho_{K-3} & \rho_{K-4} & \dots & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{K-1} \end{bmatrix}$$

2° parte: Definimos $\hat{X}_t^*$ como un estimador lineal de X_t

$$\hat{X}_t^* = \beta_1 X_{t+1} + \beta_2 X_{t+2} + \dots + \beta_{K-1} X_{t+K-1}$$



$$\hat{X}_{t+k} = \alpha_1 X_{t+k-1} + \alpha_2 X_{t+k-2} + \alpha_3 X_{t+k-3} + \dots + \alpha_{k-1} X_{t+1}$$

$$\hat{X}_t^* = \beta_1 X_{t+1} + \beta_2 X_{t+2} + \beta_3 X_{t+3} + \dots + \beta_{k-1} X_{t+k-1}$$

$$\min_{\{\beta_1, \dots, \beta_{k-1}\}} \underbrace{\mathbb{E}\left[(X_t - \hat{X}_t^*)^2 \right]}_{\mathcal{L}(X_t, \hat{X}_t^*)}$$

$$\rightarrow \frac{\partial \mathcal{L}(X_t, \hat{X}_t^*)}{\partial \beta_i} = 0 \Rightarrow \gamma_i = \dots$$

$$\begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} \\ \vdots & & & & \\ \rho_{k-2} & \rho_{k-3} & \rho_{k-4} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{k-1} \end{bmatrix}$$

De aquí, podemos concluir que $\alpha_i = \beta_i \quad \forall i \in [1, k-1]$

Wei $\rightarrow$ Time Series Analysis.

Dado lo anterior, se determina la función de autocorrelación parcial, que vamos a definir como la correlación entre $(X_t - \hat{X}_t)$ y $(X_{t+k} - \hat{X}_{t+k})$

Sea P_k la función de autocorrelación parcial entre X_t y X_{t+k}, tenemos que:

$$P_K = \frac{Cov\left(X_t - \hat{X}_t,\ X_{t+K} - \hat{X}_{t+K}\right)}{\sqrt{Var\left(X_t - \hat{X}_t\right)}\sqrt{Var\left(X_{t+K} - \hat{X}_{t+K}\right)}}$$

$$Var\left(X_{t+K} - \hat{X}_{t+K}\right) = \mathbb{E}\left[\left(X_{t+K} - \hat{X}_{t+K}\right)^2\right] - \mathbb{E}^2\left[\cancel{X_{t+K}} - \cancel{\hat{X}_{t+K}}\right] \to 0$$

$$\hat{X}_{t+K} = \alpha_1 X_{t+K-1} + \alpha_2 X_{t+K-2} + \dots + \alpha_{K-1} X_{t+1}$$

$$= \mathbb{E}\left[\left(X_{t+K} - \alpha_1 X_{t+K-1} - \alpha_2 X_{t+K-2} - \dots - \alpha_{K-1} X_{t+1}\right)^2\right] =$$

$$= \boxed{\mathbb{E}\left[X_{t+K}\left(X_{t+K} \overbrace{- \alpha_1 X_{t+K-1} - \alpha_2 X_{t+K-2} - \dots - \alpha_{K-1} X_{t+1}}^{\hat{X}_{t+K}}\right)\right]} -$$

$$\alpha_1 \mathbb{E}\left[\boxed{X_{t+K-1}\left(X_{t+K}\right.} - \left.\hat{X}_{t+K}\right)\right] - \alpha_2 \mathbb{E}\left[\boxed{X_{t+K-2}\left(X_{t+K}\right.} - \left.\hat{X}_{t+K}\right)\right]$$

$$- \alpha_3 \mathbb{E}\left[\boxed{X_{t+K-3}\left(X_{t+K}\right.} - \left.\hat{X}_{t+K}\right)\right] - \dots - \alpha_{K-1} \mathbb{E}\left[\boxed{X_{t+1}\left(X_{t+K}\right.} - \left.\hat{X}_{t+K}\right)\right]$$

Recordemos que $\gamma_i = \alpha_1 \gamma_{i-1} + \alpha_2 \gamma_{i-2} + \dots + \alpha_{K-1} \gamma_{i-K+1}$

$$- \alpha_1 \mathbb{E}\left[X_{t+K-1} X_{t+K}\right] - \alpha_2 \mathbb{E}\left[X_{t+K-2} X_{t+K}\right] - \dots - \alpha_{K-1} \mathbb{E}\left[X_{t+1} X_{t+K}\right]$$

$$\hookrightarrow - \alpha_1 \gamma_1 - \alpha_2 \gamma_2 - \alpha_3 \gamma_3 - \dots - \alpha_{K-1} \gamma_{K-1} \}$$

$$+ \alpha_1 \mathbb{E}\left[X_{t+K-1} \hat{X}_{t+K}\right] + \alpha_2 \mathbb{E}\left[X_{t+K-2} \hat{X}_{t+K}\right] + \dots + \alpha_{K-1} \mathbb{E}\left[X_{t+1} \hat{X}_{t+K}\right]$$

$$\hookrightarrow \mathbb{E}\left[\hat{X}_{t+K} \cdot \underbrace{\left[\alpha_1 X_{t+K-1} + \alpha_2 X_{t+K-2} + \dots + \alpha_{K-1} X_{t+1}\right]}_{\hat{X}_{t+K}}\right]$$

$$= \mathbb{E}\left[\hat{X}_{t+K}^2\right]$$

$$\boxed{Var\left(X_t - \hat{X}_t\right) = \gamma_0 - \alpha_1 \gamma_1 - \alpha_2 \gamma_2 - \dots - \alpha_{K-1} \gamma_{K-1}} = Var\left(X_{t+K} - \hat{X}_{t+K}\right)$$

$\hookrightarrow \hat{X}_t$ es un proceso estacionario

$\hookrightarrow \sum_{i=1}^{K-1} \alpha_i X_{t-i} \to$ es un proc. estacionario

Resultado:

$$Cov\left(X_t-\hat{X}_t,\ X_{t+k}-\hat{X}_{t+k}\right)$$

$$=\gamma_k-\alpha_1\gamma_{k-1}-\alpha_2\gamma_{k-2}-\alpha_3\gamma_{k-3}-\cdots-\alpha_{k-1}\gamma_1$$

$$\boxed{P_k=\frac{\gamma_k-\alpha_1\gamma_{k-1}-\alpha_2\gamma_{k-2}-\cdots-\alpha_{k-1}\gamma_1}{\gamma_0-\alpha_1\gamma_1-\alpha_2\gamma_2-\cdots-\alpha_{k-1}\gamma_{k-1}}}$$

Cómo encontramos α_i?

Resolviendo

$$\begin{bmatrix}\boxed{\rho_1}\\ \rho_2\\ \vdots\\ \rho_{k-1}\end{bmatrix}=\begin{bmatrix}1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2}\\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3}\\ \vdots & & & & \\ \rho_{k-2} & \rho_{k-3} & \rho_{k-4} & \cdots & 1\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_{k-1}\end{bmatrix}$$

a) $X_t = S_t + W_t$; $W_t \sim N(0,1)$

$$S_t = \begin{cases} 0 & t=1,2,\dots,100 \\ 10e^{-\frac{t-100}{20}}\cos(2\pi t/4) & t=101,102,\dots,200 \end{cases}$$

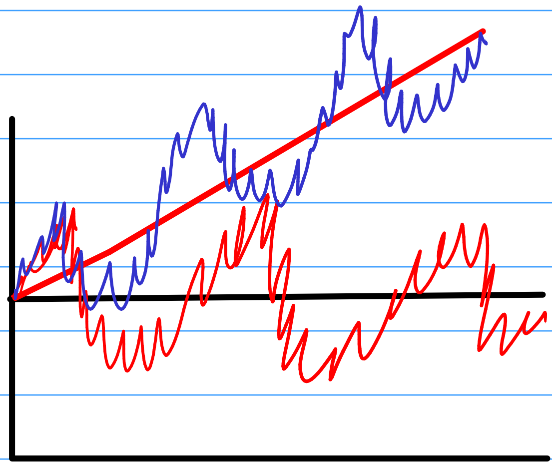
$$E[X_t] = E[S_t+W_t] = E[S_t] + \cancel{E[W_t]}^{\to 0}$$

$$\boxed{= S_t}$$

1.6)

$X_t = \beta_1 + \beta_2 t + w_t$

X_t es estacionario?



$$E[X_t] = E[\beta_1 + \beta_2 t + w_t]$$
$$= \beta_1 + \beta_2 t + 0 = \boxed{\beta_1 + \beta_2 t}, \text{ para cada } t$$

$E[X_t] = \mu \in \mathbb{R}$

↳ Como $E[X_t]$ depende de t, X_t no es estacionario.

b) $Y_t = X_t - X_{t-1} = \cancel{\beta_1} + \cancel{\beta_2} t + w_t - [\cancel{\beta_1} + \cancel{\beta_2 (t}-1) + w_{t-1}]$

$$= w_t + \beta_2 \cdot 1 - w_{t-1} = \beta_2 + (w_t - w_{t-1})$$

$E[Y_t] = \beta_2 \ \forall t$

1) $\mathbb{E}[X_t] = \mu \; ; \; \mu \in \mathbb{R}$ ✓ $\beta_2 \in \mathbb{R}$

$\mathbb{E}[X_t^2] < \infty$

$\hookrightarrow \mathbb{E}[(\beta_2 + (w_t - w_{t-1}))^2] = \overset{<\infty}{\beta_2^2} + 2\beta_2 \overset{\to 0}{\cancel{\mathbb{E}[w_t - w_{t-1}]}}$

$+ \underbrace{\mathbb{E}[(w_t - w_{t-1})^2]}_{2\sigma_w^2 < \infty} = \;\; < \infty$

$\gamma_h = Cov(y_{t+h} - y_t)$

$= \mathbb{E}[y_{t+h} \cdot y_t] - \mathbb{E}[\overset{\to \beta_2}{\cancel{y_{t+h}}}]\mathbb{E}[\overset{\to \beta_2}{\cancel{y_t}}]$

$= \mathbb{E}\left[\left[\beta_2 + (w_{t+h} - w_{t+h-1})\right]\left[\beta_2 + (w_t - w_{t-1})\right]\right] - \beta_2^2$

$= \mathbb{E}[\cancel{\beta_2^2} + \beta_2(w_t - w_{t-1}) + \beta_2(w_{t+h} - w_{t+h-1}) + (w_{t+h} - w_{t+h-1})(w_t - w$
$\cancel{-\beta_2^2}$

$= \beta_2 \overset{\to 0}{\cancel{\mathbb{E}[w_t - w_{t-1}]}} + \beta_2 \overset{\to 0}{\cancel{\mathbb{E}[w_{t+h} - w_{t+h-1}]}} + \mathbb{E}[(w_{t+h} - w_{t+h-1})(w_t - w_{t-1})]$

$= \mathbb{E}[(w_{t+h} - w_{t+h-1})(w_t - w_{t-1})]$

$= \mathbb{E}[w_{t+h} w_t - w_{t+h} w_{t-1} - w_{t+h-1} w_t + w_{t+h-1} w_{t-1}]$

$h = 0$	$h = 1$	$h = -1$
$= \sigma_w^2 + \sigma_w^2 = 2\sigma_w^2$	$= \sigma_w^2$	$= \sigma_w^2$

$\to$ En ningún caso, γ_h depende de t, sino de h $\to$ el proceso es estacionario.

[illegible]