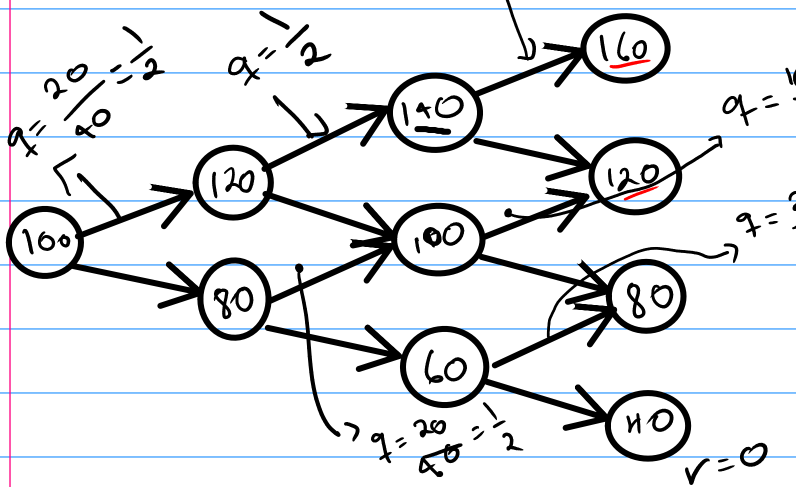
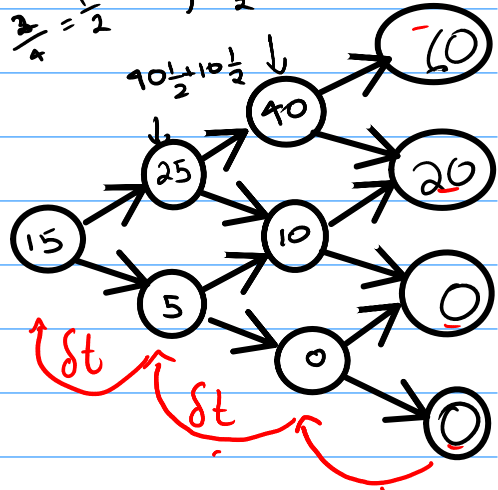
$$q = \frac{140-120}{160-120} = \frac{20}{40} = \frac{1}{2}$$

$$\text{Payoff} = (S_T - 100)^+$$

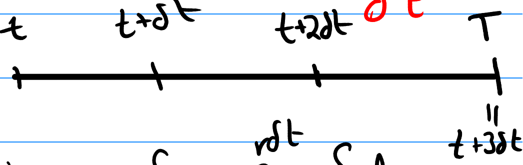
$$f = \tfrac{1}{2}(60) + \tfrac{1}{2}(20) = 40$$




$r = 0$

$$q = \frac{S - S_d}{S_u - S_d}$$




$$q_t = \frac{S_t - S_d}{S_u - S_d} \overset{\text{si } r \neq 0}{\Rightarrow} q_{t+n\delta t} = \frac{S_{t+n\delta t}\, e^{r\delta t} - S_d}{S_u - S_d}$$

$$q = \frac{\cancel{S_{t+n\delta t}}\, e^{r\delta t} - d\, \cancel{S_{t+n\delta t}}}{u\, \cancel{S_{t+n\delta t}} - d\, \cancel{S_{t+n\delta t}}} = \frac{e^{r\delta t} - d}{u - d}$$

Si $\phi > 0 \Rightarrow$ Compramos

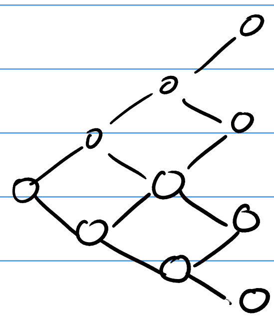
$\phi < 0 \Rightarrow$ Vendemos

$\psi > 0 \Rightarrow$ Pedimos Prestado

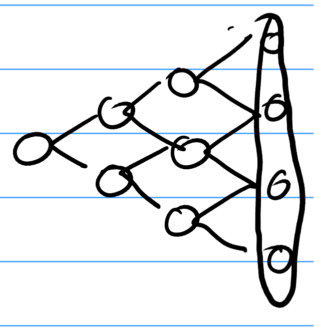
$\psi < 0 \Rightarrow$ Prestamos / invertimos

→ Programa:

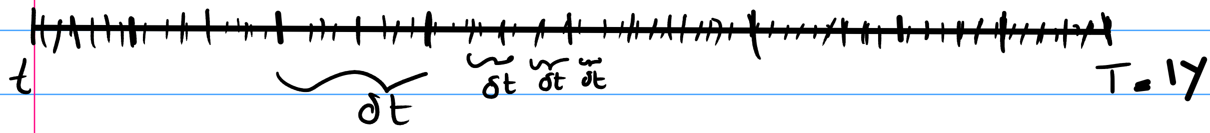
Inputs: # pasos del árbol
$S_0$
$u$
$d$
→ $r$
- Payoff



$q$



$$f = e^{-r\delta t}\left[q f_u + (1-q) f_d\right]$$



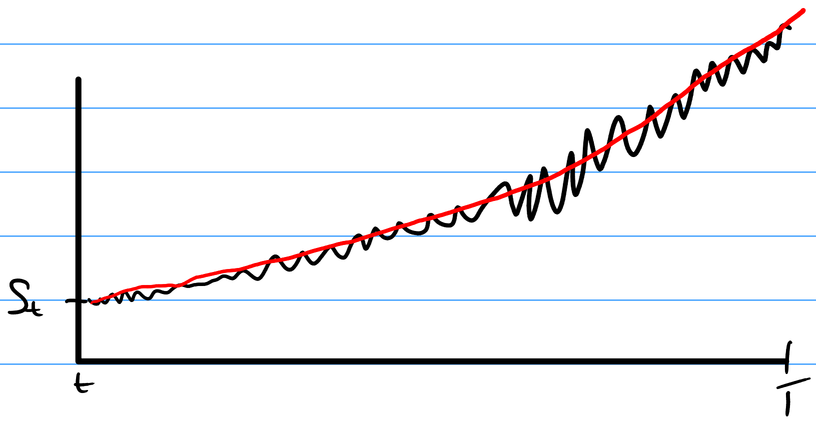

→ Si nosotros incrementamos el # de pasos en el árbol (n pasos), poco a poco $\delta t \to 0$

→ Si $\delta t \to 0$, qué pasa con el precio de la opción?

→ Recordatorio: vamos a trabajar con un árbol multiplicativo que recombina valores con $\delta t$ uniforme.

Además, vamos a suponer lo siguiente:

$$u = e^{\mu \delta t + \sigma\sqrt{\delta t}} \qquad d = e^{\mu \delta t - \sigma\sqrt{\delta t}}$$




Supongamos que conocemos la ley de probabilidad de $S_t$

$$S_u = S_t e^{\mu \delta t + \sigma\sqrt{\delta t}}$$

$$S_t \begin{cases} \xrightarrow{1/2} S_u \\ \xrightarrow{1/2} S_d \end{cases}$$

$$S_d = S_t e^{\mu \delta t - \sigma\sqrt{\delta t}}$$

$$\longrightarrow X \begin{cases} \mu \delta t + \sigma\sqrt{\delta t} \; ; \; \frac{1}{2} \\ \mu \delta t - \sigma\sqrt{\delta t} \; ; \; \frac{1}{2} \end{cases}$$

$$\mathbb{E}[X] = \mu \delta t$$

$$Var(X) = \mathbb{E}[(X - \mu\delta t)^2]$$

$$=$$

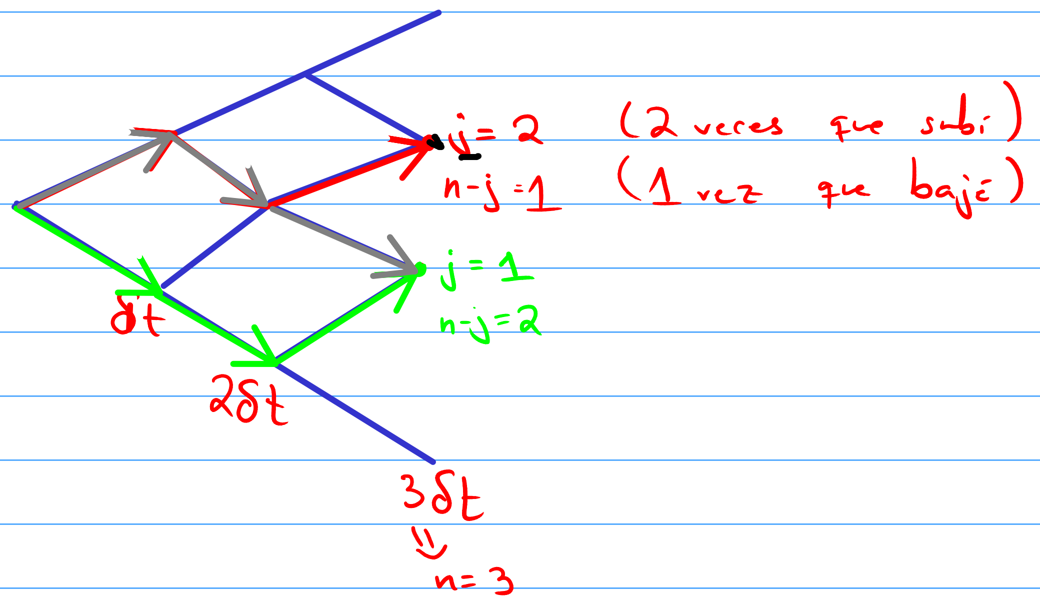
$$X = \begin{cases} \mu\delta t + \sigma\sqrt{\delta t} & ; \frac{1}{2} \\ \mu\delta t - \sigma\sqrt{\delta t} & ; \frac{1}{2} \end{cases}$$

$$\mathbb{E}[X] = \frac{1}{2}(\mu\delta t + \cancel{\sigma\sqrt{\delta t}}) + \frac{1}{2}(\mu\delta t - \cancel{\sigma\sqrt{\delta t}})$$
$$= \underline{\mu\delta t}$$

$$Y = X - \mu\delta t \begin{cases} \sigma\sqrt{\delta t} & ; \frac{1}{2} \\ -\sigma\sqrt{\delta t} & ; \frac{1}{2} \end{cases}$$

$$\mathbb{E}[Y^2] = \frac{1}{2}(\sigma^2\delta t) + \frac{1}{2}(\sigma^2\delta t) = \underline{\sigma^2\delta t}$$
$$\hookrightarrow \mathbb{E}[(X - \mu\delta t)^2] = Var(X)$$

Vamos a considerar que $t := n\delta t$, entonces para llegar a $t$, subimos en el árbol $j$ veces y bajamos en el árbol $n-j$ veces.




$$S(t) = S(n\delta t) = S_0\, u^{j} d^{n-j} = S_0\, e^{(\cancel{j}\mu\delta t + j\sigma\sqrt{\delta t})} \times e^{((n-\cancel{j})\mu\delta t - (n-j)\sigma\sqrt{\delta t})}$$

$$= S_0\, e^{(\mu t + (2j-n)\sigma\sqrt{\delta t})}$$

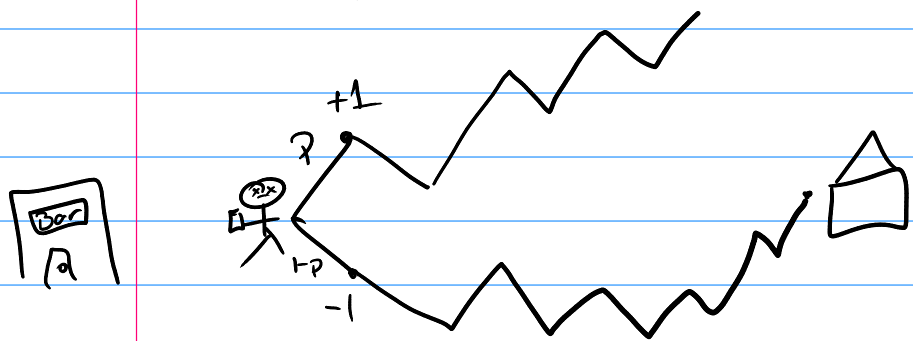
$$t = n\delta t \iff \delta t = \frac{t}{n} \Rightarrow \sqrt{\delta t} = \frac{\sqrt{t}}{\sqrt{n}}$$

¿Qué va a pasar cuando $n \to \infty$?
¿ " " " " " $\delta t \to 0$, tomando $t = n \cdot \delta t$ constante?

$$S_t = S_0 \exp\left(\mu t + \sigma \frac{\sqrt{t}}{\sqrt{n}}(2Y - n)\right)$$

$$Y = \sum_{i=1}^{n} \varepsilon_i \text{, donde } \varepsilon_i = \begin{cases} 1 ; & \frac{1}{2} \\ 0 ; & \frac{1}{2} \end{cases}$$

→ v.a. i.i.d




$$\mathbb{E}[Y] = \frac{n}{2} = \mathbb{E}\left[\sum \varepsilon_i\right] = \sum \mathbb{E}[\varepsilon_i] = \sum_{i=1}^{n} \frac{1}{2} = \frac{n}{2}$$

$$Var(Y) = Var\left(\sum_{i=1}^{n} \varepsilon_i\right) = \sum_{i=1}^{n} Var(\varepsilon_i) = \sum_{i=1}^{n} \frac{1}{4} = \frac{n}{4}$$

$$\mathbb{E}\left[\frac{Y}{n}\right] = \frac{1}{2} \qquad Var\left(\frac{Y}{n}\right) = \frac{1}{n^2} Var(Y) = \frac{1}{4n}$$

⇒ Por el teorema del límite central $\quad \frac{X - \mu}{\sigma}$

$$\underbrace{\frac{\frac{Y}{n} - \frac{1}{2}}{1/2\sqrt{n}}}_{} \xrightarrow[n \to \infty]{D} N(0,1) \qquad N(0,1)$$

$$\Uparrow D$$

$$\hookrightarrow 2\sqrt{n}\left(\frac{Y}{n} - \frac{1}{2}\right) = \frac{2Y}{\sqrt{n}} - \sqrt{n}\left(\frac{\sqrt{n}}{\sqrt{n}}\right) = \frac{2Y - n}{\sqrt{n}}$$

Recordemos que

$$S_t = S_0 \exp\left( \mu t + \frac{\sigma\sqrt{t}\,(2y-n)}{\sqrt{n}} \right)$$

cuando $n \to \infty$

$$S_t \overset{D}{\Rightarrow} S_0 \underbrace{\exp(\mu t + \sigma\sqrt{t}\, Z)}; \quad Z \sim N(0,1)$$

$$y = \exp(\mu t + \sigma\sqrt{t}) \sim \underline{\log} \text{ normal}$$

Vemos que dicho modelo funciona cuando las probabilidades son $\frac{1}{2}$; pero si no es así, ¿sigue funcionando todo?

Vamos a tomarnos un caso muy particular.

$$\begin{tikzcd} & S_u \\ S_t \arrow[ur, "q"] \arrow[dr, "1-q"'] & \\ & S_d \end{tikzcd}$$

$$q = \frac{e^{r\delta t} - e^{\mu\delta t - \sigma\sqrt{\delta t}}}{e^{\mu\delta t + \sigma\sqrt{\delta t}} - e^{\mu\delta t - \sigma\sqrt{\delta t}}}$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$

Vamos a asumir una $\delta t$ muy pequeña.

$$\Rightarrow q \approx \frac{1}{2}\left[ 1 - \sqrt{\delta t}\, \frac{(\mu - r) + \sigma^2/2}{\sigma} \right]$$

Ahora:

$$S_t = S_0 \exp\left(\mu t + \sigma\sqrt{t}\left(\frac{2y-n}{\sqrt{n}}\right)\right)$$

$$y = \sum_{i=1}^{n} \eta_i \qquad \eta_i = \begin{cases} 1 & q \\ 0 & 1-q \end{cases} \rightsquigarrow \eta_i \sim \text{Bernoulli}(q)$$

$$y \sim \text{Binomial}(n, q)$$

$$\mathbb{E}[y] = nq \quad , \quad \text{Var}(y) = nq(1-q)$$

$$\mathbb{E}\left[\frac{2y-n}{\sqrt{n}}\right] = \frac{2nq-n}{\sqrt{n}} = \sqrt{n}\,(2q-1)$$

$$\approx \sqrt{n}\left(\cancel{2}\left(\frac{1}{\cancel{2}}\left[\cancel{1} - \sqrt{\delta t}\,\frac{\mu - r + \frac{1}{2}\sigma^2}{\sigma}\right]\right) - \cancel{1}\right)$$

$$= \sqrt{n}\cdot\sqrt{\delta t}\left(-\frac{\mu - r + \frac{1}{2}\sigma^2}{\sigma}\right)$$

$$= \underbrace{-\sqrt{t}\left(\frac{\mu - r + \frac{1}{2}\sigma^2}{\sigma}\right)}_{m}$$

$$\text{Var}\left(\frac{2y-n}{\sqrt{n}}\right) = \frac{4}{n}\text{Var}(y) = \frac{4}{n}\, nq(1-q) \approx \underset{\sim}{1}$$

Como $q = \frac{1}{2}\left(1 - \underbrace{\sqrt{\delta t}}\,\frac{\mu - r + \frac{1}{2}\sigma^2}{\sigma}\right) \approx \frac{1}{2}$

$\hookrightarrow \delta t$ es muy pequeña

Por el teorema del límite central

$$\frac{\frac{2y-n}{\sqrt{n}} - m}{\sqrt{v}} \xrightarrow[n \to \infty]{D} N(0, 1)$$

$$\Leftrightarrow \frac{2y-n}{\sqrt{n}} \xrightarrow{D} N(m, v)$$
$$= N(m, 1)$$

$W \sim N(\mu, \sigma)$
$\parallel$
$\sigma V + \mu$ ; $V \sim N(0,1)$

$$S_t = S_0 \exp(\mu t + \sigma \sqrt{t}\, Z')$$
$$Z' \sim N\left(\sqrt{t}\, \frac{r-\mu-\frac{1}{2}\sigma^2}{\sigma}, 1\right)$$

$$= S_0 \exp\left(\mu t + \sigma\sqrt{t}\left[\sqrt{t}\left(\frac{r-\mu-\frac{1}{2}\sigma^2}{\sigma}\right) + Z\right]\right)$$

$Z \sim N(0,1)$

$$= S_0 \exp\left(\cancel{\mu t} + t\left(r \cancel{-\mu} - \frac{1}{2}\sigma^2\right) + \sigma\sqrt{t}\, Z\right)$$
$$= S_0 \exp\left(rt - \frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\, Z\right)$$
$$= S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}\, Z\right)$$

Recordemos que para la probabilidad $p = \frac{1}{2}$

$$S_t = S_0 \exp(\mu t + \sigma \sqrt{t} Z) \quad Z \sim N(0,1)$$

Pero, cuand usamos $p = q$

$$\underline{S_t} = S_0 \exp\left( (r - \tfrac{1}{2}\sigma^2) t + \sigma \sqrt{t} Z \right) ; \quad Z \sim N(0,1)$$

$$X = (S_T - K)^+$$

$$\underline{c = e^{-rT} \mathbb{E}[(S_T - K)^+]}$$