Recapitulando:

$$S(t) \sim \text{Log Normal}\left(\left(r - \tfrac{1}{2}\sigma^2 t\right), \sigma^2 t \right)$$

↳ cuando $\delta t \to 0$

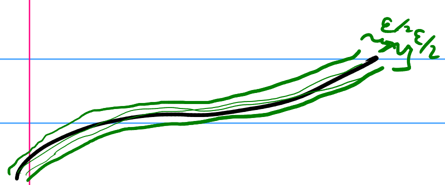
-> Solo estamos probando una convergencia "débil" (Teorema de límite central)

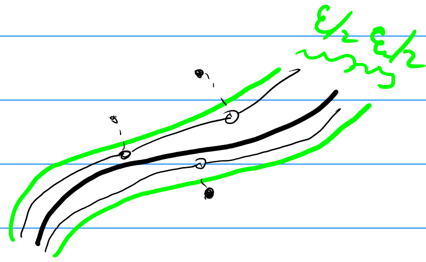
(Si $\{X_n\}_{n=1}^{\infty}$ una sucesión infinita de variables aleatorias, decimos que:

$X_n \xrightarrow{D} X$ (X_n converge en distribución/débilmente a X)

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \; ; \; F_{X_n}(x) = \mathbb{P}[X_n \le x]$$

Uniforme → Casi Segura





El conjunto de valores que salen de la banda tiene medida cero.

Alan F. Karr
Probability

$$\mathbb{P}[\lim_{n\to\infty} X_n = X] = 1$$

Convergencia en probabilidad

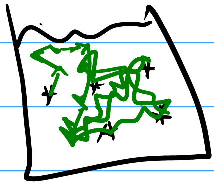
$$\lim_{n\to\infty} \mathbb{P}[|X_n - X| > \varepsilon] = 0$$

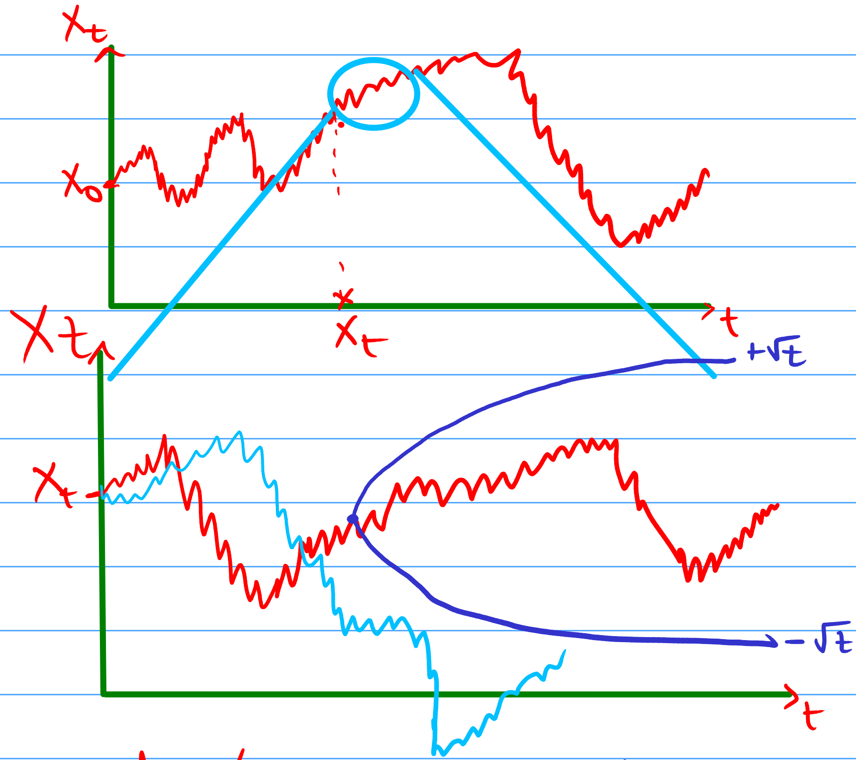
Convergencia en media cuadrada

$$\lim_{n\to\infty} \mathbb{E}[|X_n - X|^2] = 0$$

Convergencia débil

$$\underbrace{\text{Movimiento Browniano}}_{\text{Robert Brown}} \quad (\sim 1827)$$





$$\frac{dX_t}{dt} \nexists$$

$X_t \sim N(0, t)$

↳ $\sqrt{t}$

→ Tiene incrementos independientes

$$(X_{t+\delta} - X_t) \perp (X_t - X_{t-\delta})$$

$$X_t \sim N(0, \sigma^2 t)$$

Tiene incrementos estacionarios

$$X_{t+\delta} - X_t \overset{d}{=} X_\delta \sim N(0, (t+\delta-\cancel{t})\sigma^2)$$

$$X_t - X_s \sim N(0, (t-s)\sigma^2)$$

Mov. browniano estándar ($\sigma = 1$)

Louis Bachelier (1900)

- Asume que los precios de las acciones siguen un movimiento browniano
- Black, Scholes & Merton
  - Asumir un proceso lognormal para las acciones.

Lema: Supongamos que tenemos una variable aleatoria $Y \sim N(m, s^2)$, entonces para $a, b \in \mathbb{R}$

$$\mathbb{E}\left[e^{aY}\mathbb{1}_{(b,\infty)}\right] = \int_b^{\infty} e^{ay} \cdot \frac{1}{\sqrt{2\pi}\, s} \exp\left(\frac{-(y-m)^2}{2s^2}\right) dy = \circledast$$

$$= e^{am + \frac{1}{2}a^2 s^2} N(d)$$

$$d = \frac{-b + m + as^2}{s}$$

Demostración:

$$ay - \frac{(y-m)^2}{2s^2} = am + \frac{1}{2}a^2 s^2 - \frac{\left[y - (m + as^2)\right]^2}{2s^2}$$

Entonces

$$\circledast = e^{am + \frac{1}{2}a^2 s^2} \int_b^{\infty} \frac{1}{\sqrt{2\pi}\, s} \exp\left(\frac{-\left[y - (m + as^2)\right]^2}{2s^2}\right) dy$$

Definimos $u = \dfrac{y - (m + as^2)}{s}$ $\quad c = \dfrac{b - (m + as^2)}{s}$

$$du = \frac{dy}{s} \iff dy = du \cdot s$$

$$b < y < \infty \longrightarrow c < u < \infty$$

$$= e^{am+\frac{1}{2}a^2s^2} \int_c^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx$$

$$= e^{am+\frac{1}{2}a^2s^2} \left[1 - N(c)\right] = e^{am+\frac{1}{2}a^2s^2} N(-c)$$

$$-c = \frac{-b \quad + m + as^2}{2} = d \qquad \boxed{= e^{am+\frac{1}{2}a^2s^2} N(d)}$$

$\Rightarrow$ ¿Qué pasa con un call con strike K?

$\hookrightarrow P = (S_T - K)_+ \quad ; \quad S_T = S_0 \exp\left(r - \frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\, z\right)$

$z \sim N(0, 1)$

Considerando que vamos a calcula el precio del call como:

$$c = e^{-rT} \mathbb{E}\left[(S_T - K)_+\right] = e^{-rT} \mathbb{E}\left[(S_0 e^{y} - K)_+\right] =$$

donde $y \sim N(r - \frac{1}{2}\sigma^2 T, \sigma^2 T)$

$$= e^{-rT} \int_{-\infty}^{\infty} (S_0 e^{y} - K)_+ \, f_y(y) \, dy$$

$$f_y(y) = \frac{1}{\sigma\sqrt{2\pi T}} \exp\left(- \frac{y - (r - \frac{1}{2}\sigma^2 T)^2}{2\sigma^2 T}\right)$$

$$S_0 e^{y} - K > 0 \iff y > \ln\left(\frac{K}{S_0}\right) = \underline{b}$$

$$\Rightarrow e^{-rT}\int_b^\infty (S_0 e^{y} - K) f_y(y)\, dy =$$

$$= \underbrace{e^{-rT}\int_b^\infty S_0 e^{y} f_y(y)\, dy}_{A} - \underbrace{e^{-rT}\int_b^\infty K f_y(y)\, dy}_{B}$$

$$B = Ke^{-rT}\int_b^\infty f_y(y) dy = Ke^{-rT}\,\mathbb{P}[y > b]$$

$$= Ke^{-rT}\,\mathbb{P}\Big[\underbrace{\frac{y-(r-\frac{1}{2}\sigma^2 T)}{\sigma^2 T}}_{\sim N(0,1)} > \frac{b-(r-\frac{1}{2}\sigma^2 T)}{\sigma^2 T}\Big]$$

$$= Ke^{-rT}\,\mathbb{P}\Big[Z > \frac{b - r + \frac{1}{2}\sigma^2 T}{\sigma^2 T}\Big]$$

$$= Ke^{-rT}\Big[1 - \mathbb{P}\Big[Z \le \frac{b - r + \frac{1}{2}\sigma^2 T}{\sigma^2 T}\Big]\Big]$$

$$\underbrace{\ln(a) = -\ln\left(\frac{1}{a}\right)}$$

$$= Ke^{-rT}\,\mathbb{P}\Big[Z \le -\Big(\frac{b - r + \frac{1}{2}\sigma^2 T}{\sigma^2 T}\Big)\Big] \; ; \; b = \ln\left(\frac{K}{S_0}\right)$$

$$= Ke^{-rT}\,\mathbb{P}\Big[Z \le \frac{-\ln(\frac{K}{S_0}) + r - \frac{1}{2}\sigma^2 T}{\sigma^2 T}\Big] = Ke^{-rT} N(d_2)$$

$$d_2 = \frac{\ln(\frac{S_0}{K}) + r - \frac{1}{2}\sigma^2 T}{\sigma^2 T}$$

$$A = e^{-rT}\int_{b}^{\infty} S_0 e^{y} f_y(y) = S_0 e^{-rT}\int_{b}^{\infty} e^{1 \cdot y} f_y(y)$$

Por el Lema demostrado anteriormente:

$$a = 1 \; ; \quad b = \ln\left(\frac{K}{S_0}\right) \; ; \quad m = (r - \tfrac{1}{2}\sigma^2)T \quad s^2 = \sigma^2 T$$

$$\Rightarrow S_0 e^{-rT} \cdot e^{am + \frac{1}{2}a^2 s^2} N(d) \qquad d = \frac{-b + m + as^2}{s}$$

$$= S_0 e^{-\cancel{rT}} e^{\cancel{r}(-\cancel{\frac{1}{2}\sigma^2})T + \cancel{\frac{1}{2}\sigma^2}T} \cdot N\left(\frac{\ln\left(\frac{S_0}{K}\right) + (r - \frac{1}{2}\sigma^2)T + \sigma^2 T}{s}\right)$$

$$= S_0 N(d_1) \; ; \quad d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

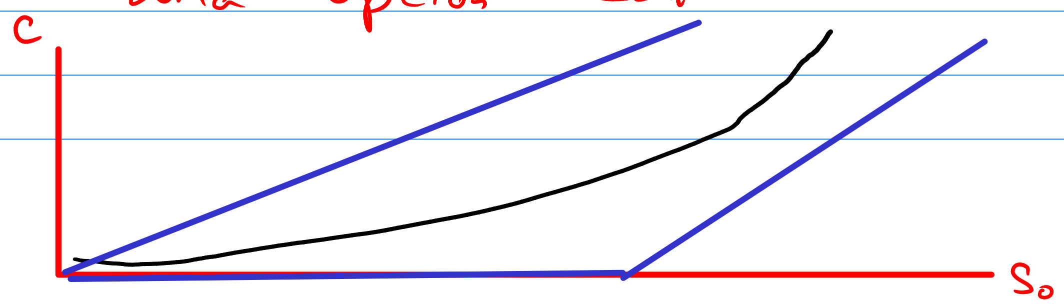
$$d_2 = d_1 - \sigma\sqrt{T}$$

$$N(x) = \int_{-\infty}^{x} f_y(y)\,dy \qquad y \sim N(0,1)$$

$$\boxed{C = S_0 N(d_1) - Ke^{-rT} N(d_2) \qquad d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \qquad d_2 = d_1 - \sigma\sqrt{T}}$$

$d_1 = 0.27$

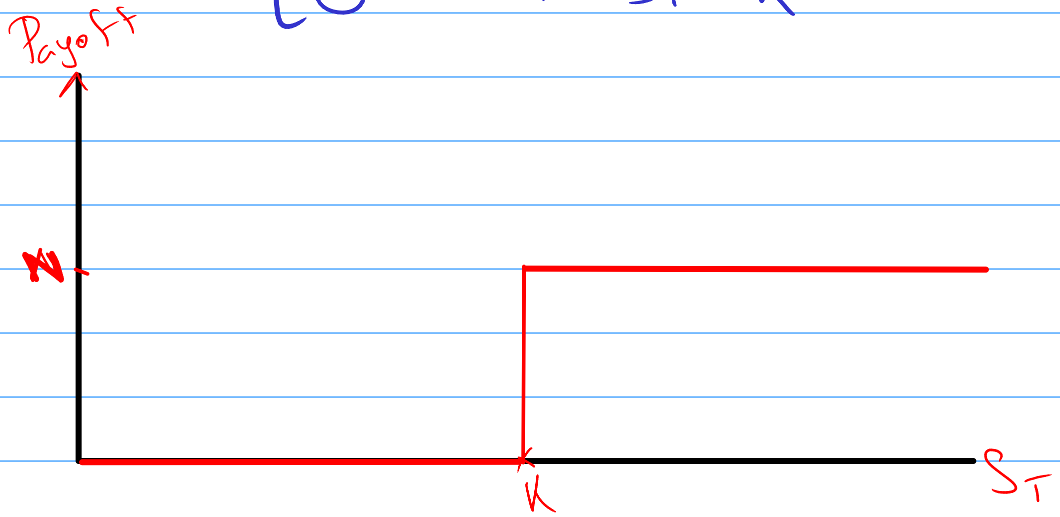
Ecuación de Black & Scholes para una opción Call



Opciones Digitales:

Supongamos que tenemos una opción cuyo payoff es de la siguiente forma:

$$\text{Payoff}\begin{cases} N & \text{si } S_T > K \\ 0 & \text{si } S_T < K \end{cases} \rightarrow N\mathbb{1}_{(S_T>K)}$$



$$c = e^{-rT}\,\mathbb{E}[\mathbb{1}_{(S_T>K)}N] = e^{-rT}\left[N\,\mathbb{P}[S_T > K] + \cancel{0\,\mathbb{P}[S_T \leq K]}^{\,0}\right]$$

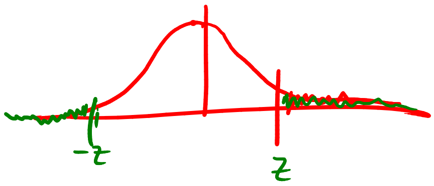
$$= N e^{-rT}\,\mathbb{P}[S_T > K] = N e^{-rT}\,\mathbb{P}[S_0 e^{y} > K]$$

$$y = (r - \tfrac{1}{2}\sigma^2 T) + \sigma\sqrt{T}\,Z$$
$$Z \sim N(0,1)$$

$$= N e^{-rT}\,\mathbb{P}\left[e^{y} > \frac{K}{S}\right] = N e^{-rT}\,\mathbb{P}\left[y > \ln\left(\frac{K}{S}\right)\right]$$

$$= N e^{-rT}\,\mathbb{P}\left[\underbrace{\frac{y - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}_{Z} > \frac{\ln\left(\frac{K}{S}\right) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right]$$

$\mathbb{P}[Z > a] = \mathbb{P}[Z < -a]$



$$= N e^{-rT} \mathbb{P}\left[Z > \frac{\ln\left(\frac{K}{S_0}\right) - \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right]$$

$$= N e^{-rT} \mathbb{P}\left[Z < -\left[\frac{\ln\left(\frac{K}{S_0}\right) - \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right]\right]$$

$$= N e^{-rT} N(d_2) \; ; \quad d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$