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Brownian Motion II
Solutions

Question 1. Show that a.s. linear Brownian motion has infinite variation, that is

$$V_B^{(1)}(t) = \sup \sum_{j=1}^{k} |B_{t_j} - B_{t_{j-1}}| = \infty$$

with probability one, where the supremum is taken over all partitions (t_j), $0 = t_0 < t_1 < \ldots < t_k = t$, of the interval $[0, t]$.

Solution. It was shown in the lecture that

$$\sup \sum_{j=1}^{k} |B_{t_j} - B_{t_{j-1}}|^2 \xrightarrow[\text{a.s.}]{k\to\infty} t,$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < \ldots < t_k = t$. We have

$$\sum_{j=1}^{k} |B_{t_j} - B_{t_{j-1}}|^2 \le V_B^{(1)}(t) \cdot \sup_j |B_{t_j} - B_{t_{j-1}}|.$$

By the uniform continuity of B on $[0, t]$ we get that as k goes to infinity, the supremum on the right hand side goes to 0 if the diameter of the partition (t_k) goes to zero. The left hand side goes to a positive t a.s., hence $V_B^{(1)}(t) = \infty$ a.s. □

Question 2. Let B be a standard linear Brownian motion. Define

$$D^*(t) = \overline{\lim}_{h\to 0} \frac{B_{t+h} - B_t}{h}, \qquad D_*(t) = \underline{\lim}_{h\to 0} \frac{B_{t+h} - B_t}{h}.$$

It was shown in the lecture that a.s., for every $t \in [0, 1]$ either $D^*(t) = +\infty$ or $D_*(t) = -\infty$ or both. Prove that

(a) for every $t \in [0, 1]$ we have $\mathbb{P}\left(B \text{ has a local maximum at } t\right) = 0$

(b) almost surely, local maxima of B exist

(c) almost surely, there exist $t_*, t^* \in [0, 1]$ such that $D^*(t^*) \le 0$ and $D_*(t_*) \ge 0$.

Solution. Fix $t \in (0,1)$. We have

$$\begin{aligned}
\mathbb{P}(t \text{ is a local maximum of } B) &= \mathbb{P}(\exists \epsilon > 0\ \forall 0 < |h| < \epsilon\ B_t - B_{t+h} \geq 0) \\
&\leq \mathbb{P}(\exists \epsilon > 0\ \forall 0 < h < \epsilon\ B_t - B_{t+h} \geq 0) \\
&= \mathbb{P}(\exists \epsilon > 0\ \forall 0 < h < \epsilon\ B_h \geq 0) \\
&= 1 - \mathbb{P}\left(\forall \epsilon > 0\ \sup_{0<h<\epsilon} B_h\right) > 0 \\
&= 1 - \mathbb{P}\left(\forall n = 1, 2, \ldots \sup_{0<h<1/n} B_h > 0\right).
\end{aligned}$$

The event

$$A = \left\{\forall n = 1, 2, \ldots \sup_{0<h<1/n} B_h > 0\right\} = \bigcap_{n=1}^{\infty} \left\{\sup_{0<h<1/n} B_h > 0\right\}$$

belongs to $\mathcal{F}_{0+} = \bigcap_{t>0} \mathcal{F}_t$. By Blumenthal's 0-1 law, $\mathbb{P}(A) \in \{0,1\}$. But

$$\mathbb{P}(A) = \lim_{n\to\infty} \mathbb{P}\left(\sup_{0<h<1/n} B_h > 0\right) \geq \lim_{n\to\infty} \mathbb{P}\left(B_{1/(2n)} > 0\right) = \frac{1}{2}.$$

Hence, $\mathbb{P}(A) = 1$ and, consequently, $\mathbb{P}(t \text{ is a local maximum of } B) = 1$.

It follows from the continuity of paths that a global maximum of B on $[0,1]$ always exists, which is also a local maximum.

If we take t^* to be a local maximum and t_* to be a local minimum, then $D^*(t^*) \leq 0$ and $D_*(t_*) \geq 0$. □

Question 3. Let B be a standard linear Brownian motion. Show that a.s.

$$\overline{\lim}_{n\to\infty} \frac{B_n}{\sqrt{n}} = +\infty \text{ and } \underline{\lim}_{n\to\infty} \frac{B_n}{\sqrt{n}} = -\infty.$$

You may want to use the Hewitt-Savage $0-1$ law which states that

Theorem (Hewitt-Savage). *Let $X_1, X_2, \ldots$ be a sequence of i.i.d. variables. An event $A = A(X_1, X_2, \ldots)$ is called* exchangeable *if $A(X_1, X_2, \ldots) \subset A(X_{\sigma(1)}, X_{\sigma(2)}, \ldots)$ for any permutation σ of the set $\{1, 2, \ldots\}$ whose support $\{k \geq 1,\ \sigma(k) \neq k\}$ is a finite set. Then for every exchangeable event A we have $\mathbb{P}((A) \in \{0,1\})$.*

Solution. Fix $c > 0$ and take $A_c = \limsup_n \{B_n > c\sqrt{n}\}$. We want to show that $\bigcap_{c=1}^{\infty} A_c$ has probability one. Plainly, $\mathbb{P}\left(\bigcap_{c=1}^{\infty} A_c\right) = \lim_{c\to\infty} \mathbb{P}(A_c)$. Let $X_n = B_n - B_{n-1}$. They are i.i.d. Notice that

$$A_c = \limsup_n \left\{ \sum_{j=1}^{n} X_j > c\sqrt{n} \right\}$$

is an exchangeable event. By the Hewitt-Savage 0-1 law we obtain that $\mathbb{P}(A_c) \in \{0, 1\}$. Since

$$\mathbb{P}(A_c) \geq \limsup_{n\to\infty} \mathbb{P}\left(B_n > c\sqrt{n}\right) = \mathbb{P}(B_1 > c) > 0,$$

we conclude that $\mathbb{P}(A_c) = 1$.

The claim about lim inf can be proved similarly. $\square$