Brownian Motion: Problem Set 2

Q1. Show that a.s. Brownian motion has infinite variation, that is

$$V_\beta^{(1)}(t) := \sup \sum_{j=1}^{k} |\beta(t_j) - \beta(t_{j-1})| = \infty,$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < \cdots < t_k = t$.

Q2. Define

$$D^*(t) := \limsup_{h\to 0} \frac{\beta(t+h) - \beta(t)}{h}, \qquad D^*(t) := \liminf_{h\to 0} \frac{\beta(t+h) - \beta(t)}{h}$$

We showed in class that a.s. for every $t \in [0,1]$ either $D^*(t) = +\infty$ or $D_*(t) = -\infty$ or both.

- **a.** Show that, for every fixed $t \in [0,1]$, $P\big(t \text{ is a local maximum of } \beta(\cdot)\ \big) = 0$.
- **b.** Almost surely, local maxima exist.
- **c.** Almost surely, there exist $t_*, t^* \in [0,1]$, such that $D^*(t^*) \leq 0$ and $D_*(t_*) \geq 0$.

Q3. Show that a.s.

$$\limsup_{n\to\infty} \frac{\beta(n)}{\sqrt{n}} = +\infty \text{ and } \liminf_{n\to\infty} \frac{\beta(n)}{\sqrt{n}} = -\infty.$$

To prove this, make use of the Hewitt-Savage 0-1 law, which states that

Theorem 1 (Hewitt-Savage)**.** *Let $X_1, X_2, ...$ a sequence of i.i.d. variables. An event $A = A(X_1, X_2, ...)$ is called exchangeable if $A(X_1, X_2, ...) \subset A(X_{\sigma_1}, X_{\sigma_2}, ...)$ for any finite permeation σ of $\{1, 2, ...\}$. Finite permutation means that there exists some finite number N_0, such that $\sigma_n = n$ for all $n \geq N_0$.*

Then, for any exchangeable set A, it holds that $P(A) \in \{0, 1\}$.