Brownian Motion: Problem Set 3

Q1. Let f be a smooth function and $\beta(\cdot)$ standard Brownian motion. Show that

$$f(t, \beta(t)) - \int_0^t (f_t + \frac{1}{2}\Delta f)(s, \beta(s))\, ds,$$

is a Martingale. Use this to write a solution for

$$\begin{aligned} u_t &= \frac{1}{2}\Delta u, \qquad t > 0, x \in \mathbb{R}^d \\ u(0, x) &= f(x) \end{aligned}$$

Q2. Let a cylinder $\mathcal{C} := B(0; 1) \times \mathbb{R}_+$, where $B(0, 1)$ is the unit disk centred at zero. Solve the problem

$$\begin{aligned} u_t &= \frac{1}{2}\Delta u, \qquad \text{in } \mathcal{C}, \\ u(0, x) &= f(x), \qquad \text{on } B(0, 1), \\ u(i, x) &= g(t, x) \qquad \text{on } (0, 2\pi] \times \mathbb{R}_+. \end{aligned}$$

Q3. Let $\beta(\cdot)$ be a d- dimensional standard Brownian motion. For which dimensions, does it hit a single point $x \in \mathbb{R}^d$, different than its starting location?

Q4. Let $f \geq 0$ be a function with compact support in the upper half plane of $\mathbb{R}^d$, i.e. $H = \{y \in \mathbb{R}^d \colon y_d \geq 0\}$. Show that

$$E_x \int_0^\tau f(\beta(t))dt = \int G(x, y)f(y)dy - \int G(x, \bar{y})f(y)dy,$$

where $G(x, y)$ is the Green's function on $\mathbb{R}^d$, τ is the hitting time of the plane $y_d = 0$ and if $y = (y_1, ..., y_{d-1}, y_d)$ then $\bar{y} = (y_1, ..., y_{d-1}, -y_d)$. This shows that the Green's function in the upper half plane is given by $G(x, y) - G(x, \bar{y})$.

Use only probabilistic arguments for the derivation. You should also explain why τ is a.s. finite. i.e. the left hand side is well defined.