Numerical Methods

Path Dependant Options and Greeks

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1 Options

1.1 Asian options

An Asian option is an option where the payoff is not determined by the underlying price at maturity but by the average underlying price over some preset period of time. For example an Asian call option might pay MAX(DAILY-AVERAGE-OVER-LAST-THREE-MONTHS(S) - X, 0). Asian options were originated in Asian markets to prevent option traders from attempting to manipulate the price of the underlying on the exercise date.

The payoff of the Asian call option is

$$\max\left(0, \frac{1}{N}\sum_{n=1}^{N} S_n - X\right),$$

where S_n are daily closing prices of the underlying and X is the strike price.

1.1.1 Monte-Carlo method for Asian options

A straight forward approach to price Asian options:

```
% asiancall.m
function [price,variance] = asiancall(S0,X,r,T,sigma,NRepl, Npts)
% Europian call pricing.
% [price, variance] = asiancall(SO,X,r,T,sigma,NRep, NSteps)
% returns the value of Asian call option using Monte-Carlo method.
%       S0     is the current asset price,
%       X      is the exercise price, r is the risk-free interest rate,
%       T      is the time to maturity of the option in years,
%       sigma  is volatility
%       NRep   is the number of samples
%       NSteps is the number of time subintervals

dt         = T/NSteps;
```

```
nudt        = (r-0.5*sigma^2)*dt;
sidt        = sigma*sqrt(dt);
Increments = nudt + sidt*randn(NRep, NSteps);
SPaths      = exp(cumsum([log(S0)*ones(NRep,1) , Increments] , 2));
DiscPayoff = exp(-r*T) *max( 0 , mean(SPaths,2) - X);
price       = mean(DiscPayoff);
variance    = var(DiscPayoff);
```

Here we first generate $NRep$ paths using

$$U_{j+1} = U_j \exp((r - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}\epsilon_j)$$

where $\epsilon_j \sim N(0,1)$. Then we find average prices along each path. We use the following formula for the price of the Asian call option:

$$C = e^{-rT} E_{\epsilon \sim N(0,1)} \max(0, \frac{1}{N}\sum_{j=1}^{N} U_j - X).$$

1.2 Barrier Options

A down-and-out call option has discounted payoff

$$\exp(-rT)\max(0, S(T) - X)1_{\min_t S(t)>B},$$

i.e. it is like a standard call option except that it pays nothing if the minimum value drops below the barrier B.

The natural numerical discretisation of this is

$$\exp(-rT)\max(0, U_N - X)1_{\min_k U_k>B},$$

where U_k is a numerical approximation of $S(t_k)$, $t_k = k\Delta t$ and $\Delta t = T/N$.

Suppose that S satisfies SDE

$$dS = a(S,t)\,dt + b(S,t)\,dW.$$

To price the Barrier option we can use Euler scheme. In this case it is possible to check that the week error for this problem is not of order $O(\Delta t)$ but of order $O(\sqrt{\Delta t})$. This is true even in the case of geometrical Brownian motion.

1.2.1 Barrier Crossing

Let $W(t)$ be just a Brownian motion with $W(0) > 0$ and $W(\Delta t) > 0$. What is the probability that $W(t) = 0$ for some $t \in (0, \Delta t)$?

If the path $W(t)$ hits a barrier at 0, it is equally likely thereafter to go up or down. Hence, by symmetry, for $x > 0$, the p.d.f. for paths with $W(\Delta t) = x$ after hitting the barrier is equal to the p.d.f. for paths with $W(\Delta t) = -x$.

Therefore, the probability that $W(t) = 0$ for some $t \in (0, \Delta t)$ is

$$\begin{aligned} P(W \text{ hits } 0 | W(0) > 0, W(\Delta t) > 0) &= \frac{\exp\left(-\frac{(-W(\Delta t) - W(0))^2}{2\Delta t}\right)}{\exp\left(-\frac{(W(\Delta t) - W(0))^2}{2\Delta t}\right)} \\ &= \exp\left(-\frac{2W(0)W(\Delta t)}{\Delta t}\right) \end{aligned}$$

The previous formula can be generalized to the case of SDE

$$dS = a(S,t)\,dt + b(S,t)\,dW$$

in the following way:

$$\begin{aligned} P(S \text{ hits Barrier } B \text{ on } (t, t+\Delta t) | S(t), S(t+\Delta t)) \quad \approx \\ \exp\left(-\frac{2(S(t)-B)(S(t+\Delta t)-B)}{b(S(t),t)^2 \Delta t}\right), \end{aligned}$$

if $(S(t) - B)(S(t+\Delta t) - B) > 0$.

By inverting this formula we can find the distribution of $\min_{t \in (t, t+\Delta t)} S(t)$ given the boundary values $S(t)$ and $S(t + \Delta t)$:

$$\min_{t \in (t, t+\Delta t)} S(t) \approx$$

$$\frac{1}{2}\left(S(t) + S(t+\Delta t) - \sqrt{(S(t) - S(t+\Delta t))^2 - 2b(S(t),t)^2 \Delta t \log \xi}\right),$$

where ξ is uniformly distributed on $[0, 1]$.

So, for pricing Barrier options by Monte-Carlo method we can use one of the following approaches:

1. “Naive” method;
2. Estimating the probability of barrier crossing;
3. Simulating minimum of $S(t)$ on $(t, t + \Delta t)$.

All three methods start the same: first we generate numerically approximations U_n of the sample paths of the solutions of SDE $dS = a(S,t)\,dt + b(S,t)\,dW$ using Milstein (or some other) scheme.

1.2.2 "Naive" approach 1.

In the naive approach we estimate the payoff function as

$$\exp(-rT) \max(0, U_N - X) 1_{\min_{n=0,\ldots,N} U_n > B}$$

As it was already mentioned this approach has the disadvantage of converging as $O(\sqrt{\Delta t})$.

1.2.3 Approach 2.

First for each interval $(n\Delta t, (n+1)\Delta t)$ compute the probability of the barrier crossing:

$$P_n = \begin{cases} 1 & \text{if } U_n \leq B \text{ or } U_{n+1} \leq B \\ \exp\left(-\frac{2(U_n-B)(U_{n+1}-B)}{b(U_n,n\Delta t)^2\Delta t}\right) & \text{otherwise} \end{cases}$$

Then the payoff function is

$$\exp(-rT)\max(0, U_N - X)\prod_{n=0}^{N-1}(1-P_n).$$

This method gives $O(\Delta t)$ convergence. Moreover, notice the payoff function depends continuously on the input data ($\{U_n\}$), this is good for "Greeks" computations.

1.2.4 Approach 3.

In this approach we will (pseudo) randomly generate minimal values of S on interval $(n\Delta t, (n+1)\Delta t)$.

For each sample $\{U_n\}$ generate N uniformly distributed random samples ξ_n and compute

$$M_n = \frac{1}{2}\left(U_n + U_{n+1} - \sqrt{(U_n - U_{n+1})^2 - 2b(U_n, n\Delta t)^2\Delta t\log\xi_n}\right).$$

Then the payoff function is

$$\exp(-rT)\max(0, U_N - X)1_{\min_{n=0,\ldots,N-1} M_n > B}$$

This method gives $O(\Delta t)$ convergence as well. However, it is not that good for "Greeks" because the payoff function depends discontinuously on the input data ($\{U_n\}$ and $\{M_n\}$).

1.3 Lookback options

A floating-strike lookback call option has discounted payoff

$$\exp(-rT)(S(T) - \min_t S(t)).$$

The pricing of this option can be done using similar method to the third approach in the previous section.

First we generate numerically approximations U_n of the sample paths of the solutions of SDE $dS = a(S,t)\,dt + b(S,t)\,dW$.

For each sample $\{U_n\}$ generate N uniformly distributed random samples ξ_n and compute

$$M_n = \frac{1}{2}\left(U_n + U_{n+1} - \sqrt{(U_n - U_{n+1})^2 - 2b(U_n, n\Delta t)^2\Delta t\log\xi_n}\right).$$

Then the payoff function is

$$\exp(-rT)(U_N - \min_{n=0,\ldots,N-1} M_n).$$

2 Greeks

2.1 Finite Difference Method

Finite differences can again be used to estimate Greeks, with all of the advantages/disadvantages discussed previously.

2.2 Likelihood ratio method

Recall that the Likelihood ratio method works as follows: Let $P(S)$ be the probability density function and assume that $P(S)$ depends on θ and the payoff function does not. The option price is

$$C = E(f) = \int f(S)P(S)\,dS.$$

Then,

$$\frac{\partial C}{\partial \theta} \quad = \quad E(f\frac{\partial \log P}{\partial \theta}).$$

Let us consider dependence of the Asian option on the volatility, i.e. we want to compute Vega of the Asian option.

For simplicity we will assume geometrical Brownian motion model, but we will not use the fact that this model is integrable.

We can estimate the price of the Asian option as

$$C = \exp(-rT)E\max(0, \frac{1}{N}\sum_{j=1}^{N} U_j - X),$$

where we compute $\{U_j\}$ using Euler scheme:

$$U_{j+1} = U_j(1 + r\Delta t + \sigma\epsilon_j\sqrt{\Delta t}).$$

Rewrite the formula for C as

$$C \quad = \quad \exp(-rT)\int f(U_0, \ldots, U_N)P(U_0, \ldots, U_N)\,dU_1 \ldots dU_N.$$

The function P is the product of the probability density functions for each time step:

$$P(U_0, \ldots, U_N) = \prod_{j=1}^{N} p_j(U_j|U_{j-1}).$$

Hence,

$$\frac{\partial}{\partial \sigma}\log P = \sum_{j=1}^{N} \frac{\partial}{\partial \sigma}\log p_j(U_j|U_{j-1}).$$

For the Euler approximation of geometrical Brownian motion p_j is $N(U_{j-1}(1+r\Delta t), U_{j-1}\sigma\sqrt{\Delta t})$ normally distributed, so

$$\begin{aligned}\frac{\partial}{\partial\sigma}\log p_j &= \frac{(U_j - U_{j-1}(1+r\Delta t))^2}{\sigma^3 U_{j-1}^2 \Delta t} - \frac{1}{\sigma} \\ &= \frac{\epsilon_j^2 - 1}{\sigma}\end{aligned}$$

So, the estimation for Vega is

$$\exp(-rT)E\left(\max(0, \frac{1}{N}\sum_{j=1}^{N} U_j - X) * \sum_{j=1}^{N} \frac{\epsilon_j^2 - 1}{\sigma}\right).$$

Note that

$$\operatorname{var}(\epsilon_j^2 - 1) = 2,$$

hence

$$\operatorname{var}\left(\max(0, \frac{1}{N}\sum_{j=1}^{N} U_j - X) * \sum_{j=1}^{N} \frac{\epsilon_j^2 - 1}{\sigma}\right) = O(N).$$

If N is large, then the variance is also large and this method does not really work. The situation becomes even worse if instead of daily average we take continuous average (i.e. the payoff function is $\max(\frac{1}{T}\int_0^T S(t)\,dt - X, 0)$). Then N is not fixed but should be taken very large.

2.3 Pathwise derivative estimate

Let us apply Pathwise derivative method to the same problem considered in the previous section, but this time we will differentiate the payoff function itself.

Recall

$$C = \exp(-rT)E\max(0, \frac{1}{N}\sum_{j=1}^{N} U_j - X),$$

so

$$\frac{\partial}{\partial\sigma}C = \exp(-rT)E\left(\frac{1}{N}\sum_{j=1}^{N}\frac{\partial}{\partial\sigma}U_j * 1_{\frac{1}{N}\sum_{j=1}^{N} U_j > X}\right).$$

Let us denote $U_j' = \frac{\partial}{\partial\sigma}U_j$.

The derivatives of U_j we can compute recursively:

$$U_{j+1} = U_j(1 + r\Delta t + \sigma\epsilon_j\sqrt{\Delta t}),$$

hence

$$U_{j+1}' = U_j'(1 + r\Delta t + \sigma\epsilon_j\sqrt{\Delta t}) + U_j\epsilon_j\sqrt{\Delta t}.$$

Observe that $U_0' = 0$.

One can check that $\operatorname{var}(U_j')$ are bounded and, therefore, there is no variance "blow-up".