Brownian Motion IV
Solutions

Question 1. Show that Donsker's theorem can be applied to bounded functions which are continuous only a.s. with respect to the Wiener measure.

Solution. To fix the notation, by $(S_n^*(t))_{t\in[0,1]}$ we mean piecewise linear paths constructed from a standard simple random walk S_n by rescaling time by n and space by $\sqrt{n}$ (that is, from $S_{\lfloor nt\rfloor}/\sqrt{n}$). By $(B(t))_{t\in[0,1]}$ we denote standard Brownian motion in $\mathbb{R}$. Donsker's principle states that

$$S_n^* \text{ convergent in distribution to } B \quad (\text{as } (C[0,1], \|\cdot\|_\infty) \text{ valued random variables}),$$

which means that for every bounded continuous function $f\colon C[0,1] \longrightarrow \mathbb{R}$ we have

$$\mathbb{E}f(S_n^*) \xrightarrow[n\to\infty]{} \mathbb{E}f(B). \tag{$\star$}$$

In applications this might be insufficient. Consider for instance the function $f(u) = \sup\{t \le 1,\ u(t) = 0\}$, $u \in C[0,1]$, that is $f(u)$ is the last zero of a path u. Plainly, f is bounded but not continuous. Indeed, looking at the piecewise linear paths u_ϵ with $u_\epsilon(0) = 0$, $u_\epsilon(1/3) = u_\epsilon(1) = 1$ and $u_\epsilon(2/3) = \epsilon$, we have that u_ϵ converges to u_0 but $f(u_\epsilon) = 0$ for $\epsilon > 0$, but $f(u_0) = 2/3$. However, if u is a path such that it changes sign in each interval $(f(u)-\delta, f(u))$, as a generic path of B does!, then f is continuous at u (why?).

This example motives the following strengthening of Donsker's principle:

for every function $f\colon C[0,1] \longrightarrow \mathbb{R}$ which is bounded and continuous for almost every Brownian path, that is, $\mathbb{P}\,(f$ is continous at $B) = 1$, we have $(\star)$.

This is however the portmanteau theorem. We shall show that for a sequence $X, X_1, X_2, \ldots$ of random variables taking values in a metric space (E, ρ) we have that the condition

$$\lim_{n\to\infty} \mathbb{P}\,(X_n \in A) = \mathbb{P}\,(X \in A) \text{ for every Borel subset } A \text{ of } E \text{ with } \mathbb{P}\,(X \in \partial A) = 0 \tag{1}$$

implies

$$\mathbb{E}f(X_n) \longrightarrow \mathbb{E}f(X) \quad \begin{array}{l}\text{for every bounded function } f\colon E \longrightarrow \mathbb{R} \\ \text{such that } \mathbb{P}\,(f \text{ is continuous at } X) = 1.\end{array} \tag{2}$$

This suffices as (1) is equivalent to the convergence in distribution of X_n to X. To show that (1) implies (2) the idea will be to approximate f with a piecewise constant function which expectation will be expressed easily in terms of probabilities that we will know converge. We assume that f is bounded, say $|f(x)| \leq K$ for every $x \in E$. Fix ϵ and choose $a_0 < \ldots < a_l$ such that $a_0 < -K$, $a_l > K$ and $a_i - a_{i-1} < \epsilon$ for $i = 1, \ldots, l$ but also $\mathbb{P}(f(X) = a_i) = 0$ for $0 \leq i \leq l$ (this is possible as there are only countably many a's for which $\mathbb{P}(f(X) = a) > 0$.) This sequence sort of discretises the image of f. Now let $A_i = f^{-1}((a_{i-1}, a_i])$ for $1 \leq i \leq l$. Then we get that $\partial A_i \subset f^{-1}(\{a_{i-1}, a_i\}) \cup D$, where D is the set of discontinuity points of f. Therefore $\mathbb{P}(X \in \partial A_i) \leq \mathbb{P}\left(X \in f^{-1}(\{a_{i-1}, a_i\}) \cup D\right) = 0$. Hence,

$$\sum_{i=1}^{l} a_i \mathbb{P}(X_n \in A_i) \underset{n\to\infty}{\longrightarrow} \sum_{i=1}^{l} a_i \mathbb{P}(X \in A_i)$$

By the choice of the a_i

$$\left| \mathbb{E} f(X_n) - \sum_{i=1}^{l} a_i \mathbb{P}(X_n \in A_i) \right| = \left| \mathbb{E} \sum_{i=1}^{l} (f(X_n) - a_i) \mathbf{1}_{\{X_n \in A_i\}} \right| \leq \epsilon$$

and the same holds with X in place of X_n. Combining these inequalities yields

$$\limsup_{n\to\infty} |\mathbb{E} f(X_n) - \mathbb{E} f(X)| \leq 2\epsilon.$$

□

Question 2. Let $(S_n)_{n\geq 0}$ be a symmetric, simple random walk.

(i) Show that there are positive constants c and C such that for every $n \geq 1$ we have

$$\frac{c}{\sqrt{n}} \leq \mathbb{P}(S_i \geq 0 \text{ for all } i = 1, 2, \ldots, n) \leq \frac{C}{\sqrt{n}}.$$

(ii) Given $a \in \mathbb{R}$ find the limit

$$\lim_{n\to\infty} \mathbb{P}\left(n^{-3/2} \sum_{i=1}^{n} S_i > a \right).$$

Solution. Let $S_0 = 0$ and $S_n = \varepsilon_1 + \ldots + \varepsilon_n$, where the ε_i are i.i.d. Bernoulli random variables, $\mathbb{P}(\varepsilon_i = 1) = 1/2 = \mathbb{P}(\varepsilon_i = -1)$.

To compute the probability

$$p_n = \mathbb{P}(\forall\ 1 \leq i \leq n,\ S_i \geq 0)$$

we look at the stopping time $\tau = \inf\{k \geq 1,\ S_k = -1\}$. Note that

$$\begin{aligned} p_n &= \mathbb{P}(S_n \geq 0, \tau > n) = \mathbb{P}(\{S_n \geq 0\} \setminus \{S_n \geq 0, \tau < n\}) \\ &= \mathbb{P}(S_n \geq 0) - \mathbb{P}(S_n \geq 0, \tau < n). \end{aligned}$$

Let $\widetilde{S}_n$ be the random walk S_n reflected at time τ with respect to the level -1, that is

$$\widetilde{S}_j = \begin{cases} S_j, & j \le \tau, \\ -2 - S_j, & j > \tau. \end{cases}$$

If $\tau < n$ then $S_n \ge 0$ if equivalent to $\widetilde{S}_n \le -2$, so $\mathbb{P}(S_n \ge 0, \tau < n) = \mathbb{P}\left(\widetilde{S}_n \le -2, \tau < n\right)$, but $\{\widetilde{S}_n \le -2\} \subset \{\tau < n\}$, therefore we get

$$p_n = \mathbb{P}(S_n \ge 0) - \mathbb{P}\left(\widetilde{S}_n \le -2\right),$$

which by symmetry and the reflection principle becomes

$$p_n = \mathbb{P}(S_n \ge 0) - \mathbb{P}(S_n \ge 2) = \mathbb{P}(S_n \in \{0,1\}) = \begin{cases} \binom{n}{n/2} 2^{-n}, & n \text{ is even}, \\ \binom{n}{(n-1)/2} 2^{-n}, & n \text{ is odd}. \end{cases}.$$

Using Stirling's formula we easily find that $cn^{-1/2} \le p_n \le Cn^{-1/2}$ for some positive constants c and C.

To find the limit

$$\lim_{n\to\infty} \mathbb{P}\left(n^{-3/2} \sum_{j=1}^{n} S_j > a\right)$$

we shall use the central limit theorem. Notice that

$$\sum_{j=1}^{n} S_j = n\varepsilon_1 + (n-1)\varepsilon_2 + \ldots + \varepsilon_n$$

which of course has the same distribution as $\sum_{j=1}^{n} j\varepsilon_j$. This is a sum of independent random variables. The variance is

$$\operatorname{Var}\left(\sum_{j=1}^{n} j\varepsilon_j\right) = \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$$

Call it σ_n^2 and let $X_j = j\varepsilon_j/\sigma_n$. We want to find

$$\lim_{n\to\infty} \mathbb{P}\left(\sum_{j=1}^{n} X_j > an^{3/2}/\sigma_n\right).$$

It is readily checked that the variables X_j satisfy Lindeberg's condition

$$\sum_{j=1}^{n} \mathbb{E}X_j \mathbf{1}_{\{|X_j|>\epsilon\}} = \sum_{j=1}^{n} j^2 \mathbf{1}_{\{j>\epsilon\sigma_n\}} \xrightarrow[n\to\infty]{} 0$$

(in fact this sequence is eventually zero, precisely for n such that $\sigma_n/n > 1/\epsilon$). Therefore, by the central limit theorem we get that

$$\lim_{n\to\infty} \mathbb{P}\left(n^{-3/2} \sum_{j=1}^{n} S_j > a\right) = \lim_{n\to\infty} \mathbb{P}\left(\sum_{j=1}^{n} X_j > an^{3/2}/\sigma_n\right) = \mathbb{P}\left(G > a\sqrt{3}\right),$$

where G is a standard Gaussian random variable.

Alternatively, using Donsker's principle we get that

$$\mathbb{P}\left(n^{-3/2}\sum_{j=1}^{n} S_j > a\right) = \mathbb{P}\left(\frac{1}{n}\sum_{j=1}^{n}\frac{S_{\frac{j}{n}\cdot n}}{\sqrt{n}} > a\right)$$

tends to

$$\mathbb{P}\left(\int_0^1 B_t dt > a\right).$$

To finish, we notice that $\int_0^1 B_t dt$ is a Gaussian random variable with mean zero and variance

$$\mathbb{E}\left(\int_0^1 B_t dt\right)^2 = \mathbb{E}\int_0^1\int_0^1 B_s B_t ds dt = \int_0^1\int_0^1 \min\{s,t\} ds dt = 1/3.$$

□

Question 3. This question discusses Doob's h-transform.

(i) Let $(X_n)_{\neq 0}$ be an irreducible Markov chain on a finite state space S with a transition matrix $[p_{ij}]_{i,j\in S}$. Let D be a subset of S, $\hat{S} = S \setminus D$ and τ the hitting time of D. Define the function

$$h(i) = \mathbb{P}\left(\tau = \infty \mid X_0 = i\right), \quad i \in \hat{S}.$$

Show that h is harmonic, that is

$$h(i) = \sum_{j\in\hat{S}} p_{ij} h(j), \qquad i \in \hat{S}.$$

Define $\hat{p}_{ij} = \frac{h(j)}{h(i)} p_{ij}$, $i, j \in \hat{S}$. Show that $[\hat{p}_{ij}]_{i,j\in\hat{S}}$ is a stochastic matrix of the transition probabilities of the chain (X_n) conditioned on never hitting D.

(ii) Let B be a standard linear Brownian motion and let τ be the hitting time of 0. Set $h(x) = \mathbb{P}_x(\tau = \infty)$. Show that B conditioned on never hitting 0 is a Markov chain with transition densities

$$\hat{p}(s,x;t,y) = \frac{h(y)}{h(x)}\big(p(s,x;t,y) - p(s,-x;t,y)\big), \qquad x,y > 0 \text{ or } x,y < 0$$

where p is the transition density of B.

(iii) Let B be a 3-dimensional standard Brownian motion. Show that the transition densities of the process $|B|$, the Euclidean norm of B, are given by $\hat{p}$ defined

Solution. (i) To get right intuitions, we shall discuss a discrete version of Doob's h transform first.

Let $X_0, X_1, \ldots$ be a Markov chain on a finite state space S with transition matrix $P = [p_{ij}]_{i,j \in S}$. Suppose it is irreducible. Fix a subset D in S and define the reaching time $\tau = \inf\{n \geq 1, X_n \in D\}$ of D. Denote $\hat{S} = S \setminus D$. We would like to know how the process (X_n) conditioned on never reaching D behaves. We define the function on $\hat{S}$

$$h(i) = \mathbb{P}\left(\tau = \infty \mid X_0 = i\right), \quad i \in \hat{S}$$

First we check that this function is *harmonic* in the sense that

$$h(i) = \sum_{j \in \hat{S}} p_{ij} h(j)$$

for every $i \in \hat{S}$. Notice that

$$\begin{aligned}
h(i) &= \mathbb{P}\left(\tau = \infty \mid X_0 = i\right) \\
&= \sum_{j \in \hat{S}} \mathbb{P}\left(\tau = \infty, X_1 = j \mid X_0 = i\right) \\
&= \sum_{j \in \hat{S}} \frac{\mathbb{P}\left(\tau = \infty, X_1 = j, X_0 = i\right)}{\mathbb{P}\left(X_0 = i\right)} \\
&= \sum_{j \in \hat{S}} \frac{\mathbb{P}\left(\tau = \infty \mid X_1 = j, X_0 = i\right) \mathbb{P}\left(X_0 = i, X_1 = j\right)}{\mathbb{P}\left(X_0 = i\right)} \\
&= \sum_{j \in \hat{S}} \mathbb{P}\left(\tau = \infty \mid X_1 = j\right) \mathbb{P}\left(X_1 = j \mid X_0 = i\right) \\
&= \sum_{j \in \hat{S}} \mathbb{P}\left(\tau = \infty \mid X_0 = j\right) \mathbb{P}\left(X_1 = j \mid X_0 = i\right) \\
&= \sum_{j \in \hat{S}} h(j) p_{ij}.
\end{aligned}$$

This harmonicity of h is equivalent to saying that the matrix $\hat{P} = [\hat{p}_{ij}]_{i,j \in \hat{S}}$ is a transition matrix of a Markov chain on $\hat{S}$, where

$$\hat{p}_{ij} = \frac{h(j)}{h(i)} p_{ij}.$$

Now we will show that this Markov chain has the same distribution as (X_n) conditioned on never reaching D. To this end, it is enough to check that for every $j_0, j_1, \ldots, j_n \in \hat{S}$ we have

$$\mathbb{P}\left(X_1 = j_1, \ldots, X_n = j_n \mid \tau = \infty, X_0 = j_0\right) = \hat{p}_{j_0,j_1} \cdot \ldots \cdot \hat{p}_{j_{n-1},j_n}.$$

Clearly,

$$\mathbb{P}\left(X_1 = j_1, \ldots, X_n = j_n \mid \tau = \infty, X_0 = j_0\right) = \frac{\mathbb{P}\left(X_1 = j_1, \ldots, X_n = j_n, \tau = \infty, X_0 = j_0\right)}{\mathbb{P}\left(\tau = \infty, X_0 = j_0\right)}.$$

The denominator is simply $\mathbb{P}(\tau=\infty \mid X_0=j_0)\,\mathbb{P}(X_0=j_0) = h(j_0)\mathbb{P}(X_0=j_0)$. Conditioning consecutively, the numerator becomes

$$\begin{aligned}
p = \mathbb{P}(\tau=\infty \mid X_n=j_n,\dots,X_0=j_0) &\cdot \mathbb{P}(X_n=j_n \mid X_{n-1}=j_{n-1},\dots,X_0=j_0) \\
&\cdot \mathbb{P}(X_{n-1}=j_{n-1} \mid X_{n-2}=j_{n-2},\dots,X_0=j_0) \\
&\cdot \dots \\
&\cdot \mathbb{P}(X_1=j_1 \mid X_0=j_0)\,\mathbb{P}(X_0=j_0).
\end{aligned}$$

Notice that $\{\tau=\infty\} = \{\forall m \geq n+1\ X_m \notin \hat{S}\} \cap \{\forall m \leq n\ X_m \notin \hat{S}\}$. Moreover, $\{\forall m \leq n\ X_m \notin \hat{S}\} \supset \{X_n=j_n,\dots,X_0=j_0\}$. Since (X_n) is stationary, this yields

$$\begin{aligned}
\mathbb{P}(\tau=\infty \mid X_n=j_n,\dots,X_0=j_0) &= \frac{\mathbb{P}(\tau=\infty, X_n=j_n,\dots,X_0=j_0)}{\mathbb{P}(X_n=j_n,\dots,X_0=j_0)} \\
&= \frac{\mathbb{P}\left(\{\forall m \geq n+1\ X_m \notin \hat{S}\}, X_n=j_n,\dots,X_0=j_0\right)}{\mathbb{P}(X_n=j_n,\dots,X_0=j_0)} \\
&= \mathbb{P}\left(\forall m \geq n+1\ X_m \notin \hat{S} \mid X_n=j_n,\dots,X_0=j_0\right) \\
&= \mathbb{P}\left(\forall m \geq 1\ X_m \notin \hat{S} \ \mid\ X_0=j_n\right) \\
&= \mathbb{P}(\tau=\infty \mid X_0=j_n) \\
&= h(j_n).
\end{aligned}$$

Therefore, the numerator p equals

$$\begin{aligned}
h(j_n)p_{j_{n-1},j_n}\cdot\dots\cdot p_{j_0,j_1}\mathbb{P}(X_0=j_0) &= \frac{h(j_n)}{h(j_{n-1})}p_{j_{n-1},j_n}\cdot\dots\cdot\frac{h(j_1)}{h(j_0)}p_{j_0,j_1}\cdot h(j_0)\mathbb{P}(X_0=j_0) \\
&= \hat{p}_{j_0,j_1}\cdot\dots\cdot\hat{p}_{j_{n-1},j_n}\cdot h(j_0)\mathbb{P}(X_0=j_0),
\end{aligned}$$

where we used the fact that

(ii) Consider standard 1-dimensional Brownian and let τ be the hitting time of $D=\{0\}$, $\tau=\inf\{t>0, B_t=0\}$. We would like to understand the process (B_t) conditioned on never hitting 0. Call this process $\hat{B}$. Set

$$h(x) = \mathbb{P}_x(\tau=\infty).$$

We know that this is a harmonic function on $\mathbb{R}\setminus\{0\}$. Define new probability $\hat{\mathbb{P}}_x$ by

$$\mathbb{P}_x(A) = \frac{\mathbb{P}_x(A,\tau=\infty)}{h(x)}.$$

($\hat{\mathbb{P}}_x$ is absolutely continuous with respect to $\mathbb{P}_x$, that is $\hat{\mathbb{E}}_x Y = \frac{1}{h(x)}\mathbb{E}Y\mathbf{1}_{\{\tau=\infty\}}$ for any bounded random variable Y). We will show that the process $\hat{B}$ is a Markov process with respect to $\hat{\mathbb{P}}_x$ with the transition probabilities

$$\hat{p}(s,x;t,y) = \frac{h(y)}{h(x)}\big(p(s,x;t,y) - p(s,-x;t,y)\big), \qquad x,y>0 \text{ or } x,y<0.$$

That is, the Markov property is

$$\mathbb{P}_x(\hat{B}_t \in A \mid \mathcal{F}_s) = \int_A \hat{p}(s, \hat{B}_s; y, t)dy$$

or, equivalently, for any bounded measurable f,

$$\hat{\mathbb{E}}_x\big(f(\hat{B}_t)|\mathcal{F}_s\big) = \int f(y)\hat{p}(s, \hat{B}_s; y, t)dy. \tag{3}$$

To see why the transition probabilities for $\hat{B}$ look as claimed, let us look at the following heuristic computation

$$\mathbb{P}_x(B_t = y \mid \tau = \infty) = \frac{\mathbb{P}_x(\tau = \infty \mid B_t = y, \tau > t)\mathbb{P}_x(B_t = y, \tau > t)}{\mathbb{P}_x(\tau = \infty)}.$$

By the Markov property of B we have that $\mathbb{P}_x(\tau = \infty \mid B_t = y, \tau > t) = h(y)$, so $\hat{p}(0, x; t, y)$ should be

$$\frac{h(y)}{h(x)}\mathbb{P}_x(B_t = y, \tau > t).$$

We have $\mathbb{P}_x(B_t = y, \tau > t) = \mathbb{P}_x(B_t = y) - \mathbb{P}_x(B_t = y, \tau \leq t)$. The first term has the meaning of $p(0, x; t, y)$. The second term by the reflection principle is $\mathbb{P}_{-x}(B_t = y, \tau \leq t) = \mathbb{P}_{-x}(B_t = y)$ which gives $p(0, -x; t, y)$.

Let us now formally prove (3). The trick is as always to first condition on $\mathcal{F}_t$. Doing this we obtain

$$\begin{aligned}\hat{\mathbb{E}}_x\big(f(\hat{B}_t)|\mathcal{F}_s\big) &= \frac{1}{h(x)}\mathbb{E}_x\big(f(B_t)\mathbf{1}_{\{\tau=\infty\}}|\mathcal{F}_s\big) = \frac{1}{h(x)}\mathbb{E}_x\Big(f(B_t)\mathbb{E}_x\big(\mathbf{1}_{\{\tau=\infty\}}|\mathcal{F}_t\big)\Big|\mathcal{F}_s\Big) \\ &= \frac{1}{h(x)}\mathbb{E}_x\Big(f(B_t)\mathbf{1}_{\{\tau>t\}}h(B_t)\Big|\mathcal{F}_s\Big)\end{aligned}$$

as using the strong Markov property for B we get

$$\mathbb{E}_x\big(\mathbf{1}_{\{\tau=\infty\}}|\mathcal{F}_t\big) = \mathbb{E}_x\big(\mathbf{1}_{\{\forall u>t\ B_u\neq 0\}}\mathbf{1}_{\{\tau>t\}}|\mathcal{F}_t\big) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}_x\big(\mathbf{1}_{\{\forall u>t\ B_u\neq 0\}}|\mathcal{F}_t\big) = \mathbf{1}_{\{\tau>t\}}h(B_t).$$

We write $\mathbf{1}_{\{\tau>t\}} = 1 - \mathbf{1}_{\{\tau\leq t\}}$ and using the reflection principle as well as the Markov property for B we get ($\tilde{B}$ is the reflected Brownian motion at τ)

$$\begin{aligned}\hat{\mathbb{E}}_x\big(f(\hat{B}_t)|\mathcal{F}_s\big) &= \frac{1}{h(x)}\Big(\mathbb{E}_x\big(f(B_t)h(B_t)\big|\mathcal{F}_s\big) - \mathbb{E}_{-x}\big(f(\tilde{B}_t)h(\tilde{B}_t)\big|\mathcal{F}_s\big)\Big) \\ &= \frac{1}{h(x)}\int \big(f(y)h(y)(p(s, B_s; t, y) - f(y)h(y)p(s, -B_s; t, y)\big)dy,\end{aligned}$$

which shows (3).

Notice that the ratio $h(y)/h(x) = \mathbb{P}_y(\tau = \infty)/\mathbb{P}_x(\tau = \infty)$ can be computed explicitly. Define the hitting time τ_a of $\partial(-a, a) = \{-a, a\}$. Fix $R > r$. Since $\mathbb{P}_x(\tau_R < \tau_r)$ is a harmonic function in $\{r < |x| < R\}$ with the boundary conditions: 0 on $\{|x| = r\}$, 1 on $\{|x| = R\}$, we get that

$$\mathbb{P}_x(\tau_R < \tau_r) = \frac{|x| - r}{R - r}.$$

Taking the ratio and letting $r \to 0$ and $R \to \infty$ yield

$$\frac{h(y)}{h(x)} = \frac{\mathbb{P}_y(\tau = \infty)}{\mathbb{P}_x(\tau = \infty)} = \frac{|y|}{|x|}.$$

(iii) As an application, we will heuristically convince ourselves that the 3 dimensional Bessel process $(|B_t|)$ (the magnitude of a standard 3 dimensional Brownian motion) is the linear Brownian motion conditioned on never hitting 0. To this end, we will find a one point density of the Bessel process

$$p(s,x;t,y) = \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\mathbb{P}\left(|B_t| \in (y-\epsilon, y+\epsilon) \mid |B_s| \in (x-\epsilon, x+\epsilon)\right)$$

and check that it matches the transition probabilities found in (ii). By the Markov property of Brownian motion as well as rotational invariance we get that $p(s,x;t,y)$ is the density g_Y of the variable

$$Y = \sqrt{(B^{(1)}_{t-s} + x)^2 + (B^{(2)}_{t-s})^2 + (B^{(3)}_{t-s})^2}$$

at y. To find it, it will suffice to find the density g_Z of the variable

$$Z = \frac{Y^2}{t-s} \sim \left(G_1 + \frac{x}{\sqrt{t-s}}\right)^2 + G_2^2 + G_3^2$$

where G_1, G_2, G_2 are i.i.d. $N(0,1)$ random variables because then

$$g_Y(y) = \frac{d}{dy}\mathbb{P}\left(Y \le y\right) = \frac{d}{dy}\mathbb{P}\left(Z \le \frac{y^2}{t-s}\right) = g_Z\left(\frac{y^2}{t-s}\right)\frac{2y}{t-s}.$$

We know that the distribution of $G_2^2 + G_3^2$ is $\chi^2(2)$ with density $\frac{1}{2}e^{-u/2}\mathbf{1}_{(0,\infty)}(u)$. If we denote by φ the density of G_1, then the density of $(G_1 + x/\sqrt{t-s})^2$ at $u > 0$ equals

$$\begin{aligned}\psi(u) &= \frac{d}{du}\mathbb{P}\left(\left(G_1 + \frac{x}{\sqrt{t-s}}\right)^2 \le u\right) = \frac{d}{du}\mathbb{P}\left(-\sqrt{u} - \frac{x}{\sqrt{t-s}} \le G_1 \le \sqrt{u} - \frac{x}{\sqrt{t-s}}\right) \\ &= \frac{d}{du}\int_{-\sqrt{u}-x/\sqrt{t-s}}^{\sqrt{u}-x/\sqrt{t-s}} \varphi = \frac{1}{2\sqrt{u}}\left(\varphi\left(\sqrt{u} - \frac{x}{\sqrt{t-s}}\right) + \varphi\left(-\sqrt{u} - \frac{x}{\sqrt{t-s}}\right)\right).\end{aligned}$$

Therefore

$$g_Z(z) = \int_0^z \frac{1}{2}e^{-u/2}\psi(z-u)du,$$

which after computing the integral becomes

$$g_Z(z) = \varphi\left(\frac{x}{\sqrt{t-s}}\right)e^{-z/2}\frac{\sqrt{t-s}}{x}\sinh\left(\frac{x}{\sqrt{t-s}}\sqrt{z}\right).$$

Thus

$$g_Y(y) = \frac{y}{x}\frac{1}{\sqrt{t-s}}\left(\varphi\left(\frac{y-x}{\sqrt{t-s}}\right) - \varphi\left(\frac{y+x}{\sqrt{t-s}}\right)\right)$$

which agrees with the transition probabilities found in (ii). $\square$