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Brownian Motion I
Solutions

Question 1. Let B be a standard linear Brownian motion. Show that for any $0 < t_1 < t_2 < \ldots < t_k$ the joint distribution of the vector $(B_{t_1}, \ldots, B(t_k))$ is Gaussian and compute the covariance matrix.

Solution. The vector $G = \left(\frac{B(t_1)}{\sqrt{t_1}}, \frac{B(t_2)-B(t_1)}{\sqrt{t_2-t_1}}, \ldots, \frac{B(t_n)-B(t_{n-1})}{\sqrt{t_n-t_{n-1}}}\right)$ has the standard Gaussian distribution. Thus, the vector $X = (B(t_1), \ldots, B(t_n))$, as a linear image of G, has a Gaussian distribution. Since $\mathbb{E}B(t_i)B(t_j) = t_i \wedge t_j$ (assuming that $B(t)$ is a standard Brownian motion, otherwise we have to subtract the mean), the covariance matrix of X equals $[t_i \wedge t_j]_{i,j\le n}$ □

Question 2. (This exercise shows that just knowing the finite dimensional distributions is not enough to determine a stochastic process.) Let B be Brownian motion and consider an independent random variable U uniformly distributed on $[0, 1]$. Show that the process

$$\tilde{B}_t = \begin{cases} B_t, & t \neq U, \\ 0, & t = U \end{cases}$$

has the same finite dimensional distributions as B but a.s. it is not continuous.

Solution. Given $0 \le t_1 < \ldots < t_n \le 1$, on the even $\{U \neq t_i,\ i = 1, \ldots, n\}$, which has probability 1, we have that $(\tilde{B}(t_1), \ldots, \tilde{B}(t_n)) = (B(t_1), \ldots, B(t_n))$, so $\tilde{B}$ and B have the same finite dimensional distributions. Since, $\mathbb{P}\left(\lim_{t\to U} \tilde{B}_t = \tilde{B}_U\right) = \mathbb{P}(B_U = 0) = \int_0^1 \mathbb{P}(B_u = 0)\,du = 0$, the process $\tilde{B}$ is *not* continuous a.s. □

Question 3. Let $B(\cdot)$ be a standard linear Brownian motion. Prove that

$$\mathbb{P}\left(\sup_{s,t\in(0,1)} \frac{|B(s) - B(t)|}{|s - t|^{1/2}} = \infty\right) = 1.$$

Solution. Consider the events

$$A_n = \left\{\left|B\left(\frac{1}{n+1}\right) - B\left(\frac{1}{n}\right)\right| \ge \sqrt{2\ln n}\left|\frac{1}{n+1} - \frac{1}{n}\right|^{1/2}\right\}.$$

They are independent. Using a usual estimate for the tail of the standard Gaussian r.v. (see, e.g., Lemma 12.9 in [P. Mörters, Y. Peres, *Brownian Motion*]),

$$\mathbb{P}(A_n) = \mathbb{P}\left(|N(0,1)| \geq \sqrt{2\ln n}\right) \geq \frac{2}{\sqrt{2\pi}} \frac{\sqrt{2\ln n}}{\sqrt{2\ln n}^2 + 1} e^{-\sqrt{2\ln n}^2/2} \geq \frac{1}{\sqrt{2\pi}} \frac{1}{n\sqrt{\ln n}},$$

so $\sum_n \mathbb{P}(A_n) = \infty$. By the Borel-Cantelli lemma, $\mathbb{P}(\limsup A_n) = 1$, i.e. with probability 1, infinitely many of A_n's occur. In particular, $\sup_{s,t\in(0,1)} |B(s) - B(t)|/|s-t|^{1/2} = \infty$ with probability 1. $\square$