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Brownian Motion VI
Solutions

Question 1. Let (B_t) be a standard one dimensional Brownian motion and τ_1 the hitting time of level 1. Show that

$$\mathbb{E}\int_0^{\tau_1} \mathbf{1}_{\{0\le B_s\le 1\}}\mathrm{d}s = 1.$$

Solution. Notice that using Fubini's theorem

$$\mu = \mathbb{E}\int_0^{\tau_1} \mathbf{1}_{\{0\le B_s\le 1\}}\mathrm{d}s = \mathbb{E}\int_0^{\infty} \mathbf{1}_{\{0\le B_s\le 1, s<\tau_1\}}\mathrm{d}s = \int_0^\infty \mathbb{P}\left(0<B_s<1, s<\tau_1\right)\mathrm{d}s.$$

The integrand equals

$$\mathbb{P}\left(0<B_s<1\right) - \mathbb{P}\left(0<B_s<1, s>\tau_1\right).$$

Let B^* be the reflected Brownian motion at τ_1. Since $B_s^* = 2 - B_s$ for $s>\tau_1$, we get from the reflection principle that

$$\mathbb{P}\left(0<B_s<1, s>\tau_1\right) = \mathbb{P}\left(1<B_s^*<2, s>\tau_1\right) = \mathbb{P}\left(1<B_s^*<2\right) = \mathbb{P}\left(1<B_s<2\right).$$

Let φ be the density of the standard Gaussian distribution and let Φ be its distribution function, $\Phi(x) = \int_{-\infty}^x \varphi$. We obtain

$$\mathbb{P}\left(0<B_s<1\right) - \mathbb{P}\left(1<B_s<2\right) = \Phi\left(\frac{1}{\sqrt{s}}\right) - \Phi(0) - \left(\Phi\left(\frac{2}{\sqrt{s}}\right) - \Phi\left(\frac{1}{\sqrt{s}}\right)\right),$$

so integrating by substitution ($t = 1/\sqrt{s}$) gives

$$\mu = \int_0^\infty \big(\mathbb{P}\left(0<B_s<1\right) - \mathbb{P}\left(1<B_s<2\right)\big)\mathrm{d}s = \int_0^\infty \big(2\Phi(t) - \Phi(2t) - \Phi(0)\big)\left(\frac{-1}{t^2}\right)'\mathrm{d}t.$$

Integrating by parts twice yields (one has to check that the boundary term vanishes each time; recall also that $\varphi'(x) = -x\varphi(x)$)

$$\mu = \int_0^\infty \big(2\varphi(t) - 2\varphi(2t)\big)\left(\frac{-1}{t}\right)'\mathrm{d}t = \int_0^\infty \big(-2t\varphi(t) + 4t\varphi(t)\big)\frac{1}{t}\mathrm{d}t = 2\int_0^\infty \varphi = 1.$$

$\square$

Question 2. Let H be a hyperplane in $\mathbb{R}^d$ passing through the origin. Let B be a d-dimensional Brownian motion and let τ be the hitting time of H. Show that for every $x \in \mathbb{R}^d$

$$\sup_{t>0} \mathbb{E}_x |B_t| \mathbf{1}_{\{t<\tau\}} < \infty.$$

Solution. We can assume that B starts at 0 and H passes through x. Moreover, by rotational invariance, we can assume that $x = (a, 0, \dots, 0)$ for some $a > 0$ so that $H = \{y \in \mathbb{R}^d,\ y_1 = a\}$. Then τ is in fact the hitting time of the first coordinate $W = B^{(1)}$ of B of level a. Write $B = (W, \bar{B})$, where $\bar{B}$ denotes the process of the last $d-1$ coordinates of B. W and $\bar{B}$ are independent standard Brownian motions. We have

$$\mathbb{E}|B_t|\mathbf{1}_{\{t<\tau\}} \le \mathbb{E}|W_t|\mathbf{1}_{\{t<\tau\}} + \mathbb{E}|\bar{B}_t|\mathbf{1}_{\{t<\tau\}}.$$

The second term is easy to handle because of independence

$$\mathbb{E}|\bar{B}_t|\mathbf{1}_{\{t<\tau\}} = \mathbb{E}|\bar{B}_t|\mathbb{E}\mathbf{1}_{\{t<\tau\}} = C\sqrt{t}\,\mathbb{P}\left(t<\tau\right),$$

where C is some positive constant which depends only on d. Using the reflection principle we get that

$$\mathbb{P}\left(t<\tau\right) = 1 - \mathbb{P}\left(|B_t| > a\right) = \mathbb{P}\left(|B_t| < a\right) = 2\int_0^{a/\sqrt{t}} \varphi < 2\frac{a}{\sqrt{t}}\varphi(0)$$

(by φ we denote the density of the standard Gaussian distribution). Therefore

$$\sup_{t>0} \mathbb{E}|\bar{B}_t|\mathbf{1}_{\{t<\tau\}} = C\sup_{t>0}\sqrt{t}\,\mathbb{P}\left(t<\tau\right) < 2Ca.$$

To handle the first term, notice that

$$\mathbb{E}|W_t|\mathbf{1}_{\{t<\tau\}} = \int_0^\infty \mathbb{P}\left(|W_t| > u, t<\tau\right) du \le a + \int_a^\infty \mathbb{P}\left(|W_t| > u, t<\tau\right) du.$$

Reflecting W at τ, we can rewrite the integrand as follows (bear in mind that $u > a$)

$$\begin{aligned}
\mathbb{P}\left(|W_t| > u, t<\tau\right) &= \mathbb{P}\left(W_t > u, t<\tau\right) + \mathbb{P}\left(W_t < -u, t<\tau\right) \\
&= \mathbb{P}\left(\varnothing\right) + \mathbb{P}\left(W_t < -u\right) - \mathbb{P}\left(W_t < -u, t>\tau\right) \\
&= \mathbb{P}\left(W_t > u\right) - \mathbb{P}\left(W_t^* > 2a+u, t>\tau\right) \\
&= \mathbb{P}\left(W_t > u\right) - \mathbb{P}\left(W_t^* > 2a+u\right) \\
&= \mathbb{P}\left(u < W_t < 2a+u\right) = \int_{u/\sqrt{t}}^{(2a+u)/\sqrt{t}} \varphi(v)dv.
\end{aligned}$$

Hence, our integral becomes

$$\int_a^\infty \mathbb{P}\left(|W_t| > u, t < \tau\right) du = \int_a^\infty \int_{u/\sqrt{t}}^{(2a+u)/\sqrt{t}} \varphi(\nu) d\nu.$$

Using Fubini's theorem we get that this equals ($|\cdot|$ of course denotes Lebesgue measure)

$$\int_0^\infty \left| \left\{ u > a,\ \nu\sqrt{t} - 2a < u < \nu\sqrt{t} \right\} \right| \varphi(\nu) d\nu \leq 2a \int_0^\infty \varphi(\nu) d\nu = a.$$

Putting these together yields

$$\mathbb{E}|W_t| \mathbf{1}_{\{t<\tau\}} \leq a + a = 2a$$

and finally

$$\sup_{t>0} \mathbb{E}|B_t| \mathbf{1}_{\{t<\tau\}} \leq 2a(1 + C).$$

□

Question 3. This is question 3.17 from [P. Mörters, Y. Peres, *Brownian Motion*]. It is left to the diligent student.