Numerical Methods

Partial Differencial equations II

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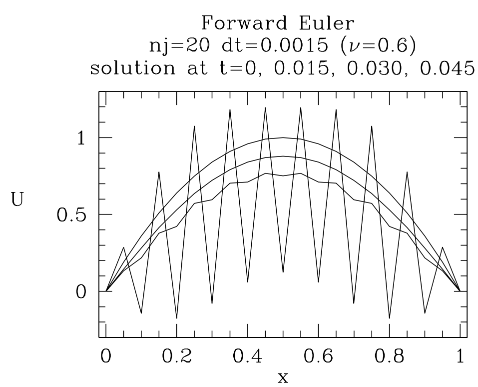
1 Stability Analysis

1.1 Motivation

In the analysis of discretization error in the explicit Euler method for the heat equation, we needed to assume $\nu = \Delta t/h^2 \leq 1/2$ in order to obtain bounds on the error. Consider now what happens to the numerical solution using the explicit Euler method when $\nu = \Delta t/h^2 > 1/2$. After a relative small number of time steps the solution develops a *numerical instability* which grows exponentially with the number of time steps.



This purpose of this chapter is to understand why this happens and to introduce stable (implicit) methods to eliminate this problem.

1.2 ODEs

Most of the issues can be understood from an analysis of time-stepping methods for the simple ODE:

$$\dot{u} = f(u) = \lambda u, \quad \text{where} \quad \lambda < 0. \tag{1}$$

We will compare the exact solution of this equation with a variety of numerical approximations, both in terms of order of accuracy and in terms of long term behaviour, i.e. numerical stability.

1.2.1 Exact solution

$$u(t) = u(0)e^{\lambda t} \tag{2}$$

1.2.2 Explicit Euler

The explicit or forward Euler time-stepping scheme is:

$$\begin{aligned} \frac{U^{n+1} - U^n}{\Delta t} &= f(U^n) \qquad (3) \\ &= \lambda U^n \qquad (4) \end{aligned}$$

which can be solved for U^{n+1} to give

$$U^{n+1} = (1 + \lambda \Delta t)U^n \tag{5}$$

1.2.3 Implicit Euler

The implicit or backward Euler time-stepping scheme is defined by:

$$\begin{aligned} \frac{U^{n+1} - U^n}{\Delta t} &= f(U^{n+1}) \qquad (6) \\ &= \lambda U^{n+1} \qquad (7) \end{aligned}$$

which when solved for U^{n+1} gives

$$U^{n+1} = \frac{1}{1 - \lambda \Delta t} U^n \tag{8}$$

Aside on terminology: Equation (6) is an *implicit* time-stepping scheme because U^{n+1} appears on the right-hand-side of this equation. Hence it is an implicit equation for U^{n+1}, the value of u at the next time step, in terms of U^n, the known value at the current time step. For the function f in this simple example case it is possible to solve for U^{n+1}, but for general f this would not be so. Compare this with the *explicit* Euler scheme (3) in which U^{n+1} is given explicitly in terms of U^n (which known) and $f(U^n)$ (which is in principle computable).

The terms *forward* and *backward* come about from a slightly different way of viewing these schemes. If the time appearing on the right-hand-side is taken as the reference point, then the derivative on the left-hand-side is a forward difference in case (3) and a backward difference in case (6)

1.2.4 Crank-Nicolson

The Crank-Nicolson scheme is given by an average of the explicit and implicit schemes:

$$\frac{U^{n+1}-U^n}{\Delta t} = \frac{f(U^n)+f(U^{n+1})}{2} = \frac{\lambda U^n + \lambda U^{n+1}}{2} \tag{9}$$

or

$$U^{n+1} = \frac{1+(\Delta t/2)\lambda}{1-(\Delta t/2)\lambda}\, U^n \tag{10}$$

1.2.5 Weighted-average

All three schemes can be generalized to an arbitrary weighted average of $f(U^n)$ and $f(U^{n+1})$:

$$\frac{U^{n+1}-U^n}{\Delta t} = (1-\theta)f(U^n) + \theta f(U^{n+1}) = (1-\theta)\lambda U^n + \theta\lambda U^{n+1} \tag{11}$$

which gives:

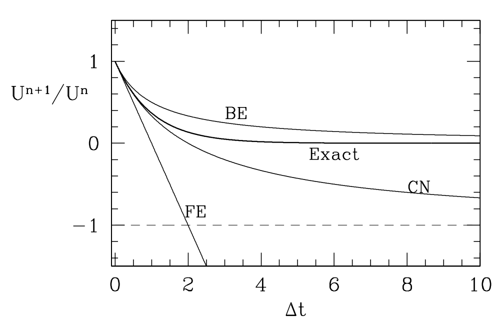
$$U^{n+1} = \frac{1+(1-\theta)\lambda\Delta t}{1-\theta\lambda\Delta t}\, U^n \tag{12}$$

where $0 \le \theta \le 1$. This is called the weighted average or θ-method.

In practice, we shall be almost exclusively interested in the explicit Euler ($\theta = 0$), implicit Euler ($\theta = 1$), and Crank-Nicolson ($\theta = 1/2$) cases.

1.2.6 One-step analysis

For each of the three schemes let us consider the dependence of U^{n+1}/U^n on Δt and compare this with the exact expression for $u(t + \Delta t)/u(t)$. The results are shown graphically for the case $\lambda = -1$.



We make the following observations.

- **All three schemes approximate the exact, exponential, relation for small Δt.**

- **The forward Euler seems particularly bad for moderate and large Δt. Importantly note that $|U^{n+1}| > |U^n|$ for $\Delta t > 2$.**

- **In many ways the backward Euler scheme appear to be the best because is never gets too far from the exact solution and has the correct limit for large Δt. Note, however, that for small Δt the backward Euler scheme approximates the exact to the same accuracy as forward Euler.**

- **The Crank-Nicolson scheme is a better approximation to the exact solution for small Δt than either of the other two methods. That is the Crank-Nicolson curve is seen to follow the exact solution better at small Δt. $|U^{n+1}| < |U^n|$ for all Δt.**

To establish these observations, we examine the schemes for both small and large Δt. For small Δt expand each form in Taylor's series to obtain:

$$\begin{aligned}
\textit{Exact: } \frac{u(t+\Delta t)}{u(t)} &= e^{\lambda \Delta t} &&= 1 + \lambda\Delta t + \frac{1}{2}\lambda^2\Delta t^2 + \frac{1}{6}\lambda^3\Delta t^3 + && (13)\\
\textit{FE: } \frac{U^{n+1}}{U^n} &= 1 + \lambda\Delta t &&= 1 + \lambda\Delta t && (14)\\
\textit{BE: } \frac{U^{n+1}}{U^n} &= (1 - \lambda\Delta t)^{-1} &&= 1 + \lambda\Delta t + \lambda^2\Delta t^2 + \ldots && (15)\\
\textit{CN: } \frac{U^{n+1}}{U^n} &= (1 - \lambda\tfrac{\Delta t}{2})^{-1}(1 + \lambda\tfrac{\Delta t}{2}) &&= 1 + \lambda\Delta t + \frac{1}{2}\lambda^2\Delta t^2 + \frac{1}{4}\lambda^3\Delta t^3 + && (16)
\end{aligned}$$

Hence we see that FE and BE agree with the exact to $O(\Delta t)$ and makes an error $O(\Delta t^2)$, whereas CN agrees with the exact to $O(\Delta t^2)$ making an error $O(\Delta t^3)$. *Thus the CN scheme is higher-order accurate in time than the other two*. Note that these are the one-time-step errors and in each case they are one power of Δt higher than the truncation or discretization error (see below).

For the large Δt we can take limits directly to obtain (for $\lambda < 0$):

$$\begin{array}{lll}
\textit{Exact:} & \lim_{\Delta t\to\infty} e^{\lambda\Delta t} & = \; 0\\
\textit{FE:} & \lim_{\Delta t\to\infty} 1 + \lambda\Delta t & = \; -\infty\\
\textit{BE:} & \lim_{\Delta t\to\infty} (1 - \lambda\Delta t)^{-1} & = \; 0\\
\textit{CN:} & \lim_{\Delta t\to\infty} (1 - \lambda\frac{\Delta t}{2})^{-1}(1 + \lambda\frac{\Delta t}{2}) & = \; -1
\end{array}$$

thereby establishing the behaviour seen in the figure for large Δt.

1.2.7 N-step analysis

Consider now the results of repeated time stepping for each of the schemes. Our primary interest is in understanding instability that arises from repeated time stepping. This can be readily established from our one-step analysis. Each scheme considered above can written as:

$$U^{n+1} = \alpha U^n \qquad (17)$$

so that:

$$U^n = \alpha^n U^0 \tag{18}$$

where:

$$\begin{aligned} \textit{FE:} \quad & \alpha = \quad 1 + \lambda\Delta t & (19)\\ \textit{BE:} \quad & \alpha = \quad (1 - \lambda\Delta t)^{-1} & (20)\\ \textit{CN:} \quad & \alpha = \quad (1 - \lambda\frac{\Delta t}{2})^{-1}(1 + \lambda\frac{\Delta t}{2}) & (21) \end{aligned}$$

These are the quantities plotted in the previous figure.

From (18) it follows that as long as $|\alpha| \leq 1$, U^n will be bounded as $n \to \infty$, in fact $U^n \to 0$ as $n \to \infty$ for $|\alpha| < 1$.

However, $|U^n|$ will diverge as $n \to \infty$ if $|\alpha| > 1$ even though the exact solution (2) decays to zero as $t \to \infty$. This is the essence of the numerical instability illustrated at the beginning of the chapter.

From the above expressions, one can easily establish $|\alpha| \leq 1$ for $0 \leq \Delta t < \infty$ for BE and CN. These schemes are said to be *unconditionally stable* because there is no requirement on Δt for stability.

For the FE case, however, it is evident that $|\alpha|$ exceeds one at $\alpha = -1$. (Recall, $\lambda < 0$, so $\alpha < 1$ for $\Delta t > 0$). Setting $\alpha = -1$ we establish the *stability limit* for the explicit Euler scheme.

$$\alpha = 1 + \lambda\Delta t = -1 \Rightarrow \Delta t = \frac{-2}{\lambda}$$

For explicit Euler scheme to be stable, the time step must be less than or equal to this minimum, i.e.

$$\boxed{\Delta t \leq \frac{-2}{\lambda}}$$

The explicit Euler scheme is said to be *conditionally stable* because the above condition must be satisfied for stability.

Note that because instability occurs for $\alpha < -1$, when instability develops it is oscillatory in time with U^{n+1} and U^n of opposite signs. This is one of the key signatures of numerical instabilities.

1.2.8 N-step discretization error

As an aside, we can apply the same expansion considered in (14-16) to establish global discretization error bounds for these schemes. The analysis is as before except that we are interested in fixing a final time t_f and requiring the number of time steps N to vary inversely with Δt such that $t_f = N\Delta t$ is constant.

For example, for the FE method we have:

$$\begin{aligned}\frac{U^N}{U^0} &= (1+\lambda\Delta t)^N && (22)\\ &= \exp(N\log(1+\lambda\Delta t)) && (23)\\ &= \exp(N(\lambda\Delta t + O(\Delta t^2))) && (24)\\ &= \exp(\lambda t_f) + t_f O(\Delta t) && (25)\end{aligned}$$

So we have that:

$$E^f = |U^N - u(t_f)| = t_f O(\Delta t). \tag{26}$$

Establishing that the discretization error for the forward Euler method is $O(\Delta t)$. Similar analyses shows that the BE method is also $O(\Delta t)$, while the CN method is $O(\Delta t^2)$.

1.3 PDEs

We return now to PDEs and consider time-stepping methods for the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{27}$$

posed on some interval in x. For the present we are not concerned with the boundary conditions imposed at the ends of the interval.

In an earlier chapter we considered the explicit Euler method for time-stepping this equation. We now wish to consider a variety of other methods, most importantly the implicit Euler and Crank-Nicolson methods. For this it useful to introduce the following notation which will be used heavily throughout the course.

The following notation will be used heavily throughout the course. Let $\mathbf{U}^n$ be the vector representing the numerical solution at time step n:

$$\mathbf{U}^n = (U_0^n, U_1^n, U_2^n, \ldots, U_j^n, \ldots, U_{J-1}^n, U_J^n)^T$$

Let $\mathbf{L}$ be the matrix representing the finite-difference representation of a linear operator $\mathcal{L}$. For now we consider $\mathcal{L} = \frac{\partial^2}{\partial x^2}$ and take $\mathbf{L}$ to be the matrix arising from the second-order center difference approximation δ_x^2/h^2. Then $\mathbf{L}$ is the following tridiagonal matrix:

$$\mathbf{L} = \frac{1}{h^2}\begin{bmatrix} & & & & & \\ \ddots & \ddots & \ddots & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & & \end{bmatrix}$$

$\mathbf{I}$ will denote the identity matrix.

With this notation we can easily express the 3 time stepping schemes as follows:

$$\text{FE:} \qquad \frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\Delta t} = \mathbf{L}\ \mathbf{U}^n \quad \Rightarrow \quad \mathbf{U}^{n+1} = (\mathbf{I} + \Delta t \mathbf{L})\ \mathbf{U}^n$$

$$\text{BE:} \qquad \frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\Delta t} = \mathbf{L}\ \mathbf{U}^{n+1} \quad \Rightarrow \quad \mathbf{U}^{n+1} = (\mathbf{I} - \Delta t \mathbf{L})^{-1}\ \mathbf{U}^n$$

$$\text{CN:} \qquad \frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\Delta t} = \frac{\mathbf{L}\ \mathbf{U}^n + \mathbf{L}\ \mathbf{U}^{n+1}}{2} \quad \Rightarrow$$
$$\mathbf{U}^{n+1} = (\mathbf{I} - \frac{\triangle t}{2}\mathbf{L})^{-1}\ (\mathbf{I} + \frac{\triangle t}{2}\mathbf{L})\ \mathbf{U}^n$$

These are the analog of the schemes discussed for ODEs with $\mathbf{L}$ playing the role of λ.

The matrices $(\mathbf{I} + \Delta t \mathbf{L})$, etc occur very frequently and so we make the following definitions:

$$\mathbf{A}_+ \equiv (\mathbf{I} + \Delta t \mathbf{L}) \qquad \mathbf{A}_- \equiv (\mathbf{I} - \Delta t \mathbf{L}) \tag{28}$$
$$\mathbf{A}_{+1/2} \equiv (\mathbf{I} + \frac{\triangle t}{2}\mathbf{L}) \qquad \mathbf{A}_{-1/2} \equiv (\mathbf{I} - \frac{\triangle t}{2}\mathbf{L}) \tag{29}$$

so that:

$$\text{FE:} \qquad \mathbf{U}^{n+1} = \mathbf{A}_+\ \mathbf{U}^n \tag{30}$$

$$\text{BE:} \qquad \mathbf{U}^{n+1} = (\mathbf{A}_-)^{-1}\ \mathbf{U}^n \tag{31}$$

$$\text{CN:} \qquad \mathbf{U}^{n+1} = (\mathbf{A}_{-1/2})^{-1}\ \mathbf{A}_{+1/2}\ \mathbf{U}^n \tag{32}$$

1.3.1 Stability analysis

All the above time-stepping schemes are of the form:

$$\mathbf{U}^{n+1} = \mathbf{A}\ \mathbf{U}^n.$$

where $\mathbf{A}$ is a matrix. We need to determine whether or not solutions to this matrix iteration grow in time and hence whether the schemes are unstable. We proceed with a separation of variables technique similar to that used to solve the heat equation and seek solutions of the form

$$U_j^n = \alpha^n \phi_j \tag{33}$$

where ϕ_j is to be determined. Let $\Phi = (\phi_0, \phi_1, \ldots, \phi_J)^T$, so that $\mathbf{U}^n = \alpha^n \Phi$.

Substituting for $\mathbf{U}^n$ gives:

$$\begin{aligned} \alpha^{n+1}\Phi &= \mathbf{A}\ (\alpha^n\Phi) && (34) \\ \alpha^{n+1}\Phi &= \alpha^n\mathbf{A}\ \Phi && (35) \\ \alpha\Phi &= \mathbf{A}\ \Phi && (36) \end{aligned}$$

So that α is an eigenvalue of $\mathbf{A}$ corresponding to eigenvector Φ. Thus the stability analysis is reduced to the problem of determining whether there are any eigenvalues α of the matrix $\mathbf{A}$ such that $|\alpha| > 1$.

We note that the eigenvalues α of $\mathbf{A}$ are related to the eigenvalues λ of $\mathbf{L}$ by the fact that the eigenvalues of any polynomial function P of $\mathbf{L}$ are simply $P(\lambda)$. The eigenfunctions are the same Φ. Thus, just as for the ODE example we have the relations:

$$\begin{aligned} &\textit{FE:} && \alpha = 1 + \lambda\Delta t && (37) \\ &\textit{BE:} && \alpha = (1 - \lambda\Delta t)^{-1} && (38) \\ &\textit{CN:} && \alpha = (1 - \lambda\frac{\Delta t}{2})^{-1}(1 + \lambda\frac{\Delta t}{2}) && (39) \end{aligned}$$

The **von Neumann** or **Fourier** method of stability analysis is based on *choosing* the eigenfunction to be trigonometric (exponential):

$$\phi_j = e^{i\beta jh} \tag{40}$$

Here we have assumed a uniform spatial grid. In the von Neumann/Fourier stability analysis, β is continuous and can take on any value. In this approach the corresponding values of α are simply computed.

First we use the von Neumann/Fourier method to find the eigenvalue of $\mathbf{L}$:

$$\begin{aligned} [\lambda\Phi]_j &= [\mathbf{L}\Phi]_j && (41) \\ \lambda\phi_j &= \frac{1}{h^2}(\phi_{j-1} - 2\phi_j + \phi_{j+1}) && (42) \\ \lambda e^{i\beta jh} &= \frac{1}{h^2}(e^{i\beta(j-1)h} - 2e^{i\beta jh} + e^{i\beta(j+1)h}) && (43) \\ \lambda &= \frac{1}{h^2}(e^{-i\beta h} - 2 + e^{i\beta h}) && (44) \\ \lambda &= \frac{2}{h^2}(\cos\beta h - 1) && (45) \end{aligned}$$

We can now immediately obtain the eigenvalues of $\mathbf{A}_+$ for the forward Euler method:

$$\begin{aligned} \alpha &= 1 + \Delta t\lambda && (46) \\ \alpha &= 1 + \frac{2\Delta t}{h^2}(\cos\beta h - 1) && (47) \end{aligned}$$

We can now ask whether or not $|\alpha| \leq 1$. Note that $(\cos\beta h - 1) \leq 0$. Hence $\alpha \leq 1$ and the only possibility for instability occurs for $\alpha < -1$. For any Δt, the minimum (most negative) value of α occurs for $\beta h = \pi$ in which case $\cos\beta h - 1 = -2$. For this worst case we then have:

$$\alpha = 1 - \frac{4\Delta t}{h^2}$$

Setting this to the limit $\alpha = -1$ and solving gives:

$$-1 = 1 - \frac{4\Delta t}{h^2} \Rightarrow \frac{\Delta t}{h^2} = \frac{1}{2}$$

For explicit Euler scheme to be stable, the time step must be less than or equal to this minimum, i.e.

$$\boxed{\Delta t \leq \frac{h^2}{2} \quad \text{or} \quad \nu = \frac{\Delta t}{h^2} \leq \frac{1}{2}}$$

We find that the explicit Euler scheme is *conditionally stable*.

I will leave it as an exercise to show that the BE and CN schemes are *unconditionally stable*.

[Note what the above computation shows that the eigenvalues of $\mathbf{L}$ are $\lambda = 2(\cos\beta h - 1)/h^2$. The maximum modulus of these eigenvalue obtains for $\beta h = \pi$ giving $\lambda = -4/h^2$. The explicit Euler stability limit of $\Delta t < -2/\lambda$ thus becomes $\Delta t < h^2/2$. This same argument could be used to show immediately, given the results from ODEs, that the BE and CN schemes are unconditionally stable.]

Finally we consider the form of the numerical instability. As before the instability occurs for $\alpha < -1$ and hence when instability develops it is oscillatory in time with U_j^{n+1} and U_j^n of opposite signs.

However, now there is the additional dependence on space given by the eigenfunction $\phi_j = \cos(\beta j h)$. The instability arises first from the eigenfunction with $\beta h = \pi$ or $\phi_j = \cos(\pi j) = (-1)^j$. This normal mode changes sign in space at every other grid point. Hence when the instability develops it is oscillatory in space as well as time. Again this is one of the key signatures of a numerical instability.

2 Dirichlet boundary condition

We consider in detail the left boundary only, the right boundary follows similarly.

The Dirichlet boundary condition is

$$u(0, t) = \gamma_0(t) \tag{48}$$

Eq. (48) dictates the numerical solution at $x = 0$ for all time: $U_0^n = \gamma_0^n \equiv \gamma_0(n\Delta t)$. Thus U_0^n does not need to be computed from the PDE and and we need not impose the PDE at $j = 0$.

The following provides a simple implementation. Consider the heat equation and let $\mathbf{L}$ be the discretization of the second derivative. Put zeros in the rows of matrix $\mathbf{L}$ at Dirichlet boundary points

$$\mathbf{L} = \frac{1}{h^2} \begin{bmatrix} 0 & 0 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & & \ddots & \\ & & & & \\ & & & & \end{bmatrix} \tag{49}$$

So $[\mathbf{LU}]_0 = 0$, $[\mathbf{LU}]_1 = (U_0 - 2U_1 + U_2)/h^2$.

Next define the boundary operator $\mathbf{B}^n$ by:

$$\mathbf{B}^n\ \mathbf{U} = \mathbf{B}^n \begin{pmatrix} U_0 \\ U_1 \\ \\ \cdots \\ \end{pmatrix} = \begin{pmatrix} \gamma_0^n \\ U_1 \\ \\ \cdots \\ \end{pmatrix} \tag{50}$$

$\mathbf{B}^n$ sets the boundary values to the corresponding BC at time $t = n\Delta t$. Note: $\mathbf{B}$ is not a linear operator if $\gamma_0 \neq 0$.

Using this operator the 3 time-stepping schemes can be written:

FE:	$\mathbf{U}^{n+1} = \mathbf{B}^{n+1}\ (\mathbf{I} + \Delta t\mathbf{L})\ \mathbf{U}^n$
BE:	$\mathbf{U}^{n+1} = (\mathbf{I} - \Delta t\mathbf{L})^{-1}\ \mathbf{B}^{n+1}\ \mathbf{U}^n$
CN:	$\mathbf{U}^{n+1} = (\mathbf{I} - \frac{\triangle t}{2}\mathbf{L})^{-1}\ \mathbf{B}^{n+1}\ (\mathbf{I} + \frac{\triangle t}{2}\mathbf{L})\ \mathbf{U}^n$

Check that $U_j^{n+1} = \gamma_0^{n+1}$ and that the PDE is imposed in the interior.

3 General Linear Parabolic Eq in 1D

Consider the general linear parabolic PDE in one "space" variable x:

$$\frac{\partial u}{\partial t}(x,t) = a(x,t)\frac{\partial^2 u}{\partial x^2}(x,t) + b(x,t)\frac{\partial u}{\partial x}(x,t) + c(x,t)u(x,t) + g(x,t) \tag{51}$$

with $a(x,t) > 0$ posed on the interval $[0,\ell]$ subject to boundary conditions as discussed in the previous lecture.

Most of the difficulties encountered in going to the general case are in the analysis of the discretization error and stability. Implementing the general case can be a simple extension of the constant coefficient heat equation.

Write the PDE as:

$$\frac{\partial u}{\partial t}(x,t) = \mathcal{L}(x,t)u(x,t) + g(x,t) \tag{52}$$

then discretize all quantities in space and time:

$$u(x,t) \to \mathbf{U}^n = \begin{pmatrix} U_0^n \\ U_1^n \\ \\ \dots \\ \\ U_J^n \end{pmatrix} \quad g(x,t) \to \mathbf{g}^n = \begin{pmatrix} g_0^n \\ g_1^n \\ \\ \dots \\ \\ g_J^n \end{pmatrix} \tag{53}$$

where $g_j^n \equiv g(jh, n\Delta t)$, and

$$\mathcal{L}(x,t) \to \mathbf{L}^n \tag{54}$$

The matrix $\mathbf{L}^n$ can be obtained from finite-difference formulas for $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial u}{\partial x}$. For second-order center-differences the matrix $\mathbf{L}$ is tridiagonal just as in our previous treatment of the heat equation:

$$\mathbf{L}^n \quad = \quad \begin{bmatrix} & & & & \\ & \ddots & & & \\ & L_{jj-1}^n & L_{jj}^n & L_{jj+1}^n & \\ & & & \ddots & \\ & & & & \end{bmatrix} \tag{55}$$

where:

$$L_{jj-1}^n = \quad \frac{a_j^n}{h^2} - \frac{b_j^n}{2h} \tag{56}$$

$$L_{jj}^n = \frac{-2a_j^n}{h^2} \qquad + c_j^n \tag{57}$$

$$L_{jj+1}^n = \quad \frac{a_j^n}{h^2} + \frac{b_j^n}{2h} \tag{58}$$

where: $a_j^n \equiv a(jh, n\Delta t)$, $b_j^n \equiv b(jh, n\Delta t)$, $c_j^n \equiv c(jh, n\Delta t)$.

The three time-stepping schemes are then:

$$\text{FE:} \quad \mathbf{U}^{n+1} = \mathbf{B}\ \{(\mathbf{I} + \Delta t \mathbf{L}^n)\ \mathbf{U}^n + \Delta t \mathbf{g}^n\}$$

$$\text{BE:} \quad \mathbf{U}^{n+1} = (\mathbf{I} - \Delta t \mathbf{L}^{n+1})^{-1}\ \mathbf{B}\ \left\{\mathbf{U}^n + \Delta t \mathbf{g}^{n+1}\right\}$$

$$\text{CN:} \quad \mathbf{U}^{n+1} = (\mathbf{I} - \frac{\triangle t}{2}\mathbf{L}^{n+1})^{-1}\ \mathbf{B}\ \left\{(\mathbf{I} + \frac{\triangle t}{2}\mathbf{L}^n)\ \mathbf{U}^n + \frac{\Delta t}{2}(\mathbf{g}^n + \mathbf{g}^{n+1})\right\}$$

where $\mathbf{B}$ represents the appropriate boundary operator.

3.1 Discretization Error

Here we provide an estimate for the discretization error for the Forward Euler scheme. We use ideas and notations from Lecture 4.

The **discretization error** e_j^n is:

$$e_j^n \equiv U_j^n - u(x_j, t_n) \tag{59}$$

where U_j^n is a solution to the finite difference equation, u and U_j^n satisfy the same initial conditions, i.e. $U_j^0 = u(x_j, 0) = u^0(x_j)$.

As before

$$E^n \equiv \|e_j^n\|_\infty = \max_j |e_j^n|$$

The Forward Euler scheme:

$$U_j^{n+1} = U_j^n + \nu_j^n(U_{j+1}^n - 2U_j^n + U_{j-1}^n) + \mu_j^n(U_{j+1}^n - U_{j-1}^n) + \Delta t c_j^n U_j^n + \Delta t g_j^n,$$

where $\nu_j^n = \frac{\Delta t}{h^2} a_j^n$ and $\mu_j^n = \frac{\Delta t}{2h} b_j^n$.

$$\begin{aligned}
e_j^{n+1} &= U_j^{n+1} - u(x_j, t_{n+1}) \\
&= e_j^n + \nu_j^n(e_{j+1}^n - 2e_j^n + e_{j-1}^n) + \\
&\quad \mu_j^n(e_{j+1}^n - e_{j-1}^n) + \Delta t c_j^n e_j^n - \Delta t T_j^n \\
&= (1 - 2\nu_j^n + \Delta t c_j^n) e_j^n \\
&\quad + (\nu_j^n + \mu_j^n) e_{j+1}^n + (\nu_j^n - \mu_j^n) e_{j-1}^n - \Delta t T_j^n
\end{aligned}$$

Then for $1 - 2\nu_j^n + \Delta t c_j^n \geq 0$ and $\nu_j^n \geq |\mu_j^n|$ the coefficients are positive and triangle inequality gives

$$|e_j^{n+1}| \quad \leq \quad (1 + \Delta t c_j^n) E^n + \Delta t |T_j^n|$$

Hence,

$$|E^{n+1}| \quad \leq \quad (1 + \Delta t C) E^n + \Delta t |T^n|$$

So, the following two inequalities should be satisfied:

$$2\frac{\Delta t}{h^2}a_j^n - \Delta t c_j \leq 1 \quad \text{and} \quad \frac{\Delta t}{2h}|b_j^n| \leq \frac{\Delta t}{h^2}a_j^n$$

The second of these leads to a rather strong constraint on the grid spacing h if $a_j^n \ll |b_j^n|$:

$$h \leq \frac{2a_j^n}{|b_j^n|}$$

which in turn leads to a strong restriction on the time step.

Essentially:

$$\Delta t \leq \frac{2a_j^n}{(b_j^n)^2}.$$

Hence, in practice the forward Euler scheme would not be used in these circumstances. Either one would use an implicit scheme, both BE and CN are unconditionally stable, or else one would use one-sided “up-wind” differences for the first derivative operator rather then the center difference.