Brownian Motion.

Q1. Let F a functional on $C(\mathbb{R})$, equipped with the supremum norm. Show that Donsker's theorem can be applied if F is $a.s.$ a continuous with respect to the Wiener measure., i.e. the assumption of continuity of F in Donker's theorem can be relaxed.

Q2. Let $(S_n)_{n\geq 0}$ be a simple, symmetric random walk on the integers.

A. Show that there exist constants C_1, C_2, such that

$$\frac{C_1}{\sqrt{n}} \leq \mathbb{P}_0\big(S_i \geq 0 \text{ for all } 1 \leq i \leq n\big) \leq \frac{C_2}{\sqrt{n}},$$

for all $n \geq 1$.

B. Write an expression for the limit

$$\lim_{n\to\infty} \mathbb{P}_0\big(n^{-3/2}\sum_{i=1}^{n} S_i > a\big),$$

for $a \in \mathbb{R}$.

Q3. (Doob's h transform) Let $\beta(\cdot)$ be a standard d-dimensional Brownian motion and D a subset of $\mathbb{R}^d$. Denote by $x(\cdot)$ the Brownian motion conditioned never to hit the set D and denote by $p_D(t,x;s,y)$ its transition probabilities. Denote also by $p(t,x;s,y)$ the transition probabilities of $\beta(\cdot)$. Show that

$$p_D(t,x;s,y) = \hat{p}_D(t,x;s,y)\frac{h_D(y)}{h_D(x)},$$

where $\hat{p}_D(t,x;s,y)$ is the transition probability of Brownian motion going from point x at time t to point y at time s without hitting the set D. $h_D(\cdot)$ is the harmonic function on $\mathbb{R}^d$ with zero boundary condition on D. Write down explicitly the transition probabilities $p_D(t,x;s,y)$. Show that the one dimensional Brownian motion condition never to hit zero has the same distribution as the modulus of a three dimensional standard Brownian motion.

State the analogous result in the case of simple random walks on $\mathbb{Z}^d$.