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Brownian Motion III
Solutions

Question 1. Let $f\colon [0,\infty)\times\mathbb{R}^d \longrightarrow \mathbb{R}$ be a smooth function and let B be standard Brownian motion in $\mathbb{R}^d$. Show that

$$M_t = f(t,B_t) - \int_0^t \left(f_t + \frac{1}{2}\Delta f\right)(s,B_s)ds$$

is a martingale. Using this, write a solution to the problem

$$\begin{cases} u_t &= \frac{1}{2}\Delta u, \qquad \text{in } (0,\infty)\times\mathbb{R}^d,\\ u(0,x) &= f(x), \qquad \text{on } \mathbb{R}^d, \end{cases}$$

where f is a given, smooth, compactly supported function (the initial condition)

Solution. In order to show that

$$M_t = f(t,B_t) - \int_0^t \left(f_t + \frac{1}{2}\Delta f\right)(s,B_s)ds$$

is a martingale, it is enough to follow closely the proof of Theorem 2.51 from [P. Mörters, Y. Peres, *Brownian Motion*].

Now we construct a solution to the problem

$$\begin{cases} u_t &= \frac{1}{2}\Delta u, \qquad t>0, x\in\mathbb{R}^d,\\ u(0,x) &= f(x), \qquad x\in\mathbb{R}^d, \end{cases}$$

where f is a given, smooth, compactly supported function (the initial condition). Let

$$u(t,x) = \mathbb{E}_x f(B_t).$$

Plainly, $u(0,x) = \mathbb{E}_x f(B_0) = f(x)$. Moreover, using the martingale property we just mentioned we have $\mathbb{E}_x M_t = \mathbb{E}_x M_0 = f(x)$, so

$$u(t,x) = f(x) + \mathbb{E}_x \int_0^t \frac{1}{2}(\Delta f)(B_s)ds.$$

Since f is compactly supported, f and all its derivatives are bounded. Therefore we can swap the integrals as well as the Laplacian and write

$$
\begin{aligned}
u(t,x) &= f(x) + \int_0^t \mathbb{E}_x \frac{1}{2}\Delta f(B_s)ds = f(x) + \int_0^t \mathbb{E}\frac{1}{2}(\Delta f)(x+B_s)ds \\
&= f(x) + \int_0^t \mathbb{E}\frac{1}{2}\Delta\big(f(x+B_s)\big)ds = f(x) + \int_0^t \frac{1}{2}\Delta\big(\mathbb{E}f(x+B_s)\big)ds \\
&= f(x) + \int_0^t \frac{1}{2}\Delta\big(\mathbb{E}_x f(B_s)\big)ds.
\end{aligned}
$$

Taking the time derivative yields

$$
u_t = \frac{1}{2}\Delta\big(\mathbb{E}_x f(B_t)\big) = \frac{1}{2}\Delta u.
$$

To come up with this solution, we could alternatively suppose that u solves the problem, observe that $M_t = u(t_0 - t, B_t)$ is a martingale which yields that

$$
u(t_0, x) = \mathbb{E}_x u(t_0, B_0) = \mathbb{E}_x M_0 = \mathbb{E}_x M_{t_0} = \mathbb{E}_x u(0, B_{t_0}) = \mathbb{E}_x f(B_{t_0}),
$$

so u is of the form $\mathbb{E}_x f(B_t)$ □

Question 2. Consider the problem

$$
\begin{cases}
u_t &= \frac{1}{2}\Delta u, & \text{in } \mathbb{R}_+ \times B(0,1), \\
u(0,x) &= f(x), & \text{on } B(0,1), \\
u(t,z) &= g(t,z), & \text{on } \mathbb{R}_+ \times \partial B(0,1).
\end{cases}
$$

(the heat equation in the cylinder $\mathbb{R}_+ \times B(0,1)$; $B(0,1) \subset \mathbb{R}^2$ is the unit disk centred at the origin), where f, g are smooth functions on $B(0,1$ and $\mathbb{R}_+ \times \partial B(0,1)$ respectively (the initial data). Show that a solution to this problem is of the form

$$
u(t,x) = \mathbb{E}_x f(B_t)\mathbf{1}_{\{t \le \tau\}} + \mathbb{E}_x g(t-\tau, B_\tau)\mathbf{1}_{\{t > \tau\}}
$$

where B is a standard planar Brownian motion and τ is the hitting time of $\partial B(0,1)$.

Solution. Suppose we have a solution u and we want to determine its form. Fix (t_0, x) in $\mathbb{R}_+ \times B(0,1)$ and consider

$$
M_t = u(t_0 - t, B_t).
$$

From Question 1 we know that this is a martingale as long as B_t stays in $B(0,1)$ so that u solves the heat equation. Therefore, defining τ to be the hitting time of $\partial B(0,1)$,

$(M_t)_{t\in[0,\tau]}$ is a martingale. Using Doob's optional stopping theorem for $0 \le t \wedge \tau$ we thus get

$$u(t_0, x) = \mathbb{E}_x M_0 = \mathbb{E}_x M_{t\wedge\tau} = \mathbb{E}_x u(t_0 - t, B_t)\mathbf{1}_{\{t\le\tau\}} + \mathbb{E}_x u(t_0 - \tau, B_\tau)\mathbf{1}_{\{t>\tau\}}.$$

Letting t go to t_0 yields

$$u(t_0, x) = \mathbb{E}_x u(0, B_{t_0})\mathbf{1}_{\{t_0\le\tau\}} + \mathbb{E}_x u(t_0 - \tau, B_\tau)\mathbf{1}_{\{t_0>\tau\}}.$$

Given initial and boundary conditions this rewrites as

$$u(t_0, x) = \mathbb{E}_x f(B_{t_0})\mathbf{1}_{\{t_0\le\tau\}} + \mathbb{E}_x g(t_0 - \tau, B_\tau)\mathbf{1}_{\{t_0>\tau\}},$$

so a solution has to be of this form.

Verifying directly that this solves the problem might not be easy. To bypass it, we could refer to the existence and uniqueness of the solutions of the heat equation.

$\square$

Question 3. Let B be a d-dimensional standard Brownian motion. For which dimensions, does it hit a single point different from its starting location?

Solution. When $d = 1$, we know the density of the hitting time of a single point. Particularly, this stopping time is a.s. finite.

Let $d \ge 2$ and fix two different points $a, x \in \mathbb{R}^d$. We will show that B_t starting at $a \in \mathbb{R}^d$, with probability one, never hits x. Let τ_r be the hitting time of the sphere $\partial B(x, r)$ (r small), let τ_R be the hitting time of the sphere $\partial B(0, R)$ (R large) and let $\tau_{\{x\}}$ be the hitting time of x. Notice that $\lim_{r\to 0} \tau_r = \tau_{\{x\}}$ and $\lim_{R\to\infty} \tau_R = \infty$. From the lecture we know that (see also Theorem 3.18 in [P. Mörters, Y. Peres, *Brownian Motion*])

$$\mathbb{P}_a(\tau_r < \tau_R) = \begin{cases} \frac{\ln R - \ln|a|}{\ln R - \ln r}, & d = 2, \\ \frac{R^{2-d} - |a|^{2-d}}{R^{2-d} - r^{2-d}}, & d \ge 3. \end{cases}$$

Therefore letting r go to zero yields

$$\mathbb{P}_a(\tau_{\{x\}} < \tau_R) = 0,$$

hence letting R go to infinity we obtain $\mathbb{P}_a(\tau_{\{x\}} < \infty) = 0$. $\square$

Question 4. Let f be a function compactly supported function in the upper half space $\{x_d \geq 0\}$ of $\mathbb{R}^d$. Show that

$$\int G(x,y)f(y)dy - \int G(x,\overline{y})f(y)dy = \mathbb{E}_x \int_0^\tau f(B_t)dt,$$

where B is a standard d-dimensional Brownian motion, τ is the hitting time of the hyperplane $H = \{x_d = 0\}$, $G(x,y)$ is the Green's function for $\mathbb{R}^d$, and $\bar{y}$ means the reflection of a point $y \in \mathbb{R}^d$ about the hyperplane H.

This shows that $G(x,y) - G(x,\bar{y})$ is the Green's function in the upper half-space.

Solution. We have (see Theorem 3.32 in [P. Mörters, Y. Peres, *Brownian Motion*])

$$\begin{aligned}
\int G(x,y)f(y)dy &= \mathbb{E}_x \int_0^\infty f(B_t)dt, \\
\int G(x,\overline{y})f(y)dy &= \int G(x,y)f(\overline{y})dy \\
&= \mathbb{E}_x \int_0^\infty f(\overline{B_t})dt.
\end{aligned}$$

The key observation is that the processes $\{B_t,\ t \geq \tau\}$ and $\{\overline{B_t},\ t \geq \tau\}$ have the same distribution. Therefore, breaking each integral on the right hand side into two, on $[0,\tau]$ and on $[\tau,\infty)$ and subtracting the above equalities, we see that two of the integrals will cancel each other, one will be zero as f is compactly supported in the upper half plane, and we will get

$$\int G(x,y)f(y)dy - \int G(x,\overline{y})f(y)dy = \mathbb{E}_x \int_0^\tau f(B_t)dt,$$

as required. □