Tomasz Tkocz 08/03/2015

Brownian Motion V
Solutions

Question 1. Let L_t be the local time at zero of linear Brownian motion. Show that $\mathbb{E}L_t = \sqrt{2t/\pi}$.

Solution. We know from the lecture that the local time at zero L_t has the same distribution as the maximum $M_t = \max_{0\le s\le t} B_t$. Therefore

$$\mathbb{E}L_t = \mathbb{E}M_t = \int_0^\infty \mathbb{P}\left(M_t > u\right) du.$$

Moreover, using the reflection principle it has been shown that $\mathbb{P}\left(M_t > u\right) = 2\mathbb{P}\left(B_t > u\right) = \mathbb{P}\left(|B_t| > u\right)$, so we get

$$\mathbb{E}L_t = \int_0^\infty \mathbb{P}\left(|B_t| > u\right) du = \mathbb{E}|B_t| = \sqrt{t}\mathbb{E}|N(0,1)| = \sqrt{\frac{2t}{\pi}}.$$

□

Question 2. Let $a < 0 < b < c$ and τ_a, τ_b, τ_cbe the hitting times of these levels for one dimensional Brownian motion. Compute

$$\mathbb{P}\left(\tau_b < \tau_a < \tau_c\right).$$

Solution. Notice that

$$\mathbb{P}\left(\tau_b < \tau_a < \tau_c\right) = \mathbb{P}\left(\tau_b < \tau_a,\ \tau_a < \tau_c\right) = \mathbb{P}\left(\tau_a < \tau_c \mid \tau_b < \tau_a\right)\mathbb{P}\left(\tau_b < \tau_a\right).$$

Take the stopping time $\tau = \tau_a \wedge \tau_b = \inf\{t > 0, B_t \in \{a, b\}\}$ and observe that $\{\tau_b < \tau_a\} = \{B_\tau = b\}$. Therefore using the strong Markov property we obtain

$$\mathbb{P}\left(\tau_a < \tau_c \mid \tau_b < \tau_a\right) = \mathbb{P}\left(\tau_a < \tau_c \mid B_\tau = b\right) = \mathbb{P}_b\{\tau_a < \tau_c\} = \mathbb{P}\left(\tau_{a-b} < \tau_{c-b}\right).$$

Recall that using Wald's lemma it is easy to find that $\mathbb{P}\left(\tau_a < \tau_b\right) = \frac{b}{b-a}$ (basically we combine the equations $\mathbb{E}B_\tau = 0$ and $\mathbb{P}\left(\tau_a < \tau_b\right) = 1 - \mathbb{P}\left(\tau_a > \tau_b\right)$). Thus we get

$$\mathbb{P}\left(\tau_b < \tau_a < \tau_c\right) = \frac{c-b}{c-b-(a-b)}\frac{-a}{b-a} = \frac{-a(c-b)}{(b-a)(c-a)}.$$

□