Numerical Methods

PDEs

1 Non Matlab exercises

1. From Lecture notes we know that the second derivative is well approximated by the following finite difference:

$$\frac{d^2u}{dx^2}(0) \simeq \frac{u(-h) - 2u(0) + u(h)}{h^2}.$$

   How would you derive this formula if you did not know it?

2. Consider the following PDE:

$$u_t = u_x$$

   where $x \in [0,1]$, $t \geq 0$. The boundary condition is periodic:

$$u|_{x=0} = u|_{x=1}.$$

   The initial condition is $u|_{t=0} = \sin(\pi x) + 10\sin(3\pi x)$.

   (a) Find the exact solution.
   (b) Design the forward Euler scheme.
   (c) Analyse the stability of the Euler scheme.
   (d) Analyse the stability of the leap-frog scheme:

$$U_j^{n+1} = U_j^{n-1} + \frac{\Delta t}{\Delta x}(U_{j+1}^n - U_{j-1}^n).$$

2 Matlab exercises

1. Consider the following PDE:

$$u_t = u_{xx}$$

   where $x \in [0,1]$, $t \geq 0$. The boundary condition is

$$u|_{x=0} = u|_{x=1} = 0.$$

   The initial condition is $u|_{t=0} = \sin(\pi x) + 10\sin(3\pi x)$.

(a) Find the exact solution of this problem. Hint: use separation variables technique. (You do not need MATLAB for this.)

(b) The Euler scheme for this equation is

$$U_j^{n+1} = U_j^n + \frac{\Delta t}{\Delta x^2}(U_{j+1}^n - 2U_j^n + U_{j-1}^n)$$

for $0 < j < J$ and

$$U_0^{n+1} = U_J^{n+1} = 0.$$

Here $J + 1$ is the number of grid points in x direction.

The theory guaranties that the approximate solution U converges to the exact solution when Δx, $\Delta t \to 0$, provided the stability condition holds:

$$\frac{\Delta t}{\Delta x^2} < \frac{1}{2}.$$

Implement the Euler scheme and numerically find an approximate solution for $t \in [0, 0.1]$. Plot a graph of the solution.

(c) Investigate what happens when the stability condition is violated.

(d) Implement the Implicit Euler scheme.

(e) Implement the Crank-Nicolson scheme.

(f) Investigate how the error converges to zero when Δx, $\Delta t \to 0$ for the all three schemes.

2. Consider the following PDE from the Non matlab section:

$$u_t = u_x$$

where $x \in [0, 1]$, $t \geq 0$. The boundary condition:

$$u|_{x=0} = u|_{x=1}.$$

The initial condition is $u|_{t=0} = \sin(\pi x) + 10\sin(3\pi x)$.

(a) Implement the forward Euler scheme.

(b) Implement the leap-frog scheme:

$$U_j^{n+1} = U_j^{n-1} + \frac{\Delta t}{\Delta x}(U_{j+1}^n - U_{j-1}^n).$$