Numerical Methods

Monte-Carlo method for Greeks

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Often, one needs to estimate some value $C(\theta)$ which depends on a parameter θ and also to find a derivative of C with respect to θ. For example, all "Greeks" are expressed in this way:

$$\begin{aligned}
\Delta &= \frac{\partial C}{\partial S_0} \\
\Gamma &= \frac{\partial^2 C}{\partial S_0^2} \\
\rho &= \frac{\partial C}{\partial r} \\
\text{Vega} &= \frac{\partial C}{\partial \sigma}
\end{aligned}$$

1 Finite Difference Method

If $C(\theta) = E(f(\theta))$ (so f depends on the parameter θ) and f is a differentiable with respect to θ function, the derivative can be approximated by one-sided finite difference

$$\begin{aligned}
\frac{dC}{d\theta}(\theta) &= \frac{C(\theta+\Delta\theta) - C(\theta)}{\Delta\theta} + O(\Delta\theta) \\
&= \frac{E(f(\theta+\Delta\theta) - E(f(\theta))}{\Delta\theta} + O(\Delta\theta).
\end{aligned}$$

or by two-sided finite difference:

$$\begin{aligned}
\frac{dC}{d\theta}(\theta) = &= \frac{C(\theta+\Delta\theta) - C(\theta-\Delta\theta)}{2\Delta\theta} + O(\Delta\theta^2) \\
&= \frac{E(f(\theta+\Delta\theta)) - E(f(\theta-\Delta\theta))}{2\Delta\theta} + O(\Delta\theta^2).
\end{aligned}$$

The variance of the second approximation is

$$\begin{aligned}
\mathrm{var}\left(\frac{f(\theta+\Delta\theta) - f(\theta-\Delta\theta)}{2\Delta\theta}\right) &= \frac{1}{4\Delta\theta^2}(\mathrm{var}(f(\theta+\Delta\theta)) + \mathrm{var}(f(\theta-\Delta\theta)) \\
&\quad -2\mathrm{cov}(f(\theta+\Delta\theta), f(\theta-\Delta\theta))) \\
&\approx \frac{1}{2\Delta\theta^2}\,\mathrm{var}(f(\theta))(1 - \mathrm{corr}(f(\theta+\Delta\theta), f(\theta-\Delta\theta))).
\end{aligned}$$

If $C(\theta + \Delta\theta)$ and $C(\theta - \Delta\theta)$ are simulated by Monte-Carlo independently, then $\mathrm{corr}(C(\theta + \Delta\theta), C(\theta - \Delta\theta)) = 0$ and the variance is

$$\frac{1}{2\Delta\theta^2}\,\mathrm{var}(f(\theta))$$

which can be very large if $\Delta\theta$ is small.

In order to minimize the variance we should maximize the correlation between $f(\theta + \Delta\theta)$ and $f(\theta - \Delta\theta)$. For monotonic functions C it can be shown that the correlation is positive if we use the same sample path set for computations of both $f(\theta + \Delta\theta)$ and $f(\theta - \Delta\theta)$. This method of variance reduction is called *path recycling*. The formula for the approximation of the derivative becomes

$$\frac{dC}{d\theta}(\theta) \quad = \quad \frac{E(f(\theta + \Delta\theta) - f(\theta - \Delta\theta))}{2\Delta\theta} + O(\Delta\theta^2).$$

In this case the variance estimate is

$$\mathrm{var}(\frac{f(\theta + \Delta\theta) - f(\theta - \Delta\theta)}{2\Delta\theta}) \approx \mathrm{var}(\frac{\partial f}{\partial\theta}(\theta)).$$

Exercise:

Implement the Finite difference method to compute delta for European call option.

1.1 Discontinuous payoffs

Consider a case of the binary options. The payoff function of cash-or-nothing binary call option is given by

$$f(S) = A\,H(S - X),$$

where H is the Heaviside function, X is the strike price and A is the amount paid.

As before, we assume the geometrical Brownian motion model and at time T S is distributed as

$$S(T) = S(0)\exp((r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\epsilon)$$

with $\epsilon \sim N(0, 1)$.

Suppose we want to compute the sensitivity of this option with respect to the strike price X (i.e. we want to compute $\frac{\partial C}{\partial X}$). For that we generate n pseudo-random numbers $\epsilon_1, \ldots, \epsilon_n$ normally $N(0, 1)$ distributed, compute corresponding $S_i(T)$ using the formula above and then

$$\begin{aligned}\frac{\partial C}{\partial X} &\approx E_n \\ &= \frac{A\exp(-rT)}{2n\Delta X}\sum_{i=1}^{n} H(S_i(T) - X - \Delta X) - H(S_i(T) - X + \Delta X).\end{aligned}$$

Notice that

- For most samples $S_i(T)$ we have

$$H(S_i(T) - X - \Delta X) - H(S_i(T) - X + \Delta X) = 0$$

  ($O(\Delta X)$ in general).

- For $O(\Delta X)$ fraction of samples we have

$$H(S_i(T) - X - \Delta X) - H(S_i(T) - X + \Delta X) = 1$$

  ($O(1)$ in general).

This gives $\mathrm{var}(E_n) \approx \frac{b}{n\Delta X}$, where b is some constant.

The mean square error of this method is a sum of the discretization error and of the MS error:

$$\text{MSE} \approx a\Delta X^4 + \frac{b}{n\Delta X}.$$

It is minimized by choosing $\Delta X = \left(\frac{b}{4an}\right)^{1/5}$, giving $\sqrt{\text{MSE}} \approx O(n^{-2/5})$ compared to usual $O(n^{-1/2})$.

2 Pathwise derivative estimate

Suppose we want to estimate delta for the standard European call option. The discounted payoff:

$$f(S) = \exp(-rT)\max(0, S - X),$$

the distribution for the underlying price at the expiry date:

$$S(0)\exp((r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\epsilon)$$

with $\epsilon \sim N(0, 1)$.

Our parameter here is $S(0)$. Using the chain rule we get

$$\frac{\partial f}{\partial S(0)} = \frac{\partial f}{\partial S(T)}\frac{\partial S(T)}{\partial S(0)}.$$

Note that $\frac{\partial f}{\partial S(T)}$ does not exists at $S(T) = X$.

All the derivatives can be evaluated:

$$\begin{aligned} \frac{\partial f}{\partial S(T)} &= \exp(-rT)1_{S(T)>X} \\ \frac{\partial S(T)}{\partial S(0)} &= \frac{S(T)}{S(0)} \end{aligned}$$

The last equality holds because $S(T)$ is linear with respect to $S(0)$.

Finally, we get:

$$\Delta = E\left(\exp(-rT)1_{S(T)>X}\frac{S(T)}{S(0)}\right).$$

Note that if the payoff function is discontinuous with respect to the parameter, this method does not work as the above example of the binary options shows. Indeed, in this case the derivative $\frac{\partial f}{\partial X}$ is zero almost everywhere and this method would give zero output.

Exercise:

Implement the Pathwise derivative method to compute delta for European call option.

3 Likelihood ratio method

Often it is not the payoff function depends on the parameter but the probability distribution does (as in the example of computing delta for the European options). In this case the following method can be used.

Let $P(S)$ be the probability density function for the final state $S(T)$. We assume that $P(S)$ depends on θ and the payoff function does not. Then

$$C = E(f) = \int f(S)P(S)\,dS.$$

Hence,

$$\begin{aligned}\frac{\partial C}{\partial \theta} &= \int f(S)\frac{\partial P(S)}{\partial \theta}\,dS \\ &= \int f(S)\frac{\partial \log(P(S))}{\partial \theta}P(S)\,dS \\ &= E(f\frac{\partial \log P}{\partial \theta}).\end{aligned}$$

In the case of the geometrical Brownian motion the probability density is

$$P(S) = \frac{1}{S\sigma\sqrt{2\pi T}}\exp\left[-\frac{1}{2}\left(\frac{\log(\frac{S}{S_0}) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)^2\right]$$

and

$$\frac{\partial \log P}{\partial S_0}(S) = \frac{\log(\frac{S}{S_0}) - (r - \frac{1}{2}\sigma^2)T}{S_0\sigma^2 T}.$$

Now delta can be estimated as

$$\Delta = E\left(\frac{\log(\frac{S}{S_0}) - (r - \frac{1}{2}\sigma^2)T}{S_0\sigma^2 T}f(S)\right).$$

Here the payoff function can be discontinuous.

Exercise:

Implement the Likelihood ratio method to compute delta for European call option. Compare volatilities for all the three methods.