Brownian Motion

Motivation: Brownian Motion is a universal object

Let $S_n = X_1 + \cdots + X_n$ be a random walk. We know that

$$\frac{S_n}{\sqrt{n}} \xrightarrow{(d)} N(0,1)$$

- What if we need more path properties, eg.

$$P(\tau_a > t) \sim ? \quad \text{where } \tau_a = \inf\{n : S_n \geq a\}$$

- $\frac{S_{[nt]}}{\sqrt{n}} \xrightarrow{d} N(0,t)$ for fixed t. What if we look at the function

$$\left\{ t \to \frac{S_{[nt]}}{\sqrt{n}} \right\} ?$$

- Brownian Motion is the scaling limit of large class of R.W.'s
- It has scaling properties
- some computations are easier
- More difficult questions about R.W.'s can be approximated, eg.

$$P(S_n \simeq x) \approx e^{-x^2/2n}$$

Definition: Brownian motion is a random function $\{B(t) : t \geq 0\}$ s.t.

① $\forall\ 0 \leq t_1 < t_2 < \cdots < t_k$ the increments

$\left(B(t_{i+1}) - B(t_i)\right)_{i \geq 1}$ are independent, $N(0, t_{i+1} - t_i)$

② $t \to B(t)$ is continuous

Question: Does B.M exist? How to construct it?

H/W: Use the definition to show that for any $0 < t_1 < \cdots < t_k$

$$(B(t_1), \ldots, B(t_k))$$

has multivariate normal distribution

H/W (finite dimensional distributions do not determine)

Let $\beta(t)$ be a B.M. & $U \sim \text{Uniform}[0,1]$

$$\tilde{\beta}(t) := \begin{cases} \beta(t), & \text{if } t \neq U \\ 0, & \text{if } t = U \end{cases}$$

Show that $\tilde{\beta}(\cdot)$ & $\beta(\cdot)$ have the same f.d.d. but $\tilde{\beta}(\cdot)$ is a.s. not continuous.

Construction of B.M. à la Lévy

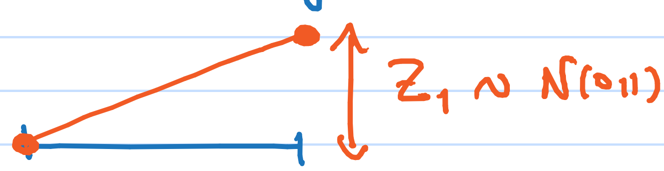
We will construct the B.M. on $[0,1]$, but the extension is easy.

$$D_n := \left\{ \frac{k}{2^n} : k = 0, 1, \dots, 2^n \right\}$$

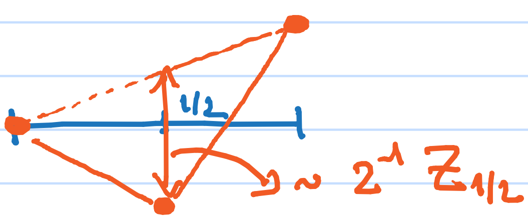
$$D := \bigcup_{n \geq 1} D_n \quad \text{the dyadic points of } [0,1]$$

We will construct a sequence of piecewise linear, continuous functions with the right f.d.d. The limit will be the B.M.

n=0



n=1

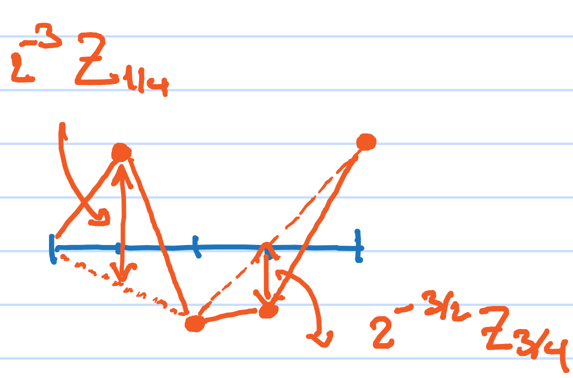


$$\beta(1/2) := \frac{\beta(0) + \beta(1)}{2} + 2^{-1} Z_{1/2}$$

$$\mathrm{Var}(\beta(1)) = \mathrm{Var}\left(\frac{\beta(0)+\beta(1)}{2}\right) + 2^{-2}\,\mathrm{Var}(Z_{1/2})$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

n=3



So, for $t = 1/4, 3/4$ we have

$$\beta(t) = \frac{\beta(t - 2^{-n}) + \beta(t + 2^{-n})}{2} + 2^{-\frac{n+1}{2}} Z_t \qquad (1)$$

Computing the variance, we will see that it is the right one at $t = 1/4, 3/4$

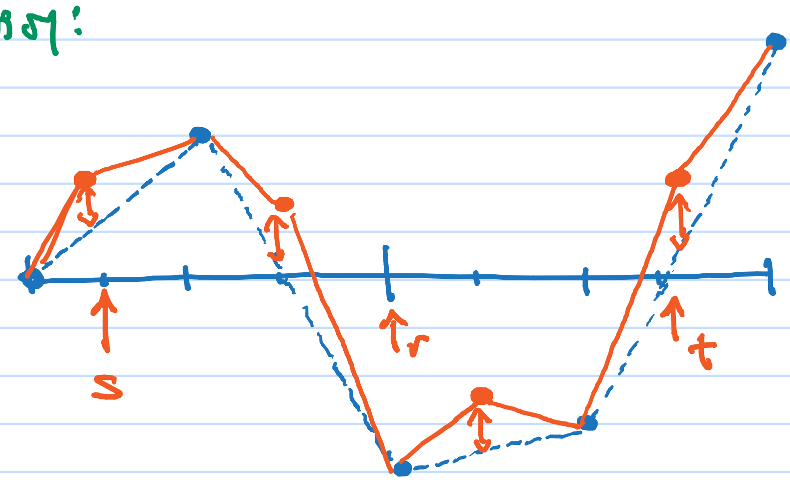
Proceed similarly at the next level of dyadic points using (1)

Need to show: 1) $B_n(\cdot)$ converges uniformly to a function with the B.M. properties.

Observations: 1) $B_n(\cdot)$ is continuous

2) $\forall\ s<r<t \in D_n$, $B_n(r)-B_n(s)$ & $B_n(t)-B_n(s)$ are independent

Explanation:



(Sort of obvious since the $B_n(r)-B_n(s)$ involves randomness only from the Gaussians between (s,r) and $B_n(t)-B_n(r)$ between (r,t).)

To check telescope $B_n(r)-B_n(s)$ with the closest to r,s dyadic points in D_{n-1}.

$$B_n(r)-B_n(s) = \left(B_n(r)-B_n(r-2^{-n})\right) + \left(B_n(r-2^{-n})-B_n(s+2^{-n})\right) + \left(B_n(s+2^{-n})-B_n(s)\right)$$

$$= \left(\frac{B_{n-1}(r)-B_{n-1}(r-2^{-n+1})}{2} + 2^{-(n+1)/2} Z_{r-2^{-n}}\right) + \left(B_{n-1}(r-2^{-n})-B_{n-1}(s+2^{-n})\right) + \left(\frac{B_{n-1}(s+2^{-n})-B_{n-1}(s-2^{-n})}{2} + 2^{-(n+1)/2} Z_s\right)$$

So the increament $B_n(r)-B_n(s)$ uses increaments on disjoint intervals at level $(n-1)$ of the construction & the Z-randomness inside (s,r). Using an induction argument we have the conclusion.

3) $B_n(r)-B_n(s) \sim N(0, r-s)$ for $r,s \in D_n$. The argument is similar: Assume that $r,s \in D_n \setminus D_{n-1}$

$$B_n(r)-B_n(s) = \left(B_n(r)-B_n(r-2^{-n})\right) + \left(B_n(r-2^{-n})-B_n(s+2^{-n})\right) + \left(B_n(s+2^{-n})-B_n(s)\right)$$

$$= \left(\frac{B_{n-1}(r+2^{-n})-B_{n-1}(r-2^{-n})}{2} + 2^{-(n+1)} Z_r\right)^{1} + \left(B_{n-1}(r-2^{-n})-B_{n-1}(s+2^{-n})\right)^{2} + \left(\frac{B_{n-1}(s+2^{-n})-B_{n-1}(s-2^{-n})}{2} + 2^{-(n+1)/2} Z_s\right)^{3}$$

By induction & Observation 2 all the $(\cdot)$ are independent Gaussians. So $B_n(r)-B_n(s)$ is Gaussian, mean 0.

Compute the variance:

$$\mathrm{Var}\left(\beta_n(r)-\beta_n(s)\right) = \mathrm{Var}\left(1\right) + \mathrm{Var}\left(2\right) + \mathrm{Var}\left(3\right) =$$

$$= \left\{ \mathrm{Var}\left(2^{-(n+1)/2} Z_r\right) + \mathrm{Var}\left(\frac{\beta_{n-1}(r+2^{-n}) - \beta_{n-1}(r-2^{-n})}{2}\right)\right\} +$$

$$\left\{ (r-2^{-n}) - (s+2^{-n}) \right\} \quad \text{by induction} \quad +$$

$$+ \left\{ \mathrm{Var}\left(2^{-(n+1)/2} Z_s\right) + \mathrm{Var}\left(\frac{\beta_{n-1}(s+2^{-n}) - \beta_{n-1}(s-2^{-n})}{2}\right)\right\}$$

use induction again

$$2^{-(n+1)} + \frac{1}{4} 2^{-n+1} + (r-s) - 2^{-n+1}$$
$$+ 2^{-(n+1)} + \frac{1}{4} 2^{-n+1}$$

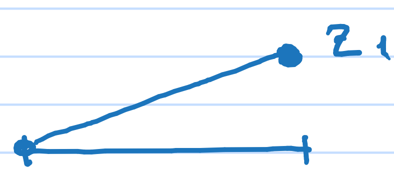
$$= r-s.$$ ■

So, if we define $\beta(t) := \beta_n(t)$ when $t \in D_n$, then we have defined a function with the correct statistics on dyadic points. Since this is a dense set, it remains to show that

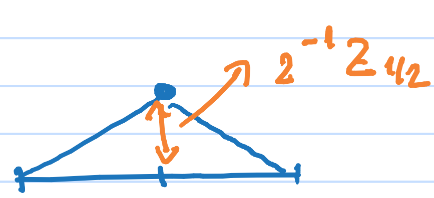
CLAIM: $(\beta_n(\cdot))$ converges in $\|\cdot\|_\infty$:

Consider the functions

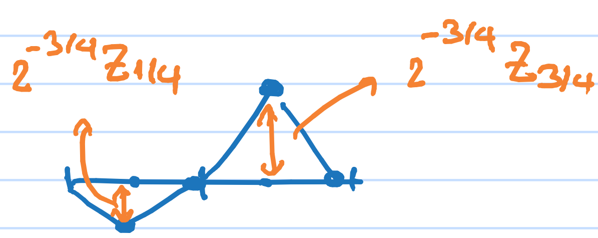
- $F_0(t) := \begin{cases} Z_1 & ; t=1 \\ 0 & ; t=0 \\ \text{linear} \end{cases}$



- $F_1(t) := \begin{cases} 2^{-1} Z_{1/2} & ; t=1/2 \\ 0 & ; t=0,1 \\ \text{linear} \end{cases}$



- $F_2 = \begin{cases} 2^{-3/2} Z_t & ; t=1/4, 3/4 \\ 0 & ; t=0, 1/2, 1 \\ \text{linear} \end{cases}$



in general

- $F_n(t) = \begin{cases} 2^{-(n+1)/2} Z_t & ; t \in D_n - D_{n-1} \\ 0 & ; t \in D_{n-1} \\ \text{linear} \end{cases}$

Claim: $\beta(t) = \sum_{i \geq 0} F_i(t) \quad \forall\, t \in D$

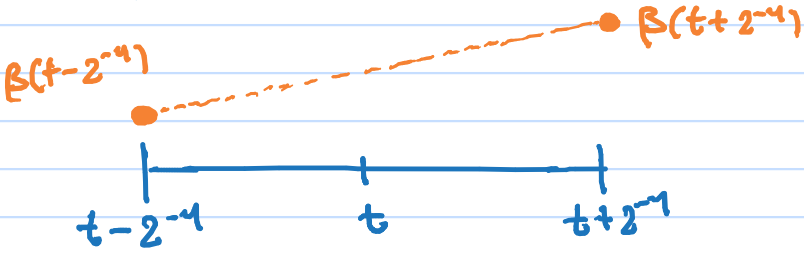
$$\left(= \sum_{i=1}^{n} F_i(t) \quad \text{if } t \in D_n \right)$$

Show this by induction! Let $t \in D_n$

$$\beta(t) = F_n(t) + \sum_{i=1}^{n-1} F_i(t)$$

if $t \in D_{n-1}$, then $F_n(t) = 0$ & $\beta(t) = \sum_{i=1}^{n-1} F_i(t)$ ✓

if $t \in D_n \setminus D_{n-1}$, then



$$\beta(t) = 2^{-(n+1)/2} Z_t + \underbrace{\sum_{i=1}^{n-1} F_i(t)}$$

$$= 2^{-(n+1)/2} Z_t + \sum_{i=1}^{n-1} \frac{1}{2}\Big(F_i(t+2^{-n}) + F_i(t-2^{-n}) \Big)$$

$$= 2^{-(n+1)/2} Z_t + \frac{1}{2}\sum_{i=1}^{n-1} F_i(t+2^{-n}) + \frac{1}{2}\sum_{i=1}^{n-1} F_i(t-2^{-n})$$

$$\overset{\text{induction}}{=} 2^{-(n+1)/2} Z_t + \frac{1}{2}\Big(\beta_{n-1}(t+2^{-n}) + \beta_{n-1}(t-2^{-n}) \Big)$$

$$\overset{\text{by def.}}{=} \beta_n(t). \qquad \blacksquare$$

CLAIM: $\sum_{n \geq 1} F_n(\cdot)$ converges in $\|\cdot\|_\infty$ - $\mathbb{P}$ a.s. (so $\beta(\cdot)$ is a.s. continuous)

for this, we need to show that $\sum_{i \geq 1} \|F_i\|_\infty < \infty$ $\mathbb{P}$.a.s

We have that $\|F_n\|_\infty \leq 2^{-\frac{n+1}{2}} \max_{t \in D_n} |Z_t|$ & estimate

$$\sum_n \mathbb{P}\Big(\max_{t \in D_n} |Z_t| > c\sqrt{n} \Big) \leq \sum_n \sum_{t \in D_n} \mathbb{P}\big(|Z_t| > c\sqrt{n}\big)$$

$$\overset{\text{check}}{\leq} \sum_t (2^n + 1) e^{-c^2 n/2} < \infty \quad \text{if} \quad c > \sqrt{2\log 2}.$$

So, by Borel-Cantelli, for all large enough n: $|Z_t| < c\sqrt{n}$

$$\leadsto \sum_{n \geq 1} \|F_n\|_\infty \leq \sum_n 2^{-\frac{n+1}{2}} c\sqrt{n} < \infty.$$

So we have shown that $\beta(\cdot)$ is a.s. continuous. It also has the right statistics:

Claim: $\beta(\cdot)$ has independent Gaussian increments

because we have shown this if $t,s \in D$. If not find

$$\left.\begin{array}{l} D \ni t_k \to t \\ D \ni s_k \to s \end{array}\right\} \rightsquigarrow \begin{array}{ccc} \beta(t_k) - \beta(s_k) & \xrightarrow[k\to\infty]{} & \beta(t) - \beta(s) \\ \|{}_d & & \\ \mathcal{N}(0, t_k - s_k) & \xrightarrow[k\to\infty]{} & \mathcal{N}(0, t-s) \end{array}$$

END OF Construction.

Alternative Approach (Kolmogorov)

Let $X \subset C(0,1)$ & $\mathcal{B}$ the σ-algebra generated by f.d.d.'s

Assume we have a family of measures $\{\mu_F\}_{\substack{F \subset [0,1] \\ \text{finite}}}$ determining the f.d.d.'s.

$$\boxed{\begin{array}{l} \textbf{Thm} \text{ If } \forall\, s,t \in [0,1] \ \exists\, \beta, \alpha > 0 \text{ s.t.} \\ \qquad \int |x-y|^\beta \mu_{s,t}(dx\,dy) \leq C\,|t-s|^{1+\alpha} \\ \text{then } \exists! \text{ measure } Q \text{ on } (X, \mathcal{B}) \text{ s.t. } Q|_F = \mu_F \ \& \\ \qquad Q\left\{ \sup_{s,t} \frac{|x(t) - x(s)|}{|t-s|^{\alpha/\beta}} < \infty \right\} = 1 \end{array}}$$

Corollary B.M. is (at least) α-Hölder continuous for any $\alpha < 1/2$

Proof Compute $\mathbb{E}\left[|\beta(t) - \beta(s)|^{2n}\right] \underset{\text{Gaussian}}{\leq} \text{const.}\ \mathbb{E}\left[|\beta(t)-\beta(s)|^2\right]^n$

$= \text{const.}\ |t-s|^n$

Kolmogorov's thm implies that

it is Hölder $\frac{n-1}{2n} \to 1/2$ as $n \to \infty$. ∎

Some basic properties of B.M.

1) **Scaling** $\forall a > 0$: $\beta(at) =^d \sqrt{a}\,\beta(t)$

because the functions $t \to \beta(at)$ are continuous

& $\beta(at) - \beta(as) \sim N(0, a(t-s)) =^d \sqrt{a}(\beta(t) - \beta(s))$

Remark: Be aware that the above scaling seems to lead to a contradiction (or confusion) :

$$\int_0^1 \beta(t)\,dt =^d \int_0^1 \sqrt{t}\,\beta(1)\,dt = \beta(1)\,\frac{2}{3}$$

What's wrong ?

2) **Inversion** Let $\beta(\cdot)$ be B.M. Then

$$\tilde\beta(t) := \begin{cases} 0, & \text{if } t = 0 \\ t\,\beta(1/t), & \text{if } t > 0 \end{cases} \quad \text{is B.M.}$$

because • check first the f.d.d.'s

$$\mathbb{E}[\tilde\beta(t)] = 0 \quad \forall t > 0$$

$$\mathbb{E}[\tilde\beta(t)\tilde\beta(s)] = ts\,\mathbb{E}[\beta(1/t)\beta(1/s)] =$$

$$= ts\left(\frac{1}{t} \wedge \frac{1}{s}\right) = t \wedge s$$

• check also continuity : Let a sequence $\mathbb{Q} \ni t_n \downarrow 0$

we have already checked that $\{\tilde\beta(t)\}_{t\in\mathbb{Q}} =^d \{\beta(t)\}_{t\in\mathbb{Q}}$

$$\mathbb{P}\left(\lim_n \tilde\beta(t_n) > \varepsilon\right) = \mathbb{P}\left(\bigcap_m \bigcup_{n > m} \{\tilde\beta(t_n) > \varepsilon\}\right)$$

$$= \lim_m \mathbb{P}\left(\bigcup_{n>m} \{\tilde\beta(t_n) > \varepsilon\}\right) = \lim_m \lim_N \mathbb{P}\left(\bigcup_{n=m+1}^N \{\tilde\beta(t_n) > \varepsilon\}\right)$$

$$= \lim_m \lim_N \mathbb{P}\left(\bigcup_{n=m+1}^N \{\beta(t_n) > \varepsilon\}\right) \quad \underset{\text{the steps}}{\overset{\text{reverse}}{=\!=\!=}}$$

$$= \mathbb{P}\left(\lim_n \beta(t_n) > \varepsilon\right) = 0 \quad (\text{by the continuity of } \beta(\cdot))$$

$\leadsto \tilde\beta(\cdot)$ is also B.M.

H/W We have shown that B.M. is a.s. Hölder with exponent α for any $\alpha < 1/2$. However

$$\mathbb{P}\left(\sup_{t,s \in [0,1]} \frac{|\beta(t)-\beta(s)|}{|t-s|^{1/2}} < \infty \right) = 0.$$

Can you strengthen this to:

"B.M. is a.s. nowhere, Hölder 1/2"

The following thm's study the modulus of continuity

Thm 1 $\exists$ $c>0$ s.t. a.s. for all h sufficiently small (depending on the realization of B.M. & all $t<1-h$ h: $\frac{|\beta(t+h)-\beta(t)|}{\sqrt{h\log 1/h}} < c$.

Thm 2 $\forall$ $c<\sqrt{2}$

$$\mathbb{P}\left(\forall \varepsilon>0, \ \exists h \in (0,\varepsilon) \ \& \ t \le 1-h : \frac{|\beta(t+h)-\beta(t)|}{\sqrt{h\log 1/h}} \geq c \right) = 1$$

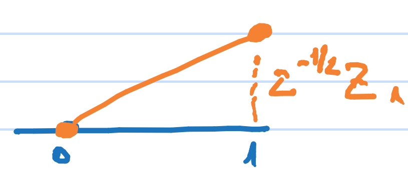
We will prove these thms using the construction of Lévy

Proof of Thm 1 We have represented B.M as

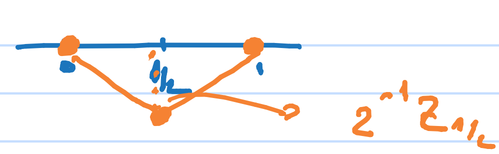
$$\beta(t) = \sum_{n=0}^{\infty} F_n(t)$$

recall:

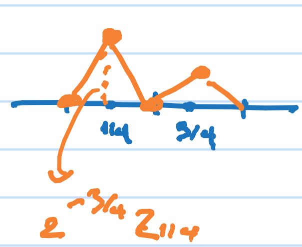
$F_0(\cdot)$:



$F_1(\cdot)$:



$F_2(\cdot)$:



We will estimate

$$\left| B(t+h) - B(t) \right| \le \sum_{n\geq 0}^{\infty} \left| F_n(t+h) - F_n(t) \right| \le \qquad (1)$$

$$\le \sum_{n=0}^{\ell} h \, \| F_n' \|_\infty + 2 \sum_{n=\ell+1}^{\infty} \| F_n \|_\infty$$

but (look at the figures)

$$\| F_n' \|_\infty \le \frac{2^{-\frac{n+1}{2}} \max_{t \in D_n} |Z_t|}{2^{-n}} < 2^{n/2} \max_{t\in D_n} |Z_t| \underset{\text{by Borel-Cantelli}}{<} c\sqrt{n}\, 2^{-n/2}$$

look at Lévy's proof
$c = \sqrt{2\log 2}$

a.s. for all large n, say $n > N_0$ with N_0 random.

Use this into (1) (after you first decompose further)

$$h \sum_{n=0}^{N_0} \|F_n'\|_\infty + h \sum_{n=N_0+1}^{\ell} c\sqrt{n}\, 2^{n/2} + 2c \sum_{n=\ell+1}^{\infty} \sqrt{n}\, 2^{-n/2}$$

$$< h \sum_{n=0}^{N_0} \|F_n'\|_\infty + \underbrace{\text{const.}\, h \sqrt{\ell}\, 2^{\ell/2} + \text{const.}\, \sqrt{\ell}\, 2^{-\ell/2}}_{\substack{\text{we choose } \ell \text{ so that this sum} \\ \text{is optimized } (\leadsto \ \ell = \log 1/h)}}$$

$$=: h \sum_{n=0}^{N_0} \|F_n'\|_\infty + \text{const.} \left(h \log \frac{1}{h} \right)^{1/2}$$

for all h small enough (depending on $N_0 = N_0(\omega)$) we have that the above is $\lesssim \text{const.} \left(h \log \frac{1}{h}\right)^{1/2}$ ■

We can tighten up this estimate by identifying the right constant c. Above the constant c was essentially the $\sqrt{2\log 2}$ obtained from the application of the Borel-Cantelli.

We can actually have the (optimal) estimate :

Refinement of Thm 1: For every $c > \sqrt{2}$ it holds that

$$\mathbb{P}\left(\overline{\lim_{h \downarrow 0}} \sup_{t < t+h} \frac{\beta(t+h) - \beta(t)}{\sqrt{h \log \frac{1}{h}}} < c \right) = 1$$

Proof: Before proving it, it is important to understand why the previous estimate is not optimal: The functions $(F_n(\cdot))_{n \geq 1}$ represent the scales within Brownian Motion & when we estimated $\sup_{t \in D_n} |Z_t|$ (& then with Borel-Cantelli) we tried to control uniformly all the scales.

The tighter estimate will come from sort of bypassing the uniform estimate & also bootstrap the $\sqrt{h \log 1/h}$ order obtained before.

Define: $\Lambda_n(m) := \Big\{ [(k-1+b)2^{-n+a}, (k+b)2^{-n+a}] :$
$a, b \in \{0, \frac{1}{m}, \dots, \frac{m-1}{m}\}$ &
$k \in \{1, 2, \dots, 2^n\} \Big\}$

$\Lambda(m) := \bigcup_n \Lambda_n(m)$

Compute:

$$\mathbb{P}\left(\sup_{k=1,\dots,2^n} \sup_{a,b \in \{0, \frac{1}{m}, \dots, \frac{m-1}{m}\}} \left| \beta((k-1+b)2^{-n+a}) - \beta((k+b)2^{-n+a}) \right| > c\sqrt{2^{-n+a} \log 2^{-n+a}} \right)$$

(c will be $> \sqrt{2}$, but arbitrary)

$$\leq 2^n m^2 \,\mathbb{P}\left(\left| \beta((k-1+b)2^{-n+a}) - \beta((k+b)2^{-n+a}) \right| > c\sqrt{2^{-n+a} \log 2^{-n+a}} \right)$$

$$= 2^n m^2 \,\mathbb{P}\left(|\beta(1)| > c\sqrt{\log 2^{-n+a}} \right)$$

$$\leq 2^n m^2 e^{-\frac{1}{2} c^2 \log 2^{n-a}} = m^2 \exp\left\{ \log 2 \left(-\frac{c^2}{2}(n-a) + n \right) \right\}$$

which is summable if $c \geq \sqrt{2}$.

$$\rightsquigarrow \mathbb{P}\left(|\beta(t) - \beta(s)| < c\sqrt{(t-s) \log \frac{1}{|t-s|}} \text{ for all } t, s \in \Lambda_n(m),\ \forall n \geq n_0 \right) = 1$$

if $c > \sqrt{2}$.

This is enough, because for arbitrary $t, s \in (0,1)$ we can interpolate with $t', s' \in \Lambda_n(m)$ for some n, m.

(this is the purpose of the bit complicated definition of $\Lambda_n(m)$ involving the a, b's.) ∎

Proof of Thm 2 (lower bound).

Want to show that $\mathbb{P}\left(\overline{\lim_{h}} \sup_{t} \frac{|\beta(t+h)-\beta(t)|}{c\sqrt{h \log 1/h}} \geq 1 \right) = 1 \quad \forall\, c < \sqrt{2}$

the probability is bounded below by

$$\mathbb{P}\left(\sup_{k,n} \frac{|\beta((k+1)e^{-n}) - \beta(ke^{-n})|}{c e^{-n/2} \sqrt{n}} \geq 1 \right)$$

Compute $\mathbb{P}\left(\frac{|\beta((k+1)e^{-n}) - \beta(ke^{-n})|}{c e^{-n/2} \sqrt{n}} < 1 \quad \forall\, k, n \right)$ & show that this is 0.

By Borel-Cantelli it suffices to estimate

$$\sum_{n} \mathbb{P}\left(\frac{|\beta((k+1)e^{-n}) - \beta(ke^{-n})|}{c e^{-n/2} \sqrt{n}} < 1 \quad \forall\, k = 0, 1, \dots, 2^n \right)$$

$$\overset{\text{independent increments}}{=} \sum_{n} \mathbb{P}\left(|\beta(1)| < c\sqrt{n} \right)^{e^n} =$$

$$= \sum_{n} \left\{ 1 - \mathbb{P}\left(|\beta(1)| > c\sqrt{n} \right) \right\}^{e^n}$$

$$\approx \sum_{n} \left(1 - e^{-\frac{c^2 n}{2}} \right)^{e^n} \overset{1-x \leq e^{-x}}{\leq} \sum_{n} \exp\left\{ -e^n e^{-\frac{c^2}{2} n} \right\} < \infty$$

if $c < \sqrt{2}$ ∎

We have, thus, proven that

(Lévy's modulus of continuity)

$$\boxed{\text{a.s.} \quad \overline{\lim_{h \downarrow 0}} \sup_{t \in [0,1]} \frac{|\beta(t+h) - \beta(t)|}{\sqrt{h \log \frac{1}{h}}} = \sqrt{2}}$$

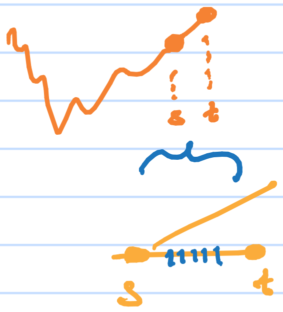
Remark: the $\overline{\lim}$ can be strengthened to $\lim$

Next we will look at the "nowhere differentiability" properties of B.M.

We will start with a warm up:

Prop: B.M. is nowhere increasing.

Proof: Suppose that there is an interval (s,t) of increase



Split this interval into an arbitrary number of subinterval $(t_i, t_{i+1})_{i=1,\dots,n-1}$ with n arbitrary.

We have:

$$\mathbb{P}\left(\beta(\cdot) \text{ is increasing in } (s,t)\right) \le$$

$$\le \mathbb{P}\left(\{\beta(t_{i+1}) > \beta(t_i)\}_{i=1\dots n}\right) =$$

$$= \mathbb{P}\left(\beta(1) > 0\right)^n \to 0 \quad \text{as } n \to 0. \quad \blacksquare$$

Thm Brownian motion is nowhere differentiable. In particular, we have that a.s. for all t

$$D^*\beta(t) := \limsup_{h \downarrow 0} \frac{\beta(t+h) - \beta(t)}{h} = +\infty$$

or

$$D_*\beta(t) := \liminf_{h \downarrow 0} \frac{\beta(t+h) - \beta(t)}{h} = -\infty$$

Proof Assume that (a.s.) there exists $t_0 \in [0,1]$ s.t.

$$-\infty < D_*\beta(t_0) \le D^*\beta(t_0) < \infty$$

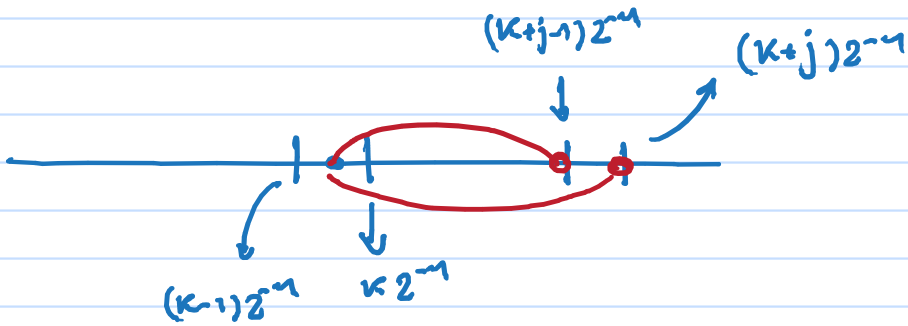
then

$$\sup_h \frac{|\beta(t+h) - \beta(t)|}{h} < M \quad \text{for some } M$$

this will imply that for any j

$$|\beta((k+j)2^{-n}) - \beta((k+j-1)2^{-n})| \le |\beta((k+j)2^{-n}) - \beta(t_0)| +$$

$$+ |\beta((k+j-1)2^{-n}) - \beta(t_0)| \le (2j+1)2^{-n} \cdot M$$



Even though this is true for all j's we will only need it for $j=1,2,3$. We have:

$$\mathbb{P}\left(\exists t_0 \text{ s.t. } \frac{|\beta(t_0+h)-\beta(t_0)|}{h} \leq M\right) \leq$$

$$\leq \mathbb{P}\Big(\forall n\ \exists k: \ |\beta((k+1+j)2^{-n}) - \beta((k+j)2^{-n})| \leq (2j+1)2^{-n}M$$

$$\text{for } j=1,2,3\Big)$$

We want to show that this is 0 & so we use Borel-Cantelli:

Compute

$$\sum_n \mathbb{P}\left(\exists k: \ |\beta((k+1+j)2^{-n}) - \beta((k+j)2^{-n})| \leq (2j+1)2^{-n}M \text{ for } j=1,2,3\right)$$

$$\leq \sum_n \sum_k \mathbb{P}\left(|\beta((k+1+j)2^{-n}) - \beta((k+j)2^{-n})| \leq (2j+1)2^{-n}M, \text{ for } j=1,2,3\right)$$

$$= \sum_n \sum_k \mathbb{P}\left(|\beta(1)| \leq 7\,2^{-n/2}M\right)^3$$

$$\leq \sum_n \sum_k \left(14M\,2^{-n/2}\right)^3 \qquad \left(\text{because } \int_{-a}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx \leq 2a\right)$$

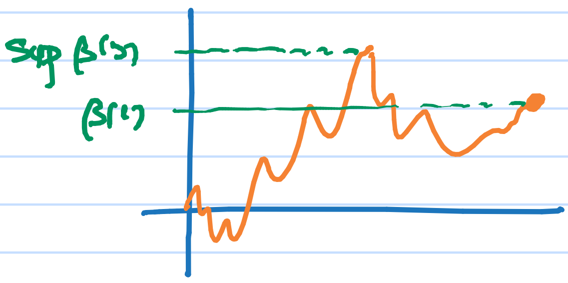
$$= \sum_n 2^n \left(14M\,2^{-n/2}\right)^3 < \infty.$$

∎

Prop $\exists\, t_* \in (0,1)$ s.t. $D^{*}(t_*) = 0$

Prof: Define the function

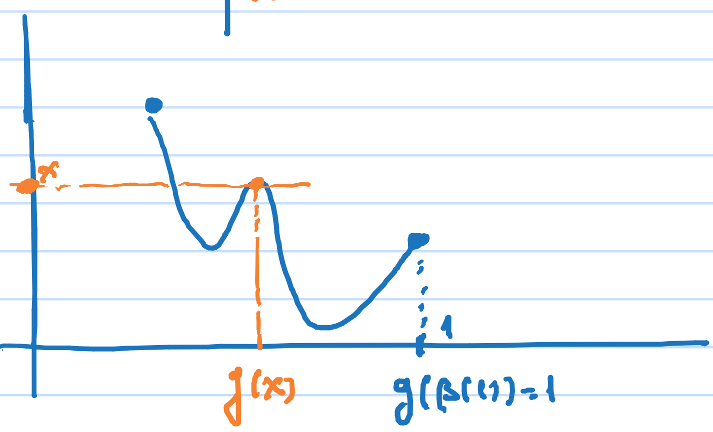
$$g(x) := \sup\{ t \in (\beta(1), \sup \beta(s)] : \beta(s) = x \}$$



Some properties of $g(\cdot)$:

① g is strictly decreasing

② g is left continuous (because if not there would accumulated times where B.M. would go arbitrarily close to a previously reached level, but would never reach it. This would violate continuity.



③ $\beta(g(x)) = x$

④ jump points of g are dense (otherwise it would violate the no increasing property – look at the second picture).

Define $V_n := \{ x : g(x-h) - g(x) > nh \text{ for some } h \in (0, n^{-1}) \}$

- V_n is open (why?)
- V_n is dense : Suppose that $\exists$ interval such that $\forall h$ & $\forall x$ in that interval $g(x-h) - g(x) \leq nh$.

Use ③ to write the inequality as

$$\underbrace{g(x-h)}_{t+\delta} - \underbrace{g(x)}_{t} \leq n\,(\beta(g(x)) - \beta(g(x-h)))$$

call $\rightsquigarrow$ for some δ $\Rightarrow$

$\Rightarrow$ $\delta \leq n\,(\beta(t) - \beta(t+\delta))$ for all t in an interval. But such an event has prob. 0 as otherwise it would violate the non-increasing property.

By the Baire Category thm

$$V = \bigcap_n V_n \quad \text{is dense}$$

This implies that for all $x \in V$,

$\exists\ x_n \uparrow x$ s.t.

$$g(x_n) - g(x) > n\,(x - x_n) =$$

$$= n\,\big(\beta(g(x)) - \beta(g(x_n))\big)$$

$$\xrightarrow[t := g(x)]{\text{set } t_n := g(x_n)} \quad t_n - t > n\,\big(\beta(t) - \beta(t_n)\big)$$

$$\Longrightarrow \quad -\frac{1}{n}(t_n - t) < \beta(t_n) - \beta(t)$$

$$\xrightarrow[n \to \infty]{\text{let}} \quad 0 \le \varlimsup_n \frac{\beta(t_n) - \beta(t)}{t_n - t} = D^*(t).$$

For $t = g(x)$ with $x \in V$, the opposite inequality

$$D^*(t) \geq 0$$

is also true: Why? ■

Quadratic Variation & Variation

Thm: let $0 = t_0^n \le t_1^n \le \dots \le t_{k_n - 1}^n \le t_{k_n}^n = t$

a nested partition, i.e. at each stage points are added.

& $\max_j (t_j^n - t_{j-1}^n) \to 0$

Then

$$D_n := \sum_{j=1}^{k_n} \big(\beta(t_j^n) - \beta(t_{j-1}^n)\big)^2 \xrightarrow[n \to \infty]{P\text{-a.s.}} t$$

Proof First, compute the expected value & then the variance.

$$\mathbb{E} D_n = \sum_{j=1}^{k_n} \mathbb{E}\big(\beta(t_j^n) - \beta(t_{j-1}^n)\big)^2 =$$

$$= \sum_{j=1}^{k_n} (t_j^n - t_{j-1}^n) = t.$$

$$\mathrm{Var}(D_n) = \sum_{j=1}^{k_n} \mathrm{Var}\left(\left(\beta(t_j^n) - \beta(t_{j-1}^n)\right)^2\right)$$

$$= \sum_{j=1}^{k_n} \mathbb{E}\left(\beta(t_j^n) - \beta(t_{j-1}^n)\right)^4 - \left(\mathbb{E}\left(\beta(t_j^n) - \beta(t_{j-1}^n)\right)^2\right)^2$$

$$= 2\sum_{j=1}^{k_n} (t_j^n - t_{j-1}^n)^2 \leq 2 \max_j (t_j^n - t_{j-1}^n) \sum_{j=1}^{k_n} (t_j^n - t_{j-1}^n)$$

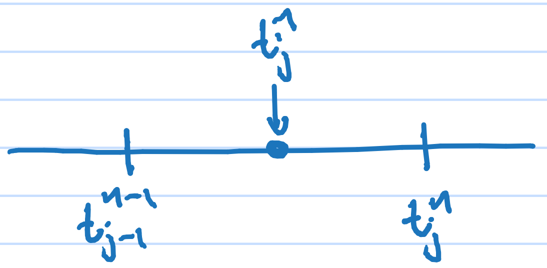
$$\to 0.$$

Remark: The above computations show that D_n converges to 0 in probability & for this we don't need the nested assumpt.
If, also, $|t_j^n - t_{j-1}^n| \to 0$ fast enough, e.g. $t_j^n = j 2^{-n}$, so that $\sum_n \sum_j (t_j^n - t_{j-1}^n)^2 < \infty$, then Borel-Cantelli implies a.s. convergence.
Interestingly, we need this nested assumption if we want the a.s. convergence but we can waive it if (t_j^n) is nested. This uses a MG argument:

a.s. convergence via MG argument: Let $X_n := \sum_{j=1}^{k_n} \left(\beta(t_j^n) - \beta(t_{j-1}^n)\right)^2$ &

$$\mathcal{G}_n := \sigma(X_n, X_{n+1}, \ldots)$$

Compute $\mathbb{E}[X_{n-1} \mid \mathcal{G}_n]$



$$= \sum_{j=1}^{k_{n-1}} \mathbb{E}\left[\left(\beta(t_j^{n-1}) - \beta(t_{j-1}^{n-1})\right)^2 \mid \mathcal{G}_n\right]$$

$$= \sum_{j=1}^{k_{n-1}} \mathbb{E}\left[\left(\left(\beta(t_j^{n-1}) - \beta(t_j^n)\right) + \left(\beta(t_j^n) - \beta(t_j)\right)\right)^2 \mid \mathcal{G}_n\right]$$

$$= \sum_{j=1}^{k_{n-1}} \mathbb{E}\left[\left(\beta(t_j^n) - \beta(t_{j-1}^{n-1})\right)^2 \mid \mathcal{G}_n\right]$$

$$+ \mathbb{E}\left[\left(\beta(t_j^n) - \beta(t_j^{n-1})\right)^2 \mid \mathcal{G}_n\right]$$

$$+ 2\underbrace{\mathbb{E}\left[\left(\beta(t_{j-1}^{n-1}) - \beta(t_j^n)\right)\left(\beta(t_j^n) - \beta(t_j^{n-1})\right) \mid \mathcal{G}_n\right]}_{= 0}$$

$$= X_n$$

We will need to explain why the cross expectation equals 0 but before let's conclude:

$$\mathbb{E}[X_{n-1} | \mathcal{G}] = X_n \quad \leadsto \quad X_n = \mathbb{E}[X_1 | \mathcal{G}_n]$$

& that $(X_n)_{n\geq 1}$ is a reverse MG. By a thm of Lévy a reverse MG has a.s. a limit, thus $\lim_n X_n$ exists a.s. & actually equals $E[X_1 | \mathcal{G}_\infty]$. For a reference regarding the backward MG convergence thm you may look at Durrett's "Probability: theory & examples".

Now, let's check why the cross expectation vanishes. Let's denote

$X := \beta(t_j^n) - \beta(t_{j-1}^{n-1})$ & $Y := \beta(t_j^n) - \beta(t_j^{n-1})$

This are independent Gaussians. We can write the cross expectation as

$$E[XY | \mathcal{G}_n] = \mathbb{E}[\mathbb{E}[XY | X_n | \mathcal{G}_n]$$ & so it suffices to show that $\mathbb{E}[XY | X_n] = 0$. This can be actually reduced to checking that $\mathbb{E}[XY | X^2+Y^2] = 0$ for X, Y independent standard normal & this is explained in class. $\blacksquare$

Ex.: Show that the nesting condition is necessary for a.s. convergence, by showing that $\exists$ (t_j^n) with mesh going to 0 s.t. $\overline{\lim_n} \sum_{j=1}^{k_n} (\beta(t_j^n) - \beta(t_{j-1}^n))^2 = \infty$ a.s.

Ex: Show that B.M. has a.s. ∞ variation

B.M. as a Markov Process

Def (d-dimensional B.M.)

If $\beta_1, \ldots, \beta_d$ are independent B.M. starting at $x_1, \ldots, x_d$

then $(\beta_1(\cdot), \beta_2(\cdot), \ldots, \beta_d(\cdot)) =: \beta(\cdot)$ is the d-dim. B.M. starting at $(x_1, \ldots, x_d)$.

Notation: P_x, $x \in \mathbb{R}^d$, will denote the distribution of d-dim B.M. starting at x (Wiener measure).

Prop Let $\{\beta(t) : t \geq 0\}$ B.M. on $\mathbb{R}^d$. Then $\forall s > 0$ $\{\beta(t+s) - \beta(s) : t \geq 0\}$ is B.M. starting at 0 & it is independent of $\{\beta(t) : 0 \leq t \leq s\}$.

Prop B.M. is Markov

proof: refer to the book of Mörters_Peres.

Def Filtration is a sequence of increasing σ-algebras

e.g. $\mathcal{F}_t = \sigma(\beta(\tau) : \tau \leq t)$

Notice that $\beta(t)$ is measurable w.r.t. $\mathcal{F}_t$ i.e. $\beta(\cdot)$ is adapted.

Germ σ-algebra: $\mathcal{F}^+(s) := \bigcap_{s > t} \mathcal{F}_t$

Thm $\{\beta(t+s) - \beta(s) : t \geq 0\}$ is independent of $\mathcal{F}^+(s)$

Proof: $\{\beta(t_i+s) - \beta(s) : i = 1 \ldots k\} = \lim_{s_n \downarrow s} \{\beta(t_i + s_n) - \beta(s_n) : i = 1 \ldots k\}$

& each of the RHS are independent of $\mathcal{F}^+(s)$.

The same will hold in the limit $s_n \downarrow s$.

Thm (Blumenthal's 0-1 law)

$\forall A \in \mathcal{F}^+(0)$ it holds that $\mathbb{P}(A) \in \{0,1\}$

Proof

Any $A \in \sigma(\beta(t): t \geq 0) = \bigcup_{\varepsilon > 0} \sigma(\beta(t): t > \varepsilon)$

but all these are independent of $\mathcal{F}^+(0)$. So

A is independent of itself $\Rightarrow$

$\mathbb{P}(A) = \mathbb{P}(A \cap A) = \mathbb{P}(A)^2 \Rightarrow \mathbb{P}(A) \in \{0,1\}$.

Thm (Tail events)

Let $x \in \mathbb{R}^d$ & $A \in \mathcal{T} := \bigcap_{t \geq 0} \sigma(\beta(s): s \geq t)$. Then

$\mathbb{P}_x(A) \in \{0,1\}$

Proof Suppose $x = 0$.

$$X(t) := \begin{cases} t\beta(1/t) & : t \neq 0 \\ 0 & : t = 0 \end{cases}$$

is a B.M. A tail event for $\beta(\cdot)$ is a germ event for $X(\cdot)$ $\Rightarrow$ it's probability is $\{0,1\}$. ■

Stopping Times

Def τ is a stopping time w.r.t. a filtration $\mathcal{F}_t$ if

$$\{\tau \leq t\} \in \mathcal{F}_t \quad \forall t.$$

Ex 1) any deterministic time is stopping time.

2) if $\{\tau_n\}$ are stopping times & $\tau_n \uparrow \tau$ then τ is a stopping time

[because $\{\tau \leq t\} = \bigcap_{n \geq 1} \{\tau_n \leq t\} \in \mathcal{F}_t$]

3) a stopping time τ w.r.t. $\mathcal{F}_t = \sigma(\beta(s): s \leq t)$ is stopping time wrt to $\mathcal{F}_t^+$ as well

[since $\mathcal{F}_t^+ \supset \mathcal{F}_t$]

4) Hitting times

Let H be a closed set, then

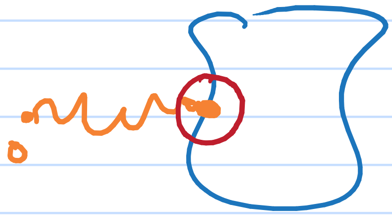
$$\tau := \inf\{t \geq 0 : \beta(t) \in H\} \text{ is } \mathcal{F}_t\text{-stopping time}$$

[because $\{\tau \leq t\} = \{\beta(s) \in H \text{ for some } s \leq t\}$

$$= \bigcap_{n} \bigcup_{s \in \mathbb{Q} \cap (0,t)} \bigcup_{x \in \mathbb{Q}^d \cap H} \{\beta(s) \in B(x, 1/n)\}$$

$$\in \mathcal{F}_t$$

picturewise



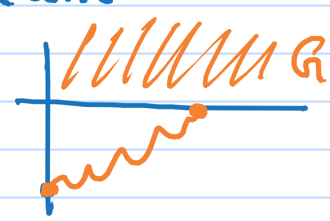
]

5) Let $G \subset \mathbb{R}^d$ open. $\tau := \inf\{t \geq 0 : \beta(t) \in G\}$

Then τ is a stopping time wrt to $\mathcal{F}_t^+$ but not necessarily wrt $\mathcal{F}_t$!

[$$\{\tau \leq t\} = \bigcap_{s > t} \{\tau < s\} = \bigcap_{s > t} \bigcup_{r \in \mathbb{Q} \cap (0,s)} \{\beta(r) \in G\} \in \mathcal{F}_t^+$$

picturewise



]

Remark: $\mathcal{F}_t^+$ is right continuous.

[You should actually think first what this statement means.

$\mathcal{F}_t^+$ is the information you obtain from look at B.M. at t^+ & right continuous should be defined as

$$\mathcal{F}_t^+ = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}^+$$

Then the right continuity should be obvious but if you want a proof, here it is:

$$\bigcap_{\varepsilon>0} \mathcal{F}^+_{t+\varepsilon} = \bigcap_{\varepsilon>0} \bigcap_{\delta>0} \mathcal{F}_{t+\varepsilon+\delta} = \bigcap_{\varepsilon'>0} \mathcal{F}_{t+\varepsilon'} = \mathcal{F}^+_t \Big].$$

Def: Let τ a stopping time. Define

$$\mathcal{F}^+(\tau) = \{A : A \cap \{\tau \le t\} \in \mathcal{F}^+_t \quad \forall\, t \ge 0\}$$

[In words: $\mathcal{F}^+(\tau)$ is the information until time τ. We use $\mathcal{F}^+$ instead of $\mathcal{F}$ because of the situation with entrance times, c.f. Ex. (5) above]

Thm (Strong Markov Property)

For every a.s. finite st. time τ the process

$$\{\beta(t+\tau) - \beta(\tau) : t \ge 0\}$$

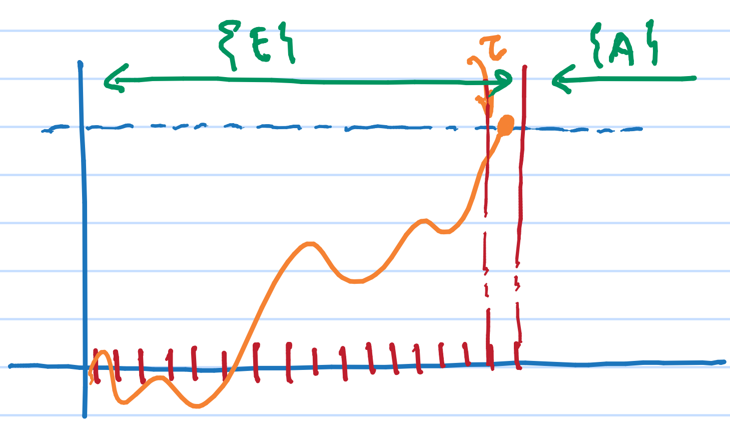
is standard B.M. independent of $\mathcal{F}^+(\tau)$

Corollary

$$E\big[f(\beta(t+\tau) : t \ge 0) \mid \mathcal{F}^+(\tau)\big] = E_{\beta(\tau)}\big[f(\beta(t) : t \ge 0)\big]$$

Proof of Thm

We start with a discretization. We define



$$\tau_n = \sum_{k \ge 1} \frac{k}{2^n} \mathbb{1}_{\tau \in (\frac{k-1}{2^n}, \frac{k}{2^n}]}$$

i.e. $\tau_n = \frac{k}{2^n}$, if $\tau \in \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right]$.

τ_n is what a myopic person would see as τ.

Let us look at $\quad (\beta(t+\tau_n) - \beta(\tau_n))$ which equals to

$$\beta(t + k2^{-n}) - \beta(k2^{-n}) \quad \text{if} \quad \tau \in \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right]$$

& let $E \in \mathcal{F}^+(\tau_n)$. We will check that

$$\{\beta(\cdot + \tau_n) - \beta(\tau_n) \in A\} \text{ is independent of } E.$$

$$\underline{P}\left(\beta(\cdot+\tau_n)-\beta(\tau_n)\in A\,;\,E\right)=\sum_{k=0}^{\infty}\underline{P}\left(\beta(\cdot+\tau_n)-\beta(\tau_n)\in A\,;\,E,\ \tau_n=k/2^n\right)$$

$$=\sum_{k=0}^{\infty}\underline{P}\left(\beta\left(\cdot+\frac{k}{2^n}\right)-\beta\left(\frac{k}{2^n}\right)\in A\,;\,E,\ \tau_n=k/2^n\right)$$

$$\underset{\text{Markov}}{=}\sum_{k=0}^{\infty}\underline{P}\left(E,\ \tau_n=k/2^n\right)\underline{P}\left(\beta(\cdot)\in A\right)$$

$$=\underline{P}(E)\,\underline{P}(\beta(\cdot)\in A)$$

$$\Rightarrow \underline{P}\left(\beta(\cdot+\tau_n)-\beta(\tau_n)\in A\,\middle|\,E\right)=\underline{P}\left(\beta(\cdot)\in A\right)$$

and this implies that $\{\beta(\cdot+\tau_n)-\beta(\tau_n)\}$ is independent of $\mathcal{F}^+(\tau_n)\supset\mathcal{F}^+(\tau)$!

Taking the limit $n\to\infty$ we have that $\tau_n\to\tau$ a.s. & by the continuity of B.M. $\{\beta(\cdot+\tau_n)-\beta(\tau_n)\}\xrightarrow{a.s.}\{\beta(\cdot+\tau)-\beta(\tau)\}$ which implies that the latter is also independent of $\mathcal{F}^+(\tau)$. ■

Q. Find a process which is Markov BUT not Strong Markov.

Brownian Motion as a MG.

(a quick reminder) Def: A process $x(\cdot)$ is a MG wrt $\mathcal{F}_t$ if

$$E[x(t)\mid\mathcal{F}_s]=x(s)\quad\text{for } t>s$$

$$\text{sub MG}:\ E[x(t)\mid\mathcal{F}_s]\geq x(s)$$

$$\text{sup MG}:\ E[x(t)\mid\mathcal{F}_s]\leq x(s)$$

Example B.M. is a MG:

for $s<t$ $E[\beta(t)\mid\mathcal{F}_s]=E[(\beta(t)-\beta(s))+\beta(s)\mid\mathcal{F}_s]=$

$=\beta(s)+E[\beta(t)-\beta(s)\mid\mathcal{F}_s]=\beta(s)$ by the Markov property.

Thm (Optional Stopping Thm)

> Let $X(\cdot)$ a continuous MG & $\sigma \le \tau$ stopping times
>
> If $(X(t\wedge\tau))_{t\ge 0}$ is a MG &
>
> $X(t\wedge\tau) \le \tilde{X}(t)$ with $\tilde{X}$ integrable
>
> then $E[X(\tau) \mid \mathcal{F}_\sigma] = X(\sigma)$.

We will not prove this but there is a nice characterization of B.M. due to Lévy as a MG:

> B.M. is the unique continuous process s.t.
>
> $$\beta(t) \quad \& \quad \beta(t)^2 - t$$
>
> are MG's

But there are many more MG's associated to B.M. The following is very useful:

Thm Let $f:\mathbb{R}^d \to \mathbb{R}$ twice continuous differentiable &

> $\beta(\cdot)$ a d-dimensional B.M. Also
>
> - $E_x \lvert f(\beta(t)) \rvert < \infty \quad \forall t \ge 0$ &
> - $E_x \int \lvert \Delta f(\beta(s)) \rvert ds < \infty \quad \forall t \ge 0$
>
> Then
>
> $$f(\beta(t)) - \frac{1}{2} \int_0^t \Delta f(\beta(s))\, ds$$
>
> is a MG

Before proving let's look at how one could guess this. To do so let's look at a SRW $(S_n)_{n\ge 1}$ & compute

$$E[f(S_n) \mid S_1, \dots, S_{n-1}] - f(S_n) = E[f(S_n) \mid S_{n-1}] - f(S_{n-1})$$

$$= \frac{1}{2} f(S_{n-1}+1) + \frac{1}{2} f(S_{n-1}-1) - f(S_n)$$

$$= \frac{1}{2} (\Delta f)(S_{n-1})$$ where here Δ is the discrete Laplacian.

Proof of Thm Let denote by $M_t := f(\beta(t)) - \frac{1}{2}\int_0^t \Delta f(\beta(r))dr$

For $s<t$

$$E_x[M_t \mid \mathcal{F}_s] = E_x[f(\beta(t)) \mid \mathcal{F}_s] - \frac{1}{2} E_x\left[\int_0^t \Delta f(\beta(r))dr \mid \mathcal{F}_s\right] \quad (1)$$

The first term equals, by the Markov property,

$$E_{\beta(s)}[f(\beta(t-s))] = \int f(y) p_{t-s}(\beta(s), y) dy \quad (2)$$

where $p_t(x,y) := \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x-y|^2}{2t}\right)$

is the heat kernel.

The second term equals

$$\frac{1}{2}\int_0^s \Delta f(\beta(r))dr + \frac{1}{2}\int_s^t E_x[\Delta f(\beta(r)) \mid \mathcal{F}_s]dr$$

$$= \frac{1}{2}\int_0^s \Delta f(\beta(r))dr + \frac{1}{2}\int_s^t \int \Delta f(y) p_{r-s}(\beta(s), y) dy$$

Integrating by parts the second integral gives

$$\frac{1}{2}\int_0^s \Delta f(\beta(r))dr + \frac{1}{2}\int_s^t \int f(y) \Delta p_{r-s}(\beta(s), y) dy$$

Keeping in mind that the heat kernel solves the heat equation

$$\left.\begin{aligned} \frac{\partial p}{\partial t} &= \frac{1}{2}\Delta p \\ p_0(x,y) &= \delta(x-y) \end{aligned}\right\}$$

gives that the above equals

$$\frac{1}{2}\int_0^s \Delta f(\beta(r))dr + \int_s^t \int f(y) \frac{\partial}{\partial r} p_{r-s}(\beta(s), y) dy\, ds$$

Interchanging the integrals equals

$$\frac{1}{2}\int_0^s \Delta f(\beta(r))dr + \int f(y) \{p_{t-s}(\beta(s), y) - \underbrace{p_0(\beta(s), y)}_{\delta(\beta(s)-y)}\} dy$$

$$= \frac{1}{2}\int_0^s \Delta f(\beta(r))dr + \int f(y) p_{t-s}(\beta(s), y) - f(\beta(s)) \quad (3)$$

putting ①,②,③ together, we see that

$$E_x[M(t)|\mathcal{F}_s] = f(\beta(s)) - \frac{1}{2}\int_0^s \Delta f(\beta(r))dr = M(s)$$

which is the MG property. ∎

This thm is important as it can provide many MG's

e.g. any f that satisfies $\Delta f = 0$ gives a MG via

$$f(\beta(t))$$

for example $f(x_1,x_2) = e^{x_1}\cos x_2$.

The thm will also provide the link to PDE's.

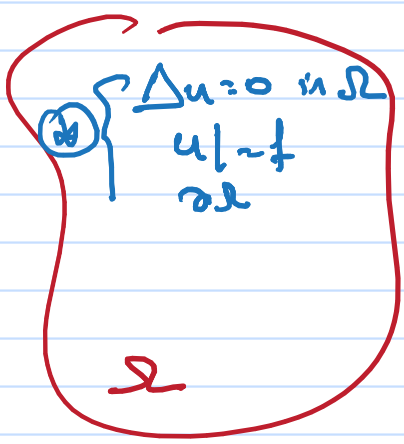
Exercise: Let $f \in C_b^{1,2}(\mathbb{R}\times\mathbb{R}^d)$. Show that

$$f(t,\beta(t)) - \int_0^t \left(\partial_t + \frac{1}{2}\Delta\right)f(r,\beta(r))dr$$

is a MG.

Use this to show that $\beta(t)^2 - t$ is MG.

Boundary Value Problems



Thm If Ω is an open subset of $\mathbb{R}^d$ with smooth boundary &

$u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a solution of bvp (∗) then

$u(x) = E_x f(\beta(\tau))$ where τ is the hitting time of $\partial\Omega$.

Proof For the moment let's assume that $\tau < \infty$ a.s. (we will check this later). We know that

$u(\beta(t)) - \frac{1}{2}\int_0^t \Delta u(\beta(s))ds$ is a MG & since $\Delta u = 0$

$u(\beta(t))$ is a MG. But also $u(\beta(t\wedge\tau))$ is a MG. So,

for any $x \in \Omega$: $u(x) = E_x u(\beta(t\wedge\tau))$ for any $t \geq 0$. Let $t \to \infty$

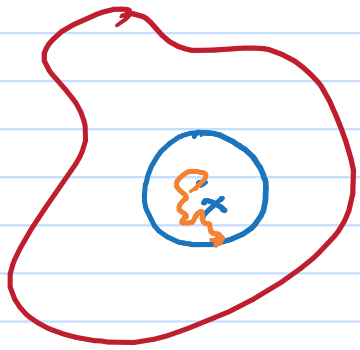
then $u(x) = \lim E_x u(\beta(t\wedge\tau)) = E_x[\lim_{t\to\infty} u(\beta(t\wedge\tau))] =$

$= E_x u(\beta(\tau))$. ∎

What about the opposite ?

Thm $u(x) := E_x f(\beta(\tau))$ is a $C^2(\Omega)\cap C(\bar\Omega)$ solution of (*).

Proof



$u(x) = E_x[f(\beta(\tau))] = E_x\left[E_x[f(\beta(\tau))\,|\,\mathcal{F}_\sigma]\right]$

where σ is the hitting time of $\partial B(x;r)$

with r arbitrary (it will be taken ~ 0).

Using the strong Markov property we write

the above as

$$u(x) = E_x\left[E_{\beta(\sigma)}[f(\beta(\tau))]\right] = E_x\left[u(\beta(\sigma))\right]$$

$$=: \int_{\partial B(x,r)} u(y)\,\pi_x(dy) = \int_{\partial B(0,r)} u(x+z)\,\pi_0(dz)$$

This mean value property will imply that u is smooth & satisfies $\Delta u(x)=0$. For this we will used the observation that $\pi_0(dz) \propto dz$ because of the symmetry of B.M. So let's check:

- $\Delta u(x)=0$: Take r small enough & expand by Taylor:

$$u(x) = \int_{\partial B(0,r)} u(x+z)\,\pi_0(dz) = \int_{\partial B(0,r)} \Big\{u(x) + \nabla u(x)\cdot z + \frac{1}{2} z^T D^2 u\, z + o(|z|^2)\Big\}\cdot \pi_0(dz).$$

$$= u(x) + \frac{1}{2}\Delta u(x)\int_{\partial B(0,r)} |z|^2\,\pi_0(dz) + o(r^2)$$

[we used the symmetry of π, which would imply that $\int_{\partial B(0,r)} z\,\pi_0(z) = 0$ & $\int D^2_{ij}u(x)\, z_i z_j\,\pi_0(dz) = 0$]

Deviding the relation by r^2 & letting $r\downarrow 0$, we obtain that

$$\Delta u(x) = 0.$$

∎

• Smoothness: Consider the integral

$$\int_{B(0,\delta)} \varphi(|z|^2)\, u(x+z)\, \pi(dz)$$

where $\varphi(\cdot)$ is a mollifier. Writing the integral in polar coordinates, we get it equals

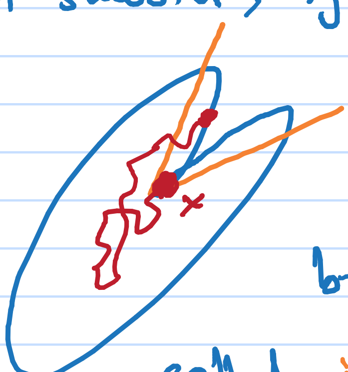
$$\text{const.} \int_0^\delta dr \cdot r^{d-1} \varphi(r^2) \int_{\partial B(0,r)} u(x+z)\, \pi(dz)$$

$$= \text{const.} \int_0^\delta dr \cdot r^{d-1} \varphi(r^2)\, u(x)$$

$$=: \text{const.}\ u(x) \qquad \forall x$$

where const. are (different) constants. Since $\varphi * u$ is smooth (this is a standard result), it follows that $u(\cdot)$ is also smooth in Ω. ▪

So, we have checked that u, as defined in this thm, is smooth & satisfies $\Delta u = 0$ in Ω & we need to check that it's continuous up to the boundary. In general, the continuity up to the boundary might fail. This can happen if the boundary is not smooth, e.g. the part $\succ$ of the boundary has the



property that we can fit no cone with center the irregural point, which will belong exclusively in the Ω^c. Such points are called irregular & have the property that

$$P_x(\tau_{\Omega^c} = 0) = 0.$$ So a B.M. that starts at x will a.s. exit somewhere far away from x.

On the other hand, if $\partial\Omega$ is smooth, then at each point in $\partial\Omega$ we can fit a cone lying exclusively in Ω^c. This point will have the property that

$$(*) \qquad \mathbb{P}_x(\tau_{\Omega^c}=0)=1$$

a cone centered at x with angle α

[why: by brownian scaling $\mathbb{P}_x(\beta(t)\in C_{x,\alpha}) = c_\alpha$

& $c_\alpha = \lim_{t\downarrow 0} \mathbb{P}_x(\beta(t)\in C_\alpha) \le \lim_{t\downarrow 0} \mathbb{P}_x(\tau\le t)$

$\le \mathbb{P}_x(\tau=0)\in\{0,1\}$ because $\tau\in \mathcal{F}_0^+$

by Blumenthal, this implies that $\mathbb{P}_x(\tau=0)=1$]

Def: Any point which satisfies $(*)$ is called regular.

The above computation shows that any cone point is regular (in particular, any point which belongs to a smooth boundary). For such points we have:

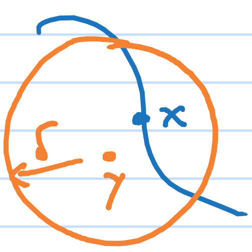
Prop. If $y\in\partial\Omega$ is regular then

$$\lim_{\Omega\ni x\to y} E_x[f(\beta(\tau))] = f(y)$$

Proof: By continuity of f, it suffices to show that

$$\lim_{\Omega\ni x\to y} \mathbb{P}_x(\tau<\infty,\ \beta(\tau)\in B(y;\delta)) = 1$$

The last probability is



for arbitrary t

$\ge \mathbb{P}_x(\tau\le t;\ \sup_{s\le t}|\beta(s)-y|<\delta)$

$\ge \mathbb{P}_x(\tau\le t) - \mathbb{P}_x(\sup_{s\le t}|\beta(s)-y|\ge\delta)$

$(**) \quad \ge \mathbb{P}_x(\tau\le t) - \mathbb{P}_x(\sup_{s\le t}|\beta(s)-x|\ge\delta/2)$

[Notice that we changed y to x in the last prob. at the cost of also changing $\delta\mapsto\delta/2$. This is justified whenever x is close enough to y]

In the lim $x\to y$ the first term is

$\ge \mathbb{P}_y(\tau\le t)$ by lower semicontinuity (to prove later)

So $(**)$ is

$$\underset{\text{limit } x\to y}{\overset{\text{in the}}{\geq}} \quad \mathbb{P}_y(\tau \le t) - \mathbb{P}_0\Big(\sup_{s<t} |\beta(s)| > \delta\Big)$$

Let now $t \to 0$: the second term converges to 0 by the continuity of B.M. & the first to 1 by the assumption that y is regular. ■

Proof of l.s.c.:

Use the Markov property to write

$$\mathbb{P}_x(\tau \le t) = \lim_{\varepsilon \downarrow 0} \mathbb{P}_x\big(\beta(s) \in \Omega^c \text{ for some } s \in (\varepsilon, t)\big)$$

$$\overset{\text{Markov}}{=} \lim_{\varepsilon \downarrow 0} \int p_\varepsilon(x,z)\, \mathbb{P}_z(\tau \le t-\varepsilon)$$

Since the heat kernel $p_\varepsilon(x,z)$ is smooth the above convolution is also a smooth function in x, while the whole integral is increasing in ε (to see this look at the line above before using the Markov).

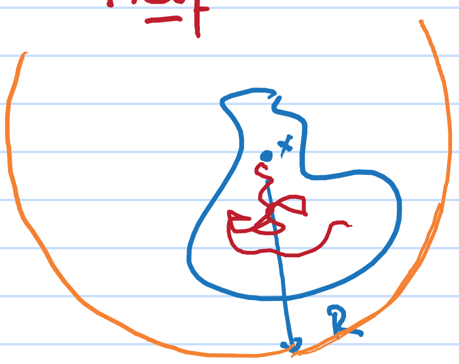
The increasing limit of continuous function is l.s.c. ■

Prop (B.M. will exit a bounded domain in finite time)

Let D a bnd domain & $\tau := \inf\{t > 0 : \beta(t) \in \partial D\}$

Then $\mathbb{P}_x(\tau < \infty) = 1 \quad \forall x \in D$

Proof



$$\mathbb{P}_x(\tau > n) = E_x\big[\mathbb{P}_x(\tau > n \mid \mathcal{F}_1)\, 1_{\tau > 1}\big]$$

$$= E_x\big[\mathbb{P}_{\beta(1)}(\tau > n-1)\, 1_{\tau > 1}\big]$$

$$\le \sup_z \mathbb{P}_z(\tau > n-1) \cdot \mathbb{P}_x(\tau > 1)$$

$$\le \sup_z \mathbb{P}_z(\tau > n-1)\, \mathbb{P}_x\big(\tau_{B(x,R)} > 1\big)$$

where $R := \operatorname{diam}(D)$ & the inequality is because if B.M. has not exited D by time 1, then it could not exit the bigger ball $B(x,R) \supset D$. But by translation invariance the last prob is independent of x: $\rho_R := \mathbb{P}_0(\tau_{B(0,R)} > 1) < 1$ & we have that

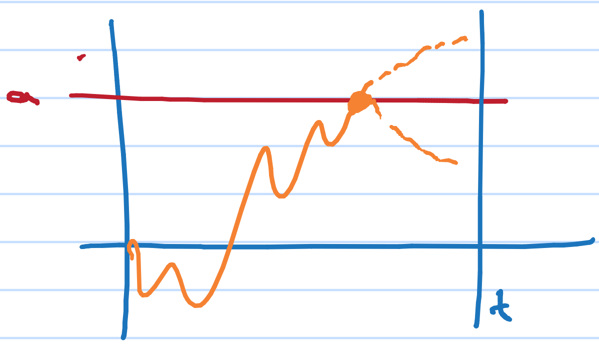
$$\sup_{x\in D} \mathbb{P}_x(\tau > n) < \alpha \sup_{x \in D} \mathbb{P}_x(\tau > n-1) < \cdots < \alpha^n$$ ■

Reflection Principle:

Let $\beta(\cdot)$ 1d B.M. Then

$$\mathbb{P}_0\left(\sup_{s\le t}\beta(s) > a\right) = 2\,\mathbb{P}_0\left(\beta(t) > a\right)$$

Proof



$$\mathbb{P}_0(\beta(t)>a) = \mathbb{P}_0(\beta(t)>a,\ \tau_a \le t)$$

$$= E_0\left[\mathbb{P}_0\left(\beta(t)>a \mid \mathcal{F}_{\tau_a}\right) 1_{\tau_a\le t}\right]$$

$$= E_0\left[\mathbb{P}_a\left(\beta(t-\tau_a)>a\right) 1_{\tau_a\le t}\right]$$

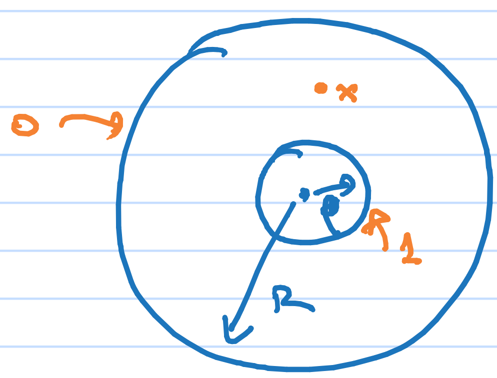
$$= \frac{1}{2} E_0\left[1_{\tau_a\le t}\right] = \frac{1}{2}\mathbb{P}_0(\tau_a\le t)$$

$$= \frac{1}{2}\mathbb{P}_0\left(\sup_{s\le t}\beta(s) > a\right) \quad \blacksquare$$

RECURRENCE & TRANSIENCE

Prop B.M is recurrent in $d=1,2$ & transient in $d\ge 3$

Proof



Let $u(x) := \mathbb{P}_x(\tau_r < \tau_R)$. We know that u is the solution to the b.v.p.

$$\begin{cases} \Delta u = 0 & \text{in } r<|x|<R \\ u=0 & \text{on } |x|=R \\ u=1 & \text{on } |x|=r \end{cases}$$

Due to spherical symmetry the solution should only depend on the radial part $u=u(|x|) := u(r)$. The Laplacian in polar coordinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{d-1}{r}\frac{\partial u}{\partial r} = 0 \iff \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial u}{\partial r}\right) = 0$$

$$\Rightarrow r^{d-1}\frac{\partial u}{\partial r} = c_1 \Rightarrow \frac{\partial u}{\partial r} = \frac{c_1}{r^{d-1}} \Rightarrow$$

$$u(r) = \begin{cases} c_1 \log r + c_2, & d=2 \\ \dfrac{c_1}{r^{d-2}} + c_2, & d\ne 2 \end{cases}$$

The boundary conditions imply that

$$u(x) = \frac{\log|x| - \log R}{\log r - \log R}, \quad \text{if } d=2$$

& $$u(x) = \frac{|x|^{2-d} - R^{2-d}}{r^{2-d} - R^{2-d}}, \quad \text{if } d \neq 2$$

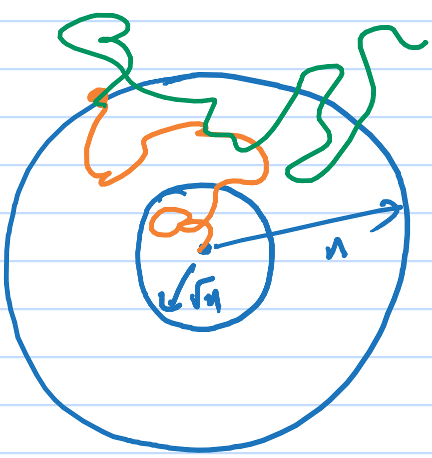
We then have that

$$\mathbb{P}_x(\tau_r < \infty) = \lim_{R \uparrow \infty} \mathbb{P}_x(\tau_r < \tau_R) =$$

$$= \lim_{R \to \infty} \begin{cases} \dfrac{\log|x| - \log R}{\log r - \log R} = 1, & \text{if } d=2 \\[2ex] \dfrac{|x|^{2-d} - R^{2-d}}{r^{2-d} - R^{2-d}} = \begin{cases} 1, & \text{if } d=1 \\ \left(\dfrac{|x|}{r}\right)^{2-d}, & \text{if } d>2 \end{cases} \end{cases}$$

$\blacksquare$

Prop For $d \geq 3$: $|\beta(t)| \to \infty$ a.s.



Let $A_n := \{ |\beta(t)| > \sqrt{n}, \ \forall t \geq \tau_n \}$

where

$$\tau_n := \inf\{t : |\beta(t)| = n\}$$

Compute for x finite

$$\mathbb{P}_x(A_n^c) = E_x\left[\mathbb{P}_{\beta(\tau_n)}(\tau_{\sqrt{n}} < \infty)\right]$$

$$= E_x\left[\left(\frac{|\beta(\tau_n)|}{\sqrt{n}}\right)^{2-d}\right]$$

$$= \left(\frac{n}{\sqrt{n}}\right)^{2-d} = \frac{1}{\sqrt{n}^{\,d-2}} \xrightarrow{n \to \infty} 0$$

We then have

$$\mathbb{P}_x(A_n \text{ i.o.}) = \mathbb{P}_x\left(\bigcap_m \bigcup_{n \geq m} A_n\right) \geq \overline{\lim_m} \, \mathbb{P}(A_m) = 1.$$

& this implies the result. $\blacksquare$

Occupation Measure

Let $B \subset \mathbb{R}^d$. The occupation measure

$$\mu(B) := \int_0^\infty \mathbb{1}_B(\beta(s))\,ds$$

is the total amount of time that B.M. spends in the set B. It should be intuitively clear that for any ball $B \subset \mathbb{R}^d$

$$\mu(B) \overset{a.s.}{=} \infty, \text{ if } d \le 2 \text{ \&}$$
$$\mu(B) \overset{a.s.}{<} \infty, \text{ if } d \ge 3.$$

Let's compute

$$E_x[\mu(B)] = E_x \int_0^\infty \mathbb{1}_B(\beta(s))\,ds =$$
$$= \int_0^\infty E_x[\mathbb{1}_B(\beta(s))]\,ds$$
$$= \int_0^\infty ds \int_B dy\, P_s(x,y) =$$
$$= \int_0^\infty ds \int dy \frac{1}{(2\pi s)^{d/2}} e^{-\frac{|x-y|^2}{2s}}$$

$$= \begin{cases} \int_B dy \,\text{const.}(d)\, |x-y|^{2-d} < \infty, \text{ if } d \ge 3 \\ \quad \text{where const.}(d) \text{ is a finite constant} \\ \infty \text{ if } d = 1,2 \text{ because } \int_0^\infty P_s(x,y)\,ds = \infty \text{ in this case.} \end{cases}$$

Let us, now, try to perform a more advanced computation. Let $D \subset \mathbb{R}^d$ a bnd. domain & $f \in C(\bar{D})$

$$u(x) = E_x \int_0^{\tau_D} f(\beta(s))\,ds.$$

Using the MG formulation, we can relate u to the solution of the b.v.p.

$$\left.\begin{aligned} \Delta u = f \quad & \text{in } D \\ u = 0 \quad & \text{on } \partial D \end{aligned}\right\}$$

But let us also do a different computation that will lead to a concept familiar from PDE's.

$$u(x) = E_x \int_0^{\tau_D} f(\beta(s))\,ds = E_x \int_0^\infty f(\beta(s))\, 1_{s<\tau_D}\, ds$$

$$= \int_0^\infty E_x\left[f(\beta(s))\, 1_{s<\tau_D} \right] ds$$

$$= \int_0^\infty ds \int dy\, f(y)\, P_x\left(\beta(s)=y\,;\, s<\tau_D\right)$$

$$= \int dy\, f(y) \underbrace{\int_0^\infty P_x\left(\beta(s)=y\,;\, s<\tau_D\right)}_{\overset{!!}{}}$$

$$=: \int dy\, f(y)\, G_D(x,y)$$

$G_D(x,y)$ is the Green's function corresponding to the Laplacian in domain D & formally is the solution to b.v.p.

$$\left.\begin{aligned} \Delta G_D(x,\cdot) = \delta_x \quad & \text{in } D \\ G_D(x,\cdot) = 0 \quad & \text{on } \partial D \end{aligned}\right\}$$

for any fixed $x \in D$. It is not easy, in general, to obtain a closed form for G_D but we can prove the following properties:

1. $G_D > 0$
2. $G_D(x,y) = G_D(y,x) \quad \forall x,y \in D$
3. $$G_D(x,y) \sim \begin{cases} |x-y|^{2-d}, & \text{if } d \geq 3 \\ -\log|x-y|, & \text{if } d=2 \\ |x-y|, & \text{if } d=1 \end{cases}$$

when $x \to y$.

Feynman-Kac formula

Thm Let $V \in C_b(\mathbb{R}^d)$. Then

$$u(t,x) := E_x\left[f(\beta(t)) \exp\left\{\int_0^t V(\beta(s))\,ds\right\}\right]$$

solves $\begin{cases} \partial_t u = \frac{1}{2}\Delta u + Vu & , \; t>0,\; x \in \mathbb{R}^d \\ u(0,x) = f \end{cases}$

Proof We first expand the exponential in the F.K. representation:

$$1 + \sum_{n=1}^{\infty} \frac{1}{n!} E_x\left[f(\beta(t)) \left(\int_0^t V(\beta(s))\,ds\right)^n\right] =$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} E_x\left[f(\beta(t)) \int_0^t \cdots \int_0^t ds_1 \cdots ds_n \, V(\beta(s_1)) \cdots V(\beta(s_n))\right]$$

$$\underset{\text{symmetry}}{=} 1 + \sum_{n=1}^{\infty} E_x\left[f(\beta(t)) \int \cdots \int_{0<s_1<\cdots<s_n<t} V(\beta(s_1)) \cdots V(\beta(s_n))\, ds_1 \cdots ds_n\right]$$

$$= 1 + \sum_{n=1}^{\infty} \int \cdots \int_{0<s_1<\cdots<s_n<t} E_x\left[V(\beta(s_1)) \cdots V(\beta(s_n))\, f(\beta(t))\right]$$

Let us denote the n^{th} term in the above series by $I_n(t,x)$ & by the change of variable $s_i = t - r_i$ we can write it as

$$I_n(t,x) = \int \cdots \int_{0<r_n<\cdots<r_1<t} E_x\left[V(\beta(t-r_1)) \cdots V(\beta(t-r_n))\, f(\beta(t))\right]$$

Use the Markov property at time $t-r_1$ to write the above as

$$I_n(t,x) = \int_{0<r_n<\cdots<r_1<t} E_x\Big[V(\beta(t-r_1))\, E_{\beta(t-r_1)}\big[V(\beta(r_1-r_2)) \cdots V(\beta(r_1-r_n))\, f(\beta(r_1))\big]\Big]\, dr_1 \cdots dr_n$$

Differentiate, now, w.r.t. t :

$$\frac{d}{dt} I_n(t,x) = \int_{0<y_n<\cdots<y_2<t} V(x)\, E_x\left[V(\beta(t-y_2))\cdots V(\beta(t-y_n))\, f(\beta(t))\right]$$

$$+ \int_{0<y_n<\cdots<y_1<t} \frac{d}{dt} E_x\left[V(\beta(t-y_1))\, E_{\beta(t-y_1)}\left[V(\beta(y_1-y_2))\cdots V(\beta(y_1-y_n))\, f(\beta(y_1))\right]\right]$$

but denoting $F(y) := V(y)\, E_y\left[V(\beta(y_1-y_2))\cdots V(\beta(y_1-y_n))\, f(\beta(y_1))\right]$

the expectation in the last integral can be compactly written as

$$E_x\left[F(\beta(t-y_1))\right]$$

& which solves the heat equation, i.e.

$$\frac{d}{dt} E_x\left[F(\beta(t-y_1))\right] = \frac{1}{2}\Delta E_x\left[F(\beta(t-y_1))\right]$$

So, we have that

$$\frac{d}{dt} I_n(t,x) = V(x)\, I_{n-1}(t,x) + \frac{1}{2}\Delta I_n(t,x)$$

Summing this over $\sum_{n=1}^{\infty} I_n(t,x)$ we have

$$\frac{\partial}{\partial t} \sum_{n=1}^{\infty} I_n(t,x) = \frac{1}{2}\Delta \sum_{n=1}^{\infty} I_n(t,x) + V(x) \sum_{n=0}^{\infty} I_n(t,x)$$

Adding the constant term 1 inside the derivatives leads to

$$\frac{\partial}{\partial t} \sum_{n=0}^{\infty} I_n(t,x) = \frac{1}{2}\Delta \sum_{n=0}^{\infty} I_n(t,x) + V(x) \sum_{n=0}^{\infty} I_n(t,x)$$

or $\displaystyle \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + Vu.$ ∎

Using the F.-K. formula we will compute:

Example (Arcsine law)

Let $\xi_t := \frac{1}{t}\int_0^{\infty} 1_{(0,\infty)}(\beta(s))\,ds$ the fraction of time 1d B.M. spends in the positive axis.

Then
$$\mathbb{P}_0(\xi_t \leq x) = \int_0^x \frac{1}{\pi\sqrt{y(1-y)}}\,dy = \frac{2}{\pi}\sin^{-1}(\sqrt{x})$$

First note that

$$\frac{1}{t}\int_0^t 1_{(0,\infty)}(\beta(s))\,ds \overset{s=ty}{=\!=} \int_0^1 1_{(0,\infty)}(\beta(ty))\,dy \overset{d}{=}$$
$$\overset{d}{=} \int_0^1 1_{(0,\infty)}(\sqrt{t}\,\beta(y))\,dy = \int_0^1 1_{(0,\infty)}(\beta(y))\,dy =: \xi_1$$

Let, now, $V(x) := 1_{(0,\infty)}(x)$. We will compute the Laplace transform of ξ_1: $E_0 \exp\{\sigma\xi_1\} = E_0\exp\{\sigma\int_0^1 1_{(0,\infty)}(\beta(y))\,dy\}$ but in order to make use of Feynman-Kac we will instead compute

$$u_\sigma(t,x) := E_x\exp\left\{\sigma\int_0^t 1_{(0,\infty)}(\beta(y))\,dy\right\}$$

which by F.K. solve

$$\begin{cases} \partial_t u_\sigma = \frac{1}{2}\Delta u_\sigma - \sigma V u \\ u_\sigma(0,x) = 1 \end{cases}$$

To solve this we use the Laplace method i.e. consider the transform

$$g_\sigma(\lambda,x) := \int_0^\infty u_\sigma(t,x)\,e^{-\lambda t}\,dt$$

This will lead to

$$(\lambda+\sigma V)g_\sigma - \frac{1}{2}\partial_x^2 g_\sigma = 1$$

$$\Leftrightarrow \begin{cases} (\lambda+\sigma)g_\sigma - \frac{1}{2}\partial_x^2 g_\sigma = 1, & x > 0 \\ \lambda g_\sigma - \frac{1}{2}\partial_x^2 g = 1, & x < 0 \end{cases}$$

Each one of the above is an ODE, which is easily solved

$$g_c(\lambda, x) = \begin{cases} \frac{1}{\lambda+c} + A e^{\sqrt{2(\lambda+c)}\,x} + B e^{-\sqrt{2(\lambda+c)}\,x} & , x>0 \\ \frac{1}{\lambda} + C e^{\sqrt{2\lambda}\,x} + D e^{-\sqrt{2\lambda}\,x} & , x<0 \end{cases}$$

The boundedness of g_c at $x = \pm\infty$ & the continuity up to first derivatives at $x=0$, determine the constants A, B, C, D. Determining the constants & looking at the solution at $x=0$, we get that

$$g_c(\lambda, 0) = \frac{1}{\sqrt{\lambda(\lambda+c)}}$$

By definition

$$g_c(\lambda, 0) = \int_0^\infty e^{-\lambda t} u_c(t, 0)\, dt = \int_0^\infty e^{-\lambda t} E_0 e^{-c\int_0^t V(\beta(s))ds} dt$$

$$= E_0 \int_0^\infty e^{-\lambda t - c\int_0^t V(\beta(s))ds} dt$$

$$= E_0 \int_0^t e^{-t(\lambda + c\xi_1)} dt$$

$$= E_0 \left[\frac{1}{\lambda + c\xi_1} \right]$$

Setting $\lambda = 1$ & expanding both sides in c we obtain

$$E_0 \xi_1^n = \frac{1}{\pi} \int_0^1 x^{n-\frac{1}{2}} (1-x)^{-\frac{1}{2}} dx$$

$$=: \int_0^1 x^n p(x) dx$$

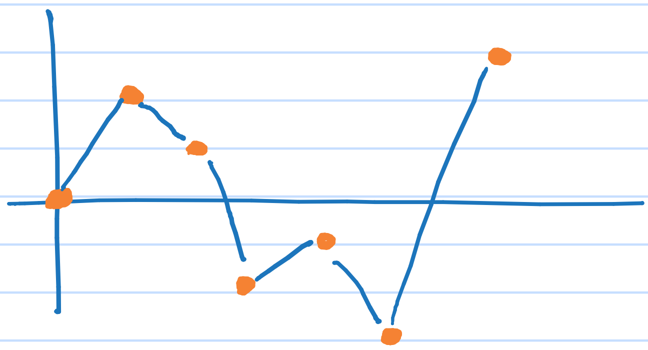
& $p(x)$ is the arcsine law. ∎

Donsker's Thm

Let $(X_n)_{n\geq 1}$ mean zero, variance 1 i.i.d. variables & set

$$S_n = \sum_{k=1}^{n} X_k \quad , n \geq 1$$

By doing a linear interpolation we can consider this sequence as a piecewise linear function, which we denote by



$$S_n^*(t) = \frac{S_{[nt]}}{\sqrt{n}} + \frac{S_{[nt]+1} - S_{[nt]}}{\sqrt{n}}$$

Donsker's theorem is a functional generalisation of CLT & says:

Thm (Donsker's Invariance Principle)

The sequence of continuous functions

$$\{S_n^*(t) : t > 0\} \xrightarrow[n\to\infty]{(d)} \{B(t) : t > 0\}$$

in $C([0,1]; \|\cdot\|_\infty)$

Before proving this we will need to recall what is meant by weak convergence. If (X_n) is a sequence of r.v.'s defined on some Polish space $\mathcal{X}$, then

$X_n \xrightarrow{d} X$ if for any $f \in C_b(\mathcal{X})$

$$\mathbb{E} f(X_n) \longrightarrow \mathbb{E} f(X).$$

We recall the following thm that provides equivalent checks for weak convergence.

Thm Let (X_n), X random variable on a metric space (E,ρ). Then the following are equivalent

1. $X_n \xrightarrow{d} X$
2. $\forall K \subset E$ closed
$$\limsup_n \mathbb{P}(X_n \in K) \leq \mathbb{P}(X \in K)$$
3. $\forall G \subset E$ open
$$\liminf_n \mathbb{P}(X_n \in G) \geq \mathbb{P}(X \in G)$$
4. $\forall A$ Borel with $\mathbb{P}(X \in \partial A) = 0$, then
$$\lim_n \mathbb{P}(X_n \in \partial A) = \mathbb{P}(X \in A)$$

By considering the distribution measures
$$\mu_n(A) := \mathbb{P}(X_n \in A)$$
we can recast the notion of weak convergence of r.v.'s to weak convergence of measures. Thus we say that
$$\mu_n \Rightarrow \mu$$
if one of the condition of the above thm is satisfied or if $\forall f \in C_b(E,\rho)$
$$\int f \, d\mu_n \to \int f \, d\mu.$$

Recall the notion of tightness: A sequence of measure is tight if no mass escapes to the boundary of the metric space or to ∞ (compare this with condition ④ in the above thm.). The mathematical definition of tightness is that $\forall \varepsilon > 0$ $\exists K_\varepsilon \subset E$ compact such that
$$\sup_n \mu_n(K_\varepsilon^c) < \varepsilon.$$

A standard thm in probability says that

> If a sequence $\{\mu_n\}$ is tight then it weakly convergent subsequence

If we want to show convergence of a sequence of measures or random variables, we first show that it is tight. Then we have to show that every subsequential limit is the same (i.e. uniqueness).

In the Donsker setting, in order to talk about tightness we first need to understand what are the compact sets of $C([0,1]; \|\cdot\|_\infty)$. Weierstrass thm tells us that these are the families of equicontinuous functions. For example Hölder continuous families with the same Hölder constant are equicontinuous. Essentially, this can reduce to checking the convergence of finite dimensional distributions, i.e. for $t_1 < t_2 < \dots < t_k$ it holds that

$$\left(S_n^*(t_1), \dots, S_n^*(t_k)\right) \xrightarrow[n\to\infty]{d} \left(\beta(t_1), \dots, \beta(t_k)\right)$$

which is just a consequence of the CLT.

Even though this is the most systematic approach to weak convergence, we will follow a different approach to prove Donsker's thm.

Before going into the proof let's look at some applications:

Example: Let $\{X_k\}_{k\geq 1}$ be i.i.d. mean 0, variance 1 & $S_n = X_1+\dots+X_n$.

Let $M_n := \max\{S_k : k\leq n\}$. Then

$$\mathbb{P}(M_n \leq x\sqrt{n}) \xrightarrow[n\to\infty]{} \sqrt{\frac{2}{\pi}}\int_x^\infty e^{-y^2/2}\,dy.$$

This is a consequence of Donsker's thm. But in order to apply it we first need to ask the question of whether

$$M_t := \max\{x(s) : s\leq t\}$$

is a continuous functional of the path $\{x(t): t>0\}$.

It is easy to see that the answer to this question is YES.

Then, Donsker implies that

$$\frac{M_n}{\sqrt{n}} : \max\left\{\frac{1}{\sqrt{n}}S_k : k\leq n\right\} \xrightarrow[n\to\infty]{d} \max\{\beta(t) : 0\leq t\leq 1\}$$

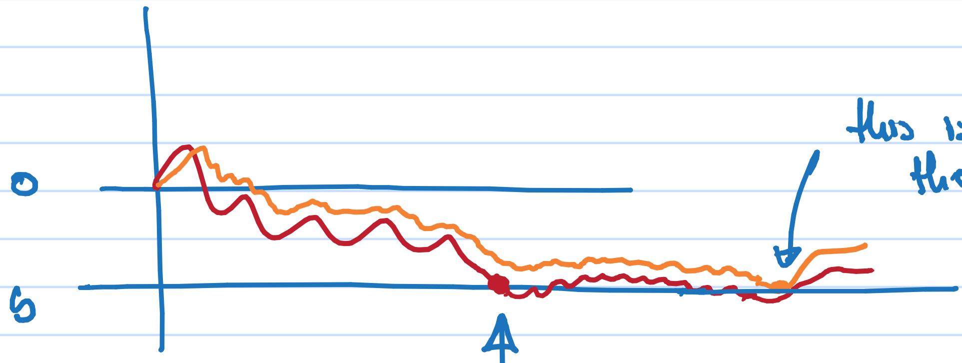
which then implies that

$$\mathbb{P}(M_n > x\sqrt{n}) \longrightarrow \mathbb{P}_0\left(\max_{0\leq t\leq 1}\beta(t) > x\right) =$$

$$= 2\,\mathbb{P}_0(\beta(1) > x) =$$

$$= \sqrt{\frac{2}{\pi}}\int_x^\infty e^{-y^2/2}\,dy.$$

Example Let $\tau_{b\sqrt{n}} := \min\{n : S_n \leq b\sqrt{n}\}$. We are tempted to say that $\tau_{b\sqrt{n}}$ converges in distribution to the law of $\inf\{t : \beta(t)\leq b\}$. But, again, in order to apply Donsker we have to ask whether τ is a continuous function of the underlying path. This is actually not the case. See for example the following figure:



The red & orange paths remain very close to each other for all times but they can have a very different first hitting time. However, even if this can certainly happen for "regular", smooth paths, it cannot happen for Brownian like paths. That is

$$\mathbb{P}^{\text{Wiener}} \left\{ x(\cdot) : \tau_b(x(\cdot)) \text{ is discontinuous} \right\} = 0.$$

In such situation Donsker's thm can be still applied, and in this case we have

$$\mathbb{P}^{\text{SRW}} \left(\tau_{b\sqrt{n}} < t \right) \to \mathbb{P}^{\text{Wiener}} \left(\tau_b \leq t \right).$$

Proof of Donsker's Thm

We will need a lemma (which will be related to Skorohod embedding thm).

Lemma Let $\{\beta(t) : t \geq 0\}$ standard B.M. Then $\forall$ r.v. X mean 0, variance 1, there exist stopping times $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots$ s.t.

(a) $\{\beta(\tau_n) : n \geq 0\} \overset{d}{=} \{S_n = X_1 + \cdots + X_n : n \geq 0\}$

(b) Consider the R.W. $(S_n)_{n\geq 0} := (\beta(\tau_n))_{n \geq 0}$ & $S_n^*(\cdot)$ the linear interpolation scaled by $\sqrt{n}$. Then

$$\mathbb{P}^W \left(\sup_{0 \leq t \leq 1} \left| \frac{\beta(nt)}{\sqrt{n}} - S_n^*(t) \right| > \varepsilon \right) \xrightarrow[n \to \infty]{} 0.$$

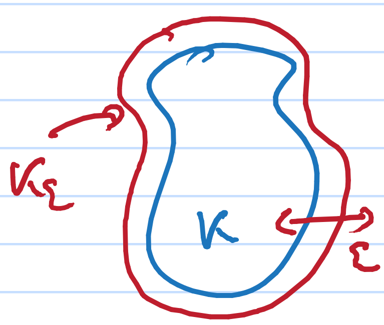
Let us assume the lemma for the moment & prove Donsker.

Proof of Donsker

Denote $W_n(t) := \frac{\beta(nt)}{\sqrt{n}}$ for $t \geq 0$. By scaling invariance $W_n(\cdot)$ has the same distribution like a standard B.M.

Let $K \subset C([0,1]; \|\cdot\|_\infty)$ closed & K_ε its ε-cover ie

$$K_\varepsilon := \{ f \in C([0,1], \|\cdot\|_\infty) : \|f-g\|_\infty < \varepsilon \ \forall g \in K \}.$$



Then

$$P^x(S_n^* \in K) = P^{W,\{\tau_n\}}(S_n^* \in K)$$

$$\leq \underbrace{P(W_n \in K_\varepsilon)}_{\substack{\| \\ P(\beta \in K_\varepsilon)}} + \underbrace{P(\|W_n - S_n^*\|_\infty \geq \varepsilon)}_{\substack{\downarrow n\to\infty \\ 0}}$$

because $W_n =^d \beta$

by the previous lemma

So, we have that

$$\limsup_{n\to\infty} P^x(S_n^* \in K) \leq P(\beta \in K_\varepsilon) \downarrow_{\varepsilon \downarrow 0} P(\beta \in K)$$

$$\left[\text{because} \quad \lim_{\varepsilon \downarrow 0} P(\beta \in K_\varepsilon) = P\Big(\beta \in \bigcap_{\varepsilon > 0} K_\varepsilon\Big) = P(\beta \in K) \right]$$

The proof is now completed by the Thm with the equivalent criteria for weak convergence. ■

The proof of the lemma (see Mörters–Peres pg. 132) we use

Thm (Skorokhod embedding)

If $\beta(\cdot)$ is standard 1d B.M & X real valued with $\mathbb{E}[X]=0$, $Var(X) < \infty$, then $\exists$ stopping time τ s.t.

- $\beta(\tau) =^d X$
- $\mathbb{E}^W[\tau] = \mathbb{E}[X^2]$

Proof of Skorohod (The Azéma-Yor version)

Let's first look at the easiest case i.e. X is mean-zero Bernoulli

$$X = \begin{cases} a, & \text{with prob. } p \\ b, & \text{with prob } 1-p \end{cases}$$

To satisfy $\mathbb{E}X = 0$ we must have $ap + b(1-p) = 0 \Rightarrow p = \frac{b}{b-a}$.

The embedding consists of stopping B.M. starting from 0 when it hits either a or b, i.e. τ is the exit time from (a,b). Since

$$\mathbb{P}_0(\beta(\tau) = a) = \frac{b}{b-a} = p$$

we have the embedding in this case.

- Assume, now, that X takes values $x_1 \le x_2 \le \dots \le x_n$ with probabilities $p_1, p_2, \dots, p_n$.

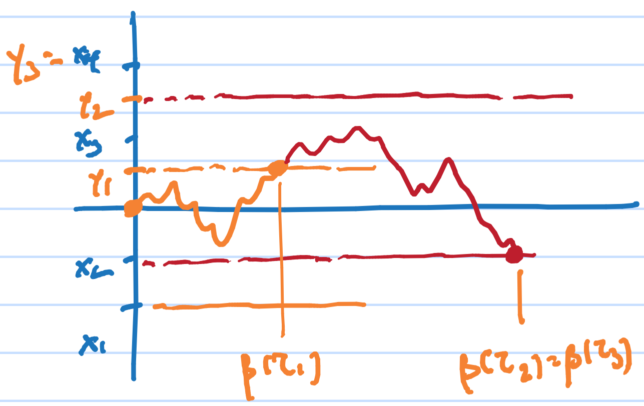
Define stopping times

$$\tau_1 := \inf\{t > 0 : \beta(t) \notin (x_1, y_1)\}$$

$$\tau_i := \inf\{t > \tau_{i-1} : \beta(t) \notin (x_i, y_i)\} \quad \text{for } i = 1, 2, \dots, n-1$$

& the sought stopping will be

$\tau := \tau_{n-1}$. Notice that $\beta(\tau)$ can take any of the values $x_1, x_2, \dots, x_{n-1}$:



It can be equal to x_1 if $\beta(\tau_1) = x_1$ in which case

$$\beta(\tau_{n-1}) = \beta(\tau_{n-2}) = \dots = \beta(\tau_2) = \beta(\tau_1) = x_1$$

or it can equal x_2 if $\beta(\tau_1) = y_1$, $\beta(\tau_2) = x_2$ in which case

$$\beta(\tau_{n-1}) = \beta(\tau_{n-2}) = \dots = \beta(\tau_2) = x_2$$

& so on.

We have already computed the probability

$$\mathbb{P}(\beta(\tau_1) = x_1) = \mathbb{P}(\beta(\tau_1) = x_1) = \mathbb{P}(\tau_{x_1} < \tau_{y_1}) = p_1$$

Let's compute

$$\mathbb{P}(\beta(\tau) = x_2) = \mathbb{P}(\beta(\tau_2) = x_2) = \mathbb{P}(\tau_{y_1} < \tau_{x_1})\,\mathbb{P}(\tau_{x_2} < \tau_{y_2})$$

$$= \frac{-x_1}{y_1 - x_1} \cdot \frac{y_2 - y_1}{y_2 - x_2}$$

Doing the computations using $y_i = \mathbb{E}[X \mid X \geq x_{i+1}]$

$$= \frac{\sum_{j \geq i+1} x_j p_j}{\sum_{j \geq i+1} p_j}$$

you see that the above equals p_2.

Similar is the computation in the general case

$$\mathbb{P}(\beta(\tau) = x_i) \qquad i = 1, \dots, n.$$

This proves the first claim of Skorohod (in the case that X a finite number of values).

- What about the claim that $\mathbb{E}\tau = \mathbb{E}\beta(\tau)^2$? This follows directly from the fact that τ is a stopping time &

$$\beta^2(t) - t \quad \text{is a MG.}$$

- What about the general case of a real variable X?

In this case we define $\Psi(x) := \mathbb{E}[X \mid X \geq x]$ &

$$\tau := \inf\{t \geq 0 : M(t) \geq \Psi(\beta(t))\}$$

where

$$M(t) := \sup_{s \leq t} \beta(s)$$

We will not check this, which can be done by approximation, but check that this definition is consistent with the above ∎

Local Times

local time is the time that 1d B.M. has spend at a point $x \in \mathbb{R}$ until time t.
Notice that this requires some proper definition! But formally we can think of the local time (denoted by $L_t(x)$)

$$L_t(x) = \int_0^t \delta_x(\beta(s))\,ds$$

Even though this would be satisfactory in physics, some care is required as it turns out that local time can only be given a meaning in dimension 1. One way to define is to use an approximation of the delta function e.g.

$$L_t(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(x-\varepsilon, x+\varepsilon)}(\beta(s))\,ds$$

& a meaning to this limit should be provided. We will follow an alternative way, that of downcrossings. We define this by first defining a sequence of stopping times:

let $a<b$ &

$$\sigma_1 := \inf\{t>0 : \beta(t)=b\}$$

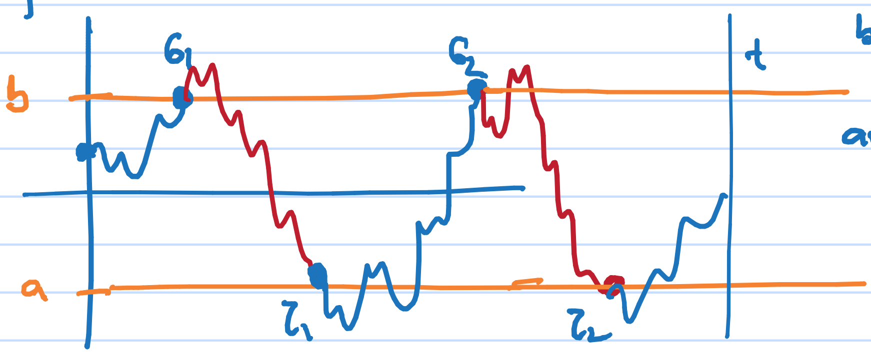
$$\tau_1 := \inf\{t>\sigma_1 : \beta(t)=a\}$$

$$\sigma_2 := \inf\{t>\tau_1 : \beta(t)=b\}$$

$$\tau_2 := \inf\{t>\sigma_2 : \beta(t)=a\}$$

$$\vdots$$

In pictures



The part of B.M. between times (σ_i, τ_i) (i.e. the red parts) are called the downcrossings.

The number of downcrossings by time t is defined by

$$D(a,b;t) := \max\{j : \tau_j \le t\}$$

We will define the local time at zero as the limit

$$\lim_{a\uparrow 0,\, b\downarrow 0} 2(b-a)\, D(a,b;t)$$

We will also look at the local time as a process & show that it is Hölder continuous, relate to PDEs with Neumann boundary condition & show that it is related the process

$$M_t = \sup_{s\le t} \beta(s).$$

Lemma: Let $a<m<b<c$, $\{\beta(t) : t\ge 0\}$ standard B.M

$T := \inf\{t>0 : \beta(t) = c\}$

- D := # downcrossings of $[a,b]$ up to T
- D_l := # downcrossings of $[a,m]$ up to T
- D_u := # downcrossings of $[m,b]$ up to T.

Then
$$D_l = X_0 + \sum_{j=1}^{D} X_j$$

$$D_u = Y_0 + \sum_{j=1}^{D} Y_j$$

where $X_0, X_1, \dots$ & $Y_0, Y_1, \dots$ are independent sequences of independent variables, independent of D

AND X_0 = # downcrossings of $[m,b]$ before first downcrossing of $[a,b]$

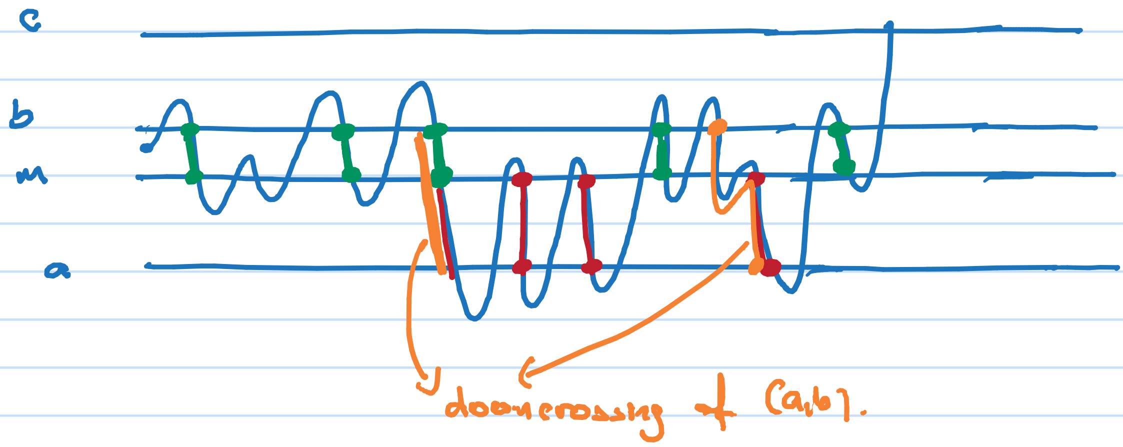
Y_0 = # downcrossings of $[a,m]$ after last downcrossing of $[a,b]$

AND

$$X_j \overset{d}{=} \text{Geom}\left\{\mathbb{N}_+ ,\ p = \frac{m-a}{b-a}\right\} \qquad j \geq 1$$

$$Y_j \overset{d}{=} \text{Geom}\left\{\mathbb{N}_+ ,\ p = \frac{b-m}{b-a}\right\}$$

Proof



The downcrossings of (a,b) decompose the path into a number of independent sub-paths. The (green) downcrossings of (m,b) is the number of attempts before a full downcrossing (orange path) of (a,b) happens & the (red) downcrossings of (a,m) are the attempts to go below level a & then above level b. The probability of a green path is $P_m(\tau_b < \tau_a) = \frac{m-a}{b-a}$. The probability of a red path is

$$P_m(\tau_a < \tau_b) = \frac{b-m}{b-a}.$$

X_j is then the number of green paths in between the j^{th} & $(j+1)^{th}$ (a,b)-downcrossing.

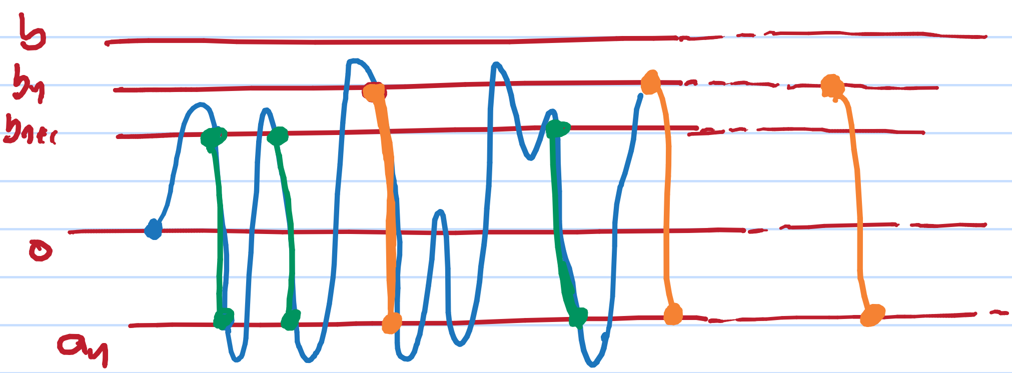
Y_j is the number of red paths in between the j^{th} & $(j+1)^{th}$ downcrossing. ∎

Lemma: For any two sequences $a_n \uparrow 0$ & $b_n \downarrow 0$, the process $\{2(b_n - a_n) D(a_n, b_n; T) : n \in \mathbb{N}\}$ is a sub-MG wrt to its own filtration (T is taken to be the hitting time of b).

Proof Assume wlog that $a_n = a_{n+1}$ & compute

$$\mathbb{E}\left[2(b_{n+1} - a_{n+1}) D(a_{n+1}, b_{n+1}, T) \,\middle|\, D_n \right] =$$

$$= \mathbb{E}\left[2(b_{n+1} - a_n) D(a_n, b_{n+1}, T) \,\middle|\, D_n \right]$$



where we denote

$$D_n := D(a_n, b_n; T)$$

#orange $= D_n$. In the notation of the previous lemma

$$D(a_{n+1}, b_{n+1}; T) = D_1 = X_0 + \sum_{j=1}^{D_n} X_j$$

so the above conditional expectation can be written as

$$\mathbb{E}\left[2(b_{n+1} - a_n)\left(X_0 + \sum_{j=1}^{D_n} X_j \right) \,\middle|\, D_n \right]$$

$$\geq 2(b_{n+1} - a_n)\, \mathbb{E}\left[\sum_{j=1}^{D_n} X_j \,\middle|\, D_n \right]$$

$$= 2(b_{n+1} - a_n) \sum_{j=1}^{D_n} \mathbb{E}[X_j \mid D_n]$$

$$= 2(b_{n+1} - a_n) \sum_{j=1}^{D_n} \mathbb{E}[X_j]$$

$$= 2\cancel{(b_{n+1} - a_n)}\, D_n \cdot \frac{b_n - a_n}{\cancel{b_{n+1} - a_n}} = 2(b_n - a_n) D_n .$$

$\blacksquare$