Numerical Methods

Paths Construction

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1 Brownian motion

Often one needs to construct a simulated discretised path of a Brownian motion (Wiener process) over an increasing sequence $\{t_i\}$, $i = 1, \ldots, n$, of points in time.

The simplest way to construct a Brownian motion path is *incremental path construction*:

$$W(t_{i+1}) = W(t_i) + \epsilon_i \sqrt{t_{i+1} - t_i},$$

where ϵ_i are normally distributed independent random variables.

1.1 Brownian Bridge

Another way to construct a sample Brownian path is by the Brownian Bridge construction. Assume that we have already constructed $W(t_i)$ and $W(t_k)$ with $t_i < t_k$. One can compute that we have for $t_j \in (t_i, t_k)$,

$$E(W_{t_j}) = \left(\frac{t_k - t_j}{t_k - t_i}\right) W_{t_i} + \left(\frac{t_j - t_i}{t_k - t_i}\right) W_{t_k}$$

and

$$\mathrm{var}(W_{t_j}) = \left(\frac{(t_j - t_i)(t_k - t_j)}{t_k - t_i}\right).$$

This allows to construct a process which gradually fills in all of the realizations: take standard normally distributed random variables ϵ_i and take $W_0 = 0$, $W_1 = \epsilon_1$, $W_{1/2} = (1/2)(W_0 + W_1) + (1/2)\epsilon_2$, $W_{1/4} = (1/2)(W_0 + W_{1/2}) + (1/\sqrt{8})\epsilon_3$ and keep continuing $W_{3/4}$, $W_{1/8}$, $W_{3/8}$, $W_{5/8}$, $W_{7/8}$ and so on.

Exercise:

Implement the Brownian Bridge construction.

2 The Euler scheme

Let us have a stochastic differential equation which paths we want to generate:

$$dX = a(X,t)\,dt + b(X,t)\,dW.$$

The Euler scheme is given by

$$U_{n+1} = U_n + a(U_n,t_n)\triangle t + b(U_n,t_n)\epsilon_n\sqrt{\triangle t},$$

where $t_n = n\triangle t$, U_n is a numerical approximation of $X(t_n)$ and $\epsilon_n \sim N(0,1)$ and independent.

For the geometrical Brownian motion

$$dS = rSdt + \sigma SdW$$

this becomes:

$$U_{n+1} = U_n(1 + r\triangle t + \sigma\epsilon_n\sqrt{\triangle t}).$$

These formulas can be generalized to multidimensional SDEs.

$$dX_i = a_i(X_1,\ldots,X_K,t)\,dt + \sum_{j=1}^{d} b_i^j(X_1,\ldots,X_K,t)\,dW_j,$$

for $i = 1,\ldots,K$ and d independent Wiener processes W_j.

The Euler scheme is

$$\begin{aligned} U_{i,n+1} &= U_{i,n} + a_i(U_{1,n},\ldots,t_n)\triangle t + \\ &+ \sum_{j=1}^{d} b_i^j(U_{1,n},\ldots,t_n)\epsilon_{i,n}\sqrt{\triangle t}, \end{aligned}$$

where as before $U_{i,n}$ is a numerical approximation of $X_i(t_n)$.

Exercise:

Implement the Euler scheme to compute the price of the European call option.

3 The Milstein scheme

The Milstein scheme for the SDE $dX = a(X,t)\,dt + b(X,t)\,dW$ is

$$\begin{aligned} U_{n+1} &= U_n + a(U_n,t_n)\triangle t + \\ &+ b(U_n,t_n)\epsilon_n\sqrt{\triangle t} + \frac{1}{2}b(U_n,t_n)b'(U_n,t_n)(\epsilon_n^2 - 1)\triangle t, \end{aligned}$$

where

$$b'(x,t) = \frac{\partial b}{\partial x}(x,t).$$

For the geometrical Brownian motion this formula becomes

$$U_{n+1} = U_n(1 + (r + \frac{1}{2}\sigma^2(\epsilon_n^2 - 1))\triangle t + \sigma\epsilon_n\sqrt{\triangle t}).$$

Exercise:

Implement the Milstein scheme to compute the price of the European call option.

4 Comparing Euler and Milstein schemes

For the geometrical Brownian motion we know the exact solution:

$$X(t + \triangle t) = X(t)\exp((r - 1/2\sigma^2)\triangle t + \sigma\epsilon\sqrt{\triangle t}),$$

where ϵ is normally distributed random variable.

Expanding this formula in the Taylor series:

$$\begin{aligned} X(t + \triangle t) &= X(t)(1 + \sigma\epsilon\sqrt{\triangle t} + (r - \frac{1}{2}\sigma^2)\triangle t + \frac{1}{2}(\sigma\epsilon\sqrt{\triangle t})^2 + O(\triangle t)) \\ &= X(t)(1 + \sigma\epsilon\sqrt{\triangle t} + (r + \frac{1}{2}\sigma(\epsilon^2 - 1))\triangle t + O(\triangle t)) \end{aligned}$$

Thus, Milstein scheme is accurate up to $O(\triangle t)$ term and Euler scheme is accurate up to $O(\sqrt{\triangle t})$.

5 Predictor-Corrector

Notice that taking an Euler step ignores the fact that a and b change their values along the path over the time step $\triangle t$. The Predictor-Corrector scheme takes this into account.

Lat us illustrate this scheme on the SDE $dX = a(X,t)\,dt + b(X,t)\,dW$. First, compute a "predictor" $\bar{U}_n$ by performing one Euler scheme:

$$\bar{U}_{n+1} = U_n + a(U_n, t_n)\triangle t + b(U_n, t_n)\epsilon_n\sqrt{\triangle t}.$$

Then the corrector is computed using the following formula:

$$\begin{aligned} U_{n+1} &= U_n + \frac{1}{2}(\bar{a}(U_n, t_n) + \bar{a}(\bar{U}_{n+1}, t_{n+1}))\triangle t + \\ &\quad \frac{1}{2}(b(U_n, t_n) + b(\bar{U}_{n+1}, t_{n+1}))\sqrt{\triangle t}, \end{aligned}$$

where

$$\bar{a}(x,t) = a(x,t) - \frac{1}{2}b(x,t)\partial_x b(x,t).$$

It can be shown that this scheme gives $O(\triangle t)$ approximation error.

Exercise:

Implement the Predictor-Corrector scheme to compute the price of the European call option.

6 Spurious paths

We know that all the paths of geometrical Brownian motion are positive (almost surely if we start with a positive value). However, generating a geometrical Brownian path using Euler (or Milstein) method will produce from time to time paths which are negative at some interval of time.

For the Euler scheme it happens when

$$\epsilon < -\frac{1+r}{\sigma\sqrt{\triangle t}}.$$

Obviously, for finite value of $\triangle t$ it is a matter of time until ϵ satisfies this inequality and U_{n+1} becomes negative.

For the geometrical Brownian motion the weight of such paths is very small and decreases as $\triangle t$ goes to zero. So, one can reject such paths.

However, in some other cases the weight of such paths does not tend to zero (e.g. $dS = rSdt + S^{\gamma}dW$, where $\gamma < \frac{1}{2}$). Usually, this means that the probability of adsorption at zero is positive. In this case, one should assume that if a path crosses zero, then it is absorbed at zero.

7 Strong and Weak convergence

Consider a SDE

$$dX = a(X,t)\,dt + b(X,t)\,dW.$$

and let U^n be its numerical approximation.

Definition 1. The numerical scheme is *strongly* convergent with order α if

$$\max_{n=0,\ldots,N} E(|X(n\triangle t) - U^n|) < K\triangle t^{\alpha}.$$

The constant K depends on $T = N\triangle t$, the SDE and the numerical scheme.

Definition 2. The numerical scheme is *weakly* convergent with order α if

$$|E(g(X(n\triangle t)) - E(g(U^n))| < K^g\triangle t^{\alpha}$$

for any polynomial g. The constant $K_{n\triangle t}$ depends on $T = n\triangle t$, g, the SDE and the numerical scheme.

In financial applications using Monte Carlo simulations the order of week convergence is usually more important.

Under certain conditions on a, b, and f the Euler scheme is

- strongly convergent of order 1/2;
- weakly convergent of order 1.

Under certain conditions on a, b, and f the Milstein scheme is

- strongly convergent of order 1;
- weakly convergent of order 1.

Exercise:

Check numerically the order of strong and weak convergence for the Euler, Milstein and Predictor-Corrector schemes.

8 Mean Square Error

Finally, how to decide whether it is better to increase the number of timesteps (reducing the weak error) or the number of paths (reducing the Monte Carlo sampling error)?

For the Euler scheme we can very roughly estimate the mean square error by the sum the error due to the variance of the Monte-Carlo estimator and the square of bias due to weak error:

$$\text{MSE} \approx \frac{a}{n} + b\triangle t^2,$$

where n here is the number of Monte-Carlo samples.

If we have N timesteps, the computational cost is proportional to $M = Nn$ and the MSE is approximately

$$\frac{a}{n} + \frac{bT^2}{N^2} = \frac{a}{n} + \frac{bT^2 n^2}{M^2}.$$

For a fixed computational cost, this is a minimum when

$$n = \left(\frac{aM^2}{2bT^2}\right)^{1/3}, \qquad N = \left(\frac{2bT^2 M}{a}\right)^{1/3}.$$

So,

$$\frac{a}{n} = \left(\frac{2a^2 bT^2}{M^2}\right)^{1/3}, \qquad b\triangle t^2 = \left(\frac{a^2 bT^2}{4M^2}\right)^{1/3},$$

and the MC term is twice as big as the bias term.