Brownian Motion.

Q1. Assuming the definition we provided in class for Brownian motion, show that for any $0 < t < t_2 < \cdots < t_k$ the joint distribution of $(B(t_1), ..., B(t_k))$ is gaussian and compute the covariance matrix.

Q2. (This exercise shows that just know the finite dimensional distributions is not enough to determine the stochastic process) Let $B(\cdot)$ Brownian motion and consider U an independent uniform random variable on $[0, 1]$. Show that the function

$$\tilde{B}(t) = \begin{cases} B(t) & , t \neq U \\ 0 & , t = U \end{cases}$$

has the same finite dimensional distributions as Brownian motion but it is a.s. not continuous.

Q3. Prove that

$$P\Big(\sup_{t,s \in (0,1)} \frac{|B(t) - B(s)|}{|t-s|^{1/2}} = \infty \Big) = 1.$$

(Give an alternative proof than the one in the class by just using Borel-Cantelli and the definition of Brownian Motion)