Numerical Methods

Pricing European and American options

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1 European options evaluation

In this section we briefly consider various practical aspects of evaluating plain vanilla European options on an example of a call option.

Let $V(S,t)$ be the price of a call option at the moment of time t and the underlying price S and let the strike price be E at the expiration T.

Is is well known that under the assumption of the geometrical Brownian motion of the underlying the option price $V(S,t)$ must satisfy the following:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$
$$V(0,t) = 0,$$
$$V(S,T) = \max(S-E,0).$$

If one of these conditions is not satisfied, it would lead to an arbitrage possibility.

The above problem can be solved analytically and an explicit formula for $V(S,t)$ can be given. However, we will ignore this and try to see how to solve this problem numerically.

First we see that the function $V(S,t)$ is defined on $[0,+\infty)$, so we should restrict ourselves to a finite interval $[0,S_{max}]$. The number S_{max} should be sufficiently large, so we can use asymptotic property for a call option price:

$$V(S,t) \approx S - E\exp(-r(T-t))$$

for large S.

Thus, we introduce a Dirichlet boundary condition at S_{max}:

$$V(S_{max},t) = S_{max} - E\exp(-r(T-t)).$$

The coefficient $\frac{1}{2}\sigma^2 S^2$ in front of the second derivative of V is very small, so to have a stable explicit Euler scheme the ratio $\Delta t/\Delta x^2$ should be extremely small. This makes the explicit Euler scheme absolutely impractical in this case.

The Crank-Nicolson scheme has a better approximation order in Δt compared with the implicit Euler scheme with the same computational effort, so, in general, it should be preferred over the other two schemes.

1.1 Reducing Black–Scholes equation to the heat equation

Consider the following variable change:

$$\begin{aligned} S &= E\exp(x) \\ t &= T - 2\tau/\sigma^2 \\ V &= Eu(x,\tau) \end{aligned}$$

After this variable change the Black–Scholes equation becomes:

$$u_\tau = u_{xx} + (k-1)u_x - ku,$$

where $k = 2r/\sigma^2$ and the initial condition becomes

$$u(x,0) = max(e^x - 1, 0)$$

Notice that

- $x \in (-\infty, +\infty)$, $\tau \geq 0$;
- there is no need for the boundary conditions;
- the equation depend only on k;
- another dimensionless parameter of the problem is $\frac{1}{2}\sigma^2 T$.

The equation can be simplified even further by the substitution

$$u = \exp(\alpha x + \beta\tau)w(x,\tau),$$

where $\alpha = -\frac{1}{2}(k-1)$ and $\beta = -\frac{1}{4}(k+1)^2$.

After this substitution the equation becomes just the heat equation

$$w_t = w_{xx}.$$

2 Free Boundary Problems

2.1 The Obstacle Problem

Consider an elastic string held at two points A and B passing smoothly over an obstacle. For a given obstacle and points A and B it should be possible to find the shape of the string. However, this means finding points P and Q between which the string makes contact with the obstacle. These points are not specified as part of the given information for the problem, but instead must be found as part of its solution. The points P and Q are **free boundary points**. In this case they divide the solution into different regions in which different equations apply.

The following conditions are required to hold (and are sufficient to determine a solution):

- **The string must be above or on the obstacle.**
- **The string must have non positive curvature (second derivative).**
- **If the string is strictly above the obstacle, the curvature must be zero.**
- **The string must be continuous.**
- **The string slope must be continuous.**

Let us consider a particular formulation of the problem in which the points A and B correspond to $x = -1$ and $x = +1$ respectively. Let $u(x)$ be the (unknown) height of the string above the x-axis and let the obstacle be represented by the function $f(x)$. We will assume that $f(-1) < 0$ and $f(+1) < 0$ so that the obstacle is below the x-axis at A and B. For simplicity assume f is twice differentiable and that $f'' < 0$ (this guarantees only one region of contact). Let x_P and x_Q correspond to the points P and Q. These are not yet known.

From the conditions given, the problem to be solved can then be specified with the following equations.

$$
\begin{array}{rl}
 & u(-1) = 0, \\
u'' = 0, & -1 < x < x_P, \\
u(x_P) = f(x_P), & u'(x_P) = f'(x_P), \\
u(x) = f(x), & x_Q < x < x_P, \\
u(x_Q) = f(x_Q), & u'(x_Q) = f'(x_Q), \\
u'' = 0, & x_Q < x < 1, \\
 & u(1) = 0.
\end{array}
$$

Again, as formulated this is a free boundary problem since the (internal) boundary points x_P and x_Q are not know *a priori*.

2.2 Linear complementarity problem

It is possible to write the preceding free boundary problem in a different form by noting that either $u'' = 0$ or $u - f = 0$ throughout the interval $-1 < x < 1$.

Thus the product is zero everywhere and the problem can be express as a **linear complementarity problem:**

$$u'' \cdot (u - f) = 0, \quad -u'' \geq 0, \quad (u - f) \geq 0,$$

subject to the boundary conditions:

$$u(-1) = u(1) = 0$$

and the continuity of u and u_x.

In this formulation the free boundary points do not appear and instead we have the constraints: $-u'' \geq 0$ **and** $(u - f) \geq 0$.

The point is that often it is easier to solve such a constrained problem than it is to solve the problem formulated explicitly in terms of a free boundary. In the linear complementarity form, the free boundary points are determined *a posteriori*.

3 The American Put

An American option valuation problem can also be formulated as a free boundary problem just as for the obstacle. The conditions to be satisfied are:

- **The option value** V **must be greater than or equal to the payoff function:** $F_p(S) = max(E - S, 0)$**.**
- **The following partial-differential inequality must hold:**

$$\frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \leq 0$$

- V **must be a continuous function of** S**.**
- $\partial V/\partial S$ **must be a continuous function of** S**.**

We divide the domain into two regions: $0 \leq S < S_f(\tau)$ and $S_f(\tau) < S < \infty$ where $S_f(\tau)$ denotes the (time-dependent) free boundary point where V first touches the payoff curve.

Assuming $S_f < E$, the derivative of the payoff function at S_f is -1. Hence continuity of the derivative requires $\partial V/\partial S(S_f) = -1$. We then have the following free boundary problem:

$$\begin{aligned} V(S,\tau) = E - S, &\qquad 0 \leq S < S_f(\tau) \\ \frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 &\qquad S_f(\tau) < S < \infty \\ V(S_f(\tau),\tau) = E - S_f(\tau) &\qquad \frac{\partial V}{\partial S}(S_f(\tau),\tau) = -1 \end{aligned}$$

(There is an additional boundary condition that $V(\infty) = 0$.)

3.1 Linear complementarity problem

As with the obstacle problem, we can formulate the American Put as a linear complementarity problem and thereby avoid explicit reference to the free boundary point S_f. In this case we have:

$$\left((\frac{\partial}{\partial \tau} + \mathcal{L})V \right)(V - F_p) = 0, \tag{1}$$

$$-\left((\frac{\partial}{\partial \tau} + \mathcal{L})V \right) \geq 0, \qquad V - F_p \geq 0, \tag{2}$$

where

$$\mathcal{L} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS\frac{\partial}{\partial S} - r$$

V and $\frac{\partial V}{\partial S}$ should be continuous and V is subject to the boundary condition that $V(\infty) = 0$.

3.2 Numerical Implementation: Explicit Euler

For the explicit Euler method the American Put does not present a problem. First let $t = T - \tau$ and use $t \geq 0$ as the dependent variable and let $V(S,t) \to u(x,t)$, $F_p(S) \to F_p(x)$. Then the above equation becomes:

$$\left((\frac{\partial}{\partial t} - \mathcal{L})u \right)(u - F_p) = 0,$$

We then discretise this equation by letting $u(x,t) \to U_j^n$, $\mathcal{L} \to \mathbf{L}$, and $F_p(x_j) = F_j$ and obtain:

$$\left(\frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\Delta t} - \mathbf{L}\mathbf{U}^n \right)\left(\mathbf{U}^{n+1} - \mathbf{F}\right) = 0$$

$$\left(\mathbf{U}^{n+1} - (\mathbf{I} + \Delta t\mathbf{L})\mathbf{U}^n)\right)\left(\mathbf{U}^{n+1} - \mathbf{F}\right) = 0$$

Taking into account possible boundary conditions this becomes:

$$\left(\mathbf{U}^{n+1} - \mathbf{B}\ (\mathbf{I} + \Delta t\mathbf{L})\mathbf{U}^n)\right)\left(\mathbf{U}^{n+1} - \mathbf{F}\right) = 0$$

In addition two inequalities must hold:

$$\mathbf{U}^{n+1} - \mathbf{B}\ (\mathbf{I} + \Delta t\mathbf{L})\mathbf{U}^n \geq 0 \qquad \mathbf{U}^{n+1} - \mathbf{F} > 0, \tag{3}$$

where it is understood that the inequalities hold component by component.

For each grid point j there are two cases:

1. either

$$U_j^{n+1} > F_j, \tag{4}$$

and

$$U_j^{n+1} = [\mathbf{B}\ (\mathbf{I} + \Delta t\mathbf{L})\mathbf{U}^n)]_j \tag{5}$$

as usual for explicit Euler.

2. or

$$U_j^{n+1} = F_j \tag{6}$$

and

$$U_j^{n+1} - [\mathbf{B}\ (\mathbf{I} + \Delta t \mathbf{L})\mathbf{U}^n)]_j \geq 0\ \Rightarrow U_j^{n+1} \geq [\mathbf{B}\ (\mathbf{I} + \Delta t \mathbf{L})\mathbf{U}^n]_j. \tag{7}$$

Hence either case 1. is satisfied or else case 2. satisfied.

Using this we can implemented the linear complementarity form of the American Put as follows:

$$\begin{aligned}
&\bar{\mathbf{U}} = \mathbf{B}\ (\mathbf{I} + \Delta t \mathbf{L})\mathbf{U}^n \\
&\text{foreach j} \\
&\quad \text{if}\ \ (\bar{U}_j > F_j)\ \ \text{then} \\
&\quad\quad U_j^{n+1} = \bar{U}_j \\
&\quad \text{else}\quad //(\bar{U}_j \leq F_j) \\
&\quad\quad U_j^{n+1} = F_j
\end{aligned}$$

Implicit Euler and Crank-Nicolson schemes for the American require more work. In particular, Implicit Euler and Crank-Nicolson schemes require iterative methods for solving linear systems.

3.3 Projected SOR

Let us recall one of the iterative methods of solving a system of linear equations, namely SOR method. Let the equation be

$$\mathbf{A}\mathbf{x} = \mathbf{b}.$$

Split the matrix $\mathbf{A}$ as follows:

$$\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U}$$

where $\mathbf{D}$ is the diagonal part of $\mathbf{A}$ and $\mathbf{L}$ is the lower triangular part of $\mathbf{A}$ (with zeros on diagonal) and $\mathbf{U}$ is the upper triangular part of $\mathbf{A}$ (with zeros on diagonal).

The SOR method is

$$\mathbf{x}^{(s+1)} = \mathbf{x}^{(s)} + \omega \left\{ \mathbf{D}^{-1} \left(-\mathbf{L}\,\mathbf{x}^{(s+1)} - \mathbf{U}\,\mathbf{x}^{(s)} + \mathbf{b} \right) - \mathbf{x}^{(s)} \right\} \tag{8}$$

where ω is called the over-relaxation parameter.

One advantage of iterative methods in general and SOR in particular over direct methods is that it is easily adaptable to other types of problems. In particular, the SOR method can be easily modified to solve constrained matrix problems, such as those which arise in discretization of American options.

Suppose we wish to solve the linear complementarity problem:

$$\mathbf{A}\,\mathbf{x} \geq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{f},$$
$$(\mathbf{A}\,\mathbf{x} - \mathbf{b})\,(\mathbf{x} - \mathbf{f}) = 0,$$

where it is understood that the inequalities hold component by component.

We simply modify the SOR procedure given by (8) to require that at each iteration $\mathbf{x}^{(s)} \geq \mathbf{f}$. This gives the **Projected SOR Method**:

$$\mathbf{x}^{(s+1)} = \max\left(\mathbf{x}^{(s)} + \omega\left\{\mathbf{D}^{-1}\left(-\mathbf{L}\,\mathbf{x}^{(s+1)} - \mathbf{U}\,\mathbf{x}^{(s)} + \mathbf{b}\right) - \mathbf{x}^{(s)}\right\}, \mathbf{f}\right) \quad (9)$$

where max applies component by component to the vector quantities,i.e. :

$$x_0^{(s+1)} = \max(x_0^{(s)} + \omega(\frac{1}{d_0}(\qquad\qquad -u_0 x_1^{(s)} + b_0) - x_0^{(s)}), f_0)$$
$$\vdots$$
$$x_j^{(s+1)} = \max(x_j^{(s)} + \omega(\frac{1}{d_j}(-l_{j-1} x_{j-1}^{(s+1)} - u_j x_{j+1}^{(s)} + b_j) - x_j^{(s)}), f_j)$$

Again this is an explicit system of equation for the the $x_j^{(s+1)}$ in terms of $x_j^{(s)}$ and as with Gauss-Seidel and SOR, the iterations may be done in place.

It is clear that if the method converges, then the solution will satisfy $\mathbf{x} \geq \mathbf{f}$, since this is forced at each iteration. One possibility is that $x_j = x_j^{(s)} = x_j^{(s-1)} = f_j$. The other possibility is that $x_j > f_j$. Then (9) implies the j-th component of $\mathbf{A}\,\mathbf{x} = \mathbf{b}$ is satisfied. $\mathbf{A}\,\mathbf{x} \geq \mathbf{b}$ also holds provided that A satisfies some extra condition. This condition holds if A is matrix arising in the Crank-Nicolson scheme. The main point from our desired application is that either the PDE is satisfied $\mathbf{A}\,\mathbf{x} = \mathbf{b}$ with $\mathbf{x} > \mathbf{f}$ or $\mathbf{x} = \mathbf{f}$.

Hence by using an iterative procedure which demands that at each iteration the constraint(s) be satisfied, we are able to self-consistently generate a solution to the constrained matrix problem. This will allow Crank-Nicolson time stepping of the Black-Scholes equations for the American put.

3.4 Crank-Nicolson scheme for American Put.

After the coordinate transformation described above the American put option valuation problem becomes:

$$\begin{aligned} (w_\tau - w_{xx})(w - g) &= 0 \\ w_\tau - w_{xx} &\geq 0 \\ w - g &\geq 0 \end{aligned}$$

where

$$g(x, \tau) = \exp(\frac{1}{4}(k+1)^2\tau)\max(\exp(\frac{1}{2}(k-1)x) - \exp(\frac{1}{2}(k+1)x), 0)$$

and $k = 2r/\sigma^2$.

The initial and (fixed) boundary conditions are

- $w(x,0) = g(x,0)$;
- w, w_x are continuous;
- $\lim_{x\to\pm\infty} w(x,\tau) = \lim_{x\to\pm\infty} g(x,\tau)$, $\tau \geq 0$.

As in the case of the European options we have to restrict the space domain to a suitable interval (x_{min}, x_{max}). For simplicity we will assume that $x_{min} = -x_{max}$. Then the boundary condition becomes:

$$\begin{aligned} w(x_{min},\tau) &= g(x_{min},\tau) \\ w(x_{max},\tau) &= g(x_{max},\tau) \end{aligned}$$

As before, we then discretise the equation by letting $w(x,t) \to \mathbf{W}_j^n$, $g(x_j,t_n) = \mathbf{G}_j^n$. Notice that this time j varies from $-J$ to J. The Crank-Nicolson scheme becomes:

$$\left(\frac{\mathbf{W}^{n+1}-\mathbf{W}^n}{\Delta t} - \frac{1}{2}\mathbf{L}(\mathbf{W}^{n+1}+\mathbf{W}^n)\right)\left(\mathbf{W}^{n+1}-\mathbf{G}^{n+1}\right) = 0$$

$$\left((\mathbf{I}-\frac{1}{2}\Delta t\mathbf{L})\mathbf{W}^{n+1} - (\mathbf{I}+\frac{1}{2}\Delta t\mathbf{L})\mathbf{W}^n\right)\left(\mathbf{W}^{n+1}-\mathbf{G}^{n+1}\right) = 0$$

where $\mathbf{L}$ is the second order approximation of the second derivative.

Taking into account the boundary conditions this becomes:

$$\left((\mathbf{I}-\frac{1}{2}\Delta t\mathbf{L})\mathbf{W}^{n+1} - \mathbf{B}^{n+1}(\mathbf{I}+\frac{1}{2}\Delta t\mathbf{L})\mathbf{W}^n\right)\left(\mathbf{W}^{n+1}-\mathbf{G}^{n+1}\right) = 0$$

where $\mathbf{B}^{n+1}$ is the usual boundary operator:

$$\begin{aligned} (\mathbf{B}^{n+1}W)_j &= \mathbf{G}_j^{n+1} && \text{if } j = \pm J; \\ (\mathbf{B}^{n+1}W)_j &= W_j && \text{otherwise.} \end{aligned}$$

In addition two inequalities must hold:

$$\begin{aligned} (\mathbf{I}-\frac{1}{2}\Delta t\mathbf{L})\mathbf{W}^{n+1} - \mathbf{B}^{n+1}(\mathbf{I}+\frac{1}{2}\Delta t\mathbf{L})\mathbf{W}^n &\geq 0 \\ \mathbf{W}^{n+1}-\mathbf{G}^{n+1} &\geq 0 \end{aligned}$$

where it is understood that the inequalities hold component by component.

Now we are ready to apply the projective SOR method. In the notation of the previous section we have the following:

$$\begin{aligned} \mathbf{A} &= (\mathbf{I}-\frac{1}{2}\Delta t\mathbf{L}), \\ \mathbf{b} &= \mathbf{B}^{n+1}(\mathbf{I}+\frac{1}{2}\Delta t\mathbf{L})\mathbf{W}^n. \end{aligned}$$

Or component wise

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -\frac{1}{2}\nu & 1+\nu & -\frac{1}{2}\nu & \dots & 0 \\ 0 & -\frac{1}{2}\nu & 1+\nu & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\nu = \Delta t / \triangle x^2$.

The vector $\mathbf{b}$ is explicitly given by the formula:

$$\begin{array}{lcll} \mathbf{b}_j & = & \mathbf{G}_j^{n+1} & \text{if } j = \pm J; \\ \mathbf{b}_j & = & (1-\nu)W_j^n + \frac{1}{2}\nu(W_{j-1}^n + W_{j+1}^n) & \text{otherwise.} \end{array}$$

The algorithm:

1. Given $\mathbf{W}^n$ compute $\mathbf{b}$ using the formula above;
2. Compute the constraint vector $\mathbf{G}^{n+1}$;
3. Start with the initial guess $\mathbf{W}^{n+1,0} = \max(\mathbf{G}^{n+1}, \mathbf{W}^n)$;
4. Given $\mathbf{W}^{n+1,k}$, in increasing j-indicial order compute $W^{n+1,k+1}$ by

$$\begin{array}{lcl} \mathbf{W}_{-J}^{n+1,k+1} & = & \mathbf{G}_{-J}^{n+1} \\ \mathbf{W}_j^{n+1,k+1} & = & \max(\mathbf{G}_j^{n+1}, \mathbf{W}_j^{n+1,k} + \\ & & \omega(\frac{1}{1+\nu}(\mathbf{b}_j + \frac{1}{2}\nu(\mathbf{W}_{j-1}^{n+1,k+1} + \mathbf{W}_{j+1}^{n+1,k})) - \mathbf{W}_j^{n+1,k})) \\ \mathbf{W}_J^{n+1,k+1} & = & \mathbf{G}_J^{n+1} \end{array}$$

5. Check if required accuracy archived. If not, go to step 4, otherwise put $\mathbf{W}^{n+1} = \mathbf{W}^{n+1,k+1}$;
6. Return to step 1 until the necessary number of time–steps have been completed.

It can be checked that the above algorithm converges (if $1 < \omega < 2$) to the solution of the American put option valuation problem solution when Δt and $\triangle x \to 0$.