Surfaces

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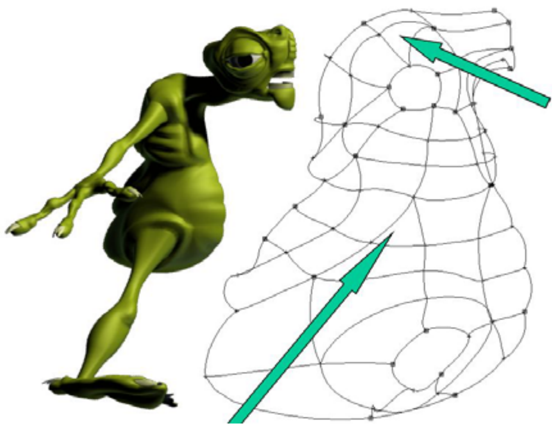
Surfaces

- Many objects we want to model are not flat:
  - Cars, animals, plants, buildings.



Accuracy/Space Trade-off

Problem

- Piecewise linear approximations require many pieces to look good (realistic, smooth, etc.).
- Set of individual surface points would take large amounts of storage.

Solution

- Higher-order formulae for coordinates on surface.
- If a simple formula won't work, subdivide surface into pieces that can be represented by simple formulae.
- May still be an approx., but uses much less storage.
- Downside: harder to specify and render.

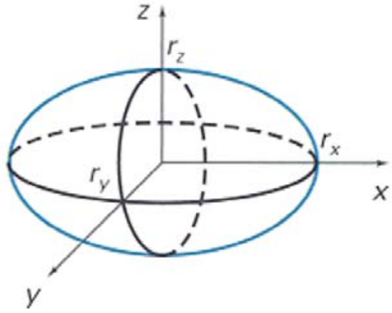
Surface Representations

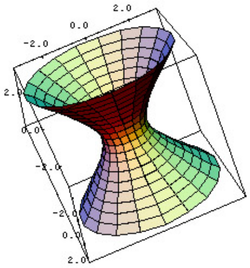
- Parametric: (x,y,z)=(f(u,v), g(u,v), h(u,v))
  - e.g. plane, sphere, cylinder, torus, bi-cubic surface, swept surface
  - parametric functions let you *iterate* over the surface by incrementing u and v
  - great for making polygon meshes, etc
  - complex for intersections: ray/surface, point-inside-boundary, etc
- Implicit: F(x,y,z) = 0
  - e.g. plane, sphere, cylinder, quadric, torus, blobby models
  - terrible for iterating over the surface
  - great for intersections, morphing

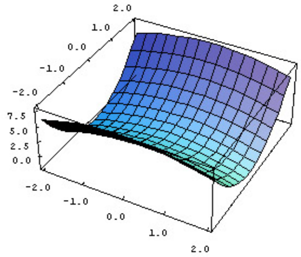
Examples

Paremetric: Ellipsoid

$$x = r_x \cos\phi \cos\theta$$
$$y = r_y \cos\phi \sin\theta$$
$$z = r_z \sin\phi$$



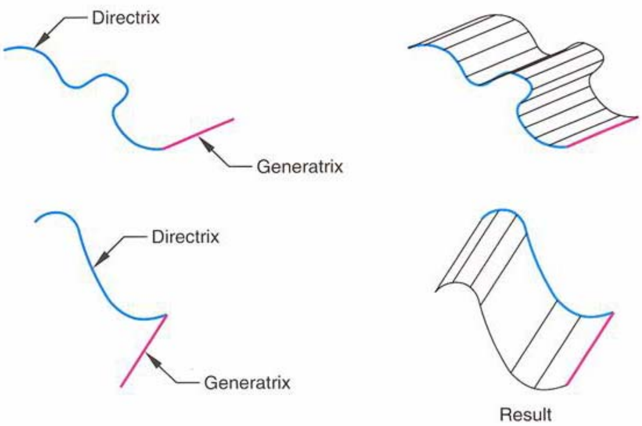




Implicit functions:
Quadrics and other

Swept Surfaces

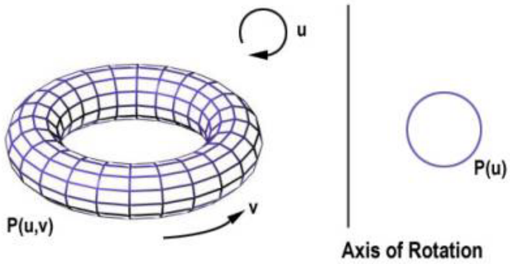
Obtained by sweeping generator enities along director entities.



Rotational Surfaces

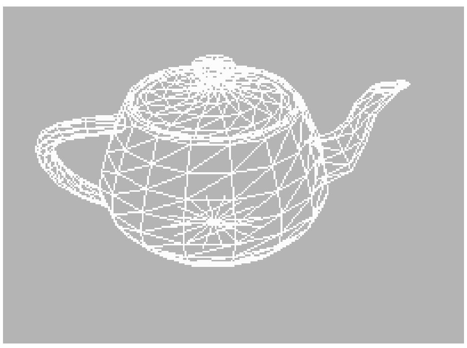
Generated by rotating a curve about an axis.

Every point of the generating curve describes a circle whose supporting plane lies orthogonally to the Axis.



Meshes

We can approximate a surface with a *polygonal mesh.*

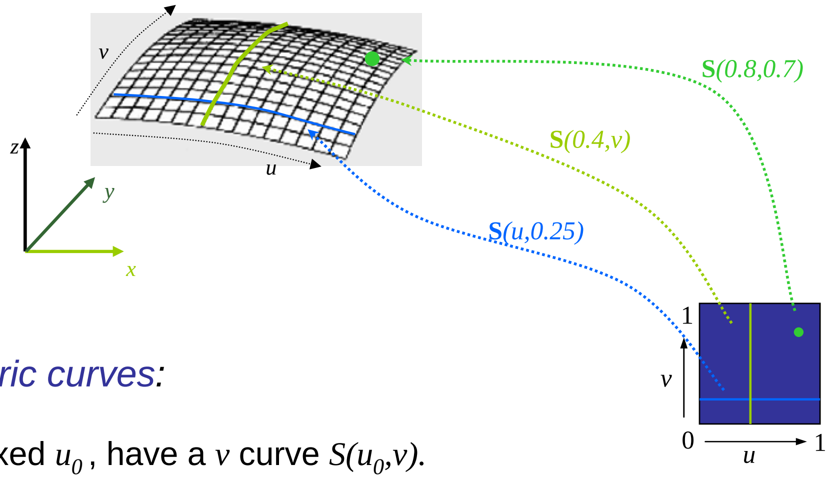


Curved Surfaces

- Remember the overview of curves:
  - Described by a series of control points.
  - A function $Q(t)$.
  - Segments joined together to form a longer curve.
- Same for surfaces, but now two dimensions
  - Described by a mesh of control points.
  - A function $S(u,v)$.
  - *Patches* joined together to form a bigger surface.

Parametric Surface Patch

- $S(u,v)$ describes a point in space for any given (u,v) pair:
  - u,v each range from 0 to 1.



- *Parametric curves:*
  - For fixed u_0, have a v curve $S(u_0,v)$.
  - For fixed v_0, have a u curve $S(u,t_0)$.
  - For any point on the surface, there are a pair of parametric curves that go through point.

Polynomial Surface Patches

- $S(s,t)$ is typically polynomial in both s and t
  - *Bilinear:*

$$S(s,t) = \mathbf{a}st + \mathbf{b}s + \mathbf{c}t + \mathbf{d}$$

$$S(s,t) = (\mathbf{a}t + \mathbf{b})s + (\mathbf{c}t + \mathbf{d}) \quad \text{-- hold } t \text{ constant} \Rightarrow \text{linear in } s$$

$$S(s,t) = (\mathbf{a}s + \mathbf{c})t + (\mathbf{b}s + \mathbf{d}) \quad \text{-- hold } s \text{ constant} \Rightarrow \text{linear in } t$$

  - *Bicubic:*

$$\begin{aligned} S(s,t) = {} & \mathbf{a}s^3t^3 + \mathbf{b}s^3t^2 + \mathbf{c}s^3t^2 + \mathbf{d}s^3 + \mathbf{e}s^2t^3 + \mathbf{f}s^2t^2 + \mathbf{g}s^2t + \mathbf{h}s^2 \\ & + \mathbf{i}st^3 + \mathbf{j}st^2 + \mathbf{k}st + \mathbf{l}s + \mathbf{m}t^3 + \mathbf{n}t^2 + \mathbf{o}t + \mathbf{p} \end{aligned}$$

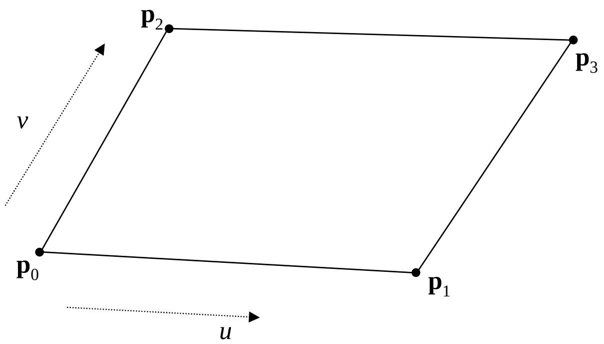
$$\begin{aligned} S(s,t) = {} & (\mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d})s^3 + (\mathbf{e}t^3 + \mathbf{f}t^2 + \mathbf{g}t + \mathbf{h})s^2 \quad \text{-- hold } t \text{ constant} \Rightarrow \text{cubic in } s \\ & + (\mathbf{i}t^3 + \mathbf{j}t^2 + \mathbf{k}t + \mathbf{l})s + (\mathbf{m}t^3 + \mathbf{n}t^2 + \mathbf{o}t + \mathbf{p}) \end{aligned}$$

$$\begin{aligned} S(s,t) = {} & (\mathbf{a}s^3 + \mathbf{e}s^2 + \mathbf{i}s + \mathbf{m})t^3 + (\mathbf{b}s^3 + \mathbf{f}s^2 + \mathbf{j}s + \mathbf{n})t^2 \quad \text{-- hold } s \text{ constant} \Rightarrow \text{cubic in } t \\ & + (\mathbf{c}s^3 + \mathbf{g}s^2 + \mathbf{k}s + \mathbf{o})t + (\mathbf{d}s^3 + \mathbf{h}s^2 + \mathbf{l}s + \mathbf{p}) \end{aligned}$$

Bilinear Patch

- A bilinear patch is defined by a control mesh with four points p_0, p_1, p_2, p_3 **defining** a (possibly-non-planar) quadrilateral.

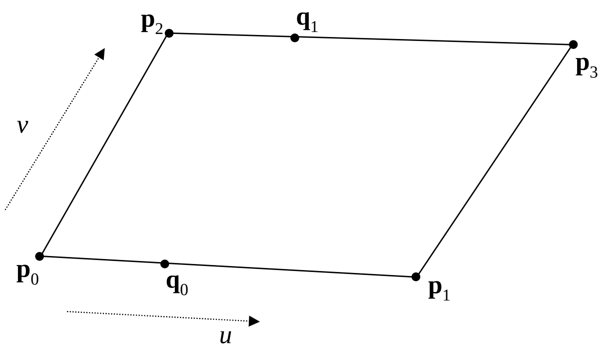
- Compute $S(u,v)$ using a two-step construction



Bilinear Patch (step 1)

- For a given value of *u*, evaluate the linear curves on the two *u*-direction edges.
- Use the same value *u* for both:

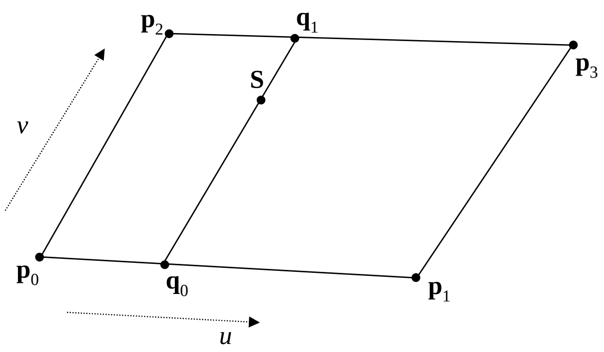
$$\mathbf{q_0} = Lerp(u, \mathbf{p_0}, \mathbf{p_1}) \qquad \mathbf{q_1} = Lerp(u, \mathbf{p_2}, \mathbf{p_3})$$



Bilinear Patch (step 2)

- Consider that $\mathbf{q_0}$, $\mathbf{q_1}$ define a line segment. Evaluate it using v to get **S.**

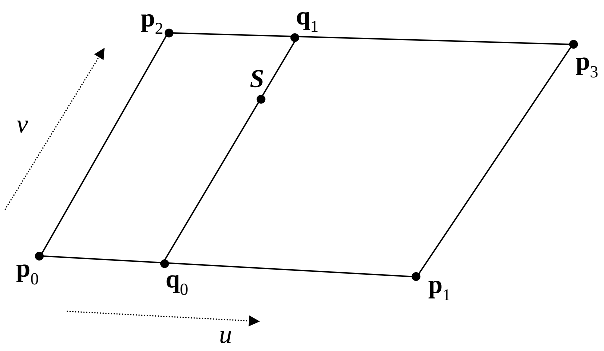
$$S = Lerp(v, \mathbf{q}_0, \mathbf{q}_1)$$



Bilinear Patch (full)

- Combining the steps, we get the full formula

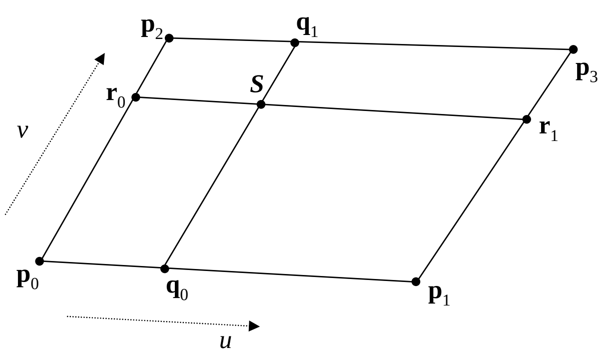
$$s(u,v) = Lerp(v, Lerp(u,\mathbf{p}_0,\mathbf{p}_1), Lerp(u,\mathbf{p}_2,\mathbf{p}_3))$$



Bilinear Patch (either order)

- It works out the same either way!

$$\boxed{\begin{aligned} s(u,v) &= Lerp(v, Lerp(u,\mathbf{p}_0,\mathbf{p}_1), Lerp(u,\mathbf{p}_2,\mathbf{p}_3)) \\ s(u,v) &= Lerp(u, Lerp(v,\mathbf{p}_0,\mathbf{p}_2), Lerp(v,\mathbf{p}_1,\mathbf{p}_3)) \end{aligned}}$$



Properties of the bilinear patch

– Interpolates the control points.

– The boundaries are straight line segments connecting the control points.

– If the all 4 points of the control mesh are co-planar, the patch is flat.

– If the points are not coplanar, get a curved surface.

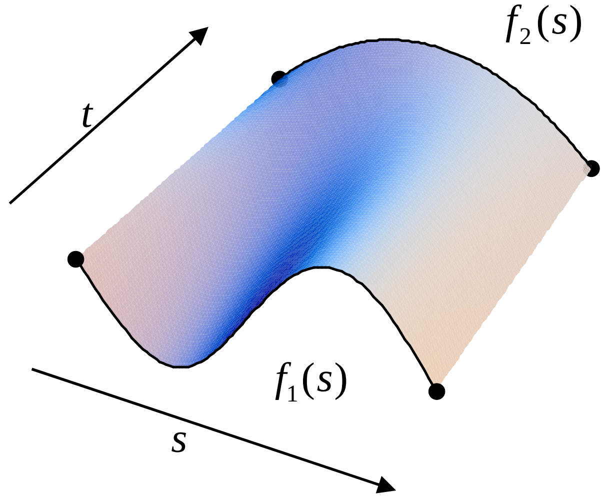
– The parametric curves are all straight line segments:

- Is a (doubly) *ruled surface:* has (two) straight lines through every point.

Ruled Surfaces

- Linear interpolation between 2 curves
  – All point lie in one line

$$f\ (s,t) = (1-t)\, f_1(s) + t\ f_2(s)$$



Particular case

Cylinder:

$$f\ (s,t) = f_1(s) + t\, d$$

$$f_1(s) = (\cos s, \sin s, 0)$$

Cone:

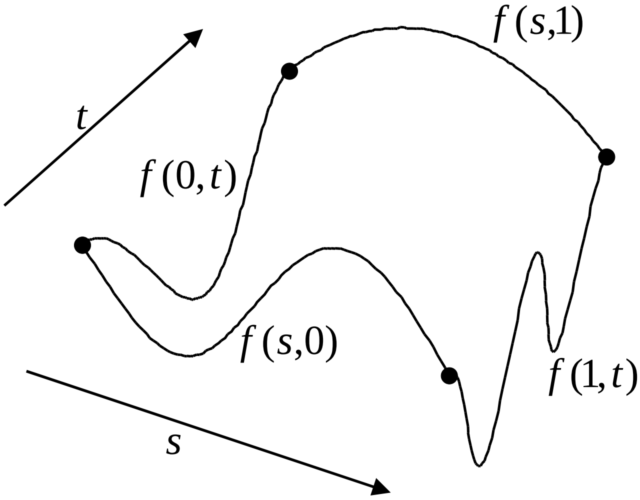
$$f\ (s,t) = (1-t)\, f_1 + t\ f_2(s)$$

$$f_2(s) = (\cos s, \sin s, 0)$$

Coons patches

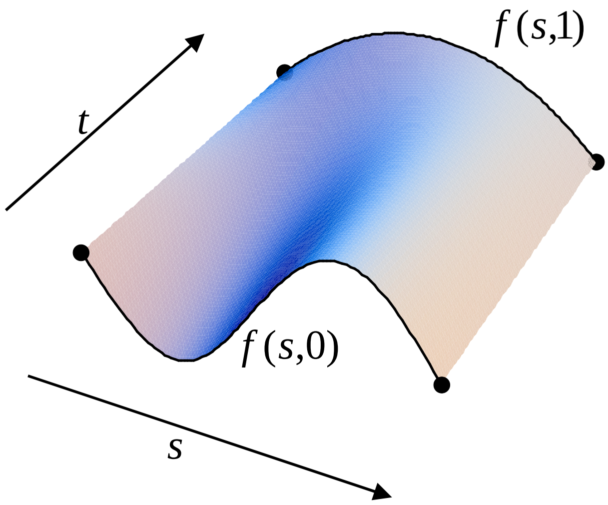
- Interpolation between 4 curves
- Build a ruled surface between pairs of curves

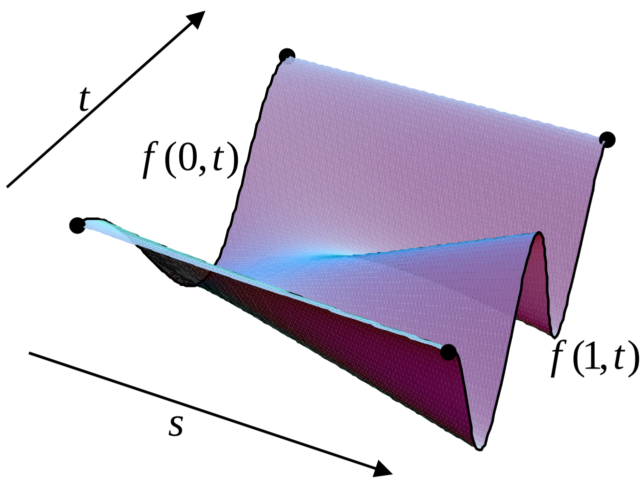
$$f_1(s,t) = (1-t)\,f(s,0) + t\,f(s,1)$$
$$f_2(s,t) = (1-s)\,f(0,t) + s\,f(1,t)$$



Coons patches

$$f_1(s,t) = (1-t)\,f(s,0) + t\,f(s,1) \qquad f_2(s,t) = (1-s)\,f(0,t) + s\,f(1,t)$$





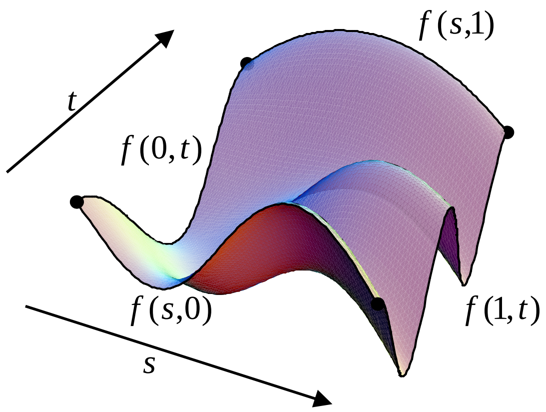
Coons patches

- “Correct” surface to make boundaries match
  - Create a linear interpolation surface between the 4 extremes:

$$f_3(s,t) = (1-s)t\ f(0,1) + s(1-t)\ f(1,0) + s\,t\ f(1,1))$$ (bilinear patch).

  - Combine surface as $f_1+f_2-f_3$

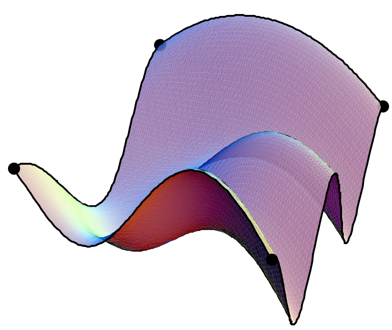
$$f_1(s,t) + f_2(s,t) - ((1-s)(1-t)\ f(0,0) + (1-s)t\ f(0,1) + s(1-t)\ f(1,0) + s\,t\ f(1,1))$$



Properties of Coons patches

$$\begin{pmatrix} f(s,0) & f(s,1) \end{pmatrix} \begin{pmatrix} 1-t \\ t \end{pmatrix} + \begin{pmatrix} 1-s & s \end{pmatrix} \begin{pmatrix} f(0,t) \\ f(1,t) \end{pmatrix} - \begin{pmatrix} 1-s & s \end{pmatrix} \begin{pmatrix} f(0,0) & f(0,1) \\ f(1,0) & f(1,1) \end{pmatrix} \begin{pmatrix} 1-t \\ t \end{pmatrix}$$

- Interpolate arbitrary boundaries
- Smoothness of surface equivalent to minimum smoothness of boundary curves
- Don’t provide higher continuity across boundaries



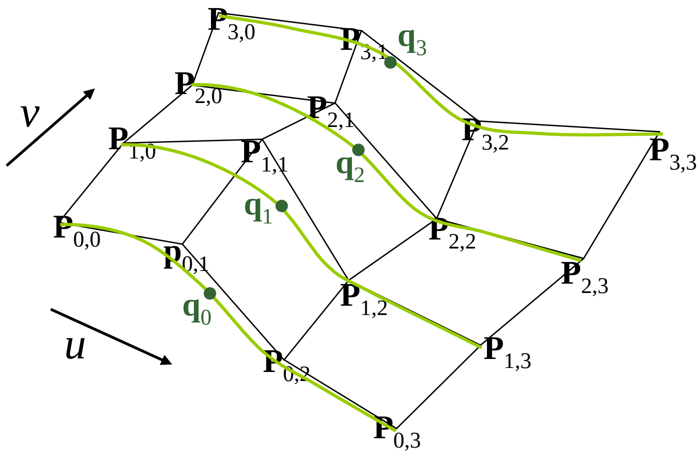
Bézier Control Mesh

- A bicubic patch has a grid of 4x4 control points:

$$P_{0,0}, \ldots, P_{0,3}$$
$$\ldots$$
$$\ldots$$
$$P_{3,0}, \ldots, P_{3,3}.$$

- Defines four Bézier curves along u and four Bézier curves along v.

- Evaluate using same approach as bilinear.



Bézier Patch Properties

- Convex hull: any point on the surface will fall within the convex hull of the control points.

- Interpolates 4 corner points.

- Approximates other 12 points, which act as "handles".

- The boundaries of the patch are the Bézier curves defined by the points on the mesh edges.

- The parametric curves are all Bézier curves.

Cubic Bezier Surfaces: Algebraic Formulation

Cubic Bezier curves can be extended to surfaces on unit squares:

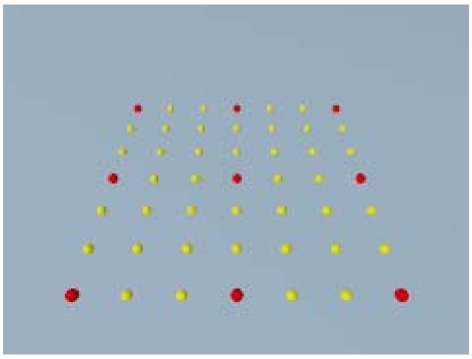
$$S(u,v) = \sum_{i=0}^{i=3} \sum_{j=o}^{j=3} B_i^3(u) B_j^3(v) P_{ij}$$

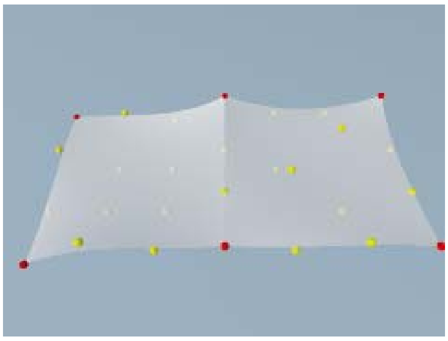
$$B_i^3(t) = \binom{3}{i} t^i (1-t)^{3-i}$$

Building a surface from Bézier patches

Building complex surfaces by putting together Bezier patches.

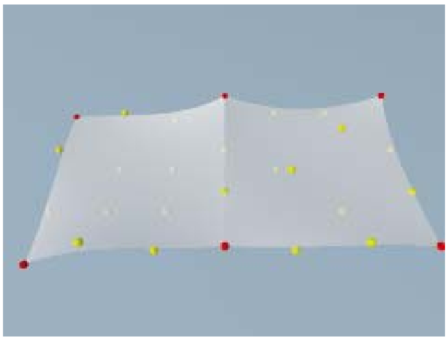
- Lay out grid of adjacent meshes.
- For C^0 continuity, must share points on the edge
  - Each edge of a Bézier patch is a Bézier curve based only on the edge mesh points.
  - So if adjacent meshes share edge points, the patches will line up exactly.

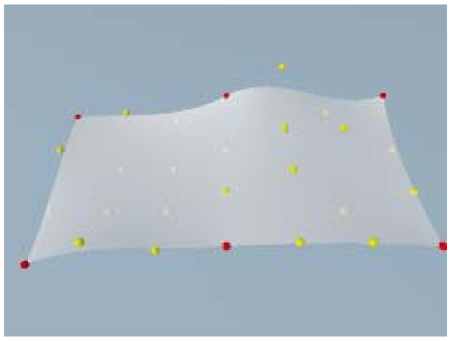




C^1 continuity across Bézier edges

- We want the parametric curves that cross each edge to have C^1 continuity:
  - So the handles must be equal-and-opposite across the edge.





B-spline patches

For the same reason as using B-spline curves:

- More uniform behavior.
- Better mathematical properties.
- Doesn't interpolate any control points.

B-Spline Surfaces

A B-spline surface $S(u,v)$, is defined by:

$$S(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{n'} B_{i,d}(u) B_{j,d'}(v) P_{i,j}$$

where:

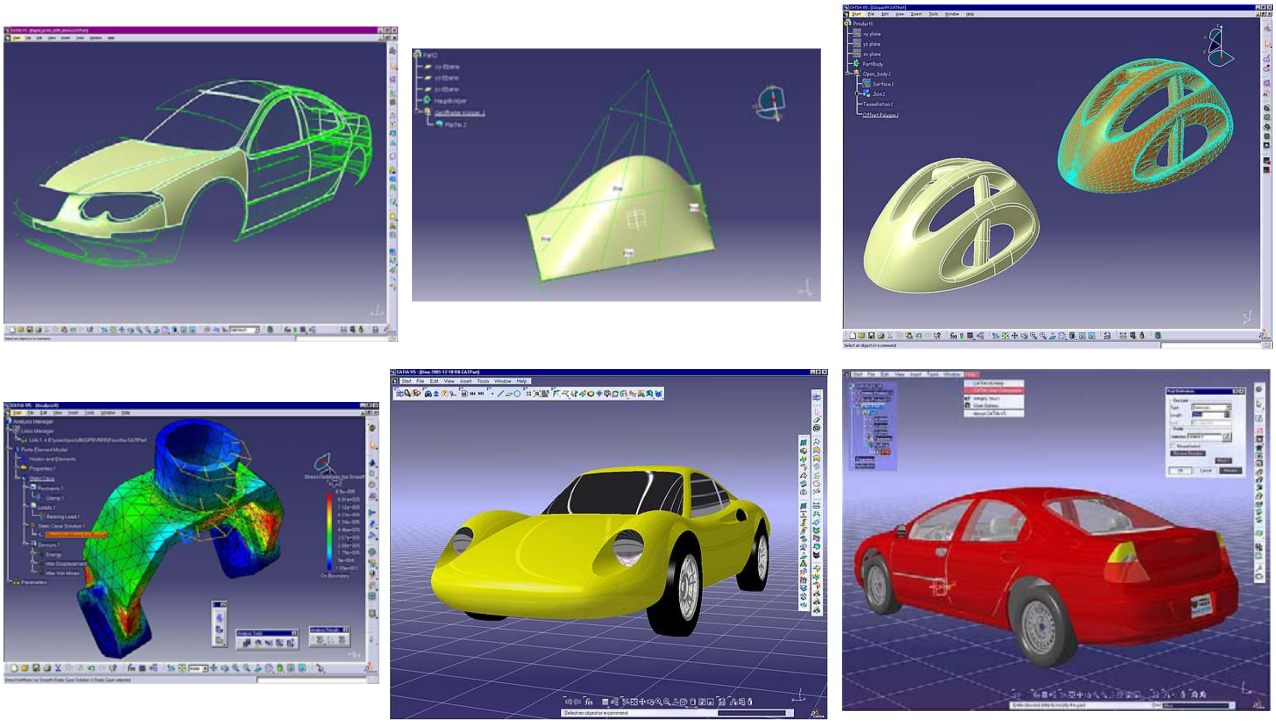
- The $P_{i,j}$ are the *(n+1)* x *(n'+1)* control points.
- *d* and *d'* are the orders in the *u* and *v* directions.
- We have two non-decreasing *knot sequences of parameters* $u_0,\ldots,u_{n+d}$ and $v_0,\ldots,v_{n'+d'}$.
- $B_{i,d}$ are the uniform B-Spline *basis* or *blending functions* of degree *d-1.*

NURBS Surfaces

$$S(u,v) = \frac{\sum_{i=0}^{n} \sum_{j=0}^{n'} w_{i,j} B_{i,d}(u) B_{j,d'}(v) P_{i,j}}{\sum_{i=0}^{n} \sum_{j=0}^{n'} w_{i,j} B_{i,d}(u) B_{j,d'}(v)}$$

- Can take on more shapes:
  - conic sections.
- Can blend, merge, … .
- Still has rectangular topology.

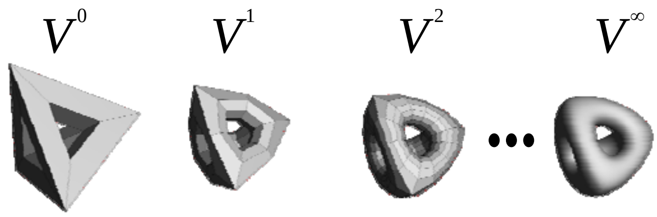
Surface examples



Subdivision Surfaces

- Defined by a control mesh and a recursive subdivision procedure
- Arbitrary mesh, not rectangular topology:
  - No u,v parameters.
- Can make surfaces with arbitrary topology or connectivity.
- Work by recursively subdividing mesh faces:
  - Per-vertex annotation for weights, corners, creases.
- good for interactive design
  - Used in particular for character animation:
  - One surface rather than collection of patches.
  - Can deform geometry without creating cracks.

The Basic Idea



- In each iteration
  - Refine a control net (mesh)
  - Increases the number of vertices / faces
- The mesh vertices converges to a limit surface
- Each subdivision scheme has:
  - Rules to calculate the locations of new vertices.
  - A method to generate the new net topology.

Surfaces

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