CHAPTER IV

Cylinder and Sphere Problems

This chapter serves to complete our stock of mathematical tools rather than to solve any new physical problems. A necessary part of the tools of a mathematical physicist are cylindrical and spherical harmonics. We shall develop these tools with the help of simple physical considerations rather than in an abstract mathematical manner. We shall connect spherical harmonics with *potential theory* (in which they first arose) and cylindrical harmonics with the *wave equation* and its simplest solution, the monochromatic wave.

§ 19. Bessel and Hankel Functions

We assume the time dependence in the wave equation (7.4) to be periodic, and write it conveniently in the form

$$e^{-i\omega t}, \qquad \omega = \text{circular frequency.} \tag{1}$$

We introduce

$$k = \frac{\omega}{c}, \qquad k = \text{wave number;} \tag{2}$$

and then write (7.4) for one and two dimensions:

$$\text{(3a)} \quad \frac{d^2u}{dx^2} + k^2 u = 0, \qquad \text{(3b)} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0.$$

Equation (3a) has the integrals

$$u = A\, e^{ikx} \quad \text{and} \quad u = B e^{-ikx}. \tag{4a}$$

Because of our choice of negative sign in (1) the first equation stands for a plane wave which progresses in the direction of the positive x-axis, the second for one which progresses in the direction of the negative x-axis. The fact that it is simpler to operate with a wave which progresses in the positive x-direction is the main reason for the choice of sign in (1). For the two-dimensional case (3b) we get

$$u = A\, e^{i(ax+by)}, \qquad a^2 + b^2 = k^2, \qquad \begin{cases} a = k\cos\alpha, \\ b = k\sin\alpha. \end{cases} \tag{4b}$$

Introducing the plane polar coordinates r, φ with

$$x = r\cos\varphi, \qquad y = r\sin\varphi,$$

we get from (4b)

$$u = A\, e^{ikr\cos(\varphi-\alpha)}. \tag{5}$$

Equation (5) represents a plane wave which progresses in the direction $\varphi = \alpha$; for $\alpha = 0$ it becomes (4a). From such solutions we can construct the general solution of (3b) by summation (integration) over with coefficients A which may depend on α.

Written in terms of r,φ equation (3b) reads

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + k^2 u = 0 \tag{6}$$

or if we set $\varrho = kr$,

$$\frac{\partial^2 u}{\partial \varrho^2} + \frac{1}{\varrho}\frac{\partial u}{\partial \varrho} + \frac{1}{\varrho^2}\frac{\partial^2 u}{\partial \varphi^2} + u = 0. \tag{6 a}$$

We seek the solutions of this equation which have the form

$$u = Z_n(\varrho)\, e^{in\varphi}, \tag{7}$$

For this purpose we set

$A = c_n e^{in\alpha}$, c_n being a constant independent of α,

and integrate with respect to α between suitable limits β and γ:

$$u = c_n \int_\beta^\gamma e^{i\varrho\cos(\varphi-\alpha)} e^{in\alpha} d\alpha. \tag{8}$$

Equation (8), unlike (5), does not represent *one* wave of direction α, but a *bundle* of waves with directions varying from $\alpha = \beta$ to $\alpha = \gamma$, which obviously satisfies the differential equation (6a). In order to bring (8) into the form (7) we write

$$\alpha = w + \varphi, \qquad w_0 = \beta - \varphi, \qquad w_1 = \gamma - \varphi, \tag{8 a}$$

Equation (8) then becomes

$$u = c_n \int_{w_0}^{w_1} e^{i\varrho\cos w} e^{inw} dw \cdot e^{in\varphi}. \tag{9}$$

The coefficient here of $e^{in\varphi}$ is a function of ϱ alone if, and only if, we remove the dependence of w_0 and w_1 on φ. This is done in (8a) by letting β and γ, and with them w_0, w_1, increase to infinity in some way. In order to accomplish this we first must investigate the convergence of the integral in (9) in the neighborhood of infinity (see Fig. 18). This is obviously a question of determining those regions of the complex w-plane in which the real part of the exponent $i\,\varrho \cos w$ of (9) *is negative.* We assume for the time being that ϱ is real and positive and set

$$w = p + i\,q, \text{ and hence } \operatorname{Re}\{i \cos w\} = \sinh q \sin p.$$

Hence for the upper half of the w-plane, $q > 0$, we have

$$\text{(10 a)} \qquad \sin p < 0, \qquad -\pi < p < 0 \mod 2\pi \text{ [1]}$$

and for the lower half of the w-plane, $q < 0$,

$$\text{(10 b)} \qquad \sin p > 0, \qquad 0 < p < \pi \mod 2\pi$$

Since conditions (10a,b) depend on only the real part p of w and not on q, we know that the regions in question are strips which are parallel to the imaginary axis. The regions for which the passage of w_0 and w_1 to infinity is permissible are shaded in Fig. 18.

If ϱ is not real and positive, say $= |\varrho| e^{i\Theta}$, then the above pattern is maintained and is only shifted by $\pm\,\Theta$ in the direction of the real axis, where the $+$ and $-$ signs are for the upper and lower half planes. In the convergence considerations of (10a,b) we merely have to replace $\sin p$ by $\sin(p \mp \Theta)$ (see the beginning of exercise IV.2).

For each choice of the limits w_0, w_1 which satisfies the stated conditions the coefficient of $e^{in\varphi}$ in (9) is a possible form of the *general cylinder function* $Z_n(\varrho)$ of (7). Substituting (7) in (6a) we see that the functions Z_n satisfy the differential equation:

$$\text{(11)} \qquad \frac{d^2 Z_n}{d\varrho^2} + \frac{1}{\varrho}\frac{dZ_n}{d\varrho} + \left(1 - \frac{n^2}{\varrho^2}\right) Z_n = 0.$$

A. The Bessel Function and its Integral Representation

Our first special choice is

$$\text{(12)} \qquad \begin{aligned} w_0 &= a + i\,\infty, \qquad -\pi < a < 0, \\ w_1 &= b + i\,\infty, \qquad \pi < b < 2\pi. \end{aligned}$$

The corresponding path of integration is denoted in Fig. 18 by w_0; the function obtained is called a *Bessel function* if the factor c_n in (9) is normalized by:

[1]Two numbers p and p' are said to be "congruent modulo 2π" (written $p \equiv p' \mod 2\pi$), if $p - p'$ is an integral multiple of 2π.

(13) $$c_n = \frac{1}{2\pi} e^{-in\pi/2}.$$

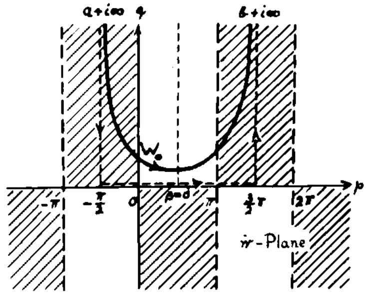


Fig. 18. Regions of the plane $w = p + iq$ in which the real part of $i\varrho \cos w$ is negative are shaded. The path of integration W_0 for the Bessel function I goes from $w_0 = a + i\infty$ to $w_1 = b + i\infty$. In addition to w we use the variable of integration $\beta = w - \pi/2$.

Using the common[2] notation I_n we obtain

(14) $$I_n(\varrho) = \frac{1}{2\pi} \int_{W_0} e^{i\varrho\cos w}\, e^{in(w-\pi/2)}\, dw.$$

The normalization (13) has been chosen so that $I_0(\varrho) = 1$ for $\varrho = 0$ and $I_n(\varrho)$ is real for arbitrary n and ϱ. The former follows from (14) if we pass to the rectangular form of W_0, which is depicted in Fig. 18 by the dotted path. We thereby cause the two partial integrals along the parts parallel to the imaginary axis (which are otherwise divergent for $\varrho \to 0$) to be complex conjugates. In order to prove this we make the substitution $w - \pi/2 = \beta$ The rectangular path W_0 is then, in terms of β,

$$-\pi + i\infty \to -\pi \to \pi \to \pi + i\infty,$$

which lies symmetric with respect to the β- axis. For real ϱ and n, $I_n(\varrho)$ decomposes into the two parts:

(15) $$I_n(\varrho) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{i(n\beta - \varrho\sin\beta)}\, d\beta - \frac{\sin n\pi}{\pi} \int_0^\infty e^{-(n\gamma + \varrho\sinh\gamma)}\, d\gamma,$$

where the second term is obtained from the integrals over the two paths $\pi \to \pi + i\infty$ and $-\pi + i\infty \to -\pi$ by the substitution $\beta = \pm\pi + i\gamma$; it does not vanish in general as it did for $n = 0$. Hence under the normalization of (14) $I_n(\varrho)$ is indeed real for real ϱ and n.

Since our integral representation converges for all values of ϱ it follows that $I_n(\varrho)$ is an *everywhere regular transcendental function* except for a single *essential singularity* at infinity and a *branch point* of order n at $\varrho = 0$ which for negative n is also a pole of the same order.

If n is an integer then the second term in (15) vanishes and we get

(16) $$I_n(\varrho) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{i(n\beta - \varrho\sin\beta)}\, d\beta,$$

[2] In the English literature one writes J_n instead of I_n and sets $I_n(\rho) = J_n(i\rho)$. We wish to reserve the letter J to denote "intensity" and we shall need no special symbol for $I_n(i\rho)$.

If we express the exponential function in terms of trigonometric functions and consider the odd and even character of the sine and cosine, then we get a representation which was given by Bessel:

$$(17) \qquad I_n(\varrho) = \frac{1}{\pi} \int_0^\pi \cos(\varrho \sin\beta - n\beta)\, d\beta .$$

We can see this directly from our original integral with respect to w. In the rectangular path W_0 the two parts which are parallel to the imaginary axis will cancel for integral values of n, and only the section of the real axis from $-\pi/2$ to $3\,\pi/2$ remains. Due to the periodicity of the integrand this can be replaced by the path from $-\pi$ to $+\pi$. We thus obtain

$$(18) \qquad I_n(\varrho) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{i\varrho\cos w}\, e^{in(w-\pi/2)} dw,$$

which agrees with (16). The integral over a complex path W_0 as in (14) has a great advantage over the real representations in that it is not limited to integral values of n but remains valid for arbitrary n. The integral (14) is first mentioned in Schläfli 1871,[3] though only with a rectangular path of integration. The following integrals (22) were first published by the author in 1896.

Since the differential equation (11) depends only on n^2 we know that if I_n is a solution then so is I_{-n}. The general solution can therefore be written in the form

$$(19) \qquad Z_n(\varrho) = c_1 I_n(\varrho) + c_2 I_{-n}(\varrho)$$

However this holds only for *non-integral* n. For *integral* n, I_n and I_{-n} are not linearly independent; we have rather

$$(19a) \qquad I_{-n}(\varrho) = (-1)^n I_n(\varrho).$$

This follows directly from (16) if in $I_{-n}(\varrho)$ we make the substitution $\beta = \pi - \beta'$.

B. The Hankel Function and its Integral Representation

As limits of integration in (9) we now choose

$$(20) \qquad \begin{aligned} w_0 &= a_1 + i\infty, & -\pi < a_1 < 0, \\ w_1 &= b_1 - i\infty, & 0 < b_1 < \pi; \end{aligned}$$

[3] For details see G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge 1922, p. 176 and 178.

and

$$(20\text{a}) \qquad \begin{aligned} w_0 &= a_2 - i\infty, \qquad & 0 < a_2 < \pi, \\ w_1 &= b_2 + i\infty, \qquad & \pi < b_2 < 2\pi. \end{aligned}$$

These paths, which are largely arbitrary and are restricted only asymptotically to the shaded regions, are denoted by W_1 and W_2 in Fig. 19. The fact that they have been drawn through the points $w = 0$ and $w = \pi$ is also arbitrary but will prove convenient later. The constant c_n is now determined by

$$(21) \qquad c_n = \frac{1}{\pi} e^{-in\pi/2}.$$

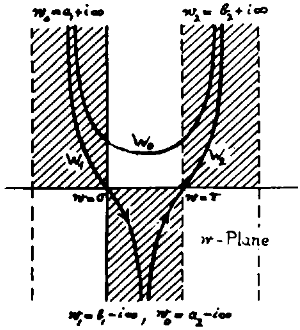


Fig. 19. The paths of integration W_1 and W_2 for H^1 and H^2. Combined in succession they are equivalent to the path W_0.

The cylindrical functions thus obtained are called the *first and second Hankel functions.*

$$(22) \qquad \begin{aligned} H_n^1(\varrho) &= \frac{1}{\pi} \int_{W_1} e^{i\varrho\cos w}\, e^{in(w-\pi/2)}\, dw, \\ H_n^2(\varrho) &= \frac{1}{\pi} \int_{W_2} e^{i\varrho\cos w}\, e^{in(w-\pi/2)}\, dw. \end{aligned}$$

They are almost more important to mathematical physicists than the Bessel functions I_n. They differ from the latter by the fact that they become infinite at $\varrho = 0$ even for positive n. This follows from the fact that the integral

$$\int_{W_1, W_2} e^{in(w-\pi/2)}\, dw$$

obtained from (22) by setting $\varrho = 0$, diverges in the infinite part of the lower half-plane.

The singularities of H^1 and H^2 at $\varrho = 0$ will be discussed in Section C. Due to their construction H^1 and H^2 are again solutions of the differential equation (11). The general integral of (11) can therefore be written in the form

$$(23) \qquad Z_n(\varrho) = C_1 H_n^1(\varrho) + C_2 H_n^2(\varrho)$$

We now want to show that the special integral I_n is obtained from this formula by setting

$$C_1 = C_2 = \frac{1}{2}$$

as is seen by looking at Fig. 19. If we traverse the paths W_1 and W_2 in succession then the lower parts cancel and the whole path contracts to W_0. Considering the new determination of c_n in (21) we have twice the amount obtained with the previous definition (13). Hence we indeed have

$$I_n(\varrho) = \frac{1}{2}\{H_n^1(\varrho) + H_n^2(\varrho)\}. \tag{24}$$

With this we compare the difference $H_n^1 - H_n^2$, which, written in terms of the variables of integration β and γ of (15), is purely imaginary for real ϱ and n. We denote this difference by $2iN_n$ and call $N_n(\varrho)$ Neumann's function:

$$N_n(\varrho) = \frac{1}{2i}\{H_n^1(\varrho) - H_n^2(\varrho)\}. \tag{25}$$

From (24) and (25) we have

$$\begin{aligned} H_n^1(\varrho) &= I_n(\varrho) + iN_n(\varrho), \\ H_n^2(\varrho) &= I_n(\varrho) - iN_n(\varrho). \end{aligned} \tag{26}$$

This decomposition of H^1, H^2 is completely analogous to the decomposition of the exponential function into its trigonometric components, as indicated in the following arrangement:

$$\begin{array}{cccc} e^{ix} & e^{-ix} & \cos x & \sin x \\ H^1(\varrho) & H^2(\varrho) & I(\varrho) & N(\varrho). \end{array}$$

We shall see in Section D that this is not only a qualitative analogy, but that asymptotically (for $\varrho \to \infty$) it holds quantitatively also. Just as we prefer the exponential imaginary representation to the trigonometric real one in descriptions of wave phenomena, so as a rule we prefer a representation in terms of Hankel functions to one in terms of Bessel and Neumann functions, especially since our complex integrals are equally convenient for all three.

For non-integral n the H^1, H^2 must be expressible in the form (19). In order to determine the coefficients c_1, c_2 we make the following observation: according to (14)

$$2\pi e^{in\pi/2} I_n(\varrho) = \int_{W_0} e^{i\varrho\cos w + inw}\, dw \tag{27}$$

and if we replace n by $-n$ and w by $-w$ (or W_0 by $-W_0$),

(28) $$-2\pi e^{-in\pi/2} I_{-n}(\varrho) = \int_{-W_0} e^{i\varrho\cos w + inw}\, dw.$$

In Fig. 20 we have W_0 and $-W_0$, drawn for convenience in rectangular shape, with their proper orientation. Their central parts from $w = -\pi/2$ to $w = +\pi/2$ cancel. There remain two rectangular paths, which, for convenience, we have deformed into paths of the type W_2 in Fig. 19. The right hand path from $\frac{\pi}{2} - i\infty$ to $\frac{3\pi}{2} + i\infty$ coincides with W_2 Let the left hand path from $-\frac{\pi}{2} + i\infty$ to $-\frac{3\pi}{2} - i\infty$ be denoted by W_2'. Adding (27) and (28) we obtain then

(29) $$2\pi\{e^{in\pi/2} I_n(\varrho) - e^{-in\pi/2} I_{-n}(\varrho)\} = \left(\int_{W_2} + \int_{W_2'}\right) e^{i\varrho\cos w + inw}\, dw.$$

Here according to (22) the integral over W_2 equals

(29a) $$\pi e^{in\pi/2} H_n^2(\varrho),$$

The integral over W_2' differs from this only in the orientation of the path and in its translation by -2π. This integral, according to (22), is then

(29b) $$-\pi e^{-3in\pi/2} H_n^2(\varrho).$$

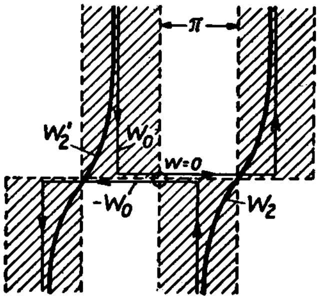


Fig. 20. The paths W_0 and $-W_0$ for I_n and I_{-n} are equivalent to the paths W_2 and W'_2 which belong to the type H^2_n.

Substituting (29a,b) in (29) we obtain

$$2[I_n(\varrho) - e^{-in\pi} I_{-n}(\varrho)] = (1 - e^{-2in\pi}) H_n^2(\varrho)$$

and hence

(30) $$H_n^2(\varrho) = \frac{e^{in\pi} I_n(\varrho) - I_{-n}(\varrho)}{i \sin n\pi}.$$

The corresponding representation for H^1 is obtained from (24):

(31) $$H_n^1(\varrho) = 2 I_n(\varrho) - H_n^2(\varrho) = \frac{e^{-in\pi} I_n(\varrho) - I_{-n}(\varrho)}{-i \sin n\pi}.$$

The coefficients c_1, c_2 for Hankel functions in equation (19) are thereby determined. We note that for real n and complex ϱ

(32) $$H_n^1(\varrho^*) = [H_n^2(\varrho)]^*, \qquad \text{hence} \qquad H_n^2(\varrho^*) = [H_n^1(\varrho)]^*.$$

Here the asterisk * stands as usual for the passage to the complex conjugate. In the derivation of (32) from (30) and (31) we use the relation $I_n(\varrho^*) = [I_n(\varrho)]^*$ which follows from (34). We further deduce from (30) and (31) that

$$H^1_{-n}(\varrho) = e^{in\pi}\, H^1_n(\varrho)\,, \quad H^2_{-n}(\varrho) = e^{-in\pi}\, H^2_n(\varrho)\,. \tag{32a}$$

and from (25), (30) and (31) that

$$N_n(\varrho) = \frac{\cos n\pi\, I_n(\varrho) - I_{-n}(\varrho)}{\sin n\pi}\,. \tag{33}$$

C. Series Expansion at the Origin

We have seen that $I_n(\varrho)$ is regular in the entire finite plane. It can therefore be expanded in ascending powers of ϱ. Indeed we see directly that the differential equation (11) is satisfied by the series:

$$I_n(\varrho) = \left(\frac{\varrho}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)} \left(\frac{\varrho}{2}\right)^{2m}. \tag{34}$$

For $n = 0$ it assumes the particularly elegant form

$$I_0(\varrho) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,m!} \left(\frac{\varrho}{2}\right)^{2m}. \tag{35}$$

which was known to Fourier. We shall demonstrate in exercise IV.1 that these series agree with the integral representation (14).

In order to obtain the series for the general cylinder function Z_n and to investigate the *singularity* at $\varrho = 0$, we proceed as in the case of ordinary linear differential equations: we write

$$Z_n = \varrho^\lambda\, (a_0 + a_1\, \varrho + a_2\, \varrho^2 + \cdots + a_k\, \varrho^k + \cdots) \tag{36}$$

and substitute this in the differential equation (11); the resulting power series must vanish term by term. The lowest power $\varrho^{\lambda-2}$ yields the determination of λ, the general term $\varrho^{\lambda+k-2}$ yields a recursion formula for a_k. We obtain

$$\lambda(\lambda-1) + \lambda - n^2 = 0, \qquad \lambda = \pm n, \tag{37}$$

and

$$\{(\lambda+k)\,(\lambda+k-1) + \lambda + k - n^2\}\, a_k + a_{k-2} = 0, \tag{37 a}$$

By the use of (37) equation (37a) can be simplified to

(37 b) $$(k^2 + 2k\lambda)a_k + a_{k-2} = 0.$$

By repeated application of this recursion formula we get for $k = 2m$

$$a_{2m} = \frac{-a_{2m-2}}{4m(m+\lambda)} = \frac{(-1)^2}{2^4}\frac{a_{2m-4}}{m(m-1)\,(m+\lambda)\,(m+\lambda-1)} = \cdots$$

If, as in (34), we choose $a_0 = 1/2^n\Gamma(\lambda+1)$ and set $a_1 = 0$ then we obtain

(38) $$a_{2m} = \frac{(-1)^m}{2^{2m+n}}\frac{1}{m!\,\Gamma(m+\lambda+1)}, \quad a_{2m+1} = 0.$$

This establishes the validity of (34). According to (37) it is equally valid for $\lambda = -n$ and for $\lambda = +n$. As mentioned on p. 88, equation (11) has the solution $I_{-n}(\varrho)$ in addition to $I_n(\varrho)$. If n has a positive real part the latter vanishes for $\varrho = 0$ with the same rapidity as ϱ^n, whereas the former becomes *infinite* with the same rapidity as ϱ^{-n}.

What we have said so far in Section C holds only when n is *non-integral.* For *integral* n, or more generally for the cases in which the difference of the two roots of (37) is integral,[4] we encounter in the solution belonging to the smaller λ a difficulty that is well known from the general theory of linear differential equations, namely, that in addition to powers with negative exponents we have *logarithmic terms.* We demonstrate this as follows.

Substituting in (37b) $\lambda = -n$ and $k = 2n$, we obtain $a_{2n-2} = 0$. Hence, tracing the recursion formula for a_{2n} backward, we see that in the series (36) for $Z_n = I_{-n}$ all the terms $a_k = a_{2n}$ vanish. This implies the relation (19a) previously established between I_n and I_{-n}.

The problem is to find a second solution of Bessel's differential equation (11) which is linearly independent of I_n. We do this by a limit consideration in which n is taken as a positive number which is arbitrarily close to an integer. Instead of applying this to the Hankel function H we apply it directly to the Neumann function N of equation (33), the decisive function for the singularity under discussion. Before passing to the limit N is given by (33); in the limit it becomes 0/0 due to equation (19a). The limiting value is determined according to De l'Hospital's rule. Denoting the integral limit of n by $\bar{n}$, we get for the denominator of (33)

$$\frac{\partial}{\partial n}\sin n\pi = \pi\cos n\pi = \pi(-1)^{\bar{n}}$$

[4] This is the case for Bessel functions in which n is half an integer. The fact that, in spite of this, the complications which are discussed in the text do not arise, will be explained in §21 C.

and for the numerator

$$-\pi \sin n\pi\, I_n(\varrho) + \cos n\pi \frac{\partial}{\partial n} I_n(\varrho) - \frac{\partial}{\partial n} I_{-n}(\varrho)$$
$$= (-1)^{\bar n} \left\{\frac{\partial}{\partial n} I_n(\varrho) - (-1)^n \frac{\partial}{\partial n} I_{-n}(\varrho)\right\}_{n\to\bar n}$$

hence

$$\pi N_{\bar n}(\varrho) = \operatorname*{Lim}_{n\to\bar n} \left\{\frac{\partial}{\partial n} I_n(\varrho) - (-1)^n \frac{\partial}{\partial n} I_{-n}(\varrho)\right\}. \tag{39}$$

Here the limit sign indicates that the differentiation with respect to n must be performed before the passage to the limit, i.e., for non-integral n. Since we are primarily interested in the neighborhood of $\varrho = 0$, we naturally use the series (34) which (for non-integral n) represents not only I_n but also I_{-n}. We compute the two parts of the right side of (39) separately.

Using the well known formula

$$\frac{d}{dx} a^x = a^x \log a$$

we obtain from the first term of the series (34)

$$\operatorname{Lim} \frac{\partial}{\partial n} I_n = \frac{1}{\Gamma(\bar n + 1)} \left(\frac{\varrho}{2}\right)^{\bar n} \left\{\log \frac{\varrho}{2} - \frac{\Gamma'(\bar n + 1)}{\Gamma(\bar n + 1)}\right\} + \cdots, \tag{40}$$

where the three dots indicate terms of higher order than $\varrho^{\bar n}$. We use the abbreviation introduced by Gauss

$$\Psi(z) = \frac{\Gamma'(z+1)}{\Gamma(z+1)} = -C + \sum_{\nu=1}^{\infty} \left(\frac{1}{\nu} - \frac{1}{z+\nu}\right) \tag{41}$$

where C is Euler's constant

$$C = \operatorname*{Lim}_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right) = 0.5772.$$

If we set

$$C = \log \gamma, \quad \gamma = 1.781, \tag{41a}$$

then using (41) and (41a) we can write for the term $\{\}$ in (40)

$$\log \frac{\varrho}{2} - \Psi(\bar n) = \log \frac{\gamma\varrho}{2} - \sum_{\nu=1}^{\bar n} \frac{1}{\nu}. \tag{41b}$$

The coefficient of the term { } in (40) is equal to $I_n(\varrho)$ except for higher powers than ϱ^n. Using (41b) we can rewrite (40) as

$$\text{(42)} \qquad \operatorname{Lim} \frac{\partial}{\partial n} I_n = \left\{\log \frac{\gamma \varrho}{2} - \sum_{\nu=1}^{\bar{n}} \frac{1}{\nu}\right\} I_{\bar{n}}(\varrho) \;\ldots$$

The three dots indicate that equation (42) is exact only up to terms of order $\varrho^{\bar{n}}$.

The computation of the second term on the right side of (39) is somewhat different. We start from

$$\text{(43)} \qquad \begin{aligned} I_{-n} = \left(\frac{\varrho}{2}\right)^{-n} &\left\{\frac{1}{\Gamma(-n+1)} - \frac{1}{1!\,\Gamma(-n+2)}\left(\frac{\varrho}{2}\right)^2 + \cdots \right. \\ &\left. + \frac{(-1)^{\bar{n}}}{\bar{n}!\,\Gamma(-n+\bar{n}+1)}\left(\frac{\varrho}{2}\right)^{2\bar{n}} + \cdots\right\}. \end{aligned}$$

By first differentiating only the term $(\varrho/2)^{-n}$ with respect to n we get as in (40)

$$-\log\frac{\varrho}{2}\left(\frac{\varrho}{2}\right)^{-n}\left\{\frac{1}{\Gamma(-n+1)} - \cdots + \frac{(-1)^{\bar{n}}}{\bar{n}!\,\Gamma(-n+\bar{n}+1)}\left(\frac{\varrho}{2}\right)^{2\bar{n}}\right\} + \cdots$$

For $n \to \bar{n}$, all the Γ's become infinite except the last. We have then:

$$\text{(44)} \qquad -\log\frac{\varrho}{2}\frac{(-1)^{\bar{n}}}{\bar{n}!}\left(\frac{\varrho}{2}\right)^{\bar{n}} = (-1)^{\bar{n}+1}\log\frac{\varrho}{2}\, I_{\bar{n}}(\varrho) + \cdots$$

On the other hand the differentiation of the term { } in (43) yields[5]

$$\text{(44a)} \quad \left(\frac{\varrho}{2}\right)^{-n}\left\{\frac{\Psi(-n)}{\Gamma(-n+1)} - \frac{\Psi(-n+1)}{1!\,\Gamma(-n+2)}\left(\frac{\varrho}{2}\right)^2 + \cdots + \frac{(-1)^{\bar{n}}\Psi(-n+\bar{n})}{\bar{n}!\,\Gamma(-n+\bar{n}+1)}\left(\frac{\varrho}{2}\right)^{2\bar{n}}\right\} + \cdots$$

The function $\Psi(z)$ has simple poles at the points $z = -1, -2, -3, \ldots$ just like $\Gamma(z+1)$. According to (41) we have in the neighborhood of the ν-th pole

$$\text{(45)} \qquad \Psi(z) = -\frac{1}{z+\nu}.$$

The development of $\Gamma(z+1)$ at the same point is[6]

$$\text{(45a)} \qquad \Gamma(z+1) = \frac{(-1)^{\nu-1}}{(\nu-1)!}\frac{1}{z+\nu}.$$

[5] In (44a) two minus signs have cancelled each other. Namely for $z = -n+1, -n+2, \ldots$ we have

$$\frac{d}{dn}\frac{1}{\Gamma(z)} = -\frac{\Gamma'(z)}{[\Gamma(z)]^2}\frac{dz}{dn} = (-1)^2\frac{\Gamma'(z)}{[\Gamma(z)]^2} = \frac{\Psi(z-1)}{\Gamma(z)}.$$

[6] For this and the previous formulas see Jahnke-Emde's tables of functions, 3rd ed., Teubner, Leipzig, 1938, p. 10, 11, and 18.

Hence

$$\frac{\Psi(z)}{\Gamma(z+1)} = (-1)^{\nu}\,(\nu-1)\,! \quad \text{for} \quad z = -\nu. \tag{45b}$$

Since $\Gamma(1) = 1$ and $\Psi(0) = -C$ we have, in the neighborhood of $z = 0$

$$\frac{\Psi(z)}{\Gamma(z+1)} = -C = -\log\gamma. \tag{45c}$$

After these preparations we can pass to the limit in (44a). According to (45) and (45a) all the terms Ψ/Γ, with the exception of the last, have the form ∞/∞ which according to (45b) can be replaced by $(-1)^{\nu}$ $(\nu-1)!$ where $\nu = \bar{n}$ in the first term, $\nu = \bar{n}-1$ in the subsequent terms. For the last term we apply (45c) and obtain

$$-\frac{(-1)^{\bar{n}}}{\bar{n}\,!}\log\gamma\left(\frac{\varrho}{2}\right)^{2\bar{n}} = (-1)^{\bar{n}}\left(\frac{\varrho}{2}\right)^{\bar{n}}\log\gamma\, I_{\bar{n}}(\varrho) + \cdots$$

We have as the limiting value of (44a) (instead of $\bar{n}$ we now write n, which is still an integer)

$$(-1)^n\left\{(n-1)!\left(\frac{\varrho}{2}\right)^{-n} + \frac{(n-2)\,!}{1\,!}\left(\frac{\varrho}{2}\right)^{-n+2} + \cdots - \log\gamma\, I_n\right\} + \cdots. \tag{46}$$

The sum of (46) and (44) now yields the second term in $\{\}$ of (39)

$$-(-1)^n \operatorname{Lim}\frac{\partial}{\partial n} I_{-n} = -(n-1)!\left(\frac{\varrho}{2}\right)^{-n} - \frac{(n-2)\,!}{1\,!}\left(\frac{\varrho}{2}\right)^{-n+2}$$
$$-\cdots + \log\frac{\gamma\varrho}{2} I_n + \cdots.$$

Combining this with (42) we obtain in (39) for $n > 0$

$$\begin{aligned}\pi N_n(\varrho) &= -(n-1)\,!\left(\frac{\varrho}{2}\right)^{-n} - \frac{(n-2)\,!}{1\,!}\left(\frac{\varrho}{2}\right)^{-n+2}\\ &\quad -\cdots + 2\log\frac{\gamma\varrho}{2} I_n - \sum_{\nu=1}^{n}\frac{1}{\nu} + \cdots.\end{aligned} \tag{47}$$

The terms on the right are written in decreasing order, the term with $(\varrho/2)^{-n}$ having highest order and the logarithmic term having lowest order. This implies a simple logarithmic singularity for $n = 0$; we have then:

$$\frac{\pi}{2} N_0(\varrho) = \log\frac{\gamma\varrho}{2} I_0 + \cdots, \tag{48}$$

or the complete form, which we state without proof

$$\frac{\pi}{2} N_0\,(\varrho) = \log\frac{\gamma\varrho}{2} I_0\,(\varrho) + 2\left(I_2\,(\varrho) - \frac{1}{2} I_4\,(\varrho) + \frac{1}{3} I_6\,(\varrho) - \cdots\right). \tag{48a}$$

According to (26) this logarithmic singularity arises in H just as it does in N. We see from this that the H_n have branch points at the origin of the complex ϱ-plane even for integral n. From (26) and (47) we see that upon continuation around the origin H_n increases by $\mp 4I_n(\varrho)$ (for details see exercise IV.2). In exercise IV.3 we shall deduce the existence of the logarithmic singularity of H_0 in a more direct, though mathematically less satisfactory, way.

D. Recursion Formulas

The $Z_n(\varrho)$ satisfy a *differential equation* in ϱ and a *difference equation* in n, for arbitrary, not necessarily integral, n. We can deduce this from our integral representation for the H and hence for arbitrary linear combinations of the H, in particular for the I and N.

Remembering that the paths of integration W_1 and W_2 in (22) are independent of n we form:

(49) $$\frac{\pi}{2}(H_{n+1} + H_{n-1}) = \int e^{i\varrho\cos w}\, e^{in(w-\pi/2)}\{\ \}_1\, dw\,,$$

(50) $$\frac{\pi}{2}(H_{n+1} - H_{n-1}) = \int e^{i\varrho\cos w}\, e^{in(w-\pi/2)}\{\ \}_2\, dw\,.$$

where

$$\{\ \}_1 = \frac{1}{2}\left(e^{i(w-\pi/2)} + e^{-i(w-\pi/2)}\right) = \sin w\,,$$

$$\{\ \}_2 = \frac{1}{2}\left(e^{i(w-\pi/2)} - e^{-i(w-\pi/2)}\right) = -i\cos w\,.$$

We may therefore write for the integrals on the right of (49), (50)

(49a) $$-\frac{1}{i\varrho}\int \frac{\partial}{\partial w}\left(e^{i\varrho\cos w}\right)\cdot e^{in(w-\pi/2)}\, dw\,,$$

(50a) $$-\frac{\partial}{\partial\varrho}\int e^{i\varrho\cos w}\cdot e^{in(w-\pi/2)}\, dw\,;$$

and by integration by parts (49a) becomes

(49b) $$\frac{n}{\varrho}\int e^{i\varrho\cos w}\cdot e^{in(w-\pi/2)}\, dw\,.$$

We now can express the right sides of (49) and (50) in terms of Hankel functions of index n. These formulas are valid for both H^1 and H^2 depending on the path of integration; we may write them directly for the general cylinder function Z, which is a linear combination of the two. We have

(51) $$Z_{n+1} + Z_{n-1} = \frac{2n}{\varrho} Z_n\,,$$

and

$$Z_{n+1} - Z_{n-1} = -2\frac{dZ_n}{d\varrho}. \tag{52}$$

These are the recursion formulas we were seeking. They hold for n integral or non-integral, positive or negative.

For $n = 0$ we get as a special case

$$Z_{-1} = -Z_{+1} \tag{51a}$$

and

$$Z_1 = -\frac{dZ_0}{d\varrho}, \tag{52a}$$

and by further specialization of (52a) we get the relation

$$I_1(\varrho) = -\frac{d}{d\varrho} I_0(\varrho). \tag{52b}$$

which could also have been obtained directly from the series (27) and (27a).

E. Asymptotic Representation of the Hankel Functions

The integrand in our representations (14) and (22) oscillates more and more rapidly with increasing ϱ, for the non-shaded regions of the w-plane with increasing amplitude, for the shaded region with amplitude decreasing to zero. As shown in Fig. 19, the paths W_1 and W_2 for H^1 and H^2 can be drawn completely in the shaded regions for real ϱ. The figures illustrating exercise IV.2 show that this is no longer the case for complex ϱ. We also see from Fig. 19 that the points $w = 0$ and $w = \pi$, at which the paths touch two non-shaded regions, will play an important role for the asymptotic computation of H^1 and H^2.

We shall develop here the *method of saddle points* in an intuitive, so to speak topographical, manner, and leave all analytic refinements and generalizations for §21. We assume

$$\varrho \text{ real} \gg 1 \qquad \text{and} \qquad n \ll \varrho. \tag{53}$$

For H^1 the path W_1 begins and ends in the shaded "low lands," and the same holds for H^2 and W_2. The deciding exponent has its extremum at

$$\sin w = 0, \qquad w = \begin{cases} 0 & \text{on } W_1 \\ \pi & \text{on } W_2. \end{cases}$$

This extremum, like all extrema of real or imaginary parts of complex functions, is not a maximum or a minimum but a saddle point. To the right and left of W_1 and W_2 at these points there tower steeply rising mountain ranges. Between them run W_1 and W_2 as *mountain passes.* The saddle point method is therefore also called the *pass method.* The altitude of the paths at $w = 0$ and $w = \pi$ is

$$|e^{i\varrho}| = 1 \qquad \text{and} \qquad |e^{-i\varrho}| = 1 .$$

What path should a mountain climber take in order to surmount the pass in the fastest possible manner? The answer is, the path of steepest ascent and descent, the so-called "drop lines." However this prescription is not binding and it may be amended for reasons of convenience (analytic reasons[7] or mountain climber's reasons). The English name "method of steepest descent" instead of pass method is therefore not entirely appropriate.

We consider a short segment of the path W_1 in the neighborhood of the crest of the path: let ds denote the arc element on this path with the orientation W_1, and let the crest itself be given by $s = 0$. We write:

$$\text{(54)} \qquad w = s\, e^{i\gamma}, \quad i \cos w = i\left(1 - \frac{s^2}{2} e^{2i\gamma}\right) = \frac{s^2}{2}\sin 2\gamma + i\left(1 - \frac{s^2}{2}\cos 2\gamma\right).$$

The level lines of the real part are perpendicular to both the level lines of the imaginary part and to the drop lines, therefore the level line of the imaginary part is at the same time the drop line of the real part which determines the altitude of the pass. In our case the level line of the imaginary part of (54) is given by

$$1 - \frac{s^2}{2}\cos 2\gamma = \text{const.}$$

with the constant equal to one, since the line must pass through the crest $s = 0$. Hence we have

$$\text{(54a)} \qquad \cos 2\gamma = 0, \qquad \gamma = \mp\frac{\pi}{4}.$$

For H^1 we must choose the minus sign for γ (see Fig. 19) whereby (54) becomes

$$\text{(54b)} \qquad w = e^{-i\pi/4}\, s, \qquad dw = e^{-i\pi/4}\, ds, \qquad i\cos w = i - \frac{s^2}{2}.$$

We substitute this in (22) and at the same time set $s = 0$ in the "slowly varying" factor $\exp\{in(w - \pi/2)\}$; the integration can obviously

[7] G. Faber, Bayr. Akad. 1922, p. 285.

be restricted to the immediate neighborhood of the pass, say to distances $< \varepsilon$. We obtain

$$H_n^1(\varrho) = \frac{1}{\pi} e^{i[\varrho - (n+\frac{1}{2})\pi/2]} \int_{-\varepsilon}^{+\varepsilon} e^{-\frac{\varrho}{2}s^2} ds. \tag{54c}$$

This integral can be reduced to the Laplace integral with the help of the substitution $s = \sqrt{\frac{2}{\varrho}}\, t$, which, at the limits of integration, becomes $\sqrt{\frac{\varrho}{2}}\varepsilon \to \infty$ and $-\sqrt{\frac{\varrho}{2}}\varepsilon \to -\infty$. We therefore have the final result:

$$H_n^1(\varrho) = \sqrt{\frac{2}{\varrho\pi}}\, e^{i[\varrho - (n+\frac{1}{2})\pi/2]}. \tag{55}$$

For H^2, where we have to use the path W_2 with the saddle point at $w = \pi$ and where in (54a) we have to choose the plus sign for γ, we obtain correspondingly

$$H_n^2(\varrho) = \sqrt{\frac{2}{\varrho\pi}}\, e^{-i[\varrho - (n+\frac{1}{2})\pi/2]}. \tag{56}$$

By taking half the sum of (55) and (56) we get

$$I_n(\varrho) = \sqrt{\frac{2}{\varrho\pi}} \cos\left[\varrho - \left(n + \frac{1}{2}\right)\frac{\pi}{2}\right]. \tag{57}$$

These asymptotic representations, though derived for real ϱ, can be continued analytically in the complex ϱ-plane (for the representation of the two H's this plane must be cut along a suitable half line because of the branching discussed at the end of Section C). On the basis of equations (55) and (56) we state: *H^1 vanishes asymptotically for* $\operatorname{Im} \varrho \to +\infty$, *$H^2$ for* $\operatorname{Im} \varrho \to -\infty$. This is the reason for the particular suitability of Hankel functions for the treatment of problems of damped oscillations. On the other hand both the Bessel function I and the Neumann function N become asymptotically infinite in both half planes.

We shall show in §21 how our asymptotic limits can be extended into asymptotic expansions and how the condition $n \ll \varrho$ of (53) can be dropped. The factor $\varrho^{-\frac{1}{2}}$ in (55) and (56) is connected with the fact that $H^1{}_0$ (or, for another choice of time dependence, $H^2{}_0$) represents, upon introduction of a coordinate z which is perpendicular to the r, φ-plane, an *expanding cylindrical wave* with source $r = 0$. Since the energy $2\pi r\, |H_0|^2$ passing through a cylinder of radius r must be independent of r (in the absence of absorption), we see that H_0 is proportional to $r^{-\frac{1}{2}}$ or in other words to $\varrho^{-\frac{1}{2}}$.

One may think of the real and imaginary parts of $H^1{}_0$ and I_0 as defining surfaces over the complex ϱ-plane. The surface of Re $\{H_0^1\}$ osculates the positive ϱ- half-plane exponentially and in the negative half plane it has exponentially rising mountain ranges separated by correspondingly deepening valleys. The surface of Im (H_0^1) behaves similarly and in addition has a narrow funnel at the origin which corresponds to the logarithmic singularity of H_0 (also of N_0: see (48)), as well as a discontinuity along the negative real axis corresponding to the branching discussed above. The The surface of $\mathrm{Re}(I_0)$consists of a mildly undulating depression flanked on both sides by rising mountain country. The undulating nature of the depression follows from the asymptotic equation (57) and indicates the existence of an infinity of roots of the equation $I_0 = 0$ along the real axis. These roots are represented in Fig. 21. The surface of Im (I_0) is very similar in appearance except that the bottom of the depression is level throughout, corresponding to the fact that I_0 is real along the real axis.

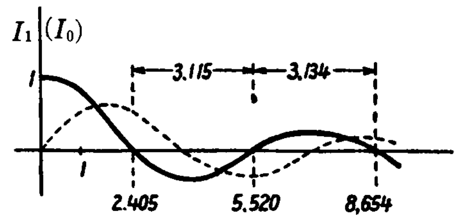


Fig. 21. Representation of I_0 (heavy line) and of I_1 (dotted line) along the real axis. The first three roots of $I_0(\varrho) = 0$.

§ 20. Heat Equalization in a Cylinder

As an excellent example for the application of the theory of Bessel functions we again consider a special problem of heat conduction. The problem was treated by Fourier, who, in fact, mentioned the functions with integral n whence they are sometimes referred to as *Fourier-Bessel functions.*

We shall treat our problem in three steps:

A. For an infinitely long cylinder and an axially symmetric initial state $f = f(r)$.

B. For an initial state which depends also on the argument $f = f(r, \varphi)$.

C. For a cylinder of finite length and general initial state $f = f(r, \varphi, z)$.

The boundary condition shall, for the sake of simplicity, always be that of isothermy

$$(1) \qquad u = 0 \qquad \text{for } r = a = \text{radius of cylinder.}$$

For the complete cylinder this is augmented by the further "boundary condition" of finality along the axis:

$$u \neq \infty \qquad \text{for} \quad r = 0. \tag{1a}$$

A. One-Dimensional Case $f = f(r)$

The equation of heat conduction is:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{k}\frac{\partial u}{\partial t}. \tag{2}$$

Making the special substitution

$$u = R(r)\, e^{-\lambda^2 k t} \tag{3}$$

we get the differential equation for R

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \lambda^2 R = 0. \tag{3a}$$

This is Bessel's differential equation (19.11) with $n = 0$ and $\varrho = \lambda r$. Its general solution can be written as:

$$Z_0 = A\, I_0(\lambda r) + B N_0(\lambda r).$$

However, the condition of finality (1a) requires that we set $B = 0$; because of (1) we must further demand that

$$I_0(\lambda a) = 0. \tag{4}$$

We already know that this equation has an infinity of roots such that the distance between consecutive roots approaches π; from (19.57) we get for the m-th root

$$\lambda_m a - \frac{\pi}{4} = \left(m - \frac{1}{2}\right)\pi, \quad \text{hence} \quad \lambda_m a = \left(m - \frac{1}{4}\right)\pi. \tag{4a}$$

This approximation is valid down to $m = 2$ with an accuracy of about 1%; for $m = 1$ we get

$$\lambda_1 a = 2{,}40 \tag{4b}$$

as compared to 2.36 from (4a) (see Fig. 21).

We have then at our disposal an infinity of solutions of (3a):

$$R(r) = A_m I_0(\lambda_m r), \qquad m = 1, 2, \ldots .$$

Correspondingly we get from (3) as the general solution of our problem

(5) $$u = \sum_{m=1}^{\infty} A_m I_0 (\lambda_m r)\, e^{-\lambda_m^2 k t}.$$

We now merely need to satisfy the initial condition

(6) $$f(r) = \sum_{m=1}^{\infty} A_m I_0 (\lambda_m r)\,.$$

A way of doing this is indicated by the treatment of the anharmonic sine series in §16. In order to emphasize the complete analogy with the equations (5) and (6) of §16 we write

$$u_n = I_0 (\lambda_n r), \qquad u_m = I_0 (\lambda_m r)$$

and then write our present equation (3a) in the form:

$$\frac{d}{dr}\left(r \frac{du_n}{dr}\right) + \lambda_n^2\, r\, u_n = 0, \qquad \frac{d}{dr}\left(r \frac{du_m}{dr}\right) + \lambda_m^2\, r\, u_m = 0.$$

Multiplying by u_m and u_n and subtracting we get as an analogue to (16.5a)

(7) $$u_m \frac{d}{dr}\left(r \frac{du_n}{dr}\right) - u_n \frac{d}{dr}\left(r \frac{du_m}{dr}\right) = (\lambda_m^2 - \lambda_n^2)\, r\, u_m\, u_n.$$

Integrating over the fundamental domain $0 < r < a$ we get as an analogue to (16.6)

(7a) $$(\lambda_m^2 - \lambda_n^2) \int_0^a r\, u_m\, u_n\, dr = r\left(u_m \frac{du_n}{dr} - u_n \frac{du_m}{dr}\right)\Big|_0^a .$$

This is *Green's theorem* applied to the two-dimensional circular region $r = a$.

The right side of (7a) vanishes for the upper limit $r = a$ on account of equation (1), for the lower limit $r = 0$ on account of the factor r and equation (1a). Since $\lambda_m \neq \lambda_n$, for $m \neq n$, we have the *orthogonality condition:*

(8) $$\int_0^a u_m\, u_n\, r\, dr = 0 \quad \text{for } m \neq n.$$

The "weighting factor" r is due to the two-dimensional element of area $r\, dr\, d\varphi$ in Green's theorem.

From (7a) we may also deduce the normalizing integral

$$N_m = \int u_m^2\, r\, dr$$

if we drop the assumption that λ_n is a root of (4). We consider λ_n

rather as a continuous variable which in the limit coincides with λ_m. Equation (7a) then represents N_m as a fraction which for $\lambda_n \to \lambda_m$ assumes the form 0/0. By differentiating the numerator and denominator with respect to λ_n and substituting $r = a$ and $r = 0$ we find, because of (1), that

$$N_m = \frac{a}{2\lambda_n}\left(\frac{du_n}{d\lambda_n}\frac{du_m}{dr}\right)_{r=a} \to \frac{a}{2\lambda_m}\left(\frac{du_m}{d\lambda_m}\frac{du_m}{dr}\right)_{r=a}. \tag{9}$$

But for $r = a$

$$\frac{du_m}{d\lambda_m} = a\, I_0'(\lambda_m a), \quad \frac{du_m}{dr} = \lambda_m I_0'(\lambda_m a).$$

Substituting this in (9) we get

$$N_m = \frac{a^2}{2}[I_0'(\lambda_m a)]^2. \tag{9a}$$

The coefficients A_m of the series (6) can now be calculated from (8) and (9) in the Fourier manner:

$$A_m N_m = \int_0^a f(r)\, I_0(\lambda_m r)\, r\, dr, \tag{10}$$

We substitute this in the series (5), thereby completing the solution of problem A.

B. Two-Dimensional Case $f = f(r, \varphi)$

We first develop $f(r, \varphi)$ in the complex Fourier series (1.12)

$$f(r,\varphi) = \sum_{n=-\infty}^{n=+\infty} C_n e^{in\varphi}, \quad C_n = C_n(r) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(r,\varphi)\, e^{-in\varphi}\, d\varphi. \tag{11}$$

Due to the two-dimensional equation (2)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = \frac{1}{k}\frac{\partial u}{\partial t} \tag{12}$$

and the generalized substitution (3)

$$u = R_n(r)\, e^{in\varphi}\, e^{-\lambda^2 kt} \tag{13}$$

we have the differential equation for $R_n(r)$

$$\frac{d^2 R_n}{dr^2} + \frac{1}{r}\frac{dR_n}{dr} + \left(\lambda^2 - \frac{n^2}{r^2}\right) R_n = 0. \tag{14}$$

This is Bessel's differential equation (19.11) with $\varrho = \lambda r$. Equation (1a) requires that the only permissible solutions be of the form $A_n I_n(\lambda r)$. On account of (1) λ must satisfy the equation $I_n(\lambda a) = 0$ which, just like $I_0(\lambda a) = 0$, has an infinity of roots:

$$\lambda_{n,1},\ \lambda_{n,2}, \ldots, \lambda_{n,m}, \ldots .$$

Each of these roots yields a particular solution of the form (13):

$$u_{n,m} = A_{n,m} I_n(\lambda_{n,m} r)\, e^{in\varphi}\, e^{-\lambda_{n,m} kt}, \tag{15}$$

and these solutions satisfy the differential equation (12). Through superposition we can construct from them the general solution of (14) which at the same time satisfies our boundary conditions:

$$u = \Sigma \Sigma u_{n,m} = \sum_{n=-\infty}^{+\infty} \sum_{m=1}^{\infty} A_{n,m} I_n(\lambda_{n,m} r)\, e^{in\varphi}\, e^{-\lambda^2_{n,m} kt}. \tag{16}$$

Here the constants $A_{n,m}$ must be chosen so that for $t = 0$ and every integer $-\infty < n < +\infty$ we have the equation

$$C_n(r) = \sum_{m=1}^{\infty} A_{n,m} I_n(\lambda_{n,m} r) \tag{17}$$

where according to (11) the left side is a known function of r. Equation (17) necessitates the development of this function in Bessel functions I_n. This is possible due to the *orthogonality* of the latter, which follows from Bessel's differential equation (14) and Green's theorem as in (7) and (7a)[8]. Using the abbreviations

$$v_m = I_n(\lambda_{n,m} r), \qquad v_l = I_n(\lambda_{n,l} r)$$

we obtain as generalization of (7)

$$(\lambda^2_{n,m} - \lambda^2_{n,l}) \int_0^a r v_m v_l\, dr = r \left(v_m \frac{dv_l}{dr} - v_l \frac{dv_m}{dr} \right) \Bigg|_0^a$$

Here, too, the right side vanishes. We thus have for $l \neq m$

$$\int_0^a v_m v_l\, r\, dr = 0. \tag{18}$$

At the same time we obtain by a passage to the limit as described in (9)

[8] In order to avoid the trivial result $0 = 0$, in the application of Green's theorem to the circle $r = a$ in the r, φ- plane we must use the two functions

$$v_{nm} = I_n(\lambda_{n,m} r)\, e^{+in\varphi} \quad \text{and} \quad v_{nl} = I_n(\lambda_{n,l} r)\, e^{-in\varphi}$$

$$N_{n,m} = \int_0^a v_m^2 r\, dr = \frac{a^2}{2}[I_n'(\lambda_{n,m} a)]^2. \tag{19}$$

The A_{nm} in (17) can now be calculated in the Fourier manner from the given $C_n(r)$ by (18) and (19) in analogy to (10). Substituting these expressions for $C_n(r)$ in (11) we obtain

$$2\pi N_{n,m} A_{n,m} = \int_0^a \int_{-\pi}^{+\pi} f(r,\varphi)\, I_n(\lambda_{n,m} r)\, e^{-in\varphi}\, r\, dr\, d\varphi\,. \tag{20}$$

which concludes the solution of (16).

C. Three-Dimensional Case $f = f(r, \varphi, z)$

Let the cylinder have the finite length h and let $0 < z < h$. We first develop $f(r,\varphi,z)$ in a Fourier series with respect to z, which, due to the boundary conditions $u = 0$ for $z = 0$ and $z = h$, becomes a pure sine series:

$$f(r,\varphi,z) = \sum_{\mu=1}^{\infty} B_\mu \sin \mu\pi \frac{z}{h}\,, \qquad B_\mu = \frac{2}{h}\int_0^h f(r,\varphi,z) \sin \mu\pi \frac{z}{h}\, dz\,; \tag{21}$$

We then develop $B_\mu = B_\mu(r,\varphi)$ in a series of $\exp(in\varphi)$:

$$B_\mu(r,\varphi) = \sum_{n=-\infty}^{+\infty} C_{\mu,n}\, e^{in\varphi}\,, \qquad C_{\mu,n} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} B_\mu\,(r,\varphi)\, e^{-in\varphi}\, d\varphi\,. \tag{22}$$

Finally we represent $C_{\mu,n} = C_{\mu,n}(r)$ as a series in the Bessel functions $I_n(\lambda r)$, which progresses according to the roots of

$$I_n(\lambda a) = 0, \qquad \lambda = \lambda_{n,m}\,, \qquad m = 1, 2, \ldots$$

Due to the three-dimensional equation of heat conduction

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

the time factor has the form

$$e^{-\alpha^2 kt} \qquad \text{with} \quad \alpha^2 = \lambda_{n,m}^2 + \left(\frac{\mu\pi}{h}\right)^2.$$

The complete solution is given by the triply infinite sum

$$(23)\quad u = \sum_{\mu=1}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} A_{\mu n m} I_n(\lambda_{n,m} r) e^{i n \varphi} \sin \mu \pi \frac{z}{h} e^{-\left[\lambda_{n,m}^2 + \left(\frac{\mu \pi}{h}\right)^2\right] k t}.$$

The coefficients A are calculated from the $N_{n,m}$ of (19) in analogy to (10) and (20)

$$(24)\quad \pi h N_{n,m} A_{\mu,n,m} = \int_0^a \int_{-\pi}^{+\pi} \int_0^h f(r, \varphi, z) I_n(\lambda_{n,m} r) e^{-i n \varphi} \sin \mu \pi \frac{z}{h} r\, dr\, d\varphi\, dz.$$

This completes the solution of (23).

In the case of a *hollow cylinder* the condition of finality (1a) is dropped. Hence in the expansion of the solution there may appear terms with N_n in addition to those with I_n (or, in other words, terms in $H^1{}_n$ and $H^2{}_n$). Heat conduction through a heating pipe is an example of this.

§ 21. More About Bessel Functions

A. Generating Function and Addition Theorems

In §19 we started from the two-dimensional wave equation $\Delta u + k^2 u = 0$ and its simplest solution, the *plane wave*

$$(1)\quad u = e^{i k x} = e^{i \varrho \cos \varphi}, \qquad \varrho = k r, \qquad k = \text{the wave number}.$$

If we develop this into a Fourier series then, due to the origin of Bessel's differential equation (19.11), the coefficients must be Bessel functions, and because of the regularity of (1) for $r = 0$ only the I functions will appear. Hence, we set the coefficient of $\exp(i n \varphi)$ equal to $c_n I_n$ and according to (1.12) we then have

$$c_n I_n(\varrho) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{i \varrho \cos \varphi} e^{-i n \varphi} d\varphi.$$

If we compare this with (19.18), in which we may replace w by $-w$, we obtain

$$c_n = e^{i n \pi/2}.$$

Hence we have the Fourier series

(2) $$e^{i\varrho\cos\varphi} = \sum_{n=-\infty}^{+\infty} e^{in\pi/2}\, I_n(\varrho)\, e^{in\varphi}$$

or upon the substitution $\psi = \varphi + \pi/2$

(2 a) $$e^{i\varrho\sin\psi} = \sum_{n=-\infty}^{+\infty} I_n(\varrho)\, e^{in\psi}.$$

In the older literature (2) is usually written in the less symmetric form

(2 b) $$e^{i\varrho\cos\varphi} = I_0(\varrho) + 2\sum_{n=1}^{\infty} i^n I_n(\varrho)\cos n\varphi.$$

The left sides of (2) and (2a) are called *generating functions of the Bessel functions with integral index.*

We now pass from the case of a plane wave to that of a *cylindrical wave* with its logarithmic source at the origin, which, according to p. 100, is represented by $H_0(\varrho)$. (We omit the upper index since the following is valid for both functions H, i.e., both for radiated and for absorbed waves.) We now shift the origin from $\varrho = 0$ to $\varrho = \varrho_0$, $\varphi = \varphi_0$ whereby $H_0(\varrho)$ goes over into

$$H_0(R), \qquad R = \sqrt{\varrho^2 + \varrho_0^2 - 2\,\varrho\,\varrho_0\cos(\varphi - \varphi_0)}.$$

If we develop this into a Fourier series with respect to $\varphi - \varphi_0$, then the coefficients must again be cylinder functions, namely functions $H_n(\varrho)$ for $\varrho < \varrho_0$ and functions $I_n(\varrho)$ for $\varrho > \varrho_0$. The latter follows from the fact that $\varrho = 0$ is now a regular point, the former from the fact that each term in the series must have the same type of radiation or absorption for $\varrho \to \infty$ as $H_0(R)$ itself. For reasons of symmetry the same consideration holds for the dependence of our coefficients on the variable ϱ_0, except that the functions I_n and H_n are interchanged since the condition $\varrho \gtrless \varrho_0$ is the same as $\varrho_0 \lessgtr \varrho$. Hence the n-th Fourier coefficient must be

$$c_n \begin{cases} I_n(\varrho_0)\, H_n(\varrho) & \text{for } \varrho > \varrho_0, \\ H_n(\varrho_0)\, I_n(\varrho) & \text{for } \varrho < \varrho_0. \end{cases}$$

The numerical factor c_n is independent of ϱ and ϱ_0 and is the same for both expansions, since the two series must go into each other continuously for $\varrho = \varrho_0$ (unless at the same time $\varphi = \varphi_0$, in which case both series diverge); it turns out to be equal to 1 if, in the case $\varrho < \varrho_0$, we pass to the limiting case of a plane wave $\varrho_0 \to \infty$, setting

$\varphi_0 = \pi$ and comparing the resulting asymptotic formula with equation (2). We thus obtain the *addition theorem:*

$$(3) \qquad H_0(R) = \begin{cases} \sum_{n=-\infty}^{+\infty} I_n(\varrho_0)\, H_n(\varrho)\, e^{in(\varphi-\varphi_0)} & \varrho > \varrho_0, \\ \sum_{n=-\infty}^{+\infty} H_n(\varrho_0)\, I_n(\varrho)\, e^{in(\varphi-\varphi_0)} & \varrho < \varrho_0. \end{cases}$$

If we consider this written for *both* Hankel functions and take half the sum then we obtain the *addition theorem for Bessel functions:*

$$(3a) \qquad I_0(R) = \sum_{n=-\infty}^{+\infty} I_n(\varrho_0)\, I_n(\varrho)\, e^{in(\varphi-\varphi_0)} \qquad \varrho \gtrless \varrho_0.$$

In the same manner we get from half the difference the addition theorem for Neumann functions, where we again have to distinguish between the cases $\varrho > \varrho_0$ and $\varrho < \varrho_0$.

Concerning (3) we note that the series in I_n corresponds to *Taylor's series* in the theory of complex functions, whereas the series in H_n corresponds to *Laurent's series.*[9] This is illustrated by the following example, in which one may replace z and z_0 by $\varrho e^{i\varphi}$ and $\varrho_0 e^{i\varphi_0}$:

$$(4) \qquad \begin{aligned} \frac{z}{z - z_0} &= \sum_{n=0}^{\infty} z_0^n z^{-n} & |z| > |z_0|, \\ \frac{z_0}{z_0 - z} &= \sum_{n=0}^{\infty} z_0^{-n} z^n & |z| < |z_0|. \end{aligned}$$

In §24 we shall develop corresponding addition theorems for spherical waves in space; there will also be a counterpart to the representation (2) of a plane wave.

B. Integral Representations in Terms of Bessel Functions

We shall give here the development of a given function $f(r)$ in terms of Bessel functions which is analogous to a representation by a Fourier integral. According to (12.11b) a function of two variables can be represented by the Fourier integral

$$(5) \qquad f(x,y) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} d\omega\, d\omega' \iint_{-\infty}^{+\infty} d\xi\, d\eta\, f(\xi,\eta)\, e^{i\omega(x-\xi)+i\omega'(y-\eta)}.$$

[9] This is further discussed, together with questions of convergence, in §2 of the author's work which was cited on p. 80. We also refer to the great work of H. Weber, *Math. Ann.* **I**, p. 1, which was a fitting beginning for that journal. It is the problem of adapting the methods of Riemann's dissertation, i.e., of adapting the theory of the differential equation $\Delta u = 0$, to the differential equation $\Delta u + k^2 u = 0$ . .

We introduce the polar coordinates:

$$\begin{aligned} x &= r\cos\varphi & \xi &= \varrho\cos\psi & \omega &= \sigma\cos\alpha \\ y &= r\sin\varphi & \eta &= \varrho\sin\psi & \omega' &= \sigma\sin\alpha \\ & & d\xi\,d\eta &= \varrho\,d\varrho\,d\psi & d\omega\,d\omega' &= \sigma\,d\sigma\,d\alpha . \end{aligned}$$

We assume the special angular dependence of $f(x,y)$:

$$f(x, y) = f(r)\, e^{in\varphi} \qquad (n \text{ integer}). \tag{6}$$

By using the relations

$$\begin{aligned} \omega\, x + \omega'\, y &= \sigma\, r\cos(\alpha - \varphi) \\ \omega\,\xi + \omega'\,\eta &= \sigma\,\varrho\cos(\psi - \alpha) \end{aligned}$$

we can transform (5) into

$$f(r)\, e^{in\varphi} = \int_0^\infty \sigma\, d\sigma \int_0^\infty f(\varrho)\,\varrho\, d\varrho\, \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{i\sigma r\cos(\alpha-\varphi)}\, d\alpha\, \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{in\psi}\, e^{-i\sigma\varrho\cos(\psi-\alpha)}\, d\psi . \tag{7}$$

In order to compare these integrals with respect to α and ψ to the representation (19.18)

$$\begin{aligned} I_n(z) &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{iz\cos\beta}\, e^{in(\beta-\pi/2)}\, d\beta \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-iz\cos\beta'}\, e^{in(\beta'+\pi/2)}\, d\beta' \qquad (\beta' = \beta - \pi) \end{aligned}$$

we multiply under the integrals of (7) by

$$e^{in(\alpha-\pi/2)} \text{ and } e^{in(-\alpha+\pi/2)},$$

and divide through by $e^{in\varphi}$. We thus obtain the simple representation

$$f(r) = \int_0^\infty \sigma\, d\sigma \int_0^\infty f(\varrho)\, I_n(\sigma r)\, I_n(\sigma\varrho)\,\varrho\, d\varrho . \tag{8}$$

In analogy to the form (4.13) of the Fourier integral theorem we can write this relation in the symmetric form:

$$\begin{aligned} f(r) &= \int_0^\infty \sigma\, d\sigma\, \varphi(\sigma)\, I_n(\sigma r), \\ \varphi(\sigma) &= \int_0^\infty \varrho\, d\varrho\, f(\varrho)\, I_n(\sigma\varrho) . \end{aligned} \tag{8a}$$

The transition from rectangular coordinates to polar coordinates in the passage from (5) to (7) requires certain conditions about the behavior of f at infinity which we do not discuss here. Equation (7) will be useful in the treatment of spherical waves in §24.

We obtain a further application of equation (8) if we let $f(r)$ degenerate to a δ-function, namely,[10] let

$$(9)\quad f(r) = \delta(r|s) = \begin{Bmatrix} 0 & \text{for } r \neq s \\ \infty & \text{for } r = s \end{Bmatrix}, \qquad \text{with} \int_{s-\varepsilon}^{s+\varepsilon} f(r)\, r\, dr = 1.$$

We then obtain from (8)

$$(9a)\qquad \int_0^\infty I_n(\sigma r)\, I_n(\sigma s)\, \sigma\, d\sigma \doteq \delta(r|s)\,.$$

This equation represents the *orthogonality* of the two functions I_n at the points r and s of the continuous domain $0 < {r \atop s} < \infty$; it is a counterpart to (20.18) in which we deal with two points m and l of the discrete λ-sequence. We shall return to the important relation (9a) in §36.

C. The Indices $n + 1/2$ and $n \pm 1/3$

Substituting $n = 1/2$ in (19.34) we get $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}, \Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\sqrt{\pi}, \ldots$

$$(10)\quad \begin{aligned} I_{\frac12}(\varrho) &= \sqrt{\frac{\varrho}{2}}\left[\frac{2}{\sqrt{\pi}} - \frac{2\cdot 2}{1\cdot 3\sqrt{\pi}}\left(\frac{\varrho}{2}\right)^2 + \frac{2\cdot 2\cdot 2}{2!\,1\cdot 3\cdot 5\cdot\sqrt{\pi}}\left(\frac{\varrho}{2}\right)^4 - \cdots\right] \\ &= \sqrt{\frac{2\varrho}{\pi}}\left(1 - \frac{\varrho^2}{3!} + \frac{\varrho^4}{5!} - \cdots\right) = \sqrt{\frac{2\varrho}{\pi}}\,\frac{\sin\varrho}{\varrho}\,; \end{aligned}$$

in the same manner we find for $n = -1/2$

$$(10a)\qquad I_{-\frac12}(\varrho) = \sqrt{\frac{2\varrho}{\pi}}\,\frac{\cos\varrho}{\varrho}\,.$$

We write generally

$$(11)\qquad I_{n+\frac12}(\varrho) = \sqrt{\frac{2\varrho}{\pi}}\,\psi_n(\varrho), \qquad \text{in particular } \psi_0 = \frac{\sin\varrho}{\varrho}.$$

From Bessel's differential equation (19.11) we get the differential equation for ψ_n

[10] We draw the reader's attention to the weighting factor r in the integral of (9). Because of this factor we no longer have $\int \delta(r|s)\, ds = 1$, but instead $\int \delta(r|s)\, r dr = 1$, as in equation (9).

(11a) $$\frac{1}{\varrho}\frac{d^2(\varrho\psi_n)}{d\varrho^2}+\left(1-\frac{n(n+1)}{\varrho^2}\right)\psi_n=0.$$

We shall meet this equation again in the theory of spherical harmonics. We now wish to show that the solutions which are finite for $\varrho=0$ can be obtained from ψ_0 by the following rule:

(12) $$\psi_n=(-\varrho)^n\left(\frac{d}{\varrho\,d\varrho}\right)^n\psi_0.$$

We start from the series (19.34). Let p be an arbitrary index (in our case we have $p = 1/2$) then we have

(13) $$\frac{I_p(\varrho)}{(\varrho/2)^p}=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(p+m+1)}\,\frac{(\varrho^2/2)^m}{2^m}.$$

We differentiate this equation m times with respect to $\varrho^2/2$. Then the right side becomes

$$\sum_{m=n}^{\infty}\frac{(-1)^m}{(m-n)!\,\Gamma(p+m+1)}\,\frac{(\varrho^2/2)^{m-n}}{2^m}.$$

By introducing a new index of summation ($\mu=m-n$, $m=n+\mu$) we get

(13a) $$\frac{(-1)^n}{2^n}\sum_{\mu=0}^{\infty}\frac{(-1)^\mu}{\mu!}\,\frac{(\varrho^2/4)^\mu}{\Gamma(\mu+n+p+1)}=\frac{(-1)^n}{2^n}\,\frac{I_{n+p}(\varrho)}{(\varrho/2)^{n+p}}.$$

At the same time the left side of (13) becomes (because of $d(\varrho^2/2)=\varrho\,d\varrho$)

(13b) $$\left(\frac{d}{\varrho\,d\varrho}\right)^n\frac{I_p(\varrho)}{(\varrho/2)^p}.$$

Comparing (13a) and (13b) we get

(13c) $$\frac{I_{n+p}(\varrho)}{\varrho^p}=(-\varrho)^n\left(\frac{d}{\varrho\,d\varrho}\right)^n\frac{I_p(\varrho)}{\varrho^p};$$

due to (10); this coincides with (12) for $p = 1/2$.

If instead of the *ratio* of $I_p(\varrho)$ and $(\varrho/2)^p$ we differentiate their *product* n times with respect to $\varrho^2/2$, then instead of (13c) we get

(13d) $$I_{-n+p}(\varrho)\cdot\varrho^p=\varrho^n\left(\frac{d}{\varrho\,d\varrho}\right)^n\{I_p(\varrho)\cdot\varrho^p\}.$$

If we again set $p = 1/2$ and apply definition (11) we get as the complement of (12)

$$\psi_{-n}(\varrho) = \varrho^{n-1}\left(\frac{d}{\varrho\, d\varrho}\right)^n \{\varrho\,\psi_0\}\,. \tag{13e}$$

From (12) and (13e) we deduce the *recursion formulas* for $\psi_{\pm n}$

$$\begin{aligned} \psi_{n+1} &= -\frac{d\psi_n}{d\varrho} + \frac{n}{\varrho}\,\psi_n\,,\\ \psi_{-n-1} &= \frac{d\psi_{-n}}{d\varrho} - \frac{n-1}{\varrho}\,\psi_{-n}\,. \end{aligned} \tag{14}$$

The corresponding formulas for $Z_{\pm n}$ were discussed in §19 D.

According to (14) we get successively from $\psi_0 = \sin\varrho/\varrho$:

$$\begin{aligned} &\psi_1 = \frac{\sin\varrho - \varrho\cos\varrho}{\varrho^2}\,, \qquad \psi_2 = \frac{3(\sin\varrho - \varrho\cos\varrho) - \varrho^2\sin\varrho}{\varrho^3}\,, \ldots\\ &\psi_{-1} = \frac{\cos\varrho}{\varrho}\,, \qquad \psi_{-2} = -\frac{\cos\varrho + \varrho\sin\varrho}{\varrho^2}\,,\\ &\psi_{-3} = \frac{3(\cos\varrho + \varrho\sin\varrho) - \varrho^2\cos\varrho}{\varrho^3}\,, \ldots \end{aligned} \tag{14a}$$

We see from this that for integral n all $\psi_{\pm n}$ can be expressed in an *elementary form with the help of the sine and cosine functions.* This representation confirms the non-logarithmic character of the half-index Bessel functions, as stated in footnote 4 of this chapter.

The "Hankel functions" ζ_n which correspond to the ψ_n are given by the equation

$$H_{n+\frac12}(\varrho) = \sqrt{\frac{2\varrho}{\pi}}\,\zeta_n(\varrho) \tag{15}$$

analogous to (11) (the upper indices 1 and 2 on both sides have been omitted). We are particularly interested in the functions ζ_0. We obtain them from the functions $H_{\frac12}$, which we get from (19.31), (19.30), and (11):

$$H^1_{\frac12} = \frac{-i\,I_{\frac12} - I_{-\frac12}}{-i} = \sqrt{\frac{2\varrho}{\pi}}\,\frac{e^{i\varrho}}{i\varrho}\,, \qquad H^2_{\frac12} = \frac{i\,I_{\frac12} - I_{-\frac12}}{i} = \sqrt{\frac{2\varrho}{\pi}}\,\frac{ie^{-i\varrho}}{\varrho}\,.$$

Hence according to (15) we have

$$\zeta^1_0 = \frac{e^{i\varrho}}{i\varrho}\,, \qquad \zeta^2_0 = \frac{e^{-i\varrho}}{-i\varrho}\,. \tag{15a}$$

Concerning the notation we make the following remark: our notation coincides with the original definition of the ψ_n in Heine's *Handbook of Spherical Harmonics* and with that used by the author in Frank-Mises. However it differs by a factor ϱ from that of other authors[11] who instead

[11] P. Debye, *Ann. Physik* **30** (1909), B. van der Pol and H. Bremmer, *Phil. Mag.* **24** (1937). Further references in G. N. Watson, Theory of Bessel Functions, p. **56**.

of (11) write:

(16) $$I_{n+\frac{1}{2}}(\varrho) = \sqrt{\frac{2}{\pi\varrho}}\,\psi_n(\varrho),$$

which is sometimes convenient.

In analogy to the equations (14a) for the ψ_n we can express the ζ_n in *elementary form with the help of* $\exp(\pm i\varrho)$. This states the fact, which will be derived in Section D, that Hankel's asymptotic expansions break off in the case of half-indices.

The differential equation for $Z_{\pm\frac{1}{3}}(kr)$ assumes an unexpectedly elegant form if we replace the independent and the dependent variable[12] by

(17) $$k\,r = \frac{2}{3}\varrho^{\frac{3}{2}}, \qquad Z_{\pm\frac{1}{3}}(k\,r) = \varrho^{-\frac{1}{2}}\,\Phi(\varrho).$$

The functions ψ_n in (16) are denoted by S_n in acoustical engineering; the corresponding C_n would, in our terminology, have to be called a "Neumann function," since it is proportional to $\zeta_n^1 - \zeta_n^2$.

[12] The direct computation would lead to lengthy transformations. We avoid them, and at the same time recognize the generalizability of the relations (17) to (20), if we start from the conformal mapping

$$x + i\,y = f(\xi + i\eta), \qquad \Delta_{xy} = \frac{1}{|f'(\xi + i\eta)|^2}\Delta_{\xi\eta},$$

through which

(1) $$\Delta_{xy}\,u + k^2 u = 0$$

goes over into

(2) $$\Delta_{\xi\eta}\;v + k^2\,|f'(\xi + i\,\eta)|^2\,v = 0;$$

where $u(x, y) \equiv v(\xi, \eta)$ (see equation (23.17) below). If in (2) we set f proportional to a power of $\xi + i\,\eta$, e.g.,

$$k\,f(\xi + i\,\eta) = \frac{2}{\mu}(\xi + i\eta)^{\mu/2}, \qquad \begin{cases} x + i\,y = r\,e^{i\varphi} \\ \xi + i\,\eta = \varrho\,e^{i\psi} \end{cases}$$

then we get

(3) $$k\,|x + i\,y| = k\,r = \frac{2}{\mu}\,|\xi + i\,\eta|^{\mu/2} = \frac{2}{\mu}\varrho^{\mu/2}, \qquad \varphi = \frac{\mu}{2}\psi,$$

and $$k\,|f'(\xi + i\,\eta)| = \varrho^{\mu/2}/\varrho.$$

The solution of (1)

$$u = I_{\mp 1/\mu}(k\,r)\;e^{i\,\varphi/\mu}$$

then goes over into the solution of (2)

(4) $$v = I_{\mp 1/\mu}\left(\frac{2}{\mu}\,\varrho^{\mu/2}\right)e^{i\,\psi/2};$$

It then becomes

$$\Phi''(\varrho) + \varrho\,\Phi(\varrho) = 0 \,. \tag{18}$$

If we write its solutions as a series with undetermined coefficients starting with ϱ^0 and ϱ^1, we get

$$\begin{aligned} \Phi_0 &= 1 - 1\cdot\frac{\varrho^3}{3!} + 1\cdot 4\cdot\frac{\varrho^6}{6!} - 1\cdot 4\cdot 7\cdot\frac{\varrho^9}{9!} + \cdots, \\ \Phi_1 &= \varrho - 2\cdot\frac{\varrho^4}{4!} + 2\cdot 5\cdot\frac{\varrho^7}{7!} - 2\cdot 5\cdot 8\cdot\frac{\varrho^{10}}{10!} + \cdots. \end{aligned} \tag{19}$$

Considering (19.34) we see that the first is proportional to $I_{-\frac{1}{3}}$, the second to $I_{+\frac{1}{3}}$, namely, that we have

$$\begin{aligned} \Phi_0(\varrho) &= 3^{-\frac{1}{3}}\,\Gamma\Big(1-\frac{1}{3}\Big)\varrho^{\frac{1}{2}}\, I_{-\frac{1}{3}}\Big(\frac{2}{3}\,\varrho^{\frac{3}{2}}\Big), \\ \Phi_1(\varrho) &= 3^{+\frac{1}{3}}\,\Gamma\Big(1+\frac{1}{3}\Big)\varrho^{\frac{1}{2}}\, I_{+\frac{1}{3}}\Big(\frac{2}{3}\,\varrho^{\frac{3}{2}}\Big). \end{aligned} \tag{20}$$

We shall meet the functions $I_{\pm\frac{1}{3}}$ again at the end of Section D.

and therefore

$$\frac{1}{\varrho}\frac{\partial}{\partial\varrho}\Big(\varrho\,\frac{\partial v}{\partial\varrho}\Big) + \frac{1}{\varrho^2}\frac{\partial^2 v}{\partial\psi^2} + \varrho^{\mu-2}\, v = 0\,.$$

Substituting $v = \varrho^{-\frac{1}{2}}\,\Phi(\varrho)\,e^{i\psi/2}$, here we get:

$$\frac{d^2\Phi}{d\varrho^2} + \varrho^{\mu-2}\,\Phi = 0, \tag{5}$$

with the easily verifiable series representations for its solutions:

$$\begin{aligned} \Phi_0 &= 1 - \frac{\varrho^{\mu}}{\mu(\mu-1)} + \frac{\varrho^{2\mu}}{\mu(\mu-1)\,2\mu(2\mu-1)} - \cdots, \\ \Phi_1 &= \varrho - \frac{\varrho^{\mu+1}}{(\mu+1)\mu} + \frac{\varrho^{2\mu+1}}{(\mu+1)\,\mu(2\mu+1)\,2\mu} - \cdots. \end{aligned} \tag{6}$$

We can relate these solutions to the solutions (4) for v; namely we have:

$$\left.\begin{matrix}\Phi_0\\ \Phi_1\end{matrix}\right\} = C_{\mp}\,\varrho^{\frac{1}{2}}\, I_{\mp 1/\mu}\Big(\frac{2}{\mu}\,\varrho^{\mu/2}\Big). \tag{7}$$

where $C_{\mp}$ are constant factors. Substituting the power series for I from (19.34), and comparing with (6) we get

$$C_- = \mu^{-1/\mu}\,\Gamma\Big(1-\frac{1}{\mu}\Big), \qquad C_+ = \mu^{1/\mu}\,\Gamma\Big(1+\frac{1}{\mu}\Big). \tag{7a}$$

For $\mu = 3$ the equations (5), (6), (7), (7a) go over into the equations (18), (19), (20) of the text.

For $\mu = 2$ equation (5) reduces to the differential equation for the trigonometric functions, and Φ_0, Φ_1 become $\cos\varrho$, $\sin\varrho$. For $\mu = 1$ our series representation breaks down, since then $\varrho = 0$ is a singular point of the differential equation (5).

D. Generalization of the Saddle-Point Method According to Debye

Although in later applications we shall in general apply only the asymptotic limiting value of the Bessel functions as determined at the end of §19, we wish to discuss here certain more general expansions due to Hankel, which progress according to negative powers of ϱ and in which the first term is the above mentioned asymptotic limiting value. Actually these series are *divergent*, being developments at an essential singularity, but they are frequently called *semi-convergent*. The first terms decrease rapidly, but from a certain term on they increase to infinity. We obviously must break off at that term in order to obtain approximation formulas.

The shortest way of obtaining these series is from the differential equation for the Hankel function, by substituting formal power series, and then computing the coefficients by setting the factor of each power equal to zero (this is obviously not completely rigorous). Considering (19.55) and (19.56) we write

$$H_n^{1,2}(\varrho) = \sqrt{\frac{2}{\pi\varrho}}\, e^{\pm i(\varrho-(n+\frac{1}{2})\pi/2)} \left(a_0 + \frac{a_1}{\varrho} + \frac{a_2}{\varrho^2} + \cdots + \frac{a_m}{\varrho^m} + \cdots\right) \tag{22}$$

and after dividing out the factor $\sqrt{2/\pi}\,\exp\{\pm i(\varrho - (n+\frac{1}{2})\,\pi/2)\}$ from the differential equation (19.11) we find the terms with $\varrho^{-m-\frac{3}{2}}$ to be

$$\begin{array}{lll}
-a_{m+1} & \mp 2i(m+\frac{1}{2})\,a_m & +(m+\frac{1}{2})(m-\frac{1}{2})\,a_{m-1}, \\
 & \pm i\,a_m & -(m-\frac{1}{2})\,a_{m-1}, \\
+a_{m+1} & & -n^2\,a_{m-1}.
\end{array}$$

where the consecutive rows correspond to the consecutive terms

$$\frac{d^2Z}{d\varrho^2}, \qquad \frac{1}{\varrho}\frac{dZ}{d\varrho}, \qquad \left(1-\frac{n^2}{\varrho^2}\right)Z$$

in (19.11). Summing the three rows we get the following *first order recursion formula*

$$\mp 2\,i\,m\,a_m = (n^2 - \{m-\tfrac{1}{2}\}^2)\,a_{m-1}. \tag{23}$$

Setting $a_0 = 1$ we get

$$\frac{a_1}{\varrho} = \frac{4n^2-1}{2^3(\mp 2i\varrho)}, \qquad \frac{a_2}{\varrho^2} = \frac{(4n^2-1)(4n^2-9)}{2^4\,2!\,(\mp 2i\varrho)^2}, \ldots \tag{24}$$

Using the symbol

$$(n, m) = \frac{(4n^2-1)(4n^2-9)\cdots(4n^2-\{2m-1\}^2)}{2^{2m}\, m!}, \quad (n, 0) = 1 \tag{25}$$

which was introduced by Hankel, we get the general formula:

$$\frac{a_m}{\varrho^m} = \frac{(n, m)}{(\mp 2i\varrho)^m}. \tag{26}$$

The series in (22) for H^1 and H^2 then assume the final form:

$$H_n^1(\varrho) = \sqrt{\frac{2}{\pi\varrho}}\, e^{i(\varrho-(n+\frac{1}{2})\pi/2)} \sum_{m=0,1,2,\ldots} \frac{(n, m)}{(-2i\varrho)^m}, \tag{27}$$

$$H_n^2(\varrho) = \sqrt{\frac{2}{\pi\varrho}}\, e^{-i(\varrho-(n+\frac{1}{2})\pi/2)} \sum_{m=0,1,2,\ldots} \frac{(n, m)}{(+2i\varrho)^m}. \tag{28}$$

Taking half their sum we get:

$$\begin{aligned} I_n = {} & \sqrt{\frac{2}{\pi\varrho}} \cos\left(\varrho-(n+\tfrac{1}{2})\pi/2\right) \sum_{m=0,2,4,\ldots} (-1)^{\frac{m}{2}} \frac{(n, m)}{(2\varrho)^m} \\ & - \sqrt{\frac{2}{\pi\varrho}} \sin\left(\varrho-(n+\tfrac{1}{2})\pi/2\right) \sum_{m=1,3,5,\ldots} (-1)^{\frac{m-1}{2}} \frac{(n, m)}{(2\varrho)^m}. \end{aligned} \tag{29}$$

In exercise IV.5 we shall apply a similar method in order to determine the leading terms of the series (27), (28) (which here were borrowed from the saddle-point method) from Bessel's differential equation with large ϱ. This method does not include the normalizing factor which remains undetermined by the differential equation.

Extensive mathematical investigations about the domain of validity of such asymptotic series exist, starting with a great work of Poincaré,[13] which we cannot discuss here. Exceptions to the divergence are the series with half-integer index $n = \nu + \frac{1}{2}$, which, according to the definition of the symbol (m,n) break off with the ν-th term and represent the Bessel function in question *exactly*. We then obtain the elementary expressions for ζ_n, ψ_n which were given in Section C.

Our considerations so far are essentially restricted by the condition $n \ll \varrho$; they fail if n becomes infinite with ϱ. The latter is the case in all optical problems which are on the border line between geometrical optics (optics of very short wavelengths) and wave optics. It was in connection with the investigation of a problem of this type, namely that of the rainbow (radius of water droplet approximately equal to

[13] *Acta Math.* **8**, 1886.

wavelength of light) that Debye[14] discovered his fundamental generalization of Hankel's asymptotic series. In order to understand its origin we first have to generalize the saddle-point method.

The exponent

$$f(w) = i\,[\varrho \cos w + n\,(w - \pi/2)] \tag{30}$$

in our representation (19.22) of the Hankel functions now depends on two large numbers ϱ and n. For convenience we take ϱ and n to be real and positive. Depending on whether n is smaller or larger than ϱ we set

(30 a) $\quad n = \varrho \cos\alpha \quad$ or $\quad$ (30b) $\quad n = \varrho \cosh\alpha;$

in addition we use the variable of integration

$$\beta = w - \pi/2. \tag{30 c}$$

as in Fig. 18.

a) *For* $n < \varrho$ we have

$$f(w) = F(\beta) = -\,i\,\varrho\,(\sin\beta - \beta\cos\alpha). \tag{31}$$

The saddle point $F'(\beta) = 0$ is given by

$$\cos\beta - \cos\alpha = 0;$$

it lies at $\beta_0 = \mp\alpha$, for H^1 and H^2 respectively. This corresponds to the previous values for the saddle points $w_0 = 0,\ \pi$, which by (30c) go into $\beta_0 = \mp\pi/2$. From (31) we get

$$F''(\beta_0) = \mp\, i\,\varrho \sin\alpha$$

which yields as the expansion of $F(\beta)$ up to the quadratic term

$$F(\beta) = \pm\, i\,\varrho\,(\sin\alpha - \alpha\cos\alpha) \mp \frac{1}{2} i\,\varrho\sin\alpha\,(\beta - \beta_0)^2. \tag{31 a}$$

Instead of β we introduce the arc length s measured from the saddle point $\beta_0 = \mp\alpha$ and set $(\beta-\beta_0)^2 = (\beta\pm\alpha)^2 = \mp\, i s^2,\ d\beta = e^{\mp i\pi/4}\,ds,$ Concerning the $\mp$- sign in the last equation we refer the reader to the discussion in (19.54b). Integrating over a neighborhood of the saddle point we get

$$H_n^{1,2}(\varrho) = \frac{1}{\pi}\int e^{F(\beta)}\,d\beta = e^{\pm i\varrho(\sin\alpha - \alpha\cos\alpha)}\,\frac{1}{\pi}\int_{-s}^{+s} e^{-\frac{\varrho}{2}\sin\alpha\cdot s^2}\,e^{\mp\frac{i\pi}{4}}\,ds. \tag{31 b}$$

[14] *Math. Ann.* **67**, 1909 and *Bayr. Akad.* 1910.

This again can be reduced to the Laplace integral. We have:

$$H_n^{1,2}(\varrho) = \sqrt{\frac{2}{\pi\,\varrho \sin\alpha}}\, e^{\pm i\varrho(\sin\alpha - \alpha\cos\alpha) \mp i\pi/4}. \tag{32}$$

In the limit $\alpha \to \pi/2$ our form (32) goes into the previous representation (19.55), (19.56).

b) The same calculation holds in the *case* $n > \varrho$ if in (30b) we replace $\cos\alpha$ by $\cosh\alpha$ and hence (31) by

$$F(\beta) = -i\,\varrho\,(\sin\beta - \beta\cosh\alpha)$$

That one of the two saddle points $\beta_0 = \pm i\alpha$, which yields the dominant term, is the one with greater altitude, namely $\beta_0 = -i\alpha$. At this point $F''(\beta_0) = \varrho \sinh\alpha$. Instead of (32) we now get

$$H_n^{1,2}(\varrho) = \sqrt{\frac{2}{\pi\,\varrho \sinh\alpha}}\, e^{\varrho(\alpha\cosh\alpha - \sinh\alpha) \mp i\pi/2}. \tag{33}$$

From these limiting values (32), (33) Debye deduced series developments of the Hankel type, which we may omit here.

c) The only remaining case is the transition case $n \sim \varrho$ in which according to (30a,b) we have $\alpha \sim 0$ and hence the representations (32), (33) fail on account of the denominators $\sqrt{\sin\alpha}$ and $\sqrt{\sinh\alpha}$. This indicates that now $F''(\beta_0)$, also approaches zero and that only the third term of the Taylor series for $F(\beta)$ is appreciably different from zero. We therefore need a better approximation in the neighborhood of the saddle point. This was carried out by Watson,[15] who instead of (31a) used an expansion which goes up to the third order in $(\beta - \beta_0)$. The Laplace integrals of the Airy type (see end of this section) which arise there can also be computed rigorously. We thus find:
in the case $n \leq \varrho$,

$$H_n^{1,2}(\varrho) = \frac{\tan\alpha}{\sqrt{3}}\, e^{\pm in(\tan\alpha - \frac{1}{3}\tan^3\alpha - \alpha) \pm i\pi/6}\, H_{\frac{1}{3}}^{1,2}\left(\frac{1}{3}\, n \tan^3\alpha\right); \tag{34}$$

in the case $n > \varrho$ (where $n = \varrho \cosh\alpha$ as in (30b)),

$$H_n^{1,2}(\varrho) = \frac{\tanh\alpha}{\sqrt{3}}\, e^{-n(\tanh\alpha + \frac{1}{3}\tanh^3\alpha - \alpha) \mp 2i\pi/3}\, H_{\frac{1}{3}}^{2,1}\left(\pm\frac{in}{3}\tanh^3\alpha\right). \tag{35}$$

Taken together the Watson formulas (34) and (35) cover the entire

[15] Chap. VIII, and in particular p. 252 of his Theory of Bessel Functions, Cambridge 1922.

asymptotic range of the Bessel-Hankel functions including the border case $n \sim \varrho$, we are now treating. In this case we are in the neighborhood of $\alpha = 0$. Hence we may replace $H_{\frac{1}{3}}$ by its limiting value for small arguments, which according to (19.31) and (19.30), is (since we may neglect $I_{\frac{1}{3}}$ as compared to $I_{-\frac{1}{3}}$)

$$H_{\frac{1}{3}}^{1,2}(z) = \mp \frac{i}{\sin \pi/3} I_{-\frac{1}{3}}(z) .$$

Since in (34) we have $z = \frac{1}{3}n \tan^3 \alpha$ we get $I_{-\frac{1}{3}}$ proportional to $1/\tan \alpha$, which cancels with the factor $\tan \alpha$ on the right side of (34); hence after the necessary contractions we get from (34)

$$H_\varrho^{1,2}(\varrho) = \frac{2}{\Gamma(\frac{2}{3})} \left(\frac{2}{9\varrho}\right)^{\frac{1}{3}} e^{\mp i\pi/3} . \tag{36}$$

The same expression is obtained from (35). As the corresponding limiting value of I we get

$$I_\varrho(\varrho) = \frac{1}{\Gamma(\frac{2}{3})} \left(\frac{2}{9\varrho}\right)^{\frac{1}{3}} . \tag{37}$$

This coincides with the original results of Debye.

We also see that if n is not too near ϱ, equations (34), (35) coincide with the Debye formulas (32), (33). For, in this case we may substitute its Hankel limiting value (19.55,56) for the function $H_{\frac{1}{3}}$ (large argument and small index):

$$H_{\frac{1}{3}}^{1,2}\left(\frac{1}{3} n \tan^3\alpha\right) = \sqrt{\frac{2\cdot 3}{\pi n \tan^3\alpha}}\, e^{\pm \frac{in}{3}\tan^3\alpha \mp i(\frac{1}{3}+\frac{1}{2})\pi/2} ,$$

whereby (34) simplifies to

$$H_n^{1,2}(\varrho) = \sqrt{\frac{2}{\pi n \tan\alpha}}\, e^{\pm in(\tan\alpha - \alpha) \mp i\pi/4} .$$

Due to $n = \varrho \cos\alpha$ this coincides with (32). In the same way one shows that (35) and (33) coincide.

Finally we have to consider the problem of the roots of the equations $H_n^{1,2}(\varrho) = 0$ for large n and ϱ. However, while up to now we assumed n and ϱ to be real, we now have to admit arbitrary complex values for n; we still may assume ϱ to be real on account of its physical meaning $(\varrho = k\,r)$. Concerning the parameter α, whose sign is undetermined in (30a), we agree that its real part is to be positive.

It would seem from (32), (33) that no roots could exist even for complex n, since the *exponential function* vanishes for no finite value of the exponent. However these representations were obtained (see p. 119 under b) by considering only that one of the two saddle points which has the *greater* altitude. If they are of *equal altitude* and if the required path of integration (leading from depression to depression) can be made to lead over both passes, then as sum of the two exponential expressions we get a *trigonometric function*, which makes the existence of roots possible.

Using the notations of equations (30) to (32) we represent the saddle points by $\beta = \mp\alpha$ and the corresponding exponential functions which appear in H by

$$e^{\pm i\varrho(\sin\alpha - \alpha\cos\alpha) \mp i\pi/4}$$

The altitude of the passes is determined by the real part of the exponent. Equal altitudes therefore mean equal real parts of the exponents, and since ϱ was assumed real, this means equal imaginary parts of $\pm(\sin\alpha - \alpha\cos\alpha)$, or in other words,

$$\text{(38)} \qquad \operatorname{Im}(\sin\alpha - \alpha\cos\alpha) = 0.$$

For small α this yields:

$$\operatorname{Im}(\alpha^3) = 0,$$

This means that α lies on one of three curves that pass through the origin and intersect there at angles of $\pi/3$, one being the real α-axis. The real axis remains a solution of (38) even for infinite α, while the other branches are continued into curves that are mirror images with respect to the imaginary α-axis.

Considering the path of integration, described in Fig. 19 for the Hankel function, we see that the path of integration for H^1 can be taken meaningfully over the two saddle points only if they lie on the branch of (38) which leads from the second to the fourth quadrants, i.e., if α has a positive real part whenever it has a negative imaginary part; hence $n = \varrho\cos\alpha$ (with real ϱ) has a positive imaginary part. On the other hand the path of integration for H^2 must lead from the third to the first quadrant, so that α has a positive, and n a negative, imaginary part.

Superimposing the contributions of both saddle points according to (32) we get the following representation for H^1

$$\text{(39)} \qquad \begin{aligned} H_n^1(\varrho) &= \sqrt{\frac{2}{\pi\varrho\sin\alpha}}\left(e^{i\varrho(\sin\alpha - \alpha\cos\alpha) - i\pi/4} - e^{-i\varrho(\sin\alpha - \alpha\cos\alpha) + i\pi/4}\right) \\ &= 2i\sqrt{\frac{2}{\pi\varrho\sin\alpha}}\sin[\varrho(\sin\alpha - \alpha\cos\alpha) - \pi/4]. \end{aligned}$$

From this we obtain the roots of the equations $H_n^1(\varrho) = 0$ directly: they are the roots of the following transcendental equation in α

(40) $\varrho(\sin\alpha - \alpha\cos\alpha) - \pi/4 = -m\pi, \qquad m = 1, 2, 3, \ldots,$

where we have to choose the negative sign on the right in order to satisfy our requirement that α be in the fourth quadrant.

For small α we obtain from (40)

$$\varrho\frac{\alpha^3}{3} = -(4m-1)\pi/4$$

and after the correct choice of the cube root of unity, we have

$$\alpha = \left[\frac{3\pi}{4\varrho}(4m-1)\right]^{\frac{1}{3}} e^{-i\pi/3}.$$

Now α is related to ϱ by equation (30a), which for small α, after we make the substitution $\cos\alpha = 1 - \frac{\alpha^2}{2}$, yields:

(41) $$n = \varrho + \frac{1}{2}\varrho^{\frac{1}{3}}\left[\frac{3\pi}{4}(4m-1)\right]^{\frac{2}{3}} e^{i\pi/3}.$$

The roots of $H_n^1(\varrho) = 0$ lie in the *positive-imaginary n-half-plane*, a fact that we shall apply later, and they are *infinite in number*. If we solve (41) with respect to ϱ then we get values in the *negative-imaginary* ϱ*- half-plane*. According to (41) n and ϱ are of the same order of magnitude, as assumed in the beginning. Hence (41) is the solution of the root problem in question.

We see that the saddle-point method is very general. It can be transferred from the treatment of the Hankel functions to that of arbitrary integrals of the form

(42) $$\int e^{F(w,\varrho,n,\ldots)}\,dw,$$

where F depends on several large numbers $\varrho, n, \ldots$ in addition to the variable of integration w, and where the path of integration W starts in a complex region in which $\lim \exp F(w, \ldots) = 0$ and leads to a similar region. In the integrals of the type (42) when the saddle point $F' = 0$ approaches a point $F'' = 0$ one encounters the same peculiarities that we encountered in the case of Hankel functions for the border line case $n \sim \varrho$. This is the case of the Airy diffraction theory of the rainbow. The phenomenon of the rainbow is in fact linked to the appearance of a turning point ($F'' = 0$) in the wave front, which in the asymptotic approximation coincides with the saddle point $F' = 0$. The calculation[16]

[16] W. Wirtinger, *Berichte des Naturw.-mediz. Vereins in Innsbruck* **23**, 97(1896); J. W. Nicholson, *Phil. Mag.* **18**, (1909).

of the "Airy integral" in question leads then to the functions $H_{\frac{1}{3}}$ or, what is the same thing, to the functions $I_{\pm\frac{1}{3}}$ just as in (34) and (35).

§ 22. Spherical Harmonics and Potential Theory

A. The Generating Function

The simplest approach to the theory of spherical harmonics is given by potential theory. We start from the so-called Newtonian potential $1/r$ and, after shifting the origin from $x = y = z = 0$ to (x_0, y_0, z_0), obtain

$$(1)\qquad \frac{1}{R} = \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} = \frac{1}{\sqrt{r^2-2rr_0\cos\vartheta+r_0^2}}\,.$$

The polar coordinates r, ϑ, φ have been chosen so that the polar axis $\vartheta = 0$ goes through the point (x_0, y_0, z_0). We then have $x_0 = y_0 = 0$, $z_0 = r_0$ and

$$(2)\qquad \begin{aligned} x &= r\sin\vartheta\cos\varphi,\\ y &= r\sin\vartheta\sin\varphi,\\ z &= r\cos\vartheta. \end{aligned}$$

We may expand (1) in ascending or descending powers of r depending on whether $r < r_0$ or $r > r_0$. If we denote the coefficient of the n-th ascending or descending power by P_n we have:

$$(3)\qquad \frac{1}{R} = \begin{cases} \dfrac{1}{r_0}\displaystyle\sum_{n=0}^{\infty}\left(\frac{r}{r_0}\right)^n P_n(\cos\vartheta) & r < r_0,\\[2ex] \dfrac{1}{r}\displaystyle\sum_{n=0}^{\infty}\left(\frac{r_0}{r}\right)^n P_n(\cos\vartheta) & r > r_0. \end{cases}$$

The P_n must be the same in both expansions since they must coincide when $r = r_0$ and $\vartheta \neq 0$. The point $r = r_0$, $\vartheta = 0$ is a singular point, the sphere $r = r_0$ playing the role here that is played by the circle of convergence of the Taylor series in the two-dimensional case.

The polynomials P_n defined by (3) are of n-th degree in $\cos\vartheta$. They are called *spherical harmonics* and we shall show that they coincide with the polynomials P_n which were introduced in §5. The function $1/R$ is called the *generating function* of spherical harmonics.

B. Differential and Difference Equation

First we want to find the differential equation of *spherical harmonics.* The fundamental equation of potential theory $\Delta u = 0$, which is satisfied by $1/R$, can be written in the form

$$\frac{1}{r}\frac{\partial^2(r u)}{\partial r^2}+\frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u}{\partial\vartheta}\right)+\frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 u}{\partial\varphi^2}=0. \tag{4}$$

Since this equation must be satisfied by each term in the series (3), we obtain from the n-th term of the first line of (3) after dividing out the common factor r^{n-2}/r_0^{n+1},

$$n(n+1)P_n+\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\frac{dP_n}{d\vartheta}\right)=0; \tag{5}$$

the same follows from the second line after factoring out r_0^n/r^{n+3}. We introduce the abbreviation

$$\cos\vartheta=\zeta=\frac{z}{r}$$

and note that

$$-\sin\vartheta\, d\vartheta=d\zeta,\qquad \sin\vartheta\frac{d}{d\vartheta}=\sin^2\vartheta\frac{d}{\sin\vartheta\, d\vartheta}=-(1-\zeta^2)\frac{d}{d\zeta}.$$

Equation (5) can then be written

$$\frac{d}{d\zeta}\left\{(1-\zeta^2)\frac{dP_n}{d\zeta}\right\}+n(n+1)P_n=0, \tag{6}$$

or

$$\left\{(1-\zeta^2)\frac{d^2}{d\zeta^2}-2\zeta\frac{d}{d\zeta}+n(n+1)\right\}P_n=0. \tag{6a}$$

We consider (6) with n replaced by l and then, following the scheme of Green's theorem, we multiply by P_l and P_n and subtract:

$$\begin{aligned}&P_l\frac{d}{d\zeta}\left\{(1-\zeta^2)\frac{dP_n}{d\zeta}\right\}-P_n\frac{d}{d\zeta}\left\{(1-\zeta^2)\frac{dP_l}{d\zeta}\right\}\\ &=\frac{d}{d\zeta}\left\{(1-\zeta^2)\left(P_l\frac{dP_n}{d\zeta}-P_n\frac{dP_l}{d\zeta}\right)\right\}=\{l(l+1)-n(n+1)\}P_lP_n.\end{aligned} \tag{7}$$

The physical range of the variable ζ is from $\zeta=-1$, to $\zeta=+1$, $(\vartheta=0)$. Integrating over this range we get the condition of orthogonality for $l\neq n$

$$\int_{-1}^{+1}P_lP_n\,d\zeta=0, \tag{8}$$

since the integral of the second line of (7) with respect to ζ vanishes unless the P become singular for $\zeta = \pm 1$ which is excluded by equation (3).

We now show that our P_n satisfy the normalizing condition (5.7)

(9) $$P_n(1) = 1$$

For $\cos\vartheta = \pm 1$ we get from (1):

$$\frac{1}{|r \mp r_0|} = \begin{cases} \frac{1}{r_0} \sum (\pm 1)^n \left(\frac{r}{r_0}\right)^n & r < r_0, \\ \frac{1}{r} \sum (\pm 1)^n \left(\frac{r_0}{r}\right)^n & r > r_0. \end{cases}$$

Comparing this with (3) we even get the somewhat more general equation:

(9a) $$P_n(\pm 1) = (\pm 1)^n .$$

Now we saw in §5 that the P_n were uniquely determined by the orthogonality (8) and the normalization (9). Hence our present definition, with the help of a generating function, leads to the same functions P_n as did the method of least squares in §5. In particular we have the representation (5.8)

(10) $$P_n(\zeta) = \frac{1}{2^n\, n!} \frac{d^n}{d\zeta^n} (\zeta^2 - 1)^n$$

and as a result, according to (5.12)

(10a) $$\int P_n^2(\zeta)\, d\zeta = \frac{1}{n + \frac{1}{2}} .$$

The P_n are even or odd functions of ζ according as n is even or odd:

(10b) $$P_n(-\zeta) = (-1)^n P_n(\zeta).$$

In addition to a differential equation with respect to the variable ζ, our generating function yields a *difference equation with respect to the index n.* We rewrite, say, the first line of equation (3) with the abbreviation $\alpha = r/r_0$:

(11) $$\frac{1}{\sqrt{\alpha^2 - 2\alpha\zeta + 1}} = \Sigma \alpha^n P_n;$$

By logarithmic differentiation with respect to α we obtain

(11 a) $$\frac{\zeta - \alpha}{\alpha^2 - 2\alpha\zeta + 1} = \frac{\Sigma n \alpha^{n-1} P_n}{\Sigma \alpha^n P_n}$$

and after cross-multiplication

$$(\zeta-\alpha)\,\Sigma\,\alpha^n P_n=(\alpha^2-2\,\alpha\,\zeta+1)\,\Sigma\,n\,\alpha^{n-1}P_n.$$

If we compare the coefficients of the same power of α on both sides, say those of α^n, we obtain

$$\zeta P_n-P_{n-1}=(n-1)\,P_{n-1}-2\,\zeta\,n\,P_n+(n+1)\,P_{n+1},$$

or

$$(n+1)\,P_{n+1}-(2\,n+1)\,\zeta\,P_n+n\,P_{n-1}=0. \tag{11 b}$$

The same *recursion formula* is, of course, obtained from the second line of (3).

By the logarithmic differentiation of (11) with respect to ζ we obtained a mixed *differential difference equation.*
Instead of (11a) we now obtain

$$\frac{\alpha}{\alpha^2-2\,\alpha\,\zeta+1}=\frac{\Sigma\,\alpha^n P_n'}{\Sigma\,\alpha^n P_n},\quad P_n'=\frac{dP_n(\zeta)}{d\zeta} \tag{11c}$$

and after cross-multiplication we get, from the coefficient of α^{n+1},

$$P_n-P_{n-1}'+2\,\zeta\,P_n'-P_{n+1}'=0. \tag{11d}$$

Multiplying this equation by $2n+1$ and adding twice the equation obtained from (11b) by differentiation with respect to ζ, we obtain

$$-(2\,n+1)\,P_n-P_{n-1}'+P_{n+1}'=0.$$

We rewrite this as the *differential recursion formula:*

$$\frac{d}{d\zeta}(P_{n+1}-P_{n-1})=(2\,n+1)\,P_n. \tag{11e}$$

C. Associated Spherical Harmonics

The potential equation (4) suggests that in addition to the particular solutions

$$u_n=\begin{Bmatrix} r^n \\ r^{-n-1}\end{Bmatrix}P_n(\cos\vartheta) \tag{12}$$

which depend only on r and ϑ we might consider also the particular solutions

(12a)
$$u_{nm} = \begin{Bmatrix} r^n \\ r^{-n-1} \end{Bmatrix} P_n^m (\cos\vartheta)\, e^{im\varphi}$$

which depend on r, ϑ and φ, by *associating* to P_n certain *spherical harmonics* P_n^m (where m is an integer assumed positive for the time being) defined by the differential equation

(13)
$$\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\,\frac{dP_n^m}{d\vartheta}\right) + \left\{n(n+1) - \frac{m^2}{\sin^2\vartheta}\right\} P_n^m = 0,$$

which follows from (4). Written in analogy to (6) and (6a) we have:

(13a)
$$\frac{d}{d\zeta}\left\{(1-\zeta^2)\frac{dP_n^m}{d\zeta}\right\} + \left\{n(n+1) - \frac{m^2}{1-\zeta^2}\right\} P_n^m = 0,$$

(13b)
$$\left\{(1-\zeta^2)\frac{d^2}{d\zeta^2} - 2\zeta\frac{d}{d\zeta} + n(n+1) - \frac{m^2}{1-\zeta^2}\right\} P_n^m = 0.$$

According to Thomson and Tait our original P_n are called *zonal spherical harmonics* and the associated ones are called *tesseral.* The lines of zeros of the former divide the surface of the sphere into latitudinal regions of different signs, those of the latter divide it into quadrangles (tesserae) of different signs which are bounded by lines of latitude and of longitude. The associated or tesseral spherical harmonics are *orthogonal* for different lower but *equal upper* indices; namely as in (7) we conclude from our differential equation, which now is (13a), that

(14)
$$\int_{-1}^{+1} P_l^m P_n^m \, d\zeta = 0 \qquad \text{for} \qquad l \neq n.$$

In order to obtain an analytic expression for P_n^m we expand at the points $\zeta = \pm 1$ (north and south poles of the unit sphere) in powers of $\zeta \mp 1$, :

$$P_n^m = (\zeta \mp 1)^\lambda \left[a_0 + a_1(\zeta \mp 1) + a_2(\zeta \mp 1)^2 + \cdots\right].$$

This is analogous to (19.36). The determination of λ in analogy to (19.37), is obtained from the differential equation (13b):

(15)
$$\lambda(\lambda - 1) + \lambda - \frac{m^2}{4} = 0, \qquad \lambda = +\frac{m}{2}.$$

(The other root $\lambda = -m/2$ must be excluded for reasons of continuity.) We unite the branches at each of the points $\zeta = \pm 1$ into

$$(1-\zeta^2)^{m/2}$$

and write

$$P_n^m = (1-\zeta^2)^{m/2}\, v = \sin^m \vartheta \cdot v. \tag{16}$$

For the v which we introduced here we obtain from (13b)

$$\left\{(1-\zeta^2)\frac{d^2}{d\zeta^2} - 2(m+1)\zeta\frac{d}{d\zeta} + [n(n+1) - m(m+1)]\right\} v = 0, \tag{17}$$

which now must be solved in terms of series which contain only *integral* powers of $\zeta \mp 1$. However we do not have to investigate these series since the required integral of (17) can be obtained in closed form from (6a). Namely if we differentiate (6a) m times with respect to ζ and apply the well known rule of differentiation

$$\frac{d^m}{d\zeta^m}\zeta\frac{d}{d\zeta} = \left\{\zeta\frac{d}{d\zeta} + \binom{m}{1}\right\}\frac{d^m}{d\zeta^m}, \tag{17a}$$

then we obtain exactly the expression $\{\}$ of (17) applied to the m-th derivative of P_n. Hence we see that we obtain a solution of (17) by setting

$$v = \frac{d^m P_n}{d\zeta^m} \tag{17b}$$

With the use of (16) and (10) we obtain a simple representation for our associated spherical harmonics and at the same time a determination of the normalizing factor which has been free up to this point:

$$P_n^m = \frac{(1-\zeta^2)^{m/2}}{2^n\, n!}\;\frac{d^{n+m}(\zeta^2-1)^n}{d\zeta^{n+m}}. \tag{18}$$

Hence for even m, P_n^m is, like P_n, a polynomial of degree n; for odd m, P_n^m is $\sqrt{1-\zeta^2}$ times a polynomial of degree $n-1$. We further see from (18) that

$$P_n^0 = P_n, \qquad P_n^m = 0 \quad \text{for} \quad m > n. \tag{18a}$$

The last statement follows from the fact that for $m > n$ the order of differentiation in (18) is greater than the degree of the differentiated polynomial.

D. On Associated Harmonics with Negative Index m

Up to now we had to assume a positive index m, for example, in (17b) we made use of differentiation of order m with respect to ζ. However our final representation (18) can be extended directly to *negative* m with $m \geq -n$. We therefore extend (18) to the $2n+1$ values $|m| \leq n$. For negative m too, the function P_n^m is a polynomial of degree n (in the same sense as for positive m). This is because the pole at $\zeta = \pm 1$ given by the factor $(1-\zeta^2)^{m/2}$ for negative m is cancelled by the second factor of (18), which has a zero there of corresponding order due to the fact that the order of differentiation has been lowered by $|m|$. In addition the P_n^m satisfy the differential equation (13) for negative m too (since (18) depends only on m^2). Hence the P_n^m for negative m can differ from the $P_n^{|m|}$ only by a constant factor C, which is best determined by the comparison of the highest powers of ζ in P_n^{-m} and P_n^{+m} as calculated from (18). We then have:

$$\frac{P_n^{-m}}{P_n^{+m}} = (1-\zeta^2)^{-m} \frac{d^{n-m}}{d\zeta^{n-m}} \zeta^{2n} \Big/ \frac{d^{n+m}}{d\zeta^{n+m}} \zeta^{2n}$$

$$\sim (-1)^m \zeta^{-2m} \frac{(2n)!}{(n+m)!} \zeta^{n+m} \Big/ \frac{(2n)!}{(n-m)!} \zeta^{n-m} = (-1)^m \frac{(n-m)!}{(n+m)!};$$

hence:

$$P_n^{-m} = C \cdot P_n^{+m}, \qquad C = (-1)^m \frac{(n-m)!}{(n+m)!}. \tag{18b}$$

This equation holds for both positive and negative m.

Our definition of the P_n^m, which departs from the older mathematical literature, has been justified by wave mechanics and will also serve to unify our expressions.[17]

Hence we have exactly $2n+1$ adjoined P_n^m, one of which coincides with P_n, the rest being pairwise equal except for the constant factor C; another difference is in the factor $\exp(i\,m\,\varphi)$ by which they are multiplied in (12a).

[17] In the older literature the upper index of P_n^m is assumed positive throughout and the φ-dependence is given by $\cos m\varphi$ or $\sin m\varphi$. It is much simpler to assume this dependence exponential, as we have done in (12a), where we also dropped the restriction to positive m.

An even greater departure from the customary definition is suggested by C. G. Darwin (*Proc. Roy. Soc. London* **115**, 1927) who appends the factor $(n-m)!$ to the right side of (18). Then (18b) simplifies to

$$P_n^m = (-1)^m P_n^{|m|} \quad \text{for} \quad m < 0.$$

But this definition implies a change in the classical expression for the Legendre polynomials $P_n = P_n^0$, which we want to avoid.

Moreover, some authors, in particular E. W. Hobson in his Theory of Spherical and Ellipsoidal Harmonics, Cambridge 1931, use the factor $(-1)^m$ in the definition of P_n^m in (18), but this is immaterial for our purposes.

E. Surface Spherical Harmonics and the Representation of Arbitrary Functions

By the most general *"surface spherical harmonic"* (introduced by Maxwell) we mean the expression

$$Y_n = \sum_{m=-n}^{+n} A_m P_n^m (\cos\vartheta)\, e^{im\varphi}. \tag{19}$$

which contains $2n+1$ arbitrary constants. Multiplied by r^n (or r^{-n-1}) Y_n yields the most general potential of order n (or $-n-1$) which is *homogeneous* in the coordinates x,y,z (Maxwell's solid harmonics). It is a combination of the special u_{nm} of (12a) which also are homogeneous in x,y,z:

$$\left.\begin{matrix} r^n \\ r^{-n-1} \end{matrix}\right\} Y_n = \sum_{m=-n}^{+n} A_m u_{nm}, \tag{19a}$$

and the general non-homogeneous solution of the potential equation (4) is represented as a sum of its homogeneous parts:

$$u = \sum_{n=0}^{\infty} \left\{\begin{matrix} r^n \\ r^{-n-1} \end{matrix}\right\} Y_n. \tag{19b}$$

By restricting this representation to the case of a sphere of radius 1 and giving the value of u on the surface as $f(\varphi, \vartheta)$ we obtain

$$f(\vartheta, \varphi) = \sum_{n=0}^{\infty} Y_n = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} A_{nm} P_n^m (\cos\vartheta)\, e^{im\varphi}. \tag{20}$$

By using the notation A_{nm} instead of the A_m of (19) we emphasize the fact that the free constants in each Y_n are independent of and different from the constants in any other Y_n. The series (19b) and (20) express the fact that the *boundary value problem of potential theory for a sphere* is solvable for an arbitrary value $f(\vartheta, \varphi)$ of the potential on the surface of the sphere, both for the interior of the sphere (factor r^n in (19b)) and for the exterior (factor r^{-n-1}). In the following section we shall treat this problem by direct construction of Green's function and thereby derive the above series again. The first rigorous proof of (20) under very general assumptions on the nature of $f(\vartheta, \varphi)$ was given by Dirichlet in 1837.

F. Integral Representation of Spherical Harmonics

We now consider a special homogeneous function of degree n in x,y,z

(21) $(z+i\,x)^n = r^n\,(\cos\vartheta + i\sin\vartheta\cos\varphi)^n = r^n\,(\zeta+\sqrt{\zeta^2-1}\cos\varphi)^n$,

which, like every function of the form $f(z+ix)$ or $f(z+iy)$ etc., obviously satisfies the equation $\Delta u = 0$. Hence the coefficient of r^n in (21) is a surface spherical harmonic Y_n. If we average it over φ, making it a pure function of ζ, we obtain our zonal spherical harmonic

$$P_n(\zeta) = \frac{1}{2\pi}\int_{-\pi}^{+\pi}(\zeta+\sqrt{\zeta^2-1}\,\cos\varphi)^n\,d\varphi. \tag{22}$$

If on the other hand we construct the m-th Fourier coefficient of Y_n as in (1.12) then we obtain the associated (tesseral) spherical harmonic

$$P_n^m(\zeta) = \frac{C}{2\pi}\int_{-\pi}^{+\pi}(\zeta+\sqrt{\zeta^2-1}\,\cos\varphi)^n\,e^{-im\varphi}\,d\varphi. \tag{23}$$

The integral representation (22) is first mentioned by Laplace in his Mécanique Céleste, Vol. V. The fact that the denominator 2π provides the correct normalization is seen by setting $\zeta=1$, for then the integrand becomes 1 and hence $P_n(1) = 1$. On the other hand we still have to determine the normalizing factor C in (23). By comparison with the normalization of (18) we find[18]

$$C = \frac{(n+m)!}{n!}\,e^{-im\pi/2}. \tag{23a}$$

If in (21) we replace n by $-n-1$, as we know is possible, we get, equivalent to (22), the representation

$$P_n(\zeta) = \frac{1}{2\pi}\int_{-\pi}^{+\pi}\frac{d\varphi}{(\zeta+\sqrt{\zeta^2-1}\,\cos\varphi)^{n+1}}, \tag{23b}$$

which is also seen to be normalized.

[18] For instance by passing to the limit $\zeta\to\infty$ in both (23) and (18), whereby, except for the factors ζ^n and $\exp(-i\,m\varphi)$, the integrand in (23) reduces to

$$(1+\cos\varphi)^n = 2^n\cos^{2n}\varphi/2 = 2^{-n}\,e^{in\varphi}(1+e^{-i\varphi})^{2n}.$$

In the binomial expansion of this expression we have to consider only the term with $e^{im\varphi}$ since all the other terms vanish upon integration with respect to φ. The factor $\exp\{-i\,m\,\pi/2\}$ in (23a) is due to the factor $\sin^m\vartheta$ in (18). See also the similar passage to the limit $\zeta\to\infty$ in (18b).

G. A Recursion Formula for the Associated Harmonics

Starting from the recursion formulas (11b) and (11e) for the zonal spherical harmonics, we differentiate (11b) m times with respect to ζ, apply rule (17a) to the middle term and multiply each term by $\sin^m \vartheta$. As a result of (18) we obtain

$$(24)\quad (n+1)\, P^m_{n+1} - (2n+1)\,\zeta\, P^m_n - m\,(2n+1)\sin\vartheta\, P^{m-1}_n + n\, P^m_{n-1} = 0.$$

On the other hand we obtain from (11e) upon $(m-1)$-fold differentiation with respect to ζ and multiplication by $\sin^m \vartheta$:

$$(25)\qquad P^m_{n+1} - P^m_{n-1} = (2n+1)\sin\vartheta\, P^{m-1}_n.$$

Eliminating the term with $\sin\vartheta$ from (24) and (25) we obtain the recursion formula

$$(26)\quad (n+1-m)\, P^m_{n+1} - (2n+1)\,\zeta\, P^m_n + (n+m)\, P^m_{n-1} = 0.$$

which is a generalization of (11b). Apparently this equation holds only for positive m, due to its derivation through m-fold differentiation. However we can verify it for negative m if we consider our general definition (18) of P^m_n and the relation (18b).

H. On the Normalization of Associated Harmonics

From (10a) we know the value of the normalizing integral for $m = 0$. We denote it by N_n or also by N^0_n. Its computation in §5 is based on the symbol $D_{k,l}$ of (5.9). We first consider the generalized form of the normalizing integral

$$(27)\qquad N^{\pm m}_n = \int_{-1}^{+1} P^m_n(\zeta)\, P^{-m}_n(\zeta)\, d\zeta.$$

Written in terms of the symbol $D_{k,l}$ we have as a result of our general definition (18)

$$(28)\qquad N^{\pm m}_n = \frac{1}{2^{2n}\, n!\, n!} \int_{-1}^{+1} D_{n+m,n} \cdot D_{n-m,n}\, d\zeta.$$

Through m-fold integration by parts we obtain, since the terms outside the sign of integration vanish for $\zeta = \pm 1$:

$$(29)\qquad N^{\pm m}_n = \frac{(-1)^m}{2^{2n}\, n!\, n!} \int_{-1}^{+1} D_{n,n} \cdot D_{n,n}\, d\zeta = (-1)^m N^0_n = \frac{(-1)^m}{n+\frac{1}{2}}.$$

Here in the last equation we have substituted the value of N_n° from (10a). The normalizing integral is usually taken as

$$N_n^m = \int_{-1}^{+1} P_n^m(\zeta)\, P_n^m(\zeta)\, d\zeta\,. \tag{30}$$

This can be deduced directly from (29) by using the relation (18b) which yields

$$N_n^m = \frac{1}{C} N_n^{\pm m} = \frac{1}{n+\frac{1}{2}} \frac{(n+m)!}{(n-m)!}\,. \tag{31}$$

Its direct computation in the manner of (28) would have been somewhat more cumbersome.

We remark that in the following chapter we shall always carry out a "normalization of the eigenfunction to 1." If we denote the associated harmonic with this normalization by Π_n^m, then we have:

$$\int_{-1}^{+1} \Pi_n^m(\zeta)\, \Pi_n^m(\zeta) d\zeta = 1\,, \tag{31 a}$$

and comparing this with (30) we get

$$\Pi_n^m = P_n^m / \sqrt{N_n^m} = P_n^m \cdot \left[(n+\tfrac{1}{2}) \frac{(n-m)!}{(n+m)!}\right]^{\frac{1}{2}}. \tag{31 b}$$

J. The Addition Theorem of Spherical Harmonics

The proof of this theorem is based on a lemma, which we shall be able to prove only with the methods of the following chapter and which shall be assumed here without proof, namely: *The surface spherical harmonic*

$$Y_n = \sum_{m=-n}^{+n} \Pi_n^m (\cos\vartheta)\, e^{im\varphi} \cdot \Pi_n^m (\cos\vartheta_0)\, e^{-im\varphi_0} \tag{32}$$

depends only on the relative position of the two points (ϑ, φ) *and* (ϑ_0, φ_0) *on the surface of the sphere,* in other words it has an invariant meaning independent of the coordinate system. If we now change the coordinate system of ϑ, φ by letting the polar axis of a new coordinate system Θ, Φ pass through the point (ϑ_0, φ_0), then the latter has the coordinate $\Theta_0 = 0$ (its Φ_0 becomes undetermined), while the Θ coordinate of the former point (ϑ, φ) is now given by

$$\cos\Theta = \cos\vartheta\cos\vartheta_0 + \sin\vartheta\sin\vartheta_0\cos(\varphi - \varphi_0) \tag{33}$$

For $\Theta_0 = 0$ all the terms in (32) vanish except those with $m = 0$. Hence the right side of (32) becomes the product of the zonal spherical harmonics $\Pi_n(\cos\Theta)$ and $\Pi_n(1)$. Due to the stated invariance of Y_n we then have

$$(34) \qquad \Pi_n(\cos\Theta)\,\Pi_n(1) = \sum_{m=-n}^{+n} \Pi_n^m(\cos\vartheta)\,\Pi_n^m(\cos\vartheta_0)\,e^{im(\varphi-\varphi_0)}.$$

This is the *symmetric form* of the addition theorem which expresses its structure in a convincingly simple form. The form which is common in the literature is obtained by expressing the Π_n^m in terms of P_n^m with the help of (31b) and (31). Equation (34) then becomes

$$(35) \qquad P_n(\cos\Theta) = \sum_{m=-n}^{+n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\vartheta)\,P_n^m(\cos\vartheta_0)\,e^{im(\varphi-\varphi_0)},$$

or written in real form

$$(36) \qquad \begin{aligned} P_n(\cos\Theta) &= P_n(\cos\vartheta)\,P_n(\cos\vartheta_0) \\ &\quad + 2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\vartheta)\,P_n^m(\cos\vartheta_0)\cos m(\varphi-\varphi_0). \end{aligned}$$

It is however evident that the true structure of the addition theorem is gradually lost in the passage (34) $\to$ (35) $\to$ (36).

Another rather transparent form of the addition theorem is obtained from (35) by replacing one of the upper indices m by $-m$ and applying (18b):

$$(37) \qquad P_n(\cos\Theta) = \sum_{m=-n}^{+n} (-1)^m P_n^m(\cos\vartheta)\,P_n^{-m}(\cos\vartheta_0)\,e^{im(\varphi-\varphi_0)}.$$

§ 23. Green's Function of Potential Theory for the Sphere. Sphere and Circle Problems for Other Differential Equations

We superimpose two principal solutions u, u' of the potential equation $\Delta u = 0$,

$$(1) \qquad \begin{aligned} u &= \frac{e}{R}, \quad R^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2, \\ u' &= \frac{e'}{R'}, \quad R'^2 = (x-\xi')^2 + (y-\eta')^2 + (z-\zeta')^2 \end{aligned}$$

and seek the level surfaces of the function $G = u - u'$, in particular the surface $G = 0$. According to (1) the latter is given by the equation $R'^2 = (e'/e)^2 R^2$, or written explicitly:

$$
\begin{aligned}
&\left(1 - \frac{e'^2}{e^2}\right)(x^2 + y^2 + z^2)\\
(2)\qquad &- 2\left\{\left(\xi' - \frac{e'^2}{e^2}\xi\right)x + \left(\eta' - \frac{e'^2}{e^2}\eta\right)y + \left(\zeta' - \frac{e'^2}{e^2}\zeta\right)z\right\}\\
&+ \xi'^2 - \frac{e'^2}{e^2}\xi^2 + \eta'^2 - \frac{e'^2}{e^2}\eta^2 + \zeta'^2 - \frac{e'^2}{e^2}\zeta^2 = 0\,.
\end{aligned}
$$

This is the equation of a *sphere*. The position of its center and the length of its radius can be calculated from (2): the center O lies on the connecting line of the "source points" $Q = (\ \xi, \eta, \zeta\ )$ and $Q' = (\ \xi', \eta', \zeta'\ )$, the radius a is obtained as the *mean proportional* of $OQ = \varrho$ and $OQ' = \varrho'$,

$$(3)\qquad \varrho\, \varrho' = a^2,$$

so that one of the source points lies in the interior, the other in the exterior of the sphere of radius a.

For further discussion we shall not use the cumbersome formula (2), but rather the elementary geometric Fig. 22.

A. Geometry of Reciprocal Radii

Fig. 22 illustrates the method of *reciprocal radii*[19] formulated in (3). We call Q' the *inverse image* of Q with respect to the sphere of radius a, or also the *electric image* (Maxwell); the notations e and e' in (1) are connected with the electric point of view. The relation between Q and Q' is symmetric: Q is the inverse image of Q'. From (3) we see in the well known manner that the points Q,Q' are harmonic with respect to the points of intersection P_1, P_2 of the line OQQ' and the sphere.

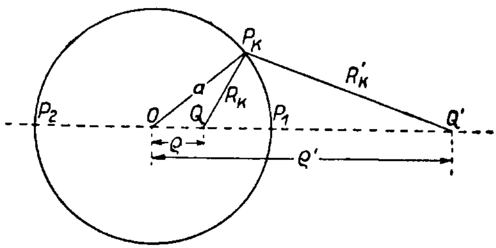


Fig. 22. Geometry of reciprocal radii. Q and Q' are transformed into each other by inversion on the sphere of radius a around the center O. The triangles OQP_k and OP_kQ' are similar.

The *method of reciprocal radii* was developed through the pioneering

[19] The term "*reciprocal*" arises from the (bad) habit of setting $a = 1$, in which case $\varrho' = 1/\varrho$. For reasons of dimensionality we consider it better to retain the radius a as a length.

work of William Thomson[20] who applied it to a wide range of problems in electro- and magnetostatics. Transformation by reciprocal radii is called *inversion* for short.

From (3) it follows that the triangle OP_kQ of the figure is similar to the triangle $OQ'P_k$, hence we have:

(4) $\frac{\varrho}{R_k} = \frac{a}{R'_k}$, where $R_k = P_kQ$ and $R'_k = P_kQ'$.

Here P_k denotes a special point on the surface of the sphere while the symbols P,R,R' are reserved for an arbitrary point P: (x,y,z) and its distances from Q and Q'. In order to determine the "image charge" e' which was introduced in (1) we compare (4) with the relation

$$\frac{e}{R_k} = \frac{e'}{R'_k}$$

which follows from (1), and is valid for every point P_k. We obtain

(5) $\frac{e'}{e} = \frac{a}{\varrho} = \frac{\varrho'}{a}$ (the latter due to equation (3)).

B. The Boundary Value Problem of Potential Theory for the Sphere, the Poisson Integral

Equation (5) brings us back to our starting point, the condition $G = u - u' = 0$; and we can now justify the notation G, which signifies *Green's function.* We have in fact

$$G = G(P, Q) = \frac{e}{R} - \frac{e'}{R'} \tag{6}$$

as Green's function for the "interior boundary value problem" which is: to find a potential U which has no singularities in the interior of the sphere for a given boundary value $\overline{U}$ on the surface of the sphere. In the same manner

$$G = G(P, Q') = \frac{e'}{R'} - \frac{e}{R} \tag{6 a}$$

is Green's function for the corresponding "exterior boundary value problem." Of the three conditions a),b),c) (see p. 50) that serve to define Green's function, we have b) satisfied on account of (5) and a) satisfied because the potential equation is self-adjoint and therefore the

[20] *Journal de Math.* **10** (1845), **12** (1847). Maxwell in his Treatise, vol. I. Chap. XI, quotes a paper in the *Cambridge and Dublin Math. Journ.* of 1848.

differential equation of G coincides with that of U. In order to satisfy condition c) of the unit source we merely have to make

$$e \text{ or } e' = -1/4\pi.$$

for the inner or outer boundary value problem, as seen from the table on p. 49.

We write (6) explicitly by introducing the spherical coordinates r, ϑ, φ for P($r = 0$ is the center of the sphere, $\vartheta = 0$ an arbitrary direction). Let the corresponding coordinates for Q and Q' be:

$$r_0, \vartheta_0, \varphi_0 \quad \text{with} \quad r_0 = \varrho,$$
$$r_0', \vartheta_0', \varphi_0' \quad \text{with} \quad r_0' = \varrho', \ \vartheta_0' = \vartheta_0, \ \varphi_0' = \varphi_0;$$

as in (22.33) we let

$$\cos\Theta = \cos\vartheta\cos\vartheta_0 + \sin\vartheta\sin\vartheta_0\cos(\varphi - \varphi_0);$$

for e'/e we use the first of the values given by (5), and we let $e = -1/4\pi$. Equation (6) then becomes

$$-4\pi G = \frac{1}{\sqrt{r^2 + \varrho^2 - 2r\varrho\cos\Theta}} - \frac{a/\varrho}{\sqrt{r^2 + \frac{a^4}{\varrho^2} - 2r\frac{a^2}{\varrho}\cos\Theta}}. \tag{7}$$

The solution for the interior boundary value problem according to the scheme of (10.12) is then

$$U(Q) = \int \bar{U}\frac{\partial G}{\partial n}\,d\sigma. \tag{8}$$

The integration on the right is with respect to the point P and is taken over the surface of the sphere, $\partial G/\partial n = \partial G/\partial r$ for $r = a$ and $d\sigma = a^2\sin\vartheta\,d\vartheta\,d\varphi$; Q is an arbitrary point in the interior of the sphere. From (7) we get the general relation

$$4\pi\frac{\partial G}{\partial r} = \frac{r - \varrho\cos\Theta}{R^3} - \frac{a}{\varrho}\,\frac{r - \frac{a^2}{\varrho}\cos\Theta}{R'^3},$$

where R and R' are as before. Hence, for the surface of the sphere, where according to (4) we have $R_k' = \frac{a}{\varrho}R_k$, we get

$$4\pi\frac{\partial G}{\partial r} = \frac{1}{R_k^3}\left\{a - \varrho\cos\Theta - \frac{\varrho^2}{a^2}\left(a - \frac{a^2}{\varrho}\cos\Theta\right)\right\} = \frac{a}{R_k^3}\left(1 - \frac{\varrho^2}{a^2}\right).$$

Therefore, if we set $\bar{U} = f(\vartheta, \varphi)$ equation (8) becomes

$$4\pi U(Q) = a^3\left(1 - \frac{\varrho^2}{a^2}\right)\iint\frac{f(\vartheta,\varphi)\sin\vartheta\,d\vartheta\,d\varphi}{(a^2 + \varrho^2 - 2a\varrho\cos\Theta)^{\frac{3}{2}}}. \tag{9}$$

This representation was deduced by Poisson in a very circuitous manner through the development of $f(\vartheta, \varphi)$ in spherical harmonics. Here we see that the direct way is through Green's function (7).

The corresponding formula for the outer boundary value problem is obtained from (6a) by setting $e' = -1/4\pi$ and taking the second value in (5) for e'/e. We have:

$$\text{(9a)} \qquad 4\pi U(Q') = a^3\left(\frac{\varrho'^2}{a^2} - 1\right) \iint \frac{f(\vartheta,\varphi)\sin\vartheta \, d\vartheta \, d\varphi}{(a^2 + \varrho'^2 - 2a\varrho' \cos\Theta)^{\frac{3}{2}}}.$$

The so-called "second boundary value problem," in which we set $\partial G/\partial n = 0$ on the surface of the sphere, can not be solved with the method of reciprocal radii.

We now wish to gain a clearer geometrical understanding of the way in which formula (9); which is analytic throughout, can, on the surface, represent an arbitrary function $f(\vartheta, \varphi)$, which is in general not analytic. For this purpose we have to consider the passage to the limit $Q \to K$ as $\varrho \to a$ In this limit the factor $1 - \varrho^2/a^2$ in front of the integral in (9) vanishes and hence only those elements of area $d\sigma$ contribute to the integral for which the denominator R^3 vanishes. The latter approaches zero for $\varrho \to a$ only when $\cos\Theta = 1$, and therefore $\vartheta = \vartheta_0$, $\varphi = \varphi_0$. Thus the only determining part is a neighborhood of that element of area which approaches Q, in other words the special value $f(\vartheta_0, \varphi_0)$ on this element of area will alone determine the limiting value. We indicate this fact by rewriting (9) in the form

$$\text{(10)} \quad 4\pi \lim_{Q \to K} U(Q) = a^3\left(1 - \frac{\varrho^2}{a^2}\right) f(\vartheta_0, \varphi_0) \int_0^{\varepsilon} \int_0^{2\pi} \frac{\sin\Theta \, d\Theta \, d\Phi}{(a^2 + \varrho^2 - 2a\varrho\cos\Theta)^{\frac{3}{2}}}.$$

In the numerator of the integrand we are allowed to replace $\sin\vartheta \, d\vartheta \, d\varphi$ by $\sin\Theta \, d\Theta \, d\Phi$; after this is done the integration can be carried out explicitly. We obtain

$$\frac{2\pi}{a\varrho}\left(\frac{1}{a - \varrho} - \frac{1}{(a^2 + \varrho^2 - 2a\varrho\cos\varepsilon)^{\frac{1}{2}}}\right).$$

Substituting this in (10) we observe that the contribution of the second term in the parentheses vanishes for $\varrho \to a$. From the first term, after dividing the denominator $a - \varrho$ into the factor $1 - \varrho^2/a^2$, we get the desired value:

$$\lim_{Q \to K} U = f(\vartheta_0, \varphi_0).$$

In the two-dimensional case (circle instead of sphere) we can carry out a simplified consideration in close analogy to (9) and (9a), where instead of (9) we get

$$2\pi U(Q) = a^2\left(1 - \frac{\varrho^2}{a^2}\right)\int \frac{f(\varphi)\,d\varphi}{a^2 + \varrho^2 - 2a\varrho\cos(\varphi - \varphi_0)}. \tag{11}$$

C. General Remarks about Transformations by Reciprocal Radii

Returning to the three-dimensional case we now wish to consider the transformation by reciprocal radii from more general viewpoints. We choose an arbitrary point 0 to be the *center of inversion* and, at the same time, the origin of a spherical polar coordinate system; we then select a sphere with a center 0 and an arbitrary radius a as the *sphere of inversion.* An arbitrary point P: (r, ϑ, φ) is transformed into a point P': $(r', \vartheta', \varphi')$. Between these points we have the relations:

$$\begin{aligned} r\,r' &= a^2, & \vartheta' &= \vartheta, & \varphi' &= \varphi; \\ dr &= -\frac{a^2}{r'^2}\,dr', & d\vartheta &= d\vartheta', & d\varphi &= d\varphi'. \end{aligned} \tag{12}$$

For the sake of completeness we also give the corresponding relations between the rectangular coordinates x,y,z and x',y',z'. Using the scheme of (22.2) we obtain from (12)

$$x' = r' \sin\vartheta' \cos\varphi' = \frac{a^2}{r}\sin\vartheta\cos\varphi = \frac{a^2}{r^2}\,x, \text{ etc.}$$

or, written in summarized form,

$$(x', y', z') = \frac{a^2}{r^2}(x, y, z); \tag{12a}$$

and conversely

$$(x, y, z) = \frac{a^2}{r'^2}(x', y', z'). \tag{12b}$$

We now seek the transformation in polar coordinates of the line element ds^2 into ds'^2. According to (12) we have

$$\begin{aligned} ds^2 &= dr^2 + r^2\,d\vartheta^2 + r^2\sin^2\vartheta\,d\varphi^2 \\ &= \left(\frac{a}{r'}\right)^4 (dr'^2 + r'^2\,d\vartheta'^2 + r'^2\sin^2\vartheta'\,d\varphi'^2) = \left(\frac{a}{r'}\right)^4 ds'^2. \end{aligned} \tag{13}$$

Hence the transformations by reciprocal radii are *conformal:* every infinitesimal triangle with sides ds is transformed into a *similar* triangle with sides ds' (ratio of sides $=(a/r')^2=(r/a)^2$). According to a theorem by Liouville these mappings are the only non-trivial conformal transformations in three-dimensional space.

The geometric characterization of our transformation consists of the fact that it transforms spheres into spheres, where the plane has to be considered as a sphere of infinite radius. A *sphere* that passes through the center of inversion is transformed into a *plane*, since the center of inversion is transformed into infinity. (Infinity in this "geometry of spheres" is a *point* and not a *plane* as it is in projective geometry.) Conversely, a plane that does not pass through the center of inversion is transformed into a sphere.

D. Spherical Inversion in Potential Theory

The next point of interest to us is the transformation of the differential parameter Δu: we start from a function $u(r, \vartheta, \varphi)$ and transform the *product* ru by reciprocal radii (r = distance from center of inversion). We denote the new function by

$$v(r', \vartheta', \varphi') = \frac{a^2}{r'}\, u\left(\frac{a^2}{r'}, \vartheta', \varphi'\right). \tag{14}$$

In other words, we transfer the value ru from the original point P (r, ϑ, φ) to the point P' with the coordinates

$$r' = \frac{a^2}{r},\ \vartheta' = \vartheta,\ \varphi' = \varphi \tag{14 a}$$

and we want to show that the differential parameter $\Delta' v$ which is calculated in terms of the coordinates r', ϑ', φ' is given by

$$\Delta' v = \left(\frac{a}{r'}\right)^4 r\, \Delta u. \tag{15}$$

Again, the reason for this relation lies in the conformality of the mapping, as indicated by the appearance in (15) of the square of the ratio of dilation $(r/a)^2$ from equation (13). Equation (15) can be proven as follows: we define the operators Δ and D as in (22.4) by the formulas

$$r^2 \Delta = \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + D, \tag{15 a}$$

$$D = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta \frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}. \tag{15 b}$$

Then, if Δ' and D' stand for the same expressions in r', ϑ', φ', we obtain from (14) and (14a)

(15 c) $$D'v = r\,Du,\quad \frac{\partial v}{\partial r'} = \frac{\partial (r\,u)}{\partial r}\frac{dr}{dr'} = -\frac{a^2}{r'^2}\frac{\partial (r\,u)}{\partial r},$$

(15 d) $$\frac{\partial}{\partial r'}\left(r'^2\frac{\partial v}{\partial r'}\right) = -a^2\frac{\partial^2 (r\,u)}{\partial r^2}\frac{dr}{dr'} = \frac{a^4}{r'^2}\frac{\partial^2 (r\,u)}{\partial r^2};$$

and hence according to (15a,c,d)

(15 e) $$r'^2 \Delta' v = \frac{a^4}{r'^2}\left(\frac{\partial^2 (r\,u)}{\partial r^2} + \frac{r\,r'^2}{a^4}Du\right) = \frac{a^4}{r'^2}\left(\frac{\partial^2 (r\,u)}{\partial r^2} + \frac{1}{r}Du\right).$$

According to equation (22.4) the expression in the last parentheses is just $r\,\Delta u$. Hence (15e) becomes

(16) $$\Delta' v = \left(\frac{a}{r'}\right)^4 r\,\Delta u,$$

which coincides with (15).[21]

If we start from a function u which satisfies the differential equation $\Delta u = 0$ *in the coordinates x,y,z, then the function* $v = ru$ *after transformation by reciprocal radii satisfies the differential equation* $\Delta v = 0$ *in the coordinates x′,y′,z′.*

This theorem (William Thomson) enables us to transfer solutions of potential problems obtained for a certain region of space S to the transformed region S'. In particular this holds for Green's function: if it is known for a region S bounded by planes with the boundary condition $G = 0$, then our theorem gives Green's function G' for an arbitrary region S' bounded by spheres where the boundary condition $G' = 0$ remains valid. Depending on the position of the center of inversion, the region S' may have diverse shapes. The totality of those regions which were treated in §17 with the help of elementary reflections now becomes a richer manifold of regions bounded by spheres, thus permitting our generalized reflection by inversion. As before, this more general reflection leads to a simple and complete covering of space. The previous condition that all face angles must be submultiples of π remains valid owing to the conformality of the mapping. Where there was an infinity of image points (e.g., plane plate) there will still be an infinity of image points (e.g., the region between spheres tangent at the center of inversion, which is the image of the plane plate). Where there was a finite number of image points (e.g., for the wedge of 60° in Fig. 17), the inversion process for the spherical problem again terminates after a finite number of steps.

Examples will be given in exercises IV.6 and IV.7, where we shall also discuss the problem of a suitable choice of the center of inversion.

[21] Here the reason for the retention of a becomes apparent: if we had $a = 1$ the dimensionality of the factor $1/r'^4$ in (16) would not be understood, whereas now the dimensional consistency is clear.

Obviously all that has been said above can be transferred to two-dimensional potential theory, where inversion in a sphere becomes inversion in a circle. At the same time the range of possible mappings is increased tremendously since every transformation $z' = f(z)$ where f is an analytic function of the complex variable $z = x + iy$ leads to a conformal mapping. The dilation ratio of the line elements is then $|df/dz|$ and (16) is replaced by

$$\Delta' v = \left|\frac{df}{dz}\right|^2 \Delta u . \tag{17}$$

E. The Breakdown of Spherical Inversion for the Wave Equation

Unfortunately, these mapping methods for the two- and three-dimensional case are *entirely restricted to potential theory.* If we were to perform a transformation by reciprocal radii on the wave equation

$$\Delta u + k^2 u = 0 \tag{18}$$

then according to (16) the factor $(a/r')^4 r$ would appear and (18) would become:

$$\Delta' v + k^2 \left(\frac{a}{r'}\right)^4 v = 0 . \tag{19}$$

Only in the potential equation ($k = 0$) does this disturbing factor $(a/r')^4$ disappear. In the wave equation this factor means that the originally homogeneous medium (k constant) appears transformed into a highly inhomogeneous medium, which, in the neighborhood of the point $r' = 0$, shows a lens-like singularity of the index of refraction. The same holds for the equation of heat conduction, which, written in our customary form with u as temperature and k as temperature conductivity, would go into

$$\Delta' v = \frac{1}{k} \left(\frac{a}{r'}\right)^4 \frac{\partial v}{\partial t} . \tag{19a}$$

This form of the equation certainly can not serve to simplify the boundary value problem for the sphere. Instead, we have to rely on the much more cumbersome method of series expansion as applied in the corresponding two-dimensional case of §20 A.

§ 24. More About Spherical Harmonics

A. The Plane Wave and the Spherical Wave in Space

The simplest solution of the three-dimensional wave equation

$$\Delta u + k^2 u = 0 \tag{1}$$

is the plane wave, e.g., a purely periodic sound wave which progresses in the z-direction

$$u = e^{ikz} = e^{i\varrho\cos\vartheta} \qquad \varrho = kr, \quad k = \text{ wave number.} \tag{2}$$

If we develop this solution in zonal spherical harmonics $P_n(\cos\vartheta)$ then the coefficients will be the $\psi_n(\varrho)$ of §21 C. This follows from the wave equation on the one hand, and the differential equation of the P_n on the other hand. Using the left side of (22.4) and the postulated independence from φ, equation (1) becomes

$$\frac{1}{r}\frac{\partial^2 ru}{\partial r^2} + \frac{1}{r^2}\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial u}{\partial\vartheta} + k^2 u = 0 .$$

Hence u can be separated into a product of P_n by a function $R(r)$ which depends only on r. Due to the differential equation (22.5) of P_n the function R must satisfy the equation

$$\frac{1}{r}\frac{d^2 rR}{dr^2} + \left(k^2 - \frac{n(n+1)}{r^2}\right)R = 0$$

which, in terms of the ϱ of equation (2), can be rewritten as:

$$\frac{1}{\varrho}\frac{d^2\varrho R}{d\varrho^2} + \left(1 - \frac{n(n+1)}{\varrho^2}\right)R = 0 . \tag{3}$$

This is the same differential equation as (21.11a); the solutions, which were continuous for $\varrho = 0$, were defined as ψ_n. Neglecting a multiplicative constant we get from (21.11):

$$R(r) = \psi_n(\varrho) = \sqrt{\frac{\pi}{2\varrho}}\, I_{n+\frac{1}{2}}(\varrho) \tag{4}$$

Similarly we obtain the linear combinations of the $\zeta^{1,2}(\varrho)$, defined in (21.15), as solutions of (3) discontinuous for $\varrho = 0$. Since the latter do not enter into the expansion of the plane wave, we have to write:

$$e^{i\varrho\cos\vartheta} = \sum_{n=0}^{\infty} c_n \psi_n(\varrho)\, P_n(\cos\vartheta) . \tag{5}$$

Here the coefficients c_n are still undetermined. They are determined from the orthogonality of the P_n. Namely, according to (22.8) and (22.10a) we get, if we again denote the variable of integration by $\zeta = \cos\vartheta$:

(6) $$c_n \psi_n(\varrho) = (n+\tfrac{1}{2}) \int_{-1}^{+1} e^{i\varrho\zeta} P_n(\zeta)\, d\zeta .$$

We now compare the asymptotic values for $\varrho \to \infty$ of the two sides. Due to the relation of ψ_n to $I_{n+\frac{1}{2}}$, we get for the left side from equation (19.57)

(6a) $$c_n \frac{\cos[\varrho-(n+1)\pi/2]}{\varrho} .$$

The integral on the right side can be expanded into a series in $1/\varrho$ through successive integrations by parts. Ignoring all higher powers of $1/\varrho$ for this integral we obtain:

(6b) $$\frac{e^{i\varrho}}{i\varrho} P_n(1) - \frac{e^{-i\varrho}}{i\varrho} P_n(-1) = \frac{1}{i\varrho}[e^{i\varrho} - (-1)^n e^{-i\varrho}] = 2 i^n \frac{\sin(\varrho - n\pi/2)}{\varrho} .$$

The coefficient of $2i^n$ here is the same as the coefficient of c_n in (6a). Substituting (6a,b) in (6) we therefore get

(6c) $$c_n = (2n+1)\, i^n .$$

Hence the expansion (5) of the plane wave assumes the final form

(7) $$e^{i\varrho\cos\vartheta} = \sum_{n=0}^{\infty} (2n+1)\, i^n \psi_n(\varrho)\, P_n(\cos\vartheta) .$$

This should be compared with the Fourier expansion (21.2b) of the two-dimensional plane wave. Just as we considered the latter as generating function of the I_n, so we may consider the three-dimensional plane wave as the *generating function of the* ψ_n . At the same time (6) and (6c) yield the following integral representation of the ψ_n :

(7a) $$2 i^n \psi_n(\varrho) = \int_{-1}^{+1} e^{i\varrho\zeta} P_n(\zeta)\, d\zeta ,$$

The next simple solution of the wave equation (1) is the *spherical wave*

(8) $$u = \frac{e^{ikr}}{ikr} = \frac{e^{i\varrho}}{i\varrho} .$$

This represents a *radiated* wave which progresses in the positive r-

direction if we give its time dependence by $\exp(-i\omega t)$. According to (21.15a) the solution (8) is identical with the solution of (3):

$$\zeta_0^1 = \sqrt{\frac{\pi}{2\varrho}} H_{\frac{1}{2}}^1(\varrho). \tag{8a}$$

which is singular at the point $r = 0$. We now transfer the source point $r = 0$ to the arbitrary point

$$Q = (r_0, \vartheta_0, \varphi_0)$$

Then (8) becomes

$$u = \frac{e^{ikR}}{ikR} = \zeta_0^1(kR) \begin{cases} R = \sqrt{r^2 + r_0^2 - 2\, r\, r_0 \cos\Theta}, \\ \cos\Theta = \cos\vartheta\cos\vartheta_0 + \sin\vartheta\sin\vartheta_0\cos(\varphi-\varphi_0). \end{cases} \tag{8b}$$

This function too can be expanded in spherical harmonics $P_n(\cos\Theta)$. Here the coefficients must again be solutions of the differential equation (3), namely,

$$\psi_n(\varrho) \quad \text{for} \quad r < r_0, \qquad \zeta_n^1(\varrho) \quad \text{for} \quad r > r_0,$$

the former, since the point $r = 0$ is now a regular point of the spherical wave, the latter, since the type of the *radiated* wave must be preserved in every term of the expansion. Owing to the symmetry of R in r and r_0 the reverse holds for the dependence on r_0. Hence in the coefficients of $P_n(\cos\Theta)$ we must have the factors

$$\zeta_n^1(\varrho_0) \quad \text{for} \quad r < r_0, \qquad \psi_n(\varrho_0) \quad \text{for} \quad r > r_0,$$

so that the expansion reads

$$\frac{e^{ikR}}{ikR} = \begin{cases} \sum_{n=0}^{\infty} c_n\, \zeta_n^1(\varrho_0)\, \psi_n(\varrho)\, P_n(\cos\Theta) & r < r_0, \\ \\ \sum_{n=0}^{\infty} c_n\, \psi_n(\varrho_0)\, \zeta_n^1(\varrho)\, P_n(\cos\Theta) & r > r_0. \end{cases} \tag{9}$$

The numerical factors c_n must be the same in both rows, since for $r = r_0$ the two rows coincide (except for the point Q, where we have $\Theta = 0$ and both series diverge). The situation here is the same as for the cylindrical wave in §21, equation (4): in the interior of the sphere we have a "Taylor series," in the exterior a series of the "Laurent type." The c_n can again be determined by passing to the limit $r \to \infty$. We get

$$R = r\left(1 - \frac{r_0}{r}\cos\Theta + \cdots\right) \to r - r_0\cos\Theta,$$

$$e^{ikR} \to e^{ikr}\, e^{-ikr_0\cos\Theta}.$$

Using (7) with $-i\varrho_0 \cos\Theta$ instead of $+i\varrho\cos\vartheta$ the left side of (9) becomes

$$\frac{e^{i\varrho}}{i\varrho} \sum (2n+1)(-i)^n \psi_n(\varrho_0) P_n(\cos\Theta).$$

Due to (21.15) and (19.55) the second line on the right side of (9) becomes in the limit

$$\sum c_n \psi_n(\varrho_0) \sqrt{\frac{\pi}{2\varrho}} \sqrt{\frac{2}{\pi\varrho}}\, e^{i[\varrho-(n+1)\pi/2]} P_n(\cos\Theta).$$

This will correspond term for term with the left side if we set

$$c_n = 2n+1. \tag{9a}$$

We may also consider this representation of the spherical wave as an *addition theorem for the function*

$$\zeta_0^1(kR) = \zeta_0^1\left(\sqrt{\varrho^2+\varrho_0^2-2\varrho\varrho_0\cos\Theta}\right).$$

If, on the left side of (9), we pass from the radiated to the absorbed spherical wave

$$\frac{e^{-ikR}}{-ikR} = \zeta_0^2\left(\sqrt{\varrho^2+\varrho_0^2-2\varrho\varrho_0\cos\Theta}\right)$$

then throughout the right side ζ^2 must be replaced by ζ^1. From half the sum of both representations we obtain the *addition theorem* for the regular "*standing wave*"

$$\psi_0(kR) = \frac{\sin kR}{kR} = \sum (2n+1)\psi_n(\varrho_0)\psi_n(\varrho) P_n(\cos\Theta), \tag{10}$$

here the distinction between $r \lessgtr r_0$ is unnecessary.

B. Asymptotic Behavior

If in the differential equation (22.13) of the associated spherical harmonics we pass to the limit

$$n\to\infty, \quad \vartheta\to 0, \quad n\vartheta\to\eta, \quad P_n^m(\cos\vartheta)\to O_m(\eta), \tag{11}$$

then we obtain

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{dO_m}{d\eta}\right) + \left(1-\frac{m^2}{\eta^2}\right) O_m = 0. \tag{11a}$$

This is the differential equation (19.11) of the cylindrical harmonic Z_m.

Since P_n^m and hence O_m is finite for $\vartheta \to 0$, the only permissible solution of (11a) is the Bessel function I_m. Hence we have

(12) $$O_m(\eta) = C_m I_m(\eta) \quad \text{with} \quad C_0 = 1.$$

The latter follows from the fact that for $m = 0$ and $\eta = 0$ we have $I_0(\eta) = 1$ on one hand and (due to (11)) $\vartheta = 0$ on the other, and hence $P_n(\cos\vartheta) = 1$ and $O_0(\eta) = 1$. In order to determine C_m for $m > 0$ too, we use (22.18), which for $\vartheta \to 0$ yields:

(12a) $$P_n^m \to \frac{\vartheta^m}{2^n n!} \lim_{\zeta \to 1} \frac{d^{n+m}}{d\zeta^{n+m}} (\zeta - 1)^n (\zeta + 1)^n.$$

We rewrite the function under differentiation in the form

$$(\zeta - 1)^n 2^n \left(1 + \frac{\zeta - 1}{2}\right)^n = \cdots + 2^n (\zeta - 1)^n \binom{n}{m} \left(\frac{\zeta - 1}{2}\right)^m + \cdots.$$

In this binomial expansion we have written only one term since the terms of lower degree vanish upon differentiation and those of higher degree vanish in the limit $\zeta \to 1$. The $(n + m)$-fold differentiation of this term yields

$$2^{n-m}(n + m)! \binom{n}{m} = 2^{n-m} \frac{n!}{m!} \frac{(n + m)!}{(n - m)!}.$$

Upon substitution in (12a) we obtain

(12b) $$P_n^m \to \frac{1}{m!} \left(\frac{\vartheta}{2}\right)^m \frac{(n + m)!}{(n - m)!}.$$

Here the last fraction has $2m$ more factors in its numerator than in its denominator; since $m \ll n$, we may identify all these factors with n to obtain

(12c) $$P_n^m \to \frac{1}{m!} \left(\frac{n\vartheta}{2}\right)^m n^m = \frac{1}{m!} \left(\frac{\eta}{2}\right)^m n^m.$$

Comparing this with (12), where we replace $I_m(\eta)$ by the first term of its power series (19.34), we obtain

(13) $$C_m = n^m.$$

Hence for $m > 0$ we must, in order to obtain I_m, divide P_n^m by n^m before passing to the limit.

The geometrical meaning of our result is as follows: The surface of the sphere can be replaced by its tangent plane for the neighborhood of the north pole $\vartheta \to 0$. The solution of the spatial wave equation, whose behavior on the sphere is determined by $P_n^m e^{im\varphi}$, thereby goes into a solution of the wave equation for the tangent plane, namely,

$I_m(\eta)\, e^{im\varphi}$, provided we perform the passage to the limit on P_n^m/n^m instead of P_n^m. The same obviously also holds for the south pole of the sphere $\vartheta \to \pi$.

Having thus treated the special cases $\vartheta \to 0$ and $\vartheta \to \pi$ we now wish to investigate the asymptotic value of P_n^m as $n \to \infty$ for a general $0 < \vartheta < \pi$. To this end we apply the saddle-point method to the integral (22.23), which we rewrite in the following complex form:

$$(14) \quad P_n^m(\zeta) = \frac{C}{2\pi i}\oint e^{n f(w)}\, dw, \qquad \text{with } w = e^{i\varphi}, \quad C = \frac{(n+m)!}{n!}\, e^{-i\, m\pi/2},$$

the latter due to (22.23a). The integration is to be taken over the unit circle of the w-plane in the positive (counterclockwise) sense; the function $f(w)$ stands for

$$(15) \qquad f(w) = \log\left\{\cos\vartheta + \frac{i}{2}\sin\vartheta\cdot(w + 1/w)\right\} - \frac{m+1}{n}\log w.$$

Hence

$$f'(w) = \frac{\frac{i}{2}\sin\vartheta\,(1 - 1/w^2)}{\cos\vartheta + \frac{i}{2}\sin\vartheta\cdot(w + 1/w)} - \frac{m+1}{nw}.$$

We therefore have two saddle points w_0, which, for $m \ll n$ and $\sin\vartheta \neq 0$, lie on the unit circle, namely,

$$w_0 = \pm 1$$

and we get

$$(15\text{a}) \qquad f''(w_0) = \sin\vartheta\, e^{\mp i(\vartheta - \pi/2)}.$$

With the same assumptions we get

$$(15\text{b}) \qquad e^{n f(w_0)} = e^{\pm i n\vartheta}\,(\pm 1)^{m+1}.$$

As in (19.54) we set for the two saddle points $w \mp 1 = s\, e^{i\gamma}$ and after applying (15a) we obtain

$$(15\text{c}) \quad f(w) - f(w_0) = f''(w_0)\,\frac{(w \mp 1)^2}{2} + \ldots = \frac{s^2}{2}\sin\vartheta\, e^{2i\gamma \mp i(\vartheta - \pi/2)}.$$

If we let

$$(15\text{d}) \quad 2i\gamma \mp i(\vartheta - \pi/2) = \pm i\pi \quad \text{and hence} \quad \gamma = \pm(\vartheta/2 + \pi/4),$$

then $f(w) - f(w_0)$ becomes real and $= -\frac{s^2}{2}\sin\vartheta$. This choice of γ means that for the saddle points we shall integrate along the line of steepest descent whose direction, according to (15d), still depends on ϑ. The two integrals then assume the common value

(16) $$\int_{-\varepsilon}^{+\varepsilon} e^{-\frac{n s^2}{2}\sin\vartheta}\,ds,$$

Due to (15d) and the relation $dw = e^{i\gamma}ds$ this must be multiplied by the factor

(16a) $$e^{i\gamma} = e^{\pm i(\vartheta/2+\pi/4)}$$

In the limit $n\to\infty$ the integral (16) can be reduced to the Laplace integral by a simple substitution and we obtain

(16b) $$\sqrt{\frac{2\pi}{n\sin\vartheta}}.$$

Due to (15c) and (16a,b) equation (14) becomes

(16c) $$P_n^m = \frac{C}{2\pi i}\sqrt{\frac{2\pi}{n\sin\vartheta}}\left(e^{i\{(n+\frac{1}{2})\vartheta+\pi/4\}} + e^{-i\{(n+\frac{1}{2})\vartheta+\pi/4+(m+1)\pi\}}\right).$$

Since we have made the assumption $m \ll n$ throughout, the value of C in (14) can be reduced to $C = n^m \exp\{-im\pi/2\}$ by the same reasoning that led from (12b) to (13). Hence C/i becomes

$$n^m\exp\{-i(m+1)\pi/2\}$$

Combining this with the two exponential functions in (16c) we obtain:

(17) $$P_n^m = n^m\sqrt{\frac{2}{\pi n\sin\vartheta}}\cos\left\{(n+\tfrac{1}{2})\vartheta - \frac{m\pi}{2} - \frac{\pi}{4}\right\}.$$

Therefore P_n^m for real n is a rapidly oscillating function of varying amplitude; the amplitude is small in the neighborhood of $\vartheta = \pi/2$ and increases symmetrically for decreasing or increasing ϑ. For $\vartheta = 0$ and π equation (17) breaks down since, according to (15a), $f''(w_0)$ vanishes and the series for $f(w) - f(w_0)$ starts with the third term (compare with the limiting case on p. **122** that led to the Airy integral). Equation (17) is then replaced by (12c).

We shall apply (17) in the appendix of Chapter VI for the case of complex n with a positive real part in which our derivation remains valid.

C. The Spherical Harmonic as an Electric Multipole

In this section we return to potential theory. Since in §22 E we were able to define the surface spherical harmonics of degree n as homogeneous potentials of degree n (or better of degree $-n-1$), it must be possible to generate them with the help of repeated differentiation "with respect to n-directions" of the elementary potential $1/R$. This is the point of view of Maxwell in Chapter IX of his treatise. We express this by the Maxwell rule:

$$(18)\quad Y_n = \frac{1}{n!}\frac{\partial}{\partial h_1}\frac{\partial}{\partial h_2}\cdots\frac{\partial}{\partial h_n}\left(\frac{1}{R}\right)\cdots \quad \begin{cases} R^2=(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2, \\ \text{Lim } x,\, y,\, z\to 0, \quad \text{Lim } R\to r, \\ r^2=\xi^2+\eta^2+\zeta^2=1\,. \end{cases}$$

The "action point" $P = (\xi, \eta, \zeta)$ is to lie on a sphere of radius 1, the "source point" $Q = (x,y,z)$ is to lie in the neighborhood of the origin. The "directional differentiations" $h_1,h_2 \ldots ,h_n$ can be performed both on the coordinates of P and on the coordinates of Q. We do the latter and then pass to the limit $x,y,z \to 0$, $R \to r$. In this way we obtain a *multipole* at Q whose order increases with the order of differentiation.

We start with the simplest case in which the directions $h_1,h_2 \ldots$ coincide, say, with the z-direction. The surface spherical harmonic which is obtained in this way is symmetric with respect to the z-axis and hence is a *zonal* spherical harmonic of the Legendre type P_n. We follow its genesis from line to line denoting the limit process of (18) by $\to$:

$$1)\qquad \frac{\partial}{\partial h_1}\frac{1}{R}=\frac{\partial}{\partial z}\frac{1}{R}=\frac{\zeta-z}{R^3}\to\zeta=P_1,$$

$$2)\qquad \frac{1}{2!}\frac{\partial}{\partial h_1}\frac{\partial}{\partial h_2}\frac{1}{R}=\frac{1}{2}\frac{\partial^2}{\partial z^2}\frac{1}{R}=\frac{1}{2}\frac{\partial}{\partial z}\frac{\zeta-z}{R^3}=-\frac{1}{2}\frac{1}{R^3}+\frac{3}{2}\frac{(\zeta-z)^2}{R^5}$$
$$\to\frac{3}{2}\zeta^2-\frac{1}{2}=P_2,$$

$$3)\qquad \frac{1}{3!}\frac{\partial}{\partial h_1}\frac{\partial}{\partial h_2}\frac{\partial}{\partial h_3}\frac{1}{R}=\frac{1}{3!}\frac{\partial^3}{\partial z^3}\frac{1}{R}=\frac{1}{3}\frac{\partial}{\partial z}\left(-\frac{1}{2}\frac{1}{R^3}+\frac{3}{2}\frac{(\zeta-z)^2}{R^5}\right)$$
$$=-\frac{3}{2}\frac{\zeta-z}{R^5}+\frac{5}{2}\frac{(\zeta-z)^3}{R^7}\to\frac{5}{2}\zeta^3-\frac{3}{2}\zeta=P_3,$$

$$4)\qquad \frac{1}{4!}\frac{\partial}{\partial h_1}\frac{\partial}{\partial h_2}\frac{\partial}{\partial h_3}\frac{\partial}{\partial h_4}\frac{1}{R}=\frac{1}{4!}\frac{\partial^4}{\partial z^4}\frac{1}{R}=\frac{1}{4}\frac{\partial}{\partial z}\left(-\frac{3}{2}\frac{\zeta-z}{R^5}+\frac{5}{2}\frac{(\zeta-z)^3}{R^7}\right)$$
$$=\frac{3}{8}\frac{1}{R^5}-\frac{15}{4}\frac{(\zeta-z)^2}{R^7}+\frac{35}{8}\frac{(\zeta-z)^4}{R^9}\to\frac{35}{8}\zeta^4-\frac{15}{4}\zeta^2+\frac{3}{8}=P_4.$$

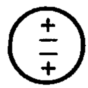
This sequence $P_1, \ldots ,P_4$, which can be completed by the zeroth derivative $P_0 = 1$ of $1/R$, coincides with the values obtained from the original definition on p. 23 (the variable x being replaced by ζ). This follows by necessity from the relation between spherical harmonics and homogeneous potentials, so that in (18) we were free to determine the normalizing factor $1/n!$ only. We note the connection between this rule and the second equation (22.3) which, after the substitution $r_0 = z$ (Q on the z-axis) and $r = 1$ (P on the unit sphere), can be written:

$$\frac{1}{R} = \sum_{m=0}^{\infty} z^m \, P_m(\cos\vartheta);$$

and hence for $z \to 0$ we indeed have

$$\frac{1}{n!}\frac{d^n}{dz^n}\frac{1}{R} = P_n(\cos\vartheta) = P_n(\zeta).$$

We list the names and symbols for the successive multipoles. In order to avoid the limit process $Q \to 0$ we replace the differentiation with respect to z (coordinate of source point) by a differentiation with respect to ζ (coordinate of action point) but with opposite sign, and interpret r as the distance $OP = \sqrt{\xi^2 + \eta^2 + \zeta^2}$.

Unipole	charge scheme		potential $\frac{1}{r}$,
Bipole	charge scheme		potential $-\frac{1}{1!}\frac{d}{d\zeta}\frac{1}{r}$,
Quadrupole	charge scheme		potential $+\frac{1}{2!}\frac{d^2}{d\zeta^2}\frac{1}{r}$.

By contracting two quadrupoles of opposite scheme we obtain the "octupole" with potential

$$-\frac{1}{3!}\frac{d^3}{d\zeta^3}\frac{1}{r}.$$

(The determination of the corresponding charge scheme is left to the reader.) By n-fold differentiation we obtain the

$$2^n\text{-pole and its potential } \frac{(-1)^n}{n!}\frac{d^n}{d\zeta^n}\frac{1}{r}.$$

In wireless telegraphy one uses the term dipole instead of bipole. Quadrupole and octupole radiations occur in atomic physics.

We now have to consider examples of differentiations with respect to *different directions*. In addition to differentiations in the z-direction we now consider differentiations in the (x,y)-plane. In order to preserve a certain degree of symmetry we consider differentiation with respect to, say, m equally spaced directions in the (x,y)-plane (in "star form," with an angle of π/m between two adjacent directions) together with $n-m$ differentiations still taken with respect to the z-direction. We thus obtain the *tesseral surface spherical harmonics*

$$P_n^m(\zeta)\,\Phi_m(\varphi), \quad \Phi_m(\varphi) = e^{\pm im\varphi}, \tag{19}$$

where instead of the two exponential functions given here we may have a linear combination, e.g., $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\varphi$. To this category (19) belong the so-called *sectorial surface spherical harmonics* (this notation too is Maxwell's) with $m = n$, which according to (22.18) is represented by

$$P_n^n \Phi_n = \sin^n \vartheta \frac{d^n P_n}{d\zeta^n} \Phi_n = \frac{(2n)!}{2^n n!} \sin^n \vartheta\, e^{\pm in\varphi}. \tag{19a}$$

We discuss this further for $n = 2$. The star formed arrangement is obtained here if we take h_1 and h_2 in the x- and y-direction. Equation (18) and what follows then yield

$$P_2^2 = \frac{1}{2}\frac{\partial^2}{\partial x\,\partial y}\frac{1}{R} = \frac{1}{2}\frac{\partial}{\partial x}\frac{\eta - y}{R^3} = \frac{3}{2}\frac{(\xi - x)(\eta - y)}{R^5}$$
$$\to \frac{3}{2}\,\xi\,\eta = \frac{3}{4}\sin^2\vartheta \sin 2\varphi,$$

which is indeed of the type (19a). In this case too we speak of a *quadrupole* (see the right hand side of the diagram below; the left side, which is placed differently in space belongs to P_2^1).

Quadrupole P_2^1 Quadrupole P_2^2.

The fact that (18) yields the complete system of the $2n + 1$ surface spherical harmonics of degree n, follows from the number of constants in (18): two directional constants for every differentiation h and one multiplicative factor.

D. Some Remarks about the Hypergeometric Function

The hypergeometric function is best defined by its differential equation:

$$z(1 - z)\,y'' + [\gamma - (\alpha + \beta + 1)\,z]\,y' - \alpha\beta\, y = 0 \tag{20}$$

From this equation we deduce the Gaussian series representation (11.10a) according to the procedure of §19 C. We set

$$y = z^\lambda (a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k + \cdots), \tag{21}$$

with undetermined exponent λ and coefficients a_k. We substitute this in (20) and set the coefficients of the initial term $z^{\lambda-1}$ and the general term $z^{\lambda+k}$ equal to zero. In this way on one hand we obtain:

$$\lambda(\lambda-1+\gamma)=0, \tag{21a}$$

and on the other hand

$$\begin{aligned}&[(\lambda+k+1)(\lambda+k)+\gamma(\lambda+k+1)]\,a_{k+1}\\ &\quad=[(\lambda+k)(\lambda+k-1)+(\alpha+\beta+1)(\lambda+k)+\alpha\beta]\,a_k.\end{aligned} \tag{21b}$$

Equation (21a) has the solutions

$$\lambda=0 \quad\text{and}\quad \lambda=1-\gamma; \tag{22a}$$

We first consider the former solution and by substituting it in (21b) obtain

$$a_{k+1}=\frac{k(k-1)+(\alpha+\beta+1)k+\alpha\beta}{(k+1)k+\gamma(k+1)}\,a_k=\frac{(\alpha+k)(\beta+k)}{(k+1)(\gamma+k)}\,a_k \tag{22b}$$

Hence if we set $a_0 = 1$ we obtain the Gauss series

$$y=y_1=F(\alpha,\beta,\gamma,z)=1+\frac{\alpha\beta}{1\cdot\gamma}z+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}z^2+\cdots. \tag{23}$$

The other solution of (22a) yields:

$$y=y_2=z^{1-\gamma}F(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma,\ z). \tag{23a}$$

There are also a large number of related representations for altered parameters α, β, γ and linearly transformed z, which coincide regionwise with (23). They have been compiled lovingly by Gauss as "relationes inter contiguas."

If we compare the differential equation (22.6a) of the zonal spherical harmonics

$$(1-\zeta^2)P''-2\zeta P'+n(n+1)P=0 \tag{24}$$

with equation (20) then we see that it is obtained from the latter by the substitution

$$z=\frac{1\mp\zeta}{2},\qquad \alpha=-n,\qquad \beta=n+1,\qquad \gamma=1$$

From this we see that P_n must coincide both with y_1 and y_2 up to a factor. Namely, we have:

$$\begin{aligned}P_n(\zeta)&=F\left(-n,n+1,1,\frac{1-\zeta}{2}\right)\\ &=(-1)^nF\left(-n,\ n+1,\ 1,\frac{1+\zeta}{2}\right),\end{aligned} \tag{24a}$$

which also yields the correct normalization $P_n(1) = 1$ for $\zeta = +1$. The series (24a) for P_n breaks off as does every hypergeometric series with negative integral α or β : since $\alpha = -n$ we have that P_n is a polynomial of degree n (the coefficients of $(1 \mp \zeta)^{n+1}$ and of all subsequent powers contain the factor $\alpha + n = -n + n = 0$ in the numerator). We remark that the series for P_n in terms of $1 - \zeta$ is simpler (since it is hypergeometric) than the series in terms of ζ. The latter reads

$$P_n = 1 + \frac{(-n)(n+1)}{1 \cdot 1} \frac{1-\zeta}{2}$$

$$(24\text{b}) \qquad + \frac{(-n)(-n+1)(n+1)(n+2)}{2!\,2!} \left(\frac{1-\zeta}{2}\right)^2 + \cdots + (-1)^n \frac{(2n)!}{n!\,n!} \left(\frac{1-\zeta}{2}\right)^n$$

$$= \sum_{p=0}^{n} (-1)^p \frac{(n+p)!}{(n-p)!} \frac{1}{p!\,p!} \left(\frac{1-\zeta}{2}\right)^p .$$

The associated P_n^m can also be represented by a hypergeometric series. We merely have to consider the general relation which is obtained from (23) by termwise differentiation:

$$(25) \qquad \frac{d}{dz} F(\alpha, \beta, \gamma, z) = \frac{\alpha \beta}{\gamma} F(\alpha + 1, \beta + 1, \gamma + 1, z).$$

Hence for positive m we obtain the representation

$$(26) \qquad \begin{aligned} P_n^m(\zeta) &= C (1 - \zeta^2)^{m/2} F\left(m - n,\ m + n + 1,\ m + 1,\ \frac{1-\zeta}{2}\right), \\ C &= \frac{(n+m)!}{2^m\, m!\,(n-m)!} . \end{aligned}$$

from (22.18). For the negative integral m this representation breaks down, but can be extended to that case by a limit process; the result then coincides with our general definition (22.18).

From the Gaussian hypergeometric function we derive the *confluent hypergeometric function,* which is of the utmost importance in wave mechanics. It depends only on two parameters α and γ since the third parameter β is subjected to the following limit process:

$$(27) \qquad \beta \to \infty,\ z \to 0,\ \beta z \to \varrho \quad (\varrho = \text{arbitrary finite number}).$$

We then obtain from (23)

$$(28) \qquad F(\alpha, \gamma, \varrho) = 1 + \frac{\alpha}{\gamma} \frac{\varrho}{1} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)} \frac{\varrho^2}{2!} + \cdots$$

and considering the fact that

$$\frac{d}{dz} = \frac{d}{d\varrho} \cdot \frac{d\varrho}{dz} = \beta \frac{d}{d\varrho}$$

we obtain the corresponding differential equation from (20) after dividing by β :

$$\varrho \frac{d^2F}{d\varrho^2} + (\gamma - \varrho)\frac{dF}{d\varrho} - \alpha F = 0. \tag{29}$$

We shall encounter this equation again in connection with the eigenfunctions of hydrogen in wave mechanics.

E. Spherical Harmonics of Non-Integral Index

Our representation must still be completed in two directions. We have been restricted so far to the case of *integral numbers n and m* and to *functions P_n^m, which were finite throughout.* Both these restrictions were suggested by the connection with potential theory.

Concerning the first point, we see that for non-integral n the hypergeometric series at the points $\zeta = \pm 1$ in ascending powers of $1 \mp \zeta$ does not break off as it does in the case of integral n. The solution, which is regular at the north pole $\zeta = +1$, diverges at the south pole $\zeta = -1$ and vice versa. Thus (24a) is valid for integral n only. The "requirement of finiteness all over the sphere" can therefore be satisfied only for integral n. The possibility of non-integral m is excluded by the requirement of uniqueness with respect to the φ-coordinate.

The type of singularity of $P_n(\zeta)$ for non-integral n can be deduced from the general theory of hypergeometric series. We prefer, however, to deduce it by direct calculation.

According to their original definition in (22.3) the P_n are the coefficients of a Taylor series which progresses in powers of $t = r/r_0$; hence for integral n:

$$P_n(\zeta) = \frac{1}{n!}\frac{d^n}{dt^n}\frac{1}{\sqrt{1-2\zeta t+t^2}} \quad \text{at} \quad t = 0. \tag{30}$$

According to Cauchy's theorem this can be written as:

$$P_n(\zeta) = \frac{1}{2\pi i}\oint \frac{dt}{t^{n+1}}\frac{1}{\sqrt{1-2\zeta t+t^2}}. \tag{30a}$$

This representation also holds for non-integral n, except that due to the many-valued character of t^{-n-1} we have to perform a branch cut, e.g., from $t = -\infty$ to $t = 0$, and that the path of integration is now a loop which starts on the negative side of the cut at $t = -\infty$, then circles the point $t = 0$ in a counterclockwise direction and ends on the positive side of the cut at $t = -\infty$. It is clear that for this definition the differential equation of P_n is satisfied regardless of whether or not n is integral. Equation (30a) defines that particular solution of the differential equation which is regular at $\zeta = 1$ and satisfies the normalizing condition $P_n(1) = 1$. Namely for $\zeta = 1$ we obtain from (30a)

$$P_n(1) = -\frac{1}{2\pi i}\oint \frac{dt}{t^{n+1}}\,\frac{1}{t-1}\,. \tag{30b}$$

The integrand now has a simple pole at $t = 1$. The loop described above can now be deformed into a path which circles the pole in a clockwise direction. According to Cauchy's theorem the integral then has the value $-2\pi i$, and hence the right side of (30b) has the required value $+1$.

For $\vartheta > 0$ the integrand of (30a) has two further branch points that are due to the square root in the denominator and lie on the unit circle of the t-plane at

$$t = e^{i\vartheta} \quad\text{and}\quad t = e^{-i\vartheta}$$

We connect these branch points by a branch cut, e.g., along the unit circle. The path of integration may not cross this cut either. For $\vartheta = \pi - \delta$, $\delta \ll 1$, the endpoints of the cut approach the negative real axis and restrict the path of integration between them. This explains why (30a) becomes singular for $\delta \to 0$, or in other words for $\vartheta \to \pi$, $\zeta \to -1$

In order to discuss this singularity we write

$$t = e^{i\pi}(1+\tau) \quad\text{and}\quad t = e^{-i\pi}(1+\tau);$$

in the neighborhood of the point $t = -1$ on the upper and lower edge of our branch cut respectively. Then, except for terms of higher order in τ and δ, the square root in (30a) becomes

$$\sqrt{1 - 2\zeta t + t^2} = \sqrt{\tau^2 + \delta^2}\,.$$

Hence for small δ only the neighborhood of $\tau = 0$ contributes to our limiting value as $\vartheta \to \pi$. We may, therefore, restrict the integration over the upper and lower edges to the small region between

$$\tau = +\varepsilon \quad\text{and}\quad \tau = -\varepsilon$$

and considering the orientation on the two edges we may write:

$$\frac{dt}{t^{n+1}} = \begin{cases} e^{+i\pi n}\,d\tau & \text{lower edge} \\ -e^{-i\pi n}\,d\tau & \text{upper edge,} \end{cases}$$

hence:

$$(31)\qquad P_n(\zeta) = \frac{e^{+i\pi n} - e^{-i\pi n}}{2\pi i} \int_{+\varepsilon}^{-\varepsilon} \frac{d\tau}{\sqrt{\tau^2+\delta^2}} = \frac{\sin n\pi}{\pi} \int_{+\varepsilon}^{-\varepsilon} \frac{d\tau}{\sqrt{\tau^2+\delta^2}}.$$

Now according to a well known formula we have

$$\int \frac{d\tau}{\sqrt{\tau^2+\delta^2}} = \log\left(\tau + \sqrt{\tau^2+\delta^2}\right)$$

for undetermined upper and lower limits of integration. Hence for the definite integral in (31) we have

$$\log\left(-\varepsilon + \sqrt{\varepsilon^2+\delta^2}\right) - \log\left(+\varepsilon + \sqrt{\varepsilon^2+\delta^2}\right)$$

and for $\delta \ll \varepsilon$

$$\log \frac{1}{2}\frac{\delta^2}{\varepsilon} - \log\left(2\varepsilon + \frac{1}{2}\frac{\delta^2}{\varepsilon}\right).$$

In the limit $\delta \to 0$ we have $\log \delta^2$ as the leading term; hence we obtain from (31)

$$(32)\qquad \lim_{\zeta=-1} P_n(\zeta) = \frac{\sin n\pi}{\pi} \log \delta^2 + \cdots,$$

The terms[22] . . . which have been omitted here reduce for $\delta \to 0$ to a finite constant which is of course independent of ε

F. Spherical Harmonics of the Second Kind

At the beginning of Section E we saw that for non-integral n two different solutions P_n of the hypergeometric differential equation exist. Only for integral n do these solutions coincide. But in this latter case, too, a second solution must exist in addition to the everywhere regular solution found above. This solution will be singular at the points $\zeta = \pm 1$ We call it a *spherical harmonic of the second kind* and denote it by Q_n.

The type of singularity can be determined from general theorems. In (21a) we saw that the quadratic equation for the exponent λ for the case of spherical harmonics $(\gamma = 1)$ has the *double root* $\lambda = 0$. By a passage to the limit we see that this indicates a logarithmic singularity for $\zeta = \pm 1$. Just as in the case of spherical harmonics of the first kind, we obtain detailed information about the *spherical harmonics*

[22] They are computed in Hobson's textbook, equation (53), p. 225, which was quoted on p. 129 above.

of the second kind Q_n from a *generating function*[23] (C. Neumann):

$$\frac{1}{\eta-\zeta}=\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)Q_n(\eta)\,P_n(\zeta)\,. \tag{33}$$

Hence the $Q_n(\eta)$ are defined as coefficients in the expansion of $1/(\eta-\zeta)$ in the P_n, and therefore they are given by the following *integral representation* (F. Neumann):

$$Q_n(\eta)=A.M.\int_{-1}^{+1}P_n(\zeta)\,\frac{d\zeta}{\eta-\zeta}\,. \tag{34}$$

In this formula the path of integration is to avoid the singular point $\zeta=\eta$ by going around it through the complex domain both to the right and to the left, and the symbol $A.M.$, which we shall omit in the future, indicates that we have to take the arithmetic mean of the two values obtained (this is identical with the so-called "principal value" of the integral). The fact that this avoidance of the singularity is not possible at the limits $\zeta=\pm 1$ implies the above-mentioned logarithmic singularity.

That $Q_n(\eta)$ satisfies the differential equation (24) (written in terms of η), is to be expected from the symmetry of the defining equation in η, ζ and Q,P, but it can also be demonstrated directly as follows: we abbreviate equation (24) to

$$L_\zeta\{P\}=0\,,\qquad L_\zeta=\frac{d}{d\zeta}(1-\zeta^2)\frac{d}{d\zeta}+n(n+1)\,,$$

and by $L_\eta\{Q\}$ we mean the analogous expression written in terms of η and Q; we further note the identity

$$L_\eta\left\{\frac{1}{\eta-\zeta}\right\}=L_\zeta\left\{\frac{1}{\eta-\zeta}\right\}.$$

Then from (34) we have

$$L_\eta\{Q_n\}=\int_{-1}^{+1}P_n(\zeta)\,L_\zeta\left\{\frac{1}{\eta-\zeta}\right\}d\zeta \tag{35}$$

We integrate by parts twice, then the terms which are due to the limits $\zeta=\pm 1$ vanish on account of the factor $1-\zeta^2$ in L_ζ and we obtain

$$L_\eta\{Q_n(\eta)\}=\int_{-1}^{+1}L_\zeta\{P_n(\zeta)\}\frac{d\zeta}{\eta-\zeta}=0\ ,\ \text{q.e.d.} \tag{36}$$

[23] Usually double this function is used, so that in (33) we have $2n+1$ instead of $n+\frac{1}{2}$. Correspondingly our $Q_n(\eta)$ differ from the customary ones by the factor 2.

It is now easy to compute the first $Q_n(\eta)$ in terms of the known $P_n(\eta)$ with the help of (34. We deduce for $|\eta| < 1$

$$\text{from } P_0 = 1: \qquad Q_0 = \log\frac{1+\eta}{1-\eta},$$

$$\text{from } P_1 = \zeta: \qquad Q_1 = -2 + \eta\log\frac{1+\eta}{1-\eta}.$$

The general law is (Christoffel):

$$Q_n(\eta) = \Pi + P_n(\eta)\log\frac{1+\eta}{1-\eta}, \tag{37}$$

where Π is a polynomial of degree $n - 1$ which is composed additively from all those P_{n-2k-1} for which the index is non-negative. Finally we obtain from (34), through m-fold differentiation with respect to η and multiplication by $(1-\eta^2)^{m/2}$ (which is analogous to $\sin^m\vartheta$),

$$Q_n^m(\eta) = (-1)^m m!\,(1-\eta^2)^{m/2}\int_{-1}^{+1}\frac{P_n(\zeta)}{(\eta-\zeta)^{m+1}}\,d\zeta. \tag{38}$$

Appendix I

Reflection on a Circular-Cylindrical or Spherical Mirror

Referring back to Fig. 8 and the notations defined there we continue the treatment of the problem which we started in §6.

a) Circular-cylindrical metal mirror. The incoming wave (electric vector which is perpendicular to the plane of the drawing) is

$$w = e^{ikr\cos\varphi} \approx I_0(kr) + 2\sum_{n=1}^{N} i^n I_n(kr)\cos n\varphi. \tag{1}$$

(see (21.2b)). This representation holds for the entire r,φ-plane and for $r = a$ it defines the function $-f(\varphi)$ in (6.4). The sum of $N+1$ terms on the right is the best approximation to w that can be obtained by the method of least squares; the fact that the coefficients in this sum are the same as those in the exact non-truncated series (21.2b) follows from the "finality" of the Fourier series. We write the radiation which is reflected (diffracted, scattered) by the mirror as the sum of $N+1$ particular solutions of the differential equation

$$\Delta u + k^2 u = 0$$

for $r < a$ in the form:

$$u = \sum_{0}^{N} C_n\frac{I_n(kr)}{I_n(ka)}\cos n\varphi; \tag{2}$$

(since the solution must be continuous for $r = 0$ only the I_n can occur in the representation; the sine terms disappear on account of the symmetry of the incoming wave with respect to $\varphi = 0$). The denominator $I_n(ka)$ is used for the sake of convenience and merely influences the meaning of the constants C_n which are as yet undetermined. The same holds for the denominators in equation (3) below.

We write the radiation which is scattered by the mirror to the outside $r > a$ as the following sum of $N + 1$ particular solutions of the wave equation:

$$v = \sum_0^N D_n \frac{H_n^1(kr)}{H_n^1(ka)} \cos n\varphi. \tag{3}$$

The time dependence of the whole process should be thought of as given by $\exp(-i\omega t)$; hence only the H^1 occur; the H^2 would correspond to absorbed waves. As we saw in §6 the boundary conditions (6.8) to (6.11) imply $C_n = D_n$ and, according to the method of least squares, the system of linear equations (6.12). The constant γ_n which occurs there is determined from (6.7), (6.11a) and the equations (3), (4) above as

$$\gamma_n = ka \left(\frac{I_n'(ka)}{I_n(ka)} - \frac{H_n^{1\prime}(ka)}{H_n^1(ka)} \right). \tag{4}$$

According to a well known theorem in the theory of linear differential equations we can rewrite this in the simpler form (see exercise IV.8)

$$\gamma_n = \frac{-2i/\pi}{I_n(ka)\, H_n^1(ka)}. \tag{5}$$

We introduce the notation:

$$a_{nm} = \int_\alpha^\pi \cos n\varphi \cos m\varphi \, d\varphi \tag{6}$$

and obtain by a simple transformation

$$\int_0^\alpha \cos n\varphi \cos m\varphi \, d\varphi = \frac{\pi}{(2)} \delta_{nm} - a_{nm}. \tag{7}$$

Here and in the following the symbol (2) stands for the number 2 when $n > 0$ and for the number 1 when $n = 0$. Then the left side of (6.12) becomes

$$\frac{\pi}{(2)} C_m + \sum_{n=0}^N a_{nm} (\gamma_n \gamma_m - 1) C_n \tag{8}$$

Setting $f(\varphi) = -w$, where w is as in equation (1), and $r = a$, we obtain for the right side of (6.12):

$$-\pi i^m I_m(ka) + \sum_{n=0}^{N} a_{nm}(2) i^n I_n(ka). \tag{8a}$$

Hence the system (6.12) becomes

$$C_m + (2) i^m I_m(ka) - \frac{(2)}{\pi} \sum_{n=0}^{N'} a_{nm} \{C_n + (2) i^n I_n(ka) - \gamma_n \gamma_m C_n\} = 0, \tag{9}$$

which must be satisfied for all $m = 0,1, \ldots, N$.

In order to discuss this system we first set $\alpha = \pi$, in other words we consider a complete circle (spatially speaking a closed, totally conductive, cylinder). According to (6) we then have $a_{nm} = 0$ and (9) yields

$$C_m = -(2) i^m I_m(ka). \tag{10}$$

This result is somewhat trivial. For, by substituting the value (10) of C_n for D_n in (3), we obtain the rigorous solution v of the corresponding scattering problem for $r > a$:

$$v = -\sum_{n=0}^{N} (2) i^n I_n(ka) \frac{H_n^1(kr)}{H_n^1(ka)} \cos n\varphi, \tag{11}$$

a radiated wave which, on the cylinder $r = a$, exactly cancels the incoming wave w of equation (1) and hence for $N \to \infty$ yields the rigorous solution of the scattering problem. In the same manner we obtain for u:

$$u = -\sum_{n=0}^{N} (2) i^n I_n(kr) \cos\varphi = -w. \tag{12}$$

This result is also trivial since in the interior of the closed circle $r = a$ we must have $u + v = 0$.

We now wish to investigate whether our equation (10) yields a useful approximation in the case $\alpha < \pi$, too. To this end we set

$$C_n = -(2) i^n I_n(ka) + \beta_n \tag{13}$$

where β_n is a correction term, and obtain from (9):

$$\beta_m - \frac{(2)}{\pi} \sum_{n=0}^{N} a_{nm} \{\beta_n(1 - \gamma_n\gamma_m) + \gamma_n\gamma_m (2) i^n I_n(ka)\} = 0. \tag{14}$$

As in (6.12) this is a system of $N + 1$ (and in the limit infinitely many) linear equations with which we can do practically nothing. However if we assume that $\pi - \alpha$ is small, that is that the cylinder has only a *narrow slit*, then a_{nm} becomes small and the product $\beta_n a_{nm}$ becomes *small of the second order.* If we neglect this term then (14) becomes simply

$$\frac{\beta_m}{\gamma_m} = -\frac{(2)}{\pi} \sum_{n=0}^{N} (2) i^n a_{nm} \gamma_n I_n(ka), \tag{15}$$

which is an explicit value for β_m and from (13) yields an explicit value for C_m.

The narrowness of the slit necessary for this consideration can be estimated by a physical consideration: its width must be *small compared to the wavelength* of the incoming radiation; only in this case does the interior field go over continuously into the zero field of the closed cylinder. Hence we must have

$$\frac{(\pi-\alpha)a}{\lambda} < 1. \tag{16}$$

This condition can be satisfied only approximately for Hertz waves. In the properly optical case this approximation breaks down. This is the reason we spoke of a "quasi-optical case" on p. 29. In the well known Hertz experiment with concave mirrors, for which we have, say, $\alpha=\pi/2$, $\lambda=$ 200 cm., a = 50 cm., equation (16) is approximately satisfied so that our system of approximation is justified.

b) The sphere segment as an acoustic reflector. In order to avoid discussions on vectors we deal with scalar acoustic waves instead of directed optical radiation. By w,u,v we mean the *velocity potentials* of the primary and secondary (reflected) radiation in the interior ($r < a$) and exterior ($r > a$) of the sphere. Let the sphere segment be given by $r=a$, $0<\vartheta<\alpha$. According to (24.7) we write w in the form

$$w=\sum_{n=0}^{N}(2n+1)\,i^n\,\psi_n(kr)\,P_n(\cos\vartheta)\,; \tag{17}$$

this is the best possible approximation of a plane wave $\exp(ikr\cos\vartheta)$ by $N+1$ spherical harmonics according to the method of least squares. We further write

$$u=\sum C_n\frac{\psi_n(kr)}{\psi_n'(ka)}\,P_n(\cos\vartheta) \qquad r<a, \tag{18}$$

$$v=\sum D_n\frac{\zeta_n(kr)}{\zeta_n'(ka)}\,P_n(\cos\vartheta) \qquad r>a, \tag{19}$$

where the constants C_n,D_n are as yet undetermined. Concerning the denominators see the remark to equation (2). The function ζ_n is the Bessel function with half-integral index which was defined in (21.15) and corresponds to the Hankel function H_n^1. On the sphere segment (which was assumed rigid) we have

$$\frac{\partial}{\partial n}(u+w)=\frac{\partial}{\partial n}(v+w)=0 \quad \text{for} \quad 0\le\vartheta<\alpha \quad \text{and } r=a,$$

and for reasons of continuity we must have

$$u=v,\ \frac{\partial u}{\partial r}=\frac{\partial v}{\partial r} \quad \text{for} \quad \alpha<\vartheta\le\pi \quad \text{and} \quad r=a\,.$$

This condition again leads to $D_n = C_n$ and to the equations:

$$\text{(20)} \qquad \begin{aligned} &\sum C_n P_n(\cos\vartheta) = f(\vartheta), && 0<\vartheta<\alpha,\\ &\sum D_n \gamma_n P_n(\cos\vartheta) = 0, && \alpha<\vartheta<\pi. \end{aligned}$$

which are analogous to (6.10) and (6.11). Here $f(\vartheta)$ stands for the value of $-\partial w/\partial(kr)$ for $r = a$,

$$\text{(20 a)} \qquad f(\vartheta) = -\left(\frac{\partial w}{\partial(kr)}\right)_{r=a}, \qquad w = e^{ikr\cos\vartheta}$$

and in analogy to (5)

$$\text{(20 b)} \qquad \gamma_n = \frac{\psi_n(ka)}{\psi_n'(ka)} - \frac{\zeta_n(ka)}{\zeta_n'(ka)}.$$

Introducing the abbreviation

$$a_{nm} = \int_\alpha^\pi P_n P_m \sin\vartheta\, d\vartheta$$

we obtain

$$\text{(21 a)} \qquad \int_0^\alpha P_n P_m \sin\vartheta\, d\vartheta = \frac{\delta_{nm}}{m+\frac{1}{2}} - a_{nm}$$

and from (20a)

$$\text{(21 b)} \qquad \begin{aligned} \int_0^\alpha f(\vartheta) P_m(\cos\vartheta)\sin\vartheta\, d\vartheta &= -\sum_{n=0}^N (2n+1)\, i^n \psi_n'(ka) \int_0^\alpha P_n P_m \sin\vartheta\, d\vartheta \\ &= -2\, i^m \psi_m'(ka) + \sum_{n=0}^N a_{nm}(2n+1)\, i^n \psi_n'(ka). \end{aligned}$$

In analogy to (6.12) the method of least squares now yields the following system of equations for the C_n:

$$\text{(22)} \qquad \sum_{n=0}^N C_n \left\{\int_0^\alpha P_n P_m \sin\vartheta\, d\vartheta + \gamma_n\gamma_m \int_\alpha^\pi P_n P_m \sin\vartheta\, d\vartheta\right\} = \int_0^\alpha f(\vartheta) P_m \sin\vartheta\, d\vartheta,$$

which holds for $m = 0,1, \ldots, N$; due to (21a,b) this becomes

$$\text{(22a)} \qquad \begin{aligned} &\frac{C_m}{m+\frac{1}{2}} - \sum_{n=0}^N a_{nm} C_n (1-\gamma_n\gamma_m) \\ &\qquad = -2\, i^m \psi_m'(ka) + \sum_{n=0}^N a_{nm}(2n+1)\, i^n \psi_n'(ka). \end{aligned}$$

which, as in equation (9), can be rearranged to

$$\text{(23)} \qquad \begin{aligned} &\frac{1}{m+\frac{1}{2}}\left[C_m + (2m+1)\, i^m \psi_m'(ka)\right] \\ &\qquad - \sum_{n=0}^N a_{nm}\{C_n + (2n+1)\, i^n \psi_n'(ka) - \gamma_n\gamma_m C_n\} = 0. \end{aligned}$$

We again start with the limiting case $\alpha = \pi$ of a closed sphere in which $a_{nm} = 0$. Then (23) yields

$$C_m = -(2m+1)\, i^m \psi'_m(ka). \tag{24}$$

Substituting this value in (19) we obtain the rigorous solution v of our acoustic problem for $r > a$, namely, the reflection of an incoming wave on a closed sphere. In the interior $r < a$ of the sphere we obtain, by substituting C_m in (18), a field u which, as it should be, is the negative of the field w of the incoming wave.

The next problem is that of a spherical surface with a circular hole in the neighborhood of $\vartheta = \pi$. By setting

$$C_m = -(2m+1)\, i^m \psi'_m(ka) + \beta_m, \tag{25}$$

and ignoring the product term of second order $a_{nm}\beta_n$ we obtain from (23)

$$\frac{1}{m+\frac{1}{2}} \frac{\beta m}{\gamma m} = \sum_{n=0}^{N}{}'(2n+1)\, i^n a_{nm} \gamma_n \psi'_n(ka), \tag{26}$$

which is an explicit computation of the correction term β_m and hence of the coefficients C_m. The reader should compare this result with the analogous result for the problem of the cylinder in equation (15). Just as the width of the slit there, so the diameter of the circular hole here must be small compared to the wavelength of the incoming radiation. Hence, here too, we can treat only a "quasi-acoustical" problem, the problem of *infra-sound*, which is very far from the more interesting problem of *ultra-sound*.

Our aim in this somewhat sketchy appendix has been to show that the method of least squares may be applied successfully even in some cases in which our condition of finality for the computation of the coefficients C_m is not satisfied.

Appendix II

Additions to the Riemann Problem of Sound Waves in §11

The purpose of this appendix is to fill a gap which we left in §11; namely we shall prove that the expression (11.10)

$$v = \left(\frac{\xi+\eta}{x+y}\right)^a F(a+1, -a, 1, z), \qquad z = -\frac{(x-\xi)(y-\eta)}{(x+y)(\xi+\eta)}, \tag{1}$$

where F stands for the hypergeometric series, satisfies the differential equation

$$M(v) = \frac{\partial^2 v}{\partial x\, \partial y} + \frac{a}{x+y}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{2av}{(x+y)^2} = 0 \tag{2}$$

which is derived from (11.2) and (11.8). Riemann was able to prove this by his general transformation theory for hypergeometric functions. However we shall proceed in an elementary fashion, by considering the function F in (1) as an unknown function and then by substituting (1) in (2) and deducing a differential equation for $F(z)$. By proving the latter identical with the differential equation (24.20) of the hypergeometric function we verify equation (1) and the determination of the parameters of the hypergeometric function which are contained in it.

First we deduce from (1)

$$\frac{\partial v}{\partial x} = \left(\frac{\xi+\eta}{x+y}\right)^a \left(\frac{-a}{x+y} F(z) + \frac{\partial z}{\partial x} F'(z)\right),$$

$$\frac{\partial v}{\partial y} = \left(\frac{\xi+\eta}{x+y}\right)^a \left(\frac{-a}{x+y} F(z) + \frac{\partial z}{\partial y} F'(z)\right)$$

and hence we obtain as the sum of the last two terms in (2)

$$\frac{a}{x+y}\left(\frac{\xi+\eta}{x+y}\right)^a \left[-\frac{2a+2}{x+y} F(z) + \left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) F'(z)\right]. \tag{3}$$

As the first term of (2) we then obtain

$$\begin{aligned} &\frac{a(a+1)(\xi+\eta)^a}{(x+y)^{a+2}} F(z) - \frac{a(\xi+\eta)^a}{(x+y)^{a+1}}\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) F'(z) \\ &\qquad + \left(\frac{\xi+\eta}{x+y}\right)^a \left(\frac{\partial^2 z}{\partial x\,\partial y} F'(z) + \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} F''(z)\right). \end{aligned} \tag{4}$$

The first two terms of (4) combine and cancel respectively with the two terms of (3). Hence (2) becomes

$$\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} F''(z) + \frac{\partial^2 z}{\partial x\,\partial y} F'(z) - \frac{a(a+1)}{(x+y)^2} F = 0. \tag{5}$$

The derivatives of z here can be expressed as follows:

$$\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = \frac{1}{(x+y)^2}(z^2 - z), \tag{6}$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{2z-1}{(x+y)^2}. \tag{7}$$

Due to (6) and (7) equation (5) becomes

$$z(1-z)F'' + (1-2z)F' + a(a+1)F = 0.$$

This is indeed the same as equation (24.20) if we substitute

$$\alpha = a+1, \quad \beta = -a, \quad \gamma = 1.$$

as in (1). This completes the discussion of the problem of §11.