CHAPTER VI

Problems of Radio

The problems of signals with electric waves have been in the foreground of applied physics since the beginning of the century. Can we understand the remarkable range of radio signals from the otherwise completely reliable Maxwell theory? The answer is both yes and no. Yes, in so far as only the known electrodynamic laws are applied. No, in so far as the ionosphere (Kenelly-Heaviside layer) plays an essential role in overcoming the curvature of the earth, and has to be added to the Maxwell wave propagation as a *deus ex machina*.

Unfortunately we shall be unable to treat the reflection processes in the ionosphere, and shall restrict ourselves to *questions of propagation in the homogeneous atmosphere and in the earth which is also assumed to be homogeneous.* We shall also have to omit the questions of the construction of transmitters and receivers, which are of such great importance for the engineer, since they do not properly belong to the domain of partial differential equations. Instead, we shall idealize the transmitter to the utmost and treat it as a *Hertz dipole* (§31). On the other hand the questions of propagation definitely belong to our domain and they will give us a complete demonstration of the usefulness of the methods which we have developed above and which we have so far applied mainly to rather artificial problems of heat conduction and of potential theory. Further demonstrations of this usefulness are given by problems in general electrodynamic oscillations. They are treated with some completeness in the textbook by Frank-Mises, Chapter XXIII, and the reader is referred to that book. If, among these problems, we again consider radio, it is because the previous representation was simplified so drastically that it could not be reconciled with practical problems. Now we shall not place our antenna-dipole on the surface of the earth, but at some distance from it, we shall treat the radiation of the horizontal antenna in more detail and demonstrate its asymptotic identity with the radiation of the vertical antenna for increasing distance from the origin, and we shall treat the radiation characteristic with respect to the terms of second order in $1/r$, etc. The energy conditions (required energy supply for prescribed antenna current, heat loss in the earth) will be discussed in the final section. We shall almost always consider the earth as a *plane*. The analytically interesting problem of the earth's curvature, which opens a further domain of application to the method of eigenfunctions, can be treated only in an appendix, since even an only moderately complete treatment of the problem of a plane earth is almost too long for us here.

§ 31. The Hertz Dipole in a Homogeneous Medium Over a Completely Conductive Earth

We assume that the reader has a knowledge of the concepts of electrodynamics and their interconnection through Maxwell's equations. Since we are not dealing with atomic physics but only with the phenomenological Maxwell theory, we shall use the system of the four units, M (meter), K (kilogram mass), S (second), Q (charge, measured in Coulombs). In this system the specific inductive capacity and the permeability are definite quantities; as usual their values in a vacuum are denoted by ε_0 and μ_0. We then have $\varepsilon_0 \mu_0 = 1/c^2$. The parasite factor 4π, which mars the customary electromagnetic equations, is suppressed in our system through the suitable choice of units, wherever it is not implied by the spherical symmetry of the problem.

A. Introduction of the Hertz Dipole

In the electrostatic case we deduce the potential of the dipole by an oriented differentiation from the fundamental potential $\Phi = 1/r$ (see §24 C); the field $\mathbf{E}$ of the dipole is then obtained from this potential by another differentiation. In the electrodynamic case Φ is replaced by the function of the spherical wave

$$(1) \qquad \Pi = \frac{1}{r} e^{ikr}, \quad \text{or more completely} \quad \Pi = \frac{1}{r} e^{i(kr-\omega t)}.$$

The notation Π is due to Hertz[1] himself. As shown by the second form of equation (1), we assume the oscillation to be *purely periodic* and *undamped in time* (this is realized for the tube transmitter).

In the abbreviated first form of (1), which we shall use in the following discussion, we have to remember that

$$(2) \qquad \Pi = -i\omega\Pi = -ikc\Pi.$$

where

[1] In his fundamental work "Die Kräfte elektrischer Schwingungen," collected works II, p. 147, which also contains the well-known force lines of the oscillating dipole.

$$\omega = \text{ circular frequency}$$

$$k = 2\pi/\lambda = \omega/c = \text{ wave number} \tag{2a}$$

$$c = \omega/k = \text{ velocity of light in a vacuum.}$$

As we know, Π satisfies the oscillation equation (7.4), which for purely periodic processes becomes the wave equation:

$$\Delta\Pi + k^2\Pi = 0. \tag{3}$$

In the electrodynamic case Π is not a *scalar* but a *vector.* Hence in the future we shall speak of the *Hertz vector* $\vec{\Pi}$. It is connected with the vector potential **A** by the simple relation

$$\vec{\Pi} = \mathbf{A}. \tag{3a}$$

Just as the individual elements of which **A** is composed have the direction of the corresponding elements of current, so our $\vec{\Pi}$ in empty space (i.e., in the absence of the earth) for a single antenna would have the direction of the antenna current. Here we assume the antenna to be *short compared to the wavelength,* that is, with both ends loaded with capacities so that the current can be considered in the same phase along the whole antenna. In representation (1) we could express the vector character of Π by multiplying the right side of (1) by a constant vector which has the direction of the antenna and, as we shall show later, the dimension of an electric momentum (charge $\times$ length). However, we shall refrain from doing this in order not to make the formulas unnecessarily cumbersome; hence we retain equation (1), although it is inconsistent from a vectorial and even a dimensional point of view. Only in §36 shall we correct this flaw. However, we wish to stress now that, due to the vector character of $\vec{\Pi}$, we have to give the Laplace operator Δ in (3) its general vector-analytic meaning

$$\Delta\vec{\Pi} = \operatorname{grad}\operatorname{div}\vec{\Pi} - \operatorname{curl}\operatorname{curl}\vec{\Pi}. \tag{3 b}$$

(see v.II, equation (3.10a)). This will be used in §32. Only in this and the following section, where we deal with one Cartesian component Π_z or Π_x at a time, can we use the ordinary Δ.

We now claim that the field **E, H** can be obtained from $\vec{\Pi}$ by the following differentiation process:

$$\mathbf{E} = k^2\vec{\Pi} + \operatorname{grad}\operatorname{div}\vec{\Pi}, \qquad \mathbf{H} = \frac{k^2}{\mu_0 i\omega}\operatorname{curl}\vec{\Pi}. \tag{4}$$

In order to prove this we must show that Maxwell's equations in a vacuum

$$\mu_0 \dot{\mathbf{H}} + \operatorname{curl} \mathbf{E} = 0\,, \tag{5}$$
$$\varepsilon_0 \dot{\mathbf{E}} - \operatorname{curl} \mathbf{H} = 0$$

are satisfied, where as in (2) we have to replace

(5 a) $\dot{\mathbf{H}}$ by $-i\omega\mathbf{H}$, $\dot{\mathbf{E}}$ by $-i\omega$ $\mathbf{E}$

Due to (4) and (5a) the left sides of (5) become:

$$\operatorname{curl}\ (-k^2 + k^2 + \operatorname{grad}\operatorname{div}) \vec{\Pi}$$

and

$$-i\omega\,\varepsilon_0(k^2 + \operatorname{grad}\operatorname{div} - \operatorname{curl}\operatorname{curl}) \vec{\Pi}.$$

Both vanish, the first due to curl grad = 0, the second due to (3) and (3b). Hence, if for $\vec{\Pi}$ we substitute (1) and determine the free constant in terms of the strength of the alternating current in the antenna, then, according to Maxwell, we have in (4) the field radiated from the antenna, valid for all distances that are large compared to $\lambda = 2\pi/k$ For the immediate neighborhood of the antenna our description breaks down owing to the excessive idealization of our antenna model. Following Hertz, we call our model an oscillating or pulsating *dipole*, since the ends of the antenna (both in this picture and in reality) carry alternating opposite charges. This extreme simplification of the antenna, which in reality is of complicated construction, may serve as an example of the degree to which physical data can be idealized in order to make them accessible to fruitful mathematical treatment.

We now pass from the case of vacuum to that of a *medium "earth" of general electromagnetic behavior:* it is still homogeneous but with arbitrary dielectric constant ε and conductivity σ; also its permeability μ will be arbitrary for the time being. The equations (1) and (3) for Π remain formally valid; however the wave number k is no longer determined by (2a) but by

$$k^2 = \varepsilon\mu\omega^2 + i\mu\sigma\omega. \tag{6}$$

At the same time (4) is replaced by:

$$\mathbf{E} = k^2 \vec{\Pi} + \operatorname{grad}\operatorname{div}\vec{\Pi}, \qquad \mathbf{H} = \frac{k^2}{\mu i \omega} \operatorname{curl} \vec{\Pi}. \tag{7}$$

As before, we prove that the corresponding generalized Maxwell equations

$$\begin{aligned} \mu \dot{\mathbf{H}} + \operatorname{curl} \mathbf{E} &= 0, \\ \varepsilon \dot{\mathbf{E}} + \sigma \mathbf{E} - \operatorname{curl} \mathbf{H} &= 0 \end{aligned} \tag{7a}$$

are satisfied. The oscillation equation, from which we obtained the wave equation by the elimination of time dependence, is obtained in analogy to (7.4):

$$\Delta \Pi = \left(\varepsilon \mu \frac{\partial^2}{\partial t^2} + \sigma \mu \frac{\partial}{\partial t}\right) \Pi. \tag{7 b}$$

B. Integral Representation of the Primary Stimulation

We first wish to bring the representation (1) of Π into the form of a *superposition of eigenfunctions.* Since we are dealing with cylindrical polar coordinates r, φ, z, we shall use the eigenfunctions u and eigen values λ of (26.3) and (26.3a) that are independent of φ ; we denote the quantity $m\pi/h$ by μ.[2]
We then have:

$$u = I_0(\lambda r) \cos \mu z, \qquad k^2 = \lambda^2 + \mu^2. \tag{8}$$

However, whereas the λ has previously been restricted to a discrete spectrum corresponding to the boundary conditions on the cylinder of finite radius, we now have a continuous spectrum $0 \leq \lambda < \infty$ corresponding to the unlimited medium (see §28). Thus, according to (8) the μ also have continuous, and in general, complex values. Furthermore, since we no longer have the boundary condition for the bases of the cylinder, we shall replace $\cos \mu z$ by $\exp(\pm \mu z)$. Hence we are looking for a representation of Π of the form

$$\Pi = \int_0^\infty F(\lambda) I_0(\lambda r) e^{\pm \mu z} d\lambda, \qquad \mu = \sqrt{\lambda^2 - k^2}, \tag{9}$$

where $F(\lambda) d\lambda$ represents the arbitrary amplitude constant by which any eigenfunction may be multiplied. Due to the altered meaning of r (cylindrical coordinate r instead of the spherical polar coordinate r in (1)) we have to rewrite the expression (1) for Π as

$$\Pi = \frac{e^{ikR}}{R}, \qquad R^2 = r^2 + z^2. \tag{10}$$

Our condition (9) then reads for $z = 0$:

[2] A confusion between this μ and the above magnetic constant μ is unlikely. The latter. moreover, will soon disappear from our formulas.

$$\frac{e^{ikr}}{r} = \int_0^\infty F(\lambda)\, I_0(\lambda r)\, d\lambda . \tag{11}$$

In order to satisfy this condition we use the integral representation of an arbitrary function by the Bessel functions of §21 B. We employ equation (8a) of that section, which for

$$f(r) = \frac{e^{ikr}}{r}, \qquad n = 0$$

becomes

$$\frac{e^{ikr}}{r} = \int_0^\infty \sigma\, \varphi(\sigma)\, I_0(\sigma r)\, d\sigma ,$$

$$\varphi(\sigma) = \int_0^\infty e^{ik\varrho}\, I_0(\sigma \varrho)\, d\varrho .$$

The first of these equations becomes identical with (11), if we make the following changes in notation

$$\sigma = \lambda, \qquad \sigma\, \varphi(\sigma) = F(\lambda), \quad \text{hence} \quad \varphi(\sigma) = F(\lambda)/\lambda ;$$

The second equation then becomes

$$F(\lambda) = \lambda \int_0^\infty e^{ik\varrho}\, I_0(\lambda \varrho)\, d\varrho , \tag{11b}$$

which is the *solution of the integral equation* (11). The integration in (11b) can be performed in an elementary fashion, if we use the representation (19.14) for I_0 with the limits of integration $\pm\pi$; namely, by reversing the order of integration we obtain

$$F(\lambda) = \frac{\lambda}{2\pi} \int_{-\pi}^{+\pi} dw \int_0^\infty e^{i\varrho(k+\lambda\cos w)}\, d\varrho = -\frac{\lambda}{2\pi i} \int_{-\pi}^{+\pi} \frac{dw}{k+\lambda\cos w} . \tag{12}$$

The last expression arises from the lower limit $\varrho = 0$ in the preceding integration with respect to ϱ; the term arising from the upper limit $\varrho = \infty$ can be made to vanish by a small deformation of the path of integration into the "shaded" region of the w-plane (see Fig. 18). The remaining integration with respect to w yields

$$\frac{2\pi}{\sqrt{k^2-\lambda^2}} . \tag{12a}$$

Hence (12) becomes:

$$F(\lambda) = \frac{\lambda}{\sqrt{\lambda^2 - k^2}} = \frac{\lambda}{\mu} \tag{13}$$

and (11) becomes:

$$\frac{e^{ikr}}{r} = \int_0^\infty I_0(\lambda r) \frac{\lambda\, d\lambda}{\mu}. \tag{13a}$$

From (10) we now obtain a corresponding representation for Π Namely, we can complete (13a) to a function of r and z, which satisfies the differential equation (1) by setting:

$$\Pi = \frac{e^{ikR}}{R} = \int_0^\infty I_0(\lambda r)\, e^{-\mu|z|} \frac{\lambda\, d\lambda}{\mu} \tag{14}$$

where $\mu = \sqrt{\lambda^2 - k^2}$ is to be taken with *positive real part,* in order to insure the convergence of the integral and its vanishing in the limit $z \to \pm\infty$. The fact that (14) coincides with (13a) for $z = 0$ insures that it also gives the correct representation of e^{ikR}/R for $z \neq 0$.

In the following section we shall transform (14) into

$$\Pi = \frac{1}{2} \int_{-\infty}^{+\infty} H_0^1(\lambda r)\, e^{-\mu|z|} \frac{\lambda\, d\lambda}{\mu} \tag{14a}$$

with a more exact determination of the path of integration, which will then be complex. Due to the asymptotic character of H_0^1, equation (14a) has the advantage over (14) in that it demonstrates that the radiation condition is satisfied, just as in (1) where the factor $\exp(+ikr)$ is adapted to the radiation condition.

C. Vertical and Horizontal Antenna for Infinitely Conductive Earth

Up to now we have dealt only with unlimited space, whether empty or filled by a homogeneous medium with the constants ε, μ, σ . We now pass to the case of the half-space $z > 0$, which, at $z = 0$, is bounded by an *infinitely conductive earth* $(\sigma \to \infty)$, in which $\mathbf{E} = 0$. Hence, due to the equality of the tangential field strength, which is required by the Maxwell theory, we know that $\mathbf{E}_{\text{tang}}$ must vanish also on the positive side of $z = 0$. According to (7) this means

$$(k^2 \vec{\Pi} + \operatorname{grad}\operatorname{div} \vec{\Pi})_{\text{tang}} = 0 \qquad \text{for } z = 0. \tag{15}$$

We satisfy this condition by adjoining the mirror images of opposite sign to the two single poles of the given dipole: Figure 27a,b serves to illustrate this.

a) Vertical antenna at a distance h above z = 0. The arrows leading from the negative to the positive charge are in the *same direction* for the original dipole and for its mirror image. Hence we write:

$$\Pi = \Pi_z = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'}, \quad \begin{cases} R^2 = r^2 + (z-h)^2, \\ R'^2 = r^2 + (z+h)^2. \end{cases} \tag{16}$$

The parallelogram on the left side of the drawing shows that charges of the two dipoles equidistant from $z = 0$ act on a hypothetical unit charge situated in the plane $z = 0$ so that the resulting force is in the z-direction. This means $\mathbf{E}_{\text{tang}} = 0$.

b) Horizontal antenna at a distance h above z = 0. The arrow of the reflected dipole has the *opposite* direction to that of the original dipole. Hence we write

$$\Pi = \Pi_x = \frac{e^{ikR}}{R} - \frac{e^{ikR'}}{R'}. \tag{17}$$

where R and R' are as before.[3] The parallelogram on the right side of the drawing shows that two associated charges of the two dipoles act on a positive unit charge in the plane $z = 0$

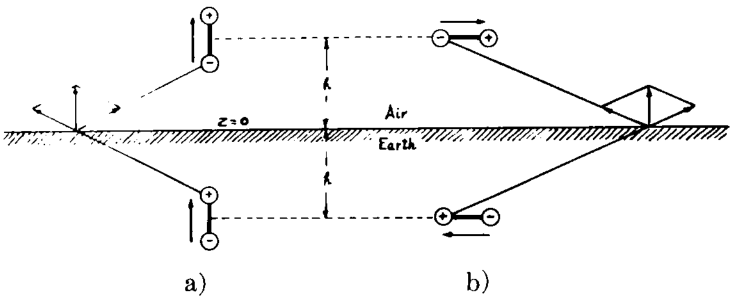

Fig. 27. Reflection by infinitely conductive earth. a) The vertical dipole. The auxiliary construction on the left shows that the horizontal components of the forces exerted by a pair of mirror image poles on a particle on the boundary plane cancel. b) The same thing is shown by the auxiliary construction on the right for the horizontal dipole. For the latter the orientations of the arrows in the original and its mirror image are *opposite*, for the vertical dipole they are *equal*.

so that the resulting force is perpendicular to the plane $z = 0$. Hence we again have $\mathbf{E}_{\text{tang}} = 0$.

[3] The opposite choice of signs in (16) and (17) indicates the vector character of Π, which is suppressed in equation (1).

In exercise VI.1 we shall compute this using the vector formula (15) for both cases a) and b).

In one respect a) and b) differ fundamentally. Namely, if we pass to the limit $h \to 0$, we obtain

$$\Pi = 2\,\frac{e^{ikR}}{R}\,, \text{ from (16) but} \qquad \Pi \equiv 0 \quad \text{from (17)}$$

Hence: a *vertical antenna located directly on the earth* for a sufficient conductivity of the soil generates the field which would be generated by the same antenna in empty space in complete absence of the earth. On the other hand a *horizontal antenna located directly on the earth* for a complete conductivity of the soil is canceled by its mirror image. *The former* made it possible to adapt the formulas and figures of Hertz' original work, which were relative to empty space, to the case of a grounded antenna (Max Abraham). In fact, we can cut the Hertz pattern of force lines of the oscillating dipole along its central plane and replace that plane by the surface of the earth. The force lines are then perpendicular to this plane and hence satisfy condition (15). *The latter*, that is, the disappearance of the horizontal antenna field for $h = 0$ as expressed by (17), decreases rapidly in importance for $h > 0$ (see the figures in §36). Indeed, the horizontal antenna is an effective means of communication even when $h < \lambda$ and the medium is sea water (a very good conductor for the comparatively long radio waves). Thus we see that for the horizontal antenna the nature of the ground and the distance from the ground play a greater role than for the vertical antenna. The formula $\vec{\Pi} = \Pi_x$ in (17) is then no longer adequate and must be generalized (see §33).

D. Symmetry Character of the Fields of Electric and Magnetic Antennas

As we have just seen, the vertical antenna gives the field of a Hertz *dipole* of strength 2 for the limit $h \to 0$, and the horizontal antenna yields a zero field. However, if in the latter case we let the antenna current increase at the rate at which h decreases, then we obtain the field of a *quadrupole*. In fact under this limit process Fig. 27b goes over into the quadrupole scheme as seen on p. 152. Replacing the amplitude factors 2 and zero by A and B we can write:

(18) Vertical antenna: $\Pi_z = A\,\dfrac{e^{ikR}}{R}$ Dipole,

Horizontal antenna: $\Pi_x = B\,\dfrac{\partial}{\partial x}\,\dfrac{e^{ikR}}{R}$ Quadrupole.

The latter representation corresponds in Fig. 27b to the combination of the pairs of poles which lie on the same vertical line to a vertical dipole and to their relative translation in the horizontal direction. This means that the horizontal antenna in the x-direction is equivalent to vertical antennas with opposite current that are mutually translated in the x-direction. We shall discuss this more closely in connection with Fig. 30. Written in polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$ the second formula (18) reads

$$\Pi_x = B\frac{x}{r}\frac{\partial}{\partial r}\frac{e^{ikR}}{R} = B\cos\varphi\,\frac{r}{R}\frac{d}{dR}\frac{e^{ikR}}{R}. \tag{18a}$$

Hence, the directions $\varphi = 0$ and $\varphi = \pi$ parallel to the antenna are preferred directions for Π_x; in the perpendicular directions $\varphi = \pm\pi/2$ Π_x vanishes. The associated direction characteristics of the horizontal antenna will be described in Fig. 29, where we shall also compute the constant B (which vanishes with increasing conductivity). On the other hand the field of the vertical antenna is symmetric with respect to the z-axis and hence its direction characteristic is a circle. From this follows the particular suitability of the horizontal antenna for directed broadcasts (see §33).

Rod antennas of vertical or horizontal direction are called *electric transmitters*. A coil traversed by an alternating current or any (circular, rectangular, etc.) closed conductor is called a *magnetic transmitter*, because then the magnetic field is concentrated in the axis of the coil (the normal of the wire loop); the customary notation is "frame antenna." In the central perpendicular of the frame a magnetic alternating current pulsates, while along the rod antenna there pulsates an electric alternating current. While the *magnetic* force lines are circles around the rod axis in the electric transmitter, in the magnetic transmitter the *electric* force lines are circles around the normal of the frame antenna (at least for distances that are large compared to the frame). These statements are correct only for the vertical electric or magnetic dipole; for an oblique or horizontal position the circular symmetry is disturbed by the conductive ground. Generally speaking the data for the magnetic transmitter are deduced from those for the electric transmitter by replacing $\mathbf{E},\mathbf{H}$ by $\mathbf{H}, -\mathbf{E}$, (for details see §35). Due to the boundary conditions for $\mathbf{E}$ (not for $\mathbf{H}$) in the case of an infinitely conductive ground, the signs in (16) and (17) are interchanged. Namely, for the magnetic Π_z (horizontal position of the plane of the frame), we have

$$\Pi_z = \frac{e^{ikR}}{R} - \frac{e^{ikR'}}{R'}, \qquad \Pi_z \equiv 0 \qquad \text{for } h \to 0 \tag{19}$$

and for the magnetic Π_x (vertical position of the plane of the frame) we have

$$(20) \qquad \Pi_x = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'}, \qquad \Pi_x = 2\,\frac{e^{ikR}}{R} \qquad \text{for } h \to 0.$$

The proof will be given in exercise VI.1. The frame antenna of type (19) is of no practical importance, the antenna of type (20) will be treated in §35. As a transmitter this latter antenna shows a marked direction in the plane of the frame (e.g., for Π_x the y,z-plane) with the same characteristic as the electric rod antenna of (18). As a receiver it is arranged so that it can be rotated around the vertical line; if it is then oriented for maximal reception its plane points to the origin of the signal and it is therefore particularly suited for range finding (see §34).

§ 32. The Vertical Antenna Over an Arbitrary Earth

Let ε and σ be the electric constants of the ground. As regards its magnetic behavior we may assume $\mu = \mu_0$, which is sufficiently close to reality and simplifies the following calculations. We write

$$(1) \qquad n^2 = \left(\varepsilon + i\,\frac{\sigma}{\omega}\right)\Big/\varepsilon_0$$

and, as in optics, we call n the "complex refractive index." The wave number k of (31.6) will, in the following discussion, be called k_E in order to distinguish it from the wave number of air for which we keep the notation k. Then according to (31.6) and (31.2a) we have

$$(2) \qquad k_E = nk.$$

We denote the altitude of the dipole antenna above the ground by h, as in (31.16).

We have to distinguish three regions:

I. Air $z > h$. In addition to the primary stimulation that becomes singular at the dipole $z = h$, $r = 0$, we have a secondary stimulation that is regular throughout due to currents induced in the ground. We write according to (31.14) and in analogy to (31.9)

$$(3) \quad \Pi_{\text{prim}} = \int_0^\infty I_0(\lambda r)\, e^{-\mu(z-h)}\,\frac{\lambda\, d\lambda}{\mu}, \qquad \Pi_{\text{sec}} = \int_0^\infty F(\lambda)\, I_0(\lambda r)\, e^{-\mu(z+h)}\,d\lambda.$$

where $F(\lambda)$ is, so to speak, the spectral distribution in the λ- continuum of the eigenfunctions, and is as yet undetermined. The factor $\exp(-\mu h)$ in the representation of Π_{sec} is convenient for what follows, and it is permissible since it is a pure function of λ and thus merely alters the meaning of $F(\lambda)$.

II. Air layer $h > z > 0$. Here, too, we have a primary and a secondary stimulation. Due to $z < h$ and according to the rule of signs of (31.14) we must write the former with a sign opposite to that in (3); the latter, being an analytic continuation, has the same form as in (3):

$$(4) \qquad \Pi_{\text{prim}} = \int_0^\infty I_0(\lambda r)\, e^{+\mu(z-h)} \frac{\lambda\, d\lambda}{\mu}, \quad \Pi_{\text{sec}} = \int_0^\infty F(\lambda)\, I_0(\lambda r)\, e^{-\mu(z+h)}\, d\lambda .$$

Equations (3) and (4) insure the continuous behavior of the Π- field at the boundary between I and II for an arbitrary choice of $F(\lambda)$.

III. Earth $0 > z > -\infty$. Here there is no primary stimulation; the Π- field — denoted by Π_E — must be continuous throughout. In order to satisfy the differential equation for earth (31.3) with k_E^2 instead of k^2, we write:

$$(5) \qquad \Pi_E = \int_0^\infty F_E(\lambda)\, I_0(\lambda r)\, e^{+\mu_E z - \mu h}\, d\lambda , \qquad \mu_E^2 = \lambda^2 - k_E^2 .$$

According to our general rule we must choose the sign of $\mu_E z$ positive since $z < 0$. The factor $\exp(-\mu h)$ is adjoined for reasons of convenience; again, this merely influences the arbitrary function $F_E(\lambda)$. The functions $F_E(\lambda)$ and $F(\lambda)$ are determined from the boundary conditions on the surface of the earth.

According to Maxwell we must require the continuity of the tangential components of **E** and **H**. These are merely

$$\mathbf{E}_r \text{ and } \mathbf{H}_\varphi .$$

Indeed, the electric force lines are in the planes through the dipole axis, the magnetic force lines are circles around this axis, and hence $\mathbf{E}_\varphi$ and $\mathbf{H}_r$ vanish. (This follows from the fact that $\Pi = \Pi_z$ is a function of r and z alone.) Now according to (31.4) and (31.7) we have

$$(6) \qquad \begin{aligned} \mathbf{E}_r &= \frac{\partial}{\partial r}\frac{\partial \Pi}{\partial z}, \quad & \mathbf{H}_\varphi &= \frac{-k^2}{\mu_0 i\omega}\frac{\partial \Pi}{\partial r} \qquad & \text{for } z > 0, \\ \mathbf{E}_r &= \frac{\partial}{\partial r}\frac{\partial \Pi_E}{\partial z}, \quad & \mathbf{H}_\varphi &= \frac{-k_E^2}{\mu_0 i\omega}\frac{\partial \Pi_E}{\partial r} \qquad & \text{for } z < 0. \end{aligned}$$

Hence the continuity conditions for $z = 0$ are:

$$\frac{\partial}{\partial r}\frac{\partial \Pi}{\partial z} = \frac{\partial}{\partial r}\frac{\partial \Pi_E}{\partial z}, \qquad k^2 \frac{\partial \Pi}{\partial r} = k_E^2 \frac{\partial \Pi_E}{\partial r} .$$

We can integrate these conditions with respect to r and the constants of integration must be zero since all expressions vanish for $r \to \infty$. If in the second equation above we replace k_E^2 by n^2k^2 according to (2) then we obtain:

$$\frac{\partial \Pi}{\partial z} = \frac{\partial \Pi_E}{\partial z}, \qquad \Pi = n^2 \Pi_E \qquad \text{for } z = 0. \tag{7}$$

On the right side of this equation we have to substitute the value of Π_E from (5) and on the left side we have to substitute the sum of Π_{prim} and Π_{sec} from (4). We thus obtain the conditions:

$$\int_0^\infty I_0(\lambda r)\, e^{-\mu h} (\lambda - \mu F - \mu_E F_E)\, d\lambda = 0\,, \tag{7 a}$$

$$\int_0^\infty I_0(\lambda r)\, e^{-\mu h} (\lambda + \mu F - n^2 \mu F_E) \frac{d\lambda}{\mu} = 0\,. \tag{7 b}$$

They are satisfied if we set

$$\begin{aligned} \mu F + \mu_E F_E &= \lambda\,, \\ \mu F - n^2 \mu F_E &= -\lambda\,. \end{aligned}$$

Hence

$$F = \frac{\lambda}{\mu}\left(1 - \frac{2\mu_E}{n^2\mu + \mu_E}\right), \qquad F_E = \frac{2\lambda}{n^2\mu + \mu_E}\,. \tag{8}$$

Thus, we have demonstrated that equations (3), (4), (5) do indeed lead to a solution of our problem with its boundary conditions. The fact that there can be no other solution is deduced from the uniqueness axiom of physical boundary value problems, which always proves reliable. Due to the meaning of n, μ, μ_E equation (8) can be written in the more symmetric form:

$$\begin{aligned} F &= \frac{\lambda}{\sqrt{\lambda^2 - k^2}} \, \frac{k_E^2\sqrt{\lambda^2 - k^2} - k^2\sqrt{\lambda^2 - k_E^2}}{k_E^2\sqrt{\lambda^2 - k^2} + k^2\sqrt{\lambda^2 - k_E^2}}\,, \\ F_E &= \frac{2\lambda k^2}{k_E^2\sqrt{\lambda^2 - k^2} + k^2\sqrt{\lambda^2 - k_E^2}}\,. \end{aligned} \tag{8 a}$$

We again write the primary stimulation in its original form e^{ikR}/R with $R^2 = r^2 + (z - h)^2$ to show that the contribution to Π_{sec} that is due to the first term of F in (8) differs from Π_{prim} only by the fact that we have to replace $-h$ by $+h$, and hence R^2 by $R'^2 = r^2 + (z + h)^2$. Then representations (3) and (4) for regions I and II can be contracted and we obtain as *the general solution of our problem* for $z > 0$ and $z < 0$:

$$\Pi = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'} - 2\int_0^\infty I_0(\lambda r)\, e^{-\mu(z+h)}\, \frac{\mu_E}{n^2\mu + \mu_E}\, \frac{\lambda\, d\lambda}{\mu},$$

(9)

$$\Pi_E = 2\int_0^\infty I_0(\lambda r)\, e^{\mu_E z - \mu h}\, \frac{\lambda\, d\lambda}{n^2\mu + \mu_E}\,.$$

If in particular we have $h = 0$ so that we can use equation (4) for the coinciding expressions e^{ikR}/R and $e^{ikR'}/R'$, then the first line of (9) can be rewritten in an elegant manner. According to previous work by the author we then have:

$$\Pi = \int_0^\infty I_0(\lambda r)\, e^{-\mu z}\, \frac{2\, n^2 \lambda\, d\lambda}{n^2\mu + \mu_E}$$

(10)

$$\Pi_E = \int_0^\infty I_0(\lambda r)\, e^{\mu_E z}\, \frac{2\lambda\, d\lambda}{n^2\mu + \mu_E}\,.$$

If, on the other hand, we consider the special case $|n| \to \infty$ of a completely conductive earth then μ_E can be neglected as compared to n^2 and the integrands in (9) will vanish. This confirms the result of the elementary reflection process in §31, equation (16):

(10 a) $$\Pi = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'}, \qquad \Pi_E = 0\,.$$

It is profitable to consider this limit process with respect to n somewhat further. To this end we replace $n^2\mu + \mu_E$ by $n^2\mu$ in the denominator of the integrand in the first equation (9), and in the numerator we write, for all values of λ that are not too large,

(10 b) $$\mu_E = \sqrt{\lambda^2 - k_E^2} = k_E\sqrt{-1 + \frac{\lambda^2}{k_E^2}} \sim -i\,k_E\,,$$

and hence $$\frac{\mu_E}{n} = -\,i\,k$$

(concerning the sign of μ_E see the figure below). Then the first equation (9) becomes:

(10 c) $$\Pi = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'} + \frac{2ik}{n}\int_0^\infty I_0(\lambda r)\, e^{-\mu(z+h)}\, \frac{\lambda\, d\lambda}{\mu^2}\,.$$

An intuitive interpretation of the latter integral[4] can be obtained as

[4] Since the denominator μ^2 vanishes at $\lambda = k$ the path of integration must be chosen in the complex λ-plane so as to avoid the point $\lambda = k$. This remark holds for the following λ-integrals, too. In the preceding integrals, starting with (31.14), we had the denominator μ, which did not destroy the convergence.

follows: corresponding to

$$R' = \sqrt{r^2 + (z+h)^2}, \qquad \frac{e^{ikR'}}{R'} = \int_0^\infty I_0(\lambda r)\, e^{-\mu(z+h)} \frac{\lambda\, d\lambda}{\mu}$$

we write

$$R'' = \sqrt{r^2 + (z+h')^2}, \qquad \frac{e^{ikR''}}{R''} = \int_0^\infty I_0(\lambda r)\, e^{-\mu(z+h')} \frac{\lambda\, d\lambda}{\mu}$$

and compute

$$\int_h^\infty \frac{e^{ikR''}}{R''}\, dh' = \int_h^\infty dh' \int_0^\infty \ldots d\lambda = \int_0^\infty I_0(\lambda r)\, e^{-\mu z} \frac{\lambda\, d\lambda}{\mu} \int_h^\infty e^{-\mu h'}\, dh'$$
$$= \int_0^\infty I_0(\lambda r)\, e^{-\mu(z+h)} \frac{\lambda\, d\lambda}{\mu^2}.$$

hence the integral in (10c) stands for the action of an imaginary continuous covering of the ray $h < h' < \infty$ with dipoles that reach from the image point $z = -h$, $r = 0$ to $z = -\infty$, $r = 0$. Hence the approximating equation (10c) can also be written as:

$$\Pi = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'} + \frac{2ik}{n} \int_h^\infty \frac{e^{ikR''}}{R''}\, dh'. \tag{10d}$$

In this connection we should remember a similar covering of a ray with imaginary source points that we used in a heat conduction problem in Fig. 15. While there we required exact satisfaction of the simple boundary condition $\partial u/\partial n + h\, u = 0$ (the h there, of course, had nothing to do with the h here), we now require approximate satisfaction of the complicated boundary conditions that arise from the juxtaposition of the air with the highly conductive earth.

From the above formulas we can deduce the field **E**, **H** by differentiation. However, we shall not write this somewhat cumbersome representation since we shall need it only in connection with the energy considerations of §36.

The integrals in (9) and (10) are not yet uniquely determined because of the square roots

$$\mu = \sqrt{\lambda^2 - k^2}, \qquad \mu_E = \sqrt{\lambda^2 - k_E^2} \tag{11}$$

that appear in them. Corresponding to the four combinations of signs of μ and μ_E, the integrand is four-valued, and its Riemann surface has four sheets. By our rule of signs in (31.14), which refers to the real part of and also applies to the real part of μ_E, one of the four sheets is singled out as a "permissible sheet." In order to insure the convergence of our integrals we demand that the path of integration at infinity shall be on the permissible sheet only. We achieve this by joining the "branch points"

$$\lambda = k \quad \text{and} \quad \lambda = k_E \tag{11a}$$

by two (essentially arbitrary) "branch cuts," which may not be intersected by the path of integration. Referring to Fig. 28 we therefore do not integrate along the real axis over the branch point $\lambda = k$, but avoid it by going into the negative imaginary half-plane and from there to infinity in, say, a direction parallel to the real λ-axis. Thus, along the path denoted by W_1 in Fig. 28 we integrate from $\lambda = 0$ to $\lambda = \infty$; this makes the meaning of the integrals in (9) and (10) precise.

But even then the representations (9) and (10) suffer from a mathematical inelegance: they are integrals with the fixed initial point $\lambda = 0$, not integrals along closed paths in the λ-plane,

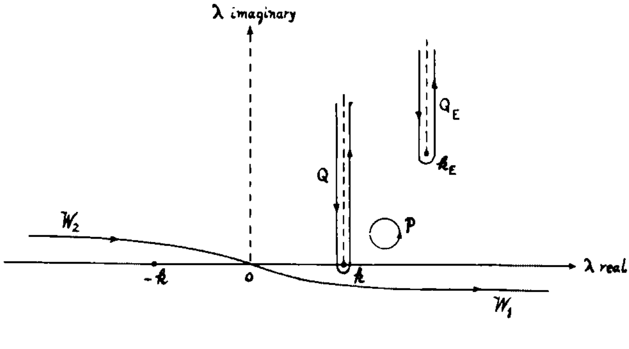

Fig. 28. The paths W_1 and $W = W_1 + W_2$ in equation (13); deformation of the path W into the loops Q and Q_E around the branch cuts and into the closed path P around the pole.

which, due to their deformability, would be much more useful. We remove this flaw by using the relation

$$I_0 = \frac{1}{2}\,(H_0^1 + H_0^2)$$

and the "semi-circuit relation" (10) of the introduction to exercise IV.2. If in the latter we set $\varrho = \lambda r$, then the preceding equation becomes

(12) $$I_0 = \frac{1}{2}\left[H_0^1(\lambda r) - H_0^1(\lambda e^{i\pi} r)\right].$$

We imagine (12) multiplied by an arbitrary function of λ^2 (indicated in the following by . . .) and by $\lambda\, d\lambda$ and integrated over W_1. Then, if we write $\lambda' = \lambda e^{i\pi}$, we obtain from the subtrahend on the right side of (12)

(12a) $$\int_{W'} H_0^1(\lambda' r) \ldots \lambda' d\lambda'.$$

Here W' is the path obtained from W_1 through reflection on the origin taken in the direction $\lambda' = 0 \rightarrow \lambda' = -\infty$, which, except for sign, is identical with the path W_2 in Fig. 28. Hence (12a) is the same as

(12b) $$\int_{W_2} H_0^1(\lambda' r) \ldots \lambda' d\lambda';$$

and from (12), if we denote the variable of integration throughout by λ and combine the paths W_1 and W_2 to $W = W_1 + W_2$, we obtain

(13) $$\int_{W_1} I_0(\lambda r) \ldots \lambda\, d\lambda = \frac{1}{2}\int_{W} H_0^1(\lambda r) \ldots \lambda\, d\lambda .$$

Thus we have achieved our purpose to replace the seemingly real integration that starts at $\lambda = 0$ in representations (9) and (10) by a complex integration over a path which closes at infinity. We consider this transformation (13) performed on all the integrals in (9) and (10). In particular we write, e.g., the primary stimulation of (31.14) and the first line of (10) in the new form

(14) $$\Pi_{\text{prim}} = \frac{1}{2}\int_{W} H_0^1(\lambda r)\, e^{-\mu|z|} \frac{\lambda\, d\lambda}{\mu},$$

(14a) $$\Pi = \int_{W} H_0^1(\lambda r)\, e^{-\mu z} \frac{n^2 \lambda\, d\lambda}{n^2 \mu + \mu_E}.$$

The attentive reader must have noticed long ago that Fig. 28 coincides with Fig. 26 (even with respect to the notation of the paths W, W_1 and the variable of integration λ), and that the present problem (determination of the function Π in space as subdivided by the surface of the earth for prescribed singularities at the dipole antenna) is summarized under the general *problem of Green's function.* Here we constructed the solution from eigenfunctions that satisfy the *radiation condition* at infinity. The fact that this condition is satisfied in the present case is made evident by the fact that in (14) and (14a) only the first Hankel function H^1 enters.[5]

We now consider the upper part of Fig. 28. Since we know that $H^1(\lambda r)$ vanishes in the infinite part of the positive imaginary half-plane, we can deform the path W into that half-plane. The path cannot be deformed across the branch cuts (11a), which it avoids by the loops Q and Q_E. However there is a further singularity of the integrand in (14) and in the analogous integrals, namely the point at which the denominator $n^2\mu + \mu_E$ vanishes. We denote it by

$$\lambda = p.$$

This corresponds to a *pole* of the integrand and must be avoided by the path of integration in a circuit P. We have not drawn the paths which join P to infinity since in the integration they cancel each other.

Of the three components Q, Q_E, P of the integral we can ignore the contribution of Q_E for large $|k_E|$, since $H^1(\lambda r)$ vanishes exponentially for great distances from the real axis. We first consider P separately, but we shall soon see that P and Q can hardly be separated.

From the defining relation for p

$$n^2\mu + \mu_E = 0 \tag{15}$$

we have

$$\sqrt{\frac{p^2 - k^2}{p^2 - k_E^2}} = -\frac{k^2}{k_E^2}, \qquad p^2 = \frac{k^2 k_E^2}{k^2 + k_E^2}, \tag{16}$$

which we can also write as

$$\frac{1}{p^2} = \frac{1}{k^2} + \frac{1}{k_E^2}. \tag{16a}$$

Due to $|k_E| \gg k$ we have approximately

$$p = k\left(1 - \frac{1}{2}\frac{k^2}{k_E^2}\right), \qquad k - p = \frac{k}{2}\frac{k^2}{k_E^2}. \tag{16b}$$

However, we wish to stress the fact that the precise value of p given by (16) or (16a) is symmetric in k and k_E.

[5] Here we have assumed a time dependence of the preferred form $\exp(-i\omega t)$. For a time dependence of the form $\exp(+i\omega t)$ we would have to make the transition from I to H^2 in (12) with the help of the semi-circuit relation (10a) in exercise (IV.2). Thus we would obtain a representation that, e.g., in (14a) is constructed from elements of the form

$$H_0^2(\lambda r)\, e^{-\mu z}\, e^{+i\omega t}$$

and hence also has the type of radiated waves.

We now compute the integral over P by applying the method of residues to (14a). Here we can let $\lambda = p$ in all the factors of the integrand of (14a), but we must replace the denominator which vanishes for $\lambda = p$ by

$$\frac{d}{d\lambda}(n^2 \mu + \mu_E) = \lambda \left(\frac{n^2}{\sqrt{\lambda^2 - k^2}} + \frac{1}{\sqrt{\lambda^2 - k_E^2}} \right),$$

taken for $\lambda = p$. We thus obtain:

$$\frac{p}{k^2} K, \qquad K = \frac{k_E^2}{\sqrt{p^2 - k^2}} + \frac{k^2}{\sqrt{p^2 - k_E^2}}; \tag{17}$$

here the new quantity K is symmetric in k and k_E. Hence, as the contribution of P to (14a) we obtain:

$$\Pi = 2\pi i \frac{k_E^2}{K} H_0^1(p\,r)\, e^{-\sqrt{p^2 - k^2}\, z}. \tag{18}$$

In the same manner, for $z < 0$ (earth, interchange of k and k_E, and reversal of the sign of z) we obtain

$$\Pi_E = 2\pi i \frac{k^2}{K} H_0^1(p\,r)\, e^{+\sqrt{p^2 - k_E^2}\, z}. \tag{18a}$$

Except for the immediate neighborhood of the transmitter, namely, for all distances $|p\,r| \gg 1$ we can replace H by the asymptotic value (19.55). We then obtain

$$\Pi = 2\sqrt{\frac{2\pi i}{pr}}\, \frac{k_E^2}{K}\, e^{ipr - \sqrt{p^2 - k^2}\, z} \qquad z \geq 0, \tag{19}$$

$$\Pi_E = 2\sqrt{\frac{2\pi i}{pr}}\, \frac{k^2}{K}\, e^{ipr + \sqrt{p^2 - k_E^2}\, z} \qquad z \leq 0. \tag{19a}$$

These formulas bear all the marks of "*surface waves*," which are mentioned in v.II in connection with the water waves or the seismic Rayleigh waves, and which have the following properties:

1. They are tied to the surface $z = 0$ and decrease in both directions from that surface; in the direction of the earth they decrease rapidly due to the coefficient $(p^2 - k_E^2)^{\frac{1}{2}}$ of z; in the direction of the air the decrease is slow at first but exponential for large z.

2 The propagation along $z = 0$ is given by

$$\frac{dr}{dt} = \frac{\omega}{p},$$

and hence depends in a symmetric manner on the material constants air and earth, as must be the case for a surface wave.

3. If for the time being we neglect the absorption in the radial direction, then the amplitude of the expressions (19), (19a) decreases as $1/\sqrt{r}$, with increasing distance from the transmitter, whereas the intensity decreases as $1/r$. This too is a criterion for the essentially two-dimensional propagation of energy in the surface $z = 0$ (see p. 100).

4. For the sake of completeness we also mention the exponential absorption in the radial direction; it is given by the real part of ipr and according to (16b) and (1),(2) it is given by

$$-\frac{kr}{2}\,\mathrm{Re}\left(\frac{i}{n^2}\right) = -\frac{kr}{2}\,\frac{\varepsilon_0\,\sigma}{\omega}\Big/\left(\varepsilon^2 + \frac{\sigma^2}{\omega^2}\right),$$

which is valid both for $z > 0$ and for $z < 0$.

For sufficiently large r, where the relative change of $r^{-\frac{1}{2}}$ is small, we can consider (19), (19a) as waves whose origin is at infinity, e.g., in the direction of the negative x-axis. These equations then become

$$\Pi = A\,k_E^2\,e^{ipx - \sqrt{p^2 - k^2}\,z}, \tag{20}$$

$$\Pi_E = A\,k^2\,e^{ipx + \sqrt{p^2 - k_E^2}\,z} \tag{20a}$$

where A is a slowly varying amplitude factor, and hence represent the so-called "*Zenneck waves.*" As early as 1907 Zenneck,[6] in great graphical and numerical detail, investigated the fields **E, H** derived from (20), (20a), and discussed the material constants of the different types of soil (also fresh and salt water). It was the main point of the author's[7] work of 1909 to show that these fields are automatically contained in the wave complex, which, according to our theory, is radiated from a dipole antenna. This fact has, of course, not been changed. What has changed is the weight which we attached to it. At the time it seemed conceivable to explain the overcoming of the earth's curvature by radio signals with the help of the character of the *surface waves;* however, we know now that this is due to the ionosphere (see the introduction to this chapter). In any case the recurrent discussion in the literature on the "reality of the Zenneck waves" seems immaterial to us.

Epstein[8] has recently shown that the surface wave P taken by itself is a solution of our problem, and hence in principle does not have to be accompanied by the wave complex represented by Q. The latter, generally speaking, has the character of spatial waves and, in contrast to (20), is represented by the formal type

$$\Pi = B\,\frac{e^{ikr}}{r}\,.$$

[6] *Ann. Physik* **23,** 846.
[7] *Ann. Physik* **28,** 665.
[8] P. S. Epstein, *Proc. Natl. Acad. Sci. U. S.*, June 1947.

Under the actual circumstances of radio communication $P+Q$ is best represented by one contour integral which goes around the near points $\lambda = p$ and $\lambda = k$, and which must be discussed with the help of the saddle-point method. This has been carried out most completely by H. Ott.[9] However, we have to forego the presentation of his results in order not to get lost in the details of the problem.

We shall consider one more general aspect and one special formula which is convenient for numerical computations.

The general aspect concerns a kind of similarity relation of radio, the introduction of "numerical distance." Measured in terms of wavelengths the radial distance traversed by a spatial wave in the time t is kr (except for a factor 2π), the distance traversed by the surface wave in the same time is equal to the real part of pr. We form the difference of these distances and introduce the quantity

$$\varrho = i\,(k-p)\,r\,. \tag{21}$$

The absolute value of ϱ is called the *numerical distance.* The quantity ϱ is a pure number whose absolute value is small compared to kr. In fact, according to (16a) we have

$$|\varrho| \sim \frac{kr}{2}\,\frac{k^2}{|k_E|^2} = \frac{kr}{2\,|n|^2}\,. \tag{21a}$$

Hence, for small values of ϱ the spatial-wave type predominates in the expression of the reception intensity; in this case the ground peculiarities have no marked influence and we can make computations using an infinite ground conductivity without introducing great errors, as was done by Abraham (see §31). For larger ϱ the rivalry between spatial and surface waves becomes apparent, as the value of ϱ in (21) was defined in terms of the difference of the two propagations. In this case the material constants of the ground are important, and indeed not only σ, but also ε. Generally speaking, equal ϱ's imply equal wave types and equal reception strengths. Thus ϱ indicates a similarity relation. The fact that for sea water, due to its relatively high conductivity, ϱ is much smaller according to (21a) (for the same absolute distance r) than it is for fresh water or for an equally level dry soil, explains the good reception at sea (the difference in reception during the day and night is, of course, due to the ionosphere).

An expansion in ascending powers of ϱ led the author, in his first investigation (1909), to a convenient approximation formula, which since has been rededuced by different authors (B. Van der Pol, K. F.

[9] *Ann. Physik* **41**, 443 (1942).

Niessen, L. H. Thomas, F. H. Murray) partly in a simpler manner. In its final form the approximation formula reads:

$$\Pi = 2\,\frac{e^{ikr}}{r}\left(1 + i\sqrt{\pi\varrho}\,e^{-\varrho} - 2\sqrt{\varrho}\;e^{-\varrho}\int\limits_0^{\sqrt{\varrho}} e^{\alpha^2}\,d\alpha\right), \tag{22}$$

It is valid for the air close to the earth.

In order to confirm the preceding remark we note: The first term, which is the dominating term for small ϱ, is of the spatial-wave type, and due to the factor 2 it corresponds to an infinitely conductive ground; the second term is of the surface-wave type and corresponds qualitatively, and for the purpose of our approximation even quantitatively, to the first equation (19); the third term represents the correction for larger ϱ. The generalization of (22) to the case of small finite distances z above the ground is

$$\Pi = 2\,\frac{e^{ikR}}{R}\left(1 + i\sqrt{\pi\varrho}\;e^{-\tau} - 2\sqrt{\varrho}\;e^{-\tau}\int\limits_0^{\sqrt{\tau}} e^{\alpha^2}\,d\alpha\right); \tag{23}$$

where

$$\tau = i(k-p)\,r\left(1 + n\,\frac{z}{r}\right)^2;$$

for $z = 0$ we have $\tau = \varrho$ and (23) becomes the same as (22).

§ 33. The Horizontal Antenna Over an Arbitrary Earth

For a horizontal antenna lying in the x-direction it seems advisable to set the Hertz vector $\vec{\Pi}$ equal to Π_x. However, as we remarked at the end of §31 C, this is possible only for an infinitely conductive ground. We start by proving this fact.

For $\vec{\Pi} = \Pi_x$ we obtain from (31.4) and (31.7)

$$\begin{aligned} &\mathbf{E}_x = k^2\,\Pi_x + \frac{\partial^2 \Pi_x}{\partial x^2}, \qquad &\mathbf{E}_y = \frac{\partial^2 \Pi_x}{\partial x\,\partial y} \qquad &z \geq 0, \\ &\mathbf{E}_x = k_E^2\,\Pi_{xE} + \frac{\partial^2 \Pi_{xE}}{\partial x^2}, \qquad &\mathbf{E}_y = \frac{\partial^2 \Pi_{xE}}{\partial x\,\partial y} \qquad &z \leq 0. \end{aligned} \tag{1}$$

where $\mathbf{E}_x$ and $\mathbf{E}_y$ must be continuous at the boundary $z = 0$. From the above formulas for $\mathbf{E}_y$ we then deduce the continuity of Π_x, which implies the continuity of $\partial^2\Pi_x/\partial x^2$. But then the formulas for $\mathbf{E}_x$ imply the equality of k^2 and k_E^2, which is a contradiction.

We resolve this contradiction by writing the Hertz vector with *two components:*

(2) $$\vec{\Pi} = (\Pi_x, \ \Pi_z);$$

then instead of (1) we have

(3)
$$\begin{aligned} \mathbf{E}_x &= k^2 \, \Pi_x + \frac{\partial}{\partial x} \operatorname{div} \vec{\Pi}, & \mathbf{E}_y &= \frac{\partial}{\partial y} \operatorname{div} \vec{\Pi} & z &\geq 0, \\ \mathbf{E}_x &= k_E^2 \, \Pi_{xE} + \frac{\partial}{\partial x} \operatorname{div} \vec{\Pi}_E, & \mathbf{E}_y &= \frac{\partial}{\partial y} \operatorname{div} \vec{\Pi}_E & z &\leq 0. \end{aligned}$$

Hence for $z = 0$:

(4) $$\operatorname{div} \vec{\Pi} = \operatorname{div} \vec{\Pi}_E$$

and

(5) $$k^2 \, \Pi_x = k_E^2 \, \Pi_{xE}.$$

For the magnetic components, according to (31.4) and (31:7), we have

$$\begin{aligned} \mathbf{H}_x &= \frac{k^2}{i\mu_0\omega} \frac{\partial \Pi_z}{\partial y}, & \mathbf{H}_y &= \frac{k^2}{i\mu_0\omega}\left(\frac{\partial \Pi_x}{\partial z} - \frac{\partial \Pi_z}{\partial x}\right) & z &\geq 0, \\ \mathbf{H}_x &= \frac{k_E^2}{i\mu_0\omega} \frac{\partial \Pi_{zE}}{\partial y}, & \mathbf{H}_y &= \frac{k_E^2}{i\mu_0\omega}\left(\frac{\partial \Pi_{xE}}{\partial z} - \frac{\partial \Pi_{zE}}{\partial x}\right) & z &\leq 0. \end{aligned}$$

From the continuity of $\mathbf{H}_x$ it follows that

(6) $$k^2 \Pi_z = k_E^2 \, \Pi_{zE}$$

and from the continuity of $\mathbf{H}_y$ it follows that

(7) $$k^2 \frac{\partial \Pi_x}{\partial z} = k_E^2 \frac{\partial \Pi_{xE}}{\partial z}.$$

Hence we have two conditions (5) and (7) for Π_x which we can write in the form

(8) $$\Pi_x = n^2 \, \Pi_{xE}, \qquad \frac{\partial \Pi_x}{\partial z} = n^2 \frac{\partial \Pi_{xE}}{\partial z},$$

After we have determined Π_x we obtain the two conditions (6) and (4) for Π_z:

(9) $$\Pi_z = n^2 \, \Pi_{zE}, \qquad \frac{\partial \Pi_z}{\partial z} - \frac{\partial \Pi_{zE}}{\partial z} = \frac{\partial \Pi_{xE}}{\partial x} - \frac{\partial \Pi_x}{\partial x}.$$

The computation of Π_x is carried out by the methods of §32. We again distinguish the three regions:

I. $\infty > z > h$, II. $h > z > 0$, III. $0 > z > -\infty$.

In I and II the function Π_x is composed of a primary and a secondary stimulation, which can be expressed exactly as in (32.3); in III we have only the secondary stimulation of the form (32.5). The conditions (8) then yield, in analogy to (32.7a,b),

$$\int_0^\infty I_0(\lambda r)\, e^{-\mu h}(\lambda - \mu F - n^2 \mu_E F_E)\, d\lambda = 0, \tag{10a}$$

$$\int_0^\infty I_0(\lambda r)\, e^{-\mu h}(\lambda + \mu F - n^2 \mu F_E)\, \frac{d\lambda}{\mu} = 0 \tag{10b}$$

and by setting the parentheses equal to zero we obtain:

$$F = \frac{\lambda}{\mu}\left(-1 + \frac{2\mu}{\mu + \mu_E}\right), \quad F_E = \frac{1}{n^2}\,\frac{2\lambda}{\mu + \mu_E}\,. \tag{11}$$

This expression for F is written so that the second term vanishes for $n \to \infty$, which is the same as $\mu_E \to \infty$, so that in the limit only the first term $F = -\lambda/\mu$ remains. If we substitute (11) in the equations (32.3,4,5), then in analogy to (32.9) we obtain the representation of the Π_x- field:

$$\begin{aligned} \Pi_x &= \frac{e^{ikR}}{R} - \frac{e^{ikR'}}{R'} + 2\int_0^\infty I_0(\lambda r)\, e^{-\mu(z+h)}\, \frac{\lambda\, d\lambda}{\mu + \mu_E}\,, \\ \Pi_{xE} &= \frac{2}{n^2}\int_0^\infty I_0(\lambda r)\, e^{+\mu_E z - \mu h}\, \frac{\lambda\, d\lambda}{\mu + \mu_E}\,. \end{aligned} \tag{12}$$

where $R^2 = r^2 + (z - h)^2$, $R'^2 = r^2 + (z + h)^2$.

If in particular $h = 0$ then we have $R' = R$ and (12) simplifies to:

$$\begin{aligned} \Pi_x &= 2\int_0^\infty I_0(\lambda r)\, e^{-\mu z}\, \frac{\lambda\, d\lambda}{\mu + \mu_E}\,, \\ \Pi_{xE} &= \frac{2}{n^2}\int_0^\infty I_0(\lambda r)\, e^{+\mu_E z}\, \frac{\lambda\, d\lambda}{\mu + \mu_E}\,. \end{aligned} \tag{12a}$$

If on the other hand we consider the special case $n \to \infty$, then we also have $|\mu_E| \to \infty$ and the integrals in (12) vanish, so that (12) reduces to

$$\Pi_x = \frac{e^{ikR}}{R} - \frac{e^{ikR'}}{R'}\,, \qquad \Pi_{xE} = 0 \tag{12b}$$

in agreement with (31.17).

The integration in equations (12) and (12a) is to be taken over the path W_1 of Fig. 28. Here again we can profitably replace this path by the closed path $W = W_1 + W_2$. If at the same time we replace I_0 by $\frac{1}{2} H_0^1$ then for vanishing h but finite k_E we obtain, in analogy to (32.14a),

$$(12\text{c}) \quad \Pi_x = \int\limits_W H_0^1(\lambda r)\, e^{-\mu z} \frac{\lambda\, d\lambda}{\mu + \mu_E}, \quad \Pi_{xE} = \frac{1}{n^2} \int\limits_W H_0^1(\lambda r)\, e^{+\mu_E z} \frac{\lambda\, d\lambda}{\mu + \mu_E}.$$

We now turn to the determination of Π_z and first consider the second condition (9). Since Π_x and Π_{xE} do not depend on x and y separately but only on $r = \sqrt{x^2 + y^2}$, we have

$$\frac{\partial \Pi_x}{\partial x} = \frac{\partial \Pi_x}{\partial r} \frac{\partial r}{\partial x} = \cos\varphi \frac{\partial \Pi_x}{\partial r}, \qquad \varphi = \sphericalangle(x, \vec{r}),$$

and a corresponding relation for $\partial \Pi_{xE}/\partial x$. From the second equation (9) it follows that Π_z must also contain the factor $\cos\varphi$. Hence we deduce that Π_z can no longer be constructed from the eigenfunctions $I_0(\lambda r)\, e^{\mp \mu z}$; it is necessary to use Bessel functions with the next higher index 1

$$I_1(\lambda r) \cos\varphi\, e^{\mp \mu z}$$

Considering the fact that Π_z should contain no primary stimulation we write:

$$(13) \quad \begin{aligned} \Pi_z &= \cos\varphi \int I_1(\lambda r)\, e^{-\mu(z+h)}\, \Phi(\lambda)\, d\lambda, \\ \Pi_{zE} &= \cos\varphi \int I_1(\lambda r)\, e^{+\mu_E z - \mu h}\, \Phi_E(\lambda)\, d\lambda. \end{aligned}$$

where Φ and Φ_E are still to be determined. The first condition (9) then yields

$$(13\text{a}) \qquad \Phi = n^2 \Phi_E.$$

The second condition (9) yields

$$(13\text{b}) \quad \begin{aligned} &-\cos\varphi \int I_1(\lambda r)\, e^{-\mu h} (\mu \Phi + \mu_E \Phi_E)\, d\lambda \\ &= \cos\varphi \left(\frac{1}{n^2} - 1\right) \int I_0'(\lambda r)\, e^{-\mu h} \frac{2\lambda^2 d\lambda}{\mu + \mu_E}. \end{aligned}$$

In fact, in the representation (12) the terms not under the integral signs vanish for $z = 0$. If we multiply the numerator and denominator of the integrand on the right side by $\mu - \mu_E$ and consider the fact that

$$\mu^2 - \mu_E^2 = k_E^2 - k^2 = k_E^2 \left(1 - \frac{1}{n^2}\right)$$

and that according to (19.52b) we have $I_0'(\varrho) = -I_1(\varrho)$, then we can contract (13b) to

$$(13\text{c}) \quad -\cos\varphi \int I_1(\lambda r)\, e^{-\mu h}\left(\mu\Phi + \mu_E\Phi_E + \frac{2}{k_E^2}(\mu - \mu_E)\,\lambda^2\right) d\lambda = 0\,.$$

From this we deduce a further relation between Φ and Φ_E :

$$(13\text{d}) \qquad \mu\,\Phi + \mu_E\,\Phi_E = -\frac{2}{k_E^2}(\mu - \mu_E)\,\lambda^2\,.$$

This, together with (13a) and (32.2), yields

$$(14) \qquad \Phi = -\frac{2\lambda^2}{k^2}\frac{\mu - \mu_E}{n^2\mu + \mu_E}\,, \qquad \Phi_E = \frac{\Phi}{n^2}\,.$$

According to (13) the final representation of Π_z is then

$$(15) \qquad \begin{aligned} \Pi_z &= -\frac{2}{k^2}\cos\varphi \int I_1(\lambda r)\, e^{-\mu(z+h)}\,\frac{\mu - \mu_E}{n^2\mu + \mu_E}\,\lambda^2\, d\lambda\,, \\ \Pi_{zE} &= -\frac{2}{k_E^2}\cos\varphi \int I_1(\lambda r)\, e^{+\mu_E z - \mu h}\,\frac{\mu - \mu_E}{n^2\mu + \mu_E}\,\lambda^2\, d\lambda\,. \end{aligned}$$

The path of integration is W_1 of Fig. 28, or, if we replace I_1 by $\frac{1}{2}H_1^1$, the path $W = W_1 + W_2$. Since the denominator in (15) coincides with that of Π_z in §32, "surface waves" also exist for the Π_z which is induced by a horizontal antenna. These surface waves correspond to the pole P in Fig. 28 and they are superimposed on the "spatial waves" or merge with them. Under the assumption $k_E \to \infty$, which implies $n \to \infty$, $\mu_E \to \infty$, the component Π_z vanishes. Hence the induced vertical component is strongly dependent on the nature of the ground and thus does not appear in the previous elementary treatment of §31.

A principal distinction of the horizontal antenna as compared to the vertical antenna is its *directed radiation*, which is implied by the factor $\cos\varphi$ in (15). The same factor is contained in the electric and magnetic field components which determine the radiation and it is a quadratic factor of the radiated energy. Later on we shall see that the component Π_x , which is free of $\cos\varphi$, in general gives no essential contribution to distant transmissions, and hence it can be neglected in the following discussion.

The solid curve in Fig. 29 represents the "direction characteristic" of the horizontal antenna. In order to obtain this curve we plot the radiated energy $\sqrt{E} = M\cos\varphi$ in a polar diagram, where M is the maximum of $\sqrt{E}$ radiated in the direction $\varphi = 0$. This curve is symmetric with respect to the direction $\varphi = \pm \pi/2$ in which there is no radiation; the radiation in the forward direction $\varphi = 0$ and in the

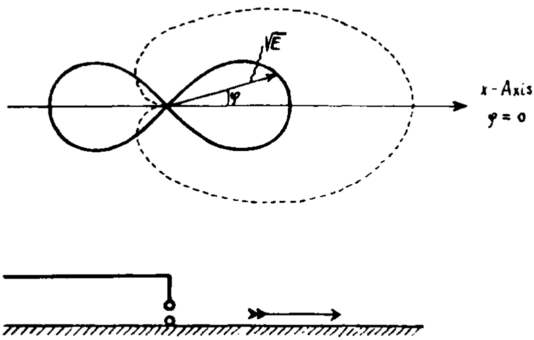

Fig. 29. Upper half: solid curve = direction characteristic of the horizontal antenna; broken curve = direction characteristic of the Marconi antenna.
Lower half: diagram of the Marconi antenna.

backward direction $\varphi = \pi$ is the same. If we combine the horizontal antenna *coherently* with a vertical antenna so that the vertical antenna alone would give the same radiation M as would be given by the horizontal antenna in the direction $\varphi = 0$ (the polar diagram would be a circle of radius M), then, we obtain as the total characteristic the curve

$$\sqrt{E} = M(1 + \cos\varphi) = \begin{cases} 2\,M & \text{for } \varphi = 0 \\ M & \text{for } \varphi = \pi/2 \\ 0 & \text{for } \varphi = \pi\,. \end{cases}$$

This characteristic is represented by the broken curve and shows a stronger directedness than the solid horizontal antenna cuive.

In the lower half of Fig. 29 we sketched an arrangement by which such a combination of horizontal and vertical antennas was realized on a large scale by Marconi (about 1906) for transatlantic communication (station Clifden in Ireland). The preferred radiation in the direction of the arrow in Fig. 29 aroused general amazement and raised the problem studied by H. von Hörschelmann,[10] in which the above theory was developed (Marconi worked only with the instinct of the ingenious experimenter). However the Clifden arrangement was somewhat cumbersome, and it has since been replaced by a more convenient combination of two or more vertical antennas (see Fig. 30).

In Fig. 30 we have drawn a horizontal antenna of the effective

[10] Dissertation, Munich 1911, *Jahresber. f. drahtl. Tel.* **5**, 14, 158 (1912).

length l, together with the current which flows through and the influx and outflux through the earth. The last two are equivalent to two

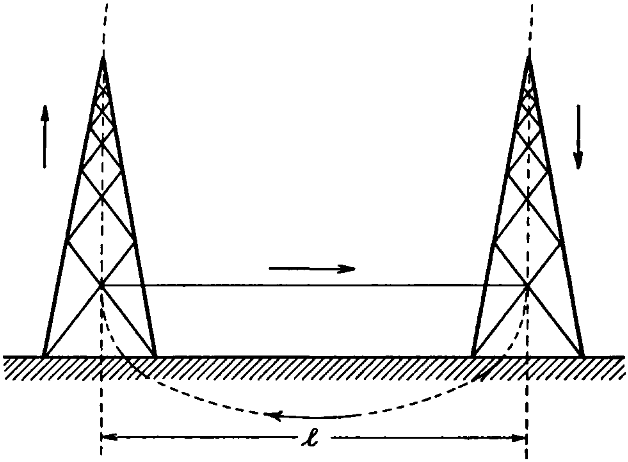

Fig. 30. Horizontal antenna with the accompanying earth currents; at a distance the effect of the two vertical antennas is the same as the effect of the horizontal antenna.

coherent vertical antennas of opposite phase, which we have indicated by towers. Their action at a distance is represented by a formula of the type

$$l \cos\varphi \frac{\partial \Pi_1}{\partial r}, \qquad \Pi_1 = \frac{e^{ikR}}{R}, \tag{16}$$

where Π_1 is the Hertz vector of the individual tower. We want to show that our theory in a rough approximation really leads to a formula of this type.

Since we are interested only in action at a distance we set $h = 0$ in (15) and in analogy to (32.10b) write

$$\mu - \mu_E \sim - \mu_E = i\,k\,n, \qquad n^2 \mu + \mu_E \sim n^2 \mu .$$

In addition we have the relation

$$\frac{\partial}{\partial r} I_0(\lambda r) = -\lambda I_1(\lambda r).$$

The first equation (15) thus becomes

$$\Pi_z = \frac{2i}{kn} \cos\varphi \frac{\partial}{\partial r} \int_0^\infty I_0(\lambda r)\, e^{-\mu z} \frac{\lambda\, d\lambda}{\mu}, \tag{16a}$$

so that we now have the primary stimulation e^{ikR}/R under the integral sign. Hence, we have actually obtained the form of equation (16); for the length of the antenna l we obtain

$$(16\mathrm{b}) \qquad |l| = \frac{2}{k\,|n|} = \frac{2\sqrt{\varepsilon_0}}{k\,|\sqrt{\varepsilon + i\,\sigma/\omega}|}\,.$$

Due to the meaning of k this length $|\,l\,|$ is of the order of magnitude of the wave length λ, but it also depends strongly on the nature of the ground; in the limit $\sigma \to \infty$ we have $l = 0$ as has been stressed before.

The same approximation method leads to an estimate of the order of magnitude of Π_x. We start from the first equation (12) and set $h = 0$ as well as $\mu + \mu_E \sim \mu_E \sim -i\,k\,n$. We then obtain

$$(16\mathrm{c}) \qquad \begin{aligned} \Pi_x = \frac{2\,i}{k\,n}\int_{W_1} I_0(\lambda\,r)\,e^{-\mu z}\,\lambda\,d\lambda &= -\frac{2\,i}{k\,n}\frac{\partial}{\partial z}\int_{W_1} I_0(\lambda\,r)\,e^{-\mu z}\,\frac{\lambda\,d\lambda}{\mu} \\ &= -\frac{2\,i}{k\,n}\frac{\partial}{\partial z}\frac{e^{ikR}}{R}\,. \end{aligned}$$

Now we have

$$\frac{\partial}{\partial z}\frac{e^{ikR}}{R} = \frac{z}{R}\frac{d}{dR}\frac{e^{ikR}}{R}\,, \qquad \frac{\partial}{\partial r}\frac{e^{ikR}}{R} = \frac{r}{R}\frac{d}{dR}\frac{e^{ikR}}{R}\,.$$

The ratio of these latter quantities is z/r, and hence is very small in the neighborhood of the surface of the earth at a great distance from the transmitter. According to (16c) and (16a) $-\Pi_x$ and Π_z have the same ratio. Hence we have

$$(17) \qquad |\Pi_x| \ll |\Pi_z|\,.$$

This fact has been mentioned before but is proved here for the first time.

The result is very remarkable: *The primary stimulation* Π_x *serves only to give rise to the secondary stimulation* Π_z. *The transmission at a distance is caused by* Π_z *alone.* Only in the immediate neighborhood of the transmitter, due to the prescribed pole of Π_x, does Π_z have an effect which outweighs that of Π_x. *At a great distance the field of transmission of a horizontal antenna has the same character as the field of transmission of a vertical antenna,* except for the φ-dependence which indicates the primary origin from a horizontal antenna. *In both cases the signals for large distances are best received with a vertical antenna;* a horizontal antenna would be unsuited as a receiver, since the horizontal component of the induced field is always small compared to the vertical component, even for a moderately conductive ground.

These results are generally known in practice, but they can hardly be understood without our theory which takes the nature of the soil into account.

We note that approximations (16a) and (16c) can be considered as the first terms of an expansion in ascending powers of the numerical distance ϱ. Just as we had to complement the term e^{ikR}/R by terms dependent on ϱ for the vertical antenna, so now we must correct (16a) and (16c) by terms dependent on ϱ.

§ 34. Errors in Range Finding for an Electric Horizontal Antenna

In navigation, range finding means the location of that direction from which a signal reaches the receiver. As an ideal *receiver* for radio signals we have the frame antenna, which was described at the end of §31, and which will be investigated in greater detail in §35. We consider the receiving antenna rotatable around a vertical axis. As for navigation, we assume the receiver to be at sea, near the surface of the earth. We assume the *transmitter* to be a horizontal antenna. Then, corresponding to the directional characteristic of Fig. 29, we not only expect maximal reception on all points of the x-axis for an x-directed transmitter, but at every point (x,y) on the earth we expect a maximal reception in the r-direction from which the signal comes, and no reception in the φ-direction. In reality things are not that simple because, in addition to the principal radiation of the order $1/r$, the horizontal antenna also emits radiation of the order $1/r^2$.

In order to prove this last fact we have to carry the approximation of the field one step further than we did in the equations of the preceding section. Namely, equations (33.16a) and (33.16c) yield $\operatorname{div} \vec{\Pi} = 0$ and hence, since $\Pi_x = 0$, for $h = 0$ and $z = 0$ they yield a field $\mathbf{E}_z$ perpendicular to the surface of the earth. We now compute $\operatorname{div} \vec{\Pi}$ with greater precision. We obtain Π_x from (33.12c) with $h = 0$, and Π_z from (33.15) by setting $h = 0$ and replacing I_1 by $\frac{1}{2} H_1^1$. Then we obtain

$$\frac{\partial \Pi_x}{\partial x} = -\cos\varphi \int_W H_1^1(\lambda r)\, e^{-\mu z} \frac{\lambda^2 \, d\lambda}{\mu + \mu_E},$$

$$\frac{\partial \Pi_z}{\partial z} = \cos\varphi \int_W H_1^1(\lambda r)\, e^{-\mu z} \frac{\mu}{k^2} \frac{\mu - \mu_E}{n^2 \mu + \mu_E} \lambda^2 \, d\lambda ,$$

$$\operatorname{div} \vec{\Pi} = -\cos\varphi \int_W H_1^1(\lambda r)\, e^{-\mu z} \left(\frac{1}{\mu + \mu_E} - \frac{\mu}{k^2} \frac{\mu - \mu_E}{n^2 \mu + \mu_E} \right) \lambda^2 \, d\lambda ;$$

and by a simple contraction:

$$
(1)\qquad
\begin{aligned}
\operatorname{div} \vec{\Pi} &= -\cos\varphi \int_W H_1^1(\lambda r)\, e^{-\mu z} \frac{\lambda^2 d\lambda}{n^2\mu + \mu_E} \\
&= \cos\varphi \frac{\partial}{\partial r} \int_W H_0^1(\lambda r)\, e^{-\mu z} \frac{\lambda\, d\lambda}{n^2\mu + \mu_E} .
\end{aligned}
$$

According to (32.14a) the last integral is nothing else than the Π-field of a *vertical* antenna divided by n^2. Since we assume $z = 0$ we may represent this field by (32.22). It even suffices to use the first term of (32.22), which we can write as

$$
(2)\qquad \operatorname{div} \vec{\Pi} = \frac{2}{n^2} \cos\varphi \frac{\partial}{\partial r} \frac{e^{ikr}}{r} .
$$

It is now profitable to use polar coordinates. Then we obtain from (2), if we neglect the terms with $(kr)^{-3}$,

$$
(3)\qquad
\begin{aligned}
\operatorname{grad}_r \operatorname{div} \vec{\Pi} &= \frac{\partial}{\partial r} \operatorname{div} \vec{\Pi} = \frac{2}{n^2} \cos\varphi \frac{\partial^2}{\partial r^2} \frac{e^{ikr}}{r} \\
&= -\frac{2k^2}{n^2} \cos\varphi \left(1 - \frac{2}{ikr}\right) \frac{e^{ikr}}{r} , \\
\operatorname{grad}_\varphi \operatorname{div} \vec{\Pi} &= \frac{1}{r} \frac{\partial}{\partial \varphi} \operatorname{div} \vec{\Pi} = -\frac{2}{n^2} \sin\varphi \frac{1}{r} \frac{\partial}{\partial r} \frac{e^{ikr}}{r} \\
&= +\frac{2k^2}{n^2} \frac{\sin\varphi}{ikr} \frac{e^{ikr}}{r} ;
\end{aligned}
$$

and from (31.4) we obtain:

$$
(4)\qquad
\begin{aligned}
\mathbf{E}_r &= \cos\varphi\, \mathbf{E}_x + \sin\varphi\, \mathbf{E}_y = k^2 \cos\varphi\, \Pi_x + \operatorname{grad}_r \operatorname{div} \vec{\Pi} , \\
\mathbf{E}_\varphi &= -\sin\varphi\, \mathbf{E}_x + \cos\varphi\, \mathbf{E}_y = -k^2 \sin\varphi\, \Pi_x + \operatorname{grad}_\varphi \operatorname{div} \vec{\Pi} .
\end{aligned}
$$

We still have to estimate Π_x. With the approximation in (33.16c) we would obtain $\Pi_x = 0$ for $z = 0$; a more exact computation yields, if we again ignore the terms with $(kr)^{-3}$,

$$
(5)\qquad k^2 \Pi_x = \frac{2}{n^2} \frac{1}{r} \frac{\partial}{\partial r} \frac{e^{ikr}}{r} = -\frac{2k^2}{n^2} \frac{1}{ikr} \frac{e^{ikr}}{r} .
$$

Hence we obtain from (3),(4),(5)

$$
\begin{aligned}
\mathbf{E}_r &= -\frac{2k^2}{n^2} \cos\varphi \left(1 - \frac{1}{ikr}\right) \frac{e^{ikr}}{r}, \\
\mathbf{E}_\varphi &= +\frac{4k^2}{n^2} \frac{\sin\varphi}{ikr} \frac{e^{ikr}}{r} ;
\end{aligned}
$$

and thus, due to $kr \gg 1$,

$$
\tag{6}
\begin{aligned}
\mathbf{E}_r &= -\frac{2k^2}{n^2}\cos\varphi\,\frac{e^{ikr}}{r},\\
\mathbf{E}_\varphi &= +\frac{4k^2}{n^2}\frac{\sin\varphi}{ikr}\frac{e^{ikr}}{r}.
\end{aligned}
$$

From this we conclude that for $\varphi = 0$ we have $\mathbf{E}_\varphi = 0$ and that the horizontal antenna field is in the r-direction. Therefore —

A rotatable frame antenna situated *on the extension of the transmitting antenna* shows the strongest reception *in the direction of the transmitting antenna,* as we had expected from the start.

On the other hand for $\varphi = \pm\pi/2$ we have

$$
\tag{7}
\mathbf{E}_r = 0, \qquad \mathbf{E}_\varphi = \frac{4k^2}{n^2}\frac{e^{ikr}}{ikr^2}.
$$

A rotatable receiving antenna situated *on the perpendicular to the transmitting antenna* shows a misdirection. In the position of maximal reception the receiving antenna does not point in the direction of the transmitting antenna, but in a direction perpendicular to it, which is *parallel to the transmitting antenna.* However, the reception is very weak, being of the order $1/r^2$; this explains the fact that in Fig. 29, where we considered only terms of the order $1/r$, this reception was *zero.*

Generally we may denote $\mathbf{E}_r$ as "correct direction" and $\mathbf{E}_\varphi$ as "misdirection." The latter, as in (7), is entirely due to terms of the order $1/r^2$.

For an arbitrary φ the "relative misdirection" in our approximation is, according to (6),

$$
\left|\frac{\mathbf{E}_\varphi}{\mathbf{E}_r}\right| = \frac{2}{kr}\tan\varphi.
$$

It increases to infinity as φ approaches $\pi/2$, which means that for that value the correct direction vanishes, corresponding to $\mathbf{E}_r = 0$ in (7).

The practical engineer is in error if he considers such misdirections the result of mistakes in the construction of the transmitting or the receiving antenna. As we have seen these misdirections are in the nature of things. Certain other misdirections called "after effects," which are due to reflections on the ionosphere, will not be discussed here.

§ 35. The Magnetic or Frame Antenna

The *frame antenna* can be used not only for range finding but also for directed transmission. In both cases the plane of the loop is taken perpendicular to the surface of the earth and the normal to this plane will be taken as the x-axis. For rectangular forms the loop consists of two pairs of coherent vertical and horizontal antennas of opposite phase, similar to the scheme in Fig. 30.

In §31 D we called such an antenna *magnetic*, no matter what the shape of the loop. Our frame antenna, which is situated in the y,z-plane, is equivalent to a *magnetic dipole* in the x-direction; its primary action can be represented by a Hertz vector $\vec{\Pi}_{\text{prim}} = \Pi_x$. Due to the presence of the earth this Hertz vector becomes a general vector $\vec{\Pi}$.

The relation between $\vec{\Pi}$ and the electromagnetic field in a vacuum is the same as in (31.4), *but we must replace* **E**, **H**, ε_0, μ_0 by **H**, $-$**E**, μ_0, ε_0 In fact this interchange transforms the Maxwell equations (31.5) into themselves. Thus, in a vacuum, as counterpart to (31.4) we have:

$$\text{(1)} \qquad \mathbf{H} = k^2 \vec{\Pi} + \operatorname{grad}\operatorname{div}\vec{\Pi}, \quad -\mathbf{E} = \frac{k^2}{\varepsilon_0 i\omega}\operatorname{curl}\vec{\Pi} = -\mu_0 i\omega \operatorname{curl}\vec{\Pi}$$

and in the earth, as counterpart to (31.7) we have:

$$\text{(2)} \qquad \mathbf{H} = k_E^2 \vec{\Pi} + \operatorname{grad}\operatorname{div}\vec{\Pi}; \qquad -\mathbf{E} = \frac{k_E^2}{\varepsilon i\omega}\operatorname{curl}\vec{\Pi},$$

where we have as before:

$$\text{(2a)} \qquad k_E^2 = \varepsilon\mu_0\omega^2 + i\sigma\mu_0\omega, \quad k^2 = \varepsilon_0\mu_0\omega^2 = \omega^2/c^2 .$$

The vector Π again satisfies the differential equation (31.3).

The boundary conditions for $z = 0$ force us to consider $\vec{\Pi}$ as a vector with two components

$$\vec{\Pi} = (\Pi_x, \Pi_z).$$

just as in the case of the electric horizontal antenna. Indeed, we have

$$\text{(3)} \qquad \Pi_z = \Pi_{zE}, \qquad\qquad \text{(4)} \qquad \frac{\partial \Pi_x}{\partial z} = \frac{\partial \Pi_{xE}}{\partial z}$$

due to the continuity of $\mathbf{E}_{\text{tang}}$,

$$\text{(5)} \quad \operatorname{div}\vec{\Pi} = \operatorname{div}\vec{\Pi}_E, \qquad\qquad \text{(6)} \quad k^2\Pi_x = k_E^2\Pi_{xE}$$

due to the continuity of $\mathbf{H}_{\text{tang}}$.

Hence, we have two conditions (4) and (6) for Π_x, and two further conditions (3) and (5) that determine Π_z from the known Π_x. Conditions (4) and (6) are exactly the same as the conditions (32.7) for the vertical antenna. Hence, we can apply the previous representations (32.9) *et seq.* directly to our Π_x. Written in the form (32.14a) as specialized for $h = 0$, these representations read:

$$(7)\qquad \begin{aligned} \Pi_x &= \int_W H_0^1(\lambda r)\, e^{-\mu z} \frac{n^2 \lambda\, d\lambda}{n^2 \mu + \mu_E}, \\ \Pi_{xE} &= \int_W H_0^1(\lambda r)\, e^{+\mu_E z} \frac{\lambda\, d\lambda}{n^2 \mu + \mu_E}. \end{aligned}$$

For the same reasons as in the case of the electric horizontal antenna, we write Π_z in the form (33.13) that contains $\cos\varphi$. However, due to condition (3), the functions Φ and Φ_E will now be equal; their common value is determined from (5):

$$\Phi = \frac{\mu - \mu_E}{n^2 \mu + \mu_E} \frac{\lambda^2}{k^2}.$$

Hence by setting $h = 0$ and replacing I by H we obtain from (33.15)

$$(8)\qquad \begin{aligned} \Pi_z &= -\frac{\cos\varphi}{k^2} \int_W H_1^1(\lambda r)\, e^{-\mu z} \frac{\mu - \mu_E}{n^2 \mu + \mu_E} \lambda^2\, d\lambda, \\ \Pi_{zE} &= -\frac{\cos\varphi}{k^2} \int_W H_1^1(\lambda r)\, e^{+\mu_E z} \frac{\mu - \mu_E}{n^2 \mu + \mu_E} \lambda^2\, d\lambda. \end{aligned}$$

However, in contrast to the electric horizontal antenna, we may now neglect Π_z as compared to Π_x, so that in the discussion of the field and of its directional characteristic we shall consider the component Π_x alone.

According to (1) we then have

$$(9)\qquad \mathbf{E}_x = 0, \quad \mathbf{E}_y = \mu_0 i \omega \frac{\partial \Pi_x}{\partial z}, \quad \mathbf{E}_z = -\mu_0 i \omega \frac{\partial \Pi_x}{\partial y}.$$

Now we obtained the first line of (7) from the representation (32.14a), which for small numerical distances was approximated by (32.23). Applying the latter to (7) we obtain

$$(10)\qquad \Pi_x = 2 \frac{e^{ikR}}{R}(1 + \cdots), \qquad R = \sqrt{r^2 + z^2}.$$

This agrees with the representation (31.20) for an infinitely conductive ground. From (9) and (10) for $z = 0$, we now obtain

$$\mathbf{E}_x = \mathbf{E}_y = 0, \qquad \mathbf{E}_z = -2\mu_0 i \omega \frac{y}{r} \frac{d}{dr} \frac{e^{ikr}}{r} = 2\mu_0 \omega k \sin\varphi \frac{e^{ikr}}{r}.$$

For the *directional characteristic* in the sense of p. 261 we obtain

$$(11)\qquad \sqrt{E} = M \sin\varphi,$$

where E is the radiated energy and M is the maximum of $\sqrt{E}$ that is radiated in the direction $\varphi = \pm \pi/2$ (M is proportional to r^{-1}).

We compare (11) with the elongated directional characteristic for the horizontal antenna in Fig. 29. The two curves are identical except for the interchange of $\sin \varphi$ and $\cos \varphi$, in accordance with the remark at the beginning of this section about the current in the frame and the horizontal antenna. The interchange of $\sin \varphi$ and $\cos \varphi$ is obviously due to the fact that while our horizontal antenna had the direction of the x-axis the plane of our frame antenna was situated perpendicular to the x-axis.

Hence, the frame antenna has its *maximal radiation in the plane of its frame* ($\varphi = \pm \pi/2$), just as the horizontal antenna has the maximal radiation in its own direction ($\varphi = 0$ and $\varphi = \pi$). Correspondingly, the frame antenna has *maximal reception* if its plane is situated *in the direction of the incoming wave.* Since this plane was assumed throughout to be the y,z-plane, the signal for maximal reception comes from the y-direction with dominating electric z-component (perpendicular to the ground) and magnetic x-component (perpendicular to the plane of the frame). Then the *electric z-component induces an electric current in the frame* or, as we may also put it, *the magnetic x-component stimulates the magnetic dipole of the frame.* Thus, the frame acts as a *magnetic receiver,* just as previously it acted as a *magnetic transmitter.*

Incidentally, in range finding we do not try for *maximal* reception but for *minimal* reception, which yields the more precise measurements, as in all zero methods of measuring in physics. The frame is then in the x,z-plane instead of the y,z-plane. The normal to the frame then points in the y-direction, i.e., *in the direction of the incoming signal.*

§ 36. Radiation Energy and Earth Absorption

In discussing certain energy questions we abandon the domain **E, H** of the field strengths that permit superposition, and turn to the quadratic quantity of energy flow

$$\mathbf{S} = [\mathbf{E}\mathbf{H}]$$

It now no longer suffices to consider the complex representation of the field under omission of the time factor $\exp(-i\omega t)$; instead we must multiply the real field components themselves. However, the complications which this brings with it can be eliminated by *averaging* over space and time. The mean values will be even simpler than our representation of the field so far, since due to the *orthogonality of the eigenfunctions,* the Bessel functions drop out of the representation and are replaced by more or less elementary functions.

Most important for our purposes is the total energy flow, integrated over a horizontal plane in the air:

$$S = \int \mathbf{S}_z \, d\sigma = \int (\mathbf{E}_r \, \mathbf{H}_\varphi - \mathbf{E}_\varphi \, \mathbf{H}_r) \, d\sigma. \tag{1}$$

Corresponding to whether this plane lies above ($z > h$) or below ($z < 0$) the dipole antenna ($z = 0$), we denote the energy flow (1) by S_+ or S_-. Both S_- and S_+ are taken relative to the positive z-direction. The energy that effectively[11] enters the earth in the *negative* z-direction is then given by $-S_-$, which, for the time being, is to be taken over the plane $z = 0$. However, we see that instead of this plane $z = 0$ we can use an arbitrary plane $z < h$, and in particular the planes $z = h - \varepsilon$, $\varepsilon \to 0$, for the computation of S_- (the space between two such planes is free from absorption and there is no noticeable energy loss in the direction of infinity). Since all energy which effectively enters the earth is transformed into Joule heat, the function $-S_-$ at the same time represents the total *thermal absorption of the earth* per unit of time. On the other hand S_+ taken over the planes $z = h + \varepsilon$, $\varepsilon \to 0$, measures the total radiation into the air above the plane $z = h$ per unit of time. We call S_+ the *effective radiation.* Hence

$$W = S_+ - S_- \tag{1a}$$

is the energy needed by the antenna per unit of time if we can neglect all energy losses in the antenna; or, in other words, it is the *power* needed by the antenna (the letter W reminds us of "watt"). In the following discussion we shall have to do mainly with this quantity W.

A. For the vertical antenna we had $\mathbf{E}_\varphi = 0$ and $\mathbf{H}_r = 0$. If we denote the expressions for $\mathbf{E}_r$ and $\mathbf{H}_\varphi$, which so far were complex, by E_r and H_φ and adjoin the time dependence, then (1) written explicitly becomes:

$$S = \frac{1}{4} \iint (E_r \, e^{-i\omega t} + E_r^* \, e^{+i\omega t}) \, (H_\varphi \, e^{-i\omega t} + H_\varphi^* \, e^{+i\omega t}) \, r \, dr \, d\varphi .$$

Upon averaging over time the terms involving $\exp(\pm 2 \, i \, \omega \, t)$ drop out, and, if from now on we understand S to be the mean value, we obtain

$$S = \frac{1}{4} \iint (E_r \, H_\varphi^* + E_r^* \, H_\varphi) \, r \, dr \, a\varphi .$$

[11] "Effective entry" means "excess of influx over outflux." The outgoing reflected radiation is of course automatically included in S_-.

Owing to the independence of the field from the φ- coordinate we can write this in the form

$$(2) \qquad S = \frac{\pi}{2}\int_0^\infty (E_r H_\varphi^* + E_r^* H_\varphi)\, r\, dr = \pi\,\mathrm{Re}\left\{\int_0^\infty E_r^* H_\varphi\, r\, dr\right\}.$$

For the computation of S_+ we take E_r and H_φ from (32.3) and for S_- we take them from (32.4). These expressions differ only by the signs of Π_{prim} in (32.3) and (32.4). We obtain

$$(3) \qquad E_r = \frac{\partial^2 \Pi}{\partial r\, \partial z} = \int_0^\infty I_1(\lambda r)\, f_1(\lambda, z)\, \lambda\, d\lambda .$$

$$(4) \qquad H_\varphi = \frac{-k^2}{\mu_0\, i\omega}\frac{\partial \Pi}{\partial r} = \frac{-k^2}{\mu_0\, i\omega}\int_0^\infty I_1(l\, r)\, f_2(l, z)\, l\, dl,$$

with

$$(5) \qquad f_1(\lambda, z) = \pm\,\lambda\, e^{-\mu|z-h|} + \mu\, F(\lambda)\, e^{-\mu(z+h)},$$

$$(6) \qquad f_2(l, z) = -\frac{l}{\mu_l}\, e^{-\mu_l|z-h|} - F(l)\, e^{-\mu_l(z+h)},$$

where $F(\lambda)$ and $F(l)$ are determined by (32.8). The fact that in (4) and (6) we used a variable of integration different from λ and hence had to replace μ by $\mu_l = \sqrt{l^2 - k^2}$, will prove useful in what follows. Using (3) and (4), equation (2) can be rewritten as follows:

$$(7) \qquad \frac{\mu_0\omega}{\pi k^2} S_\pm = \mathrm{Re}\left\{ i\int_0^\infty f_1^*(\lambda, z)\, \lambda\, d\lambda \int_0^\infty f_2(l, z)\, l\, dl \int_0^\infty I_1(\lambda r)\, I_1(l r)\, r\, dr \right\}.$$

Here we can apply the *orthogonality relation* (21.9a), which we write in our present notation for the special case $n = 1$:

$$(8) \qquad \int_0^\infty I_1(\lambda r)\, I_1(l r)\, r\, dr = \delta(\lambda | l).$$

Hence, the right-most integral in (7) vanishes for all values of l except for $l = \lambda$, so that the middle integration in (7) yields $f_2(\lambda, z)$ (see the footnote on p. 111). Thus (7) reduces to the simple integral

$$(9) \qquad \frac{\mu_0\,\omega}{\pi k^2} S_\pm = \mathrm{Re}\left\{ i \int_0^\infty f_1^*(\lambda, z)\, f_2(\lambda, z)\, \lambda\, d\lambda \right\}.$$

A further simplification is obtained if we let the planes $z = h \pm \varepsilon$

approach the position of the dipole antenna, that is

$$|z-h| = \varepsilon \ll h, \quad z+h \sim 2h$$

Then instead of (5) and (6) we have

$$f_1(\lambda) = \pm \lambda e^{-\mu\varepsilon} + \mu F(\lambda) e^{-2\mu h}, \tag{10}$$

$$f_2(\lambda) = -\frac{\lambda}{\mu} e^{-\mu\varepsilon} - F(\lambda) e^{-2\mu h}. \tag{11}$$

The product $f_1^*(\lambda) f_2(\lambda)$ in (9) is thus the sum of four terms. However, when we pass to the difference $S_+ - S_-$ only two terms remain, namely, those that correspond to the two signs of $f_1^*(\lambda)$ in (10). Applying the definition (1a) we obtain

$$\frac{\mu_0 \omega}{2\pi k^3} W = \mathrm{Re}\left\{-i \int_0^\infty e^{-(\mu+\mu^*)\varepsilon} \frac{\lambda^3 d\lambda}{\mu}\right\} + \mathrm{Re}\left\{-i \int_0^\infty F(\lambda) e^{-2\mu h} \lambda^2 d\lambda\right\}, \tag{12}$$

where due to $\varepsilon \ll h$ we may neglect $\mu^* \varepsilon$ as compared to $2\mu h$ in the exponential function under the second integral sign. The first integral in (12) is easily evaluated. For $\lambda > k$ both μ and, of course, $\mu+\mu^*$ are real. Hence the real part of $-i$ times the integral from k to ∞ vanishes. Only the integral from 0 to k in which we may pass to the limit $\varepsilon = 0$ remains. Using the variable of integration μ instead of λ we obtain[12]

$$\mathrm{Re}\left\{-i \int_0^k \frac{\lambda^3 d\lambda}{\mu}\right\} = \mathrm{Re}\left\{-i \int_{-ik}^0 (\mu^2 + k^2)\, d\mu\right\} = \frac{2}{3} k^3. \tag{13}$$

Concerning the second term in (12) we first consider the term $F(\lambda) = \lambda/\mu$ in (32.8) which does not vanish for $|k_E| \to \infty$, and thus compute:

$$\mathrm{Re}\left\{-i \int_0^\infty e^{-2\mu h} \frac{\lambda^3 d\lambda}{\mu}\right\}. \tag{14}$$

Due to the real character of μ for $\lambda > k$ we again need consider only the integral from $\lambda = 0$ to $\lambda = k$. Written in terms of the variable μ, with the abbreviation $\zeta = 2kh$ (14) becomes

$$\mathrm{Re}\left\{-i \int_{-ik}^0 e^{-\mu\zeta/k} (k^2 + \mu^2)\, d\mu\right\} = k^3\, \mathrm{Re}\left\{\left(1 + \frac{d^2}{d\zeta^2}\right) \frac{e^{i\zeta} - 1}{i\zeta}\right\}.$$

[12] Due to the sign of μ we must follow the prescriptions concerning the "permissible sheet of the Riemann surface" in Fig. 28.

By evaluating the real part we obtain:

$$k^3\left(\frac{\sin\zeta}{\zeta}+\frac{d^2}{d\zeta^2}\frac{\sin\zeta}{\zeta}\right)=2\,k^3\,\frac{\sin\zeta-\zeta\cos\zeta}{\zeta^3}. \tag{15}$$

Combining (12), (13) and (15) we have

$$W=\frac{2\pi k^5}{\mu_0\omega}\left(\frac{2}{3}+2\,\frac{\sin\zeta-\zeta\cos\zeta}{\zeta^3}+K\right),\quad \zeta=2\,k\,h, \tag{16}$$

where K stands for the remaining contribution of $F(\lambda)$ for $|k_E|\neq\infty$, which was not yet considered in (14), namely,

$$K=\frac{1}{k^3}\ \mathrm{Re}\left\{i\int_0^\infty\frac{2\,\mu_E}{n^2\mu+\mu_E}\ e^{-2\mu h}\,\frac{\lambda^3\,d\lambda}{\mu}\right\}. \tag{17}$$

In connection with (16) we note that the first two terms on the right, which are independent of the nature of the ground, could have been deduced with the help of the apparatus of §31. However, the correction term K can be computed only with the help of our complete theory. We defer the discussion of these formulas to Section C.

B. For the *horizontal antenna* the formulas become more complicated due to the combined action of Π_x and Π_z, but with the help of the orthogonality relation (8) we finally obtain simplifications similar to those obtained for the vertical antenna. We shall merely outline the necessary computations. Instead of (2) we now have

$$S=\frac{1}{4}\ \mathrm{Re}\iint\left(E_r\,H_\varphi^*-E_\varphi\,H_r^*\right)r\,dr\,d\varphi \tag{2a}$$

and instead of (3) and (4) we have for the time being

$$E_r=k^2\cos\varphi\,\Pi_x+\frac{\partial}{\partial r}\operatorname{div}\vec{\Pi},\quad E_\varphi=-k^2\sin\varphi\,\Pi_x+\frac{1}{r}\frac{\partial}{\partial\varphi}\operatorname{div}\vec{\Pi}, \tag{3a}$$

$$H_\varphi=\frac{-k^2}{i\,\mu_0\,\omega}\left(-\cos\varphi\,\frac{\partial\Pi_x}{\partial z}+\frac{\partial\Pi_z}{\partial r}\right),\ H_r=\frac{-k^2}{i\,\mu_0\,\omega}\left(\sin\varphi\,\frac{\partial\Pi_x}{\partial z}+\frac{1}{r}\,\frac{\partial\Pi_z}{\partial\varphi}\right). \tag{4a}$$

Since, according to (34.1) and (33.15), $\operatorname{div}\Pi$ and Π_z are proportional to $\cos\varphi$ while according to (33.12) Π_x is independent of φ, we conclude from (3a) and (4a) that E_r and H_φ contain the factor $\cos\varphi$, while E_φ and H_r contain the factor $\sin\varphi$. We then can carry out the integration with respect to φ in (2a), and instead of (7) we obtain a triple integral with respect to λ, l and r that has a somewhat complicated structure. However, if we form the difference $W = S_+ - S_-$ then the formulas become much simpler, since only the term that arises from the primary stimulation Π_x has alternate signs. If we also use the Bessel differential equation for the elimination of the derivatives of I_0, then we obtain

$$\frac{W}{\pi k c} = \mathrm{Re}\left\{-i\int_0^\infty \lambda\, d\lambda\, \Lambda \int_0^\infty l\, dl\, e^{-\varepsilon \mu^*_l} \int_0^\infty r\, dr\, I_0(\lambda r)\, I_0(l r)\right\}, \tag{5a}$$

$$\Lambda = \frac{2k^2 - \lambda^2}{\mu}\left(e^{-\varepsilon\mu} - e^{-2h\mu}\right) + 2\,\frac{\lambda^2 - 2\mu\mu_E}{n^2\mu + \mu_E}\, e^{-2h\mu}. \tag{6a}$$

where ε is as before. Due to the orthogonality relation (8) this reduces to the simple integral:

$$\frac{W}{\pi k c} = \mathrm{Re}\left\{-i\int_0^\infty e^{-\varepsilon\mu^*}\Lambda\,\lambda\, d\lambda\right\}. \tag{7a}$$

The integration of the term in (6a) that is independent of k_E can again be carried out as in (13) and (14), and again we need consider only the interval $0 < \lambda < k$. Thus, instead of (16) we obtain

$$W = \frac{2\pi k^5}{\mu_0\omega}\left(\frac{2}{3} - \frac{\sin\zeta}{\zeta} + \frac{\sin\zeta - \zeta\cos\zeta}{\zeta^3} + L\right) \tag{16a}$$

where ζ is as before and

$$L = \frac{1}{k^3}\,\mathrm{Re}\left\{i\int_0^\infty e^{-2\mu h}\,\frac{2\mu\mu_E - \lambda^2}{n^2\mu + \mu_E}\,\lambda\, d\lambda\right\}. \tag{17a}$$

The expression (16a) is free of Bessel functions, just as (16) is (see the beginning of this section). F. Renner has drawn my attention to the fact that the expressions (16) and (16a) can also be obtained by a process that may be more familiar to practical engineers, and that we shall discuss in exercise VI.3. However this process yields only the power $W = S_+ - S_-$ and not the values of S_+ and S_- separately, and the latter are of considerable practical interest.

C. *Discussion.* We first consider the principal terms of the equations (16) and (16a), neglecting for the time being the correction terms K and L:

$$\begin{aligned} &\frac{2}{3} + 2\,\frac{\sin\zeta - \zeta\cos\zeta}{\zeta^3}, \\ &\frac{2}{3} - \frac{\sin\zeta}{\zeta} + \frac{\sin\zeta - \zeta\cos\zeta}{\zeta^3}. \end{aligned} \tag{18}$$

For $\zeta \to \infty$ they assume the common value 2/3. Due to $\zeta = 2kh$ the limit $\zeta = \infty$ is the same as $h = \infty$ Indeed, for $h = \infty$ the earth has no influence on the radiation of the antenna and the vertical and horizontal antennas must act in the same way. In both cases the total power is transformed into radiation. Correspondingly the equations (16) and (16a) yield the common limit

$$W = \frac{4\pi}{3}\frac{k^5}{\mu_0\omega} = \frac{4\pi}{3}\frac{k^4}{\mu_0 c}. \tag{18a}$$

This is identical with a formula given by Hertz[13] for the radiation of his dipole (freely oscillating in space). We note that the factor k^4 corresponds to the reciprocal fourth power of the wavelength in Rayleigh's law of scattering, which does actually arise from the superposition of a large number of distant dipoles that are distributed over the atmosphere and that are stimulated to radiation by the incoming sun rays.

If we expand the expressions (18) in ascending powers of ζ and then pass to the limit $h = 0$ so that the expansion breaks off with the term ζ^0, then we obtain:

$$\text{(18b)}\qquad \begin{aligned} &\frac{2}{3}+\frac{2}{3}+\cdots = \frac{4}{3} = 2\cdot\frac{2}{3} \text{ for the vertical antenna,}\\ &\frac{2}{3}-1+\frac{1}{3}\cdots = 0\cdot\frac{2}{3} \text{ for the horizontal antenna.}\end{aligned}$$

We can better understand the factors 2 and 0 on the right here with the help of Fig. 27 in §31: through reflection on an infinitely conductive earth the radiation of the vertical antenna doubles for $h = 0$, the radiation of the horizontal antenna is canceled by its mirror image. However we must remember that we have neglected the correction terms K and L in (18). This disregard of K and L means that simultaneously with passage to the limit $h \to 0$ we also let $k_E \to \infty$.

Figure 31 gives a general representation of the expressions (18). Above the axis of abscissas we have marked the values of ζ below it the corresponding values of h. The figure shows that both for the vertical and the horizontal antenna the passage to the limit 2/3 is through continued oscillation around this limit. The distance between the abscissas of two consecutive extrema measured in the h scale is, for both curves, approximately equal to half a wavelength; this corresponds to the interference between the incoming radiation and the radiation which is reflected by the infinitely conductive ground.

In addition, for both curves we have traced a first correction by

[13] In the work quoted on p. 237; the formula can be found on p. 160 of his collected works v.II. In comparing (18a) with Hertz' formula we have to take into consideration the dimensionality factor which will be determined in (22) below.

broken lines as given by the terms K and L in (17) and (17a). The value $k/\mid k_E \mid = 1/100$ that we use corresponds to the case of sea water

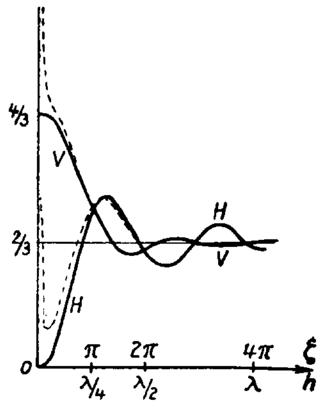

Fig. 31. The power needed by a dipole antenna for different altitudes h above the ground. V = vertical, H = horizontal antenna, the solid curves are for an infinitely conductive ground, the broken curves are for sea water.

for a 40-m. wavelength. We note that the ordinates of the broken curves increase steeply as h tends to zero; of course for finite h the difference in the ordinates from the limiting case $k_E \to \infty$ increases as k_E decreases. We are dealing here with a quite complicated double limit process, which reminds us of the double limit process in the Gibbs phenomenon of §2: if we first let $k_E = \infty$ and then $h \to 0$, we end up with the finite ordinates 4/3 and 0. However, if we stop at a finite value of k_E and first let $h \to 0$, then we end with an infinite ordinate which remains the same if afterwards we let $k_E \to \infty$.

What is the physical meaning of the infinite increase of W? *It does not add to the effective radiation S_+ but gets lost as earth heat* $- S_-$. In fact the Joule heat generated in the ground per unit of volume[14] for a fixed antenna current increases with increasing k_E, whereas the effective radiation remains finite. In order to prove this fact we should have to discuss the formulas for S separately, together with the correction terms K and L, and this would lead us too far afield.[15]

D. *Normalization to a given antenna current.* We have developed the entire theory of this chapter without consideration of the physical dimensions of the quantities introduced. This omission must be corrected now.

[14] Since the volume in which Joule heat is generated decreases with increasing $|k_E|$ (skin effect), we see that despite the statement in the text there is no heat loss in the limit $|k_E| \to \infty$.

[15] We refer the reader to the investigation by A. Sommerfeld and F. Renner, Strahlungsenergie und Erdabsorption bei Dipolantennen, *Ann. Physik* **41** (1942), where one also finds details concerning the concepts of radiation resistance and the form factor for a finite length of the antenna, which are customary in technology.

In the formula (31.1) we made the Hertz dipole factor equal to 1. In reality this factor is a denominate number, whose dimension is obtained from the relation between Π and $\mathbf{E}$ in (31.4). According to (31.4) Π has the dimension $\mathbf{E} \times M^2$. Since, according to (31.1), Π would have the dimension $1/r$, which is the same as M^{-1}, we obtain for the coefficient of Π, which we set equal to 1, the dimension $\mathbf{E} \times M^3$. We compare this factor with the Maxwell dielectrical translation $\mathbf{D} = \varepsilon\mathbf{E}$, which has the dimension of charge per unit of area, that is Q/M^2, where Q is the dimensional symbol for charge (see p. 237). Hence, written for the special case of the vacuum, we have the dimensional equation:

$$(19) \qquad \varepsilon_0\, \mathbf{E} = \frac{Q}{M^2}, \qquad \text{hence} \qquad \mathbf{E} M^3 = \frac{QM}{\varepsilon_0}.$$

where QM is an electric momentum that we set equal to el. In Hertz' original model e was the charge of one particle which oscillated with respect to a resting charge $-e$ and beyond it.

Now what takes the place of this momentum in the case of the short antenna described on p. 237 that is loaded with end capacities? The current j_t that flows in the antenna must by assumption be constant over the whole antenna at every moment. We write it in the form:

$$(20) \qquad j_t = j \sin \omega t = j \ \mathrm{Re}\{i\, e^{-i\omega t}\}.$$

The corresponding charges of the end capacities shall be

$$e_t = e \cos \omega t \quad \text{and} \quad = -e \cos \omega t .$$

According to the general relation

$$j_t = \frac{d}{dt} e_t$$

we must have $e = j/\omega$. At the time $t = 0$, when the current is zero, the charges of the end capacities are $\pm e$. Since these capacities are at the distance l of the length of the antenna they represent an electric momentum of magnitude

$$(21) \qquad e\, l = \frac{j\, l}{\omega}.$$

We have to substitute this product el for the momentum QM in (19). In addition we have to append to (19) the factor $1/4\pi$ obtained from the comparison of the field (31.4) in the neighborhood of the dipole with the field of the antenna current. Thus we obtain for the dimensionality factor to be appended to our Π:

$$\frac{jl}{4\pi\omega\varepsilon_0}. \tag{22}$$

Both the radiation S and the power W must be multiplied by the square of this factor. Using the relation

$$\frac{\omega}{k} = c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

we obtain instead of (16)

$$W = \frac{1}{8\pi} k^2 l^2 \sqrt{\frac{\mu_0}{\varepsilon_0}}\, j^2 \left(\frac{2}{3} + 2\,\frac{\sin\zeta - \zeta\cos\zeta}{\zeta^3} + K\right). \tag{23}$$

This formula gives the power in watts in a dimensionally consisten manner. Indeed, $\sqrt{\frac{\mu_0}{\varepsilon_0}}$ has the dimension of resistance and the numerical value $120\pi = 377\,\Omega$. In our system, which is based on the unit of electricity $Q = 1$ coulomb, j is to be measured in amperes. Since kl has the dimension zero, W is expressed directly in units of power: ΩA^2 = watt.

After multiplication by the same factor, equation (16a) for the horizontal antenna becomes dimensionally correct; we find

$$W = \frac{1}{8\pi} k^2 l^2 \sqrt{\frac{\mu_0}{\varepsilon_0}}\, j^2 \left(\frac{2}{3} - \frac{\sin\zeta}{\zeta} + \frac{\sin\zeta - \zeta\cos\zeta}{\zeta^3} + L\right). \tag{23a}$$

Appendix

Radio Waves on the Spherical Earth

The earth will be assumed to be totally conductive (e.g., everywhere covered with sea water). We are dealing with a vertical antenna near the surface of the earth. The direction of the antenna is taken as the axis $\vartheta = 0$ of a polar coordinate system r, ϑ, φ; the distance of the antenna from the center of the earth is denoted by r_0; the radius of the earth by $a < r_0$. The field then consists of the components $\mathbf{E}_r, \mathbf{E}_\vartheta, \mathbf{H}_\varphi$ and is independent of φ. We now want to deduce this field from a scalar solution u of the wave equation.

The Hertz vector Π is not suitable for the representation of this field, since it satisfies not the simple wave equation $\Delta\Pi + k^2\Pi = 0$, but the more complicated form (31.3b) which holds for curvilinear coordinates. It is more convenient to start from the magnetic component $\mathbf{H}_\varphi = H e^{-i\omega t}$.

Using the second equation (31.5) we compute $\mathbf{E}_{r,\vartheta} = E_{r,\vartheta}\, e^{-i\omega t}$ from H

$$-i\,\omega\,\varepsilon_0\,E_r = \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta H), \quad i\,\omega\,\varepsilon_0\,E_\vartheta = \frac{1}{r}\frac{\partial}{\partial r}(r\,H); \tag{1}$$

then, according to this scheme and from the φ-component of the first equation (31.5) we obtain

$$i\,\omega\,\mu_0\,r\,H = \frac{\partial}{\partial r}(r\,E_\vartheta) - \frac{\partial}{\partial\vartheta}E_r = \frac{1}{i\,\omega\,\varepsilon_0}\left\{\frac{\partial^2}{\partial r^2}(rH) + \frac{1}{r}\frac{\partial}{\partial\vartheta}\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta H)\right\}.$$

Hence H satisfies the differential equation

$$\frac{\partial^2}{\partial r^2}(rH) + \frac{1}{r}\frac{\partial}{\partial\vartheta}\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta H) + k^2\,r\,H = 0. \tag{2}$$

We can transform this equation into the wave equation $\Delta u + k^2 u = 0$ by making H proportional to $\partial u/\partial\vartheta$; for convenience we set in particular

$$H = i\,\omega\,\varepsilon_0\frac{\partial u}{\partial\vartheta}. \tag{2a}$$

Then (2) becomes

$$i\,\omega\,\varepsilon_0\,r\frac{\partial}{\partial\vartheta}\left\{\frac{1}{r}\frac{\partial^2 r\,u}{\partial r^2} + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u}{\partial\vartheta}\right) + k^2\,u\right\} = 0. \tag{3}$$

The first two terms in $\{\quad\}$ are equal to Δu according to the above mentioned scheme. Hence if we choose u as a solution of

$$\Delta u + k^2\,u = 0 \tag{4}$$

then according to (1) and (2a) the electromagnetic field is completely described by

$$E_r = -\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u}{\partial\vartheta}\right), \qquad E_\vartheta = \frac{1}{r}\frac{\partial^2(r\,u)}{\partial\vartheta\,\partial r}, \qquad H = i\,\omega\,\varepsilon_0\frac{\partial u}{\partial\vartheta}. \tag{5}$$

The boundary condition $E_\vartheta = 0$ on the surface of the completely conductive earth is satisfied if we set

$$\frac{\partial r u}{\partial r} = 0 \quad \text{for} \quad r = a. \tag{5a}$$

In addition we have the condition that u is to behave as a unit source at the point $r = r_0$, $\vartheta = 0$, which means the existence of a radially directed dipole of the $\mathbf{E}$-field.

In this form the problem can be solved according to the method of §28 with the equation (22) of that section for $G(P,Q)$, the only difference being that then we had the boundary condition $u = 0$ instead of our present condition (5a). However, this implies only a change in the constant A. Whereas from (28.18) and the condition $u = 0$ we obtained the value (28.18a) for A, we now obtain from (5a)

$$A = -\left\{\frac{\partial}{\partial r} r\,\psi_n(k\,r) \Big/ \frac{\partial}{\partial r} r\,\zeta_n(k\,r)\right\}_{r=a} \tag{6}$$

where here and in what follows ζ_n stands for ζ_n^1.

From the solution (28.22) modified in this way we first deduce the simplified formula for the limiting case $r_0 = a$, in which the antenna is directly on the ground. From (6) and (28.18) we obtain for this case

$$u_n(k\,r_0) = u_n(k\,a) = \psi_n(k\,a) - \frac{\psi_n(k\,a) + k\,a\,\psi_n'(k\,a)}{\zeta_n(k\,a) + k\,a\,\zeta_n'(k\,a)}\,\zeta_n(k\,a)$$

$$= k\,a\,\frac{\psi_n(k\,a)\,\zeta_n'(k\,a) - \zeta_n(k\,a)\,\psi_n'(k\,a)}{\zeta_n(k\,a) + k\,a\,\zeta_n'(k\,a)}\,.$$

According to exercise IV.8, equation II, the numerator of this fraction reduces to $i/(ka)^2$, so that if we abbreviate the denominator by $\xi_n(k\,a)$ we obtain

$$u_n(k\,a) = \frac{i/k\,a}{\xi_n(k\,a)}\,, \qquad \xi_n(k\,a) = \zeta_n(k\,a) + k\,a\,\zeta_n'(k\,a)\,. \tag{6a}$$

Substituting this in the first line of (28.22) we obtain for G, which is our present u:

$$u = \frac{k}{4\pi i}\sum_{n=0}^{\infty}(2\,n+1)\,P_n(\cos\vartheta)\,\frac{\zeta_n(k\,r)}{\xi_n(k\,a)}\,, \tag{7}$$

This equation holds for all values $a < r < \infty$ and $0 \leq \vartheta \leq \pi$; the domain of validity of the lower line of the same equation (28.22) has now reduced to zero. This result (7) agrees with the previous treatment of this case by Frank-Mises (except for a factor which depends on our present definition of the unit source). In addition, the results there for arbitrary earth can be deduced here by a suitable extension of §28 (continuation into the interior of the sphere instead of the boundary condition on the surface).

If, further, we wished to treat the *horizontal* antenna on the spherical earth, then we should have to introduce, in addition to u, a function v which arises from the interchange of **H** and **E**, and we should obtain a representation for v that is similar to (7) but somewhat more complicated.[16]

However, the convergence of the series (7) is very poor, like that

[16] This was done by P. Debye in his dissertation, Munich 1908; *Ann. Physik* **30**, 67 (1909). See also Frank-Mises, Chapter XX, §4.

of Green's general representation in Chapter V. In order to see this for the present case we merely have to note that because of the ratio of earth radius to wavelength the numbers ka and kr are >1000. As long as n is of moderate magnitude, Hankel's asymptotic values for ζ are valid and they show that the ratio ζ_n/ξ_n in (7) is nearly independent of n. We should have to use more than 1000 terms of the series until the Debye asymptotic approximations (21.32) became valid; and only the latter can bring about a real convergence of the series.

In order to obtain a usable computation of u, we apply a method which was first applied successfully to our problem by G. N. Watson,[17] and which we shall find to be connected with the method developed in Appendix II to Chapter V. Namely, we transform the sum (7) into a *complex integral.*

To this end and on the basis of the relation

$$P_n(\cos\vartheta) = (-1)^n P_n(-\cos\vartheta).$$

which is valid for *integral* (and *only for integral*) n, we first rewrite the series in (7) in the form

$$\sum_{n=0}^{\infty}(2n+1)(-1)^n P_n(-\cos\vartheta)\frac{\zeta_n(kr)}{\xi_n(ka)} \tag{8}$$

We then replace n by a complex variable ν and we trace a loop **A** in the ν-plane of Fig. 32 that surrounds all the points

$$\nu = 0, 1, 2, 3, \ldots n, \ldots \tag{8a}$$

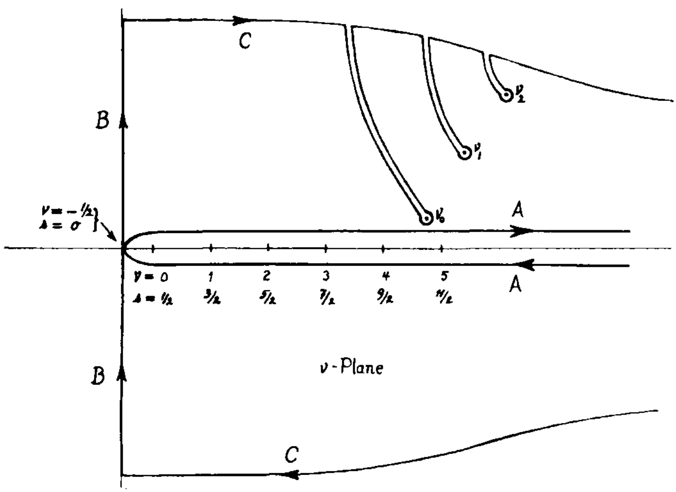

Fig. 32. Deformation of the loop **A** around the real axis. The curve **B** runs parallel to the imaginary axis of the ν-plane. The connection **C** between **B** and **A** must be considered situated at infinity.

[17] *Proc. Roy. Soc. London* **95** (1918).

in a clockwise direction. Over this loop we take the integral

$$\int \frac{2\nu+1}{2i\sin\nu\pi} P_\nu(-\cos\vartheta) \frac{\zeta_\nu(kr)}{\xi_\nu(ka)} d\nu, \tag{9}$$

which is obtained from the general term in (8) by interchanging n and ν, suppressing the factor $(-1)_n$, and appending the denominator $\sin\nu\pi$. As on p. 215, P_ν does not stand for the Legendre polynomial, but for the hypergeometric function

$$P_\nu(x) = F\left(-\nu, \nu+1, 1, \frac{1-x}{2}\right) \tag{9a}$$

(which is identical with the Legendre polynomial only for integral ν). Now the integrand of (9) has poles of first order at all the zeros of $\sin\nu\pi$; the zeros that lie inside the loop **A** are the points (8a) and in the neighborhood of the point $\nu = n$ we have as a first approximation:

$$\sin\nu\pi = \sin n\pi + (\nu - n)\pi\cos n\pi = (-1)^n\pi(\nu - n).$$

Hence the residue of the first fraction in (9) becomes

$$\frac{2n+1}{2i\pi}(-1)^n$$

and by computing the integral (9) as $-2\pi i$ times the sum of all residues we obtain

$$-\sum_{n=0}^{\infty}(2n+1)(-1)^n P_n(-\cos\vartheta)\frac{\zeta_n(kr)}{\xi_n(ka)}, \tag{10}$$

which is identical with (8) except for sign.

The next step consists in a deformation of the path **A**. We note that the hypergeometric series in (9a) is a *symmetric* function of its first two arguments. Hence we have for all (including complex) indices ν:

$$P_\nu = P_{-\nu-1}. \tag{11}$$

With the notation

$$\nu = s - \tfrac{1}{2} \tag{11a}$$

equation (11) becomes

$$P_{s-\frac{1}{2}} = P_{-s-\frac{1}{2}}. \tag{11b}$$

Hence $P_{s-\frac{1}{2}}$ is an *even* function of s.

This also holds for the last factor of the integrand in (9). In order to prove this we start from the representation (19.22) of H^1, which is valid for arbitrary indices; if we denote the index by s we have:

$$H_s^1(\varrho) = \frac{1}{\pi}\int_{W_1} e^{i\varrho\cos w}\, e^{is(w-\pi/2)}\, dw.$$

If we replace w by $-w$, s by $-s$ and reverse the orientation of W_1 we obtain:

$$H^1_{-s}(\varrho) = \frac{1}{\pi}\int_{W_1} e^{i\varrho\cos w}\, e^{is(w+\pi/2)}dw = e^{is\pi} H^1_{+s}(\varrho).$$

If we multiply this equation by $\sqrt{\pi/2\varrho}$ and use (21.15) in order to pass from H to ζ then we obtain

$$\zeta_{-s-\frac{1}{2}}(\varrho) = e^{is\pi}\,\zeta_{s-\frac{1}{2}}(\varrho). \tag{11c}$$

The same relation holds for the quantity $\xi\,(k\,a)$ of (6a):

$$\xi_{-s-\frac{1}{2}}(k\,a) = e^{is\pi}\,\xi_{s-\frac{1}{2}}(k\,a). \tag{11d}$$

By division we find that the quotient

$$\frac{\zeta_{s-\frac{1}{2}}(kr)}{\xi_{s-\frac{1}{2}}(ka)} \tag{12}$$

is also an *even* function of s.

Finally the first fraction in the integrand of (9) written in terms of s is

$$\frac{2s}{-2\,i\cos s\pi}, \tag{13}$$

and therefore is an *odd* function of s.

We now deform the loop A into a straight line B (which is parallel to the imaginary axis of the ν- plane and passes through the point $s = 0$, i.e., $\nu = -\tfrac{1}{2}$) and two paths C (which are at a great distance from the real axis and, so to speak, join the ends of B with those of A). The poles that have to be considered in this connection will be discussed later. For the moment we show that the integrals over the paths B and C *vanish*.

For the path B this follows directly from the odd character of the integrand of (9) as written in terms of the variable s. In order to show the same thing for the path C we investigate the factor ζ_ν/ξ_ν of the integrand for large values of ν. We start from the series (19.34)

$$I_\nu(\varrho) = \frac{(\varrho/2)^\nu}{\Gamma(\nu+1)}\,(1 - \cdots),$$

where all the terms indicated by ... can be neglected for $|\nu| > \varrho$. According to Stirling's formula we have

$$\Gamma(\nu+1) = \sqrt{2\,\pi\,\nu}\; e^{-\nu}\,\nu^\nu,$$

hence

$$I_\nu(\varrho) = \frac{1}{\sqrt{2\,\pi\,\nu}}\left(\frac{e\,\varrho}{2\,\nu}\right)^\nu;$$

and for general complex ν:

(14) $$I_{-\nu}(\varrho) = \frac{1}{\sqrt{-2\pi\nu}} \left(\frac{e\varrho}{-2\nu}\right)^{-\nu}.$$

From this follows:

$$\frac{I_\nu(\varrho)}{I_{-\nu}(\varrho)} = (-1)^{\nu+\frac{1}{2}} \left(\frac{e\varrho}{2\nu}\right)^{2\nu}.$$

This last quantity approaches zero if the real part of ν approaches plus infinity. Hence, in the representation (19.31) we can neglect I_ν as compared to $I_{-\nu}$. If from H_ν we pass to

$$\zeta_\nu = \sqrt{\frac{\pi}{2\varrho}}\, H_{\nu+\frac{1}{2}}$$

and from $\zeta_\nu(kr)$ we pass to the quotient of two ζ- functions, then, from (14) we obtain:

(15) $$Z = \frac{\zeta_\nu(kr)}{\zeta_\nu(ka)} = \left(\frac{a}{r}\right)^{\nu+1}.$$

Since $a/r < 1$ the quantity in (15) vanishes if the real part of $\nu+1$ approaches plus infinity, as is the case on both parts of **C**. The same statement holds for the quotient $\zeta_\nu(kr)/\xi_\nu(ka)$, which according to (6a) and (14) can be written in the form

$$\xi_\nu(ka) = \zeta_\nu(ka)\left\{1 + \varrho\,\frac{\zeta'_\nu(\varrho)}{\zeta_\nu(\varrho)}\right\}_{\varrho=ka} = \zeta_\nu(ka)\,\{-\nu\}.$$

From this we see that the third factor of the integrand in (9) vanishes. The first factor vanishes due to the denominator $\sin\nu\pi$. The fact that the second factor vanishes follows from (24.17) which holds for an arbitrary complex index of the spherical harmonic. Hence our original path **A** can indeed be deformed through the infinite part of the half plane in which the real part of ν is positive.

However, in this deformation the path cannot cross the poles of the integrand:

(15a) $$\xi_\nu(ka) = 0,\quad \nu = \nu_0, \nu_1, \nu_2, \ldots$$

We shall now investigate their position more closely. For the neighborhood of the m-th root we write:

(15b) $$\xi_\nu(ka) = (\nu - \nu_m)\,\eta_\nu(ka), \qquad \eta_\nu = \left(\frac{\partial \xi_\nu}{\partial \nu}\right)_{\nu=\nu_m}.$$

Then by forming residues we obtain from (9)

(16) $$\pi \sum_{\nu=\nu_0,\nu_1,\nu_2,\ldots} \frac{2\nu+1}{\sin\nu\pi}\, P_\nu(-\cos\vartheta)\, \frac{\zeta_\nu(kr)}{\eta_\nu(ka)}.$$

Now, except for sign, the integral (9) is identical with the series (10) and the latter in turn is identical, except for a constant factor, with the solution (7) of the sphere problem. Hence the series (16) also represents the solution of the sphere problem, and suppressing the immaterial constant factor we may write:

$$u = \sum_{\nu} \frac{2\nu+1}{\sin \nu\pi} P_\nu(-\cos\vartheta) \frac{\zeta_\nu(kr)}{\eta_\nu(ka)}. \tag{17}$$

We see that the passage from the series (7), which is summed over integral n, to the series (17), which is summed over the complex ν, is obtained by *forming residues in a complex integral twice.*

Before we proceed with the discussion of (17) we return for a moment to Appendix II of Chapter V. There, too, we were dealing with a series summed over integral n and one summed over complex non-integral ν, namely, the series (1) and (3). We now show that there, too, the identity of the two series can be proved by forming residues in a complex integral twice. Written in analogy to (9) this integral is

$$\begin{aligned} &\int \frac{2\nu+1}{2i\sin\nu\pi} P_\nu(-\cos\vartheta)\, u_\nu(k,r_0)\, \zeta_\nu(kr)\, d\nu \qquad r > r_0, \\ &\int \frac{2\nu+1}{2i\sin\nu\pi} P_\nu(-\cos\vartheta)\, \zeta_\nu(kr_0)\, u_\nu(k,r)\, d\nu \qquad r < r_0, \end{aligned} \tag{18}$$

with
$$u_\nu(k,r) = \frac{\psi_\nu(kr)\,\zeta_\nu(ka) - \psi_\nu(ka)\,\zeta_\nu(kr)}{\zeta_\nu(ka)}.$$

We see that the poles of the integrands under consideration are the zeros of the denominator

$$\zeta_\nu(ka) = 0, \qquad \nu = \nu_1, \nu_2, \nu_3, \ldots \nu_m \ldots, \tag{18a}$$

which is common to the functions $u_\nu(k,r)$ and $u_\nu(k,r_0)$. The corresponding residues of $u_\nu(k,r)$ and $u_\nu(k,r_0)$ are

$$-\frac{\psi_\nu(ka)\,\zeta_\nu(kr)}{\eta_\nu(ka)} \quad \text{and} \quad -\frac{\psi_\nu(ka)\,\zeta_\nu(kr_0)}{\eta_\nu(ka)} \tag{18b}$$

where, in contrast to (15b), we have

$$\eta_\nu = \left(\frac{\partial\zeta_\nu}{\partial\nu}\right)_{\nu=\nu_m}. \tag{18c}$$

The original path of integration for the integrals in (18) is again the path A of Fig. 32. As in that figure, we can deform the path into the sum of the contours around the points $\nu = \nu_1, \nu_2, \ldots$, since here too the paths B ,C make no contribution to the integral. Thus we obtain as the *common representation for the two integrals* (18)

$$(19) \qquad -\sum_{\nu} \frac{2\nu+1}{2i\sin\nu\pi} P_\nu(-\cos\vartheta) \frac{\psi_\nu(ka)}{\eta_\nu(ka)} \zeta_\nu(kr_0)\zeta_\nu(kr).$$

It can be seen that this coincides with the series (9), Appendix II to Chapter V, by considering the relation (II) from exercise (IV.8)

$$\zeta_\nu(\varrho)\psi_\nu'(\varrho) - \zeta_\nu'(\varrho)\psi_\nu(\varrho) = i/\varrho^2,$$

This relation yields the fact that for $\varrho = ka$ and $\zeta_\nu(ka) = 0$ the quantity $\psi_\nu(ka)$ is inversely proportional to $\zeta_\nu'(ka)$.

Thus the novel series of Appendix II can be derived by complex integration from ordinary series summed over integral n. In particular, this derivation shows the mathematical reason for the remarkable fact stressed on p. 217 paragraph 1, that the two n-series which are different for $r \gtrless r_0$ merge into the same ν-series (19).

In order to conclude our discussion of the spherical earth problem we must first show that the roots (15a) lie in the first quadrant of the ν-plane (just as the roots of (18a)). In (15a) we had the roots of the transcendental equation:

$$(20) \qquad \xi_\nu(ka) = (\zeta_\nu + \varrho\,\zeta_\nu')_{\varrho=ka} = 0.$$

For ζ_ν we again must use the special trigonometric form (11a) on p. 218 (both saddle points of equal altitude), since, for the general exponential form of ζ_ν, equation (20) can have no roots at all. Hence we can take $d\zeta_\nu/d\varrho$ from (11e), p. 219. We then obtain

$$(20\,a) \qquad \xi_\nu(ka) = \frac{i}{\varrho\sqrt{\sin\alpha}}(\sin z + \varrho\sin\alpha\cos z), \quad z = \varrho(\sin\alpha - \alpha\cos\alpha) - \frac{\pi}{4}.$$

Due to the quantity $\varrho = ka$ the second term in the parentheses (20a) is dominant. Hence, the roots of $\xi_\nu = 0$ (in contrast to the roots of $\zeta_\nu = 0$ on p. 219) are given with sufficient accuracy by

$$(20\,b) \qquad \cos z = 0, \qquad z = -\left(m+\frac{1}{2}\right)\pi \qquad \sin z = (-1)^m.$$

From this we obtain, in analogy to (21.40),

$$(21) \qquad \nu = \varrho\cos\alpha = ka\left(1-\frac{\alpha^2}{2}\right) = ka\left\{1+\frac{1}{2}(4m+1)^{\frac{2}{3}}\left(\frac{3\pi}{4ka}\right)^{\frac{2}{3}} e^{i\pi/3}\right\},$$

so that to each $m = 0,1,2,\ldots$ a root ν_m does indeed correspond in the first quadrant of the ν-plane.

Since the absolute values of the ν_m are large numbers (because $ka \gg 1$), we can compute $P_\nu(-\cos\vartheta)$ in (17) from the asymptotic equation (24.17). This equation can be written in the form

$$P_\nu(-\cos\vartheta) = \sqrt{\frac{1}{2\pi\nu\sin\vartheta}}\, e^{-i(\nu+\frac{1}{2})(\pi-\vartheta)+i\pi/4}\,.$$

if we neglect an exponentially small part. With the same accuracy we have

$$\sin\nu\pi = e^{-i\nu\pi}/2i.$$

Hence we can write in (17)

$$\frac{P_\nu(-\cos\vartheta)}{\sin\nu\pi} = \sqrt{\frac{2i}{\pi\nu\sin\vartheta}}\, e^{i(\nu+\frac{1}{2})\vartheta}\,. \tag{22}$$

We now specialize the factor ζ/η in (17) to the neighborhood of he surface of the earth, that is, we set $r = a$. According to equation (11a) on p. 218 with $\sin z = (-1)^m$ we now have

$$\zeta_\nu = \frac{i}{\varrho\sqrt{\sin\alpha}}\,(-1)^m$$

and if we restrict ourselves to the principal term of (20a) we obtain from (15b)

$$\eta_\nu = \frac{-i}{\varrho\sqrt{\sin\alpha}}\,(-\varrho\sin\alpha\,(-1)^m)\,\frac{\partial z}{\partial\nu}\,.$$

In analogy to p. 219 we have $\partial z/\partial\nu = -\alpha$. Hence,

$$\frac{\zeta_\nu}{\eta_\nu} = \frac{1}{\varrho\,\alpha\sin\alpha} = \frac{1}{ka\,\alpha\sin\alpha} \sim \frac{1}{ka\,\alpha^2}\,. \tag{23}$$

Substituting (22) and (23) in (17) we finally obtain

$$u = \frac{\sqrt{2i}}{ka}\sum_{\nu_m}\frac{2\nu+1}{\sqrt{\pi\nu\sin\vartheta}}\, e^{i(\nu+\frac{1}{2})\vartheta}\,\alpha^{-2}\,. \tag{24}$$

The last factor α^{-2} depends on the index of summation m; in fact we have as in (21)

$$\alpha^2 = (4m+1)^{\frac{2}{3}}\left(\frac{3\pi}{4ka}\right)^{\frac{2}{3}} e^{-2i\pi/3}\,.$$

In the first factor under the summation sign of (24) we may replace ν by the first term of the last member of (21), which is independent of m. Thus (24) simplifies to

$$u = \cdots (\sin\vartheta)^{-\frac{1}{2}} \sum_{\nu_m} (4\,m + 1)^{-\frac{1}{3}}\, e^{i(\nu_m + \frac{1}{2})\vartheta}\,. \tag{25}$$

where the terms which are independent of m and ϑ are denoted by . . . . In our original series (7), summed over n, we would have had to consider more than 1000 terms. In our present series, summed over m, convergence is so rapid, due to the exponential dependence of the terms on $i\nu_m\vartheta$ and to the increase of ν_m indicated in (21), that we may break off at the first or second term. Because of the positive imaginary part of ν the increase of ν_m indicates an exponential damping of the radio signals with increasing distance along the surface of the earth; the factor $(\sin\vartheta)^{-\frac{1}{2}}$ indicates an increase of intensity at the antipodal point $\vartheta = \pi$ of the transmitter.[18]

We have omitted all numerical details since our formulas are of no importance for radio communication due to the predominant role of the ionosphere. However, our formulas are of interest for the general method of Green's function in Appendix II to Chapter V and they show the power of this method for a special example.[19]

[18] The apparent infinity of (22) for $\vartheta = \pi$ contradicts our general condition of continuity, but this need not disturb us since the equation (24.17) which was used in (22) loses its validity at the points $\vartheta = \pi$ and $\vartheta = 0$. A more precise investigation of the point $\vartheta = \pi$ leads to a kind of Poisson diffraction phenomenon of finite intensity (see J. Gratiatos, Dissertation, Munich; *Ann. Physik* **86** (1928)).

[19] I wish to mention the fact, communicated to me by Mr. Whipple on the occasion of a friendly visit by English physicists, that Watson's results can be deduced directly without the use of complex integration. I may venture the guess that the particular physical considerations which were made for this case are contained in the general method of our Appendix II to Chapter V.