CHAPTER II

Introduction to Partial Differential Equations

§ 7. How the Simplest Partial Differential Equations Arise

The potential equation

$$(1) \qquad \Delta u = 0 \qquad \text{or} \qquad (1\text{a}) \qquad \Delta u = -(4\pi)\varrho$$

is known in the *theory of gravitation* as the expression of the field-action approach, as opposed to the action-at-a-distance approach of Newton. The *Laplace operator* is defined as

$$(2) \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \text{div grad}.$$

The same equations (1) and (1a) are fundamental for *eloctrostatic* and *magnetic fields*, (1) in empty space, (1a) in the presence of a source of density ϱ the factor 4π in (1a) has been put in parentheses since it can be removed by a proper choice of units.

Equation (1) appears also in the *hydrodynamics* of incompressible and irrotational fluids, u standing for the velocity potential. We also mention the two-dimensional potential equation

$$(3) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

as the basis of Riemannian function theory, which we may characterize as the "field theory" of the analytic functions $f(x + iy)$.

Equally well known is the wave equation

$$(4) \qquad \Delta u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

It is fundamental in acoustics (c = velocity of sound). It is also fundamental in the electrodynamics of variable fields (c = velocity of light), and therefore in optics. In the special theory of relativity one may write (4) as the four-dimensional potential equation

$$\text{(5)} \qquad \Box\, u = 0 \qquad \text{with} \qquad \Box = \sum_{k=1}^{4} \frac{\partial^2}{\partial x_k^2}$$

by introducing the fourth coordinate x_4 (or x_0) $= ict$ in addition to the three spatial coordinates x_1,x_2,x_3. For an oscillating membrane we have (4) with two spatial dimensions, for an oscillating string we have one spatial dimension. In the latter case we write

$$\text{(6)} \qquad \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{or sometimes} \qquad \text{(6a)} \qquad \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0,$$

setting, for the time being, $y = ct$ (not $y = ict$). Neither membrane nor string has a proper elasticity; the constant c is computed from the tension imposed from outside and from the density per unit of area or of length.

In the general theory of elasticity one has, as a special case, the differential equation for the transverse vibrations of a thin disc

$$\text{(7)} \qquad \Delta\,\Delta\, u = -\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad \Delta\,\Delta = \frac{\partial^4}{\partial x^4} + 2\,\frac{\partial^4}{\partial x^2\,\partial y^2} + \frac{\partial^4}{\partial y^4};$$

for reasons of dimensionality c here does not stand for the velocity of sound in the elastic material, as it does in acoustics, but is computed from the elasticity, density, and thickness of the disc. Analogously, the differential equation of an oscillating elastic rod is

$$\text{(8)} \qquad \frac{\partial^4 u}{\partial x^4} = -\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

This will be derived in exercise II.1, where the resulting characteristic frequencies will be compared with the acoustic frequencies of open and of covered pipes.

As a third type we add to the differential equations of states of equilibrium ((1) to (3)), and of oscillating processes ((4) to (8)), those of *equalization processes*. As their chief representative we shall here consider *heat conduction* (equalization of energy differences). We remark, however, that *diffusion* (equalization of differences of material densities), *fluid friction* (equalization of impulse differences), and pure *electric conduction* (equalization of differences of potential), follow the same pattern.

Let $\mathbf{G}$ be a vector of the magnitude and direction of the heat flow and let the initial point P be surrounded by an element of volume $d\tau$. Then $\operatorname{div} \mathbf{G}\, d\tau$ is the outflow of heat energy from $d\tau$ per unit of time. A decrease per unit of time in the amount of heat in $d\tau$, which we shall denote by $-\partial Q/\partial t$, corresponds to this. We then have

$$\text{div}\,\mathbf{G}\,d\tau = -\frac{\partial Q}{\partial t}. \tag{9}$$

Our heat conductor is here considered to be a rigid body so that we can neglect expansion; heat content is then the same as energy content. Now every increase dQ in heat causes an increase in the temperature of $d\tau$, every decrease $-dQ$ in heat causes a decrease in temperature. Denoting the temperature by u, we have

$$dQ = c\,dm\,du, \qquad dm = \varrho\,d\tau. \tag{10}$$

c being the specific heat (for a rigid body we need not distinguish between c_v and c_p). The factor dm is due to the fact that c is related to the unit of mass.

From (9) and (10) we get

$$\text{div}\,\mathbf{G} = -c\,\varrho\,\frac{\partial u}{\partial t}. \tag{11}$$

We now apply *Fourier's law*, which determines the relation between $\mathbf{G}$ and u. It states that for an isotropic medium

$$\mathbf{G} = -\varkappa\,\text{grad}\,u: \tag{12}$$

the flow of heat is in the direction of decreasing temperature and is proportional to the rate of this decrease. The factor of proportionality $\varkappa$ is called the *heat conductivity*.

Introducing (12) in (11) we get the differential equation of *heat conduction*

$$\Delta u = \frac{1}{k}\frac{\partial u}{\partial t}, \qquad k = \frac{\varkappa}{c\,\varrho}. \tag{13}$$

k is called the *temperature conductivity*.

Fourier's law was adapted to the case of *diffusion* by the physiologist Fick. Here u stands for the concentration of dissolved matter in the solvent, $\mathbf{G}$ for the *material flow* of the dissolved matter, and k for the *diffusion coefficient*. In the case of *inner friction* of an incompressible fluid, k stands for the *kinematic viscosity*, and (13) is the Navier-Stokes equation for laminar flow (i.e., flow in a fixed direction). Owing to the tensor character of this process equation (12) has no general validity here. The analogue of Fourier's law in the *electric* case is Ohm's law. Here u stands for the *potential*, $\mathbf{G}$ for the *specific electric current* (the current per unit of area of the conductor), and k for the *specific resistance of the* *conductor*. Equation (13) is of the type of Maxwell's equations in the case of pure Ohm conduction.

Schrödinger's equation of wave mechanics belongs formally to the same scheme, in particular in the force-free case, to which we restrict ourselves here:

$$\Delta u = \frac{2\,m}{i\,\hbar}\,\frac{\partial u}{\partial t} \begin{cases} \hbar = \text{Planck's constant divided by } 2\pi \\ m = \text{mass of the particle.} \end{cases} \tag{14}$$

However, owing to the fact that the real constant, k, of (13) is replaced here by the imaginary constant $i\,\hbar/2\,m$, equation (14) describes an oscillation rather than an equalization process. We see this in the passage to the case of periodicity in time, if we set

$$u = \psi\, e^{-i\omega t}, \qquad \omega = \frac{W}{\hbar}, \qquad W = \text{energy of the state.} \tag{14 a}$$

Then (14) becomes

$$\Delta\psi + C\,\psi = 0, \quad C = \frac{2\,m}{\hbar^2}\,W. \tag{15}$$

This is the same form as we would obtain from the wave equation (4) if we set $u = \psi \cdot \exp(-i\omega t)$ and let $C = \omega^2/c^2$

The so-called case of *linear heat conduction*, with the thermal state depending on only one variable x, will be treated in detail in the following chapter. In order to compare its differential equation with (3) and (6a), we write it in the form:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0, \quad y = k\,t. \tag{16}$$

Looking back on this sketchy survey one notices a family resemblance among the differential equations of physics. This stems from the *invariance under rotation and translation*, which must be demanded for the case of isotropic and homogeneous media. The differential operator of second order implied by this invariance is just the Laplace Δ. In the case of space-time invariance of relativity this is replaced by the corresponding four-dimensional $\Box$ of (15). For the case of an *anisotropic medium*, Δ must be replaced by a sum of all second derivatives with factors determined from the crystal constants. For the case of an *inhomogeneous medium* these factors will also be functions of position. We shall deal with such generalized differential expressions in the beginning of the next section.

The fact that we are dealing throughout with *partial differential* *equations* is due to the *field-action approach*, which is the basis of present day physics, according to which only neighboring elements of space can influence each other.

§ 8. Elliptic, Hyperbolic and Parabolic Type. Theory of Characteristics

We restrict ourselves to the case of two independent variables, x and y. The most general form of a *linear partial differential equation of second order* is then:

$$L(u) \equiv A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0. \tag{1}$$

$A,B, \ldots ,F$ being given functions of x and y having sufficiently many derivatives. For the present we may even consider the far more general equation:

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} = \Phi\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, x, y\right), \tag{2}$$

where Φ need not be linear in u, $\partial u/\partial x$, $\partial u/\partial y$.

We now investigate the conditions for the solvability of the following problem, which is put first in the mathematical theory of partial differential equations, although in the physical applications it is of secondary importance compared to certain boundary value problems considered later.

Let Γ be a given curve in the xy-plane along which both u and the derivative $\partial u/\partial n$ of u in the direction of the normal are prescribed. Does a solution of (2) that satisfies these initial conditions exist?

Preliminary remark: If u is given on Γ then so is $\partial u/\partial s$; but from $\partial u/\partial s$ and $\partial u/\partial n$ one can calculate $\partial u/\partial x$ and $\partial u/\partial y$. Therefore both u and its first derivatives are known on Γ.

We introduce the following abbreviations, which are common in the theory of surfaces:

$$p = \frac{\partial u}{\partial x}, \quad q = \frac{\partial u}{\partial y},$$
$$r = \frac{\partial^2 u}{\partial x^2}, \quad s = \frac{\partial^2 u}{\partial x\,\partial y}, \quad t = \frac{\partial^2 u}{\partial y^2}.$$

Written in terms of r,s,t equation (2) reads:

$$Ar + 2Bs + Ct = \Phi. \tag{3}$$

Furthermore the following relations are valid in general, and therefore hold on Γ

$$dp = r\,dx + s\,dy, \tag{3a}$$

$$dq = s\,dx + t\,dy. \tag{3b}$$

Now, since p and q are known on Γ, equations (3) and (3a,b) constitute three linear equations for the determination of r,s,t on the curve. The determinant of this system is

$$\Delta = \begin{vmatrix} A & 2B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = A\,dy^2 - 2B\,dx\,dy + C\,dx^2.$$

Only when this determinant Δ is different from zero can r,s,t be calculated from (3), (3a), and (3b). However, in general, two directions, $dy{:}dx$, exist for every point (x,y), for which this is not the case. Therefore two (real or conjugate complex) families of curves exist on which $\Delta = 0$, and which, according to Monge, are called *characteristics*.[1] They are the dotted lines of Fig. 9. Along each of these characteristics it is in general impossible to solve for r,s,t in terms of u,p,q. We shall therefore demand as a necessary condition for the solvability of our problem, that Γ shall be *nowhere tangent to a characteristic*. The opposite case, in which Γ coincides with one of the characteristics, will be discussed in §9A in connection with D'Alembert's solution.

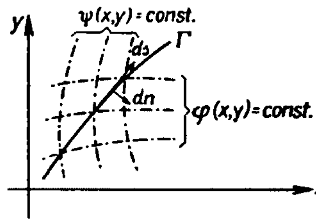

Fig. 9. The curve Γ, along which u and $\partial u/\partial n$ are given, and the two families of characteristics $\xi = \varphi(x,y) = \text{const.}$ and $\eta = \psi(x,y) = \text{const.}$

When the condition $\Delta \neq 0$ is satisfied, a solution of the differential equation in the neighborhood of Γ must exist. Then the higher derivatives can be calculated in exactly the same way as the second derivatives. Let us consider, say, the third derivatives:

$$r_x = \frac{\partial^3 u}{\partial x^3}, \quad s_x = \frac{\partial^3 u}{\partial x^2 \partial y} = r_y, \quad t_x = \frac{\partial^3 u}{\partial x \partial y^2} = s_y, \quad t_y = \frac{\partial^3 u}{\partial y^3}.$$

Differentiating (3) and (3a,b) with respect to x, we get:

$$\begin{aligned} A r_x + 2B s_x + C t_x &= \Phi_x + \cdots \\ r_x\,dx + s_x\,dy &= dr, \\ s_x\,dx + t_x\,dy &= ds. \end{aligned}$$

[1] A geometrically intuitive introduction of characteristics is given, e.g., by B. Baule in v. VI (Partielle Differentialgleichungen) of his Mathematik des Naturforschers und Ingenieurs, Hirzel, Leipzig 1944.

On the right . . . represents terms that contain no third derivatives, and therefore contain only known quantities. The determinant of this system is again Δ. The same holds for equations obtained by differentiation with respect to y. Our condition is therefore sufficient for the computability of the third and all higher derivatives. Therefore u can be expanded in a Taylor series at every point of Γ and the coefficients are uniquely determined by the boundary conditions on Γ.

We now turn to the discussion of the equation of characteristics

$$A\,dy^2 - 2\,B\,dx\,dy + C\,dx^2 = 0, \tag{4}$$

where we restrict ourselves to an arbitrarily chosen neighborhood in the xy-plane,[2] and distinguish between the following cases:

1) $A\,C - B^2 > 0$ *elliptic type* in which the characteristics are conjugate complex.

2) $A\,C - B^2 < 0$ *hyperbolic type* in which the characteristics form two distinct families.

3) $A\,C - B^2 = 0$ *parabolic type* in which only one real family of characteristics exists.

Each of the three types can be brought into a special normal form in which the equations of the characteristics are utilized for the introduction of new coordinates. Let these equations be

$$\varphi(x, y) = \text{const. and } \psi(x, y) = \text{const.} \tag{4a}$$

respectively. Then through the transformation

$$\xi + i\eta = \varphi(x, y), \qquad \xi - i\eta = \psi(x, y) \tag{5}$$

one obtains the normal form for the *elliptic* type,

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = X\left(u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \xi, \eta\right); \tag{5a}$$

through the transformation

$$\xi = \varphi(x, y), \qquad \eta = \psi(x, y) \tag{6}$$

one obtains the normal form for the *hyperbolic* type,

$$\frac{\partial^2 u}{\partial \xi\,\partial \eta} = X\left(u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \xi, \eta\right); \tag{6a}$$

[2] When A,B,C depend on x,y, then the equation may obviously be of different types for different neighborhoods of the xy-plane.

and through

$$\xi = \varphi(x, y) = \psi(x, y), \qquad \eta = x$$

one obtains the normal form for the *parabolic* type,

$$\text{(7a)} \qquad \frac{\partial^2 u}{\partial \eta^2} = X\left(u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \xi, \eta\right).$$

Before proving this, we compare the above forms (5a), (6a) and (7a) with the equations (7.3), (7.6a), and (7.16), i.e., with the two-dimensional potential equation, the equation of the vibrating string, and the equation of linear heat conduction. We observe that the left hand sides of (5a) and (7.3) coincide except for the letters used to denote the independent variables. The analogous relation holds between (7a) and (7.16). In (6a) we only have to perform the simple transformation

$$\text{(8)} \qquad \xi = \frac{1}{2}(\xi' + \eta'), \qquad \eta = \frac{1}{2}(\xi' - \eta')$$

with the inverse

$$\text{(8a)} \qquad \xi' = \xi + \eta, \qquad \eta' = \xi - \eta$$

we obtain

$$\frac{\partial^2 u}{\partial \xi\, \partial \eta} = \frac{\partial^2 u}{\partial \xi'^2} - \frac{\partial^2 u}{\partial \eta'^2},$$

which establishes the essential equality of the left hand sides of (6a) and (7.6a). Hence *the two-dimensional potential equation, the equation of the vibrating string and the equation of linear heat conduction are the simplest examples of the elliptic, the hyperbolic, and of the parabolic types*, respectively.

Starting with the treatment of the *hyperbolic case*, we first show that (6a) is obtained from the initial equation (2) through the transformation (6). From (6) we obtain for the first derivatives

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\varphi_x + \frac{\partial u}{\partial \eta}\psi_x, \qquad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\varphi_y + \frac{\partial u}{\partial \eta}\psi_y$$

where the subscripts again denote differentiation. From this we obtain for the second derivatives

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial \xi^2}\varphi_x^2 + 2\frac{\partial^2 u}{\partial \xi\,\partial \eta}\varphi_x\psi_x + \frac{\partial^2 u}{\partial \eta^2}\psi_x^2 + \cdots \\
\frac{\partial^2 u}{\partial x\,\partial y} &= \frac{\partial^2 u}{\partial \xi^2}\varphi_x\varphi_y + \frac{\partial^2 u}{\partial \xi\,\partial \eta}(\varphi_x\psi_y + \varphi_y\psi_x) + \frac{\partial^2 u}{\partial \eta^2}\psi_x\psi_y + \cdots \\
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 u}{\partial \xi^2}\varphi_y^2 + 2\frac{\partial^2 u}{\partial \xi\,\partial \eta}\varphi_y\psi_y + \frac{\partial^2 u}{\partial \eta^2}\psi_y^2 + \cdots
\end{aligned}$$

where the three dots stand for terms containing only first derivatives. Multiplying the last three equations by A, $2B$ and C, respectively, and adding, we obtain for the left side of (2):

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial \xi^2}\left(A\varphi_x^2 + 2B\varphi_x\varphi_y + C\varphi_y^2\right)\\
(9)\qquad &+ 2\frac{\partial^2 u}{\partial \xi\,\partial \eta}\left(A\varphi_x\psi_x + B(\varphi_x\psi_y + \varphi_y\psi_x) + C\varphi_y\psi_y\right)\\
&+ \frac{\partial^2 u}{\partial \eta^2}\left(A\psi_x^2 + 2B\psi_x\psi_y + C\psi_y^2\right) + \cdots
\end{aligned}
$$

But here the coefficients of $\partial^2 u/\partial\xi^2$ and $\partial^2 u/\partial\eta^2$ vanish, since for the family of characteristics φ = const. we have

$$\varphi_x\,dx + \varphi_y\,dy = 0,$$

Hence on introducing the ratio $dx{:}dy$ into (4) we get

$$A\varphi_x^2 + 2B\varphi_x\varphi_y + C\varphi_y^2 = 0. \tag{10}$$

The derivatives of ψ must satisfy the same equation. Hence (9) indeed reduces to the hyperbolic normal form (6a) if we transfer the coefficient of $\partial^2 u/\partial\xi\partial\eta$ in (9) to the other side of the equation.

Since in the *parabolic case* we have $\eta = x$, we must substitute in (9)

$$\psi(x, y) = x, \quad \text{and hence} \quad \psi_x = 1, \quad \psi_y = 0, \tag{11}$$

whereas (10) still holds for φ_x, φ_y. The first term in (9) therefore vanishes. Owing to (11) the coefficient of the second term reduces to $A\varphi_x + B\varphi_y$ which also vanishes since $AC - B^2 = 0$ makes the left side of (10) a perfect square, so that (10) can be rewritten as $(A\varphi_x + B\varphi_y)^2/A = 0$. Considering (9) and (11) the third term finally becomes simply

$$A\frac{\partial^2 u}{\partial\eta^2},$$

which is the parabolic normal form (7a).

The *elliptic case* need not be treated separately. It can be reduced to the hyperbolic case by a transformation analogous to (8a):

$$\xi' = \xi + i\eta, \quad \eta' = \xi - i\eta.$$

§ 9. Differences Among Hyperbolic, Elliptic, and Parabolic Differential Equations. The Analytic Character of Their Solutions

The problem of integration, which is illustrated in Fig. 9, is applied in physics only to the case of hyperbolic differential equations; for elliptic differential equations it is replaced by an entirely different kind of problem, the *boundary value problem.* For the time being, we shall discuss this profound difference only sketchily and refer the reader to the following sections for a more precise treatment.

A. Hyperbolic Differential Equations

As the simplest example we use the equation of the vibrating string, which, written in its normal form, is

$$\frac{\partial^2 u}{\partial\xi\,\partial\eta} = 0, \quad \xi = x + y, \quad \eta = x - y, \quad y = ct. \tag{1}$$

Here the characteristics are the lines ξ = const., η = const., which in Fig. 10 are drawn at 45° angles with the x- and y-axes. The general solution of (1) is the sum of a function of ξ and a function of η:

$$u = F_1(\xi) + F_2(\eta). \tag{2}$$

Because of the meaning of ξ and η this is *d'Alembert's solution* (see V. II, §13). For the sake of simplicity, let us consider u as being given on segments AB and AD of two of the characteristics. This determines u in the entire rectangle $ABCD$. We could calculate the value of u at P by passing in the directions of the characteristics to P_1 and P_2, and substituting into (2) the values $F_1(\xi)$, $F_2(\eta)$ which are given at these points. *The values along two intersecting characteristics determine the function everywhere. For example, any discontinuities of the given functions on the characteristics would be continued into the interior of ABCD.* Thus the solution need not be an *analytic function*[3] of x and y over its domain of definition.

In physics one is given the values of u *and* of $\partial u/\partial y$ along a segment of length l on the x-axis (l = length of string):

$$u = u(x, 0) \qquad \text{and} \qquad \frac{\partial u}{\partial y} = \frac{1}{c}\frac{\partial u}{\partial t} = v(x, 0).$$

This segment corresponds to the curve Γ of Fig. 9, on which, too, u and $\partial u/\partial n$ were given, and it satisfies the requirement of not being tangent to any characteristic.

In order to apply the conclusions drawn from (2) to our present problem, we have to calculate F_1 and F_2 from our given $u(x,0)$, $v(x,0)$. This is done with the help of the following equations, which are immediate consequences of (2):

[3] A function of two real variables x,y is called analytic in a certain domain, if in some neighborhood of each point (x_0, y_0) of this domain it can be represented as a power series in $x - x_0$ and $y - y_0$.

$$u(x,0) = F_1(x) + F_2(x), \qquad F_1(x) = \frac{1}{2}\left\{u(x,0) + \int v(x,0)\,dx\right\},$$
$$v(x,0) = F_1'(x) - F_2'(x). \qquad F_2(x) = \frac{1}{2}\left\{u(x,0) - \int v(x,0)\,dx\right\}.$$

We conclude: *the given initial values, together with any possible discontinuities, are continued along the characteristics.* The solution, $u(x,y)$ is in general *not an analytic function* of x and y. It is determined *only within the rectangle of characteristics determined by the length of string l* as shown in Fig. 10.

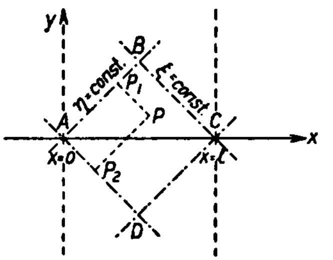

Fig. 10. The vibrating string of length l and the square of characteristics determined by its end point.

However, from a physical point of view, the solution must be determined from the initial time on, i.e., for all $y > 0$. This indicates that, in addition to the initial values, certain boundary values must be prescribed at the ends of the string. These are the stringing conditions $u = 0$ for $x = 0$ and $x = l$. Just as for all x such that $0 < x < l$ *two* values (u and $\partial u/\partial y$) had to be given, so for all $y > 0$, *two* values are given. This is due to the fact that our differential equation is of second order in both variables x,y and the only difference is that both values along the x-axis are given at the *same* point $(x,0)$ whereas the values along the y-direction are given at the different points $(0,y)$ and (l,y). The only exceptions to this rule of two necessary boundary conditions are the characteristics on which, as we saw above, *one value* (F_1 *or* F_2) *is sufficient.*

We shall show in §11 that these results, which we have established for the case of the vibrating string, can be extended to all cases of hyperbolic type.

B. Elliptic Differential Equations

Here the characteristics are imaginary and therefore have no direct bearing on the problems we are going to treat. These problems do not deal with an arc Γ, as in Fig. 9, but rather with a *closed region* S of the real xy-plane. On the boundary of S, u *or* $\partial u/\partial n$ (or a linear combination of u and $\partial u/\partial n$) will be given but not *both* u and $\partial u/\partial n$ as in the hyperbolic case. Discontinuities of the boundary values are *not* continued into the interior of S, but only into the imaginary domain, and the function u is *analytic* everywhere in the interior of S.

These are known theorems from the theory of functions (two-dimensional potential theory). Their proof for arbitrary linear elliptic differential equations will be given in the following section.

The analogue to d'Alembert's solution (2) is given in potential theory by

$$u = f_1(x + i\,y) + f_2(x - i\,y),$$

where, in order that u be real, we must set $f_2 = f_1^*$, i.e., f_2 conjugate[4] to f_1 We may also write:

$$u = \mathrm{Re}[f(z)], \tag{3}$$

where f is an arbitrary analytic function of the complex variable $z = x + iy$. However, this general solution of the equation $\Delta u = 0$ does not help us (at least not directly) in the general solution of our boundary value problem.

C. Parabolic Differential Equations

Here the two families of characteristics have degenerated into *one*. In the special case of the normal form of the equation of linear heat conduction this is the family of lines parallel to the x-axis. Only *one* boundary condition should be given on these characteristics just as in the case of hyperbolic differential equations (see p. 42). We can also see this directly from (7.16): here $\partial u/\partial y$ is determined uniquely if u is given as a function of x for some fixed y. From physical considerations one sees this in the following manner: the thermal behavior of a rod of length l is determined once and for all as soon as its initial temperature is given together with conditions for the ends of the rod (the lateral surface of the rod must be considered adiabatically closed, if heat is to flow only in the x-direction).

We shall see in §12, that the temperature distribution of the rod becomes *an analytic function of x and y* for arbitrary — even discontinuous — initial temperature. To this extent, therefore, the parabolic type resembles the elliptic type. However, the problem is not relative to a bounded region, but rather, as in the hyperbolic case, relative to a strip, i.e., a region which is infinite in one direction. The parabolic type, therefore, occupies a middle position between the elliptic and the hyperbolic types.

[4] We use the notation f^* instead of $\overline{f}$, which is more common in mathematical literature, since we want to reserve the use of the bar for mean values in time. Re and Im stand for the real and imaginary part respectively.

§10. Green's Theorem and Green's Function for Linear, and, in Particular, for Elliptic Differential Equations

In (8.1) we had the general form of a linear differential equation of second order. In order to retain a common expression for the three types, we shall not transform this system into its canonical form for the time being.

A. Definition of the Adjoint Differential Expression

We now have to introduce the seemingly rather formal concept of *the differential form $M(v)$ which is adjoint to $L(u)$*. It is defined by the requirement that the expression $v\,L(u) - u\,M(v)$ be *generally integrable* or as we may put it, that it be a *kind of divergence.*
We demand, namely

$$v\,L(u) - u\,M(v) = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}. \tag{1}$$

The problem is to determine M and X,Y as functions of v and of u,v respectively.[5]

We shall use the following identities:

$$vA\frac{\partial^2 u}{\partial x^2} - u\frac{\partial^2 A v}{\partial x^2} = \frac{\partial}{\partial x}\left(Av\frac{\partial u}{\partial x} - u\frac{\partial A v}{\partial x}\right), \dots \tag{2}$$

$$v\,B\frac{\partial^2 u}{\partial x\,\partial y} - u\frac{\partial^2 Bv}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(v\,B\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(u\frac{\partial Bv}{\partial x}\right) = \cdots \tag{2a}$$

$$Dv\frac{\partial u}{\partial x} - u\frac{\partial}{\partial x}(-Dv) = \frac{\partial}{\partial x}(Duv), \dots \tag{3}$$

Here the three dots (. . .) indicate the fact that (2) and (3) remain valid if we replace x by y and A,D by C,E respectively, and that on the right side of (2a) we may use the symmetric expression obtained by interchanging x and y. From this we get:

$$M(v) = \frac{\partial^2 Av}{\partial x^2} + 2\frac{\partial^2 Bv}{\partial x\,\partial y} + \frac{\partial^2 Cv}{\partial y^2} - \frac{\partial Dv}{\partial x} - \frac{\partial Ev}{\partial y} + Fv, \tag{4}$$

$$\begin{aligned} X &= A\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + B\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \left(D - \frac{\partial A}{\partial x} - \frac{\partial B}{\partial y}\right)uv,\\ Y &= B\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + C\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \left(E - \frac{\partial B}{\partial x} - \frac{\partial C}{\partial y}\right)uv. \end{aligned} \tag{5}$$

[5] The operation of divergence is properly defined only for a *vector*. Since, as equation (5) will show, X and Y are not vector components, we speak of "a kind of divergence."

Obviously X and Y are determined only up to quantities X_0, Y_0, whose divergence vanishes. We can therefore change the terms in (5): we may add $-\partial\Phi/\partial y$ to X and $+\partial\Phi/\partial x$ to Y, where Φ is an arbitrary function of x,y as well as of u,v.

We see that the relation between L and M is *reciprocal:* $L(v)$ is the adjoint differential form to $M(u)$.

Of particular importance for mathematical physics are those differential expressions for which $L(u) = M(u)$. They are called *self-adjoint.* By comparing (4) with (8.1) we get the condition of self-adjointness

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} = D, \qquad \frac{\partial B}{\partial x} + \frac{\partial C}{\partial y} = E. \tag{6}$$

B. Green's Theorem for an Elliptic Differential Equation in its Normal Form

We now consider a region S with boundary curve C in the xy-plane and integrate (1) over S. We denote the element of area of S by $d\sigma$, and the line element of C by ds; let the orientation be counter-clockwise (see Fig. 11).

Applying Gauss' theorem[6] we get

$$\begin{aligned}\int_S [v\,L(u) - u\,M(v)]\,d\sigma &= \int_S \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) d\sigma \\ &= \int_C \{X\cos(n, x) + Y\cos(n, y)\}\,ds.\end{aligned} \tag{7}$$

This is the *general formulation of Green's theorem* which is valid for all three types. Setting $A = C = 1$, $B = 0$, we specialize it to the case of the *elliptic type* in *normal form.* We then have:

$$\begin{aligned}\int_S [v\,L(u) - u\,M(v)]\,d\sigma &= \int_C \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds \\ &\quad + \int_C \{D\cos(n, x) + E\cos(n, y)\}\,u\,v\,ds.\end{aligned} \tag{7 a}$$

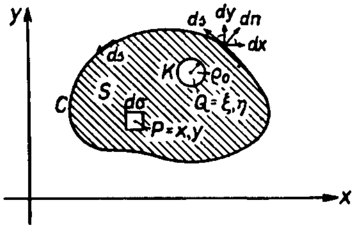

Fig. 11. Illustrating Green's theorem for an elliptic differential equation. The integration with respect to $d\sigma$ is extended over the domain S between the boundary curve C and the circle K of radius ϱ_0 which contains the unit source at Q.

[6] It states, when applied to a two-dimensional vector **A** with components X, Y, that $\int \operatorname{div} \mathbf{A}\,d\sigma = \int \mathbf{A}_n\,ds$,

This is a generalization of *Green's theorem* of potential theory

$$\int (v\,\Delta u - u\,\Delta v)\,d\sigma = \int \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)ds,$$

which is obtained from (7a) by setting $D = E = 0$. (The fact that in potential theory also $F = 0$ is of no importance here.)

We shall meet another form of Green's theorem in exercise 112.

If, in the interior of S, u and v satisfy the equations

$$L(u) = 0, \quad M(v) = 0$$

then the left hand sides of the equations (7), (7a) vanish. These equations, therefore, become

$$(7\,\mathrm{b}) \qquad 0 = \int_C \left\{X\cos(n, x) + Y\cos(n, y)\right\} ds$$

$$(7\,\mathrm{c}) \qquad 0 = \int_C \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds + \int_C \left\{D\cos(n, x) + E\cos(n, y)\right\} u\,v\,ds.$$

However, this holds only if u and v and the derivatives which appear here are *continuous* throughout S. If v has a discontinuity at the point $Q = (\xi, \eta)$, then it must be excluded from the domain of integration, just as in all applications of Green's theorem. We therefore surround Q by a curve K, which we choose to be a circle of arbitrarily small radius ρ_0. The integration in (7b,c) must then be taken over both boundaries K and C:

$$\int_K \ldots ds + \int_C \ldots ds = 0,$$

where the orientation is opposite on the two curves and the direction n is to the exterior of S.

If for K we use (7c) and for C we use (7b) we get:

$$(8) \qquad \begin{aligned} &\int_K \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds + \int_K u\,v\left\{D\cos(n, x) + E\cos(n, y)\right\} ds \\ &\qquad = -\int_C \left\{X\cos(n, x) + Y\cos(n, y)\right\} ds. \end{aligned}$$

or, written in terms of coordinates:

$$\int \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) d\sigma = \int [X\cos(n, x) + Y\cos(n, y)]\,ds.$$

We apply this formally in (7) to our "pseudovector" X, Y.

C. Definition of a Unit Source and of the Principal Solution

We shall assume that the discontinuity of v at Q consists of a "*unit source.*" By this we mean the following: the yield q of a source Q is defined as the outward gradient of its field v. If we denote the distance from Q by ϱ, we have

$$q = \int_K \frac{\partial v}{\partial \varrho} ds \tag{9}$$

where K has the same meaning as before. Assuming that in the immediate neighborhood of the source v depends only on ϱ, we get

$$q = \int_{\varphi=-\pi}^{+\pi} \frac{dv}{d\varrho} \varrho \, d\varphi = 2\pi \varrho \frac{dv}{d\varrho}. \tag{9a}$$

A unit source is therefore given by:

$$\frac{dv}{d\varrho} = \frac{1}{2\pi\varrho}, \qquad v = \frac{1}{2\pi} \log \varrho + \text{const} \qquad \text{for} \quad \varrho \to 0. \tag{9b}$$

For arbitrary ϱ we write:

$$v = U \log \varrho + V, \qquad \varrho = \sqrt{(x-\xi)^2 + (y-\eta)^2}, \tag{10}$$

where U and V are analytic functions of x,y and ξ, η such that U becomes $1/2\pi$ for $(x, y) \to (\xi, \eta)$.

A function of this kind we call a *principal solution* of the differential equation $M(v) = 0$. In the same way we shall speak of a principal solution of the adjoint equation $L(u) = 0$. Since the latter also corresponds to a unit source it will have the same form (10), although in general U and V will be different functions. Here too we can assume U and V to be analytic as long as the coefficients D,E,F in the differential equation are analytic. In the case of the potential equation $\Delta u = 0$ our principal solution corresponds essentially to the logarithmic potential, where we have for all ϱ

$$v = \frac{1}{2\pi} \log \varrho. \tag{10a}$$

D. The Analytic Character of the Solution of an Elliptic Differential Equation

We return now to equation (8). Substituting (10) in (8), we see that only the term with

$$\frac{\partial v}{\partial n} = -\frac{\partial v}{\partial \varrho} = -\frac{U}{\varrho} + \cdots = -\frac{1}{2\pi\varrho} + \cdots$$

contributes to the integral over K, while all the other terms on the left side of (8) have zeros of the same order as $\varrho \log \varrho$ or of higher order. Since u is continuous at Q and the perimeter of K is $2\pi\varrho_0$, we obtain for the left side of (8):

$$-\int_K u \frac{\partial v}{\partial n} ds = \frac{u_Q}{2\pi\varrho_0} \int_K ds = u_Q;$$

and equation (8) becomes

(11) $$u_Q = -\int_C \{X \cos(n, x) + Y \cos(n, y)\} ds.$$

The most interesting aspect of this formula is its dependence upon ξ, η which is brought about by the terms v, $\partial u/\partial x$, $\partial v/\partial y$ that enter in X and Y and are given analytically by (10). When Q lies in the *interior* of S (*not* on the boundary), then $\log \varrho$ is a regular analytic function, since the point P $= (x,y)$ in the integration is restricted to the boundary curve C and does not coincide with Q. *Therefore* $u_Q = u(\xi, \eta)$ *is an analytic function of* ξ,η *in the interior of* S. This holds whether or not the boundary values u, $\partial u/\partial x$, $\partial u/\partial y$ are analytic; in any case the dependence of the integrand on x,y disappears upon integration with respect to ds. Even discontinuities of the boundary values are averaged out. *Discontinuities on the boundary are not continued into the interior of* S. (The characteristics are imaginary.) This proves the assertion of §9B.

For a self-adjoint differential equation in its normal form we have, according to (6), $D = E = 0$. Using the form (7c) of the line integral we get from (11)

(11a) $$u_Q = \int_C \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) ds.$$

Using expression (10a) for v, we get for the special case of the *potential equation:*

(11b) $$u_Q = u(\xi, \eta) = \frac{1}{2\pi} \int_C \left(u \frac{\partial \log \varrho}{\partial n} - \log \varrho \frac{\partial u}{\partial n}\right) ds.$$

E. The Principal Solution for an Arbitrary Number of Dimensions

We restrict ourselves here to the case of the potential equation. The three-dimensional analogue of (9b) is

$$\frac{\partial v}{\partial r} = \frac{1}{4\pi r^2}, \qquad v = -\frac{1}{4\pi r} + \text{const.}$$

(r = distance from the source at Q, $4\pi r^2$ = surface area of the sphere.) This is essentially[7] the so-called "Newtonian potential."

In the four-dimensional case we have equation (7.5). This yields the principal solution:

$$\frac{\partial v}{\partial R} = \frac{1}{2\pi^2 R^3}, \qquad v = -\frac{1}{4\pi^2 R^2} + \text{const.}$$

where R is the distance from Q and $2\pi^2 R^3$ is the surface of the hypersphere. The following table shows the decreasing orders of infinity at the source with decreasing number of dimensions. For the dimension one v is continuous at the source. In fact, the potential equation in one dimension is $d^2v/dx^2 = 0$ which yields $dv/dx =$ const. The constant will have different values C_1 and C_2 on the right and left side of the source respectively. This follows from the condition that it be a unit source so that $C_1 - C_2 = 1$. The discontinuity has passed from v to the gradient of v. (See exercise II.3).

Dimension	4	3	2	1
grad v . . .	$\frac{1}{2\pi^2 R^3}$	$\frac{1}{4\pi r^2}$	$\frac{1}{2\pi\varrho}$	C_1 or C_2
v	$-\frac{1}{4\pi^2 R^2}$	$-\frac{1}{4\pi r}$	$-\frac{1}{2\pi}\log\frac{1}{\varrho}$	continuous

F. Definition of Green's Function for Self-Adjoint Differential Equations

We now deal with the boundary value problem of §9. This question is by no means settled by the construction of the principal solution. We first consider the simplest case of self-adjointness. In order to calculate u at the point Q in equation (11a) we must know *both u and $\partial u/\partial n$* on C, whereas in the boundary value problem we are given *either u or $\partial u/\partial n$*.

[7] The denominator 4π corresponds to the "rational units" of electrodynamics.

Our problem is now to modify the principal solution v, so as to eliminate $\partial u/\partial n$ (or u) from (11a). We call this modified function of the two pairs of variables x,y and ξ, η Green's function and denote it by $G(P,Q)$. It has to satisfy the following conditions:

a) $L(G) = 0$ in the interior of C,

b) $\quad G = 0$ (or $\partial G/\partial n = 0$) on C,

c) $\lim\limits_{P\to Q} G(P,Q) \to \frac{1}{2\pi}\log\varrho$ (condition of unit source).

Conditions a) and c) are the same as for the original v, but condition b) has been added. Replacing v by G in (11a) we get

$$u_Q = \int u \frac{\partial G}{\partial n} ds \qquad \text{or} \qquad \left(u_Q = -\int \frac{\partial u}{\partial n} G\, ds\right). \tag{12}$$

This solves the boundary value problem in both cases (for given u or $\partial u/\partial n$). However, due to condition b), the construction of G itself requires the solution of a boundary value problem. But this problem is simpler than the general boundary value problem, and we shall see that in special cases it can be solved in an elegant way with the help of a reflection process. On the other hand G, unlike u, is not regular in the interior of C, but like v is a function with a prescribed unit source.

Equation (12) reduces the boundary value problem to a simple quadrature. Green's function plays the same role in the general theory of integral equations. It is called there the "resolving kernel."

Another interesting property of G which follows from the conditions a),b),c) is the *reciprocity relation*

d) $\quad G(P,Q) = G(Q,P)$.

It expresses the interchangeability of *source-point and action-point*, so to speak, the interchangeability of *cause and effect*.

In order to prove d) we substitute in (7a)

$$M = L, \quad u = G(I,P), \quad v = G(I,Q).$$

The point $I = (x_1,y_1)$ shall be called "point of integration." Since u becomes infinite for $I = P$, and v for $I = Q$, these points must be excluded from the integration by infinitesimal circles K_P and K_Q. According to a), integration over the region bounded by these circles and by C makes the left side of (7a) equal to 0; also, according to b), the integral over C on the right side of (7a) becomes equal to 0. There only remain the line integrals over K_P and K_Q which, according to c), yield:

$$u_Q \int\limits_{K_Q} \frac{ds}{2\pi \varrho_Q} - v_P \int\limits_{K_P} \frac{ds}{2\pi \varrho_P} = G(Q, P) - G(P, Q).$$

Since this must vanish d) is proved.

Equation (12) is the solution of the boundary problem for the homogeneous equation $L(u) = 0$. We now consider the solution of the non-homogeneous differential equation

$$L(u) = \varrho \tag{13}$$

where $(\varrho x, y)$ is an arbitrary continuous point-function in S with continuous first and second derivatives. Substituting $v = G(P, Q)$ in (7a) we get for the first term on the left side:

$$\int\limits_S \varrho\, G(P, Q)\, d\sigma_P$$

which is added to the term in (12). Instead of (12) we get

$$u_Q = \int\limits_S \varrho\, G\, d\sigma + \int\limits_C u \frac{\partial G}{\partial n}\, ds \tag{13a}$$

or, if $\partial u/\partial n$ instead of u is given on C:

$$u_Q = \int\limits_S \varrho\, G\, d\sigma - \int\limits_C \frac{\partial u}{\partial n} G\, ds. \tag{13b}$$

These formulas apply to every self-adjoint differential expression in its normal form $L(u) = \Delta u + Fu$, in particular to the ordinary wave equation ($F = k^2 =$ Const.) and to the potential equation ($F = 0$).

In the case of a *non-self-adjoint* differential form $L(u)$ equations (12) and (13*a*,*b*) remain valid. But, as we see from (7a), G must satisfy the adjoint equation $M(G) = 0$ in the variables x,y; also, condition b) must be changed somewhat. Instead of a) and b) we now have:

a′) $$M(G) = 0,$$

b′) $$G = 0 \left(\text{or } \frac{\partial G}{\partial n} - G\{D \cos(n, x) + E \cos(n, y)\} = 0\right).$$

Condition c) remains valid. However the reciprocity law d) now reads

d′) $$G(P, Q) = H(Q, P).$$

Here H is Green's function for the *adjoint* equation to $M = 0$, and hence is satisfies the equation $L(H) = 0$ in the coordinates of Q.

§11. Riemann's Integration of the Hyperbolic Differential Equation

The normal form of a linear differential equation of second order of hyperbolic type is obtained from (8.1) by setting $A = C = 0, B = 1/2$:

$$L(u) = \frac{\partial^2 u}{\partial x\, \partial y} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + F u = 0. \tag{1}$$

Its adjoint differential equation is according to (10.4):

$$M(v) = \frac{\partial^2 v}{\partial x\, \partial y} - \frac{\partial D\, v}{\partial x} - \frac{\partial E\, v}{\partial y} + F v = 0. \tag{2}$$

At the same time one obtains from (10.5)

$$\begin{aligned} X &= \frac{1}{2}\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + D\,u\,v, \\ Y &= \frac{1}{2}\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + E\,u\,v. \end{aligned} \tag{3}$$

Substituting (1),(2),(3) in (10.7) we get:

$$\int_S \left\{v\,L(u) - u\,M(v)\right\} d\sigma = \int_C \left\{X\cos(n, x) + Y\cos(n, y)\right\} ds. \tag{4}$$

In order to obtain an integration of the hydrodynamic equations Riemann chose as the region S the "triangle" PP_1P_2 of Fig. 12, with a boundary consisting of the segments of characteristics PP_1 and PP_2 and of the arc P_1P_2 of the curve Γ. u and $\partial u/\partial n$ are given on Γ, which implies that $\partial u/\partial x$, $\partial u/\partial y$ are given on Γ (see p. 36). The curve Γ must satisfy the condition that it be tangent to no characteristic (p. 37). The function v in (4) is determined according to Riemann by the conditions:

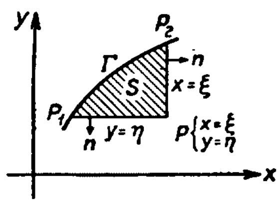

Fig. 12. Riemann's integration of a hyperbolic differential equation in its normal form with the help of the characteristic function v.

(5a) $M(v) = 0$ in S with respect to the variables x,y;

(5b) $v = 1$ at the point P with coordinates $x=\xi$, $y=\eta$;

(5c) $\dfrac{\partial v}{\partial y} - D\,v = 0$ on the characteristic $x = \xi$,

$\dfrac{\partial v}{\partial x} - E\,v = 0$ on the characteristic $y = \eta$.

We add the following remarks.

1. It would not be possible to replace (5b) by a condition of discontinuity as in (10.10), since a hyperbolic equation does not admit isolated singularities (every singularity is continued along the characteristics). For this reason we no longer call v the "principal solution" or "Green's function" but call it the "characteristic function."

2. The conditions (5c) prescribe only *one* condition each on the characteristics $x = \xi$ and $y = \eta$, whereas on Γ two conditions have to be given for u. This corresponds to the fact that the characteristics are an exception with respect to the boundary conditions that are to be prescribed on them; we saw this in §9 in connection with the equation of the vibrating string. If we call the boundary value problem along two intersecting characteristics a *boundary value problem of the second kind,* in contrast to a *boundary value problem of the first kind* along a general curve Γ, then we can say that Riemann's method consists of *the reduction of a boundary value problem of the first kind to a much simpler boundary value problem of the second kind.*

Substituting condition (5a) in (4) and remembering that $L(u) = 0$ we get

$$0 = \int_{\Gamma} \cdots + \int_{P_1}^{P} \cdots + \int_{P}^{P_2} \cdots \tag{6}$$

In the last integral $\cos(n,y) = 0$ and only the X term remains. We transform the term with $\partial u/\partial y$ by integration by parts:

$$\frac{1}{2}\int_{P}^{P_2} v \frac{\partial u}{\partial y} dy = \frac{1}{2} v\, u \Big|_{P}^{P_2} - \frac{1}{2}\int_{P}^{P_2} u \frac{\partial v}{\partial y} dy .$$

Combining this with the other terms we get

$$\int_{P}^{P_2} X\, dy = \frac{1}{2}(v\, u)_{P_2} - \frac{1}{2}(v\, u)_P - \int_{P}^{P_2} u\left(\frac{\partial v}{\partial y} - Dv\right) dy . \tag{6a}$$

For the middle integral of (6) where $\cos(n,x) = 0$ and $\cos(n,y) = -1$ (n is the outer normal) we get in an analogous manner:

$$-\int_{P_1}^{P} Y\, dx = \frac{1}{2}(v u)_{P_1} - \frac{1}{2}(v\, u)_P + \int_{P_1}^{P} u\left(\frac{\partial v}{\partial x} - E\, v\right) dx . \tag{6b}$$

The integrals on the right sides of (6a) and (6b) vanish on account of condition (5c). If we consider (5b) equation (6) becomes,

(7) $$u_P = \int_\Gamma \{X\cos(n,x) + Y\cos(n,y)\}\,ds + \frac{1}{2}\{(v\,u)_{P_1} + (v\,u)_{P_2}\}.$$

The value of u at an arbitrary point P is given here in terms of the values of u and its first derivatives on Γ as they enter in X and Y (u_{P_1} and u_{P_2} are among those values). We state: Equation (7), reduces the boundary value problem of the *first* kind for u to the problem of the computation of v, that is to a boundary value problem of the *second* kind which is given by the conditions (5a,b,c).

The computation of v is not difficult. It is particularly easy in the hydrodynamic example that was treated by Riemann. In that case we have[8]

(8) $$D = E = -\frac{a}{x+y}, \qquad F = 0.$$

Condition (5c) implies

$$x = \xi: \qquad \frac{1}{v}\frac{\partial v}{\partial y} = -\frac{a}{\xi + y}, \qquad v = C_1(\xi + y)^{-a},$$

$$y = \eta: \qquad \frac{1}{v}\frac{\partial v}{\partial x} = -\frac{a}{x+\eta}, \qquad v = C_2(x+\eta)^{-a}.$$

Both these conditions and condition (5b) are satisfied if we set:

(9) $$C_1 = C_2 = (\xi+\eta)^a, \qquad v = \left(\frac{\xi+\eta}{x+y}\right)^a.$$

In order to satisfy (5a) Riemann modifies (9) as follows:

(10) $$v = \left(\frac{\xi+\eta}{x+y}\right)^a F(a+1, -a, 1, z), \qquad z = -\frac{(x-\xi)\,(y-\eta)}{(x+y)\,(\xi+\eta)},$$

where

(10a) $$F(\alpha, \beta, \gamma, z) = 1 + \frac{\alpha\beta}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \cdots$$

is the hypergeometric series. In §24D we shall see some of the function-theoretic properties of this series, and in appendix II of Chapter IV we shall prove that v in (10) satisfies condition (5a), in other words, that it is a solution of the equation $M(v) = 0$. We note that on the characteristics we have $x = \xi$ or $y = \eta$, and therefore $z = 0$ and $F = 1$, which makes (10) identical with (9). Equation (10) for v and equation (7) for u solve our hyberbolic boundary value problem completely.

[8]The constant a which enters in (8) is expressed simply in terms of the exponent in the anisotropic equation of state from which the hydrodynamic problem is derived.

§12. Green's Theorem in Heat Conduction. The Principal Solution of Heat Conduction

The differential equation (7.14) of heat conduction

$$L(u) = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0, \quad y = kt, \tag{1}$$

is *not* self-adjoint. The adjoint equation of (1) is:

$$M(v) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial y} = 0. \tag{2}$$

We see this from (9.4) if we substitute the values of (1)

$$B = C = D = F = 0, A = 1, E = -1;$$

from (9.5) we get

$$X = v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}, \quad Y = -uv. \tag{3}$$

Just as in the elliptic and the hyperbolic case we get *Green's theorem for linear heat conduction* from (1),(2),(3) by integration over the interior and the boundary of a bounded domain in the x,y-plane. However, since x represents a spatial measurement and y a time measurement, we shall not consider here a region with curved boundary, but only such regions whose boundaries consist of segments parallel to the x- or y-axis, as that shown in Fig. 13.

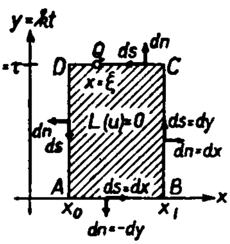

Fig. 13. Reduction of the general boundary value problem of heat conduction to the principal solution V for a rod with endpoints x_0 and x_1. The unit heat pole is at Q and has the coordinates $x = \xi, y = \eta$.

Along the side AB of the figure we have $ds = dx$, $dn = -dy$, $\cos(n,x) = 0$, $\cos(n,y) = -1$ and therefore

$$\int_A^B \{X\cos(n, x) + Y\cos(n; y)\}\,ds = -\int_A^B Y\,dx.$$

The same thing holds for the side CD which is also parallel to the x-axis and where the signs of both dx and $\cos(n,y)$ are reversed. Correspondingly we have for the sides BC and AD parallel to the y-axis

$$\int_B^C \{X \cos(n, x) + Y \cos(n, y)\}\, ds = + \int_B^C X\, dy\,.$$

Using the values of L, M, X, Y given in (1),(2),(3) we get the following form of Green's theorem:

$$\begin{aligned} &\int \left\{ v\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y}\right) - u\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial y}\right)\right\} dx\, dy \\ &= \int u\, v\, dx + \int \left(v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x}\right) dy\,, \end{aligned} \tag{4}$$

where the first integral on the right side is taken over the two sides of the rectangle that are parallel to the x-axis and the second integral is taken over the other sides.

Formula (4) also represents Green's theorem for *two-dimensional* or *three-dimensional* heat conduction if we perform the following replacements:

$$\text{(5a)} \qquad dx \quad \text{by} \quad \begin{cases} d\sigma & \text{(two-dimensional case)} \\ d\tau & \text{(three-dimensional case)} \end{cases}$$

$$\text{(5b)} \qquad \frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial x}, \quad dy \ \text{by} \ \frac{\partial u}{\partial n}, \frac{\partial v}{\partial n}, \begin{cases} dy\ ds & \text{(two-dimensional case)} \\ dy\ d\sigma & \text{(three-dimensional case)} \end{cases}$$

$$\text{(5c)} \qquad \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial^2 v}{\partial x^2} \quad \text{by} \quad \Delta u,\ \Delta v\,.$$

In the three-dimensional case we integrate over a four-dimensional cylinder whose base is the three-dimensional heat conductor and whose generatrix is parallel to the time-axis; the integration in the second term on the right, which is indicated in (5b) by dy and is now replaced by integration with respect to $dy\, d\sigma = k\, dt\, d\sigma$, is extended over the three-dimensional lateral surface of this cylinder.

Before we apply these general formulas we must decide how we want to define the analogue of the "principal solution" of (10.9). We shall see that the "unit source" will have to be replaced by a "heat pole of strength one."

We first consider the case of linear heat conduction and its differenrial equation $L = 0$; the passage to the adjoint equation $M = 0$ and to the two-and three-dimensional cases will then be easy.

Let the heat conductor be infinite in both directions and let its temperature for $t = 0$ be given as a function of x:

$$u = f(x), \quad -\infty < x < +\infty.$$

We represent $f(x)$ by a Fourier integral as in (4.8):

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} f(\xi)\, e^{i\omega(x-\xi)} d\xi. \tag{6}$$

In order to obtain a solution of equation (1) we must merely complete $\exp[i\,\omega(x-\xi)]$ to the product

$$\varphi(y)\, e^{i\,\omega(x-\xi)}. \tag{7}$$

Substituting this in (1) we get:

$$-\omega^2 \varphi(y) = \frac{d\varphi(y)}{dy}, \qquad \varphi(y) = C\, e^{-\omega^2 y}.$$

Here $C = 1$ on account of the obvious condition $\varphi(0) = 1$. We therefore replace in (6)

$$\exp[i\,\omega(x-\xi)] \qquad \text{by} \qquad \exp[i\,\omega(x-\xi) - \omega^2 y]. \tag{7 a}$$

This seemingly complicates the Fourier integral (6) but in reality it makes it much simpler. In (6) $f(x)$ must converge to 0 "sufficiently rapidly" in order that the integral with respect to ξ will converge, and this integration must be performed *before* the much simpler integration with respect to ω, which would otherwise not converge. But now the order of integration is reversible and $f(x)$ is less restricted in its behavior at infinity. The new factor $\varphi(y) = \exp(-\omega^2 y)$ serves as convergence factor[9] for all $y > 0$.

Combining (6) and (7) and substituting $y = kt$ we get:

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\xi)\, d\xi \int_{-\infty}^{+\infty} e^{-\omega^2 kt + i\,\omega(x-\xi)} d\omega. \tag{8}$$

We abbreviate the exponent in (8) by $-\alpha\,\omega^2 + \beta\,\omega$ and complete the square:

$$-\alpha\,\omega^2 + \beta\,\omega = -\alpha\left(\omega - \frac{\beta}{2\,\alpha}\right)^2 + \frac{\beta^2}{4\,\alpha}.$$

[9] See the author's dissertation, Königsberg 1891: "Die willkürlichen Funktionen der mathematischen Physik" where the general case of the limit for $t \to 0$ of a Fourier integral with a convergence factor is considered. The function $f(x)$ may then have, for example, "an infinity of maxima and minima" or arbitrary discontinuities.

Substituting $p = \omega - \beta/2\alpha$: we get:

$$\text{(9)} \qquad \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\omega^2 k t + i\omega(x-\xi)}\,d\omega = \frac{1}{2\pi}\, e^{-\frac{(x-\xi)^2}{4kt}} \int_{-\infty}^{+\infty} e^{-\alpha p^2} dp\,.$$

We have the well known formula for the Laplace integral:

$$\int_{-\infty}^{+\infty} e^{-p^2} dp = \sqrt{\pi}\,, \quad \text{and therefore} \quad \int_{-\infty}^{+\infty} e^{-\alpha p^2} dp = \sqrt{\frac{\pi}{\alpha}}\,.$$

Denoting the right side of (9) by U we get

$$\text{(10)} \qquad U = \frac{1}{\sqrt{4\pi k t}}\, e^{-\frac{(x-\xi)^2}{4kt}}$$

Equation (8) then becomes

$$\text{(10a)} \qquad u(x,t) = \int_{-\infty}^{+\infty} f(\xi)\, U\, d\xi\,.$$

We note that the initial temperature at the point $x = \xi$ spreads in space-time independently of the initial temperature at all other points. (This is due to the linearity of the differential equation which permits the superimposition of solutions.) For $t \to 0$ we have $u(x,t) \to f(x)$ and (10a) becomes:

$$f(x) = \int_{-\infty}^{+\infty} f(\xi)\, U\, d\xi\,.$$

This shows that U has the "character of a δ function." As on p. 27 this means that U vanishes in the limit $t \to 0$ for all values of $x \neq \xi$ and becomes infinite at $x = \xi$ so that

$$\text{(10b)} \qquad \int_{x-\epsilon}^{x+\epsilon} U\, d\xi = 1$$

(These properties of U are easily seen from (10).) Ignoring the distinction between heat-energy and temperature we may say that U describes the space-time behavior of a *unit heat-source* or of a *heat-pole of strength* 1.

For the case of a general initial time $t = \tau$ we get instead of (10):

$$\text{(10c)} \qquad U = \{4\pi k\,(t-\tau)\}^{-\frac{1}{2}} \exp\left\{-\frac{(x-\xi)^2}{4k\,(t-\tau)}\right\}$$

For the special case of a heat-pole at $\xi = 0$ we get

$$U = (4\pi kt)^{-\frac{1}{2}} \exp\left\{-\frac{x^2}{4kt}\right\}. \tag{10d}$$

Before discussing the deeper meaning of these formulas we shall generalize them to two and three dimensions.

We noted the possibilities of generalizing Fourier's double integral to quadruple and sextuple integrals at the end of §4. We perform this generalization by writing instead of (6):

$$f(x, y) = \frac{1}{2\pi}\int d\omega \int f(\xi, y)\, e^{i\omega(x-\xi)}\, d\xi \tag{11}$$

and

$$f(\xi, y) = \frac{1}{2\pi}\int d\omega' \int f(\xi, \eta)\, e^{i\omega'(y-\eta)}\, d\eta; \tag{11a}$$

Combining (11) and (11a) we get

$$f(x, y) = \frac{1}{(2\pi)^2}\int d\omega \int d\omega' \iint f(\xi, \eta)\, e^{i\omega(x-\xi)+i\omega'(y-\eta)} d\xi\, d\eta. \tag{11b}$$

The same process which led from (6) to (10a) leads for the two-dimensional case from (11b) to

$$u(x, y, t) = \iint_{-\infty}^{+\infty} f(\xi, \eta)\, U\, d\xi\, d\eta \tag{12}$$

where U is the product of two factors of the form (10):

$$U = (4\pi kt)^{-1} e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4kt}} \tag{13}$$

In the three-dimensional case U is the product of three factors of the form (10):

$$U = (4\pi kt)^{-\frac{3}{2}} e^{-\frac{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}{4kt}} \tag{14}$$

Equations (13) and (14) stand for *unit heat-poles in the plane and in space.* Equations (10), (13), (14) indicate the connection between *heat-conduction* and *probability.*

We compare (10d) with the Gaussian law of error

$$dW = \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha x^2}\, dx. \tag{15}$$

Here dW is the probability of an error between x and $x + dx$ in a measuring process whose precision is given by the "precision factor" α. In our case this factor is $(4kt)^{-1}$; "infinite precision" is given by $t = 0$ which means absolute concentration of heat in the point $x = 0$; "decreasing precision" corresponds to increasing t. Fig. 14 shows the well known bell-shaped curves which for decreasing α give the behavior of U for increasing t. The function U in (10d) is equal to the "probability density" dW/dx.

In an analogous manner we compare (13) to the measuring of a position in the xy-plane whose exact position is given by (ξ, η); and (14) with the measurement of a space point with the true position (ξ, η, ζ). The precision-factor in both cases is $\alpha = (4kt)^{-1}$ as before. This suggests that the physical reason for heat-conduction is of a *statistical* — not *dynamical* — nature. This becomes apparent in the treatment of heat conduction in the kinetic theory of gases (or, more correctly, the "statistical theory of gases"). Connected with this is the following fact which we discuss for the spatial case of equation (14). For $t = 0$ the total heat-energy is concentrated at the point (ξ, η, ζ), but after an arbitrarily short time we have a non-vanishing temperature U at a distant point (x,y,z). Hence *heat expands with infinite velocity*. This is impossible from the point of view of dynamics where no velocity may exceed c.

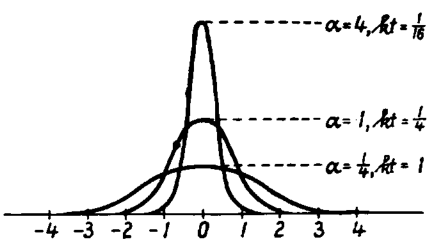

Fig. 14. The Gaussian error curve for the precision factors $\alpha = 4, 1, 1/4$, being at the same time the principal solution of heat conduction for $kt = 1/16, 1/4, 1$.

From §7, p. 34 we know that diffusion, electric conduction, and viscosity satisfy the same differential equation as heat conduction. Here too the statistical approach is clear. Diffusion is based on the *Brownian motion* in a solvent of the individual dissolved molecules, and the statistical origin is ascertained both by theory and experiment. The electron theory of metals shows that upon electrical conduction the electrons are diffused through the grid of metal molecules etc.

The function U of the equations (10), (13), (14) is the *principal solution of the differential equation* $L(u) = 0$. We now wish to transform it into the principal solution V of the *adjoint equation* $M(v) = 0$. Comparing (1) and (2) we see that this is done by reversing the sign of $y = kt$; we shall also reverse the sign of $y_0 = k\tau$ so that the heat pole will again be situated at the point $x = \xi$, $t = \tau$. Thus we obtain from (10c):

$$V = \{4\pi k(\tau - t)\}^{-\frac{1}{2}} \exp\left\{-\frac{(x-\xi)^2}{4k(\tau - t)}\right\} \tag{16}$$

V has an essential singularity for $t = \tau$ and is defined only for the past of τ, i.e., for $t < \tau$, in contrast to the principal solution of U which is regular only for the future of τ, i.e., for $t > \tau$.

We return to Green's theorem (4). Setting $v = V$ and taking for u a solution of $L(u) = 0$ we get:

$$\int u\, V\, dx + k\int\left(V\frac{\partial u}{\partial x} - u\frac{\partial V}{\partial x}\right)dt = 0\,. \tag{17}$$

The two integrals are extended over the sides of the rectangle of Fig. 13, the first over the two horizontal sides, the second over the two vertical sides.

Since V too is a "δ function" the first integral taken over the side $t = \tau$ yields $-u_Q$. If we decompose the second integral into the two components which correspond to the two rod ends x_0 and x_1 and denote this by $\Sigma^{(x_1,\, x_0)}$ we get:

$$u_Q = \int_{x_0}^{x_1} u\, V_0\, dx + k\sum{}^{(x_1,\, x_0)}\int_0^{\tau}\left(V\frac{\partial u}{\partial x} - u\frac{\partial V}{\partial x}\right)dt\,. \tag{18}$$

Here V_0 is V for $t = 0$.

This representation of u is general since the source point Q can be situated at the *arbitrary* point $x = \xi$, $t = \tau$. However it does not yet solve the boundary value problem of §9C, since in addition to the initial values of u it assumes that the boundary values of u *and* of $\partial u/\partial x$ are given at the endpoints, whereas in the boundary value problem only u *or* $\partial u/\partial x$ may be prescribed. In order to solve the boundary value problem we must replace V in (18) by *Green's function G* which *satisfies the condition $G = 0$ at the endpoints* and thereby makes the term containing $\partial u/\partial x$ in (18) vanish. We shall see in the next chapter how G can be constructed from V by a reflection process. Exercise II.4 contains an application of (18) to laminar fluid friction.

The above considerations can be transferred immediately to the two- and three-dimensional cases. As remarked above in connection with (5a, b), we merely have to extend the integration in the first integral of (18) over the base, and in the second integral over the lateral surface of the three- or four- dimensional space-time cylinder. However the construction of Green's function G by a reflection process for the two and three dimensional problems will succeed only in exceptional cases (see §17).

On the other hand equation (18) (in terms of V, not G) suffices to insure the analytic character of u, since the coordinates ξ, τ (or ξ, η, τ or ξ, η, ζ, τ) of Q appear on the right side only in the principal solution V, that is, only in analytic form. The solutions of our parabolic differential equation are *analytic* in the interior of their domain just as in the elliptic case. However, the domain here is not bounded as in the elliptic case, but is an infinite strip (as was pointed out at the end of §9). From this latter point of view the parabolic boundary value problem resembles the hyperbolic one.