Chapter I

Fourier Series and Integrals

Fourier's Théorie analytique de la chaleur[1] is the bible of the mathematical physicist. It contains not only an exposition of the trigonometric series and integrals named after Fourier, but the general boundary value problem is treated in an exemplary fashion for the typical case of heat conduction.

In mathematical lectures on Fourier series emphasis is usually put on the concept of arbitrary function, on its continuity properties and its singularities (accumulation points of an infinity of maxima and minima). This point of view becomes immaterial in the physical applications. For, the initial or boundary values of functions considered here, partially because of the atomistic nature of matter and of interaction, must always be taken as smoothed mean values, just as the partial differential equations in which they enter arise from a statistical averaging of much more complicated elementary laws. Hence we are concerned with relatively simple idealized functions and with their approximation with "least possible error." What is meant by the latter is explained by Gauss in his "Method of Least Squares." We shall see that it opens a simple and rigorous approach not only to Fourier series but to all other series expansions of mathematical physics in spherical and in cylindrical harmonics, or generally in eigenfunctions.

§ 1. Fourier Series

Let an arbitrary function $f(x)$ be given in the interval $-\pi \leq x \leq +\pi$; this function may, e.g., be an empirical curve determined by sufficiently many and sufficiently accurate measurements. We want to approximate it by the sum of $2n+1$ trigonometric terms

$$
\begin{aligned}
S_n(x) = A_0 &+ A_1 \cos x + A_2 \cos 2x + \cdots + A_n \cos nx \\
&+ B_1 \sin x + B_2 \sin 2x + \cdots + B_n \sin nx
\end{aligned}
\tag{1}
$$

[1] Jean Baptiste Fourier, 1768–1830. His book on the conduction of heat appeared in 1822 in Paris. Fourier also distinguished himself as an algebraist, engineer, and writer on the history of Egypt, where he had accompanied Napoleon.

The influence of his book even outside France is illustrated by the following quotation: "Fourier's incentive kindled the spark in (the then 16-year-old) William Thomson as well as in Franz Neumann." (F. Klein, Vorlesungen über die Geschichte der Mathematik im 19. Jahrhundert, v. I, p. 233.)

By what criterion shall we choose the coefficients A_k, B_k at our disposal? We shall denote the error term $f(x) - S_n(x)$ by $\varepsilon_n(x)$; thus

$$f(x) = S_n(x) + \varepsilon_n(x). \tag{2}$$

Following Gauss we consider the *mean square error*

$$M = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \varepsilon_n^2 \, dx \tag{3}$$

and reduce M to a minimum through the choice of the A_k, B_k.

To this we further remark that the corresponding measure of the total error formed with the first power of ε_n would not be suitable, since arbitrarily large positive and negative errors could then cancel each other and would not count in the total error. On the other hand the use of the absolute value $|\varepsilon_n|$ under the integral sign in place of ε_n^2 would be inconvenient because of its non-analytic character.[2]

The requirement that (3) be a minimum leads to the equations

$$\begin{aligned} -\frac{\partial M}{\partial A_k} &= \frac{1}{\pi} \int_{-\pi}^{+\pi} \{f(x) - S_n(x)\} \cos k x \, dx = 0, \quad k = 0, 1, 2, \ldots, n \\ -\frac{\partial M}{\partial B_k} &= \frac{1}{\pi} \int_{-\pi}^{+\pi} \{f(x) - S_n(x)\} \sin k x \, dx = 0, \quad k = 1, 2, \ldots, n. \end{aligned} \tag{4}$$

These are exactly $2n + 1$ equations for the determination of the $2n + 1$ unknowns A, B. A favorable feature here is that each individual coefficient A or B is determined directly and is not connected recursively with the other A, B. We owe this to the *orthogonality relations* that exist among trigonometric functions:[3]

$$\int \cos k x \sin l x \, dx = 0, \tag{5}$$

$$\left.\begin{aligned} &\int \cos k x \cos l x \, dx \quad (5a) \\ &\int \sin k x \sin l x \, dx \quad (5b) \end{aligned}\right\} = 0, \; k \neq l.$$

[2] A completely different approach is taken by the great Russian mathematician Tchebycheff in the approximation named after him. He considers not the *mean* but the *maximal* $|\varepsilon_n|$ appearing in the interval of integration, and makes this a minimum through the choice of the coefficients at his disposal.

[3] Here and below all integrals are to be taken from $-\pi$ to $+\pi$. In order to justify the word "orthogonality" we recall that two vectors $\mathbf{u}, \mathbf{v}$ which are orthogonal in Euclidean three dimensional, or for that matter n-dimensional space, satisfy the condition that their scalar product

In order to prove them it is not necessary to write down the cumbersome addition formulae of trigonometric functions, but to think rather of their connection with the exponential functions $e^{\pm ikx}$ and $e^{\pm ilx}$. The integrands of (5a,b) consist then of only four terms of the form $\exp\{\pm i(k+l)x\}$ or $\exp\{\pm i(k-l)x\}$, all of which vanish upon integration unless $l = k$. This proves (5a,b). The fact that (5) is valid even without this restriction follows from the fact that for $l = k$ it reduces to

$$\frac{1}{4i}\int (e^{2ikx} - e^{-2ikx})\,dx = 0$$

In a similar manner one obtains the values of (5a,b) for $l = k>0$ (only the product of $\exp(ikx)$ and $\exp(-ikx)$ contributes to it): this value simply becomes equal to π; for $l = k = 0$ the value of the integral in (5a) obviously equals 2π. We therefore can replace (5a,b) by the single formula which is valid also for $l = k>0$

$$\frac{1}{\pi}\int \cos kx \cos lx\,dx = \frac{1}{\pi}\int \sin kx \sin lx\,dx = \delta_{kl} \tag{6}$$

with the usual abbreviation

$$\delta_{kl} = \begin{cases} 0 \ldots l \neq k \\ 1 \ldots l = k > 0. \end{cases}$$

Equation (6) for $k = l$ is called the *normalizing condition.* It is to be augmented for the exceptional case $l = k = 0$ by the trivial statement

$$\frac{1}{2\pi}\int dx = 1. \tag{6a}$$

If we now substitute (5),(6) and (6a) in (4) then in the integrals with S_n all terms except the k-th vanish, and we obtain directly *Fourier's representation of coefficients:*

$$\begin{aligned} A_k &= \frac{1}{\pi}\int f(x)\cos kx\,dx \\ B_k &= \frac{1}{\pi}\int f(x)\sin kx\,dx \end{aligned}\Bigg\} k>0, \tag{7}$$

$$A_0 = \frac{1}{2\pi}\int f(x)\,dx.$$

$$(\mathbf{u}\,\mathbf{v}) = \sum_1^N \mathbf{u}_i\,\mathbf{v}_i = 0$$

vanish. The integrals appearing in (5) can be considered as sums of this same type with infinitely many terms. See the remarks in §26 about so-called "Hilbert space."

Our approximation S_n is hereby determined completely. If, e.g. $f(x)$ were given empirically then the integrations (7) would have to be carried out numerically or by machine.[4]

From (7) one sees directly that for an even function $f(-x) = f(+x)$, all B_k vanish, whereas for an odd function, $f(-x) = -f(+x)$, all A_k, including A_0, vanish. Hence the former is approximated by a *pure cosine series*, the latter by a *pure sine series.*

The accuracy of the approximation naturally increases with the number of constants A,B at our disposal, i.e., with increasing n. Here the following fortunate fact should be stressed: since the A_k,B_k for $k<n$ are independent of n, the previously calculated A_k,B_k remain unchanged by the passage from n to $n+1$, and only the coefficients A_{n+1}, B_{n+1} have to be newly calculated. The A_k,B_k, once found, are *final.*

There is nothing to prevent us from letting n grow indefinitely, that is, to perform the passage to the limit $n \to \infty$. The finite series considered so far thereby goes over into an *infinite Fourier series.* The following two sections will deal with its convergence.

More complicated than the question of convergence is that of the *completeness* of the system of functions used here as basis. It is obvious that if in the Fourier series one of the terms, e.g., the k-th cosine term, were omitted, then the function $f(x)$ could no longer be described by the remaining terms with arbitrary accuracy; even in passing to the limit $n \to \infty$ a finite error $A_k \cos kx$ would remain. To take an extremely simple case, if one attempted to express $\cos nx$ by an incomplete series of all cosine terms with $k< n$ and $k > n$, then all A_k would vanish because of orthogonality and the error would turn out to be $\cos nx$ itself. Of course it would not occur to anyone to disturb the regularity of a system like that of the trigonometric functions by the omission of one term. But in more general cases such considerations of mathematical esthetics need not be compelling.

What the mathematicians teach us on this question with their *relation of completeness* is in reality no more than is contained in the basis of the method of least squares. One starts, namely, with the remark that a system of functions say $\varphi_0, \varphi_1, \ldots, \varphi_k, \ldots,$ can be complete only if *for every continuous function* $f(x)$ the mean error formed according to (3) goes *to zero* in the limit $n \to \infty$. It is assumed that the system of φ is orthogonal and normalized to 1, that is

$$(8) \qquad \int \varphi_k \varphi_l \, dx = 0, \qquad \int \varphi_k^2 \, dx = 1,$$

[4] Integrating machines that serve in Fourier analysis are called "harmonic analyzers." The most perfect of these is the machine of Bush and Caldwell; it can be used also for the integration of arbitrary simultaneous differential equations; see *Phys. Rev.* **38**, 1898 (1931).

which implies that the expansion coefficients A_k are simply

$$A_k = \int f(x)\,\varphi_k(x)\,dx. \tag{9}$$

Let the limits of integration in this and the preceding integrals be a and b so that the length of the interval of expansion is $b - a$. One then forms according to (3)

$$(b-a)\,M = \int \Big(f - \sum_{k=0}^{n} A_k \varphi_k\Big)^2 dx = \int f^2\,dx - 2\sum_{k=0}^{n} A_k \int f\varphi_k\,dx + \sum_{k=0}^{n} A_k^2.$$

Equation (8) has been used in the last term here. According to (9) the middle term equals twice the last term except for sign. Hence

$$\lim_{n\to\infty} (b-a)\,M = \int f^2\,dx - \sum_{k=0}^{\infty} A_k^2$$

and one requires, as remarked above, that for every continuous function

$$\sum A_k^2 = \int f^2\,dx. \tag{10}$$

This is the mathematical formulation of the *relation of completeness* which is so strongly emphasized in the literature. It is obvious that it can hardly be applied as a practical criterion. Also, since it concerns only the mean error, it says nothing on the question of whether the function f is really represented everywhere by the Fourier series (see also §3, p. 15).

In this introductory section we have followed the historical development in deducing the *finality of the Fourier coefficients* from the *orthogonality of the trigonometric functions.* In §4 we shall demonstrate, for the typical case of spherical harmonics, that, conversely, orthogonality can be deduced quite generally from our requirement of *finality.* From our point of view of approximation this seems to be the more natural approach. In any case it should be stressed at this point that *orthogonality* and *requirement of finality* imply each other and can be replaced by each other.

Finally, we want to translate our results into a form that is mathematically more perfect and physically more useful. We carry this out for the case of infinite Fourier series, remarking however, that the following is valid also for a truncated series — actually the more general and rigorous case.

We write, replacing the variable of integration in (7) by ξ:

$$f(x) = \frac{1}{2\pi}\int f(\xi)\,d\xi +$$
$$\frac{1}{\pi}\sum_{k=1}^{\infty}\int f(\xi)\cos k\xi\,d\xi\cdot\cos kx + \frac{1}{\pi}\sum_{k=1}^{\infty}\int f(\xi)\sin k\xi\,d\xi\cdot\sin kx$$
$$= \frac{1}{2\pi}\int f(\xi)\,d\xi + \frac{1}{\pi}\sum_{k=1}^{\infty}\int f(\xi)\cos k(x-\xi)\,d\xi$$
$$= \frac{1}{2\pi}\left\{\int f(\xi)\,d\xi + \sum_{k=1}^{\infty}\left(\int f(\xi)\,e^{ik(x-\xi)}d\xi + \int f(\xi)\,e^{-ik(x-\xi)}d\xi\right)\right\}.$$

In the last term we can consider the summation for positive k in $\exp\{-ik(x-\xi)\}$ to be the summation for the corresponding negative values of k in $\exp\{+ik(x-\xi)\}$. We therefore replace this term by

$$\sum_{k=-1}^{-\infty}\int f(\xi)\,e^{ik(x-\xi)}\,d\xi = \sum_{k=-\infty}^{-1}\int f(\xi)\,e^{ik(x-\xi)}\,d\xi\,.$$

Then the uncomfortable exceptional position of the term $k = 0$ is removed: it now fits between the positive and negative values of k and we obtain

$$f(x) = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty}\int f(\xi)\,e^{ik(x-\xi)}d\xi\,. \tag{11}$$

Finally, introducing the Fourier coefficients C_k, which are *complex* even for real $f(x)$:

$$f(x) = \sum_{k=-\infty}^{+\infty} C_k e^{ikx},\quad C_k = \frac{1}{2\pi}\int f(\xi)\,e^{-ik\xi}\,d\xi. \tag{12}$$

The relations among the C's and the A's and B's defined by (7), are given by

$$C_k = \begin{cases}\frac{1}{2}(A_k - i\,B_k), & k > 0\,,\\[2mm] \frac{1}{2}(A_{|k|} + i\,B_{|k|}), & k < 0\,,\end{cases} \tag{13}$$
$$C_0 = A_0\,.$$

Our complex representation (12) is obviously simpler than the usual real representation; it will be of special use to us in the theory of Fourier integrals.

If we extend our representation, originally intended for the interval

$-\pi < x < +\pi$, to the intervals $x > \pi$ and $x < -\pi$ then we obtain continued periodic repetitions of the branch between $-\pi$ and $+\pi$; in general they do not constitute the analytic continuation of our original function $f(x)$. In particular the periodic function, thus obtained will have *discontinuities* for the odd multiples of $\pm\pi$, unless we happen to have $f(-\pi) = f(+\pi)$. The next section deals with the investigation of the error arising at such a point.

§ 2. Example of a Discontinuous Function. Gibbs' Phenomenon and Non-Uniform Convergence

Let us consider the function

$$f(x) = \begin{cases} +1 \text{ for } 0 < x < \pi \\ -1 \text{ for } -\pi < x < 0. \end{cases} \tag{1}$$

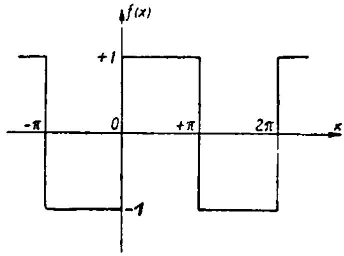

Fig. 1. The chain of segments $y = \pm 1$ for positive and negative $|x| < \pi$ and its periodic repetition represented by the Fourier series.

We sketch it in Fig. 1 with its periodic repetitions completed by the vertical connecting segments of length 2 at the points of discontinuity $x = 0, \pm\pi, \pm 2\pi, \ldots$, whereby it becomes a "meander line." Our function f is odd, its Fourier series consists therefore solely of sine terms as pointed out in (1.7). The coefficients can best be calculated from equation (1.12), which yields

$$\begin{aligned} C_k &= \frac{1}{2\pi}\left(\int_0^{\pi} e^{-ik\xi}\,d\xi - \int_{-\pi}^{0} e^{-ik\xi}\,d\xi\right) \\ &= \frac{1}{2\pi}\left(\frac{e^{-ik\pi}-1}{-ik} - \frac{1-e^{+ik\pi}}{-ik}\right) = \frac{(-1)^k - 1}{-i\pi k} = \begin{cases} -\dfrac{2i}{\pi k} \cdots k \text{ odd} \\ 0 \ldots k \text{ even} \end{cases} \end{aligned} \tag{1a}$$

This implies according to (1.13):

$$B_k = \frac{4}{\pi k}, \quad k = 1, 3, 5, \ldots$$

We obtain the following sine series:

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right). \tag{2}$$

One may imagine the upheaval caused by this series when it was first constructed by Fourier. A discontinuous chain formed through the superposition of an infinite sequence of only the simplest continuous functions! Without exaggeration one may say that this series has contributed greatly to the development of the general concept of real function. We shall see below that it also served to deepen the concept of convergence of series.

In order to understand how the series manages to approximate the discontinuous sequence of steps, we draw[5] in Fig. 2 the approximating functions S_1, S_3, S_5 defined by (1.1) together with $S_\infty = f(x)$.

$$S_1 = \frac{4}{\pi}\sin x, \quad S_3 = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right),$$
$$S_5 = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3\,x + \frac{1}{5}\sin 5\,x\right).$$

S_1 has its maximum value

$$y = 4/\pi = 1.27,$$

at $x = \pi/2$, and hence rises 27% above the horizontal line $y = 1$, which is to be described. S_3 has a minimum value at the same point and hence

$$y = \frac{4}{\pi}\left(1 - \frac{1}{3}\right) = 0.85\,,$$

stays 15% below the straight line to be described. In addition S_3 also has maxima at $\pi/4$ and $3\,\pi/4$, which lie 20% above that line. (The reader is invited to check this!) S_5 on the other hand has a maximum of

$$y = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5}\right) = 1.10\,,$$

at $x = \pi/2$ which is too high by only 10%. A flat minimum on either side is followed by two steeper maxima situated near $x = 0$ and $x = \pi$. In general the maxima and minima of S_{2n+1} lie *between* those of S_{2n-1}(see exercise I.1).

All that has been said here about the stepwise approximation of the line $y = +1$, is of course equally valid for its mirror image $y = -1.$ It too is approximated by *successive oscillations,* so that the approximating curve S_n swings n times above and $n + 1$ times below the line segment which is to be represented. The oscillations in the *middle part* of the line segment decrease with increasing n; at the *points of discontinuity*

[5] In the lectures at this point abundant use was made of colored chalk. Since this unfortunately is impossible in print, both S_∞ and the approximation S_5, which are the most important for us, are drawn in bolder lines.

$x = 0, \pm\pi, \pm 2\pi, \ldots$, where there is no systematic decrease of the maxima, the approximating curves approach the vertical jumps of discontinuity. The picture of an approximating curve of very large n therefore looks the way it has been pictured schematically in Fig. 3.

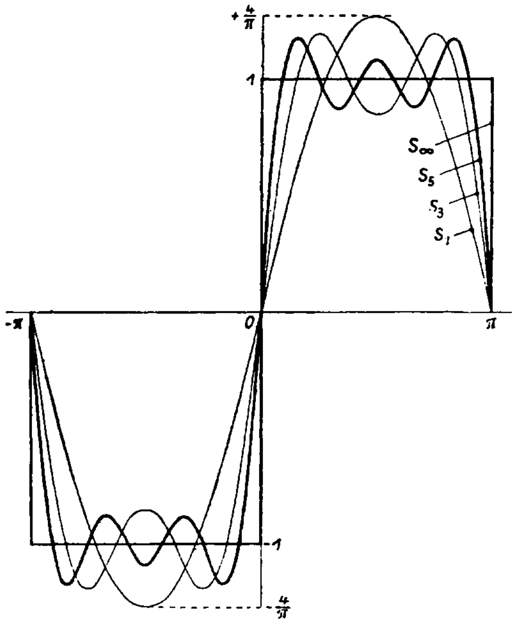

Fig. 2. The approximations of the chain S_∞; the maxima and minima lie at equally spaced values of x, respectively between those of the preceding approximation.

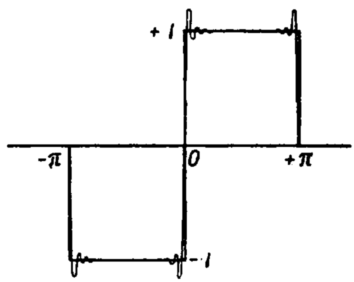

Fig. 3. An approximation S of very high order for the illustration of Gibbs' phenomenon.

We now consider more closely the behavior of $S_{2n+1}(x)$ for large n at one of the jumps, e.g., for $x = 0$. To this end we rewrite the original formula for S_{2n+1} in integral form (an integral usually being easier to discuss than a sum). This is done in the following steps:

$$S_{2n+1} = \frac{4}{\pi} \sum_{k=0}^{n} \frac{\sin(2k+1)x}{2k+1} = \frac{4}{\pi} \sum_{k=0}^{n} \int_0^x \cos(2k+1)\xi \, d\xi$$

$$= \frac{2}{\pi} \int_0^x \left\{ \sum_{k=0}^{n} e^{(2k+1)i\xi} + \sum_{k=0}^{n} e^{-(2k+1)i\xi} \right\} d\xi .$$

After factoring out $\exp(\pm i\xi)$ from the two sums of the last line they become geometric series in increasing powers of $\exp(\pm 2i\xi)$ which can be summed in the familiar manner. Therefore, one obtains

$$S_{2n+1} = \frac{2}{\pi}\int_0^x \left\{e^{i\xi}\frac{1-e^{2i(n+1)\xi}}{1-e^{2i\xi}} + e^{-i\xi}\frac{1-e^{-2i(n+1)\xi}}{1-e^{-2i\xi}}\right\}d\xi. \tag{3}$$

By further factorization these two fractions can be brought to the common form (except for the sign of i):

$$e^{\pm in\xi}\frac{\sin(n+1)\xi}{\sin\xi}. \tag{3 a}$$

In this way (3) goes over into

$$S_{2n+1} = \frac{2}{\pi}\int_0^x \frac{2\cos(n+1)\xi\,\sin(n+1)\xi}{\sin\xi}\,d\xi. \tag{3b}$$

Finally for sufficiently small x we can replace $\sin\xi$ in the denominator by ξ; the corresponding simplification in the numerator would not be permissible since ξ there is accompanied by the large factor $n+1$. We obtain therefore for (3a), if we introduce the new variable of integration u and the new argument v,

$$S_{2n+1} = \frac{2}{\pi}\int_0^v \frac{\sin u}{u}\,du \ldots \qquad \begin{cases} u = 2(n+1)\xi, \\ v = 2(n+1)x. \end{cases} \tag{4}$$

From this the following conclusion may be drawn: if for finite n we set $x = 0$ then v vanishes and $S_{2n+1} = 0$. If now we allow n to increase toward infinity, the relation $S_{2n+1} = 0$ holds in the limit. Hence

$$\lim_{n\to\infty}\lim_{x\to 0} S_{2n+1} = 0. \tag{4a}$$

But if for $x > 0$ we first allow n to approach infinity, then v becomes infinite, and, according to a fundamental formula that will be treated in exercise I.5: $S_{2n+1} = 1$. If we then allow x to decrease towards zero, the value $S_{2n+1} = 1$ holds also for the limit $x = 0$; hence

$$\lim_{x\to 0}\lim_{n\to\infty} S_{2n+1} = 1. \tag{4b}$$

The two limiting processes therefore are not interchangeable. If the function $f(x)$ to be represented were continuous at the point $x = 0$, then the order of passage to the limit would be immaterial, and in contrast to (4a,b) one would have

$$\text{(4c)} \qquad \operatorname*{Lim}_{n\to\infty}\ \operatorname*{Lim}_{x\to 0} S_{2n+1} = \operatorname*{Lim}_{x\to 0}\ \operatorname*{Lim}_{n\to\infty} S_{2n+1} = f(v).$$

This, however, does not exhaust by any means the peculiarities contained in equation (4); in order to develop them we introduce the frequently tabulated[6] "integral sine"

$$\text{(5)} \qquad Si(v) = \int_0^v \frac{\sin u}{u}\, du$$

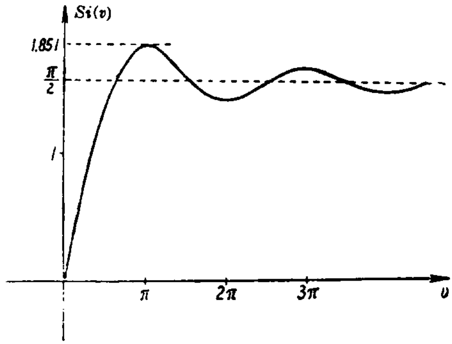

Fig. 4. Graphic representation of the integral sine.

and represent its general form in Fig. 4. It can be described as follows: for small values of v, where $\sin u$ can be set equal to u, we have *proportionality with v;* for large values of v we have *asymptotic approach to* $\pi/2$; in between we have successively *decreasing oscillations* with maxima and minima at $v = \pi, 2\pi, 3\pi, \ldots$, as can be seen from (5); the ordinate of the first and greatest maximum is 1.851 according to the above mentioned tables. To the associated abscissa of the Si-curve there corresponds in the original variable x, owing to the relation $v = 2(n+1)\,x$, the infinite sequence of points

$$\text{(6)} \qquad \ldots, x_n = \frac{\pi}{2(n+1)}, \qquad x_{n+1} = \frac{\pi}{2(n+2)}, \ldots$$

at which according to (4) the approximations $S_{2n+1}, S_{2n+3}, \ldots$ have the fixed value:

$$\text{(7)} \qquad S = \frac{2}{\pi}\,1{,}851 = 1.18.$$

This value, which exceeds $y = 1$ by 18%, is at the same time the *upper limit* of the range of values given by our approximations. Its *lower limit,* $S = -1.18$ is assumed when we approach zero from the negative side in the sequence of points $-x_n, -x_{n+1}, \ldots$. Each point of the range

[6] E.g. B. Jahnke-Emde, Funktionentafeln, Teubner, Leipzig, 3d edition, 1938.

between -1.18 and $+1.18$ can be obtained by a special manner of passing to the limit; e.g., the points $S = 0$ and $S = 1$ are obtained in the manner described in (4a) and (4b).

This behavior of the approximating functions, in particular the appearance of an excess over the range of discontinuity ± 1, is called *Gibbs' phenomenon.* (Willard Gibbs, 1844 to 1906, was one of America's greatest physicists, and simultaneously with Boltzmann, was the founder of statistical mechanics.) Gibbs' phenomenon appears wherever a discontinuity is approximated. One then speaks of the *non-uniform convergence* of the approximation process.

We still want to convince ourselves that actually every point between $S = 1.18$ and $S = -1.18$ can be obtained if we *couple* the two passages to the limit in a suitable fashion. According to (6), this coupling consists in setting $x(n+1)$ or, what comes to the same thing, setting x_n equal to the fixed value $\pi/2$. If instead we take the more general value, q, then from (4) we obtain $v = 2q$, and (4) and (5) together yield

$$S_{2n+1} = \frac{2}{\pi} Si(2q),$$

where $Si(2q)$ can assume all values between 0 and 1.851 with varying positive q, as can be seen directly from Fig. 4. Correspondingly for negative q one obtains all values between 0 and -1.851. The passages to the limit that have thus been coupled yield not only the approach of our approximating function to the discontinuity from -1 to $+1$, but also an excess beyond it, i.e., Gibbs' phenomenon.

In addition to these basic statements we want to deduce some formal mathematical facts from our Fourier representation (2). In particular we substitute $x=\pi/2$ therein and obtain the famous *Leibniz series*

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \tag{8}$$

This series converges slowly; we obtain more rapidly convergent representations for the powers of π if we integrate (2) repeatedly. For the following refer to Fig. 5 below.

By restricting ourselves to the interval $0 < x < \pi$, we write

$$\frac{\pi}{4} = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots \tag{9}$$

instead of (2). Integration from 0 to x yields:

$$\frac{\pi}{4}x = 1 - \cos x + \frac{1}{3^2}(1 - \cos 3x) + \frac{1}{5^2}(1 - \cos 5x) + \cdots \tag{10}$$

Hence for $x = \pi/2$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \tag{11}$$

Subtracting (10) from (11) we get:

$$\frac{\pi}{4}\left(\frac{\pi}{2} - x\right) = \cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots \tag{12}$$

By another integration from 0 to x this becomes

$$\frac{\pi}{8}(\pi x - x^2) = \sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + \cdots \tag{13}$$

Hence for $x = \pi/2$, as an *analogue to the Leibniz series*

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots \tag{14}$$

We integrate (13) once more with respect to x and set $x = \pi/2$:

$$\frac{\pi}{8}\left(\pi\frac{x^2}{2} - \frac{x^3}{3}\right) = 1 - \cos x + \frac{1}{3^4}(1 - \cos 3x) + \frac{1}{5^4}(1 - \cos 5x) + \cdots \tag{15}$$

$$\frac{\pi^4}{3 \cdot 32} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots \tag{16}$$

Finally we subtract (15) from (16) and have

$$\frac{\pi}{8}\left(\frac{\pi^3}{12} - \frac{\pi x^2}{2} + \frac{x^3}{3}\right) = \cos x + \frac{1}{3^4}\cos 3x + \frac{1}{5^4}\cos 5x + \cdots \tag{17}$$

The series (11) and (16) range only over the *odd* numbers. The series ranging over the *even* numbers are respectively equal to 1/4 and 1/16 of the sums ranging over *all* integers. If we denote the latter by Σ_2 and Σ_4 respectively, then we have

$$\frac{\pi^2}{8} + \frac{1}{4}\sum\nolimits_2 = \sum\nolimits_2 \quad \text{and} \quad \frac{\pi^4}{3 \cdot 32} + \frac{1}{16}\sum\nolimits_4 = \sum\nolimits_4,$$

hence

$$\sum\nolimits_2 = \frac{\pi^2}{6} \quad \text{and} \quad \sum\nolimits_4 = \frac{\pi^4}{90}. \tag{18}$$

This value of Σ_4 was needed in the derivation of Stefan's law of radiation or Debye's law for the energy content of a fixed body. The trigonometric series (12), (13), (17) will be useful examples in the following sections. The higher analogues to the "Leibniz series" (8) and (14) as well as those to Σ_2 and Σ_4 will be computed in exercise I.2.

§ 3. On the Convergence of Fourier Series

We are going to prove the following theorem: If a function $f(x)$, together with its first $n-1$ derivatives is continuous and differentiable between $-\pi$ and $+\pi$ inclusive, and the n-th derivative, is differentiable over the same interval except possibly at a finite number of points $x = x_i$ where it may have bounded discontinuities (i.e., finite jumps), then the coefficients A_k, B_k of its Fourier expansion approach zero at least as fast as k^{-n-1} as $k \to \infty$.

The stipulation "inclusive" in referring to the boundaries of the interval has here the following meaning: every function which is represented by a Fourier series is periodic in nature. An adequate picture of its argument would therefore not be the straight line segment from $-\pi$ to $+\pi$, but a circle closing at $x = \pm\pi$. It is this fact to which the *continuity* of f and its first $n-1$ derivatives at the point $x = \pm\pi$ refers. This point is in no way distinguished from the interior points of the interval, just as it is immaterial whether we denote the boundaries of the interval by $-\pi, +\pi$ or, e.g., by $\frac{\pi}{4}, \frac{9\pi}{4}$ etc.

For the proof of this theorem it is convenient to use the complex form (1.12)

$$(1) \qquad f(x) = \sum_{-\infty}^{+\infty} C_k e^{ikx}, \qquad\qquad (1a) \qquad 2\pi C_k = \int_{-\pi}^{+\pi} f(\xi)\, e^{-ik\xi}\, d\xi$$

From (1a) one obtains through integration by parts

$$(2) \qquad 2\pi C_k = \frac{1}{-ik} f(\xi) e^{-ik\xi} \Big|_{-\pi}^{+\pi} + \frac{1}{ik} \int_{-\pi}^{+\pi} f'(\xi)\, e^{-ik\xi}\, d\xi .$$

Here the first term on the right side vanishes because of the postulated continuity of f; the second term can again be transformed by integration by parts. After n iterations of the same process one obtains

$$(3) \qquad 2\pi (ik)^n C_k = \int_{-\pi}^{+\pi} f^{(n)}(\xi)\, e^{-ik\xi}\, d\xi .$$

Because of the discontinuities of $f^{(n)}(x)$ at $x = x_l$, this integral has to be divided into partial integrals between $x = x_l$ and $x = x_{l+1}$; let the jumps of $f^{(n)}$ at the points of discontinuity be denoted by Δ_l^n. Equation (3) written explicitly then reads:

$$2\pi(ik)^n C_k = \sum_l \int_{x_l}^{x_{l+1}} f^{(n)}(\xi)\, e^{-ik\xi} d\xi, \tag{3a}$$

where the point $x = \pm\pi$ may be contained among the points $x = x_l$ By one more partial integration (3a) becomes

$$2\pi(ik)^n C_k = \frac{1}{-ik} \sum_l \Delta_l^n\, e^{-ikx_l} + \frac{1}{ik} \sum_l \int_{x_l}^{x_{l+1}} f^{(n+1)}(\xi)\, e^{-ik\xi} d\xi. \tag{4}$$

Considering the fact that the discontinuities Δ_l^n were assumed to be bounded and that $f^{(n)}$ was assumed to be differentiable between the points of discontinuity, one sees from (4) that C_k vanishes at least to the same order as k^{-n-1} when one lets $k \to \infty$. For special relations between the Δ_l^n or for special behavior of $f^{(n+1)}(\xi)$, the order of vanishing could become even higher.

This theorem is valid for negative k too. This implies that it is valid also for the *real Fourier coefficients* A_k, B_k $(k > 0)$, since according to (1.13) they are expressible in terms of the C_k with positive and negative k.

A special consequence of our theorem is that an *analytic function* of period 2π (such a function is continuous and periodic together with all its derivatives) has Fourier coefficients that decrease faster than any power of $1/k$ with increasing k. An example of this would be an arbitrary polynomial in $\sin x$ and $\cos x$. This is represented by a *finite* Fourier series with as many terms as required by the degree of the polynomial, so that all higher Fourier coefficients are equal to zero. Another example is given by the elliptic ϑ series, which we shall meet in a heat conduction problem in §15; its Fourier coefficients C_k decrease as fast as $e^{-\alpha k^2}$.

It further follows from our theorem that the sum ΣA_k^2 which appears in the relation of completeness converges like Σk^{-2} for every function $f(x)$ which has a finite number of jumps and which is differentiable everywhere else (case $n = 0$ of our theorem). An example of this is given by our function (2.1) where ΣA_k^2 converges, although ΣA_k diverges. This function also shows that the relation of completeness does not insure representability of the function at every point (this has already been noted on p. 5). Namely, if we sharpen definition (2.1) by putting $f = 1$ for $x \geq 0$ and $f = -1$ for $x < 0$, then f if not represented by the Fourier series (2.2) at the point $x = 0$, for there the series converges to 0.

A further illustration of our theorem is given by the sine and cosine series which were derived at the end of the last section. The expressions of the functions which are represented by these series were valid only for the interval $0 < x < \pi$. We complete these expressions by adjoining the corresponding expressions for the interval $-\pi < x < 0$. The latter are obtained simply from the remark that the cosine series are even functions of x, and the sine series are odd. The expressions thus obtained are written below inside the $\{\ \}$ to the right of the semicolon. We therefore complete the equations (2.9), (2.12), (2.15), (2.17) as follows:

$$(5)\ \left\{\frac{\pi}{4};\ -\frac{\pi}{4}\right\} = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots$$

$$(6)\ \left\{\frac{\pi}{4}\left(\frac{\pi}{2} - x\right);\quad \frac{\pi}{4}\left(\frac{\pi}{2} + x\right)\right\} = \cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots$$

$$(7)\ \left\{\frac{\pi}{8}(\pi x - x^2);\ \frac{\pi}{8}(\pi x + x^2)\right\} = \sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + \cdots$$

$$(8)\ \left\{\frac{\pi}{8}\left(\frac{\pi^3}{12} - \frac{\pi x^2}{2} + \frac{x^3}{3}\right);\ \frac{\pi}{8}\left(\frac{\pi^3}{12} - \frac{\pi x^2}{2} - \frac{x^3}{3}\right)\right\} = \cos x + \frac{1}{3^4}\cos 3x + \cdots.$$

Here the functions which are represented possess successively stronger continuity properties: in (5) the function possesses discontinuities at the points $x = 0$ and $x = \pm\pi$, in (6) the function is continuous but the first derivative is discontinuous, in (7) the function and its first derivative are continuous but the second derivative is discontinuous; in (8) the function and its first two derivatives are continuous but the third derivative is not. The discontinuity arising in each case is the same as that of the function in (5) and it appears at the same points $x = 0$ and $x = \pm\pi$ corresponding to the fact that each succeeding function was obtained from the previous one by integration.

Figure 5 illustrates this. Its curves 0,1,2,3 represent the left sides of (5),(6),(7),(8). The discontinuity of the tangent to the curve 1 at $x = 0$ strikes the eye; the discontinuity of the curvature of 2 at $x = 0$ can be deduced from the behavior of the two mirror image parabolas which meet there. Curve 3 consists of two cubic parabolas, that osculate with continuous curvature. The scale, which for convenience has been chosen differently for the different curves, can be seen by the ordinates of the maximal values which have been inserted on the right hand side.

The *increasing continuity* of our curves 0 to 3 has its counterpart in the *increasing rate of convergence* of the Fourier series on the right sides of eqs. (5) to (8): in (5) we have a decrease of the coefficients like $1/k$, in general, in accord with our theorem, we have a decrease with k^{-n-1}, where n is the order of the first discontinuous derivative of the represented function.

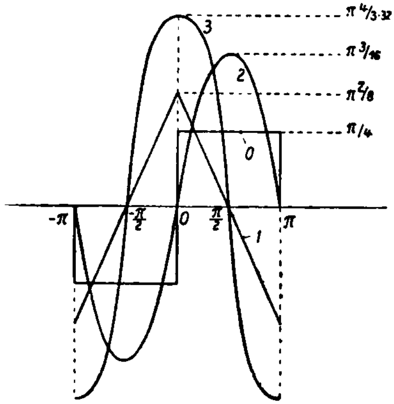

Fig. 5. Four curves 0,1,2,3, obtained by successive integration. Increasing continuity at $x = 0$: 0 discontinuous in the ordinate, 1 in the tangent, 2 in the curvature, 3 in the third derivative.

The convergence of Fourier series stands in a marked contrast to that of Taylor series. The former depends only on the continuity of the function to be represented and its derivatives on the real axis, the latter depends also on the position of the singularities in the complex domain. (Indeed the singular point nearest the origin of expansion in the complex plane determines the radius of convergence of the Taylor series.) Accordingly the principles of the two expansions are basically different: for Fourier series we have an *oscillating approach* over the entire range of the interval of representation, for Taylor series we have an *osculating approach* at its origin. We shall return to this in §6.

§ 4. Passage to the Fourier Integral

The interval of representation $-\pi < x < \pi$ can be changed in many ways. Not only can it be displaced, as remarked on p. 14, but also its length can be changed, e.g., to $-a < z < +a$ for arbitrary a. This is done by the substitution

$$x = \frac{\pi z}{a}, \tag{1}$$

which transforms (1.7) into

$$\left.\begin{matrix} A_k \\ B_k \end{matrix}\right\} = \frac{1}{a}\int_{-a}^{+a} f(z) \begin{matrix}\cos\\ \sin\end{matrix} \frac{\pi k z}{a}\, dz, \qquad A_0 = \frac{1}{2a}\int_{-a}^{+a} f(z)\, dz. \tag{2}$$

In the more convenient complex way of writing (1.12), one then has

$$f(z) = \sum_{-\infty}^{+\infty} C_k\, e^{i\frac{\pi}{a}kz}, \qquad C_k = \frac{1}{2a}\int_{-a}^{+a} f(\zeta)\, e^{-i\frac{\pi}{a}k\zeta}\, d\zeta. \tag{3}$$

We may obviously consider also the more general interval $b < z < c$, by substituting

$$(4) \qquad x = \alpha z + \beta, \qquad \alpha = \frac{2\pi}{c-b}, \qquad \beta = -\pi \frac{c+b}{c-b}$$

The formulas (2) then become

$$(5) \qquad \left.\begin{matrix} A_k \\ B_k \end{matrix}\right\} = \frac{2}{c-b} \int_b^c f(z) \begin{matrix}\cos\\ \sin\end{matrix} k(\alpha z + \beta)\, dz, \qquad A_0 = \frac{1}{c-b} \int_b^c f(z)\, dz.$$

In this connection we mention some "pure sine and cosine series" that appear in Fourier's work. One considers a function $f(x)$ which is given only in the interval $0 < x < \pi$ say, and which is to be continued to the negative side in an odd or even manner. For example, one gets for odd continuation

$$f(x) = \sum_{k=1}^{\infty} B_k \sin k x, \qquad B_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin k x\, dx,$$

See also exercise I.3.

Starting from (3) we take a to be very large. The sequence of values

$$\omega_k = \frac{\pi}{a} k$$

then becomes dense, for which reason we shall write ω instead of ω_k from now on. For the difference of two consecutive ω_k we write correspondingly

$$d\omega = \frac{\pi}{a}, \qquad \frac{1}{a} = \frac{d\omega}{\pi}.$$

If in (3) we replace the symbols z, ζ by the previous ones x, ξ then we obtain

$$(6) \qquad C_k = \frac{d\omega}{2\pi} \int_{-a}^{+a} f(\xi)\, e^{-i\omega\xi}\, d\xi.$$

For the moment we avoid calling the limits of this integral $-\infty$ and $+\infty$

Introducing (6) into the infinite series (3) for $f(x)$, replacing the summation by integration, and denoting the limits of integration for the time being by $\pm\Omega$, we get:

$$f(x) = \lim_{\Omega\to\infty} \lim_{a\to\infty} \frac{1}{2\pi} \int_{-\Omega}^{+\Omega} e^{i\omega x}\, d\omega \int_{-a}^{+a} f(\xi)\, e^{-i\omega\xi}\, d\xi. \tag{7}$$

The order of passage to the limit indicated here is obviously necessary: if the passage to the limit $\Omega\to\infty$ were carried out first, we would obtain the completely meaningless integral

$$\int_{-\infty}^{+\infty} e^{i\omega(x-\xi)}\, d\omega$$

On the other hand $f(\xi)$ must vanish for $\xi\to\pm\infty$ in order that the first limit for $a\to\infty$ have a meaning. We do not have to investigate how fast $f\to 0$ in order that the other passage to the limit be possible, since for all suitably formulated physical problems this convergence to 0 will be "sufficiently rapid."

After this preliminary discussion we shall further abbreviate the more exact form of (7) by writing:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} f(\xi)\, e^{i\omega(x-\xi)}\, d\xi. \tag{8}$$

From this we pass to the *real* form of the Fourier integral (8) as it is commonly given in the literature. We set

$$e^{i\omega(x-\xi)} = \cos\omega\,(x-\xi) + i\sin\omega\,(x-\xi).$$

Here the sine is an odd function of ω, and hence vanishes on integration from $-\infty$ to $+\infty$; the cosine, being even in ω, yields twice the integral taken from 0 to ∞. We therefore have

$$f(x) = \frac{1}{\pi}\int_{0}^{+\infty} d\omega \int_{-\infty}^{+\infty} f(\xi)\cos\omega\,(x-\xi)\, d\xi, \tag{9}$$

by which we do not wish to imply that the real form is better or simpler than our complex form (8). We can write instead of (9):

$$f(x) = \int_0^\infty a(\omega)\cos\omega x\, d\omega + \int_0^\infty b(\omega)\sin\omega x\, d\omega \tag{10}$$

where

$$(10\text{a})\quad a(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(\xi)\cos\omega\,\xi\,d\xi, \qquad b(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(\xi)\sin\omega\,\xi\,d\xi.$$

In particular $b(\omega)$ must vanish if $f(x)$ is even, $a(\omega)$ if $f(x)$ is odd. We then have corresponding to the above "pure cosine or sine series," a "pure cosine or sine integral." One or the other can be produced whenever $f(x)$ is given only for $x > 0$, by continuing $f(x)$ as an even or odd function to the negative side. We then write explicitly:

for even continuation

$$(11\text{a})\quad f(x) = \int_0^\infty a(\omega)\cos\omega\,x\,d\omega, \qquad a(\omega) = \frac{2}{\pi}\int_0^\infty f(\xi)\cos\omega\,\xi\,d\xi,$$

for odd continuation

$$(11\text{b})\quad f(x) = \int_0^\infty b(\omega)\sin\omega\,x\,d\omega, \qquad b(\omega) = \frac{2}{\pi}\int_0^\infty f(\xi)\sin\omega\,\xi\,d\xi.$$

The usefulness of this procedure will become apparent to us in some particular problems of heat conduction below.

We denoted the variable of integration by ω deliberately. In general one denotes the *frequency* in oscillation processes by ω. Let us therefore, for the time being, think of x as the *time coordinate;* then in equation (10) we have the *decomposition of an arbitrary process in time, $f(x)$, into its harmonic components.* In the *Fourier integral* one is concerned with a *continuous spectrum,* which ranges over all frequencies from $\omega = 0$ to $\omega = \infty$ in the *Fourier series* with a *discrete spectrum,* consisting of a fundamental tone plus harmonic overtones. Here the following fact must be kept in mind: when a physicist determines the spectrum of a process with a suitable spectral apparatus, he finds only the *amplitude* belonging to the frequency ω, while the phase of the partial oscillations remains unknown to him. In our notation the amplitude corresponds to the quantity

$$c(\omega) = \sqrt{a^2(\omega) + b^2(\omega)},$$

the phase, $\gamma(\omega)$, is given by the ratio b/a. The relation between these various quantities is best given as

$$(12)\qquad c(\omega)\,e^{i\gamma(\omega)} = a(\omega) + i\,b(\omega).$$

The Fourier integral which describes the process completely uses both quantities a and b, i.e., *both* amplitude *and* phase. The observable spectrum therefore yields, so to speak, only half the information which is contained in the Fourier integral.

This is noted markedly in the "Fourier analysis of crystals," which is so successfully carried out nowadays. Here only the *intensities* of the crystal reflexes, i.e., the squares of the *amplitudes*, can be observed; for a complete knowledge of the crystal structure one would have to know the *phases* too. This defect can only be partially removed by symmetry considerations.

In exercise I.4 we shall deal with the spectra of diverse oscillation processes as examples for the theory of the Fourier integral and at the same time as completion of the spectral theory.

Once more we return to the complex form of the Fourier integral and split it into two parts

$$(13) \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(\omega)\, e^{i\omega x}\, d\omega, \qquad \varphi(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-ix\omega}\, dx.$$

which together are equivalent to (8). Disregarding the splitting of the denominator 2π into $\sqrt{2\pi}\cdot\sqrt{2\pi}$, which was done mainly for reasons of symmetry, and disregarding the notation of the variable of integration in the second equation, we have $\varphi(\omega)$ identical with the quantity $a(\omega) - i\,b(\omega)$ defined in (10a); it therefore contains information concerning both the amplitude and the phase of the oscillating process $f(x)$.

Moreover (10) shows that the two functions f and φ have a *reciprocal* relation: one is determined by the other, whether we regard f as known and φ as unknown or conversely, and the determination in each case is by "integral equations" of exactly the same character. One says that one function is the *Fourier transform* of the other. In (13) we have a particularly elegant formulation of Fourier's integral theorem.

So far we have spoken only of functions $f(x)$ of *one* variable. It is obvious that a function of several variables can be developed into a Fourier series or integral with respect to any one of the variables. By developing with respect to x,y,z for example we obtain a triply infinite Fourier series and sixfold Fourier integrals. We do not wish to write here the somewhat lengthy formulas since we shall have ample opportunity to explain them in their applications.

§ 5. Development by Spherical Harmonics

We do not claim that the path we shall pursue is the most convenient approach to the theory of spherical harmonics; but it proceeds immediately from the discussion of §1, needs no preparation from the theory of differential equations, and leads to interesting points of view on far reaching generalizations.

We consider the problem: Approximate a function $f(x)$ given in the interval $-1 < x < +1$ by a sequence of polynomials $P_0, P_1, P_2, \ldots P_k, \ldots P_n$ of degrees $0, 1, 2, \ldots k, \ldots n$ in the *manner which is the best possible from the point of view of the method of least squares.* We form an n-th approximation of the form

$$(1) \qquad S_n = \sum_{k=0}^{n} A_k P_k$$

and reduce the mean error

$$(2) \qquad M = \frac{1}{2}\int_{-1}^{+1} [f(x) - S_n]^2 \, dx$$

to a minimum through choice of A_k, just as in (1.3). This leads to the $n + 1$ equations:

$$(3) \qquad \int_{-1}^{+1} [f(x) - S_n] \, P_k \, dx = 0, \quad k = 0, 1, \ldots, n.$$

just as in (1.4). This minimal requirement we complete by a requirement concerning the amount of calculation that will be needed: the coefficients A_k which are to be calculated from (3) in the n-th approximation, shall also be valid in the $(n + 1)$-st and in all subsequent approximations; they shall represent the *final* A_k for all $k \leq n$, and the finer approximations are to complete their determination by yielding the A_k for $k > n$. In §1, p. 4 this finality of the A_k resulted from the known orthogonality of the trigonometric functions. Here, conversely, the *requirement of finality* will be seen to imply the *orthogonality* of the P_k.

The proof is very simple. Equation (3), written explicitly, reads (we omit in the following the limits of integration ± 1):

$$(4) \quad A_0 \int P_0 P_k \, dx + A_1 \int P_1 P_k \, dx + \cdots + A_n \int P_n P_k \, dx = \int f(x) \, P_k \, dx.$$

Since the right side is independent of n and the A_i are to be final, this equation retains its validity for the $(n + 1)$-st approximation S_{n+1}, except that on the left side we add the term

$$A_{n+1} \int P_{n+1} P_k \, dx$$

Equation (4) implies that this term must vanish, and since A_{n+1} does not vanish (except for special choice of $f(x)$), the integral must vanish for all k for which (4) is valid, i.e., for all $k \leq n$. But this implies that P_{n+1} is *orthogonal* to $P_0, P_1, \ldots P_n$ for arbitrary n. Hence, if we take P_0 and P_1 orthogonal to each other, our requirement of finality implies the *general condition of orthogonality*

$$\int P_n P_m \, dx = 0 , \qquad m \neq n . \tag{5}$$

Using (5) we obtain from (4)

$$A_k \int P_k^2 \, dx = \int f(x) \, P_k(x) \, dx . \tag{6}$$

The A_k are therefore determined individually if we add a convention about the *normalizing integral* on the left side of (6). The most obvious procedure would be to set it directly equal to 1, and indeed we shall do this in the general theory of characteristic functions. Here we prefer to follow historical usage and require instead that

$$P_n \, (1) = 1 . \tag{7}$$

This normalizing condition has an advantage in that, as we shall see, all the coefficients in P_n become rational numbers.

We now pass to the recursive calculation of $P_0, P_1, P_2, \ldots$ from (5) and (7). P_0 is a constant, which according to (7), must be set equal to 1. In the linear function $P_1 = a\,x + b$ we see from (5), after setting $n = 0$ and $m = 1$, that $b = 0$ and from (7) that $a = 1$. After setting $P_2 = a\,x^2 + b\,x + c$ we obtain

$$\int P_2 P_0 \, dx = \frac{2}{3} a + 2c = 0 ; \quad \text{hence} \quad c = -\frac{a}{3} ;$$

$$\int P_2 P_1 \, dx = \frac{2}{3} b = 0 ; \qquad \text{hence} \quad b = 0 ;$$

Therefore $P_2 = a\,(x^2 - \frac{1}{3})$ and by (7)

$$a = \frac{3}{2}, \qquad P_2 = \frac{3}{2} x^2 - \frac{1}{2} .$$

Correspondingly we find

$$P_3 = \frac{5}{2} x^3 - \frac{3}{2} x , \qquad P_4 = \frac{35}{8} x^4 - \frac{15}{4} x^2 + \frac{3}{8} \text{ , etc.}$$

The P_n are therefore completely determined by our two requirements, the P_{2n} as even, the P_{2n+1} as odd polynomials with rational coefficients.

More transparent than the recursive process is the following explicit representation:

$$P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n} (x^2 - 1)^n. \tag{8}$$

We see that $P_n(x)$ as defined by (8) satisfies condition (7) as follows: for $x \to 1$ we have to carry out the n-fold differentiation solely for the factor $(x - 1)^n$, whereby we obtain $n!$; the factor $(x + 1)^n$ becomes equal to 2^n; equation (8) therefore does imply that $P_n(1) = 1$.

It remains to be proven that (8) satisfies the orthogonality condition (5), which is equivalent to our "condition of finality." To this end we introduce the notation

$$D_{k,l} = \frac{d^k}{dx^k}(x^2 - 1)^l \tag{9}$$

and write the left side of (5) (suppressing the constant factor which is immaterial here) as

$$\int_{-1}^{+1} D_{nn}\, D_{mm}\, dx\,,$$

where we take, say, $m > n$. We now reduce the order of differentiation of the second factor D_{mm} by integration by parts; this increases the order of differentiation of D_{nn}. The terms which fall outside the integral sign will vanish for $x = \pm 1$, since in $D_{m-1,m}$ according to (9) one factor $x^2 - 1$ remains. Repeating this process we get

$$\begin{aligned} &\int D_{n,n} \cdot D_{m,m}\, dx = -\int D_{n+1,n} \cdot D_{m-1,m}\, dx = \\ &\int D_{n+2,n} \cdot D_{m-2,m}\, dx = \cdots = (-1)^n \int D_{2n,n} \cdot D_{m-n,m}\, dx. \end{aligned} \tag{10}$$

Here according to (9) $D_{2n,n}$ is a constant, namely $(2n)!$ Hence

$$\begin{aligned} \int D_{n,n} \cdot D_{m,m}\, dx &= (-1)^n (2n)! \int D_{m-n,m}\, dx \\ &= (-1)^n (2n)!\, D_{m-n-1,m} \Big|_{-1}^{+1}. \end{aligned} \tag{11}$$

This vanishes, since the number $m - n - 1$ of differentiations that still remain to be carried out is less than the number m of factors $x - 1$ and $x + 1$ which are to be differentiated. This deduction is valid for $m = n + 1$, too, and fails only for $m = n$. The orthogonality is therefore proved for all $m \neq n$.

At the same time the method just used provides a way of calculating the normalizing integral of (6):

$$\int P_k^2\,dx = \left(\frac{1}{2^k\,k!}\right)^2 \int D_{k,\,k} \cdot D_{k,\,k}\,dx\,.$$

Using the first line of (11) for $m = n = k$, we obtain

$$\int P_k^2\,dx = \frac{(-1)^k\,(2\,k)!}{(2^k\,k!)^2} \int D_{0,\,k}\,dx = \frac{(2\,k)!}{(2\cdot4\cdot6\,\ldots\,2\,k)^2} \int (1 - x^2)^k\,dx\,.$$

The numerical factor in front of the last integral is

$$z = \frac{1\cdot3\cdot5\cdot\,\ldots\,\cdot(2k-1)}{2\cdot4\cdot6\cdot\,\ldots\,\cdot2k}\,;$$

under the substitution $x = \cos\vartheta$ the integral itself goes over into the well known form

$$\int_0^\pi \sin^{2k+1}\vartheta\,d\vartheta = 2\cdot\frac{2\cdot4\cdot6\ldots2k}{3\cdot5\cdot7\ldots(2k+1)} = \frac{2}{2k+1}\cdot\frac{1}{z}\,.$$

Therefore, one obtains

$$\int P_k^2\,dx = \frac{2}{2k+1} = \frac{1}{k+\frac{1}{2}}\,. \tag{12}$$

Equation (6) then gives

$$A_k = (k + {}^1/_2) \int f(x)\,P_k(x)\,dx. \tag{13}$$

Substituting this in equation (1) of the n-th approximation S_n and letting $n \to \infty$ we get (assuming convergence and the completeness of the system of functions P):

$$f(x) = \sum_{k=0}^{\infty} (k + {}^1/_2) \int_{-1}^{+1} f(\xi)\,P_k(\xi)\,d\xi \cdot P_k(x)\,. \tag{14}$$

The two assumptions just mentioned can be justified here, just as in the case of Fourier series, by consideration of the limiting value of the mean square error. The k-th approximating function has k zeros in the interval of approximation just as before, except that now they are not equally spaced. The approach to the given function, f, proceeds, here too, through more and more frequent *oscillations*. Also, we find Gibbs' phenomenon at the points of discontinuity, etc.

§ 6. Generalizations: Oscillating and Osculating Approximations. Anharmonic Fourier Analysis. An Example of Non-Final Determination of Coefficients

The following question suggests itself: Why are the two series different, despite the identical nature of the approximation processes? Since we saw that the form of the $P_n(x)$ was completely determined by our approximation requirements, we might think, e.g., that the pure cosine series (expansion of an even function) would go over into a series of spherical harmonics, if in the former we set $\cos\varphi = x$, because then $\cos k\varphi$ becomes a polynomial of degree k in x just like $P_k(x)$, and the interval of expansion $0 < \varphi < \pi$ becomes the interval $+1 > x > -1$. *But the individual infinitesimal elements of this interval receive a different weight g in each case* since

$$d\varphi = -\frac{dx}{\sqrt{1-x^2}}\,.$$

Whereas in the Fourier approximation we associate the same weight with all $d\varphi$, the endpoints $x = \pm 1$ of the interval in the x scale seem to be favored since $g(x) = 1/\sqrt{1-x^2}$. At these points the function is better approximated than at the middle of the interval. The opposite is obviously the case for approximations by spherical harmonics which, translated to the φ-scale, discriminate against the endpoints of the interval since $g(\varphi) = \sin\varphi$. Pictorially speaking, in the case of Fourier series, one deals with a uniformly weighted unit semicircle between $\varphi = 0$ and π, which, under orthogonal projection on the diameter between $x = -1$ and $+1$, yields a non-uniform density; on the other hand the case of spherical harmonics deals with a uniformly weighted diameter, which corresponds to a non-uniformly weighted semicircle.

A. Oscillating and Osculating Approximation

These different *distributions of weight g* (that is, densities) are the factors that, in conjunction with the delimitation of the interval of expansion, distinguish among the different series expansions common in mathematical physics. Here we only mention the expansions in Hermite- and Laguerre-polynomials because of their importance for wave mechanics. We shall not concern ourselves here with their formal representation — they can be obtained from the requirement of a best possible calculation of the coefficients satisfying a condition of finality, just as in the case of spherical harmonics. (See exercise I.6, where the usual normalizations are given; orthogonality would again be the necessary result of these requirements.) We restrict ourselves here to a tabulation of the most important characteristics of both polynomial series:

	HERMITE	LAGUERRE
Interval	$-\infty < x < +\infty$	$0 < x < \infty$
Weight $g(x)$	e^{-x^2}	e^{-x}
Orthogonality condition for $m \neq n$	$\int_{-\infty}^{+\infty} H_n H_m e^{-x^2} dx = 0$	$\int_0^{\infty} L_n L_m e^{-x} dx = 0$

For these series, just as for Fourier series and spherical harmonics, the approach to the given function, f, is through closer and closer *oscillations.* However, from the calculus we know a series whose character is *osculating* rather than *oscillating*, namely the *Taylor series.* In the case of Taylor series the consecutive approximations S_n osculate the curve to be represented in such a way that at a given point S_n has the same derivatives as f up to and including $f^{(n)}$ The graphic representation of the power series of sin x (Fig. 6) demonstrates this without further explanation.

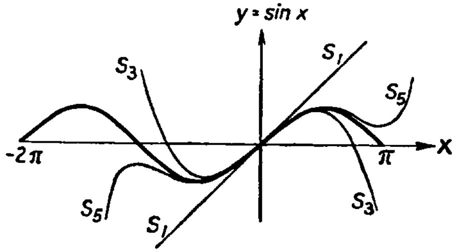

Fig. 6. The Taylor expansion of sin x (heavy line) and its approximations

$$S_1 = x,\ S_3 = x - \frac{x^3}{3!},$$
$$S_5 = x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Here the total accuracy is concentrated at a single point. Following Dirac we can express this succinctly as follows: $g(x)$ has degenerated into a δ function. Dirac defines, as an analogue to the algebraic symbol δ_{kl} of (1.6), a highly discontinuous function $\delta(x \mid x_0)$

$$\delta(x \mid x_0) = \begin{cases} 0 & x \neq x_0, \\ \infty & x = x_0, \end{cases} \qquad \int_{x_0-\varepsilon}^{x_0+\varepsilon} \delta(x \mid x_0)\, dx = 1 \tag{1}$$

for arbitrary ε For the Taylor series of Fig. 6, where x_0 has been set equal to 0, we get

$$g(x) = \delta(x \mid 0). \tag{1a}$$

B. ANHARMONIC FOURIER ANALYSIS

Whereas in §1–3 we considered only Fourier series which proceed according to *harmonic* (integral) overtones of a fundamental tone, we now consider the problem of expanding an arbitrary function $f(x)$ in the interval $0 < x < \pi$ into a series of the form

(2) $$f(x) = B_1 \sin \lambda_1 x + B_2 \sin \lambda_2 x + B_3 \sin \lambda_3 x + \cdots$$

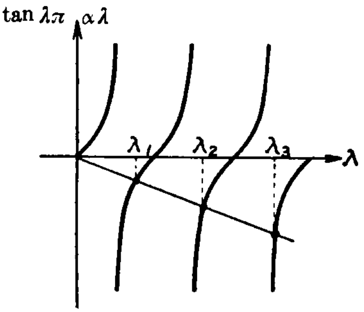

Fig. 7. Diagram of the transcendental equation $\tan \lambda\pi = \alpha\lambda$ $\alpha < 0$. In the ordinate both $y = \tan \lambda\pi$ and $y = \alpha\lambda$ have been drawn. The intersections yield the roots, λ_k, of the equation. $\lambda_0 = 0$ is not to be considered as a root; for $n \to \infty$ we get asymptotically $\lambda_n = n - \frac{1}{2}$.

where the λ_k are given as the roots of a transcendental equation, e.g.,

(2a) $$\tan \lambda \pi = \alpha \lambda$$

(α being an arbitrary number). We do this for use in problems of heat conduction (see §16). The fact that (2a) has infinitely many roots is seen directly from Fig. 7 where λ has been drawn as the abscissa and both $\tan \lambda\pi$ and $\alpha \lambda$ as ordinates. We shall meet another equation of character similar to (2a) in exercise II.1.

We first show that the functions $\sin \lambda_k x$ form an orthogonal system with weighting factor $g(x) = 1$, i.e., that

(3) $$\int_0^\pi \sin \lambda_k x \sin \lambda_l x \, dx = 0 \ldots k \neq l. \qquad k \neq l.$$

In fact, by passing from the product of sines to the cosines of the sums and differenees, we obtain for the left hand of (3)

$$\frac{\lambda_k \lambda_l}{\lambda_k^2 - \lambda_l^2} \cos \lambda_k \pi \cos \lambda_l \pi \left(\frac{\tan \lambda_k \pi}{\lambda_k} - \frac{\tan \lambda_l \pi}{\lambda_l} \right),$$

where the expression inside the brackets now vanishes because of (2a). In the same manner we find for $k = l$

(3a) $$\int_0^\pi \sin^2 \lambda_k x \, dx = \frac{\pi}{2}\left(1 - \frac{1}{\lambda_k \pi} \sin \lambda_k \pi \cdot \cos \lambda_k \pi\right).$$

This calculation of (3) and (3a) which is based on special trigonometric identities, will receive a less formal treatment in §16 where it will be reduced to an application of Green's theorem.

From (3) and (3a) one obtains the following value for the expansion coefficients B_k in (1):

$$B_k = \frac{2}{\pi} \int_0^{\pi} \frac{f(x) \sin \lambda_k x}{1 - \frac{\sin 2\lambda_k \pi}{2\lambda_k \pi}} dx. \tag{3b}$$

This value for B_k is *final* in the sense of p. 22, since it is independent of n and minimizes the mean square error of the approximation

$$S_n = \sum_{k=1}^{n} B_k \sin \lambda_k x$$

At the same time this settles the question of convergence and completeness, if for $n \to \infty$ the mean square error approaches zero.

C. An Example of a Non-Final Determination of Coefficients

As preparation for an optical (or rather "quasi-optical") application, we shall consider a much more involved case in which the requirement of finality is *not* satisfied. Let us consider a metal mirror in the shape of a circular cylinder (see Fig. 8). The electric vector of the total oscillation, which we take as perpendicular to the plane of the drawing, is composed of the incoming wave, represented on the mirror by

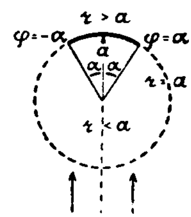

Fig. 8. Reflection of an incoming "quasioptical" wave on a circular cylinder mirror of opening $\varphi = \alpha$ and radius $r = a$.

$$w = -f(\varphi), \quad -\alpha < \varphi < +\alpha, \quad r = a \tag{4}$$

and of the reflected (refracted, scattered) wave. Let the latter be represented by:

$$u = u(r, \varphi), \quad -\pi < \varphi < +\pi, \quad r < a \text{, inner field,}$$

$$v = v(r, \varphi), \quad -\pi < \varphi < +\pi, \quad r > a \text{, outer field.}$$

We then have to demand

$$u + w = v + w = 0 \quad \text{for } r = a \text{ and } |\varphi| < \alpha, \tag{5}$$

(6) $$u = v, \quad \frac{\partial u}{\partial r} = \frac{\partial v}{\partial r} \qquad \text{for } r = a \text{ and } |\varphi| > \alpha,$$

the former on account of the assumed infinite conductivity of the metal mirror, the latter on account of the required continuous passage from the inner to the outer field.

Assuming w to be symmetric with respect to the axis of the mirror (as, for example in the case of a plane wave proceeding in that direction), we write[7]

(7) $$\begin{aligned} u &= \sum_n C_n\, g_n(r) \cos n\varphi, \\ v &= \sum_n D_n\, h_n(r) \cos n\varphi. \end{aligned}$$

g_n and h_n will turn out to be Bessel and Hankel functions, respectively (see §19); they can be chosen so that

$$g_n(a) = h_n(a) = 1$$

Equation (5) and the first equation (6) then imply

(8) $$\sum_n C_n \cos n\varphi = \sum_n D_n \cos n\varphi = f(\varphi) \qquad |\varphi| < \alpha$$

and

(9) $$\sum_n C_n \cos n\varphi = \sum_n D_n \cos n\varphi \qquad |\varphi| > \alpha.$$

respectively. From these two equations it follows that

$$\sum_n (C_n - D_n) \cos n\varphi = 0 \quad \text{for all } \varphi,$$

hence, whether the preceding summations are extended over all integers n or only over the first N integers (the more general case), we have

$$D_n = C_n.$$

This satisfies (9) while (8) still requires

(10) $$\sum_n C_n \cos n\varphi = f(\varphi) \quad \text{for } |\varphi| < \alpha.$$

In addition to this we have to satisfy the second equation (6) which on account of (7) reads:

[7] In view of the notations to be used in Chapter IV, it is advisable here to change the index of summation from k to n. For the previous n we shall write N, and instead of l we shall use m.

$$(11) \qquad \sum_n C_n \gamma_n \cos n\varphi = 0 \qquad \text{for } |\varphi| > \alpha$$

$$(11a) \qquad \gamma_n = a\left(\frac{dg_n(r)}{dr} - \frac{dh_n(r)}{dr}\right)_{r=a}$$

We add the factor a before the parentheses here, as we may by (11), in order to make γ_n a pure number. Equations (10) and (11) together determine the C_n.

Here the way is again shown by the method of least squares. We consider the square errors corresponding to the equations (10) and (11)

$$\int_0^\alpha \left(f(\varphi) - \sum_{n=0}^N C_n \cos n\varphi\right)^2 d\varphi \quad \text{and} \quad \int_\alpha^\pi \left(\sum_{n=0}^N C_n \gamma_n \cos n\varphi\right)^2 d\varphi .$$

The sum of these two is to be minimized through choice of the C_n. By differentiation with respect to the C_n this yields a system of $N+1$ linear equations for $C_0, \ldots C_n, \ldots C_N$, of which the $(m+1)$-st equation is:

$$(12) \qquad \sum_{n=0}^N C_n \left\{\int_0^\alpha \cos n\varphi \cos m\varphi \, d\varphi + \gamma_n \gamma_m \int_\alpha^\pi \cos n\varphi \cos m\varphi \, d\varphi\right\} = \int_0^\alpha f(\varphi) \cos m\varphi \, d\varphi .$$

If we pass to the limit $N \to \infty$ we obtain an *infinite system of linear equations for the infinitely many unknowns* C_n, which are in general of no interest to us. We must postpone further treatment of this problem until appendix I of Chapter IV, for only there shall we have the necessary values of the parameters γ_n. The corresponding spatial problem, where we have a spherical segment instead of a circular cylinder segment, would lead in the limit $N \to \infty$ to an infinite system of linear equations, in which $P_n(\cos \vartheta)$ would replace $\cos n\varphi$ (by ϑ we denote here the angle measured from the axis of symmetry of the spherical mirror). This problem too will be treated in appendix I of Chapter IV. At present we call attention only to the difference in method between those problems in which the method of least squares leads to a definitive calculation of the *individual coefficients* C, and those problems in which the "requirement of finality" is not satisfied and in which therefore, *the totality of the* C_n *must be determined from the totality of minimality conditions.*