# Problems & Solutions

for

# Statistical Physics of Fields

*Updated July 2008*

by

Mehran Kardar
Department of Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139, USA

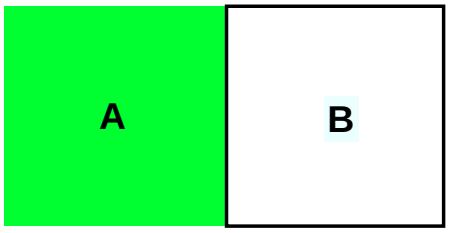
# Table of Contents

**Solutions to problems from chapter 1- Collective Behavior, From Particles to Fields**

**1.** *The binary alloy:* A binary alloy (as in $\beta$ brass) consists of $N_A$ atoms of type $A$, and $N_B$ atoms of type $B$. The atoms form a simple cubic lattice, each interacting only with its six nearest neighbors. Assume an attractive energy of $-J$ $(J > 0)$ between like neighbors $A - A$ and $B - B$, but a repulsive energy of $+J$ for an $A - B$ pair.

(a) What is the minimum energy configuration, or the state of the system at zero temperature?

• The minimum energy configuration has as little A-B bonds as possible. Thus, at zero temperature atoms A and B *phase separate*, e.g. as indicated below.




(b) Estimate the total interaction energy assuming that the atoms are randomly distributed among the $N$ sites; i.e. each site is occupied independently with probabilities $p_A = N_A/N$ and $p_B = N_B/N$.

• In a mixed state, the average energy is obtained from

$$\begin{aligned} E &= (\text{number of bonds}) \times (\text{average bond energy}) \\ &= 3N \cdot \left(-Jp_A^2 - Jp_B^2 + 2Jp_Ap_B\right) \\ &= -3JN\left(\frac{N_A - N_B}{N}\right)^2 . \end{aligned}$$

(c) Estimate the mixing entropy of the alloy with the same approximation. Assume $N_A, N_B \gg 1$.

• From the number of ways of randomly mixing $N_A$ and $N_B$ particles, we obtain the mixing entropy of

$$S = k_B \ln\left(\frac{N!}{N_A!N_B!}\right).$$

Using Stirling's approximation for large $N$ $(\ln N! \approx N \ln N - N)$, the above expression can be written as

$$S \approx k_B \left(N \ln N - N_A \ln N_A - N_B \ln N_B\right) = -Nk_B \left(p_A \ln p_A + p_B \ln p_B\right).$$

(d) Using the above, obtain a free energy function $F(x)$, where $x = (N_A - N_B)/N$. Expand $F(x)$ to the fourth order in $x$, and show that the requirement of convexity of $F$ breaks down below a critical temperature $T_c$. For the remainder of this problem use the expansion obtained in (d) in place of the full function $F(x)$.

- In terms of $x = p_A - p_B$, the free energy can be written as

$$\begin{aligned} F &= E - TS \\ &= -3JNx^2 + Nk_BT\left\{\left(\frac{1+x}{2}\right)\ln\left(\frac{1+x}{2}\right) + \left(\frac{1-x}{2}\right)\ln\left(\frac{1-x}{2}\right)\right\}. \end{aligned}$$

Expanding about $x = 0$ to fourth order, gives

$$F \simeq -Nk_BT\ln 2 + N\left(\frac{k_BT}{2} - 3J\right)x^2 + \frac{Nk_BT}{12}x^4.$$

Clearly, the second derivative of $F$,

$$\frac{\partial^2 F}{\partial x^2} = N\left(k_BT - 6J\right) + Nk_BTx^2,$$

becomes negative for $T$ small enough. Upon decreasing the temperature, $F$ becomes concave first at $x = 0$, at a critical temperature $T_c = 6J/k_B$.

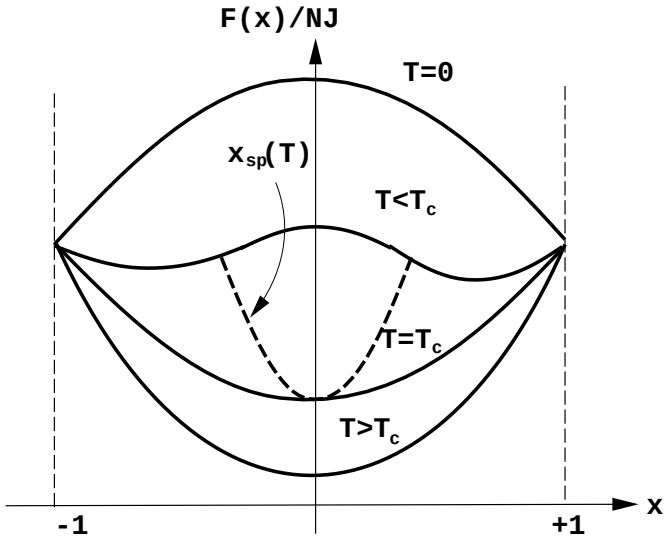
(e) Sketch $F(x)$ for $T > T_c$, $T = T_c$, and $T < T_c$. For $T < T_c$ there is a range of compositions $x < |x_{sp}(T)|$ where $F(x)$ is not convex and hence the composition is locally unstable. Find $x_{sp}(T)$.

- The function $F(x)$ is concave if $\partial^2 F/\partial x^2 < 0$, *i.e.* if

$$x^2 < \left(\frac{6J}{k_BT} - 1\right).$$

This occurs for $T < T_c$, at the spinodal line given by

$$x_{sp}(T) = \sqrt{\frac{6J}{k_BT} - 1},$$




as indicated by the dashed line in the figure below.

(f) The alloy globally minimizes its free energy by separating into $A$ rich and $B$ rich phases of compositions $\pm x_{eq}(T)$, where $x_{eq}(T)$ minimizes the function $F(x)$. Find $x_{eq}(T)$.

• Setting the first derivative of $dF\left(x\right)/dx = Nx\left\{(k_BT - 6J) + k_BTx^2/3\right\}$, to zero yields the equilibrium value of

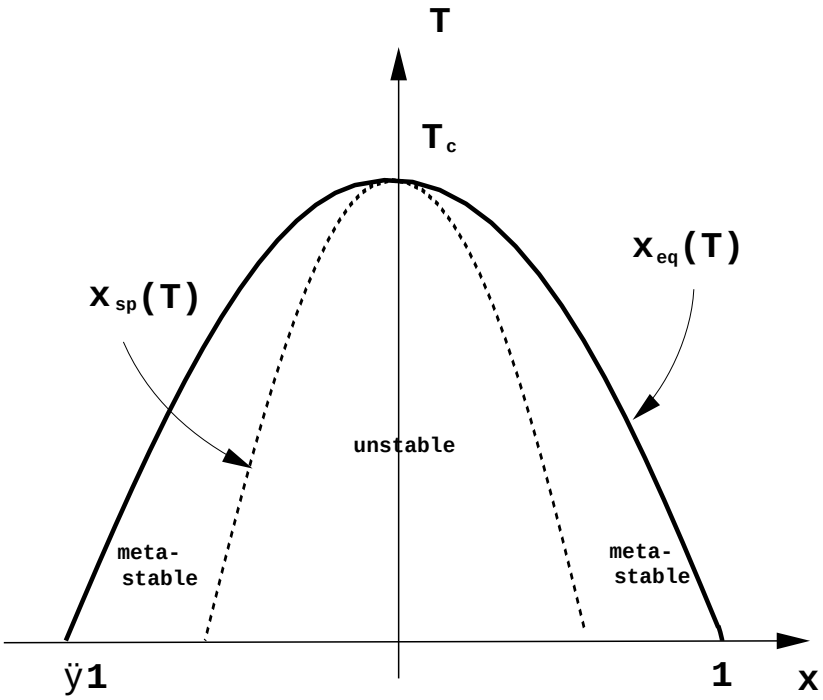
$$x_{eq}\left(T\right) = \begin{cases} \pm\sqrt{3}\sqrt{\dfrac{6J}{k_BT} - 1} & \text{for } \; T < T_c \\ 0 & \text{for } \; T > T_c \end{cases} .$$

(g) In the $(T, x)$ plane sketch the phase separation boundary $\pm x_{eq}(T)$; and the so called spinodal line $\pm x_{sp}(T)$. (The spinodal line indicates onset of metastability and hysteresis effects.)

• The spinodal and equilibrium curves are indicated in the figure above. In the interval between the two curves, the system is locally stable, but globally unstable. The formation of ordered regions in this regime requires nucleation, and is very slow. The dashed area is locally unstable, and the system easily phase separates to regions rich in A and B.

********

**2.** *The Ising model of magnetism:* The local environment of an electron in a crystal sometimes forces its spin to stay parallel or anti-parallel to a given lattice direction. As a model of magnetism in such materials we denote the direction of the spin by a single




variable $\sigma_i = \pm 1$ (an Ising spin). The energy of a configuration $\{\sigma_i\}$ of spins is then given by

$$\mathcal{H} = \frac{1}{2}\sum_{i,j=1}^{N} J_{ij}\sigma_i\sigma_j - h\sum_i \sigma_i \quad ;$$

where $h$ is an external magnetic field, and $J_{ij}$ is the interaction energy between spins at sites $i$ and $j$.

(a) For $N$ spins we make the drastic *approximation* that the interaction between all spins is the same, and $J_{ij} = -J/N$ (the equivalent neighbor model). Show that the energy can now be written as $E(M,h) = -N[Jm^2/2 + hm]$, with a magnetization $m = \sum_{i=1}^{N}\sigma_i/N = M/N$.

• For $J_{ij} = -J/N$, the energy of each configuration is only a function of $m = \sum_i \sigma_i/N$, given by

$$\begin{aligned} E\,(M,h) &= -\frac{J}{2N}\sum_{i,j=1}^{N}\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i \\ &= -N\frac{J}{2}\left(\sum_{i=1}^{N}\sigma_i/N\right)\left(\sum_{j=1}^{N}\sigma_j/N\right) - Nh\left(\sum_{i=1}^{N}\sigma_i/N\right) \\ &= -N\left(\frac{J}{2}m^2 + hm\right). \end{aligned}$$

(b) Show that the partition function $Z(h,T) = \sum_{\{\sigma_i\}} \exp(-\beta\mathcal{H})$ can be re-written as $Z = \sum_M \exp[-\beta F(m,h)]$; with $F(m,h)$ easily calculated by analogy to problem (1). For the remainder of the problem work only with $F(m,h)$ expanded to 4th order in $m$.

• Since the energy depends only on the *number* of up spins $N_+$, and not on their configuration, we have

$$\begin{aligned} Z(h,T) &= \sum_{\{\sigma_i\}} \exp(-\beta\mathcal{H}) \\ &= \sum_{N_+=0}^{N} (\text{number of configurations with } N_+ \text{ fixed}) \cdot \exp\left[-\beta E(M,h)\right] \\ &= \sum_{N_+=0}^{N} \left[\frac{N!}{N_+!\,(N-N_+)!}\right] \exp\left[-\beta E(M,h)\right] \\ &= \sum_{N_+=0}^{N} \exp\left\{-\beta\left[E(M,h) - k_B T \ln\left(\frac{N!}{N_+!\,(N-N_+)!}\right)\right]\right\} \\ &= \sum_{M} \exp\left[-\beta F(m,h)\right]. \end{aligned}$$

By analogy to the previous problem ($N_+ \leftrightarrow N_A$, $m \leftrightarrow x$, $J/2 \leftrightarrow 3J$),

$$\frac{F(m,h)}{N} = -k_B T \ln 2 - hm + \frac{1}{2}(k_B T - J)\, m^2 + \frac{k_B T}{12} m^4 + \mathcal{O}\left(m^5\right).$$

(c) By saddle point integration show that the actual free energy $F(h,T) = -kT \ln Z(h,T)$ is given by $F(h,T) = \min[F(m,h)]_m$. When is the saddle point method valid? Note that $F(m,h)$ is an analytic function but not convex for $T < T_c$, while the true free energy $F(h,T)$ is convex but becomes non-analytic due to the minimization.

• Let $m^*(h,T)$ minimize $F(m,h)$, *i.e.* $\min\left[F(m,h)\right]_m = F(m^*,h)$. Since there are $N$ terms in the sum for $Z$, we have the bounds

$$\exp\left(-\beta F(m^*,h)\right) \leq Z \leq N \exp\left(-\beta F(m^*,h)\right),$$

or, taking the logarithm and dividing by $-\beta N$,

$$\frac{F(m^*,h)}{N} \geq \frac{F(h,T)}{N} \geq \frac{F(m^*,h)}{N} + \frac{\ln N}{N}.$$

Since $F$ is extensive, we have therefore

$$\frac{F\left(m^{*},h\right)}{N}=\frac{F\left(h,T\right)}{N}$$

in the $N\rightarrow\infty$ limit.

(d) For $h=0$ find the critical temperature $T_c$ below which spontaneous magnetization appears; and calculate the magnetization $\overline{m}(T)$ in the low temperature phase.

- From the definition of the *actual* free energy, the magnetization is given by

$$\overline{m}=-\frac{1}{N}\frac{\partial F\left(h,T\right)}{\partial h},$$

*i.e.*

$$\overline{m}=-\frac{1}{N}\frac{dF\left(m,h\right)}{dh}=-\frac{1}{N}\left\{\frac{\partial F\left(m,h\right)}{\partial h}+\frac{\partial F\left(m,h\right)}{\partial m}\frac{\partial m}{\partial h}\right\}.$$

Thus, if $m^{*}$ minimizes $F\left(m,h\right)$, *i.e.* if $\partial F\left(m,h\right)/\partial m\big|_{m^{*}}=0$, then

$$\overline{m}=-\frac{1}{N}\left.\frac{\partial F\left(m,h\right)}{\partial h}\right|_{m^{*}}=m^{*}.$$

For $h=0$,

$$m^{*2}=\frac{3\left(J-k_{B}T\right)}{k_{B}T},$$

yielding

$$T_{c}=\frac{J}{k_{B}},$$

and

$$\overline{m}=\begin{cases}\pm\sqrt{\dfrac{3\left(J-k_{B}T\right)}{k_{B}T}} & \text{if } \ T<T_{c}\\ 0 & \text{if } \ T>T_{c}\end{cases}.$$

(e) Calculate the singular (non-analytic) behavior of the response functions

$$C=\left.\frac{\partial E}{\partial T}\right|_{h=0},\quad\text{and}\quad\chi=\left.\frac{\partial\overline{m}}{\partial h}\right|_{h=0}.$$

- The hear capacity is given by

$$C=\left.\frac{\partial E}{\partial T}\right|_{h=0,m=m^{*}}=-\frac{NJ}{2}\frac{\partial m^{*2}}{\partial T}=\begin{cases}\dfrac{3NJT_{c}}{2T^{2}} & \text{if } \ T<T_{c}\\ 0 & \text{if } \ T>T_{c}\end{cases},$$

*i.e.* $\alpha = 0$, indicating a discontinuity. To calculate the susceptibility, we use

$$h = (k_B T - J)\,\overline{m} + \frac{k_B T}{3}\overline{m}^3.$$

Taking a derivative with respect to $h$,

$$1 = \left(k_B T - J + k_B T \overline{m}^2\right)\frac{\partial \overline{m}}{\partial h},$$

which gives

$$\chi = \left.\frac{\partial \overline{m}}{\partial h}\right|_{h=0} = \begin{cases} \dfrac{1}{2k_B\,(T_c - T)} & \text{if } \; T < T_c \\ \dfrac{1}{k_B\,(T - T_c)} & \text{if } \; T > T_c \end{cases}.$$

From the above expression we obtain $\gamma_\pm = 1$, and $A_+/A_- = 2$.

********

**3.** *The lattice–gas model:* Consider a gas of particles subject to a Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N} \frac{\vec{p}_i\ ^2}{2m} + \frac{1}{2}\sum_{i,j}\mathcal{V}(\vec{r}_i - \vec{r}_j), \quad \text{in a volume } \; V.$$

(a) Show that the grand partition function $\Xi$ can be written as

$$\Xi = \sum_{N=0}^{\infty}\frac{1}{N!}\left(\frac{e^{\beta\mu}}{\lambda^3}\right)^N \int \prod_{i=1}^{N} d^3\vec{r}_i \exp\left[-\frac{\beta}{2}\sum_{i,j}\mathcal{V}(\vec{r}_i - \vec{r}_j)\right].$$

- The grand partition function is calculated as

$$\begin{aligned}
\Xi &= \sum_{N=0}^{\infty}\frac{e^{N\beta\mu}}{N!}Z_N \\
&= \sum_{N=0}^{\infty}\frac{e^{N\beta\mu}}{N!}\int\prod_{i=1}^{N}\frac{d^3p_i d^3r_i}{h^3}e^{-\beta\mathcal{H}} \\
&= \sum_{N=0}^{\infty}\frac{e^{N\beta\mu}}{N!}\left(\prod_{i=1}^{N}\int\frac{d^3p_i}{h^3}e^{-\beta p_i^2/2m}\right)\int\prod_{i=1}^{N}d^3r_i\exp\left(-\frac{\beta}{2}\sum_{i,j}\mathcal{V}_{ij}\right) \\
&= \sum_{N=0}^{\infty}\frac{1}{N!}\left(\frac{e^{N\beta}}{\lambda^3}\right)^N\int\prod_{i=1}^{N}d^3r_i\exp\left(-\frac{\beta}{2}\sum_{i,j}\mathcal{V}_{ij}\right),
\end{aligned}$$

where $\lambda^{-1} = \sqrt{2\pi m k_B T}/h$.

(b) The volume $V$ is now subdivided into $\mathcal{N} = V/a^3$ cells of volume $a^3$, with the spacing $a$ chosen small enough so that each cell $\alpha$ is either empty or occupied by one particle; i.e. the cell occupation number $n_\alpha$ is restricted to 0 or 1 ($\alpha = 1, 2, \cdots, \mathcal{N}$). After approximating the integrals $\int d^3\vec{r}$ by sums $a^3 \sum_{\alpha=1}^{\mathcal{N}}$, show that

$$\Xi \approx \sum_{\{n_\alpha=0,1\}} \left(\frac{e^{\beta\mu}a^3}{\lambda^3}\right)^{\sum_\alpha n_\alpha} \exp\left[-\frac{\beta}{2}\sum_{\alpha,\beta=1}^{\mathcal{N}} n_\alpha n_\beta \mathcal{V}(\vec{r}_\alpha - \vec{r}_\beta)\right].$$

- Since

$$\int \prod_{i=1}^{N} d^3 r_i \exp\left(-\frac{\beta}{2}\sum_{i,j}\mathcal{V}_{ij}\right) \approx a^{3N} {\sum}' \exp\left\{-\frac{\beta}{2}\sum_{\alpha,\beta=1}^{\mathcal{N}} n_\alpha n_\beta \mathcal{V}\left(\vec{r}_\alpha - \vec{r}_\beta\right)\right\} \cdot N!,$$

where the primed sum is over the configurations $\{n_\alpha = 0, 1\}$ with fixed $N$, and

$$N = \sum_{\alpha=1}^{\mathcal{N}} n_\alpha,$$

we have

$$\Xi \approx \sum_{\{n_\alpha=0,1\}} \left(\frac{e^{\beta\mu}a^3}{\lambda^3}\right)^{\sum_\alpha n_\alpha} \exp\left\{-\frac{\beta}{2}\sum_{\alpha,\beta=1}^{\mathcal{N}} n_\alpha n_\beta \mathcal{V}\left(\vec{r}_\alpha - \vec{r}_\beta\right)\right\}.$$

(c) By setting $n_\alpha = (1 + \sigma_\alpha)/2$ and approximating the potential by $\mathcal{V}(\vec{r}_\alpha - \vec{r}_\beta) = -J/\mathcal{N}$, show that this model is identical to the one studied in problem (2). What does this imply about the behavior of this imperfect gas?

- With $n_\alpha = (1 + \sigma_\alpha)/2$, and $\mathcal{V}\left(\vec{r}_\alpha - \vec{r}_\beta\right) = -J/\mathcal{N}$,

$$\Xi = \sum_{\{n_\alpha=0,1\}} \exp\left\{\left(\beta\mu + 3\ln\frac{a}{\lambda}\right)\sum_{\alpha=1}^{\mathcal{N}}\left(\frac{1+\sigma_\alpha}{2}\right) + \frac{\beta J}{2\mathcal{N}}\sum_{\alpha,\beta=1}^{\mathcal{N}}\left(\frac{1+\sigma_\alpha}{2}\right)\left(\frac{1+\sigma_\beta}{2}\right)\right\}.$$

Setting $m \equiv \sum_\alpha \sigma_\alpha/\mathcal{N}$, $h' = \frac{1}{2}\left(\mu + \frac{3}{\beta}\ln\frac{a}{\lambda} + \frac{J}{2}\right)$, and $J' = J/4$, the grand partition function is written

$$\Xi = \text{const.} \sum_{\{n_\alpha=0,1\}} \exp\left\{\mathcal{N}\beta\left(J'm^2/2 + h'm\right)\right\}.$$

The phase diagram of the lattice-gas can thus be mapped onto the phase diagram of the Ising model of problem 2. In particular, at a chemical potential $\mu$ such that $h' = 0$, there is a continuous "condensation" transition at a critical temperature $T_c = J/4k_B$. (Note that

$$m = \sum_\alpha \sigma_\alpha / \mathcal{N} = \sum_\alpha (2n_\alpha - 1) / \mathcal{N} = 2a^3\rho - 1,$$

where $\rho = N/V$ is the density of the gas.)

• The manifest equivalence between these three systems is a straightforward consequence of their mapping onto the same (Ising) Hamiltonian. However, there is a more subtle equivalence relating the critical behavior of systems that cannot be so easily mapped onto each other due to the Universality Principle.

********

---

**4.** *Surfactant condensation:* $N$ surfactant molecules are added to the surface of water over an area $A$. They are subject to a Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N} \frac{\vec{p}_i{}^2}{2m} + \frac{1}{2}\sum_{i,j} \mathcal{V}(\vec{r}_i - \vec{r}_j),$$

where $\vec{r}_i$ and $\vec{p}_i$ are two dimensional vectors indicating the position and momentum of particle $i$.

(a) Write down the expression for the partition function $Z(N, T, A)$ in terms of integrals over $\vec{r}_i$ and $\vec{p}_i$, and perform the integrals over the momenta.

• The partition function is obtained by integrating the Boltzmann weight over phase space, as

$$Z(N,T,A) = \int \frac{\prod_{i=1}^N d^2\vec{p}_i d^2\vec{q}_i}{N! h^{2N}} \exp\left[-\beta \sum_{i=1}^{N} \frac{p_i^2}{2m} - \beta \sum_{i<j} \mathcal{V}(\vec{q}_i - \vec{q}_j)\right],$$

with $\beta = 1/(k_B T)$. The integrals over momenta are simple Gaussians, yielding

$$Z(N,T,A) = \frac{1}{N!}\,\frac{1}{\lambda^{2N}} \int \prod_{i=1}^{N} d^2\vec{q}_i \exp\left[-\beta \sum_{i<j} \mathcal{V}(\vec{q}_i - \vec{q}_j)\right],$$

where as usual $\lambda = h/\sqrt{2\pi m k_B T}$ denotes the thermal wavelength.

The inter–particle potential $\mathcal{V}(\vec{r})$ is infinite for separations $|\vec{r}| < a$, and attractive for $|\vec{r}| > a$ such that $\int_a^\infty 2\pi r dr \mathcal{V}(r) = -u_0$.

(b) Estimate the total non–excluded area available in the positional phase space of the system of $N$ particles.

• To estimate the joint phase space of particles with excluded areas, add them to the system one by one. The first one can occupy the whole area $A$, while the second can explore only $A - 2\Omega$, where $\Omega = \pi a^2$. Neglecting three body effects (i.e. in the dilute limit), the area available to the third particle is $(A - 2\Omega)$, and similarly $(A - n\Omega)$ for the $n$-th particle. Hence the joint excluded volume in this dilute limit is

$$A(A-\Omega)(A-2\Omega)\cdots(A-(N-1)\Omega) \approx (A - N\Omega/2)^N,$$

where the last approximation is obtained by pairing terms $m$ and $(N - m)$, and ignoring order of $\Omega^2$ contributions to their product.

(c) Estimate the total *potential* energy of the system, *assuming a uniform density* $n = N/A$. Using this potential energy for all configurations allowed in the previous part, write down an approximation for $Z$.

• Assuming a *uniform density* $n = N/A$, an average attractive potential energy, $\bar{U}$, is estimated as

$$\begin{aligned}\bar{U} =& \frac{1}{2}\sum_{i,j} \mathcal{V}_{\text{attr.}}(\vec{q}_i - \vec{q}_j) = \frac{1}{2}\int d^2\vec{r}_1 d^2\vec{r}_2 n(\vec{r}_1) n(\vec{r}_2) \mathcal{V}_{\text{attr.}}(\vec{r}_1 - \vec{r}_2) \\ \approx& \frac{n^2}{2} A \int d^2\vec{r}\, \mathcal{V}_{\text{attr.}}(\vec{r}) \equiv -\frac{N^2}{2A}\, u_0.\end{aligned}$$

Combining the previous results gives

$$Z(N,T,A) \approx \frac{1}{N!}\,\frac{1}{\lambda^{2N}}\,(A - N\Omega/2)^N \exp\left[\frac{\beta u_0 N^2}{2A}\right].$$

(d) The surface tension of water without surfactants is $\sigma_0$, approximately independent of temperature. Calculate the surface tension $\sigma(n,T)$ in the presence of surfactants.

• Since the work done is changing the surface area is $dW = \sigma dA$, we have $dF = -TdS + \sigma dA + \mu dN$, where $F = -k_B T \ln Z$ is the free energy. Hence, the contribution of the surfactants to the surface tension of the film is

$$\sigma_s = -\left.\frac{\partial \ln Z}{\partial A}\right|_{T,N} = -\frac{N k_B T}{A - N\Omega/2} + \frac{u_0 N^2}{2A^2},$$

which is a two-dimensional variant of the familiar van der Waals equation. Adding the (constant) contribution in the absence of surfactants gives

$$\sigma(n,T) = \sigma_0 - \left.\frac{\partial \ln Z}{\partial A}\right|_{T,N} = -\frac{Nk_BT}{A - N\Omega/2} + \frac{u_0N^2}{2A^2}.$$

(e) Show that below a certain temperature, $T_c$, the expression for $\sigma$ is manifestly incorrect. What do you think happens at low temperatures?

• Thermodynamic stability requires $\delta\sigma\delta A \geq 0$, i.e. $\sigma$ must be a monotonically increasing function of $A$ at any temperature. This is the case at high temperatures where the first term in the equation for $\sigma_s$ dominates, but breaks down at low temperatures when the term from the attractive interactions becomes significant. The critical temperature is obtained by the usual conditions of $\partial\sigma_s/\partial A = \partial^2\sigma_s/\partial A^2 = 0$, i.e. from

$$\begin{cases} \left.\dfrac{\partial\sigma_s}{\partial A}\right|_T = \dfrac{Nk_BT}{(A - N\Omega/2)^2} - \dfrac{u_0N^2}{A^3} = 0 \\ \left.\dfrac{\partial^2\sigma_s}{\partial A^2}\right|_T = -\dfrac{2Nk_BT}{(A - N\Omega/2)^3} + \dfrac{3u_0N^2}{A^4} = 0 \end{cases},$$

The two equations are simultaneously satisfied for $A_c = 3N\Omega/2$, at a temperature

$$T_c = \frac{8u_0}{27k_B\Omega}.$$

As in the van der Waals gas, at temperatures below $T_c$, the surfactants separate into a high density (liquid) and a low density (gas) phase.

(f) Compute the heat capacities, $C_A$ and write down an expression for $C_\sigma$ without explicit evaluation, due to the surfactants.

• The contribution of the surfactants to the energy of the film is given by

$$E_s = -\frac{\partial \ln Z}{\partial\beta} = 2N \times \frac{k_BT}{2} - \frac{u_0N^2}{2A}.$$

The first term is due to the kinetic energy of the surfactants, while the second arises from their (mean-field) attraction. The heat capacities are then calculated as

$$C_A = \left.\frac{dQ}{dT}\right|_A = \left.\frac{\partial E}{\partial T}\right|_A = Nk_B,$$

and

$$C_\sigma = \left.\frac{dQ}{dT}\right|_\sigma = \left.\frac{\partial E}{\partial T}\right|_\sigma - \sigma \left.\frac{\partial A}{\partial T}\right|_\sigma .$$

********

**5.** *Critical behavior of a gas:* The pressure $P$ of a gas is related to its density $n = N/V$, and temperature $T$ by the truncated expansion

$$P = k_B T n - \frac{b}{2} n^2 + \frac{c}{6} n^3 \quad ,$$

where $b$ and $c$ are assumed to be positive, temperature independent constants.

(a) Locate the critical temperature $T_c$ below which this equation must be invalid, and the corresponding density $n_c$ and pressure $P_c$ of the critical point. Hence find the ratio $k_B T_c n_c / P_c$.

• Mechanical stability of the gas requires that any spontaneous change in volume should be opposed by a compensating change in pressure. This corresponds to $\delta P \delta V < 0$, and since $\delta n = -(N/V^2)\delta V$, any equation of state must have a pressure that is an increasing function of density. The transition point between pressure isotherms that are monotonically increasing functions of $n$, and those that are not (hence manifestly incorrect) is obtained by the usual conditions of $dP/dn = 0$ and $d^2P/dn^2 = 0$. Starting from the cubic equation of state, we thus obtain

$$\begin{aligned} \frac{dP}{dn} &= k_B T_c - b n_c + \frac{c}{2} n_c^2 = 0 \\ \frac{d^2 P}{dn^2} &= -b + c n_c = 0 \end{aligned} \quad .$$

From the second equation we obtain $n_c = b/c$, which substituted in the first equation gives $k_B T_c = b^2/(2c)$. From the equation of state we then find $P_c = b^3/(6c^2)$, and the dimensionless ratio of

$$\frac{k_B T_c n_c}{P_c} = 3.$$

(b) Calculate the isothermal compressibility $\kappa_T = -\frac{1}{V} \left.\frac{\partial V}{\partial P}\right|_T$, and sketch its behavior as a function of $T$ for $n = n_c$.

• Using $V = N/n$, we get

$$\kappa_T(n) = -\frac{1}{V} \left.\frac{\partial V}{\partial P}\right|_T = \frac{1}{n} \left.\frac{\partial P}{\partial n}\right|_T^{-1} = \left[n \left(k_B T - b n + c n^2/2\right)\right]^{-1} .$$

For $n = n_c$, $\kappa_T(n_c) \propto (T - T_c)^{-1}$, and diverges at $T_c$.

(c) On the critical isotherm give an expression for $(P - P_c)$ as a function of $(n - n_c)$.

• Using the coordinates of the critical point computed above, we find

$$\begin{aligned} P - P_c &= -\frac{b^3}{6c^2} + \frac{b^2}{2c}n - \frac{b}{2}n^2 + \frac{c}{6}n^3 \\ &= \frac{c}{6}\left(n^3 - 3\frac{b}{c}n^2 + 3\frac{b^2}{c^2}n - \frac{b^3}{c^3}\right) \\ &= \frac{c}{6}\left(n - n_c\right)^3 . \end{aligned}$$

(d) The instability in the isotherms for $T < T_c$ is avoided by phase separation into a liquid of density $n_+$ and gas of density $n_-$. For temperatures close to $T_c$, these densities behave as $n_\pm \approx n_c\,(1 \pm \delta)$. Using a Maxwell construction, or otherwise, find an implicit equation for $\delta(T)$, and indicate its behavior for $(T_c - T) \to 0$. (Hint: Along an isotherm, variations of chemical potential obey $d\mu = dP/n$.)

• According to the Gibbs–Duhem relation, the variations of the intensive variables are related by $SdT - VdP + Nd\mu = 0$, and thus along an isotherm ($dT = 0$) $d\mu = dP/n = \partial P/\partial n|_T dn/n$. Since the liquid and gas states are in coexistence they should have the same chemical potential. Integrating the above expression for $d\mu$ from $n_-$ to $n_+$ leads to the so-called Maxwell construction, which reads

$$0 = \mu(n_+) - \mu(n_-) = \int_{n_-}^{n_+} \frac{dP}{n} = \int_{n_c(1-\delta)}^{n_c(1+\delta)} dn \left(\frac{k_B T - bn + cn^2/2}{n}\right).$$

Performing the integrals gives the equation

$$0 = k_B T \ln\left(\frac{1+\delta}{1-\delta}\right) - bn_c(2\delta) + \frac{c}{4}n_c^2\left[(1+\delta)^2 - (1-\delta)^2\right] = k_B T \ln\left(\frac{1+\delta}{1-\delta}\right) - 2k_B T_c \delta,$$

where for the find expression, we have used $n_c = b/c$ and $k_B T_c = b^2/(2c)$. The implicit equation for $\delta$ is thus

$$\delta = \frac{T}{2T_c}\ln\left(\frac{1+\delta}{1-\delta}\right) \approx \frac{T}{T_c}\left(\delta - \delta^3 + \cdots\right).$$

The leading behavior as $(T_c - T) \to 0$ is obtained by keeping up to the cubic term, and given by

$$\delta \approx \sqrt{1 - \frac{T_c}{T}}.$$

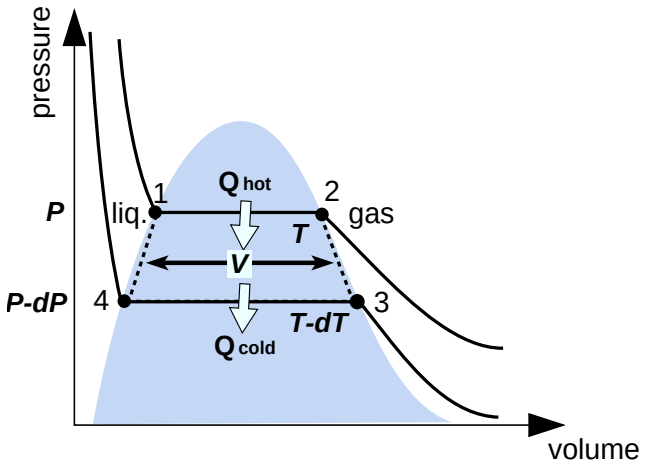
(e) Now consider a gas obeying Dieterici's equation of state:

$$P(v-b) = k_B T \exp\left(-\frac{a}{k_B T v}\right),$$

where $v = V/N$. Find the ratio $Pv/k_BT$ at its critical point.

- The critical point is the point of inflection, described by

$$\left.\frac{\partial P}{\partial v}\right|_{T_c,N} = 0, \qquad \text{and} \qquad \left.\frac{\partial^2 P}{\partial v^2}\right|_{T_c,N} = 0.$$




The first derivative of $P$ is

$$\begin{aligned}\left.\frac{\partial P}{\partial v}\right|_{T_c,N} =& \frac{\partial}{\partial v}\left[\frac{k_BT}{v-b}\exp\left(-\frac{a}{k_BTv}\right)\right] = \frac{k_BT}{v-b}\exp\left(-\frac{a}{k_BTv}\right)\left(\frac{a}{k_BTv^2}-\frac{1}{v-b}\right)\\ =& P\left(\frac{a}{k_BTv^2}-\frac{1}{v-b}\right),\end{aligned}$$

while a second derivative gives

$$\begin{aligned}\left.\frac{\partial^2 P}{\partial v^2}\right|_{T_c,N} &= \frac{\partial}{\partial v}\left[P\left(\frac{a}{k_BTv^2}-\frac{1}{v-b}\right)\right]\\ &= \frac{\partial P}{\partial v}\left(\frac{a}{k_BTv^2}-\frac{1}{v-b}\right) - P\left(\frac{2a}{k_BTv^3}-\frac{1}{(v-b)^2}\right).\end{aligned}$$

Therefore $v_c$ and $T_c$ are determined by

$$\frac{a}{k_BT_cv_c^2}-\frac{1}{v_c-b}=0, \qquad \text{and} \qquad \frac{2a}{k_BT_cv_c^3}-\frac{1}{(v_c-b)^2}=0,$$

with the solutions

$$v_c = 2b, \qquad \text{and} \qquad k_B T_c = \frac{a}{4b}.$$

The critical pressure is

$$P_c = \frac{k_B T_c}{v_c - b} \exp\left(-\frac{a}{k_B T_c v_c}\right) = \frac{a}{4b^2} e^{-2},$$

resulting in the ratio

$$\frac{P_c v_c}{k_B T_c} = 2e^{-2} \approx 0.27.$$

Note that for the van der Waals gas

$$\frac{P_c v_c}{k_B T_c} = \frac{3}{8} = 0.375,$$

while for some actual gases

$$\left(\frac{P_c v_c}{k_B T_c}\right)_{\text{water}} = 0.230, \qquad \text{and} \quad \left(\frac{P_c v_c}{k_B T_c}\right)_{\text{Argon}} = 0.291.$$

(f) Calculate the isothermal compressibility $\kappa_T$ for $v = v_c$ as a function of $T - T_c$ for the Dieterici gas.

- The isothermal compressibility is defined by

$$\kappa_T \equiv -\frac{1}{v} \left.\frac{\partial v}{\partial P}\right|_{T,N},$$

and from part (a), given by

$$\left.\frac{\partial P}{\partial v}\right|_{T_c,N} = P\left(\frac{a}{k_B T v^2} - \frac{1}{v - b}\right).$$

Expanding this expression, at $v = v_c$, in terms of $t \equiv k_B T - k_B T_c$ (for $T > T_c$), yields

$$\left.\frac{\partial P}{\partial v}\right|_{T_c,N} \approx P_c\left(\frac{a}{(a/4b + t)\, 4b^2} - \frac{1}{b}\right) \approx -\frac{P_c}{b}\frac{4bt}{a} = -\frac{2P_c}{v_c k_B T_c} t,$$

and thus

$$\kappa_T = \frac{k_B T_c}{2P_c}\frac{1}{t} = \frac{be^2}{2k_B(T - T_c)}.$$

Note that expanding any analytic equation of state will yield the same simple pole for the divergence of the compressibility.

(g) On the Dieterici critical isotherm expand the pressure to the lowest non-zero order in $(v - v_c)$.

- Perform a Taylor–series expansion along the critical isotherm $T = T_c$, as

$$P(v, T_c) = P_c + \left.\frac{\partial P}{\partial v}\right|_{T_c, v_c} (v - v_c) + \frac{1}{2!} \left.\frac{\partial^2 P}{\partial v^2}\right|_{T_c, v_c} (v - v_c)^2 + \frac{1}{3!} \left.\frac{\partial^3 P}{\partial v^3}\right|_{T_c, v_c} (v - v_c)^3 + \cdots .$$

The first two terms are zero at the critical point, and

$$\begin{aligned} \left.\frac{\partial^3 P}{\partial v^3}\right|_{T_c, v_c} &= -P_c \frac{\partial}{\partial v} \left( \frac{2a}{k_B T_c v^3} - \frac{1}{(v-b)^2} \right) \\ &= -P_c \left( \frac{6a}{k_B T_c v_c^4} - \frac{2}{(v_c - b)^3} \right) \\ &= -\frac{P_c}{2b^3}. \end{aligned}$$

Substituting this into the Taylor expansion for $P(v, T_c)$, results in

$$P(v, T_c) = P_c \left( 1 - \frac{(v - v_c)^3}{12 b^3} \right),$$

which is equivalent to

$$\frac{P}{P_c} - 1 = \frac{2}{3} \left( \frac{v}{v_c} - 1 \right)^3 .$$

********

**6.** *Magnetic thin films:* A crystalline film (simple cubic) is obtained by depositing a finite number of layers $n$. Each atom has a three component (Heisenberg) spin, and they interact through the Hamiltonian

$$-\beta \mathcal{H} = \sum_{\alpha=1}^{n} \sum_{<i,j>} J_H \vec{s}_i^{\,\alpha} \cdot \vec{s}_j^{\,\alpha} + \sum_{\alpha=1}^{n-1} \sum_{i} J_V \vec{s}_i^{\,\alpha} \cdot \vec{s}_i^{\,\alpha+1}.$$

(The unit vector $\vec{s}_i^{\,\alpha}$ indicates the spin at site $i$ in the $\alpha$th layer. The subscript $< i, j >$ indicates that the spin at $i$ interacts with its 4 nearest-neighbors, indexed by $j$ on the square lattice on the same layer.) A mean–field approximation is obtained from the variational density $\rho_0 \propto \exp\left(-\beta \mathcal{H}_0\right)$, with the trial Hamiltonian

$$-\beta \mathcal{H}_0 = \sum_{\alpha=1}^{n} \sum_{i} \vec{h}^{\,\alpha} \cdot \vec{s}_i^{\,\alpha}.$$

(Note that the most general single–site Hamiltonian may include the higher order terms $L^{\alpha}_{c_1,\cdots,c_p} s^{\alpha}_{c_1} \cdots s^{\alpha}_{c_1}$, where $s_c$ indicates component $c$ of the vector $\vec{s}$.)

(a) Calculate the partition function $Z_0\left(\left\{\vec{h}^{\,\alpha}\right\}\right)$, and $\beta F_0 = -\ln Z_0$.

• The partition function $Z_0$ is obtained by integrating over all angles as

$$\begin{aligned} Z_0 &= \prod_{i,\alpha}\left[\int d\vec{s}_i^{\,\alpha} e^{\vec{h}^{\,\alpha}\cdot\vec{s}_i^{\,\alpha}}\right] = \prod_{i,\alpha}\left[\int_0^{2\pi} d\phi \int_0^{\pi} d\cos\theta e^{h_\alpha \cos\theta}\right] \\ &= \prod_{i,\alpha}\left[2\pi\int_{-1}^{1} dx e^{h_\alpha x}\right] = \prod_{i,\alpha}\left[2\pi\frac{e^{h_\alpha}-e^{-h_\alpha}}{h_\alpha}\right] \\ &= \left(\frac{4\pi\sinh h_\alpha}{h_\alpha}\right)^N, \end{aligned}$$

where $N$ is the number of sites in each layer.

(b) Obtain the magnetizations $m_\alpha = \left|\langle\vec{s}_i^{\,\alpha}\rangle_0\right|$, and $\langle\beta\mathcal{H}_0\rangle_0$, in terms of the *Langevin function* $\mathcal{L}(h) = \coth(h) - 1/h$.

• From the partition function, we easily obtain the magnetizations as

$$\vec{m}_\alpha = \langle\vec{s}_i^{\,\alpha}\rangle_0 = \frac{1}{N}\frac{\partial \ln Z_0}{\partial \vec{h}^{\,\alpha}} = \hat{h}^{\,\alpha}\frac{\partial}{\partial h_\alpha}\ln\frac{\sinh h_\alpha}{h_\alpha} = \hat{h}^{\,\alpha}\left[\coth h_\alpha - 1/h_\alpha\right] = \hat{h}^{\,\alpha}\mathcal{L}(h_\alpha).$$

(c) Calculate $\langle\beta\mathcal{H}\rangle_0$, with the (reasonable) assumption that all the variational fields $\left(\left\{\vec{h}^{\,\alpha}\right\}\right)$ are parallel.

• For the zeroth order weight the spins at different sites are independent random variables, and

$$\langle\beta\mathcal{H}\rangle_0 = J_H\sum_{\alpha=1}^{n}\sum_{<i,j>}\langle\vec{s}_i^{\,\alpha}\rangle_0\cdot\left\langle\vec{s}_j^{\,\alpha}\right\rangle_0 + J_V\sum_{\alpha=1}^{n-1}\sum_i\langle\vec{s}_i^{\,\alpha}\rangle_0\cdot\left\langle\vec{s}_i^{\,\alpha+1}\right\rangle_0.$$

Noting that $\langle\vec{s}_i^{\,\alpha}\rangle_0 = m_\alpha\hat{h}$ with all spins on average pointing in the same direction, and that each spin has 4 neighbors in the plane on a square lattice, we obtain

$$\begin{aligned} \langle\beta\mathcal{H}\rangle_0 &= J_H\frac{4N}{2}\sum_{\alpha=1}^{n} m_\alpha m_\alpha + J_V N\sum_{\alpha=1}^{n-1} m_\alpha m_{\alpha+1} \\ &= N\sum_{\alpha=1}^{n}\left(2J_H m_\alpha^2 + J_V m_\alpha m_{\alpha+1}\right), \end{aligned}$$

with $m_{n+1} = 0$.

(d) The exact free energy, $\beta F = -\ln Z$, satisfies the Gibbs inequality (see below), $\beta F \leq \beta F_0 + \langle \beta\mathcal{H} - \beta\mathcal{H}_0 \rangle_0$. Give the self-consistent equations for the magnetizations $\{m_\alpha\}$ that optimize $\beta\mathcal{H}_0$. How would you solve these equations numerically?

• According to the Gibbs inequality $\beta F \leq N\Psi(\{m_\alpha\})$, with

$$\Psi(\{m_\alpha\}) = \sum_\alpha \left[ -\ln\left(\frac{4\pi \sinh h_\alpha}{h_\alpha}\right) + h_\alpha m_\alpha - 2J_H m_\alpha^2 - J_V m_\alpha m_{\alpha+1} \right],$$

with $m_\alpha = \mathcal{L}(h_\alpha)$. The best variational choice for $\{h_\alpha\}$ is obtained by minimizing $\Psi$, and thus

$$\frac{\partial \Psi}{\partial m_\alpha} = -\mathcal{L}(h_\alpha)\frac{dh_\alpha}{dm_\alpha} + m_\alpha \frac{dh_\alpha}{dm_\alpha} + h_\alpha - 4J_H m_\alpha - J_V\left(m_{\alpha+1} + m_{\alpha-1}\right) = 0.$$

After canceling the first two terms, the above equation can be re-written as

$$h_\alpha = \mathcal{L}^{-1}(m_\alpha) = 4J_H m_\alpha + J_V\left(m_{\alpha+1} + m_{\alpha-1}\right),$$

which is the effective field acting on a site in layer $\alpha$.

(e) Find the critical temperature, and the behavior of the magnetization *in the bulk* by considering the limit $n \to \infty$. (Note that $\lim_{m\to 0} \mathcal{L}^{-1}(m) = 3m + 9m^3/5 + \mathcal{O}(m^5)$.)

• When examining the bulk, we can drop the index $\alpha$, and using the expansion of $\mathcal{L}^{-1}(m)$ we obtain

$$3m + \frac{9}{5}m^3 + \cdots = (4J_H + J_V)m.$$

The condition for criticality is the equality of the linear terms, i.e. $4J_H + 2J_V = 3$. Setting $J_H = \hat{J}_H/k_BT$, and $J_V = \hat{J}_V/k_BT$, this gives $k_BT_c = (4\hat{J}_H + 2\hat{J}_V)/3$. For $T < T_c$, we have

$$\frac{9}{5}m^3 = 3\left(\frac{T_c}{T} - 1\right) \equiv 3t, \qquad \Longrightarrow \qquad m = \pm\sqrt{5t/3},$$

i.e. the usual saddle-point exponent of $\beta = 1/2$ is recovered.

(f) By linearizing the self-consistent equations, show that the critical temperature of film depends on the number of layers $n$, as $kT_c(n \gg 1) \approx kT_c(\infty) - J_V\pi^2/(3n^2)$.

• For a finite number of layers, the linearized recursion relations are

$$\mathcal{L}^{-1}(m_\alpha) = 3m_\alpha + \mathcal{O}(m^3) = 4J_H m_\alpha + J_V\left(m_{\alpha+1} + m_{\alpha-1}\right).$$

These equations have to be supplemented by boundary conditions $m_0 = m_{n+1} = 0$. The solution to the linearized equations takes the form

$$m_\alpha = \mu \sin\left(\frac{\alpha\pi}{n+1}\right),$$

which gives the profile of magnetization at the critical point. It can be checked that this is a solution to the linearized equations, by noting that

$$\begin{aligned} 3\mu \sin\left(\frac{\alpha\pi}{n+1}\right) + \cdots &= 4J_H \mu \sin\left(\frac{\alpha\pi}{n+1}\right) + J_V \mu \left[\sin\left(\frac{(\alpha+1)\pi}{n+1}\right) + \sin\left(\frac{(\alpha-1)\pi}{n+1}\right)\right] \\ &= 4J_H \mu \sin\left(\frac{\alpha\pi}{n+1}\right) + J_V \mu \left[2\sin\left(\frac{\alpha\pi}{n+1}\right)\cos\left(\frac{\pi}{n+1}\right)\right]. \end{aligned}$$

Thus the linear terms match is

$$4J_H + 2J_V \cos\left(\frac{\pi}{n+1}\right) = 3,$$

giving the critical temperature of the finite system as

$$k_B T_c(n) = \left[\hat{J}_H + 2J_V \cos\left(\frac{\pi}{n+1}\right)\right]/3 \approx k_B T_c(\infty) - \frac{\pi^2 \hat{J}_v}{3n^2} + \cdots,$$

where the last expression is obtained by expanding the cosine for large $n$.

(g) Derive a continuum form of the self-consistent equations, and keep terms to cubic order in $m$. Show that the resulting non-linear differential equation has a solution of the form $m(x) = m_{\rm bulk} \tanh(kx)$. What circumstances are described by this solution?

• The continuum limit is obtained by replacing the discrete layer number $\alpha$ with a continuous coordinate $z$, such that $m_\alpha \to m(x)$, and

$$(m_{\alpha+1} + m_{\alpha-1}) \approx 2m(x) + \frac{d^2 m}{dx^2} + \cdots.$$

In this limit the discrete set of equations are replaced with the differential equation

$$(4J_H + 2J_V)\, m(x) + J_V \frac{d^2 m}{dx^2} + \cdots = 3m + \frac{9}{5} m^3 + \cdots.$$

To get the profile of a large system close to criticality the leading terms included above are sufficient. It is in general not possible to give closed form solutions to non-linear equations. However, it is easy to check that $m(x) = m_{\rm bulk} \tanh(kx)$ satisfies this equation, using

$$\frac{dm}{dx} = m_{\rm bulk} k \left(1 - \tanh^2 kx\right), \quad \text{and} \quad \frac{d^2 m}{dx^2} = -2 m_{\rm bulk} k^2 \tanh(kx) \left(1 - \tanh^2 kx\right).$$

Substituting in the differential equation gives

$$(4J_H + 2J_V)\, m_{\rm bulk} \tanh(kx) - 2m_{\rm bulk} k^2 J_V \left(\tanh kx - \tanh^3 kx\right) = 3m_{\rm bulk} \tanh(kx) + \frac{9}{5} m_{\rm bulk}^3 \tanh^3 kx.$$

Since the equality must hold irrespective of $x$, we must equate the coefficients of $\tanh kx$ and $\tanh^3 kx$ at each point. From the terms of order $\tanh kx$, we get

$$(4J_H + 2J_V) - 2k^2 J_V = 0, \quad \Longrightarrow \quad k^2 = \frac{3 - (4J_H + 2J_V)}{2J_V},$$

while at order of $\tanh^3 kx$, we get

$$m_{\rm bulk}^2 = \frac{10}{9} k^2 J_V = \frac{5}{9} \left[3 - (4J_H + 2J_V)\right].$$

The latter agrees with the bulk magnetization obtained earlier, validating the consistency of this solution. This profile corresponds to a domain wall in the system, located at $x = 0$, separating a down magnetized phase at $x \to -\infty$, and an up magnetized phase at $x \to +\infty$.

(h) How can the above solution be modified to describe a *semi–infinite* system? Obtain the critical behaviors of the healing length $\lambda \sim 1/k$.

• As a first approximation to a semi–infinite system, we can take one half of the above solution, say for $x > 0$. The magnetization is small (almost zero) at the surface, and recovers to the bulk value over a distance $\lambda \sim 1/k \propto t^{-1/2}$. Note that the divergence of the healing length is precisely the same as the bulk correlation length.

(i) Show that the magnetization of the surface layer vanishes as $|T - T_c|$.

• A more accurate answer is obtained by noting that in the discrete system, the coupling to neighboring layers comes from $m_{\alpha-1} + m_{\alpha+1}$. For the equation at the surface layer, we have $m_{-1} + m_1$, and we must set $m_{-1} = 0$. Thus a more accurate solution is actually shifted by one lattice spacing, as $m(x \geq 0) = m_{\rm bulk} \tanh[k(x + a)]$ for a lattice spacing $a$. Expanding the tanh gives

$$m_0 = m_{\rm surface} \approx m_{\rm bulk} ka \propto t.$$

The result in (f) illustrates a quite general result that the transition temperature of a finite system of size $L$, approaches its asymptotic (infinite–size) limit from below, as $T_c(L) = T_C(\infty) - A/L^{1/\nu}$, where $\nu$ is the exponent controlling the divergence of the correlation length. However, some liquid crystal films appeared to violate this behavior. In fact, in these films the couplings are stronger on the surface layers, which thus order before the bulk. For a discussion of the dependence of $T_c$ on the number of layers in this case, see H. Li, M. Paczuski, M. Kardar, and K. Huang, Phys. Rev. B **44**, 8274 (1991).

********

**Solutions to problems from chapter 2- Statistical Fields**

**1.** *Cubic invariants:* When the order parameter $m$, goes to zero discontinuously, the phase transition is said to be first order (discontinuous). A common example occurs in systems where symmetry considerations do not exclude a cubic term in the Landau free energy, as in

$$\beta\mathcal{H} = \int d^d\mathbf{x} \left[ \frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 + cm^3 + um^4 \right] \qquad (K, c, u > 0).$$

(a) By plotting the energy density $\Psi(m)$, for uniform $m$ at various values of $t$, show that as $t$ is reduced there is a discontinuous jump to $\overline{m} \neq 0$ for a positive $\overline{t}$ in the saddle–point approximation.

• To simplify the algebra, let us rewrite the energy density $\Psi(m)$, for uniform $m$, in terms of the rescaled quantity

$$m_r = \frac{u}{c}m.$$

In this way, we can eliminate the constant parameters $c$, and $u$, to get the expression of the energy density as

$$\Psi_r(m_r) = \frac{1}{2}t_r m_r^2 + m_r^3 + m_r^4,$$

where we have defined

$$\Psi_r = \left(\frac{c^4}{u^3}\right)\Psi, \qquad \text{and} \qquad t_r = \left(\frac{u}{c^2}\right)t.$$

To obtain the extrema of $\Psi_r$, we set the first derivative with respect to $m_r$ to zero, i.e.

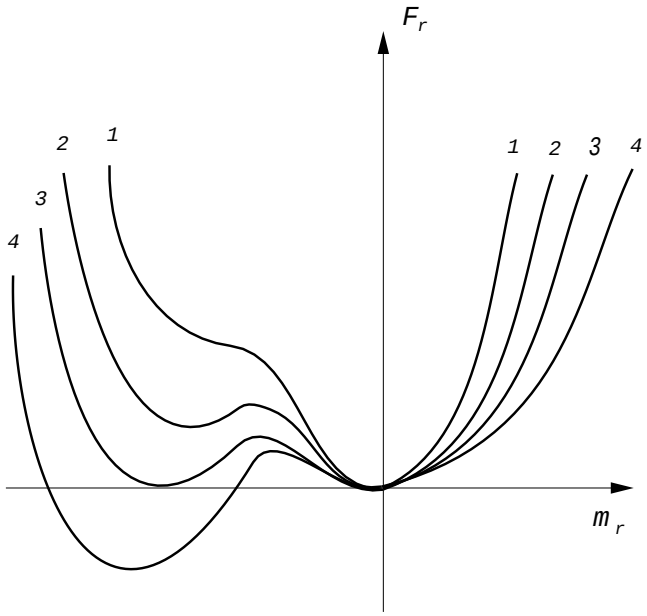
$$\frac{d\Psi_r(m_r)}{dm_r} = m_r\left(t_r + 3m_r + 4m_r^2\right) = 0.$$

The trivial solution of this equation is $m_r^* = 0$. But if $t_r \leq 9/16$, the derivative vanishes also at $m_r^* = (-3 \pm \sqrt{9 - 16t_r})/8$. Provided that $t_r > 0$, $m_r^* = 0$ is a minimum of the function $\Psi_r(m_r)$. In addition, if $t_r < 9/16$, $\Psi_r(m_r)$ has another minimum at

$$m_r^* = -\frac{3 + \sqrt{9 - 16t_r}}{8},$$

and a maximum, located in between the two minima, at

$$m_r^* = \frac{-3 + \sqrt{9 - 16t_r}}{8}.$$




The accompanying figure depicts the behavior of $\Psi_r(m_r)$ for different values of $t_r$.

1. For $t_r > 9/16$, there is only one minimum $m_r^* = 0$ .
2. For $0 < \overline{t}_r < t_r < 9/16$, there are two minima, but $\Psi_r(m_r^*) > \Psi_r(0) = 0$.
3. For $0 < t_r = \overline{t}_r$, $\Psi_r(m_r^*) = \Psi_r(0) = 0$.
4. For $0 < t_r < \overline{t}_r$, $\Psi_r(m_r^*) < \Psi_r(0) = 0$.

The discontinuous transition occurs when the local minimum at $m_r^* < 0$ becomes the absolute minimum. There is a corresponding jump of $m_r$, from $m_r^* = 0$ to $m_r^* = \overline{m}_r$, where $\overline{m}_r = m_r^*(t_r = \overline{t}_r)$.

(b) By writing down the two conditions that $\overline{m}$ and $\overline{t}$ must satisfy at the transition, solve for $\overline{m}$ and $\overline{t}$.

• To determine $\overline{m}_r$ and $\overline{t}_r$, we have to simultaneously solve the equations

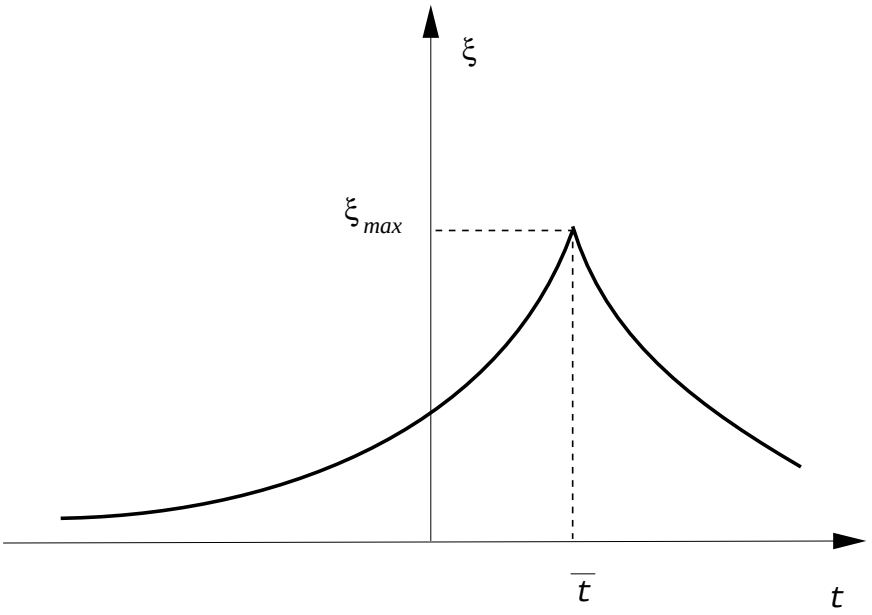
$$\frac{d\Psi_r(m_r)}{dm_r} = 0, \qquad \text{and} \qquad \Psi_r(m_r) = \Psi_r(0) = 0.$$

Excluding the trivial solution $m_r^* = 0$, from

$$\begin{cases} t_r + 3m_r + 4m_r^2 = 0 \\ \dfrac{t_r}{2} + m_r + m_r^2 = 0 \end{cases},$$

we obtain $\overline{t}_r = -\overline{m}_r = 1/2$, or in the original units,

$$\overline{t} = \frac{c^2}{2u}, \qquad \text{and} \qquad \overline{m} = -\frac{c}{2u}.$$




(c) Recall that the correlation length $\xi$ is related to the curvature of $\Psi(m)$ at its minimum by $K\xi^{-2} = \partial^2\Psi/\partial m^2|_{eq.}$. Plot $\xi$ as a function of $t$.

•

Likewise, the equilibrium value of $m = m_{eq}$ in the original units equals to

$$m_{eq} = \begin{cases} 0 & \text{for} \quad t > \overline{t} = \dfrac{c^2}{2u}, \\ -\left(\dfrac{c}{u}\right)\dfrac{3+\sqrt{9-16ut/c^2}}{8} & \text{for} \quad t < \overline{t}. \end{cases}$$

The correlation length $\xi$, is related to the curvature of $\Psi(m)$ at its equilibrium minimum by

$$K\xi^{-2} = \left.\frac{\partial^2\Psi}{\partial m^2}\right|_{m_{eq}} = t + 6cm_{eq} + 12um_{eq}^2,$$

which is equal to

$$\xi = \begin{cases} \left(\dfrac{K}{t}\right)^{1/2} & \text{if} \quad t > \overline{t}, \\ \left(-\dfrac{K}{2t+3cm_{eq}}\right)^{1/2} & \text{if} \quad t < \overline{t}. \end{cases}$$

(To arrive to the last expression, we have used$d\Psi(m)/dm|_{m=m_{eq}} = 0$.)

$$\xi_{max} = \xi(\overline{t}) = \frac{\sqrt{2Ku}}{c}.$$

A plot of $\xi$ as a function of $t$ is presented here. Note that the correlation length $\xi$, is finite at the discontinuous phase transition, attaining a maximum value of

********

**2.** *Tricritical point:* By tuning an additional parameter, a second order transition can be made first order. The special point separating the two types of transitions is known as a tricritical point, and can be studied by examining the Landau–Ginzburg Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x} \left[ \frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 + um^4 + vm^6 - hm \right],$$

where $u$ can be positive or negative. For $u < 0$, a positive $v$ is necessary to ensure stability.

(a) By sketching the energy density $\Psi(m)$, for various $t$, show that in the saddle–point approximation there is a first-order transition for $u < 0$ and $h = 0$.

- If we consider $h = 0$, the energy density $\Psi(m)$, for uniform $m$, is

$$\Psi(m) = \frac{t}{2}m^2 + um^4 + vm^6.$$

As in the previous problem, to obtain the extrema of $\Psi$, let us set the first derivative with respect to $m$ to zero. Again, provided that $t > 0$, $\Psi(m)$ has a minimum at $m^* = 0$. But the derivative also vanishes for other nonzero values of $m$ as long as certain conditions are satisfied. In order to find them, we have to solve the following equation

$$t + 4um^2 + 6vm^4 = 0,$$

from which,

$$m^{*2} = -\frac{u}{3v} \pm \frac{\sqrt{4u^2 - 6tv}}{6v}.$$

Thus, we have real and positive solutions provided that

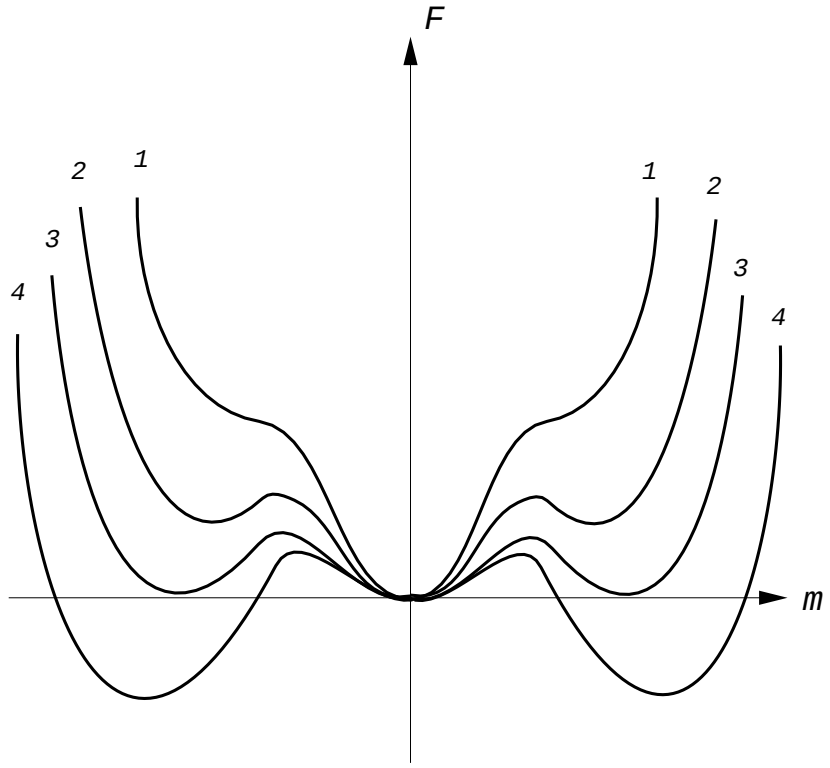
$$u < 0, \qquad \text{and} \qquad t < \frac{2u^2}{3v}.$$

Under these conditions $\Psi(m)$ has another two minima at

$$m^{*2} = \frac{|u|}{3v} + \frac{\sqrt{4u^2 - 6tv}}{6v},$$

and two maxima at

$$m^{*2} = \frac{|u|}{3v} - \frac{\sqrt{4u^2 - 6tv}}{6v},$$




as depicted in the accompanying figure.

The different behaviors of the function $\Psi(m)$ are as follows:

1. For $t > 2u^2/3v$, there is only one minimum $m^* = 0$.
2. For $0 < \overline{t} < t < 2u^2/3v$, there are three minima, but $\Psi(\pm m^*) > \Psi(0) = 0$.
3. For $0 < t = \overline{t}$, $\Psi(\pm m^*) = \Psi(0) = 0$.
4. For $0 < t < \overline{t}$, $\Psi(\pm m^*) < \Psi(0) = 0$.

There is a thus discontinuous phase transition for $u < 0$, and $t = \overline{t}(u)$.

(b) Calculate $\overline{t}$ and the discontinuity $\overline{m}$ at this transition.

• To determine $\overline{t}$, and $\overline{m} = m^*(t = \overline{t})$, we again have to simultaneously solve the equations

$$\frac{d\Psi(m)}{dm^2} = 0, \qquad \text{and} \qquad \Psi(m^2) = \Psi(0) = 0,$$

or equivalently,

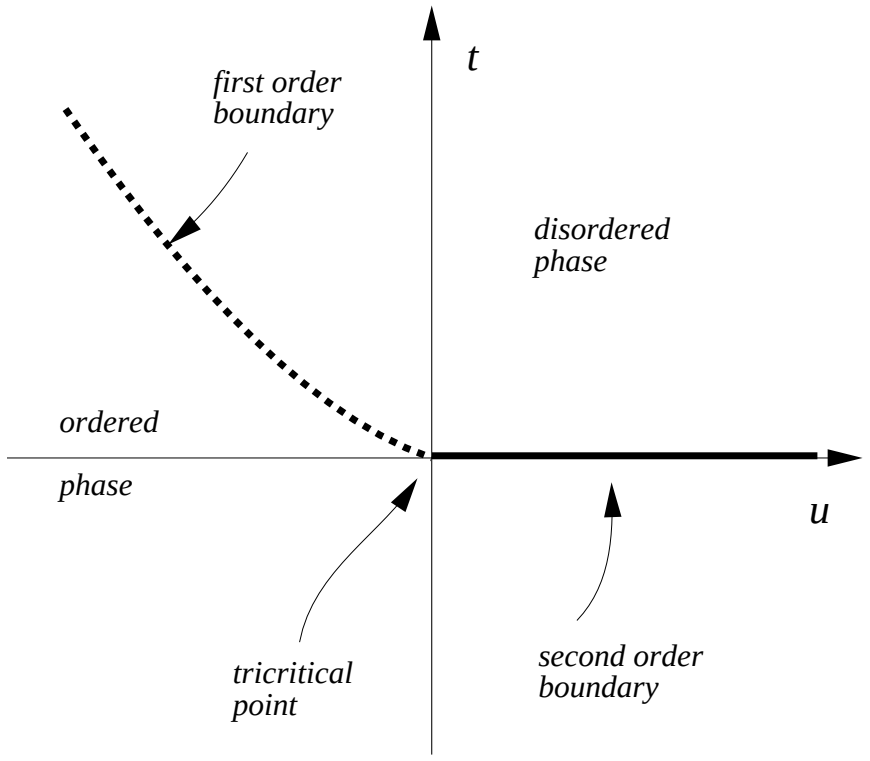
$$\begin{cases} \dfrac{t}{2} + 2um^2 + 3vm^4 = 0 \\ \dfrac{t}{2} + um^2 + vm^4 = 0 \end{cases},$$

from which we obtain

$$\overline{t} = \frac{u^2}{2v}, \qquad \text{and} \quad \overline{m}^2 = -\frac{u}{2v} = \frac{|u|}{2v}.$$

(c) For $h = 0$ and $v > 0$, plot the phase boundary in the $(u, t)$ plane, identifying the phases, and order of the phase transitions.

• In the $(u, t)$ plane, the line $t = u^2/2v$ for $u < 0$, is a first-order phase transition boundary. In addition, the line $t = 0$ for $u > 0$, defines a second-order phase transition boundary, as indicated in the accompanying figure.




(d) The special point $u = t = 0$, separating first– and second–order phase boundaries, is a *tricritical* point. For $u = 0$, calculate the tricritical exponents $\beta$, $\delta$, $\gamma$, and $\alpha$, governing the singularities in magnetization, susceptibility, and heat capacity. (Recall: $C \propto t^{-\alpha}$; $\overline{m}(h = 0) \propto t^{\beta}$; $\chi \propto t^{-\gamma}$; and $\overline{m}(t = 0) \propto h^{1/\delta}$.)

• For $u = 0$, let us calculate the tricritical exponents $\alpha$, $\beta$, $\gamma$, and $\delta$. In order to calculate $\alpha$ and $\beta$, we set $h = 0$, so that

$$\Psi(m) = \frac{t}{2} m^2 + v m^6.$$

Thus from

$$\left.\frac{\partial \Psi}{\partial m}\right|_{\overline{m}} = \overline{m}\left(t + 6v\overline{m}^4\right) = 0,$$

we obtain,

$$\overline{m} = \begin{cases} 0 & \text{for} \quad t > \overline{t} = 0, \\ \left(-\dfrac{t}{6v}\right)^{1/4} & \text{for} \quad t < 0 \end{cases},$$

resulting in,

$$\overline{m}(h=0) \propto t^{\beta}, \qquad \text{with} \qquad \beta = \frac{1}{4}.$$

The corresponding free energy density scales as

$$\Psi(\overline{m}) \sim \overline{m}^{\,6} \propto (-t)^{3/2}.$$

The tricritical exponent $\alpha$ characterizes the non-analytic behavior of the heat capacity $C \sim (\partial^2\Psi/\partial T^2)|_{h=0,\overline{m}}$, and since $t \propto (T - T_c)$,

$$C \sim \left.\frac{\partial^2\Psi}{\partial t^2}\right|_{h=0,\overline{m}} \propto t^{-\alpha}, \qquad \text{with} \qquad \alpha = \frac{1}{2}.$$

To calculate the tricritical exponent $\delta$, we set $t = 0$ while keeping $h \neq 0$, so that

$$\Psi(m) = vm^6 - hm.$$

Thus from

$$\left.\frac{\partial\Psi}{\partial m}\right|_{\overline{m}} = 6v\overline{m}^5 - h = 0,$$

we obtain,

$$\overline{m} \propto h^{1/\delta}, \qquad \text{with} \qquad \delta = 5.$$

Finally, for $h \neq 0$ and $t \neq 0$,

$$\left.\frac{\partial\Psi}{\partial m}\right|_{\overline{m}} = t\overline{m} + 6v\overline{m}^5 - h = 0,$$

so that the susceptibility scales as

$$\chi = \left.\frac{\partial\overline{m}}{\partial h}\right|_{h=0} \propto |t|^{-1}, \qquad \text{for both} \qquad t < 0 \qquad \text{and} \qquad t > 0,$$

i.e. with the exponents $\gamma_{\pm} = 1$.

********

**3.** *Transverse susceptibility:* An $n$–component magnetization field $\vec{m}(\mathbf{x})$ is coupled to an external field $\vec{h}$ through a term $-\int d^d\mathbf{x}\ \vec{h}\cdot\vec{m}(\mathbf{x})$ in the Hamiltonian $\beta\mathcal{H}$. If $\beta\mathcal{H}$ for $\vec{h} = 0$

is invariant under rotations of $\vec{m}(\mathbf{x})$; then the free energy density ($f = -\ln Z/V$) only depends on the absolute value of $\vec{h}$; i.e. $f(\vec{h}) = f(h)$, where $h = |\vec{h}|$.

(a) Show that $m_\alpha = \langle \int d^d\mathbf{x} m_\alpha(\mathbf{x})\rangle/V = -h_\alpha f'(h)/h$.

• The magnetic work is the product of the magnetic field and the magnetization density, and appears as the argument of the exponential weight in the (Gibbs) canonical ensemble. We can thus can "lower" the magnetization $M = \int d^d x m_\alpha(\mathbf{x})$ "inside the average" by taking derivatives of the (Gibbs) partition function with respect to $h_\alpha$, as

$$\begin{aligned} m_\alpha &= \frac{1}{V}\left\langle \int d^d x m_\alpha(\mathbf{x}) \right\rangle = \frac{1}{V}\frac{\int \mathcal{D}\mathbf{m}(\mathbf{x})\left(\int d^d x' m_\alpha(\mathbf{x}')\right) e^{-\beta\mathcal{H}}}{\int \mathcal{D}\mathbf{m}(\mathbf{x})\, e^{-\beta\mathcal{H}}} \\ &= \frac{1}{V}\frac{1}{Z}\frac{1}{\beta}\frac{\partial}{\partial h_\alpha}\int \mathcal{D}\mathbf{m}(\mathbf{x})\, e^{-\beta\mathcal{H}} = \frac{1}{\beta V}\frac{\partial}{\partial h_\alpha}\ln Z = -\frac{\partial f}{\partial h_\alpha}. \end{aligned}$$

For an otherwise rotationally symmetric system, the (Gibbs) free energy depends only on the magnitude of $\mathbf{h}$, and using

$$\frac{\partial h}{\partial h_\alpha} = \frac{\partial\sqrt{h_\beta h_\beta}}{\partial h_\alpha} = \frac{1}{2}\frac{2\delta_{\alpha\beta}h_\beta}{\sqrt{h_\beta h_\beta}} = \frac{h_\alpha}{h},$$

we obtain

$$m_\alpha = -\frac{\partial f}{\partial h_\alpha} = -\frac{df}{dh}\frac{\partial h}{\partial h_\alpha} = -f'\frac{h_\alpha}{h}.$$

(b) Relate the susceptibility tensor $\chi_{\alpha\beta} = \partial m_\alpha/\partial h_\beta$, to $f''(h)$, $\vec{m}$, and $\vec{h}$.

• The susceptibility tensor is now obtained as

$$\begin{aligned} \chi_{\alpha\beta} &= \frac{\partial m_\alpha}{\partial h_\beta} = \frac{\partial}{\partial h_\beta}\left(-\frac{h_\alpha}{h}f'(h)\right) = -\frac{\partial h_\alpha}{\partial h_\beta}\frac{1}{h}f' - \frac{\partial h^{-1}}{\partial h_\beta}h_\alpha f' - \frac{h_\alpha}{h}\frac{\partial f'}{\partial h_\beta} \\ &= -\left(\delta_{\alpha\beta} - \frac{h_\alpha h_\beta}{h^2}\right)\frac{f'}{h} - \frac{h_\alpha h_\beta}{h^2}f''. \end{aligned}$$

In order to express $f'$ in terms of the magnetization, we take the magnitude of the result of part (a),

$$m = |f'(h)| = -f'(h),$$

from which we obtain

$$\chi_{\alpha\beta} = \left(\delta_{\alpha\beta} - \frac{h_\alpha h_\beta}{h^2}\right)\frac{m}{h} + \frac{h_\alpha h_\beta}{h^2}\frac{dm}{dh}.$$

(c) Show that the transverse and longitudinal susceptibilities are given by $\chi_t = m/h$ and $\chi_\ell = -f''(h)$; where $m$ is the magnitude of $\vec{m}$.

• Since the matrix $\left(\delta_{\alpha\beta} - h_\alpha h_\beta / h^2\right)$ removes the projection of any vector along the magnetic field, we conclude

$$\begin{cases} \chi_\ell = -f''(h) = \dfrac{dm}{dh} \\ \chi_t = \dfrac{m}{h} \end{cases} .$$

Alternatively, we can choose the coordinate system such that $h_i = h\delta_{i1}$ $(i = 1, \ldots, d)$, to get

$$\begin{cases} \chi_\ell = \chi_{11} = \left(\delta_{11} - \dfrac{h_1 h_1}{h^2}\right)\dfrac{m}{h} - \dfrac{h_1 h_1}{h^2} f''(h) = \dfrac{dm}{dh} \\ \chi_t = \chi_{22} = \left(\delta_{11} - \dfrac{h_2 h_2}{h^2}\right)\dfrac{m}{h} - \dfrac{h_2 h_2}{h^2} f''(h) = \dfrac{m}{h} \end{cases} .$$

(d) Conclude that $\chi_t$ diverges as $\vec{h} \to 0$, whenever there is a spontaneous magnetization. Is there any similar a priori reason for $\chi_\ell$ to diverge?

• Provided that $\lim_{h\to 0} m \neq 0$, the transverse susceptibility clearly diverges for $h \to 0$. There is no similar reason, on the other hand, for the longitudinal susceptibility to diverge. In the saddle point approximation of the Landau–Ginzburg model, for example, we have

$$tm + 4um^3 + h = 0,$$

implying (since $4um^2 = -t$ at $h = 0$, for $t < 0$) that

$$\chi_\ell|_{h=0} = \left(\frac{dh}{dm}\right)^{-1}\bigg|_{h=0} = (t - 3t)^{-1}, \qquad i.e. \qquad \chi_\ell = \frac{1}{2\,|\mathrm{t}|},$$

at zero magnetic field, in the ordered phase ($t < 0$).

**NOTE:** Another, more pictorial approach to this problem is as follows. Since the Hamiltonian is invariant under rotations about $\mathbf{h}$, $\mathbf{m}$ must be parallel to $\mathbf{h}$, *i.e.*

$$m_\alpha = \frac{h_\alpha}{h}\varphi(h),$$

where $\varphi$ is some function of the magnitude of the magnetic field. For simplicity, let $\mathbf{h} = h\mathbf{e}_1$, with $\mathbf{e}_1$ a unit vector, implying that

$$\mathbf{m} = m\mathbf{e}_1 = \varphi(h)\,\mathbf{e}_1.$$

The longitudinal susceptibility is then calculated as

$$\chi_\ell = \frac{\partial m_1}{\partial h_1}\bigg|_{\mathbf{h}=h\mathbf{e}_1} = \frac{dm}{dh} = \varphi'(h).$$

To find the transverse susceptibility, we first note that if the system is perturbed by a small external magnetic field $\delta h \mathbf{e}_2$, the change in $m_1$ is, by symmetry, the same for $\delta h > 0$ and $\delta h < 0$, implying

$$m_1 \left( h\mathbf{e}_1 + \delta h \mathbf{e}_2 \right) = m_1 \left( h\mathbf{e}_1 \right) + \mathcal{O}\left( \delta h^2 \right).$$

Hence

$$\left. \frac{\partial m_1}{\partial h_2} \right|_{\mathbf{h}=h\mathbf{e}_1} = 0.$$

Furthermore, since $\mathbf{m}$ and $\mathbf{h}$ are parallel,

$$\frac{m_1 \left( h\mathbf{e}_1 + \delta h \mathbf{e}_2 \right)}{h} = \frac{m_2 \left( h\mathbf{e}_1 + \delta h \mathbf{e}_2 \right)}{\delta h},$$

from which

$$m_2 \left( h\mathbf{e}_1 + \delta h \mathbf{e}_2 \right) = \frac{m_1 \left( h\mathbf{e}_1 \right)}{h} \delta h + \mathcal{O}\left( \delta h^3 \right),$$

yielding

$$\chi_t = \left. \frac{\partial m_2}{\partial h_2} \right|_{\mathbf{h}=h\mathbf{e}_1} = \frac{m}{h}.$$

********

---

**4.** *Superfluid $He^4$–$He^3$ mixtures:* The superfluid $He^4$ order parameter is a complex number $\psi(\mathbf{x})$ . In the presence of a concentration $c(\mathbf{x})$ of $He^3$ impurities, the system has the following Landau–Ginzburg energy

$$\beta \mathcal{H}[\psi, c] = \int d^d\mathbf{x} \left[ \frac{K}{2} \left| \nabla \psi \right|^2 + \frac{t}{2} \left| \psi \right|^2 + u \left| \psi \right|^4 + v \left| \psi \right|^6 + \frac{c(\mathbf{x})^2}{2\sigma^2} - \gamma c(\mathbf{x}) |\psi|^2 \right],$$

with positive $K$, $u$ and $v$.

(a) Integrate out the $He^3$ concentrations to find the effective Hamiltonian, $\beta \mathcal{H}_{\text{eff}}[\psi]$, for the superfluid order parameter, given by

$$Z = \int \mathcal{D}\psi \exp\left( -\beta \mathcal{H}_{\text{eff}}[\psi] \right) \equiv \int \mathcal{D}\psi \mathcal{D}c \exp\left( -\beta \mathcal{H}[\psi, c] \right).$$

• The integration over $c(\mathbf{x})$ is merely a Gaussian integral, which at each point gives

$$\int dc(\mathbf{x}) \exp\left( -\frac{c(\mathbf{x})^2}{2\sigma^2} + \gamma c(\mathbf{x}) |\psi|^2 \right) \propto \exp\left( +\frac{1}{2} \gamma^2 \sigma^2 |\psi|^4 \right).$$

This modifies the quartic term in the Hamiltonian, resulting in

$$-\beta\mathcal{H}_{\rm eff}[\psi] = \int d^d\mathbf{x}\left[\frac{K}{2}\left|\nabla\psi\right|^2 + \frac{t}{2}\left|\psi\right|^2 + \left(u - \frac{\gamma^2\sigma^2}{2}\right)\left|\psi\right|^4 + v\left|\psi\right|^6\right].$$

(b) Obtain the phase diagram for $\beta\mathcal{H}_{\rm eff}[\psi]$ using a saddle point approximation. Find the limiting value of $\sigma^*$ above which the phase transition becomes discontinuous.

• As long as the quartic term is positive, in the saddle point approximation there is a *continuous* phase transition at $t = 0$, separating a disordered phase for $t > 0$, and a superfluid phase for $t < 0$. The fourth order term becomes negative for $\sigma > \sigma^* = \sqrt{2u}/\gamma$, and the phase transition is then discontinuous.

(c) The discontinuous transition is accompanied by a jump in the magnitude of $\psi$. How does this jump vanish as $\sigma \to \sigma^*$?

• The saddle point solution minimizes the above energy, leading to

$$t\overline{\psi} + 4\tilde{u}\overline{\psi}^3 + 6v\overline{\psi}^5 = 0,$$

where $\tilde{u} = u - \gamma^2\sigma^2/2 \propto (\sigma^* - \sigma)$. At the first order boundary, the resulting energy must equal to that of the disordered phase with $\overline{\psi} = 0$, resulting in a second equation

$$\frac{t}{2}\overline{\psi}^2 + \tilde{u}\overline{\psi}^4 + v\overline{\psi}^6 = 0.$$

Eliminating $t$ between the above two equations gives

$$\overline{\psi}^2 = -\frac{\tilde{u}}{2v} \propto (\sigma - \sigma^*),$$

i.e. the discontinuity in $\overline{\psi}$ along the boundary vanishes as $\sqrt{\sigma - \sigma^*}$.

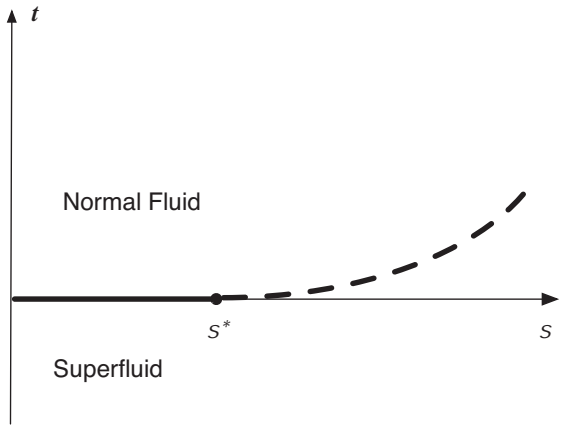
(d) Show that the discontinuous transition is accompanied by a jump in $He^3$ concentration.

• From the initial Gaussian Hamiltonian for $c(\mathbf{x})$, it is easy to see that

$$\langle c(\mathbf{x})\rangle = \gamma\sigma^2|\psi|^2.$$

Hence a discontinuity in $\psi$ is necessarily accompanied by a discontinuity in the density $c(\mathbf{x})$. (Note that the ensemble for $He^3$ is grand canonical, thus allowing this change in density.)

(e) Sketch the phase boundary in the $(t, \sigma)$ coordinates, and indicate how its two segments join at $\sigma^*$.




• Using the above value of $\overline{\Psi}^2$ in one of the equations determining the transition point, one finds

$$\overline{t} = -4\left(u - \frac{\sigma^2\gamma^2}{2}\right)\overline{\Psi}^2 - 6v\overline{\Psi}^4 = \frac{\gamma^4}{8v}\left(\sigma^2 - \sigma^{*2}\right)^2,$$

i.e. the discontinuous line joins the continuous portion quadratically, as illustrated in the figure

(f) Going back to the original joint probability for the fields $c(\mathbf{x})$ and $\Psi(\mathbf{x})$, show that $\langle c(\mathbf{x}) - \gamma\sigma^2|\Psi(\mathbf{x})|^2\rangle = 0$.

• By grouping the quadratic forms involving $c(\mathbf{x})$, the joint probability can be written as

$$e^{-\beta\mathcal{H}} = e^{-\int d^d\mathbf{x}\left[\frac{K}{2}|\nabla\Psi|^2 + \frac{t}{2}|\Psi|^2 + \left(u - \frac{\sigma^2\gamma^2}{2}\right)|\Psi|^4 + v|\Psi|^6 + \frac{\left(c(\mathbf{x}) - \gamma\sigma^2|\Psi(\mathbf{x})|^2\right)^2}{2\sigma^2}\right]} .$$

We can change variables to $c'(\mathbf{x}) = c(\mathbf{x}) - \gamma\sigma^2|\Psi(\mathbf{x})|^2$, whose average is clearly zero by symmetry, and hence

$$\langle c(\mathbf{x})\rangle = \gamma\sigma^2\langle|\Psi(\mathbf{x})|^2\rangle.$$

(g) Show that $\langle c(\mathbf{x})c(\mathbf{y})\rangle = \gamma^2\sigma^4\langle|\Psi(\mathbf{x})|^2|\Psi(\mathbf{y})|^2\rangle$, for $\mathbf{x} \neq \mathbf{y}$.

• Since the field $c'(\mathbf{x})$ is not coupled to any other field at a different location in space, $\langle c'(\mathbf{x})c'(\mathbf{y})\rangle = 0$, and $\langle c'(\mathbf{x})|\Psi(\mathbf{y})^2|\rangle = 0$. Hence,

$$\langle c(\mathbf{x})c(\mathbf{y})\rangle = \langle\left(c'(\mathbf{x}) + \gamma\sigma^2|\Psi(\mathbf{x})|^2\right)\left(c'(\mathbf{y}) + \gamma\sigma^2|\Psi(\mathbf{y})|^2\right)\rangle = \gamma^2\sigma^4\langle|\Psi(\mathbf{x})|^2|\Psi(\mathbf{y})|^2\rangle.$$

(h) Qualitatively discuss how $\langle c(\mathbf{x})c(0)\rangle$ decays with $x = |\mathbf{x}|$ in the disordered phase.

• In the disordered phase $\langle \Psi(\mathbf{x})\Psi(0)\rangle$ decays exponentially, as $e^{-x/\xi}$. From the result of the previous part, we expect

$$\langle c(\mathbf{x})c(0)\rangle \sim \gamma^2\sigma^4 \exp\left(-\frac{2x}{\xi}\right).$$

(i) Qualitatively discuss how $\langle c(\mathbf{x})c(0)\rangle$ decays to its asymptotic value in the ordered phase.
• In the ordered phase, $\langle \Psi(\mathbf{x})\Psi(0)\rangle \sim 1/x^{d-2}$ due to the Goldstone mode. Hence in $d = 3$ dimensions,

$$\langle c(\mathbf{x})c(0)\rangle \sim 1/x^2.$$

********

**5.** *Crumpled surfaces:* The configurations of a crumpled sheet of paper can be described by a vector field $\vec{r}(\mathbf{x})$, denoting the position in three dimensional space, $\vec{r} = (r_1, r_2, r_3)$, of the point at location $\mathbf{x} = (x_1, x_2)$ on the flat sheet. The energy of each configuration is assumed to be invariant under translations and rotations of the sheet of paper.

(a) Show that the two lowest order (in derivatives) terms in the quadratic part of a Landau–Ginzburg Hamiltonian for this system are:

$$\beta\mathcal{H}_0[\vec{r}] = \sum_{\alpha=1,2} \int d^2\mathbf{x} \left[\frac{t}{2}\partial_\alpha\vec{r}\cdot\partial_\alpha\vec{r} + \frac{K}{2}\partial_\alpha^2\vec{r}\cdot\partial_\alpha^2\vec{r}\right].$$

• Since the energy in the sheet is invariant under translations, it cannot depend on the vector $\vec{r}$, but only on its gradients such as $\partial_\alpha\vec{r}$, and $\partial_\alpha\partial_\beta\vec{r}$. To ensure rotational invariance, each component $r_i$ must be contracted (paired and summed over) with another $r_i$. Similarly to ensure isotropy in $\mathbf{x} = (x_1, x_2)$, each derivative $\partial_\alpha$ must be paired with another one. Taking advantage of contractions is a good way to find invariants. Thus using the summation notation, and ensuring proper contractions, the lowest order terms in the effective Hamiltonian are

$$\beta\mathcal{H}_0[\vec{r}] = \int d^2\mathbf{x} \left[\frac{t}{2}\partial_\alpha r_i\partial_\alpha r_i + \frac{K}{2}\partial_\alpha\partial_\alpha r_i\partial_\beta\partial_\beta r_i\right].$$

(b) Write down the lowest order terms (there are two) that appear at the quartic level.

• The same procedure allows us to construct 4-th order terms in $\vec{r}$: There should be two $r_i$, two $r_j$ and derivatives that are contracted. The two possible terms are

$$\beta\mathcal{H}'[\vec{r}] = \int d^2\mathbf{x}\,[u\partial_\alpha r_i\partial_\alpha r_i\partial_\beta r_j\partial_\beta r_j + v\partial_\alpha r_i\partial_\beta r_i\partial_\alpha r_j\partial_\beta r_j]\,.$$

(c) Discuss what happens when $t$ changes sign, assuming that quartic terms provide the required stability (and $K > 0$).

• The above theory has the same structure as the Landau-Ginzburg Hamiltonian, which a more complicated order parameter ($\partial_\alpha r_i$ in place of $m_i$). As in the Landau-Ginzburg model, when $t$ become negative, the order parameter is no longer zero, but the weight is maximized for a finite value of $\partial_\alpha r_i$. This implies that the sheet of paper gets stretched along a specific direction. This is known as the crumpling (or uncrumpling) transition. The sheet of paper is in a crumpled state for $t > 0$, while for $t < 0$, it opens up in some arbitrary direction.

********

**Solutions to problems from chapter 3 - Fluctuations**

**1.** *Spin waves:* In the XY model of $n = 2$ magnetism, a unit vector $\vec{s} = (s_x, s_y)$ (with $s_x^2 + s_y^2 = 1$) is placed on each site of a $d$–dimensional lattice. There is an interaction that tends to keep nearest–neighbors parallel, i.e. a Hamiltonian

$$-\beta\mathcal{H} = K \sum_{<ij>} \vec{s}_i \cdot \vec{s}_j \qquad .$$

The notation $< ij >$ is conventionally used to indicate summing over all *nearest–neighbor* pairs $(i, j)$.

(a) Rewrite the partition function $Z = \int \prod_i d\vec{s}_i \exp(-\beta\mathcal{H})$, as an integral over the set of angles $\{\theta_i\}$ between the spins $\{\vec{s}_i\}$ and some arbitrary axis.

- The partition function is

$$Z = \int \prod_i d^2\vec{s}_i \exp\left(K \sum_{\langle ij\rangle} \vec{s}_i \cdot \vec{s}_j\right) \delta\left(\vec{s}_i{}^2 - 1\right).$$

Since $\vec{s}_i \cdot \vec{s}_j = \cos(\theta_i - \theta_j)$, and $d^2\vec{s}_i = ds_i d\theta_i s_i = d\theta_i$, we obtain

$$Z = \int \prod_i d\theta_i \exp\left(K \sum_{\langle ij\rangle} \cos(\theta_i - \theta_j)\right).$$

(b) At low temperatures ($K \gg 1$), the angles $\{\theta_i\}$ vary slowly from site to site. In this case expand $-\beta\mathcal{H}$ to get a quadratic form in $\{\theta_i\}$.

- Expanding the cosines to quadratic order gives

$$Z = e^{N_b K} \int \prod_i d\theta_i \exp\left(-\frac{K}{2} \sum_{\langle ij\rangle} (\theta_i - \theta_j)^2\right),$$

where $N_b$ is the total number of bonds. Higher order terms in the expansion may be neglected for large $K$, since the integral is dominated by $|\theta_i - \theta_j| \approx \sqrt{2/K}$.

(c) For $d = 1$, consider $L$ sites with periodic boundary conditions (i.e. forming a closed chain). Find the normal modes $\theta_q$ that diagonalize the quadratic form (by Fourier transformation), and the corresponding eigenvalues $K(q)$. Pay careful attention to whether the modes are real or complex, and to the allowed values of $q$.

- For a chain of $L$ sites, we can change to Fourier modes by setting

$$\theta_j = \sum_q \theta(q) \frac{e^{iqj}}{\sqrt{L}}.$$

Since $\theta_j$ are real numbers, we must have

$$\theta(-q) = \theta(q)^*,$$

and the allowed $q$ values are restricted, for periodic boundary conditions, by the requirement of

$$\theta_{j+L} = \theta_j, \quad \Rightarrow \quad qL = 2\pi n, \quad \text{with} \quad n = 0, \pm 1, \pm 2, \ldots, \pm\frac{L}{2}.$$

Using

$$\theta_j - \theta_{j-1} = \sum_q \theta(q) \frac{e^{iqj}}{\sqrt{L}} \left(1 - e^{-iq}\right),$$

the one dimensional Hamiltonian, $\beta\mathcal{H} = \frac{K}{2}\sum_j (\theta_j - \theta_{j-1})^2$, can be rewritten in terms of Fourier components as

$$\beta\mathcal{H} = \frac{K}{2} \sum_{q,q'} \theta(q)\theta(q') \sum_j \frac{e^{i\left(q+q'\right)j}}{L} \left(1 - e^{-iq}\right)\left(1 - e^{-iq'}\right).$$

Using the identity $\sum_j e^{i\left(q+q'\right)j} = L\delta_{q,-q'}$, we obtain

$$\beta\mathcal{H} = K \sum_q |\theta(q)|^2 [1 - \cos(q)].$$

(d) Generalize the results from the previous part to a $d$–dimensional simple cubic lattice with periodic boundary conditions.

- In the case of a $d$ dimensional system, the index $j$ is replaced by a vector

$$j \mapsto \mathbf{j} = (j_1, \ldots, j_d),$$

which describes the lattice. We can then write

$$\beta\mathcal{H} = \frac{K}{2} \sum_{\mathbf{j}} \sum_{\alpha} \left(\theta_{\mathbf{j}} - \theta_{\mathbf{j}+\mathbf{e}_\alpha}\right)^2,$$

where $\mathbf{e}_\alpha$'s are unit vectors $\{\mathbf{e}_1 = (1, 0, \cdots, 0), \cdots, \mathbf{e}_d = (0, \cdots, 0, 1)\}$, generalizing the one dimensional result to

$$\beta\mathcal{H} = \frac{K}{2} \sum_{\mathbf{q},\mathbf{q}'} \theta(\mathbf{q})\, \theta(\mathbf{q}') \sum_\alpha \sum_{\mathbf{j}} \frac{e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{j}}}{L^d} \left(1 - e^{-i\mathbf{q}\cdot\mathbf{e}_\alpha}\right) \left(1 - e^{-i\mathbf{q}'\cdot\mathbf{e}_\alpha}\right).$$

Again, summation over $\mathbf{j}$ constrains $\mathbf{q}$ and $-\mathbf{q}'$ to be equal, and

$$\beta\mathcal{H} = K \sum_{\mathbf{q}} |\theta(\mathbf{q})|^2 \sum_\alpha [1 - \cos(q_\alpha)].$$

(e) Calculate the contribution of these modes to the free energy and heat capacity. (Evaluate the *classical* partition function, i.e. do not quantize the modes.)

- With $K(\mathbf{q}) \equiv 2K \sum_\alpha [1 - \cos(q_\alpha)]$,

$$Z = \int \prod_{\mathbf{q}} d\theta(\mathbf{q}) \exp\left[-\frac{1}{2} K(\mathbf{q}) |\theta(\mathbf{q})|^2\right] = \prod_{\mathbf{q}} \sqrt{\frac{2\pi}{K(\mathbf{q})}},$$

and the corresponding free energy is

$$F = -k_B T \ln Z = -k_B T \left[\text{constant} - \frac{1}{2} \sum_{\mathbf{q}} \ln K(\mathbf{q})\right],$$

or, in the continuum limit (using the fact that the density of states in $\mathbf{q}$ space is $(L/2\pi)^d$),

$$F = -k_B T \left[\text{constant} - \frac{1}{2} L^d \int \frac{d^d q}{(2\pi)^d} \ln K(\mathbf{q})\right].$$

As $K \sim 1/T$ at $T \to \infty$, we can write

$$F = -k_B T \left[\text{constant}' + \frac{1}{2} L^d \ln T\right],$$

and the heat capacity per site is given by

$$C = -T \frac{\partial^2 F}{\partial T^2} \cdot \frac{1}{L^d} = \frac{k_B}{2}.$$

This is because there is one degree of freedom (the angle) per site that can store potential energy.

(f) Find an expression for $\langle \vec{s}_0 \cdot \vec{s}_{\mathbf{x}} \rangle = \Re\langle \exp[i\theta_{\mathbf{x}} - i\theta_0] \rangle$ by adding contributions from different Fourier modes. Convince yourself that for $|\mathbf{x}| \to \infty$, only $\mathbf{q} \to \mathbf{0}$ modes contribute appreciably to this expression, and hence calculate the asymptotic limit.

• We have

$$\theta_{\mathbf{x}} - \theta_{\mathbf{0}} = \sum_{\mathbf{q}} \theta(\mathbf{q}) \frac{e^{i\mathbf{q}\cdot\mathbf{x}} - 1}{L^{d/2}},$$

and by completing the square for the argument of the exponential in $\left\langle e^{i(\theta_{\mathbf{x}} - \theta_{\mathbf{0}})} \right\rangle$, *i.e.* for

$$-\frac{1}{2} K(\mathbf{q}) |\theta(\mathbf{q})|^2 + i\theta(\mathbf{q}) \frac{e^{i\mathbf{q}\cdot\mathbf{x}} - 1}{L^{d/2}},$$

it follows immediately that

$$\left\langle e^{i(\theta_{\mathbf{x}} - \theta_{\mathbf{0}})} \right\rangle = \exp\left\{ -\frac{1}{L^d} \sum_{\mathbf{q}} \frac{\left| e^{i\mathbf{q}\cdot\mathbf{x}} - 1 \right|^2}{2K(\mathbf{q})} \right\} = \exp\left\{ -\int \frac{d^d q}{(2\pi)^d} \frac{1 - \cos(\mathbf{q}\cdot\mathbf{x})}{K(\mathbf{q})} \right\}.$$

For $x$ larger than 1, the integrand has a peak of height $\sim x^2/2K$ at $q = 0$ (as it is seen by expanding the cosines for small argument). Furthermore, the integrand has a first node, as $q$ increases, at $q \sim 1/x$. From these considerations, we can obtain the leading behavior for large $x$:

• In $d = 1$, we have to integrate $\sim x^2/2K$ over a length $\sim 1/x$, and thus

$$\left\langle e^{i(\theta_{\mathbf{x}} - \theta_{\mathbf{0}})} \right\rangle \sim \exp\left( -\frac{|x|}{2K} \right).$$

• In $d = 2$, we have to integrate $\sim x^2/2K$ over an area $\sim (1/x)^2$. A better approximation, at large $x$, than merely taking the height of the peak, is given by

$$\begin{aligned} \int \frac{d^d q}{(2\pi)^d} \frac{1 - \cos(\mathbf{q}\cdot\mathbf{x})}{K(\mathbf{q})} &\approx \int \frac{dq d\varphi q}{(2\pi)^2} \frac{1 - \cos(qx\cos\varphi)}{Kq^2} \\ &= \int \frac{dq d\varphi}{(2\pi)^2} \frac{1}{Kq} - \int \frac{dq d\varphi}{(2\pi)^2} \frac{\cos(qx\cos\varphi)}{Kq}, \end{aligned}$$

or, doing the angular integration in the first term,

$$\int \frac{d^d q}{(2\pi)^d} \frac{1 - \cos(\mathbf{q}\cdot\mathbf{x})}{K(\mathbf{q})} \approx \int^{1/|x|} \frac{dq}{2\pi} \frac{1}{Kq} + \text{subleading in } x,$$

resulting in

$$\left\langle e^{i(\theta_{\mathbf{x}} - \theta_{\mathbf{0}})} \right\rangle \sim \exp\left( -\frac{\ln|x|}{2\pi K} \right) = |x|^{-\frac{1}{2\pi K}}, \quad \text{as } x \to \infty.$$

• In $d \geq 3$, we have to integrate $\sim x^2/2K$ over a volume $\sim (1/x)^3$. Thus, as $x \to \infty$, the $x$ dependence of the integral is removed, and

$$\left\langle e^{i(\theta_{\mathbf{x}}-\theta_{\mathbf{0}})} \right\rangle \to \text{constant},$$

implying that correlations don't disappear at large $x$.

The results can also be obtained by noting that the fluctuations are important only for small $q$. Using the expansion of $K(\mathbf{q}) \approx Kq^2/2$, then reduces the problem to calculation of the Coulomb Kernel $\int d^d\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{x}}/q^2$, as described in the preceding chapter.

(g) Calculate the transverse susceptibility from $\chi_t \propto \int d^d\mathbf{x} \langle \vec{s}_0 \cdot \vec{s}_{\mathbf{x}} \rangle_c$. How does it depend on the system size $L$?

• We have

$$\left\langle e^{i(\theta_{\mathbf{x}}-\theta_{\mathbf{0}})} \right\rangle = \exp\left\{ - \int \frac{d^dq}{(2\pi)^d} \frac{1-\cos(\mathbf{q}\cdot\mathbf{x})}{K(\mathbf{q})} \right\},$$

and, similarly,

$$\left\langle e^{i\theta_{\mathbf{x}}} \right\rangle = \exp\left\{ - \int \frac{d^dq}{(2\pi)^d} \frac{1}{2K(\mathbf{q})} \right\}.$$

Hence the connected correlation function

$$\langle \vec{s}_{\mathbf{x}} \cdot \vec{s}_{\mathbf{0}} \rangle_c = \left\langle e^{i(\theta_{\mathbf{x}}-\theta_{\mathbf{0}})} \right\rangle_c = \left\langle e^{i(\theta_{\mathbf{x}}-\theta_{\mathbf{0}})} \right\rangle - \left\langle e^{i\theta_{\mathbf{x}}} \right\rangle \left\langle e^{i\theta_{\mathbf{0}}} \right\rangle,$$

is given by

$$\langle \vec{s}_{\mathbf{x}} \cdot \vec{s}_{\mathbf{0}} \rangle_c = e^{-\int \frac{d^dq}{(2\pi)^d} \frac{1}{K(\mathbf{q})}} \left\{ \exp\left[ \int \frac{d^dq}{(2\pi)^d} \frac{\cos(\mathbf{q}\cdot\mathbf{x})}{K(\mathbf{q})} \right] - 1 \right\}.$$

In $d \geq 3$, the $x$ dependent integral vanishes at $x \to \infty$. We can thus expand its exponential, for large $x$, obtaining

$$\langle \vec{s}_{\mathbf{x}} \cdot \vec{s}_{\mathbf{0}} \rangle_c \sim \int \frac{d^dq}{(2\pi)^d} \frac{\cos(\mathbf{q}\cdot\mathbf{x})}{K(\mathbf{q})} \approx \int \frac{d^dq}{(2\pi)^d} \frac{\cos(\mathbf{q}\cdot\mathbf{x})}{Kq^2} = \frac{1}{K} C_d(x) \sim \frac{1}{K|x|^{d-2}}.$$

Thus, the transverse susceptibility diverges as

$$\chi_t \propto \int d^dx \langle \vec{s}_{\mathbf{x}} \cdot \vec{s}_{\mathbf{0}} \rangle_c \sim \frac{L^2}{K}.$$

(h) In $d = 2$, show that $\chi_t$ only diverges for $K$ larger than a critical value $K_c = 1/(4\pi)$.

• In $d = 2$, there is no long range order, $\langle \vec{s}_{\mathbf{x}} \rangle = 0$, and

$$\langle \vec{s}_{\mathbf{x}} \cdot \vec{s}_{\mathbf{0}} \rangle_c = \langle \vec{s}_{\mathbf{x}} \cdot \vec{s}_{\mathbf{0}} \rangle \sim |x|^{-1/(2\pi K)} .$$

The susceptibility

$$\chi_t \sim \int^L d^2x \, |x|^{-1/(2\pi K)} ,$$

thus converges for $1/(2\pi K) > 2$, for $K$ below $K_c = 1/(4\pi)$. For $K > K_c$, the susceptibility diverges as

$$\chi_t \sim L^{2-2K_c/K}.$$

********

**2.** *Capillary waves:* A reasonably flat surface in $d$–dimensions can be described by its height $h$, as a function of the remaining $(d-1)$ coordinates $\mathbf{x} = (x_1, ...x_{d-1})$. Convince yourself that the generalized "area" is given by $\mathcal{A} = \int d^{d-1}\mathbf{x}\sqrt{1 + (\nabla h)^2}$. With a surface tension $\sigma$, the Hamiltonian is simply $\mathcal{H} = \sigma\mathcal{A}$.

(a) At sufficiently low temperatures, there are only slow variations in $h$. Expand the energy to quadratic order, and write down the partition function as a functional integral.

• For a surface parametrized by the height function

$$x_d = h\left(x_1, \ldots, x_{d-1}\right),$$

an area element can be calculated as

$$dA = \frac{1}{\cos\alpha} dx_1 \cdots dx_{d-1},$$

where $\alpha$ is the angle between the $d^{\text{th}}$ direction and the normal

$$\vec{n} = \frac{1}{\sqrt{1 + (\nabla h)^2}} \left( -\frac{\partial h}{\partial x_1}, \ldots, -\frac{\partial h}{\partial x_{d-1}}, 1 \right)$$

to the surface ($n^2 = 1$). Since, $\cos\alpha = n_d = \left[1 + (\nabla h)^2\right]^{-1/2} \approx 1 - \frac{1}{2}(\nabla h)^2$, we obtain

$$\mathcal{H} = \sigma\mathcal{A} \approx \sigma \int d^{d-1}x \left\{ 1 + \frac{1}{2}(\nabla h)^2 \right\},$$

and, dropping a multiplicative constant,

$$Z = \int \mathcal{D}h\left(\mathbf{x}\right) \exp\left\{-\beta\frac{\sigma}{2}\int d^{d-1}x\left(\nabla h\right)^2\right\}.$$

(b) Use Fourier transformation to diagonalize the quadratic Hamiltonian into its normal modes $\{h_{\mathbf{q}}\}$ (capillary waves).

• After changing variables to the Fourier modes,

$$h\left(\mathbf{x}\right) = \int \frac{d^{d-1}q}{\left(2\pi\right)^{d-1}} h\left(\mathbf{q}\right) e^{i\mathbf{q}\cdot\mathbf{x}},$$

the partition function is given by

$$Z = \int \mathcal{D}h\left(\mathbf{q}\right) \exp\left\{-\beta\frac{\sigma}{2}\int \frac{d^{d-1}q}{\left(2\pi\right)^{d-1}} q^2 \left|h\left(\mathbf{q}\right)\right|^2\right\}.$$

(c) What symmetry breaking is responsible for these Goldstone modes?

• By selecting a particular height, the ground state breaks the translation symmetry in the $d^{\text{th}}$ direction. The transformation $h\left(\mathbf{x}\right) \to h\left(\mathbf{x}\right) + \xi\left(\mathbf{x}\right)$ leaves the energy unchanged if $\xi\left(\mathbf{x}\right)$ is constant. By continuity, we can have an arbitrarily small change in the energy by varying $\xi\left(\mathbf{x}\right)$ arbitrarily slowly.

(d) Calculate the height–height correlations $\langle\big(h(\mathbf{x}) - h(\mathbf{x}')\big)^2\rangle$.

• From

$$h\left(\mathbf{x}\right) - h\left(\mathbf{x}'\right) = \int \frac{d^{d-1}q}{\left(2\pi\right)^{d-1}} h\left(\mathbf{q}\right)\left(e^{i\mathbf{q}\cdot\mathbf{x}} - e^{i\mathbf{q}\cdot\mathbf{x}'}\right),$$

we obtain

$$\left\langle\left(h\left(\mathbf{x}\right) - h\left(\mathbf{x}'\right)\right)^2\right\rangle = \int \frac{d^{d-1}q}{\left(2\pi\right)^{d-1}} \frac{d^{d-1}q'}{\left(2\pi\right)^{d-1}} \left\langle h\left(\mathbf{q}\right) h\left(\mathbf{q}'\right)\right\rangle \left(e^{i\mathbf{q}\cdot\mathbf{x}} - e^{i\mathbf{q}\cdot\mathbf{x}'}\right)\left(e^{i\mathbf{q}'\cdot\mathbf{x}} - e^{i\mathbf{q}'\cdot\mathbf{x}'}\right).$$

The height-height correlations thus behave as

$$\begin{aligned} G\left(\mathbf{x}-\mathbf{x}'\right) &\equiv \left\langle\left(h\left(\mathbf{x}\right) - h\left(\mathbf{x}'\right)\right)^2\right\rangle \\ &= \frac{2}{\beta\sigma}\int \frac{d^{d-1}q}{\left(2\pi\right)^{d-1}} \frac{1-\cos\left[\mathbf{q}\cdot\left(\mathbf{x}-\mathbf{x}'\right)\right]}{q^2} = \frac{2}{\beta\sigma} C_{d-1}\left(\mathbf{x}-\mathbf{x}'\right). \end{aligned}$$

(e) Comment on the form of the result (d) in dimensions $d = 4$, 3, 2, and 1.

• We can now discuss the asymptotic behavior of the Coulomb Kernel for large $|\mathbf{x}-\mathbf{x}'|$, either using the results from problem 1(f), or the exact form given in lectures.

• In $d \geq 4$, $G(\mathbf{x}-\mathbf{x}') \to$ constant, and the surface is *flat*.

• In $d = 3$, $G(\mathbf{x}-\mathbf{x}') \sim \ln|\mathbf{x}-\mathbf{x}'|$, and we come to the surprising conclusion that there are no asymptotically flat surfaces in three dimensions. While this is technically correct, since the logarithm grows slowly, very large surfaces are needed to detect appreciable fluctuations.

• In $d = 2$, $G(\mathbf{x}-\mathbf{x}') \sim |\mathbf{x}-\mathbf{x}'|$. This is easy to comprehend, once we realize that the interface $h(x)$ is similar to the path $x(t)$ of a random walker, and has similar ($x \sim \sqrt{t}$) fluctuations.

• In $d = 1$, $G(\mathbf{x}-\mathbf{x}') \sim |\mathbf{x}-\mathbf{x}'|^2$. The transverse fluctuation of the 'point' interface are very big, and the approximations break down as discussed next.

(f) By estimating typical values of $\nabla h$, comment on when it is justified to ignore higher order terms in the expansion for $\mathcal{A}$.

• We can estimate $(\nabla h)^2$ as

$$\frac{\left\langle \left(h(\mathbf{x})-h(\mathbf{x}')\right)^2 \right\rangle}{(\mathbf{x}-\mathbf{x}')^2} \propto |\mathbf{x}-\mathbf{x}'|^{1-d}.$$

For dimensions $d \geq d_\ell = 1$, the typical size of the gradient decreases upon coarse-graining. The gradient expansion of the area used before is then justified. For dimensions $d \leq d_\ell$, the whole idea of the gradient expansion fails to be sensible.

********

**3.** *Gauge fluctuations in superconductors:* The Landau–Ginzburg model of superconductivity describes a complex superconducting order parameter $\Psi(\mathbf{x}) = \Psi_1(\mathbf{x}) + i\Psi_2(\mathbf{x})$, and the electromagnetic vector potential $\vec{A}(\mathbf{x})$, which are subject to a Hamiltonian

$$\beta\mathcal{H} = \int d^3\mathbf{x} \left[ \frac{t}{2}|\Psi|^2 + u|\Psi|^4 + \frac{K}{2} D_\mu \Psi D_\mu^* \Psi^* + \frac{L}{2}(\nabla \times A)^2 \right].$$

The gauge-invariant derivative $D_\mu \equiv \partial_\mu - ieA_\mu(\mathbf{x})$, introduces the coupling between the two fields. (In terms of Cooper pair parameters, $e = e^*c/\hbar$, $K = \hbar^2/2m^*$.)

(a) Show that the above Hamiltonian is invariant under the *local gauge symmetry*:

$$\Psi(\mathbf{x}) \mapsto \Psi(x)\exp\left(i\theta(\mathbf{x})\right), \qquad \text{and} \qquad A_\mu(\mathbf{x}) \mapsto A_\mu(\mathbf{x}) + \frac{1}{e}\partial_\mu\theta.$$

• Under a local gauge transformation, $\beta\mathcal{H} \longmapsto$

$$\int d^3x \Bigg\{ \frac{t}{2} |\Psi|^2 + u |\Psi|^4 + \frac{K}{2} \left[ (\partial_\mu - ieA_\mu - i\partial_\mu\theta) \Psi e^{i\theta} \right] \left[ (\partial_\mu + ieA_\mu + i\partial_\mu\theta) \Psi^* e^{-i\theta} \right] + \frac{L}{2} \left( \nabla \times \vec{A} + \nabla \times \frac{1}{e} \nabla\theta \right)^2 \Bigg\}.$$

But this is none other than $\beta\mathcal{H}$ again, since

$$(\partial_\mu - ieA_\mu - i\partial_\mu\theta) \Psi e^{i\theta} = e^{i\theta} (\partial_\mu - ieA_\mu) \Psi = e^{i\theta} D_\mu \Psi,$$

and

$$\nabla \times \frac{1}{e} \nabla\theta = 0.$$

(b) Show that there is a saddle point solution of the form $\Psi(\mathbf{x}) = \overline{\Psi}$, and $\vec{A}(\mathbf{x}) = 0$, and find $\overline{\Psi}$ for $t > 0$ and $t < 0$.

• The saddle point solutions are obtained from

$$\frac{\delta\mathcal{H}}{\delta\Psi^*} = 0, \quad \Longrightarrow \quad \frac{t}{2}\Psi + 2u\Psi |\Psi|^2 - \frac{K}{2} D_\mu D_\mu \Psi = 0,$$

and

$$\frac{\delta\mathcal{H}}{\delta A_\mu} = 0, \quad \Longrightarrow \quad \frac{K}{2} \left( -ie\Psi D_\mu^* \Psi^* + ie\Psi^* D_\mu \Psi \right) - L\epsilon_{\alpha\beta\mu}\epsilon_{\alpha\gamma\delta}\partial_\beta\partial_\gamma A_\delta = 0.$$

The ansatz $\Psi(\mathbf{x}) = \overline{\Psi}$, $\vec{A} = 0$, clearly solves these equations. The first equation then becomes

$$t\overline{\Psi} + 4u\overline{\Psi} \left|\overline{\Psi}\right|^2 = 0,$$

yielding (for $u > 0$) $\overline{\Psi} = 0$ for $t > 0$, whereas $\left|\overline{\Psi}\right|^2 = -t/4u$ for $t < 0$.

(c) For $t < 0$, calculate the cost of fluctuations by setting

$$\begin{cases} \Psi(\mathbf{x}) = \left(\overline{\Psi} + \phi(\mathbf{x})\right) \exp\left(i\theta(\mathbf{x})\right), \\ A_\mu(\mathbf{x}) = a_\mu(\mathbf{x}), \quad \text{(with } \partial_\mu a_\mu = 0 \text{ in the Coulomb gauge)} \end{cases}$$

and expanding $\beta\mathcal{H}$ to quadratic order in $\phi$, $\theta$, and $\vec{a}$.

• For simplicity, let us choose $\overline{\Psi}$ to be real. From the Hamiltonian term

$$D_\mu \Psi D_\mu^* \Psi^* = \left[ (\partial_\mu - iea_\mu) \left(\overline{\Psi} + \phi\right) e^{i\theta} \right] \left[ (\partial_\mu + iea_\mu) \left(\overline{\Psi} + \phi\right) e^{-i\theta} \right],$$

we get the following quadratic contribution

$$\overline{\Psi}^2 (\nabla\theta)^2 + (\nabla\phi)^2 - 2e\overline{\Psi}^2 a_\mu \partial_\mu \theta + e^2\overline{\Psi}^2 |\vec{a}|^2 .$$

The third term in the above expression integrates to zero (as it can be seen by integrating by parts and invoking the Coulomb gauge condition $\partial_\mu a_\mu = 0$). Thus, the quadratic terms read

$$\beta\mathcal{H}^{(2)} = \int d^3x \Bigg\{ \left(\frac{t}{2} + 6u\overline{\Psi}^2\right)\phi^2 + \frac{K}{2}(\nabla\phi)^2 + \frac{K}{2}\overline{\Psi}^2 (\nabla\theta)^2 + \frac{K}{2}e^2\overline{\Psi}^2 |\vec{a}|^2 + \frac{L}{2}(\nabla\times\vec{a})^2 \Bigg\}.$$

(d) Perform a Fourier transformation, and calculate the expectation values of $\langle|\phi(\mathbf{q})|^2\rangle$, $\langle|\theta(\mathbf{q})|^2\rangle$, and $\langle|\vec{a}(\mathbf{q})|^2\rangle$.

• In terms of Fourier transforms, we obtain

$$\beta\mathcal{H}^{(2)} = \sum_{\mathbf{q}} \Bigg\{ \left(\frac{t}{2} + 6u\overline{\Psi}^2 + \frac{K}{2}q^2\right)|\phi(\mathbf{q})|^2 + \frac{K}{2}\overline{\Psi}^2 q^2 |\theta(\mathbf{q})|^2 + \frac{K}{2}e^2\overline{\Psi}^2 |\vec{a}(\mathbf{q})|^2 + \frac{L}{2}(\mathbf{q}\times\vec{a})^2 \Bigg\}.$$

In the Coulomb gauge, $\mathbf{q} \perp \vec{a}(\mathbf{q})$, and so $[\mathbf{q}\times\vec{a}(\mathbf{q})]^2 = q^2|\vec{a}(\mathbf{q})|^2$. This diagonal form then yields immediately (for $t < 0$)

$$\left\langle|\phi(\mathbf{q})|^2\right\rangle = \left(t + 12u\overline{\Psi}^2 + Kq^2\right)^{-1} = \frac{1}{Kq^2 - 2t},$$
$$\left\langle|\theta(\mathbf{q})|^2\right\rangle = \left(K\overline{\Psi}^2 q^2\right)^{-1} = -\frac{4u}{Ktq^2},$$
$$\left\langle|\vec{a}(\mathbf{q})|^2\right\rangle = 2\left(Ke^2\overline{\Psi}^2 + Lq^2\right)^{-1} = \frac{2}{Lq^2 - Ke^2 t/4u} \qquad (\vec{\mathrm{a}} \text{ has 2 components}).$$

Note that the gauge field, "mass-less" in the original theory, acquires a "mass" $Ke^2t/4u$ through its coupling to the order parameter. This is known as the Higgs mechanism.

********

**4.** *Fluctuations around a tricritical point:* As shown in a previous problem, the Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x}\left[\frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 + um^4 + vm^6\right],$$

with $u = 0$ and $v > 0$ describes a tricritical point.

(a) Calculate the heat capacity singularity as $t \to 0$ by the saddle point approximation.

• As already calculated in a previous problem, the saddle point minimum of the free energy $\vec{m} = \overline{m}\hat{e}_\ell$, can be obtained from

$$\left.\frac{\partial \Psi}{\partial m}\right|_{\overline{m}} = \overline{m}\,\left(t + 6v\overline{m}^{\,4}\right) = 0,$$

yielding,

$$\overline{m} = \begin{cases} 0 & \text{for} \quad t > \overline{t} = 0 \\ \left(-\dfrac{t}{6v}\right)^{1/4} & \text{for} \quad t < 0 \end{cases}.$$

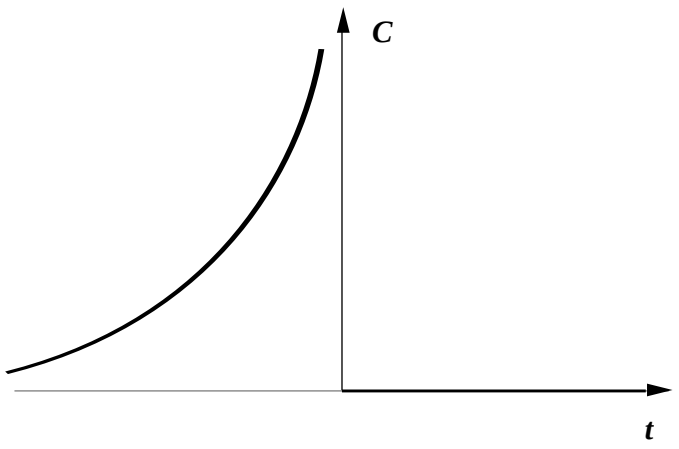
The corresponding free energy density equals to

$$\Psi(\overline{m}) = \frac{t}{2}\overline{m}^{\,2} + v\overline{m}^{\,6} = \begin{cases} 0 & \text{for} \quad t > 0 \\ -\dfrac{1}{3}\dfrac{(-t)^{3/2}}{(6v)^{1/2}} & \text{for} \quad t < 0 \end{cases}.$$

Therefore, the singular behavior of the heat capacity is given by

$$C = C_{s.p.} \sim -T_c\,\left.\frac{\partial^2 \Psi}{\partial t^2}\right|_{\overline{m}} = \begin{cases} 0 & \text{for} \quad t > 0 \\ \dfrac{T_c}{4}(-6vt)^{-1/2} & \text{for} \quad t < 0 \end{cases},$$

as sketched in the figure below.




(b) Include both longitudinal and transverse fluctuations by setting

$$\vec{m}(\mathbf{x}) = \big(\overline{m} + \phi_\ell(\mathbf{x})\big)\hat{e}_\ell + \sum_{\alpha=2}^{n} \phi_t^\alpha(\mathbf{x})\hat{e}_\alpha,$$

and expanding $\beta\mathcal{H}$ to quadratic order in $\phi$.

- Let us now include both longitudinal and transversal fluctuations by setting

$$\vec{m}(\mathbf{x}) = (\overline{m} + \phi_\ell(\mathbf{x}))\hat{e}_\ell + \sum_{\alpha=2}^{n} \phi_t^\alpha(\mathbf{x})\hat{e}_\alpha,$$

where $\hat{e}_\ell$ and $\hat{e}_\alpha$ form an orthonormal set of $n$ vectors. Consequently, the free energy $\beta\mathcal{H}$ is a function of $\phi_\ell$ and $\phi_t$. Since $\overline{m}\hat{e}_\ell$ is a minimum, there are no linear terms in the expansion of $\beta\mathcal{H}$ in $\phi$. The contributions of each factor in the free energy to the quadratic term in the expansion are

$$(\nabla\vec{m})^2 \Longrightarrow (\nabla\phi_\ell)^2 + \sum_{\alpha=2}^{n} (\nabla\phi_t^\alpha)^2,$$

$$(\vec{m})^2 \Longrightarrow (\phi_\ell)^2 + \sum_{\alpha=2}^{n} (\phi_t^\alpha)^2,$$

$$(\vec{m})^6 = ((\vec{m})^2)^3 = (\overline{m}^2 + 2\overline{m}\phi_\ell + \phi_\ell^2 + \sum_{\alpha=2}^{n} (\phi_t^\alpha)^2)^3 \Longrightarrow 15\overline{m}^{\,4}(\phi_\ell)^2 + 3\overline{m}^{\,4} \sum_{\alpha=2}^{n} (\phi_t^\alpha)^2.$$

The expansion of $\beta\mathcal{H}$ to second order now gives

$$\beta\mathcal{H}(\phi_\ell, \phi_t^\alpha) = \beta\mathcal{H}(0,0) + \int d^d\mathbf{x} \left\{ \left[ \frac{K}{2}(\nabla\phi_\ell)^2 + \frac{\phi_\ell^2}{2}\left(t + 30v\overline{m}^4\right) \right] + \sum_{\alpha=2}^{n} \left[ \frac{K}{2}(\nabla\phi_t^\alpha)^2 + \frac{(\phi_t^\alpha)^2}{2}\left(t + 6v\overline{m}^4\right) \right] \right\}.$$

We can formally rewrite it as

$$\beta\mathcal{H}(\phi_\ell, \phi_t^\alpha) = \beta\mathcal{H}(0,0) + \beta\mathcal{H}_\ell(\phi_\ell) + \sum_{\alpha=2}^{n} \beta\mathcal{H}_{t_\alpha}(\phi_t^\alpha),$$

where $\beta\mathcal{H}_i(\phi_i)$, with $i = \ell, t_\alpha$, is in general given by

$$\beta\mathcal{H}_i(\phi_i) = \frac{K}{2} \int d^d\mathbf{x} \left[ (\nabla\phi_i)^2 + \frac{\phi_i^2}{\xi_i^2} \right],$$

with the inverse correlation lengths

$$\xi_\ell^{-2} = \begin{cases} \dfrac{t}{K} & \text{for} \quad t > 0 \\ \dfrac{-4t}{K} & \text{for} \quad t < 0 \end{cases},$$

and

$$\xi_{t_\alpha}^{-2} = \begin{cases} \dfrac{t}{K} & \text{for} \quad t > 0 \\ 0 & \text{for} \quad t < 0 \end{cases} .$$

As shown in the lectures for the critical point of a magnet, for $t > 0$ there is no difference between longitudinal and transverse components, whereas for $t < 0$, there is no restoring force for the Goldstone modes $\phi_t^\alpha$ due to the rotational symmetry of the *ordered* state.

(c) Calculate the longitudinal and transverse correlation functions.

• Since in the harmonic approximation $\beta\mathcal{H}$ turns out to be a sum of the Hamiltonians of the different fluctuating components $\phi_\ell$, $\phi_t^\alpha$, these quantities are independent of each other, i.e.

$$\langle \phi_\ell \phi_t^\alpha \rangle = 0, \qquad \text{and} \qquad \langle \phi_t^\gamma \phi_t^\alpha \rangle = 0 \quad \text{for } \alpha \neq \gamma.$$

To determine the longitudinal and transverse correlation functions, we first express the free energy in terms of Fourier modes, so that the probability of a particular fluctuation configuration is given by

$$\mathcal{P}(\{\phi_\ell, \phi_t^\alpha\}) \propto \prod_{\mathbf{q},\alpha} \exp\left\{ -\frac{K}{2}\left(q^2 + \xi_\ell^{-2}\right) |\phi_{\ell,\mathbf{q}}|^2 \right\} \cdot \exp\left\{ -\frac{K}{2}\left(q^2 + \xi_{t_\alpha}^{-2}\right) |\phi_{t,\mathbf{q}}^\alpha|^2 \right\}.$$

Thus, as it was also shown in the lectures, the correlation function is

$$\langle \phi_\alpha(\mathbf{x}) \phi_\beta(0) \rangle = \frac{\delta_{\alpha,\beta}}{VK} \sum_{\mathbf{q}} \frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{\left(q^2 + \xi_\alpha^{-2}\right)} = -\frac{\delta_{\alpha,\beta}}{K} I_d(\mathbf{x}, \xi_\alpha),$$

therefore,

$$\langle \phi_\ell(\mathbf{x}) \phi_\ell(0) \rangle = -\frac{1}{K} I_d(\mathbf{x}, \xi_\ell),$$

and

$$\langle \phi_t^\alpha(\mathbf{x}) \phi_t^\beta(0) \rangle = -\frac{\delta_{\alpha,\beta}}{K} I_d(\mathbf{x}, \xi_{t_\alpha}).$$

(d) Compute the first correction to the saddle point free energy from fluctuations.

• Let us calculate the first correction to the saddle point free energy from fluctuations. The partition function is

$$\begin{aligned} Z &= e^{-\beta\mathcal{H}(0,0)} \int \mathcal{D}\phi(\mathbf{x}) \, \exp\left\{ -\frac{K}{2} \int d^d\mathbf{x} \left[ (\nabla\phi)^2 + \xi^{-2}\phi^2 \right] \right\} \\ &= e^{-\beta\mathcal{H}(0,0)} \int \prod_{\mathbf{q}} d\phi_{\mathbf{q}} \, \exp\left\{ -\frac{K}{2} \sum_{\mathbf{q}} \left(q^2 + \xi^{-2}\right) \phi_{\mathbf{q}} \phi_{\mathbf{q}}^* \right\} \quad , \\ &= \prod_{\mathbf{q}} \left[ K\left(q^2 + \xi^{-2}\right) \right]^{-1/2} = \exp\left\{ -\frac{1}{2} \sum_{\mathbf{q}} \left( Kq^2 + K\xi^{-2} \right) \right\} \end{aligned}$$

and the free energy density equals to

$$\beta f = \frac{\beta \mathcal{H}(0,0)}{V} + \begin{cases} \dfrac{n}{2}\displaystyle\int \frac{d^d\mathbf{q}}{(2\pi)^d} \ln\left(Kq^2+t\right) & \text{for} \quad t>0 \\ \dfrac{1}{2}\displaystyle\int \frac{d^d\mathbf{q}}{(2\pi)^d} \ln\left(Kq^2-4t\right) + \dfrac{n-1}{2}\displaystyle\int \frac{d^d\mathbf{q}}{(2\pi)^d} \ln\left(Kq^2\right) & \text{for} \quad t<0 \end{cases} .$$

Note that the first term is the saddle point free energy, and that there are $n$ contributions to the free energy from fluctuations.

(e) Find the fluctuation correction to the heat capacity.

• As $C = -T(d^2f/dT^2)$, the fluctuation corrections to the heat capacity are given by

$$C - C_{s.p.} \propto \begin{cases} \dfrac{n}{2}\displaystyle\int \frac{d^d\mathbf{q}}{(2\pi)^d} \left(Kq^2+t\right)^{-2} & \text{for} \quad t>0 \\ \dfrac{16}{2}\displaystyle\int \frac{d^d\mathbf{q}}{(2\pi)^d} \left(Kq^2-4t\right)^{-2} & \text{for} \quad t<0 \end{cases} .$$

These integrals change behavior at $d = 4$. For $d > 4$, the integrals diverge at large $\mathbf{q}$, and are dominated by the upper cutoff $\Delta \simeq 1/a$. That is why fluctuation corrections to the heat capacity add just a constant term on each side of the transition, and the saddle point solution keeps its qualitative form. On the other hand, for $d < 4$, the integrals are proportional to the corresponding correlation length $\xi^{4-d}$. Due to the divergence of $\xi$, the fluctuation corrections diverge as

$$C_{fl.} = C - C_{s.p.} \propto K^{-d/2}|t|^{d/2-2}.$$

(f) By comparing the results from parts (a) and (e) *for* $t < 0$ obtain a Ginzburg criterion, and the upper critical dimension for validity of mean–field theory at a tricritical point.

• To obtain a Ginzburg criterion, let us consider $t < 0$. In this region, the saddle point contribution already diverges as $C_{s.p.} \propto (-vt)^{-1/2}$, so that

$$\frac{C_{fl.}}{C_{s.p.}} \propto (-t)^{\frac{d-3}{2}} \left(\frac{v}{K^d}\right)^{1/2} .$$

Therefore at $t < 0$, the saddle point contribution dominates the behavior of this ratio provided that $d > 3$. For $d < 3$, the mean field result will continue being dominant far enough from the critical point, i.e. if

$$(-t)^{d-3} \gg \left(\frac{K^d}{v}\right), \qquad \text{or} \qquad |t| \gg \left(\frac{K^d}{v}\right)^{1/(d-3)} .$$

Otherwise, i.e. if

$$|t| < \left(\frac{K^d}{v}\right)^{1/(d-3)},$$

the fluctuation contribution to the heat capacity becomes dominant. The upper critical dimension for the tricritical point is then $d = 3$.

(g) A generalized multicritical point is described by replacing the term $vm^6$ with $u_{2n}m^{2n}$. Use simple power counting to find the upper critical dimension of this multicritical point.

• If instead of the term $vm^6$ we have a general factor of the form $u_{2n}m^{2n}$, we can easily generalize our results to

$$\overline{m} \propto (-t)^{1/(2n-2)}, \qquad \Psi(\overline{m}) \propto (-t)^{n/(n-1)}, \qquad C_{s.p.} \propto (-t)^{n/(n-1)-2}.$$

Moreover, the fluctuation correction to the heat capacity for any value of $n$ is the same as before

$$C_{fl.} \propto (-t)^{d/2-2}.$$

Hence the upper critical dimension is, in general, determined by the equation

$$\frac{d}{2} - 2 = \frac{n}{n-1} - 2, \qquad \text{or} \qquad d_u = \frac{2n}{n-1}.$$

********

---

**5.** *Coupling to a 'massless' field:* Consider an $n$-component vector field $\vec{m}(\mathbf{x})$ coupled to a scalar field $A(\mathbf{x})$, through the effective Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x} \left[\frac{K}{2}(\nabla\vec{m})^2 + \frac{t}{2}\vec{m}^2 + u(\vec{m}^2)^2 + e^2\vec{m}^2A^2 + \frac{L}{2}(\nabla A)^2\right],$$

with $K$, $L$, and $u$ positive.

(a) Show that there is a saddle point solution of the form $\vec{m}(\mathbf{x}) = \overline{m}\hat{e}_\ell$ and $A(x) = 0$, and find $\overline{m}$ for $t > 0$ and $t < 0$.

• The saddle point solution, assuming uniform $\vec{m}(\mathbf{x}) = \overline{m}\hat{e}_\ell$ and $A(x) = \overline{A}$, is

$$\ln Z = -V\min\left[\frac{t}{2}\overline{m}^2 + u\overline{m}^4 + e^2\overline{m}^2\overline{A}^2\right].$$

Since the last term is non-negative, it is minimized for $\overline{A} = 0$, while minimizing with respect to $\overline{m}$ yields

$$\overline{m} = \begin{cases} 0 & \text{for } t > 0 \\ \sqrt{-t/4u} & \text{for } t < 0 \end{cases} .$$

(b) Sketch the heat capacity $C = \partial^2 \ln Z / \partial t^2$, and discuss its singularity as $t \to 0$ in the saddle point approximation.

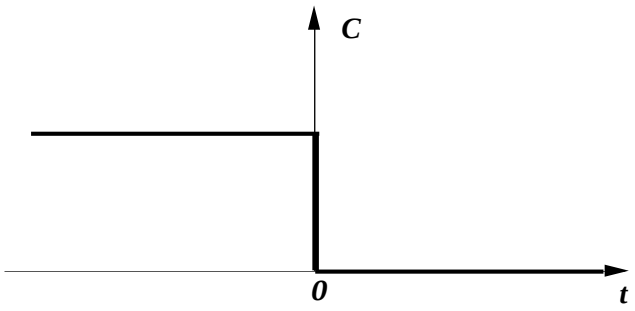
- From the saddle point free energy

$$f(t) = -\frac{\ln Z}{V} = \begin{cases} 0 & \text{for } t > 0 \\ -t^2/16u & \text{for } t < 0 \end{cases} ,$$

we obtain a heat capacity

$$C \propto \frac{\partial^2 f}{\partial t^2} = \begin{cases} 0 & \text{for } t > 0 \\ 1/8u & \text{for } t < 0 \end{cases} .$$

The heat capacity has a discontinuity at the transition, corresponding to $\alpha = 0$.




(c) Include fluctuations by setting

$$\begin{cases} \vec{m}(\mathbf{x}) = \big(\overline{m} + \phi_\ell(\mathbf{x})\big)\hat{e}_\ell + \phi_t(\mathbf{x})\hat{e}_t, \\ A(\mathbf{x}) = a(\mathbf{x}), \end{cases}$$

and expanding $\beta\mathcal{H}$ to quadratic order in $\phi$ and $a$.

- From $\vec{m}(\mathbf{x}) = (\overline{m} + \phi_\ell(\mathbf{x}))\hat{e}_\ell + \phi_t(\mathbf{x})\hat{e}_t$, we obtain

$$\begin{aligned} m^2 &= \overline{m}^2 + 2\overline{m}\phi_\ell + \phi_\ell^2 + \phi_t^2, \\ m^4 &= \overline{m}^4 + 4\overline{m}^3\phi_\ell + 6\overline{m}^2\phi_\ell^2 + 2\overline{m}^2\phi_t^2 + \mathcal{O}\left(\phi^3\right). \end{aligned}$$

After substituting the above (and $A = a$) in $\beta\mathcal{H}$, the linear terms vanish at the minimum, and the second order terms give

$$\beta\mathcal{H}_2 = \int d^d\mathbf{x}\left[\frac{K}{2}(\nabla\phi_\ell)^2 + \frac{t+12u\overline{m}^2}{2}\phi_\ell^2\right] + \int d^d\mathbf{x}\left[\frac{K}{2}(\nabla\phi_t)^2 + \frac{t+4u\overline{m}^2}{2}\phi_t^2\right] + \int d^d\mathbf{x}\left[\frac{L}{2}(\nabla a)^2 + \frac{2e^2\overline{m}^2}{2}a^2\right] + \mathcal{O}\left(\phi^3\right).$$

(d) Find the correlation lengths $\xi_\ell$, and $\xi_t$, for the longitudinal and transverse components of $\phi$, for $t > 0$ and $t < 0$.

- The longitudinal and transverse correlation lengths are given by

$$\xi_\ell^{-2} = \frac{t+12u\overline{m}^2}{K}, \quad \Longrightarrow \quad \xi_\ell = \begin{cases} \sqrt{\dfrac{K}{t}} & \text{for } t > 0 \\ \sqrt{\dfrac{-K}{2t}} & \text{for } t < 0 \end{cases},$$

and

$$\xi_t^{-2} = \frac{t+4u\overline{m}^2}{K}, \quad \Longrightarrow \quad \xi_t = \begin{cases} \sqrt{\dfrac{K}{t}} & \text{for } t > 0 \\ \infty & \text{for } t < 0 \end{cases}.$$

(e) Find the correlation length $\xi_a$ for the fluctuations of the scalar field $a$, for $t > 0$ and $t < 0$.

- The correlation length for the scalar field is

$$\xi_a^{-2} = \frac{2e^2\overline{m}^2}{L}, \quad \Longrightarrow \quad \xi_a = \begin{cases} \infty & \text{for } t > 0 \\ \sqrt{\dfrac{-2uL}{e^2t}} & \text{for } t < 0 \end{cases}.$$

(f) Calculate the correlation function $\langle a(\mathbf{x})a(\mathbf{0})\rangle$ for $t > 0$.

- Since the gauge field has no correlation length for $t > 0$, we have

$$\langle a(\mathbf{x})a(\mathbf{0})\rangle = \int \frac{d^d\mathbf{q}}{(2\pi)^d}e^{i\mathbf{q}\cdot\mathbf{x}}\left\langle|a(\mathbf{q})|^2\right\rangle = \int \frac{d^d\mathbf{q}}{(2\pi)^d}e^{i\mathbf{q}\cdot\mathbf{x}}\frac{1}{Lq^2} = -\frac{1}{L}C_d(\mathbf{x}).$$

The function $C_d(\mathbf{x})$ satisfies the Coulomb relation $\nabla^2 C_d = \delta^d(\mathbf{x})$. Integrating the above relation over a sphere of radius $x$ in $d$-dimensions yields

$$S_d x^{d-1}\frac{dC_d}{dx} = 1, \qquad C_d(x) = \frac{x^{2-d}}{S_d(2-d)} + c,$$

where $S_d = 2\pi^{d/2}/(d/2-1)!$ is the $d$-dimensional solid angle, and $c$ is a constant of integration. Up to this unknown constant, we thus get

$$\langle a(\mathbf{x})a(\mathbf{0})\rangle = -\frac{x^{2-d}}{LS_d(2-d)}.$$

(g) Compute the correction to the saddle point free energy $\ln Z$, from fluctuations. (You can leave the answer in the form of integrals involving $\xi_\ell$, $\xi_t$, and $\xi_a$.)

• The fluctuation corrections due to the three terms in $\beta\mathcal{H}_2$ are independent, and adding them up (after performing the Gaussian integrals) yields

$$\begin{aligned}-\frac{\ln Z}{V} &= \frac{t}{2}\overline{m}^2 + u\overline{m}^4 + \frac{1}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\ln\left[K\left(q^2+\xi_\ell^{-2}\right)\right] \\ &\quad + \frac{1}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\ln\left[K\left(q^2+\xi_t^{-2}\right)\right] + \frac{1}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\ln\left[L\left(q^2+\xi_a^{-2}\right)\right].\end{aligned}$$

(h) Find the fluctuation corrections to the heat capacity in (b), again leaving the answer in the form of integrals.

• For $t > 0$, the fluctuation corrections to the saddle point come from the 2 components of the field $m \sim \phi$, and the field $a$, resulting in

$$-\frac{\ln Z(t>0)}{V} = \frac{2}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\ln\left(Kq^2+t\right) + \frac{1}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\ln\left(Lq^2\right).$$

For $t < 0$, the various terms can be summed to

$$\begin{aligned}-\frac{\ln Z(t<0)}{V} &= -\frac{t^2}{16u} + \frac{1}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\ln\left(Kq^2-2t\right) \\ &\quad + \frac{1}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\ln\left(Kq^2\right) + \frac{1}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\ln\left(Lq^2-\frac{te^2}{2u}\right).\end{aligned}$$

After taking two derivatives with respect to $t$, we obtain

$$C \propto \frac{1}{V}\frac{\partial^2 \ln Z}{\partial t^2} = \begin{cases} \displaystyle\int \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{1}{(Kq^2+t)^2} & \text{for } t>0 \\ \displaystyle\frac{1}{8u} + 2\int \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{1}{Kq^2-2t} + \frac{e^4}{8u^2}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{1}{\left(Lq^2-\frac{te^2}{2u}\right)^2} & \text{for } t<0 \end{cases}.$$

(i) Discuss the behavior of the integrals appearing above schematically, and state their dependence on the correlation length $\xi$, and cutoff $\Lambda$, in different dimensions.

• The corrections to the heat capacity depend on the integral

$$I_d(\xi) \equiv \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{1}{(q^2+\xi^{-2})^2},$$

which has dimensions of $[q]^{d-4}$. In dimensions $d > 4$, the integral is dominated by the upper cut-off $\Lambda$, and its leading behavior is proportional to $\Lambda^{d-4}$. For $2 < d < 4$, the integral is convergent as $\Lambda \to \infty$ for finite $\xi$, and its most singular part scales as $\xi^{4-d}$, which diverges as $t \to 0$.

(j) What is the critical dimension for the validity of saddle point results, and how is it modified by the coupling to the scalar field?

• In dimensions $d \leq 4$, the fluctuation corrections to the saddle point diverge as $\xi \to \infty$. The scalar field fluctuations make an additional contribution, which also diverges for $d \leq 4$, and thus does not modify the upper critical dimension.

********

**6.** *Random magnetic fields:* Consider the Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x} \left[ \frac{K}{2} (\nabla m)^2 + \frac{t}{2} m^2 + u\, m^4 - h(\mathbf{x}) m(\mathbf{x}) \right] \quad ,$$

where $m(\mathbf{x})$ and $h(\mathbf{x})$ are scalar fields, and $u > 0$. The random magnetic field $h(\mathbf{x})$ results from frozen (quenched) impurities that are independently distributed in space. For simplicity $h(\mathbf{x})$ is assumed to be an independent Gaussian variable at each point $\mathbf{x}$, such that

$$\overline{h(\mathbf{x})} = 0, \qquad \text{and} \qquad \overline{h(\mathbf{x})h(\mathbf{x}')} = \Delta\delta^d(\mathbf{x}-\mathbf{x}'), \tag{1}$$

where the over-line indicates (*quench*) averaging over all values of the random fields. The above equation implies that the Fourier transformed random field $h(\mathbf{q})$ satisfies

$$\overline{h(\mathbf{q})} = 0, \qquad \text{and} \qquad \overline{h(\mathbf{q})h(\mathbf{q}')} = \Delta(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}'). \tag{2}$$

(a) Calculate the quench averaged free energy, $\overline{f_{sp}} = \overline{\min\{\Psi(m)\}_m}$, *assuming a saddle point solution with uniform magnetization* $m(\mathbf{x}) = m$. (Note that with this assumption, the random field disappears as a result of averaging and has no effect at this stage.)

• Assuming a uniform solution $m(\mathbf{x}) = \overline{m}$ to the saddle point of the weight, we obtain

$$\overline{f_{sp}} = -\frac{\overline{\ln Z}}{V} = \min\left[\frac{t}{2}m^2 + um^4 - m\overline{h(\mathbf{x})}\right]_m .$$

Since the random field averages to zero, the saddle point magnetization is obtained as in the case without random fields as

$$t\overline{m}+4u\overline{m}^{\,3} = 0, \quad \Rightarrow \quad \overline{m} = \begin{cases} 0 & \text{for } t > 0 \\ \pm\sqrt{-t/4u} & \text{for } t > 0 \end{cases}, \quad \Rightarrow \quad \overline{f_{sp}} = \begin{cases} 0 & \text{for } t > 0 \\ -t^2/16u & \text{for } t > 0 \end{cases}.$$

(b) Include fluctuations by setting $m(\mathbf{x}) = \overline{m} + \phi(\mathbf{x})$, and expanding $\beta\mathcal{H}$ to second order in $\phi$.

• To second order, we find

$$\beta\mathcal{H} = V\overline{f_{sp}} + \int d^d\mathbf{x}\left[\frac{K}{2}\left(\nabla\phi\right)^2 + \frac{t+12u\overline{m}^{\,2}}{2}\phi^2 - h(\mathbf{x})\phi(\mathbf{x})\right].$$

(c) Express the energy cost of the above fluctuations in terms of the Fourier modes $\phi(\mathbf{q})$.

• In terms of Fourier modes

$$\beta\mathcal{H} = V\overline{f_{sp}} + \frac{K}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\left[\left(q^2+\xi^{-2}\right)|\phi(\mathbf{q})|^2 - h(\mathbf{q})\phi(-\mathbf{q})\right],$$

where we have introduced a correlation length $\xi$ such that

$$\frac{K}{\xi^2} = \begin{cases} t & \text{for } t > 0 \\ -2t & \text{for } t > 0 \end{cases}.$$

(d) Calculate the mean $\langle\phi(\mathbf{q})\rangle$, and the variance $\left\langle|\phi(\mathbf{q})|^2\right\rangle_c$, where $\langle\cdots\rangle$ denotes the usual thermal expectation value *for a fixed* $h(\mathbf{q})$.

• For fixed $h(\mathbf{q})$, the probability for fluctuations in amplitudes of the independent Fourier mode is given as a product of Gaussian forms

$$\mathcal{P}[\phi(\mathbf{q})] = \prod_{\mathbf{q}} \exp\left\{-\frac{K(q^2+\xi^{-2})}{2}|\phi(\mathbf{q})|^2 - h(\mathbf{q})\phi(-\mathbf{q})\right\}.$$

Hence we can read off the mean and variance of the fluctuations as

$$\langle\phi(\mathbf{q})\rangle = \frac{h(\mathbf{q})}{K\left(q^2+\xi^{-2}\right)}, \qquad \text{and} \qquad \left\langle|\phi(\mathbf{q})|^2\right\rangle_c = \frac{1}{K\left(q^2+\xi^{-2}\right)}.$$

(e) Use the above results, in conjunction with Eq.(2), to calculate the quench averaged scattering line shape $S(q) = \overline{\langle|\phi(\mathbf{q})|^2\rangle}$.

- From the definitions of $S(\mathbf{q})$ and the variance, we get

$$\begin{aligned} S(\mathbf{q}) &= \overline{|\langle\phi(\mathbf{q})\rangle|^2} + \overline{\langle|\phi(\mathbf{q})|^2\rangle_c} = \frac{\overline{|h(\mathbf{q})|^2}}{K^2\left(q^2+\xi^{-2}\right)^2} + \frac{1}{K\left(q^2+\xi^{-2}\right)} \\ &= \frac{\Delta}{K^2\left(q^2+\xi^{-2}\right)^2} + \frac{1}{K\left(q^2+\xi^{-2}\right)}. \end{aligned}$$

Note that the result is the sum of a Lorentzian and a squared–Lorentzian. This is a common signature of scattering from random systems.

(f) Perform the Gaussian integrals over $\phi(\mathbf{q})$ to calculate the fluctuation corrections, $\delta f[h(\mathbf{q})]$, to the free energy.

$$\left(\text{Reminder}: \qquad \int_{-\infty}^{\infty} d\phi d\phi^* \exp\left(-\frac{K}{2}|\phi|^2 + h^*\phi + h\phi^*\right) = \frac{2\pi}{K}\exp\left(\frac{|h|^2}{2K}\right) \qquad \right)$$

- Including the Gaussian fluctuations, the partition function is given by

$$Z[h] = e^{-V\overline{f_{sp}}} \prod_{\mathbf{q}\geq 0} \int d\phi(\mathbf{q}) d\phi(\mathbf{q})^* \exp\left[-\frac{K\left(q^2+\xi^{-2}\right)}{2}|\phi(\mathbf{q})|^2 + h(-\mathbf{q})\phi(\mathbf{q}) + h(\mathbf{q})\phi(\mathbf{q})^*\right]$$

where $\phi(\mathbf{q})^* = \phi(-\mathbf{q})$. Hence, for a given configuration of the field $h$, the fluctuation–corrected free energy is

$$f[h] = -\frac{\ln Z[h]}{V} = \overline{f_{sp}} + \sum_{\mathbf{q}} \left\{ -\frac{1}{2}\ln\left[K\left(q^2+\xi^{-2}\right)\right] + \frac{|h(\mathbf{q})|^2}{2K\left(q^2+\xi^{-2}\right)} \right\}.$$

(g) Use Eq.(2) to calculate the corrections due to the fluctuations in the previous part to the quench averaged free energy $\overline{f}$. (Leave the corrections in the form of two integrals.)

- Taking the average of the last result over $h$ gives

$$\overline{f} = \overline{f_{sp}} - \frac{1}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d} \ln\left[K\left(q^2+\xi^{-2}\right)\right] + \frac{\Delta}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{1}{K\left(q^2+\xi^{-2}\right)}.$$

(h) Estimate the singular $t$ dependence of the integrals obtained in the fluctuation corrections to the free energy.

• The first integral has dimensions of $[q]^d$, and since the relevant $t$-dependent momentum is $q_\xi \sim \xi^{-1} \propto \sqrt{t}$, the most singular part of this integral behaves as $t^{d/2}$. The second integral has dimensions of $[q^{d-2}]$, and by similar reasoning leads to a singular form of $t^{d/2-1}$.

(i) Find the upper critical dimension, $d_u$, for the validity of saddle point critical behavior.

• The singular form of the saddle point solution scales as $t^2$, and we ask if the fluctuation corrections can dominate this singularity. The fluctuation correction that results from random fields (proportional to $\Delta$) is always more important than the standard contribution from thermal fluctuations. The former masks the saddle point singularity if $d/2 - 1 \leq 2$, i.e. for dimensions less that the upper critical dimension of $d_u = 6$.

********

**7.** *Long–range interactions:* Consider a continuous spin field $\vec{s}(\mathbf{x})$, subject to a long–range ferromagnetic interaction

$$\int d^d\mathbf{x} d^d\mathbf{y} \frac{\vec{s}(\mathbf{x}) \cdot \vec{s}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+\sigma}},$$

as well as short-range interactions.

(a) How is the quadratic term in the Landau-Ginzburg expansion modified by the presence of this long-range interaction? For what values of $\sigma$ is the long-range interaction dominant?

• The Fourier transform of the above interaction gives a quadratic term proportional to $|q|^\sigma$. Such a term is relevant, and dominates the standard $q^2$ term that arises from short-range interactions, provided that $\sigma < 2$.

(b) By estimating the magnitude of thermally excited Goldstone modes (or otherwise), obtain the lower critical dimension $d_\ell$ below which there is no long–range order.

• Goldstone modes are 'massless', and governed by a quadratic term proportional to $|q|^\sigma |\phi(\mathbf{q})|^2$ in the Fourier transformed Hamiltonian. Hence $\left\langle |\phi(\mathbf{q})|^2 \right\rangle \propto |q|^{-\sigma}$, and in real-space

$$\left\langle \phi(\mathbf{x})^2 \right\rangle \propto \int d^d\mathbf{q} \frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{|q|^\sigma}.$$

The above integral diverges with system size for dimensions $d \leq d_\ell = \sigma$. Due to increased stiffness of the Goldstone modes, the lower critical dimension is reduced from 2.

(c) Find the upper critical dimension $d_u$, above which saddle point results provide a correct description of the phase transition.

• From the above analysis, it is apparent that under a change of scale by a factor of $b$, the naive dimension of the field is $\zeta = b^{(d-\sigma)/2}$. The quartic term in the Landau Ginzburg equation thus scales as

$$u \to u' = b^{y_u^0} u + \mathcal{O}(u^2), \qquad \text{where} \qquad y_u^0 = d - 4 \times \frac{d-\sigma}{2} = 2\sigma - d.$$

The upper critical dimension is then identified as $d_u = 2\sigma$, below which the quartic term is relevant, invalidating the results from a Gaussian analysis.

********

**8.** *Ginzburg criterion along the magnetic field direction:* Consider the Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x} \left[ \frac{K}{2}(\nabla\vec{m})^2 + \frac{t}{2}\vec{m}^2 + u(\vec{m}^2)^2 \right] \quad ,$$

describing an $n$–component magnetization vector $\vec{m}(\mathbf{x})$, with $u > 0$.

(a) In the saddle point approximation, the free energy is $f = \min\{\Psi(m)\}_m$. Indicate the resulting phase boundary in the $(h, t)$ plane, and label the phases. ($h$ denotes the magnitude of $\vec{h}$.)

(b) Sketch the form of $\Psi(m)$ for $t < 0$, on both sides of the phase boundary, and for $t > 0$ at $h = 0$.

(c) For $t$ and $h$ close to zero, the spontaneous magnetization can be written as $\overline{m} = t^\beta g_m(h/t^\Delta)$. Identify the exponents $\beta$ and $\Delta$ in the saddle point approximation.

*For the remainder of this problem set $t = 0$.*

(d) Calculate the transverse and longitudinal susceptibilities at a finite $h$.

(e) Include fluctuations by setting $\vec{m}(\mathbf{x}) = \big(\overline{m} + \phi_\ell(\mathbf{x})\big)\hat{e}_\ell + \vec{\phi}_t(\mathbf{x})\hat{e}_t$, and expanding $\beta\mathcal{H}$ to second order in the $\phi$s. ($\hat{e}_\ell$ is a unit vector parallel to the average magnetization, and $\hat{e}_t$ is perpendicular to it.)

(f) Calculate the longitudinal and transverse correlation lengths.

(g) Calculate the first correction to the free energy from these fluctuations. (The scaling form is sufficient.)

(h) Calculate the first correction to magnetization, and to longitudinal susceptibility from the fluctuations.

(i) By comparing the saddle point value with the correction due to fluctuations, find the upper critical dimension, $d_u$, for the validity of the saddle point result.

(j) For $d < d_u$ obtain a Ginzburg criterion by finding the field $h_G$ below which fluctuations are important. (You may ignore the numerical coefficients in $h_G$, but the dependances on $K$ and $u$ are required.)

********

**Solutions to problems from chapter 4 - The Scaling Hypothesis**

**1.** *Scaling in fluids:* Near the liquid–gas critical point, the free energy is assumed to take the scaling form $F/N = t^{2-\alpha} g(\delta\rho/t^{\beta})$, where $t = |T - T_c|/T_c$ is the reduced temperature, and $\delta\rho = \rho - \rho_c$ measures deviations from the critical point density. The leading singular behavior of any thermodynamic parameter $Q(t, \delta\rho)$ is of the form $t^x$ on approaching the critical point along the isochore $\rho = \rho_c$; or $\delta\rho^y$ for a path along the isotherm $T = T_c$. Find the exponents $x$ and $y$ for the following quantities:

- Any homogeneous thermodynamic quantity $Q(t, \delta\rho)$ can be written in the scaling form

$$Q(t, \delta\rho) = t^{x_Q} g_Q\left(\frac{\delta\rho}{t^{\beta}}\right).$$

Thus, the leading singular behavior of $Q$ is of the form $t^{x_Q}$ if $\delta\rho = 0$, i.e. along the critical isochore. In order for any $Q$ to be independent of $t$ along the critical isotherm as $t \to 0$, the scaling function for a large enough argument should be of the form

$$\lim_{x\to\infty} g_Q(x) = x^{x_Q/\beta},$$

so that

$$Q(0, \delta\rho) \propto (\delta\rho)^{y_Q}, \qquad \text{with} \qquad y_Q = \frac{x_Q}{\beta}.$$

(a) The internal energy per particle $\langle H\rangle/N$, and the entropy per particle $s = S/N$.

- Let us assume that the free energy per particle is

$$f = \frac{F}{N} = t^{2-\alpha} g\left(\frac{\delta\rho}{t^{\beta}}\right),$$

and that $T < T_c$, so that $\frac{\partial}{\partial T} = -\frac{1}{T_c}\frac{\partial}{\partial t}$. The entropy is then given by

$$s = -\left.\frac{\partial f}{\partial T}\right|_V = \frac{1}{T_c}\left.\frac{\partial f}{\partial t}\right|_\rho = \frac{t^{1-\alpha}}{T_c} g_S\left(\frac{\delta\rho}{t^{\beta}}\right),$$

so that $x_S = 1 - \alpha$, and $y_S = (1 - \alpha)/\beta$. For the internal energy, we have

$$f = \frac{\langle\mathcal{H}\rangle}{N} - Ts, \qquad \text{or} \qquad \frac{\langle\mathcal{H}\rangle}{N} \sim T_c\ s(1 + t) \sim t^{1-\alpha} g_{\mathcal{H}}\left(\frac{\delta\rho}{t^{\beta}}\right),$$

therefore, $x_{\mathcal{H}} = 1 - \alpha$ and $y_{\mathcal{H}} = (1 - \alpha)/\beta$.

(b) The heat capacities $C_V = T\partial s/\partial T\,|_V$, and $C_P = T\partial s/\partial T\,|_P$.

- The heat capacity at constant volume

$$C_V = T\left.\frac{\partial S}{\partial T}\right|_V = -\left.\frac{\partial s}{\partial t}\right|_\rho = \frac{t^{-\alpha}}{T_c} g_{C_V}\left(\frac{\delta\rho}{t^\beta}\right),$$

so that $x_{C_V} = -\alpha$ and $y_{C_V} = -\alpha/\beta$.

To calculate the heat capacity at constant pressure, we need to determine first the relation $\delta\rho(t)$ at constant $P$. For that purpose we will use the thermodynamic identity

$$\left.\frac{\partial \delta\rho}{\partial t}\right|_P = -\frac{\left.\frac{\partial P}{\partial t}\right|_\rho}{\left.\frac{\partial P}{\partial \delta\rho}\right|_t}.$$

The pressure $P$ is determined as

$$P = -\frac{\partial F}{\partial V} = \rho^2 \frac{\partial f}{\partial \delta\rho} \sim \rho_c^2\, t^{2-\alpha-\beta} g_P\left(\frac{\delta\rho}{t^\beta}\right),$$

which for $\delta\rho \ll t^\beta$ goes like

$$P \propto t^{2-\alpha-\beta}\left(1 + A\frac{\delta\rho}{t^\beta}\right), \quad \text{and consequently} \quad \begin{cases} \left.\frac{\partial P}{\partial t}\right|_\rho \propto t^{1-\alpha-\beta} \\ \left.\frac{\partial P}{\partial \delta\rho}\right|_t \propto t^{2-\alpha-2\beta} \end{cases}.$$

In the other extreme of $\delta\rho \gg t^\beta$,

$$P \propto \delta\rho^{(2-\alpha-\beta)/\beta}\left(1 + B\frac{t}{\delta\rho^{1/\beta}}\right), \quad \text{and} \quad \begin{cases} \left.\frac{\partial P}{\partial t}\right|_\rho \propto \delta\rho^{(1-\alpha-\beta)/\beta} \\ \left.\frac{\partial P}{\partial \delta\rho}\right|_t \propto \delta\rho^{(2-\alpha-2\beta)/\beta} \end{cases},$$

where we have again required that $P$ does not depend on $\delta\rho$ when $\delta\rho \to 0$, and on $t$ if $t \to 0$.

From the previous results, we can now determine

$$\left.\frac{\partial \delta\rho}{\partial t}\right|_P \propto \begin{cases} t^{\beta-1} & \Longrightarrow \delta\rho \propto t^\beta \\ \delta\rho^{(\beta-1)/\beta} & \Longrightarrow t \propto \delta\rho^{1/\beta} \end{cases}.$$

From any of these relationships follows that $\delta\rho \propto t^\beta$, and consequently the entropy is $s \propto t^{1-\alpha}$. The heat capacity at constant pressure is then given by

$$C_P \propto t^{-\alpha}, \qquad \text{with} \qquad x_{C_P} = -\alpha \qquad \text{and} \qquad y_{C_P} = -\frac{\alpha}{\beta}.$$

(c) The isothermal compressibility $\kappa_T = \partial\rho/\partial P\,|_T\,/\rho$, and the thermal expansion coefficient $\alpha = \partial V/\partial T\,|_P\,/V$.

Check that your results for parts (b) and (c) are consistent with the thermodynamic identity $C_P - C_V = TV\alpha^2/\kappa_T$.

• The isothermal compressibility and the thermal expansion coefficient can be computed using some of the relations obtained previously

$$\kappa_T = \frac{1}{\rho}\frac{\partial\rho}{\partial P}\bigg|_T = \frac{1}{\rho_c}\frac{\partial P}{\partial\rho}\bigg|_T^{-1} = \frac{1}{\rho_c^3}t^{\alpha+2\beta-2}g_\kappa\left(\frac{\delta\rho}{t^\beta}\right),$$

with $x_\kappa = \alpha + 2\beta - 2$, and $y_\kappa = (\alpha + 2\beta - 2)/\beta$. And

$$\alpha = \frac{1}{V}\frac{\partial V}{\partial T}\bigg|_P = \frac{1}{\rho T_c}\frac{\partial\rho}{\partial t}\bigg|_P \propto t^{\beta-1}\ ,$$

with $x_\alpha = \beta - 1$, and $y_\alpha = (\beta - 1)/\beta$. So clearly, these results are consistent with the thermodynamic identity,

$$(C_P - C_V)(t,0) \propto t^{-\alpha}, \qquad \text{or} \qquad (C_P - C_V)(0,\delta\rho) \propto \delta\rho^{-\alpha/\beta},$$

and

$$\frac{\alpha^2}{\kappa_T}(t,0) \propto t^{-\alpha}, \qquad \text{or} \qquad \frac{\alpha^2}{\kappa_T}(0,\delta\rho) \propto \delta\rho^{-\alpha/\beta}.$$

(d) Sketch the behavior of the latent heat per particle $L$, on the coexistence curve for $T < T_c$, and find its singularity as a function of $t$.

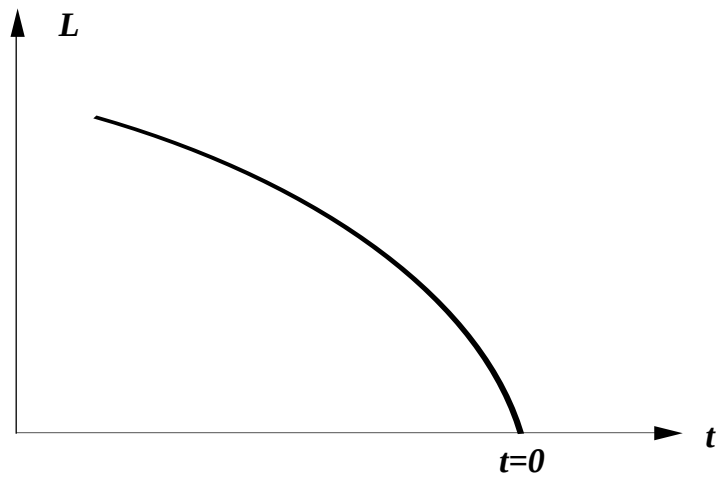
• The latent heat

$$L = T(s_+ - s_-)$$

is defined at the coexistence line, and as we have seen before

$$Ts_\pm = t^{1-\alpha}g_s\left(\frac{\delta\rho_\pm}{t^\beta}\right).$$

The density difference between the two coexisting phases is the order parameter, and vanishes as $t^\beta$, as do each of the two deviations $\delta\rho_+ = \rho_c - 1/v_+$ and $\delta\rho_- = \rho_c - 1/v_-$ of the gas and liquid densities from the critical critical value. (More precisely, as seen in (b), $\delta\rho|_{P=\text{constant}} \propto t^\beta$.) The argument of $g$ in the above expression is thus evaluated at




a finite value, and since the latent heat goes to zero on approaching the critical point, we get

$$L \propto t^{1-\alpha}, \qquad \text{with} \qquad x_L = 1 - \alpha.$$

********

**2.** *The Ising model:* The differential recursion relations for temperature $T$, and magnetic field $h$, of the Ising model in $d = 1 + \epsilon$ dimensions are

$$\begin{cases} \dfrac{dT}{d\ell} = -\epsilon T + \dfrac{T^2}{2} & , \\ \dfrac{dh}{d\ell} = dh & . \end{cases}$$

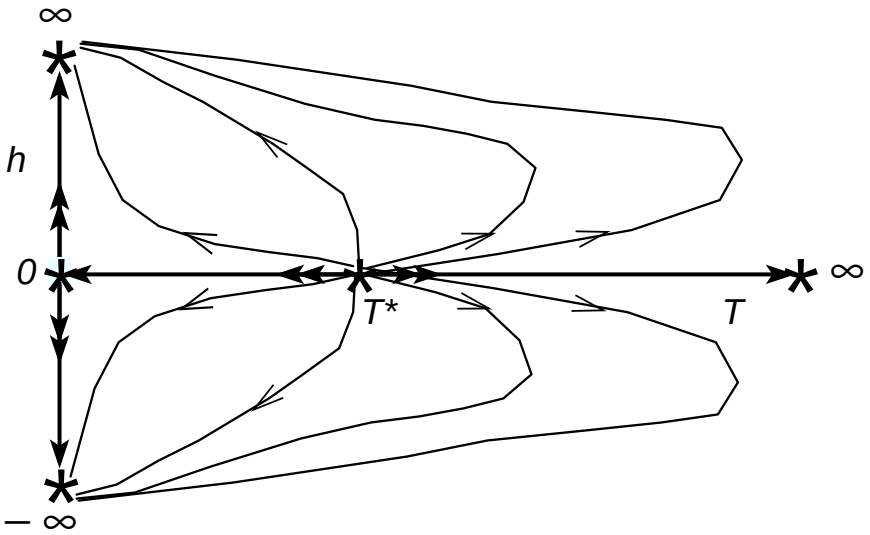
(a) Sketch the renormalization group flows in the $(T, h)$ plane (for $\epsilon > 0$), marking the fixed points along the $h = 0$ axis.

• The fixed points of the flow occur along the $h = 0$ axis, which is mapped to itself under RG. On this axis, there are three fixed points: **(i)** $T^* = 0$, is the stable sink for the low temperature phase. **(ii)** $T^* \to \infty$, is the stable sink for the high temperature phase. **(iii)** There is a critical fixed point at $(T^* = 2\epsilon, h^* = 0)$, which is unstable. All fixed points are unstable in the field direction.

(b) Calculate the eigenvalues $y_t$ and $y_h$, at the critical fixed point, to order of $\epsilon$.

• Linearizing $T = T^* + \delta T$, around the critical fixed point yields

$$\begin{cases} \dfrac{d\delta T}{d\ell} = -\epsilon\,\delta T + T^*\delta T = \epsilon\delta T \\ \dfrac{dh}{d\ell} = (1+\epsilon)h \end{cases}, \quad \Longrightarrow \quad \begin{cases} y_t = +\epsilon \\ y_h = 1+\epsilon \end{cases} .$$




(c) Starting from the relation governing the change of the correlation length $\xi$ under renormalization, show that $\xi(t,h) = t^{-\nu} g_\xi\left(h/|t|^\Delta\right)$ (where $t = T/T_c - 1$), and find the exponents $\nu$ and $\Delta$.

• Under rescaling by a factor of $b$, the correlation length is reduced by $b$, resulting in the homogeneity relation

$$\xi(t,h) = b\xi(b^{y_t}t, b^{y_h}h).$$

Upon selecting a rescaling factor such that $b^{y_t}t \sim 1$, we obtain

$$\xi(t,h) = t^{-\nu} g_\xi\left(h/|t|^\Delta\right),$$

with

$$\nu = \frac{1}{y_t} = \frac{1}{\epsilon}, \qquad \text{and} \qquad \Delta = \frac{y_h}{y_t} = \frac{1}{\epsilon} + 1.$$

(d) Use a hyperscaling relation to find the singular part of the free energy $f_{\text{sing.}}(t,h)$, and hence the heat capacity exponent $\alpha$.

• According to hyperscaling

$$f_{\text{sing.}}(t,h) \propto \xi(t,h)^{-d} = t^{d/y_t} g_f\left(h/|t|^\Delta\right).$$

Taking two derivatives with respect to $t$ leads to the heat capacity, whose singularity for $h = 0$ is described by the exponent

$$\alpha = 2 - d\nu = 2 - \frac{1+\epsilon}{\epsilon} = -\frac{1}{\epsilon} + 1.$$

(e) Find the exponents $\beta$ and $\gamma$ for the singular behaviors of the magnetization and susceptibility, respectively.

- The magnetization is obtained from the free energy by

$$m = -\left.\frac{\partial f}{\partial h}\right|_{h=0} \sim |t|^{\beta}, \quad \text{with} \quad \beta = \frac{d - y_h}{y_t} = 0.$$

(There will be corrections to $\beta$ at higher orders in $\epsilon$.) The susceptibility is obtained from a derivative of the magnetization, or

$$\chi = -\left.\frac{\partial^2 f}{\partial h^2}\right|_{h=0} \sim |t|^{-\gamma}, \quad \text{with} \quad \gamma = \frac{2y_h - d}{y_t} = \frac{1+\epsilon}{\epsilon} = \frac{1}{\epsilon} + 1.$$

(f) Starting the relation between susceptibility and correlations of local magnetizations, calculate the exponent $\eta$ for the critical correlations ($\langle m(\mathbf{0})m(\mathbf{x})\rangle \sim |\mathbf{x}|^{-(d-2+\eta)}$).

- The magnetic susceptibility is related to the connected correlation function via

$$\chi = \int d^d\mathbf{x} \left\langle m(\mathbf{0})m(\mathbf{x})\right\rangle_c .$$

Close to criticality, the correlations decay as a power law $\langle m(\mathbf{0})m(\mathbf{x})\rangle \sim |\mathbf{x}|^{-(d-2+\eta)}$, which is cut off at the correlation length $\xi$, resulting in

$$\chi \sim \xi^{(2-\eta)} \sim |t|^{-(2-\eta)\nu}.$$

From the corresponding exponent identity, we find

$$\gamma = (2-\eta)\nu, \quad \Longrightarrow \quad \eta = 2 - y_t\gamma = 2 - 2y_h + d = 2 - d = 1 - \epsilon.$$

(g) How does the correlation length diverge as $T \to 0$ (along $h = 0$) for $d = 1$?

- For $d = 1$, the recursion relation for temperature can be rearranged and integrated, i.e.

$$\frac{1}{T^2}\frac{dT}{d\ell} = \frac{1}{2}, \quad \Longrightarrow \quad d\left(-\frac{2}{T}\right) = d\ell.$$

We can integrate the above expression from a low temperature with correlation length $\xi(T)$ to a high temperature where $1/T \approx 0$, and at which the correlation length is of the order of the lattice spacing, to get

$$-\frac{2}{T} = \ln\left(\frac{\xi}{a}\right) \quad \Longrightarrow \quad \xi(T) = a\exp\left(\frac{2}{T}\right).$$

********

---

**3.** *The nonlinear $\sigma$ model* describes $n$ component unit spins. As we shall demonstrate later, in $d = 2$ dimensions, the recursion relations for temperature $T$, and magnetic field $h$, are

$$\begin{cases} \dfrac{dT}{d\ell} = \dfrac{(n-2)}{2\pi}T^2 & , \\ \dfrac{dh}{d\ell} = 2h & . \end{cases}$$

(a) How does the correlation length diverge as $T \to 0$?

• We can solve for $d\ell$ in the above two recursion relations, and integrate to obtain $\ell$ as

$$\begin{cases} d\ell = \dfrac{2\pi}{(n-2)}\dfrac{dT}{T^2}, & \ell = \dfrac{2\pi}{(n-2)}\left(\dfrac{1}{T} - \dfrac{1}{T(\ell)}\right) \\ d\ell = \dfrac{dh}{2h}, & \ell = \dfrac{1}{2}\ln\left(\dfrac{h(\ell)}{h}\right) \end{cases}, \quad \Rightarrow \quad \begin{cases} T(\ell)^{-1} = T^{-1} - \dfrac{(n-2)\ell}{2\pi} \\ h(\ell) = he^{2\ell} \end{cases}.$$

The correlation length is then obtained from ($b = e^{\ell}$)

$$\xi(T,h) = e^{\ell}\xi\left[T(\ell), h(\ell)\right].$$

Starting from $T$ and $h$ close to zero, renormalize until a value of $T(\ell^*) \sim 1$, which occurs for $\ell^* = 2\pi/[(n-2)T]$, leading to

$$\xi(T,h) = \exp\left[\frac{2\pi}{(n-2)T}\right] g_1\left(h\exp\left[\frac{4\pi}{(n-2)T}\right]\right).$$

(b) Write down the singular form of the free energy as $T, h \to 0$.

• Since $d = 2$, the singular part of the free energy scales as $\xi^{-2}$, i.e.

$$f_{sing}(T,h) \propto \xi^{-2} = \exp\left[-\frac{4\pi}{(n-2)T}\right] g_2\left(h\exp\left[\frac{4\pi}{(n-2)T}\right]\right).$$

(c) How does the susceptibility $\chi$, diverge as $T \to 0$ for $h = 0$?

• The susceptibility is obtained from the second derivative of the free energy as

$$\chi = \left.\frac{\partial^2 f}{\partial h^2}\right|_{h=0} \sim \exp\left[\frac{4\pi}{(n-2)T}\right].$$

********

**4.** *Coupled scalars:* Consider the Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x}\left[\frac{t}{2}m^2 + \frac{K}{2}(\nabla m)^2 - hm + \frac{L}{2}(\nabla^2\phi)^2 + v\nabla m.\nabla\phi\right],$$

coupling two one component fields $m$ and $\phi$.

(a) Write $\beta\mathcal{H}$ in terms of the Fourier transforms $m(\mathbf{q})$ and $\phi(\mathbf{q})$.

• Since all terms are quadratic, and $\int d^d\mathbf{x}A(\mathbf{x})B(\mathbf{x}) = \int \frac{d^d\mathbf{q}}{(2\pi)^d}A(\mathbf{q})B(-\mathbf{q})$, we can immediately write down

$$\beta\mathcal{H} = \int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d}\left[\frac{t+Kq^2}{2}|m(\mathbf{q})|^2 + \frac{Lq^4}{2}|\phi(\mathbf{q})|^2 + vq^2 m(\mathbf{q})\phi(-\mathbf{q})\right] - hm(\mathbf{q}=0).$$

(b) Construct a renormalization group transformation as in the text, by rescaling distances such that $\mathbf{q}' = b\mathbf{q}$; and the fields such that $m'(\mathbf{q}') = \tilde{m}(\mathbf{q})/z$ and $\phi'(\mathbf{q}') = \tilde{\phi}(\mathbf{q})/y$. Do not evaluate the integrals that just contribute a constant additive term.

• For an RG transformation, we subdivide the modes in Fourier space into two parts: One set with $\Lambda/b < |\mathbf{q}| \leq \Lambda$ which we integrate over, and a second set with $0 < |\mathbf{q}| \leq \Lambda/b$ which we keep. Since for this Gaussian Hamiltonian the two sets of modes are not coupled, the integration in trivial, and apart from a constant $V\delta f_b$, we obtain the same Hamiltonian for the retained modes ($\tilde{m}(\mathbf{q})$ and $\tilde{\phi}(\mathbf{q})$) as in the previous part, except that the integration range is now up to $\Lambda/b$. The second step of the RG transformation is to remove this difference by the rescaling $\mathbf{q} = \mathbf{q}'/b$. In the third step of RG, we shall renormalize the fields according to $m'(\mathbf{q}') = \tilde{m}(\mathbf{q}')/z$, and $\phi'(\mathbf{q}) = \tilde{\phi}(\mathbf{q}')/y$, to get the final renormalized Hamiltonian

$$\beta\mathcal{H}' = V\delta f_b + \int_0^\Lambda \frac{d^d\mathbf{q}'}{(2\pi)^d}\left[\frac{z^2b^{-d}t + Kz^2b^{-d-2}q'^2}{2}|m'(\mathbf{q}')|^2 + \frac{Ly^2b^{-d-4}q'^4}{2}|\phi'(\mathbf{q}')|^2 \right.$$
$$\left. + vzyb^{-d-2}q'^2 m'(\mathbf{q}')\phi'(-\mathbf{q}')\right] - hzm'(\mathbf{q}'=0).$$

The resulting recursion relations are

$$\begin{cases} t' = z^2 b^{-d} t \\ K' = z^2 b^{-d-2} K \\ L' = y^2 b^{-d-4} L \\ v' = zy b^{-d-2} v \\ h' = zh \end{cases} .$$

(c) There is a fixed point such that $K' = K$ and $L' = L$. Find $y_t$, $y_h$ and $y_v$ at this fixed point.

• We can choose $z^2 = b^{d+2}$ such that $K' = K$, and $y^2 = b^{d+4}$ such that $L' = L$. The recursion relations then become

$$\begin{cases} t' = b^2 t \\ v' = bv \\ h' = b^{1+d/2} h \end{cases} , \quad \Longrightarrow \quad \begin{cases} y_t = 2 \\ y_v = 1 \\ y_h = 1 + d/2 \end{cases} .$$

(d) The singular part of the free energy has a scaling from $f(t, h, v) = t^{2-\alpha} g(h/t^{\Delta}, v/t^{\omega})$ for $t, h, v$ close to zero. Find $\alpha, \Delta$, and $\omega$.

• The RG transformation preserves the partition function, and thus

$$\ln Z(t, h, v) = \ln Z(t', h', v'), \quad \Longrightarrow \quad V f(t, h, v) = V' f(t', h', v').$$

Since $V' = V/b^d$, we get the homogeneous form

$$f(t, h, v) = b^{-d} f(b^{y_t} t, ^{y_h} h, ^{y_v} v).$$

Setting $b^{y_t} t = 1$ with $b = t^{-1/y_t}$, yields

$$f(t, h, v) = t^{-d/y_t} f\left(1, h/t^{y_h/y_t}, v/t^{y_v/y_t}\right) \equiv t^{2-\alpha} g\left(h/t^{\Delta}, v/t^{\omega}\right),$$

with

$$\begin{cases} \alpha = 2 - \dfrac{d}{y_t} = 2 - \dfrac{d}{2} \\ \Delta = \dfrac{y_h}{y_t} = \dfrac{1}{2} + \dfrac{d}{4} \\ \omega = \dfrac{y_v}{y_t} = \dfrac{1}{2} \end{cases} .$$

(e) There is another fixed point such that $t' = t$ and $L' = L$. What are the relevant operators at this fixed point, and how do they scale?

• We can choose $z^2 = b^d$ such that $t' = t$, and $y^2 = b^{d+4}$ such that $L' = L$. The recursion relations then become

$$\begin{cases} K' = b^{-2}K \\ v' = v \\ h' = b^{d/2}h \end{cases} , \quad \Longrightarrow \quad \begin{cases} y_K = -2 \quad \text{(irrelevant)} \\ y_v = 0 \quad \text{(marginal)} \\ y_h = d/2 \quad \text{(relevant)} \end{cases} .$$

********

**Solutions to problems from chapter 5 - Perturbative Renormalization Group**

**1.** *Longitudinal susceptibility:* While there is no reason for the longitudinal susceptibility to diverge at the mean-field level, it in fact does so due to fluctuations in dimensions $d < 4$. This problem is intended to show you the origin of this divergence in perturbation theory. There are actually a number of subtleties in this calculation which you are instructed to ignore at various steps. You may want to think about why they are justified.

Consider the Landau–Ginzburg Hamiltonian:

$$\beta\mathcal{H} = \int d^d\mathbf{x} \left[ \frac{K}{2}(\nabla\vec{m})^2 + \frac{t}{2}\vec{m}^2 + u(\vec{m}^2)^2 \right] \quad ,$$

describing an $n$–component magnetization vector $\vec{m}(\mathbf{x})$, in the ordered phase for $t < 0$.

(a) Let $\vec{m}(\mathbf{x}) = \big(\overline{m}+\phi_\ell(\mathbf{x})\big)\hat{e}_\ell + \vec{\phi}_t(\mathbf{x})\hat{e}_t$, and expand $\beta\mathcal{H}$ *keeping all terms in the expansion.*

• With $\vec{m}\,(\mathbf{x}) = (\bar{m} + \phi_\ell\,(\mathbf{x}))\,\hat{e}_\ell + \vec{\phi}_t\,(\mathbf{x})\,\hat{e}_t$, and $\bar{m}$ the minimum of $\beta\mathcal{H}$,

$$\begin{aligned}\beta\mathcal{H} =& V\left(\frac{t}{2}\bar{m}^2 + u\bar{m}^4\right) + \int d^dx \Bigg\{ \frac{K}{2}\left[(\nabla\phi_\ell)^2 + \left(\nabla\vec{\phi}_t\right)^2\right] + \left(\frac{t}{2} + 6u\bar{m}^2\right)\phi_\ell^2 \\ &+ \left(\frac{t}{2} + 2u\bar{m}^2\right)\vec{\phi}_t{}^2 + 4u\bar{m}\left(\phi_\ell^3 + \phi_\ell\vec{\phi}_t{}^2\right) + u\left[\phi_\ell^4 + 2\phi_\ell^2\vec{\phi}_t{}^2 + \left(\vec{\phi}_t{}^2\right)^2\right] \Bigg\}.\end{aligned}$$

Since $\bar{m}^2 = -t/4u$ in the ordered phase ($t < 0$), this expression can be simplified, upon dropping the constant term, as

$$\begin{aligned}\beta\mathcal{H} = \int d^dx \Bigg\{ &\frac{K}{2}\left[(\nabla\phi_\ell)^2 + \left(\nabla\vec{\phi}_t\right)^2\right] - t\phi_\ell^2 + 4u\bar{m}\left(\phi_\ell^3 + \phi_\ell\vec{\phi}_t{}^2\right) \\ &+u\left[\phi_\ell^4 + 2\phi_\ell^2\vec{\phi}_t{}^2 + \left(\vec{\phi}_t{}^2\right)^2\right] \Bigg\}.\end{aligned}$$

(b) Regard the quadratic terms in $\phi_\ell$ and $\vec{\phi}_t$ as an unperturbed Hamiltonian $\beta\mathcal{H}_0$, and the lowest order term coupling $\phi_\ell$ and $\vec{\phi}_t$ as a perturbation $U$; i.e.

$$U = 4u\overline{m}\int d^d\mathbf{x}\phi_\ell(\mathbf{x})\vec{\phi}_t(\mathbf{x})^2.$$

Write $U$ in Fourier space in terms of $\phi_\ell(\mathbf{q})$ and $\vec{\phi}_t(\mathbf{q})$.

- We shall focus on the cubic term as a perturbation

$$U = 4u\bar{m} \int d^d x \phi_\ell \left(\mathbf{x}\right) \vec{\phi}_t \left(\mathbf{x}\right)^2 ,$$

which can be written in Fourier space as

$$U = 4u\bar{m} \int \frac{d^d q}{\left(2\pi\right)^d} \frac{d^d q'}{\left(2\pi\right)^d} \phi_\ell \left(-\mathbf{q} - \mathbf{q}'\right) \vec{\phi}_t \left(\mathbf{q}\right) \cdot \vec{\phi}_t \left(\mathbf{q}'\right) .$$

(c) Calculate the Gaussian (bare) expectation values $\langle \phi_\ell(\mathbf{q}) \phi_\ell(\mathbf{q}') \rangle_0$ and $\langle \phi_{t,\alpha}(\mathbf{q}) \phi_{t,\beta}(\mathbf{q}') \rangle_0$, and the corresponding momentum dependent susceptibilities $\chi_\ell(\mathbf{q})_0$ and $\chi_t(\mathbf{q})_0$.

- From the quadratic part of the Hamiltonian,

$$\beta \mathcal{H}_0 = \int d^d x \frac{1}{2} \left\{ K \left[ \left(\nabla \phi_\ell\right)^2 + \left(\nabla \vec{\phi}_t\right)^2 \right] - 2t \phi_\ell^2 \right\} ,$$

we read off the expectation values

$$\begin{cases} \langle \phi_\ell \left(\mathbf{q}\right) \phi_\ell \left(\mathbf{q}'\right) \rangle_0 = \dfrac{\left(2\pi\right)^d \delta^d \left(\mathbf{q} + \mathbf{q}'\right)}{Kq^2 - 2t} \\ \langle \phi_{t,\alpha} \left(\mathbf{q}\right) \phi_{t,\beta} \left(\mathbf{q}'\right) \rangle_0 = \dfrac{\left(2\pi\right)^d \delta^d \left(\mathbf{q} + \mathbf{q}'\right) \delta_{\alpha_\beta}}{Kq^2} \end{cases} ,$$

and the corresponding susceptibilities

$$\begin{cases} \chi_\ell \left(\mathbf{q}\right)_0 = \dfrac{1}{Kq^2 - 2t} \\ \chi_t \left(\mathbf{q}\right)_0 = \dfrac{1}{Kq^2} \end{cases} .$$

(d) Calculate $\langle \vec{\phi}_t(\mathbf{q}_1) \cdot \vec{\phi}_t(\mathbf{q}_2) \;\; \vec{\phi}_t(\mathbf{q}_1') \cdot \vec{\phi}_t(\mathbf{q}_2') \rangle_0$ using Wick's theorem. (Don't forget that $\vec{\phi}_t$ is an $(n-1)$ component vector.)

- Using Wick's theorem,

$$\begin{aligned} &\left\langle \vec{\phi}_t \left(\mathbf{q}_1\right) \cdot \vec{\phi}_t \left(\mathbf{q}_2\right) \vec{\phi}_t \left(\mathbf{q}_1'\right) \cdot \vec{\phi}_t \left(\mathbf{q}_2'\right) \right\rangle_0 \equiv \langle \phi_{t,\alpha} \left(\mathbf{q}_1\right) \phi_{t,\alpha} \left(\mathbf{q}_2\right) \phi_{t,\beta} \left(\mathbf{q}_1'\right) \phi_{t,\beta} \left(\mathbf{q}_2'\right) \rangle_0 \\ &= \langle \phi_{t,\alpha} \left(\mathbf{q}_1\right) \phi_{t,\alpha} \left(\mathbf{q}_2\right) \rangle_0 \langle \phi_{t,\beta} \left(\mathbf{q}_1'\right) \phi_{t,\beta} \left(\mathbf{q}_2'\right) \rangle_0 + \langle \phi_{t,\alpha} \left(\mathbf{q}_1\right) \phi_{t,\beta} \left(\mathbf{q}_1'\right) \rangle_0 \langle \phi_{t,\alpha} \left(\mathbf{q}_2\right) \phi_{t,\beta} \left(\mathbf{q}_2'\right) \rangle_0 \\ &+ \langle \phi_{t,\alpha} \left(\mathbf{q}_1\right) \phi_{t,\beta} \left(\mathbf{q}_2'\right) \rangle_0 \langle \phi_{t,\alpha} \left(\mathbf{q}_2\right) \phi_{t,\beta} \left(\mathbf{q}_1'\right) \rangle_0 . \end{aligned}$$

Then, from part (c),

$$\left\langle \vec{\phi}_t\left(\mathbf{q}_1\right)\cdot\vec{\phi}_t\left(\mathbf{q}_2\right)\vec{\phi}_t\left(\mathbf{q}_1'\right)\cdot\vec{\phi}_t\left(\mathbf{q}_2'\right)\right\rangle_0 = \frac{(2\pi)^{2d}}{K^2}\Bigg\{ (n-1)^2\,\frac{\delta^d\left(\mathbf{q}_1+\mathbf{q}_2\right)\delta^d\left(\mathbf{q}_1'+\mathbf{q}_2'\right)}{q_1^2 q_1'^2} + (n-1)\,\frac{\delta^d\left(\mathbf{q}_1+\mathbf{q}_1'\right)\delta^d\left(\mathbf{q}_2+\mathbf{q}_2'\right)}{q_1^2 q_2^2} + (n-1)\,\frac{\delta^d\left(\mathbf{q}_1+\mathbf{q}_2'\right)\delta^d\left(\mathbf{q}_1'+\mathbf{q}_2\right)}{q_1^2 q_2^2}\Bigg\},$$

since $\delta_{\alpha\alpha}\delta_{\beta\beta} = (n-1)^2$, and $\delta_{\alpha\beta}\delta_{\alpha\beta} = (n-1)$.

(e) Write down the expression for $\langle\phi_\ell(\mathbf{q})\phi_\ell(\mathbf{q}')\rangle$ to second-order in the perturbation $U$. Note that since $U$ is odd in $\phi_\ell$, only two terms at the second order are non–zero.

• Including the perturbation $U$ in the calculation of the correlation function, we have

$$\left\langle \phi_\ell\left(\mathbf{q}\right)\phi_\ell\left(\mathbf{q}'\right)\right\rangle = \frac{\left\langle \phi_\ell\left(\mathbf{q}\right)\phi_\ell\left(\mathbf{q}'\right)e^{-U}\right\rangle_0}{\left\langle e^{-U}\right\rangle_0} = \frac{\left\langle \phi_\ell\left(\mathbf{q}\right)\phi_\ell\left(\mathbf{q}'\right)\left(1-U+U^2/2+\cdots\right)\right\rangle_0}{\left\langle\left(1-U+U^2/2+\cdots\right)\right\rangle_0}.$$

Since $U$ is odd in $\phi_\ell$, $\langle U\rangle_0 = \langle\phi_\ell\left(\mathbf{q}\right)\phi_\ell\left(\mathbf{q}'\right)U\rangle_0 = 0$. Thus, after expanding the denominator to second order,

$$\frac{1}{1+\langle U^2/2\rangle_0+\cdots} = 1-\left\langle\frac{U^2}{2}\right\rangle_0+\mathcal{O}\left(U^3\right),$$

we obtain

$$\left\langle \phi_\ell\left(\mathbf{q}\right)\phi_\ell\left(\mathbf{q}'\right)\right\rangle = \left\langle \phi_\ell\left(\mathbf{q}\right)\phi_\ell\left(\mathbf{q}'\right)\right\rangle_0 + \frac{1}{2}\left(\left\langle \phi_\ell\left(\mathbf{q}\right)\phi_\ell\left(\mathbf{q}'\right)U^2\right\rangle_0 - \left\langle \phi_\ell\left(\mathbf{q}\right)\phi_\ell\left(\mathbf{q}'\right)\right\rangle_0\left\langle U^2\right\rangle_0\right).$$

(f) Using the form of $U$ in Fourier space, write the correction term as a product of two 4–point expectation values similar to those of part (d). Note that only connected terms for the longitudinal 4–point function should be included.

• Substituting for $U$ its expression in terms of Fourier transforms from part (b), the fluctuation correction to the correlation function reads

$$\begin{aligned} G_F\left(\mathbf{q},\mathbf{q}'\right) &\equiv \left\langle \phi_\ell\left(\mathbf{q}\right)\phi_\ell\left(\mathbf{q}'\right)\right\rangle - \left\langle \phi_\ell\left(\mathbf{q}\right)\phi_\ell\left(\mathbf{q}'\right)\right\rangle_0 \\ &= \frac{1}{2}\left(4u\bar{m}\right)^2\int\frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_1'}{(2\pi)^d}\frac{d^dq_2'}{(2\pi)^d}\Big\langle \phi_\ell\left(\mathbf{q}\right)\phi_\ell\left(\mathbf{q}'\right)\phi_\ell\left(-\mathbf{q}_1-\mathbf{q}_2\right)\vec{\phi}_t\left(\mathbf{q}_1\right)\cdot\vec{\phi}_t\left(\mathbf{q}_2\right) \\ &\times\phi_\ell\left(-\mathbf{q}_1'-\mathbf{q}_2'\right)\vec{\phi}_t\left(\mathbf{q}_1'\right)\cdot\vec{\phi}_t\left(\mathbf{q}_2'\right)\Big\rangle_0 - \frac{1}{2}\left\langle \phi_\ell\left(\mathbf{q}\right)\phi_\ell\left(\mathbf{q}'\right)\right\rangle_0\left\langle U^2\right\rangle_0, \end{aligned}$$

*i.e.* $G_F(\mathbf{q},\mathbf{q}')$ is calculated as the connected part of

$$\frac{1}{2}(4u\bar{m})^2\int\frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_1'}{(2\pi)^d}\frac{d^dq_2'}{(2\pi)^d}\left\langle\phi_\ell(\mathbf{q})\,\phi_\ell(\mathbf{q}')\,\phi_\ell(-\mathbf{q}_1-\mathbf{q}_2)\,\phi_\ell(-\mathbf{q}_1'-\mathbf{q}_2')\right\rangle_0$$
$$\times\left\langle\vec{\phi}_t(\mathbf{q}_1)\cdot\vec{\phi}_t(\mathbf{q}_2)\,\vec{\phi}_t(\mathbf{q}_1')\cdot\vec{\phi}_t(\mathbf{q}_2')\right\rangle_0,$$

where we have used the fact that the unperturbed averages of products of longitudinal and transverse fields factorize. Hence

$$\begin{aligned}G_F(\mathbf{q},\mathbf{q}') &= \frac{1}{2}(4u\bar{m})^2\int\frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_1'}{(2\pi)^d}\frac{d^dq_2'}{(2\pi)^d}\left\langle\vec{\phi}_t(\mathbf{q}_1)\cdot\vec{\phi}_t(\mathbf{q}_2)\,\vec{\phi}_t(\mathbf{q}_1')\cdot\vec{\phi}_t(\mathbf{q}_2')\right\rangle_0\\ &\quad\times\Big\{\left\langle\phi_\ell(\mathbf{q})\,\phi_\ell(-\mathbf{q}_1-\mathbf{q}_2)\right\rangle_0\left\langle\phi_\ell(\mathbf{q}')\,\phi_\ell(-\mathbf{q}_1'-\mathbf{q}_2')\right\rangle_0\\ &\quad+\left\langle\phi_\ell(\mathbf{q})\,\phi_\ell(-\mathbf{q}_1'-\mathbf{q}_2')\right\rangle_0\left\langle\phi_\ell(\mathbf{q}')\,\phi_\ell(-\mathbf{q}_1-\mathbf{q}_2)\right\rangle_0\Big\}\\ &= 2\times\frac{1}{2}(4u\bar{m})^2\int\frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_1'}{(2\pi)^d}\frac{d^dq_2'}{(2\pi)^d}\left\langle\vec{\phi}_t(\mathbf{q}_1)\cdot\vec{\phi}_t(\mathbf{q}_2)\,\vec{\phi}_t(\mathbf{q}_1')\cdot\vec{\phi}_t(\mathbf{q}_2')\right\rangle_0\\ &\quad\times\left\langle\phi_\ell(\mathbf{q})\,\phi_\ell(-\mathbf{q}_1-\mathbf{q}_2)\right\rangle_0\left\langle\phi_\ell(\mathbf{q}')\,\phi_\ell(-\mathbf{q}_1'-\mathbf{q}_2')\right\rangle_0.\end{aligned}$$

Using the results of parts (c) and (d) for the two and four points correlation functions, and since $u^2\bar{m}^2=-ut/4$, we obtain

$$\begin{aligned}G_F(\mathbf{q},\mathbf{q}') &= 4u(-t)\int\frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_1'}{(2\pi)^d}\frac{d^dq_2'}{(2\pi)^d}\frac{(2\pi)^{2d}}{K^2}\Bigg\{(n-1)^2\frac{\delta^d(\mathbf{q}_1+\mathbf{q}_2)\,\delta^d(\mathbf{q}_1'+\mathbf{q}_2')}{q_1^2q_1'^2}\\ &\quad+(n-1)\frac{\delta^d(\mathbf{q}_1+\mathbf{q}_1')\,\delta^d(\mathbf{q}_2+\mathbf{q}_2')+\delta^d(\mathbf{q}_1+\mathbf{q}_2')\,\delta^d(\mathbf{q}_1'+\mathbf{q}_2)}{q_1^2q_2^2}\Bigg\}\\ &\quad\times\frac{(2\pi)^d\,\delta^d(\mathbf{q}-\mathbf{q}_1-\mathbf{q}_2)}{Kq^2-2t}\frac{(2\pi)^d\,\delta^d(\mathbf{q}'-\mathbf{q}_1'-\mathbf{q}_2')}{Kq^2-2t},\end{aligned}$$

which, after doing some of the integrals, reduces to

$$\begin{aligned}G_F(\mathbf{q},\mathbf{q}') &= \frac{4u(-t)}{K^2}\Bigg\{(n-1)^2\frac{\delta^d(\mathbf{q})\,\delta^d(\mathbf{q}')}{4t^2}\left(\int\frac{d^dq_1}{q_1^2}\right)^2\\ &\quad+2(n-1)\frac{\delta^d(\mathbf{q}+\mathbf{q}')}{(Kq^2-2t)^2}\int\frac{d^dq_1}{q_1^2(\mathbf{q}+\mathbf{q}_1)^2}\Bigg\}.\end{aligned}$$

(g) Ignore the disconnected term obtained in (d) (i.e. the part proportional to $(n-1)^2$), and write down the expression for $\chi_\ell(\mathbf{q})$ in second order perturbation theory.

• From the dependence of the first term (proportional to $\delta^d(\mathbf{q})\,\delta^d(\mathbf{q}')$), we deduce that this term is actually a correction to the unperturbed value of the magnetization, i.e.

$$\overline{m} \to \overline{m}\left[1 - \frac{2(n-1)u}{Kt}\left(\int \frac{d^d q_1}{q_1^2}\right)\right],$$

and does not contribute to the correlation function at non-zero separation. The spatially varying part of the connected correlation function is thus

$$\langle \phi_\ell(\mathbf{q})\,\phi_\ell(\mathbf{q}')\rangle = \frac{(2\pi)^d\,\delta^d(\mathbf{q}+\mathbf{q}')}{Kq^2-2t} + \frac{8u(-t)}{K^2}(n-1)\frac{\delta^d(\mathbf{q}+\mathbf{q}')}{(Kq^2-2t)^2}\int \frac{d^d q_1}{q_1^2(\mathbf{q}+\mathbf{q}_1)^2},$$

leading to

$$\chi_\ell(\mathbf{q}) = \frac{1}{Kq^2-2t} + \frac{8u(-t)}{K^2}\frac{(n-1)}{(Kq^2-2t)^2}\int \frac{d^d q_1}{(2\pi)^d}\frac{1}{q_1^2(\mathbf{q}+\mathbf{q}_1)^2}.$$

(h) Show that for $d<4$, the correction term diverges as $q^{d-4}$ for $q\to 0$, implying an infinite longitudinal susceptibility.

• In $d>4$, the above integral converges and is dominated by the large $q$ cutoff $\Lambda$. In $d<4$, on the other hand, the integral clearly diverges as $q\to 0$, and is thus dominated by small $q_1$ values. Changing the variable of integration to $\mathbf{q}_1' = \mathbf{q}_1/q$, the fluctuation correction to the susceptibility reads

$$\chi_\ell(\mathbf{q})_F \sim q^{d-4}\int_0^{\Lambda/q}\frac{d^d q_1'}{(2\pi)^d}\frac{1}{q_1'^2(\hat{\mathbf{q}}+\mathbf{q}_1')^2} = q^{d-4}\int_0^{\infty}\frac{d^d q_1'}{(2\pi)^d}\frac{1}{q_1'^2(\hat{\mathbf{q}}+\mathbf{q}_1')^2} + \mathcal{O}\left(q^0\right),$$

which diverges as $q^{d-4}$ for $q\to 0$.

**NOTE:** For a translationally invariant system,

$$\langle \phi(\mathbf{x})\,\phi(\mathbf{x}')\rangle = \varphi(\mathbf{x}-\mathbf{x}'),$$

which implies

$$\begin{aligned}\langle \phi(\mathbf{q})\,\phi(\mathbf{q}')\rangle &= \int d^d x d^d x' e^{i\mathbf{q}\cdot\mathbf{x}+i\mathbf{q}'\cdot\mathbf{x}'}\langle \phi(\mathbf{x})\,\phi(\mathbf{x}')\rangle \\ &= \int d^d(x-x')\,d^d x' e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')+i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{x}'}\varphi(\mathbf{x}-\mathbf{x}') \\ &= (2\pi)^d\,\delta^d(\mathbf{q}+\mathbf{q}')\,\psi(\mathbf{q}).\end{aligned}$$

Consider the Hamiltonian

$$-\beta\mathcal{H}' = -\beta\mathcal{H} + \int d^d x h(\mathbf{x})\,\phi(\mathbf{x}) = -\beta\mathcal{H} + \int \frac{d^d q}{(2\pi)^d} h(\mathbf{q})\,\phi(-\mathbf{q}),$$

where $-\beta\mathcal{H}$ is a translationally invariant functional of $\phi$ (a one-component field for simplicity), independent of $h(\mathbf{x})$. We have

$$m(\mathbf{x}=\mathbf{0}) = \langle\phi(\mathbf{0})\rangle = \int \frac{d^d q}{(2\pi)^d}\,\langle\phi(\mathbf{q})\rangle,$$

and, taking a derivative,

$$\frac{\partial m}{\partial h(\mathbf{q})} = \int \frac{d^d q'}{(2\pi)^d}\,\langle\phi(\mathbf{q}')\,\phi(\mathbf{q})\rangle.$$

At $h=0$, the system is translationally invariant, and

$$\left.\frac{\partial m}{\partial h(\mathbf{q})}\right|_{h=0} = \psi(\mathbf{q}).$$

Also, for a uniform external magnetic field, the system is translationally invariant, and

$$-\beta\mathcal{H}' = -\beta\mathcal{H} + h\int d^d x\phi(\mathbf{x}) = -\beta\mathcal{H} + h\phi(\mathbf{q}=\mathbf{0}),$$

yielding

$$\chi = \frac{\partial m}{\partial h} = \int \frac{d^d q'}{(2\pi)^d}\,\langle\phi(\mathbf{q}')\,\phi(\mathbf{q}=\mathbf{0})\rangle = \psi(\mathbf{0}).$$

********

**2.** *Crystal anisotropy:* Consider a ferromagnet with a tetragonal crystal structure. Coupling of the spins to the underlying lattice may destroy their full rotational symmetry. The resulting anisotropies can be described by modifying the Landau–Ginzburg Hamiltonian to

$$\beta\mathcal{H} = \int d^d\mathbf{x}\left[\frac{K}{2}(\nabla\vec{m})^2 + \frac{t}{2}\vec{m}^2 + u\left(\vec{m}^2\right)^2 + \frac{r}{2}m_1^2 + v\,m_1^2\,\vec{m}^2\right],$$

where $\vec{m} \equiv (m_1, \cdots, m_n)$, and $\vec{m}^2 = \sum_{i=1}^n m_i^2$ ($d = n = 3$ for magnets in three dimensions). Here $u > 0$, and *to simplify calculations we shall set* $v = 0$ throughout.

(a) For a fixed magnitude $|\vec{m}|$; what directions in the $n$ component magnetization space are selected for $r > 0$, and for $r < 0$?

• $r > 0$ discourages ordering along direction 1, and leads to order along the remaining $(n-1)$ directions.

$r < 0$ encourages ordering along direction 1.

(b) Using the saddle point approximation, calculate the free energies ($\ln Z$) for phases uniformly magnetized *parallel* and *perpendicular* to direction 1.

• In the saddle point approximation for $\vec{m}(\mathbf{x}) = m\hat{e}_1$, we have

$$\ln Z_{sp} = -V \min \left[ \frac{t+r}{2} m^2 + u m^4 \right]_m ,$$

where $V = \int d^d\mathbf{x}$, is the system volume. The minimum is obtained for

$$(t+r)\overline{m} + 4u\overline{m}^{\,3} = 0, \qquad \Longrightarrow \qquad \overline{m} = \begin{cases} 0 & \text{for } t+r > 0 \\ \sqrt{-(t+r)/4u} & \text{for } t+r < 0 \end{cases} .$$

For $t + r < 0$, the free energy is given by

$$f_{sp} = -\frac{\ln Z_{sp}}{V} = -\frac{(t+r)^2}{16u}.$$

When the magnetization is perpendicular to direction 1, i.e. for $\vec{m}(\mathbf{x}) = m\hat{e}_i$ for $i \neq 1$, the corresponding expressions are

$$\ln Z_{sp} = -V \min \left[ \frac{t}{2} m^2 + u m^4 \right]_m , \quad t\overline{m} + 4u\overline{m}^{\,3} = 0, \quad \overline{m} = \begin{cases} 0 & \text{for } t > 0 \\ \sqrt{-t/4u} & \text{for } t < 0 \end{cases} ,$$

and the free energy for $t < 0$ is

$$f_{sp} = -\frac{t^2}{16u}.$$

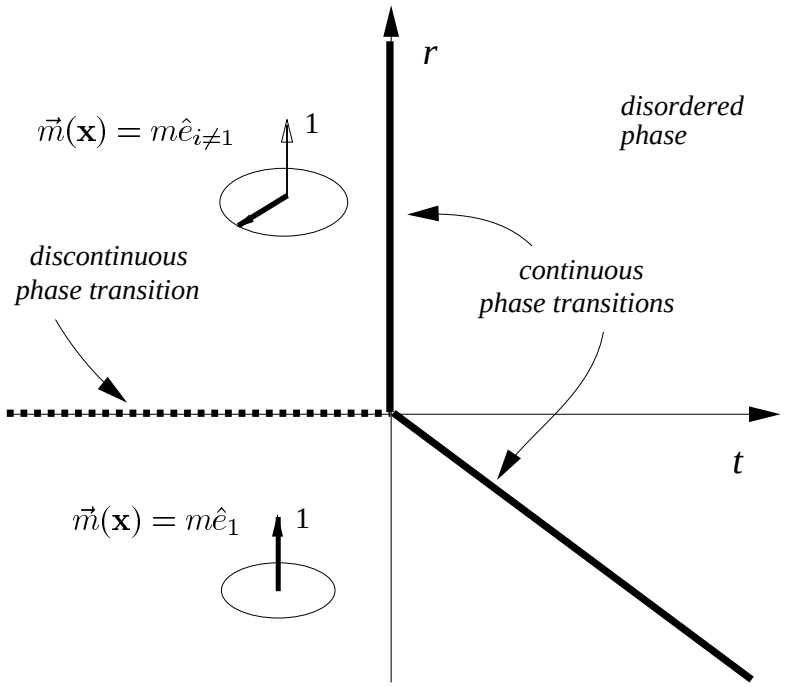
(c) Sketch the phase diagram in the $(t, r)$ plane, and indicate the phases (type of order), and the nature of the phase transitions (continuous or discontinuous).

• The saddle point phase diagram is sketched in the figure.

(d) Are there Goldstone modes in the ordered phases?

• There are no Goldstone modes in the phase with magnetization aligned along direction 1, as the broken symmetry in this case is discrete. However, there are $(n-2)$ Goldstone modes in the phase where magnetization is perpendicular to direction 1.

(e) For $u = 0$, and positive $t$ and $r$, calculate the unperturbed averages $\langle m_1(\mathbf{q}) m_1(\mathbf{q}') \rangle_0$ and $\langle m_2(\mathbf{q}) m_2(\mathbf{q}') \rangle_0$, where $m_i(\mathbf{q})$ indicates the Fourier transform of $m_i(\mathbf{x})$.




• The Gaussian part of the Hamiltonian can be decomposed into Fourier modes as

$$\beta\mathcal{H}_0 = \int \frac{d^d\mathbf{q}}{(2\pi)^d}\left[\frac{K}{2}q^2|\vec{m}(\mathbf{q})|^2 + \frac{t+r}{2}|m_1(\mathbf{q})|^2 + \sum_{i=2}^{n}\frac{t}{2}|m_i(\mathbf{q})|^2\right].$$

From this form we can easily read off the covariances

$$\begin{cases} \langle m_1(\mathbf{q})m_1(\mathbf{q}')\rangle_0 = \dfrac{(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')}{t+r+Kq^2} \\ \langle m_2(\mathbf{q})m_2(\mathbf{q}')\rangle_0 = \dfrac{(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')}{t+Kq^2} \end{cases}.$$

(f) Write the fourth order term $\mathcal{U} \equiv u\int d^d\mathbf{x}(\vec{m}^2)^2$, in terms of the Fourier modes $m_i(\mathbf{q})$.
• Substituting $m_i(\mathbf{x}) = \int \frac{d^d\mathbf{q}}{(2\pi)^d}\exp(i\mathbf{q}\cdot\mathbf{x})m_i(\mathbf{q})$ in the quartic term, and integrating over $\mathbf{x}$ yields

$$\mathcal{U} = u\int d^d\mathbf{x}\left(\vec{m}^2\right)^2 = u\int \frac{d^dq_1d^dq_2d^dq_3}{(2\pi)^{3d}}\sum_{i,j=1}^{n} m_i(\mathbf{q}_1)m_i(\mathbf{q}_2)m_j(\mathbf{q}_3)m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3).$$

(g) Treating $\mathcal{U}$ as a perturbation, calculate the *first order* correction to $\langle m_1(\mathbf{q})m_1(\mathbf{q}')\rangle$. (You can leave your answers in the form of some integrals.)

• In first order perturbation theory $\langle\mathcal{O}\rangle = \langle\mathcal{O}\rangle_0 - (\langle\mathcal{O}\mathcal{U}\rangle_0 - \langle\mathcal{O}\rangle_0 \langle\mathcal{U}\rangle_0)$, and hence

$$\begin{aligned}\langle m_1(\mathbf{q})m_1(\mathbf{q}')\rangle &= \langle m_1(\mathbf{q})m_1(\mathbf{q}')\rangle_0 - u\int \frac{d^dq_1 d^dq_2 d^dq_3}{(2\pi)^{3d}} \sum_{i,j=1}^{n} \\ &\qquad \left(\langle m_1(\mathbf{q})m_1(\mathbf{q}')m_i(\mathbf{q}_1)m_i(\mathbf{q}_2)m_j(\mathbf{q}_3)m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\rangle_0^c\right) \\ =&\frac{(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')}{t+r+Kq^2}\left\{1 - \frac{u}{t+r+Kq^2}\int\frac{d^d\mathbf{k}}{(2\pi)^d}\left[\frac{4(n-1)}{t+Kk^2} + \frac{4}{t+r+Kk^2} + \frac{8}{t+r+Kk^2}\right]\right\}\end{aligned}$$

The last result is obtained by listing all possible contractions, and keeping track of how many involve $m_1$ versus $m_{i\neq 1}$. The final result can be simplified to

$$\langle m_1(\mathbf{q})m_1(\mathbf{q}')\rangle = \frac{(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')}{t+r+Kq^2}\left\{1 - \frac{u}{t+r+Kq^2}\int\frac{d^d\mathbf{k}}{(2\pi)^d}\left[\frac{n-1}{t+Kk^2} + \frac{3}{t+r+Kk^2}\right]\right\}$$

(h) Treating $\mathcal{U}$ as a perturbation, calculate the *first order* correction to $\langle m_2(\mathbf{q})m_2(\mathbf{q}')\rangle$.

• Similar analysis yields

$$\begin{aligned}\langle m_2(\mathbf{q})m_2(\mathbf{q}')\rangle &= \langle m_2(\mathbf{q})m_2(\mathbf{q}')\rangle_0 - u\int \frac{d^dq_1 d^dq_2 d^dq_3}{(2\pi)^{3d}} \sum_{i,j=1}^{n} \\ &\qquad \left(\langle m_2(\mathbf{q})m_2(\mathbf{q}')m_i(\mathbf{q}_1)m_i(\mathbf{q}_2)m_j(\mathbf{q}_3)m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\rangle_0^c\right) \\ =&\frac{(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')}{t+Kq^2}\left\{1 - \frac{u}{t+Kq^2}\int\frac{d^d\mathbf{k}}{(2\pi)^d}\left[\frac{4(n-1)}{t+Kk^2} + \frac{4}{t+r+Kk^2} + \frac{8}{t+Kk^2}\right]\right\} \\ =&\frac{(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')}{t+Kq^2}\left\{1 - \frac{u}{t+Kq^2}\int\frac{d^d\mathbf{k}}{(2\pi)^d}\left[\frac{n+1}{t+Kk^2} + \frac{1}{t+r+Kk^2}\right]\right\}.\end{aligned}$$

(i) Using the above answer, identify the inverse susceptibility $\chi_{22}^{-1}$, and then find the transition point, $t_c$, from its vanishing to first order in $u$.

• Using the fluctuation–response relation, the susceptibility is given by

$$\begin{aligned}\chi_{22} &= \int d^d\mathbf{x}\,\langle m_2(\mathbf{x})m_2(\mathbf{0})\rangle = \int\frac{d^d\mathbf{q}}{(2\pi)^d}\,\langle m_2(\mathbf{q})m_2(\mathbf{q}=0)\rangle \\ &= \frac{1}{t}\left\{1 - \frac{u}{t}\int\frac{d^d\mathbf{k}}{(2\pi)^d}\left[\frac{n+1}{t+Kk^2} + \frac{1}{t+r+Kk^2}\right]\right\}.\end{aligned}$$

Inverting the correction term gives

$$\chi_{22}^{-1} = t + 4u\int\frac{d^d\mathbf{k}}{(2\pi)^d}\left[\frac{n+1}{t+Kk^2} + \frac{1}{t+r+Kk^2}\right] + O(u^2).$$

The susceptibility diverges at

$$t_c = -4u \int \frac{d^d\mathbf{k}}{(2\pi)^d} \left[ \frac{n+1}{Kk^2} + \frac{1}{r+Kk^2} \right] + O(u^2).$$

(j) Is the critical behavior different from the isotropic $O(n)$ model in $d < 4$? In RG language, is the parameter $r$ *relevant* at the $\mathcal{O}(n)$ fixed point? In either case indicate the universality classes expected for the transitions.

• The parameter $r$ changes the symmetry of the ordered state, and hence the universality class of the disordering transition. As indicated in the figure, the transition belongs to the $O(n-1)$ universality class for $r > 0$, and to the Ising class for $r < 0$. Any RG transformation must thus find $r$ to be a relevant perturbation to the $O(n)$ fixed point.

********

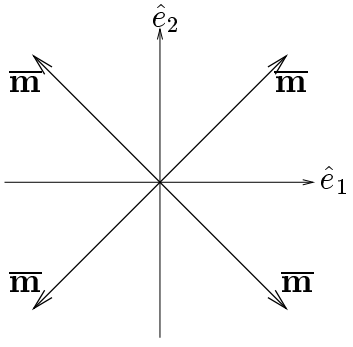
**3.** *Cubic anisotropy– Mean-field treatment:* Consider the modified Landau–Ginzburg Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x} \left[ \frac{K}{2}(\nabla\vec{m})^2 + \frac{t}{2}\vec{m}^2 + u(\vec{m}^2)^2 + v\sum_{i=1}^{n} m_i^4 \right] \quad ,$$

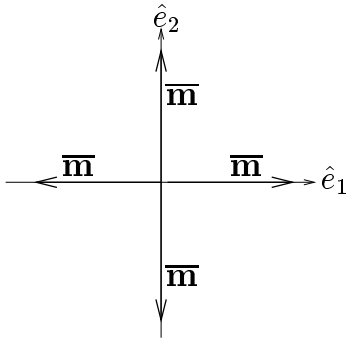
for an $n$–component vector $\vec{m}(\mathbf{x}) = (m_1, m_2, \cdots, m_n)$. The "cubic anisotropy" term $\sum_{i=1}^{n} m_i^4$, breaks the full rotational symmetry and selects specific directions.

(a) For a fixed magnitude $|\vec{m}|$; what directions in the $n$ component magnetization space are selected for $v > 0$ and for $v < 0$? What is the degeneracy of easy magnetization axes in each case?

• In the figures below, we indicate the possible directions of the magnetization which are selected depending upon the sign of the coefficient $v$, for the simple case of $n = 2$:




$v > 0$ diagonal order




$v < 0$ cubic axis order

This qualitative behavior can be generalized for an $n$-component vector: For $v > 0$, $\overline{\mathbf{m}}$ lies along the diagonals of a $n$-dimensional hypercube, which can be labelled as

$$\overline{\mathbf{m}} = \frac{\overline{m}}{\sqrt{n}}(\pm 1, \pm 1, \ldots, \pm 1),$$

and are consequently $2^n$-fold degenerate. Conversely, for $v < 0$, $\overline{\mathbf{m}}$ can point along any of the cubic axes $\hat{e}_i$, yielding

$$\overline{\mathbf{m}} = \pm \overline{m} \hat{e}_i,$$

which is $2n$-fold degenerate.

(b) What are the restrictions on $u$ and $v$ for $\beta\mathcal{H}$ to have finite minima? Sketch these regions of stability in the $(u, v)$ plane.

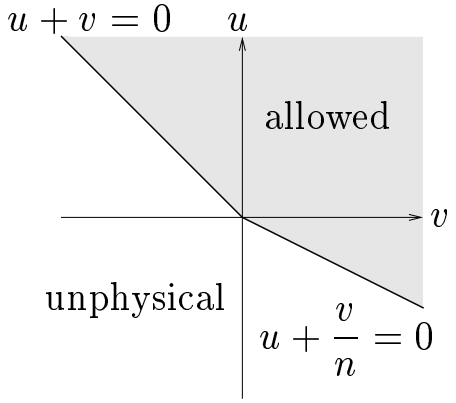
• The Landau-Ginzburg Hamiltonian for each of these cases evaluates to

$$\begin{cases} \beta\mathcal{H} = \dfrac{t}{2}\overline{m}^{\,2} + u\overline{m}^{\,4} + \dfrac{v}{n}\overline{m}^{\,4}, & \text{if} \quad v > 0 \\ \beta\mathcal{H} = \dfrac{t}{2}\overline{m}^{\,2} + u\overline{m}^{\,4} + v\overline{m}^{\,4}, & \text{if} \quad v < 0 \end{cases},$$

implying that there are finite minima provided that

$$\begin{cases} u + \dfrac{v}{n} > 0, & \text{for} \quad v > 0, \\ u + v > 0, & \text{for} \quad v < 0. \end{cases}$$

Above, we represent schematically the distinct regions in the $(u, v)$ plane.

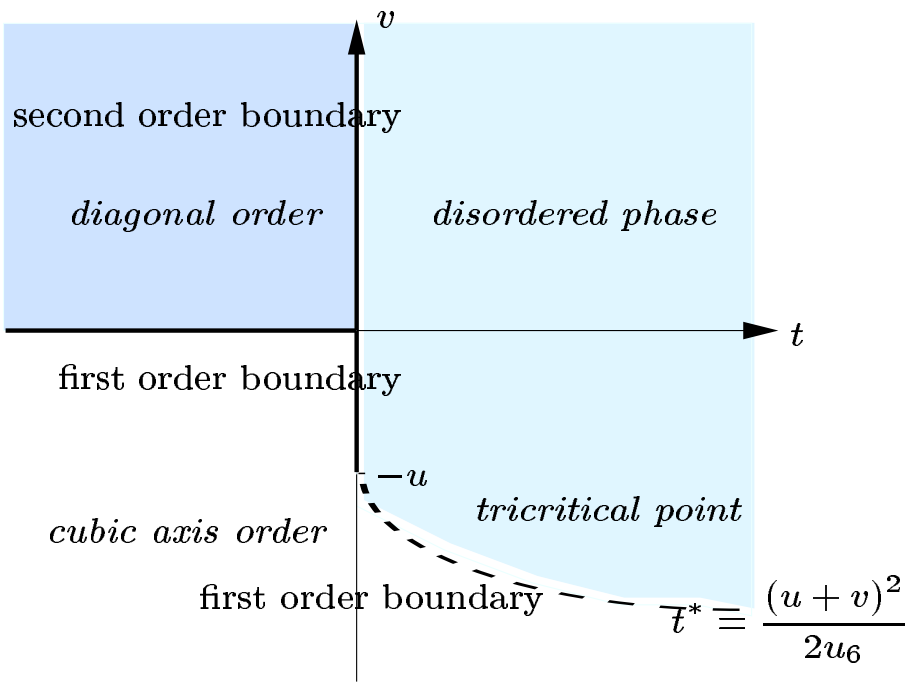


(c) In general, higher order terms (e.g. $u_6(\vec{m}^2)^3$ with $u_6 > 0$) are present and ensure stability in the regions not allowed in part (b); (as in case of the tricritical point discussed in earlier problems). With such terms in mind, sketch the saddle point phase diagram in the $(t, v)$ plane for $u > 0$; clearly identifying the phases, and order of the transition lines.

• We need to take into account higher order terms to ensure stability in the regions not allowed in part b). There is a tricritical point which can be obtained after simultaneously solving the equations

$$\left.\begin{aligned} t + 4(u+v)m^2 + 6u_6 m^4 = 0 \\ t + 2(u+v)m^2 + 2u_6 m^4 = 0 \end{aligned}\right\}, \quad \Longrightarrow \quad t^* = \frac{(u+v)^2}{2u_6}, \quad \overline{m}^2 = -\frac{(u+v)^2}{2u_6}.$$

The saddle point phase diagram in the $(t, v)$ plane is then as follows:




(d) Are there any Goldstone modes in the ordered phases?

• There are no Goldstone modes in the ordered phases because the broken symmetry is discrete rather than continuous. We can easily calculate the estimated value of the transverse fluctuations in Fourier space as

$$\langle \phi_t(\mathbf{q})\phi_t(-\mathbf{q})\rangle = \frac{(2\pi)^d}{Kq^2 + \frac{vt}{u+v}},$$

from which we can see that indeed these modes become massless only when $v = 0$, i.e., when we retrieve the $O(n)$ symmetry.

********

**4.** *Cubic anisotropy– $\varepsilon$–expansion:*

(a) By looking at diagrams in a second order perturbation expansion in both $u$ and $v$ show that the recursion relations for these couplings are

$$\begin{cases} \dfrac{du}{d\ell} = \varepsilon u - 4C \left[ \ (n+8)u^2 + 6uv \ \right] \\ \dfrac{dv}{d\ell} = \varepsilon v - 4C \left[ \ 12uv + 9v^2 \ \right] \end{cases} ,$$

where $C = K_d \Lambda^d/(t + K\Lambda^2)^2 \approx K_4/K^2$, is approximately a constant.

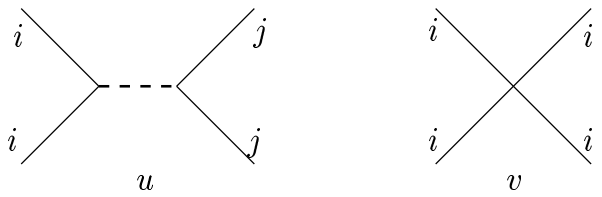
• Let us write the Hamiltonian in terms of Fourier modes

$$\begin{aligned} \beta\mathcal{H} = & \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{t + Kq^2}{2} \ \vec{m}(\mathbf{q}) \cdot \vec{m}(-\mathbf{q}) \\ & + u \int \frac{d^d\mathbf{q}_1 d^d\mathbf{q}_2 d^d\mathbf{q}_3}{(2\pi)^{3d}} \ m_i(\mathbf{q}_1) m_i(\mathbf{q}_2) \ m_j(\mathbf{q}_3) m_j(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \\ & + v \int \frac{d^d\mathbf{q}_1 d^d\mathbf{q}_2 d^d\mathbf{q}_3}{(2\pi)^{3d}} \ m_i(\mathbf{q}_1) m_i(\mathbf{q}_2) \ m_i(\mathbf{q}_3) m_i(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3), \end{aligned}$$

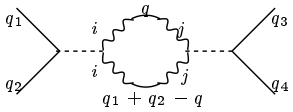
where, as usual, we assume summation over repeated indices. In analogy to problem set 6, after the three steps of the RG transformation, we obtain the renormalized parameters:

$$\begin{cases} t' = b^{-d} z^2 \tilde{t} \\ K' = b^{-d-2} z^2 \tilde{K} \\ u' = b^{-3d} z^4 \tilde{u} \\ v' = b^{-3d} z^4 \tilde{v} \end{cases} ,$$

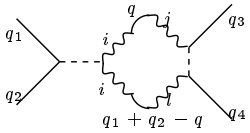
with $\tilde{t}$, $\tilde{K}$, $\tilde{u}$, and $\tilde{v}$, are the parameters in the coarse grained Hamiltonian. The dependence of $\tilde{u}$ and $\tilde{v}$, on the original parameters can be obtained by looking at diagrams in a second order perturbation expansion in both $u$ and $v$. Let us introduce diagrammatic representations of $u$ and $v$, as



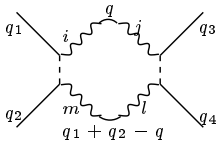
Contributions to u



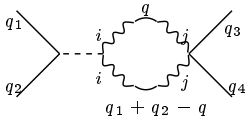

$$2\times 2\times 2n\frac{u^2}{2}\frac{K_d\Lambda^d}{(t+K\Lambda^2)^2}\delta l$$




$$2\times 2\times 4\times 2\frac{u^2}{2}\frac{K_d\Lambda^d}{(t+K\Lambda^2)^2}\delta l$$

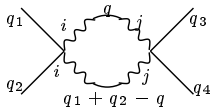



$$4\times 4\times 2\frac{u^2}{2}\frac{K_d\Lambda^d}{(t+K\Lambda^2)^2}\delta l$$

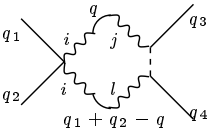



$$2\times 6\times 2uv\frac{K_d\Lambda^d}{(t+K\Lambda^2)^2}\delta l$$

Contributions to v




$$6\times 6\times 2\frac{v^2}{2}\frac{K_d\Lambda^d}{(t+K\Lambda^2)^2}\delta l$$




$$6\times 4\times 2uv\frac{K_d\Lambda^d}{(t+K\Lambda^2)^2}\delta l$$

where, again we have set $b=e^{\delta\ell}$. The new coarse grained parameters are

$$\begin{cases}\tilde{u}=u-4C[(n+8)u^2+6uv]\delta\ell\\ \tilde{v}=v-4C[9v^2+12uv]\delta\ell\end{cases},$$

which after introducing the parameter $\epsilon=4-d$, rescaling, and renormalizing, yield the recursion relations

$$\begin{cases}\dfrac{du}{d\ell}=\epsilon u-4C[(n+8)u^2+6uv]\\ \dfrac{dv}{d\ell}=\epsilon v-4C[9v^2+12uv]\end{cases}.$$

(b) Find all fixed points in the $(u,v)$ plane, and draw the flow patterns for $n<4$ and $n>4$. Discuss the relevance of the cubic anisotropy term near the stable fixed point in each case.

• From the recursion relations, we can obtain the fixed points $(u^*,v^*)$. For the sake of simplicity, from now on, we will refer to the rescaled quantities $u=4Cu$, and $v=4Cv$, in terms of which there are four fixed points located at

$$\begin{cases}u^*=v^*=0 & \text{Gaussian fixed point}\\ u^*=0\quad v^*=\dfrac{\epsilon}{9} & \text{Ising fixed point}\\ u^*=\dfrac{\epsilon}{(n+8)}\quad v^*=0 & \mathcal{O}(\text{n}) \text{ fixed point}\\ u^*=\dfrac{\epsilon}{3n}\quad v^*=\dfrac{\epsilon(n-4)}{9n} & \text{Cubic fixed point}\end{cases}.$$

Linearizing the recursion relations in the vicinity of the fixed point gives

$$A = \frac{d}{d\ell}\begin{pmatrix}\delta u \\ \delta v\end{pmatrix}_{u^*,v^*} = \begin{pmatrix}\epsilon - 2(n+8)u^* - 6v^* & -6u^* \\ -12v^* & \epsilon - 12u^* - 18v^*\end{pmatrix}\begin{pmatrix}\delta u \\ \delta v\end{pmatrix}.$$

As usual, a positive eigenvalue corresponds to an unstable direction, whereas negative ones correspond to stable directions. For each of the four fixed points, we obtain:

1. *Gaussian fixed point*: $\lambda_1 = \lambda_2 = \epsilon$, i.e., this fixed point is doubly unstable for $\epsilon > 0$, as

$$A = \begin{pmatrix}\epsilon & 0 \\ 0 & \epsilon\end{pmatrix}.$$

2. *Ising fixed point*: This fixed point has one stable and one unstable direction, as

$$A = \begin{pmatrix}\frac{\epsilon}{3} & 0 \\ -4\frac{\epsilon}{3} & -\epsilon\end{pmatrix},$$

corresponding to $\lambda_1 = \epsilon/3$ and $\lambda_2 = -\epsilon$. Note that for $u = 0$, the system decouples into $n$ noninteracting 1-component Ising spins.

3. *O(n) fixed point*: The matrix

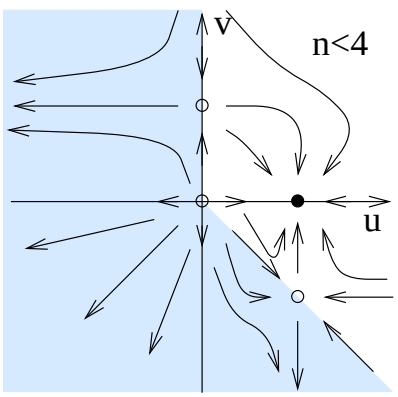
$$A = \begin{pmatrix}-\epsilon & -6\frac{\epsilon}{(n+8)} \\ 0 & \frac{(n-4)}{(n+8)}\epsilon\end{pmatrix},$$

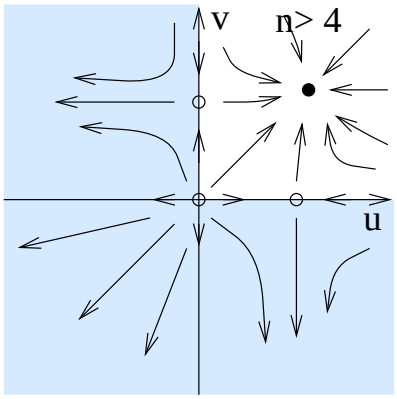
has eigenvalues $\lambda_1 = -\epsilon$, and $\lambda_2 = \epsilon(n-4)/(n+8)$. Hence for $n > 4$ this fixed point has one stable an one unstable direction, while for $n < 4$ both eigendirections are stable. This fixed point thus controls the critical behavior of the system for $n < 4$.

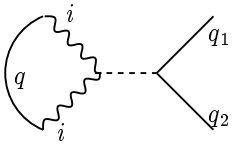
4. *Cubic fixed point*: The eigenvalues of

$$A = \begin{pmatrix}-\frac{(n+8)}{3} & -2 \\ -4\frac{(n-4)}{3} & 4-n\end{pmatrix}\frac{\epsilon}{n},$$

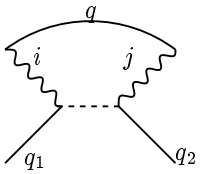
are $\lambda_1 = \epsilon(4-n)/3n$, $\lambda_2 = -\epsilon$. Thus for $n < 4$, this fixed point has one stable and one unstable direction, and for $n > 4$ both eigendirections are stable. This fixed point controls critical behavior of the system for $n > 4$.



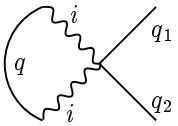




$$2nu\frac{K_d\Lambda^d}{(t+K\Lambda^2)}\delta l$$




$$4u\frac{K_d\Lambda^d}{(t+K\Lambda^2)}\delta l$$




$$6v\frac{K_d\Lambda^d}{(t+K\Lambda^2)}\delta l$$

$$\tilde{t} = t+4\frac{K_d\Lambda^d}{(t+K\Lambda^2)}[(n+2)u+3v]$$

$$\frac{dt}{dl} = 2t+4\frac{K_d\Lambda^d}{(t+K\Lambda^2)}[(n+2)u+3v]$$

In the $(u,v)$ plane, $v^* = 0$ for $n < 4$, and the cubic term is irrelevant, i.e., fluctuations restore full rotational symmetry. For $n > 4$, $v$ is relevant, resulting in the following flows:

(c) Find the recursion relation for the reduced temperature, $t$, and calculate the exponent $\nu$ at the stable fixed points for $n < 4$ and $n > 4$.

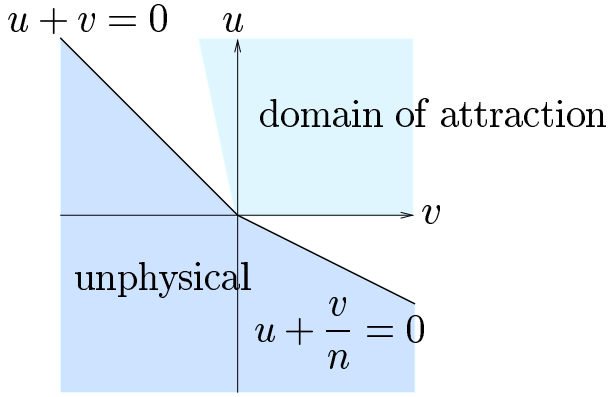
• Up to linear order in $\epsilon$, the following diagrams contribute to the determination of $\tilde{t}$:

After linearizing in the vicinity of the stable fixed points, the exponent $y_t$ is given by

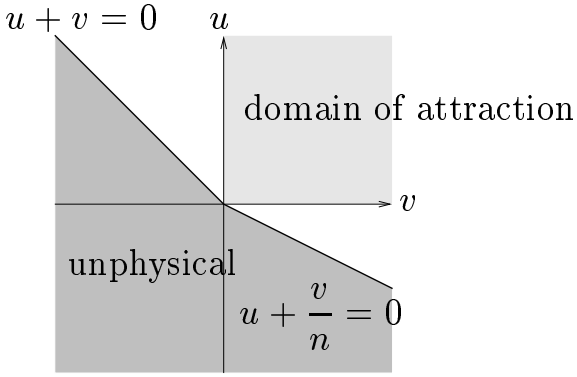
$$y_t = 2 - 4C[(n+2)u^* + 3v^*], \Longrightarrow \nu = \frac{1}{y_t} = \begin{cases} \frac{1}{2} + \frac{(n+2)}{4(n+8)}\epsilon + \mathcal{O}(\epsilon^2) & \text{for} \quad n < 4 \\ \frac{1}{2} + \frac{(n-1)}{6n}\epsilon + \mathcal{O}(\epsilon^2) & \text{for} \quad n > 4 \end{cases}.$$

(d) Is the region of stability in the $(u, v)$ plane calculated in part (b) of the previous problem enhanced or diminished by inclusion of fluctuations? Since in reality higher order terms will be present, what does this imply about the nature of the phase transition for a small negative $v$ and $n > 4$?

• All fixed points are located within the allowed region calculated in 1b). However, not all flows starting in classically stable regions are attracted to stable fixed point. If the RG flows take a point outside the region of stability, then fluctuations decrease the region of stability. The domains of attraction of the fixed points for $n < 4$ and $n > 4$ are indicated in the following figures:




$n < 4$




$n > 4$

Flows which are not originally located within these domains of attraction flow into the unphysical regions. The coupling constants $u$ and $v$ become more negative. This is the signal of a fluctuation induced first order phase transition, by what is known as the Coleman–Weinberg mechanism. Fluctuations are responsible for the change of order of the transition in the regions of the $(u,v)$ plane outside the domain of attraction of the stable fixed points.

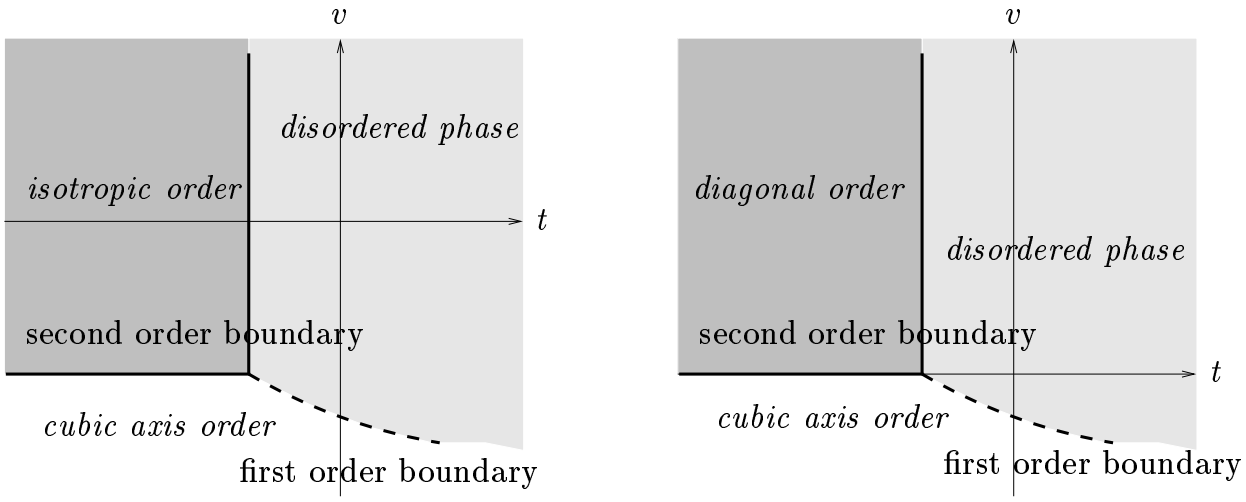
(e) Draw schematic phase diagrams in the $(t, v)$ plane $(u > 0)$ for $n > 4$ and $n < 4$, identifying the ordered phases. Are there Goldstone modes in any of these phases close to the phase transition?

• From the recursion relation obtained in 2c) for the parameter $t$, we obtain the following non-trivial $t^*$

$$t^* = -\frac{1}{2}[(n+2)u^* + 3v^*] \propto -\epsilon$$

Therefore, the phase diagrams in the $(t,v)$ plane is schematically represented as

As mentioned above, only when $n < 4$ fluctuations restore the full rotational symmetry. The parameter $v$ is renormalized to zero, and there are Goldstone modes at the $(u,v)$ plane, but only near the second order phase transition, where $K\xi_t^{-2} = tv/(u+v) \to 0$. In the ordered phases, the renormalized value of $v$ is finite, albeit small, indicating that there are no Goldstone modes.




********

**5.** *Exponents:* Two critical exponents at second order are,

$$\begin{cases} \nu = \dfrac{1}{2} + \dfrac{(n+2)}{4(n+8)}\,\epsilon + \dfrac{(n+2)(n^2+23n+60)}{8(n+8)^3}\,\epsilon^2 \quad , \\ \eta = \dfrac{(n+2)}{2(n+8)^2}\,\epsilon^2 \quad . \end{cases}$$

Use scaling relations to obtain $\epsilon$–expansions for two or more of the remaining exponents $\alpha,\ \beta,\ \gamma,\ \delta$ and $\Delta$. Make a table of the results obtained by setting $\epsilon = 1,\ 2$ for $n = 1,\ 2$ and 3; and compare to the best estimates of these exponents that you can find by other sources (series, experiments, etc.).

********

---

**6.** *Anisotropic criticality:* A number of materials, such as liquid crystals, are anisotropic and behave differently along distinct directions, which shall be denoted parallel and perpendicular, respectively. Let us assume that the $d$ spatial dimensions are grouped into $n$ parallel directions $\mathbf{x}_\parallel$, and $d-n$ perpendicular directions $\mathbf{x}_\perp$. Consider a one–component field $m(\mathbf{x}_\parallel, \mathbf{x}_\perp)$ subject to a Landau–Ginzburg Hamiltonian, $\beta\mathcal{H} = \beta\mathcal{H}_0 + U$, with

$$\beta\mathcal{H}_0 = \int d^n\mathbf{x}_\parallel d^{d-n}\mathbf{x}_\perp \left[\frac{K}{2}(\nabla_\parallel m)^2 + \frac{L}{2}(\nabla_\perp^2 m)^2 + \frac{t}{2}m^2 - hm\right],$$

and

$$U = u\int d^n\mathbf{x}_\parallel d^{d-n}\mathbf{x}_\perp\ m^4\ .$$

*(Note that $\beta\mathcal{H}$ depends on the* **first** *gradient in the $\mathbf{x}_\parallel$ directions, and on the* **second** *gradient in the $\mathbf{x}_\perp$ directions.)*

(a) Write $\beta\mathcal{H}_0$ in terms of the Fourier transforms $m(\mathbf{q}_\parallel, \mathbf{q}_\perp)$.

• In terms of the Fourier modes $m(\mathbf{q}) = \int d^d\mathbf{x} e^{i\mathbf{q}\cdot\mathbf{x}} m(\mathbf{x})$,

$$\beta\mathcal{H}_0 = \int \frac{d^n\mathbf{q}_\parallel d^{d-n}\mathbf{q}_\perp}{(2\pi)^d} \left( \frac{Kq_\parallel^2 + Lq_\perp^4 + t}{2} \right) |m(\mathbf{q})|^2 - hm(\mathbf{q}=0).$$

(b) Construct a renormalization group transformation for $\beta\mathcal{H}_0$, by rescaling coordinates such that $\mathbf{q}'_\parallel = b\ \mathbf{q}_\parallel$ and $\mathbf{q}'_\perp = c\ \mathbf{q}_\perp$ and the field as $m'(\mathbf{q}') = m(\mathbf{q})/z$. *Note that parallel and perpendicular directions are scaled differently.* Write down the recursion relations for $K,\ L,\ t$, and $h$ in terms of $b,\ c$, and $z$. (The exact shape of the Brillouin zone is immaterial at this stage, and you do not need to evaluate the integral that contributes an additive constant.)

• The unperturbed Hamiltonian describes a set of independent modes labelled by a wavevector $\mathbf{q}$. We can integrate out the modes that correspond to short distance fluctuations, and then rescale the parameters as suggested above to get

$$(\beta\mathcal{H}_0)' = \delta f_b^0 + \int \frac{d^n\mathbf{q}'_\parallel d^{d-n}\mathbf{q}'_\perp}{(2\pi)^d} b^{-n} c^{-(d-n)} \left( \frac{Kb^{-2}q'^2_\parallel + Lc^{-4}q'^4_\perp + t}{2} \right) z^2 |m'(\mathbf{q}')|^2 - zhm'(\mathbf{q}'=0).$$

The renormalized bare Hamiltonian has the same form as $\beta\mathcal{H}_0$, with the renormalized parameters

$$\begin{cases} K' = b^{-(n+2)} c^{-(d-n)} z^2 K \\ L' = b^{-n} c^{-(d-n+4)} z^2 L \\ t' = b^{-n} c^{-(d-n)} z^2 t \\ h' = zh \end{cases} .$$

(c) Choose $c(b)$ and $z(b)$ such that $K' = K$ and $L' = L$. At the resulting fixed point calculate the eigenvalues $y_t$ and $y_h$ for the rescalings of $t$ and $h$.

• The requirements of $K' = K$ and $L' = L$ lead to $z^2 = b^{n+2}c^{d-n}$ and $z^2 = b^n c^{d-n+4}$, respectively. Consistency of these equations requires

$$c = b^{1/2}, \qquad \text{and} \qquad z = b^{1+\frac{d+n}{4}}.$$

Inserting these results for $c$ and $z$ in the recursion relations for $t$ and $h$, we find

$$\begin{cases} t' = b^{-n-\frac{d-n}{2}+2+\frac{d+n}{2}} t = b^2 t \\ h' = zh = b^{1+\frac{d+n}{4}} h \end{cases}, \quad \Longrightarrow \quad \begin{cases} y_t = 2 \\ h_h = 1 + \dfrac{d+n}{4} \end{cases} .$$

(d) Write the relationship between the (singular parts of) free energies $f(t,h)$ and $f'(t',h')$ in the original and rescaled problems. Hence write the unperturbed free energy in the homogeneous form $f(t,h) = t^{2-\alpha} g_f(h/t^{\Delta})$, and identify the exponents $\alpha$ and $\Delta$.

• The RG transformation preserves the partition function $Z$, and the free energy density is $f \sim \ln Z/V$. Under the rescaling transformation the volume changes to $V' = b^{-n} c^{-(d-n)} V$, and hence

$$f(t,h) = b^{-n-\frac{d-n}{2}} f(t',h') = b^{-\frac{d+n}{2}} f\left(b^2 t, b^{1+\frac{d+n}{4}} h\right).$$

We can now rescale to a point at which $b^2 t \sim 1$, at which point

$$f(t,h) = t^{\frac{d+n}{4}} g\left(\frac{h}{t^{\frac{1}{2}+\frac{d+n}{8}}}\right),$$

corresponding to

$$\alpha = 2 - \frac{d+n}{4}, \qquad \text{and} \qquad \Delta = \frac{1}{2} + \frac{d+n}{8}.$$

(e) How does the unperturbed zero–field susceptibility $\chi(t, h=0)$, diverge as $t \to 0$?

• Taking two derivatives of the above free energy with respect to $h$, we obtain

$$\chi \sim \frac{\partial^2 f}{\partial h^2} \sim t^{\frac{d+n}{4}} \left(t^{\frac{1}{2}+\frac{d+n}{8}}\right)^2 g_\chi \left(\frac{h}{t^{\frac{1}{2}+\frac{d+n}{8}}}\right) \sim t^{-1} \quad (\text{for } h=0) \quad .$$

The zero-field susceptibility diverges with an exponent $\gamma = 1$ at $t = 0$. (The same result can also be obtained by examining the two point correlation function.)

*In the remainder of this problem set $h = 0$, and treat $U$ as a perturbation.*

(f) In the unperturbed Hamiltonian calculate the expectation value $\langle m(q) m(q') \rangle_0$, and the corresponding susceptibility $\chi_0(q) = \langle |m_q|^2 \rangle_0$, where $q$ stands for $(\mathbf{q}_\parallel, \mathbf{q}_\perp)$.

• By examining the variance of the unperturbed Gaussian weight, we observe that

$$\langle m(q) m(q') \rangle_0 = (2\pi)^d \delta^d(q+q') \chi_0(q), \qquad \text{with} \qquad \chi_0(q) = \frac{1}{t + K q_\parallel^2 + L q_\perp^4}.$$

(g) Write the perturbation $U$, in terms of the normal modes $m(q)$.

• After writing each of the four factors of $m(\mathbf{x})$ in $U$ in terms of $m(q)$, and integrating over all $\mathbf{x}$, we obtain

$$U = u \int dq_1 dq_2 dq_3 m(q_1) m(q_2) m(q_3) m(-q_1 - q_2 - q_3),$$

where $dq \equiv d^n \mathbf{q}_\parallel d^{d-n} \mathbf{q}_\perp / (2\pi)^d$.

(h) Using RG, or any other method, find the upper critical dimension $d_u$, for validity of the Gaussian exponents.

• Under the rescaling steps outlined in the earlier parts of the problem, the coefficient of the quartic perturbation changes to lowest order as

$$u' = u(bc)^{-3} z^4 = u b^{-3n} b^{-\frac{3}{2}(d-n)} b^{4+(d+n)} \equiv u b^{y_u}, \quad \text{with} \quad y_u = 4 - \frac{d+n}{2}.$$

The upper critical dimension is identified as the point at which $y_u = 0$, and thus

$$d_u = 8 - n.$$

(i) Write down the expansion for $\langle m(q) m(q') \rangle$, to first order in $U$, and reduce the correction term to a product of two point expectation values.

• Averages in perturbation theory are calculated as $\langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle_0 - (\langle \mathcal{O} U \rangle_0 - \langle \mathcal{O} \rangle_0 \langle U \rangle_0) + \cdots$. Using the expression for $U$ in terms of Fourier modes, we find

$$\begin{aligned} \langle m(q) m(q') \rangle =& \langle m(q) m(q') \rangle_0 \\ & - u \int dq_1 dq_2 dq_3 \langle m(q) m(q') m(q_1) m(q_2) m(q_3) m(-q_1 - q_2 - q_3) \rangle_0^c + O(u^2), \end{aligned}$$

where the superscript 'c' in the first order correction stands for the cumulant. Using Wick's theorem the average of the six factors of $m$ can be reduced to 12 equivalent pairwise contractions, such that

$$\begin{aligned} \langle m(q) m(q') \rangle =& \langle m(q) m(q') \rangle_0 \\ & -12u \int dq_1 dq_2 dq_3 \langle m(q) m(q_1) \rangle_0 \langle m(q') m(q_2) \rangle_0 \langle m(q_3) m(-q_1 - q_2 - q_3) \rangle_0 + O(u^2). \end{aligned}$$

(j) Write down the expression for $\chi(q)$, in first order perturbation theory, and identify the transition point $t_c$ at order of $u$. (Do not evaluate any integrals explicitly.)

• The two point expectation value has the form $\langle m(q)m(q')\rangle = (2\pi)^d\delta^d(q+q')\chi(q)$, with

$$\chi(q) = \frac{1}{t+Kq_\parallel^2+Lq_\perp^4} - \frac{12u}{(t+Kq_\parallel^2+Lq_\perp^4)^2}\int dk \frac{1}{t+Kk_\parallel^2+Lk_\perp^4} + O(u^2)$$
$$= \left[t+Kq_\parallel^2+Lq_\perp^4 + 12u\int dk \frac{1}{t+Kk_\parallel^2+Lk_\perp^4} + O(u^2)\right]^{-1}.$$

The inverse susceptibility $\chi(q=0)$ should diverge at the critical point, $t=t_c$, which is thus identified as

$$t_c = -12u\int \frac{d^n k_\parallel d^{d-n}k_\perp}{(2\pi)^d}\frac{1}{Kk_\parallel^2+Lk_\perp^4} + O(u^2).$$

********

**7.** *Long–range interactions* between spins can be described by adding a term

$$\int d^d\mathbf{x}\int d^d\mathbf{y} J(|\mathbf{x}-\mathbf{y}|)\vec{m}(\mathbf{x})\cdot\vec{m}(\mathbf{y}),$$

to the usual Landau–Ginzburg Hamiltonian.

(a) Show that for $J(r)\propto 1/r^{d+\sigma}$, the Hamiltonian can be written as

$$\beta\mathcal{H} = \int \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{t+K_2q^2+K_\sigma q^\sigma+\cdots}{2}\,\vec{m}(\mathbf{q})\cdot\vec{m}(-\mathbf{q})$$
$$+u\int \frac{d^d\mathbf{q}_1 d^d\mathbf{q}_2 d^d\mathbf{q}_3}{(2\pi)^{3d}}\vec{m}(\mathbf{q}_1)\cdot\vec{m}(\mathbf{q}_2)\,\vec{m}(\mathbf{q}_3)\cdot\vec{m}(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\quad.$$

• After changing variables to $\mathbf{R}=(\mathbf{x}+\mathbf{y})/2$, $\mathbf{r}=(\mathbf{x}-\mathbf{y})/2$, and writing the coarse-grained spin as a sum of Fourier modes, the long-range interaction term becomes

$$\int d^d\mathbf{x}d^d\mathbf{y}\frac{\vec{m}(\mathbf{x})\cdot\vec{m}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+\sigma}} = \int d^d\mathbf{R}d^d\mathbf{r}\frac{1}{r^{d+\sigma}}\int^\Lambda \frac{d^d\mathbf{q}_1 d^d\mathbf{q}_2}{(2\pi)^{2d}}\vec{m}(\mathbf{q}_1)\cdot\vec{m}(\mathbf{q}_2)\mathrm{e}^{i\mathbf{q}_1\cdot(\mathbf{R}+\mathbf{r})}\mathrm{e}^{i\mathbf{q}_2\cdot(\mathbf{R}-\mathbf{r})}$$
$$= \int^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d}\vec{m}(\mathbf{q})\cdot\vec{m}(-\mathbf{q})\int d^d\mathbf{r}\frac{1}{|\mathbf{r}|^{d+\sigma}}\mathrm{e}^{2i\mathbf{q}\cdot\mathbf{r}} = I\int^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d}q^\sigma\vec{m}(\mathbf{q})\cdot\vec{m}(-\mathbf{q}).$$

The final expression is obtained by a rescaling of $\mathbf{r}=\mathbf{x}/|\mathbf{q}|$, and $I=\int d^d\mathbf{x}e^{2ix_1}/|\mathbf{x}|^{d+\sigma}$ is independent of $q=|\mathbf{q}|$. Including the ever-present short-range parts, we can write the full Hamiltonian as

$$\beta\mathcal{H} = \int \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{t+K_2q^2+K_\sigma q^\sigma+\dots}{2}\,\vec{m}(\mathbf{q})\cdot\vec{m}(-\mathbf{q})$$
$$+u\int \frac{d^d\mathbf{q}_1 d^d\mathbf{q}_2 d^d\mathbf{q}_3}{(2\pi)^{3d}}\vec{m}(\mathbf{q}_1)\cdot\vec{m}(\mathbf{q}_2)\;\vec{m}(\mathbf{q}_3)\cdot\vec{m}(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\quad.$$

(b) For $u = 0$, construct the recursion relations for $(t, K_2, K_\sigma)$ and show that $K_\sigma$ is irrelevant for $\sigma > 2$. What is the fixed Hamiltonian in this case?

• For $u = 0$, the Gaussian integral over the outer-most shell will give us an additive constant. Subsequent length $q' = bq$, and spin $m' = m/z$, rescalings lead to the following recursion relations

$$\begin{cases} t' = b^{-d} z^2 t \\ K_2' = b^{-d-2} z^2 K_2 \\ K_\sigma' = b^{-d-\sigma} z^2 K_\sigma \end{cases} .$$

If $\sigma > 2$, the fixed point $(t^*, K_2^*, K_\sigma^*) = (0, 0, K_\sigma)$ (obtained by choosing $K_\sigma' = K_\sigma$) has no basin of attraction, as

$$\begin{cases} t' = b^\sigma t \\ K_2' = b^{\sigma-2} K_2 \end{cases}$$

indicate two relevant directions; consequently, this fixed point does not describe the usual critical point of the system. On the other hand, the fixed point $(t^*, K_2^*, K_\sigma^*) = (0, K_2, 0)$, for which $z^2 = b^{d+2}$, leads to

$$\begin{cases} t' = b^2 t \\ K_\sigma' = b^{2-\sigma} K_\sigma \end{cases} ,$$

i.e., it has one relevant and one irrelevant direction, and is therefore, the fixed point controlling standard critical behavior of the system. In conclusion, if $\sigma > 2$, $K_\sigma$ is irrelevant, and the fixed Hamiltonian in this case is

$$\beta \mathcal{H}^* = \frac{K_2}{2} \int \frac{d^d \mathbf{q}}{(2\pi)^d} \, q^2 \; \vec{m}(\mathbf{q}) \cdot \vec{m}(-\mathbf{q}).$$

(c) For $\sigma < 2$ and $u = 0$, show that the spin rescaling factor must be chosen such that $K_\sigma' = K_\sigma$, in which case $K_2$ is irrelevant. What is the fixed Hamiltonian now?

• Conversely, for $\sigma < 2$, $K_\sigma$ is the only relevant parameter at the fixed point $(t^*, K_2^*, K_\sigma^*) = (0, 0, K_\sigma)$, which is the one controlling the critical behavior in this case. At this point, the spin scaling factor $z$, is chosen such that $z^2 = b^{d+\sigma}$, and the fixed Hamiltonian is

$$\beta \mathcal{H}^* = \frac{K_\sigma}{2} \int \frac{d^d \mathbf{q}}{(2\pi)^d} \, q^\sigma \; \vec{m}(\mathbf{q}) \cdot \vec{m}(-\mathbf{q}).$$

(d) For $\sigma < 2$, calculate the generalized Gaussian exponents $\nu$, $\eta$, and $\gamma$ from the recursion relations. Show that $u$ is irrelevant, and hence the Gaussian results are valid, for $d > 2\sigma$.

• The divergence of the correlation length $\xi$, is controlled by the exponent $\nu = 1/y_t$. Since $\xi' = \xi_o t'^{\,-\nu} = b^{-y_t\nu}\xi$, and $\xi' = \xi/b$, the exponent $\nu$ is then

$$\nu = 1/\sigma.$$

For $\sigma < 2$, the Gaussian correlation function in Fourier space is

$$\chi(q) = (t + K_\sigma q^\sigma)^{-1},$$

and consequently,

$$\chi = \lim_{q\to 0}\chi(q) = t^{-1}, \qquad \text{with} \qquad \gamma = 1.$$

In addition, at the critical point

$$\chi(q) \sim \frac{1}{q^{2-\eta}}, \qquad \text{so} \qquad \eta = 2 - \sigma.$$

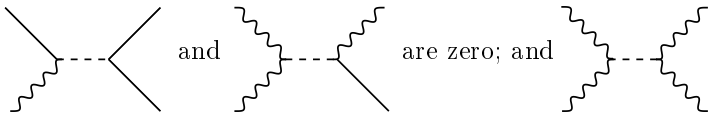
If $u \neq 0$, at first order it will be renormalized to

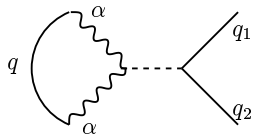
$$u' = b^{-3d}z^4 u = b^{2\sigma-d}u,$$

indicating that $u$ is irrelevant and the Gaussian results are valid provided that $d > 2\sigma$.

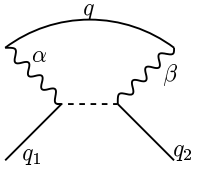
(e) For $\sigma < 2$, use a perturbation expansion in $u$ to construct the recursion relations for $(t, K_\sigma, u)$ as in the text.

• Let us obtain the recursion relations for $(t, K_\sigma, u)$. As shown in the text, the first order diagrams




$$u \times 2n\int_{\Lambda/b}^{\Lambda}\frac{d^dq}{(2\pi)^d}\frac{1}{(t + K_\sigma q^\sigma)} = 2nu\frac{K_d\Lambda^d}{(t + K_\sigma\Lambda^\sigma)}\delta l$$




$$u \times 4\int_{\Lambda/b}^{\Lambda}\frac{d^dq}{(2\pi)^d}\frac{1}{(t + K_\sigma q^\sigma)} = 4u\frac{K_d\Lambda^d}{(t + K_\sigma\Lambda^\sigma)}\delta l$$

The above two diagrams contribute to the coarse-grained parameter $t$. Their respective contributions are obtained after considering $b = \mathrm{e}^{\delta l}$, and expanding the exponential to linear order in the infinitesimal $\delta l$. The coarse-grained parameter $\tilde{t}$ is then

$$\tilde{t} = t + 2\frac{(2n+4)uK_d\Lambda^d}{t + K_\sigma\Lambda^\sigma}\delta l + \mathcal{O}(u^2).$$

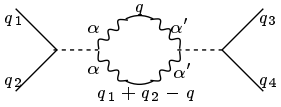
At this order, the other parameters in the coarse grained Hamiltonian remain unchanged ($\tilde{u} = u, \tilde{K}_2 = K_2$).

Taking into account the results of section 1.(b)-(c), after the usual steps 2 and 3 of the RG transformation, we obtain the following differential recursion relations for the infinitesimal rescaling factor $b \sim 1 + \delta l$:

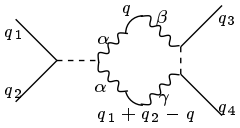
$$\begin{cases} t' = b^\sigma \tilde{t} \\ K_2' = b^{\sigma-2} K_2 \\ u' = b^{2\sigma-d} u \end{cases} , \qquad \Longrightarrow \qquad \begin{cases} \dfrac{dt}{dl} = \sigma t + 4u\dfrac{(n+2)K_d\Lambda^d\delta l}{t + K_\sigma\Lambda^\sigma} + \mathcal{O}(u^2) \\ \dfrac{dK_2}{dl} = (\sigma - 2)K_2 + \mathcal{O}(u^2) \\ \dfrac{du}{dl} = (2\sigma - d)u + \mathcal{O}(u^2) \end{cases} .$$

For $\sigma < 2$ and $d > 2\sigma$, the Gaussian fixed point, which is the only solution of these equations, has only one relevant direction, describing correctly the phase transition. However, if $d < 2\sigma$, there are two relevant directions, and we have to calculate the recursion relations at the next order to find other non-trivial fixed points.

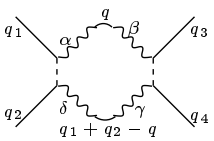
At second order in $u$, there are several new interactions proportional to $m^2$, $m^4$, $m^2q^2$, which will appear in the coarse grained Hamiltonian. But of all these, we are essentially interested in the second order diagrams which contribute to the renormalization of $\vec{m}^4$. As shown in the text, these are:




$$\frac{u^2}{2} 2 \times 2 \times 2n \int_{\Lambda/b}^{\Lambda} \frac{d^dq}{(2\pi)^d} \frac{1}{(t + K_\sigma q^\sigma)(t + K_\sigma |q_1 + q_2 - q|^\sigma)} = 4nu^2 \frac{K_d\Lambda^d}{(t + K_\sigma\Lambda^\sigma)^2}\delta l$$




$$\frac{u^2}{2} 2 \times 2 \times 4 \times 2 \int_{\Lambda/b}^{\Lambda} \frac{d^dq}{(2\pi)^d} \frac{1}{(t + K_\sigma q^\sigma)(t + K_\sigma |q_1 + q_2 - q|^\sigma)} = 16u^2 \frac{K_d\Lambda^d}{(t + K_\sigma\Lambda^\sigma)^2}\delta l$$




$$\frac{u^2}{2} 4 \times 4 \times 2 \int_{\Lambda/b}^{\Lambda} \frac{d^dq}{(2\pi)^d} \frac{1}{(t + K_\sigma q^\sigma)(t + K_\sigma |q_1 + q_2 - q|^\sigma)} = 16u^2 \frac{K_d\Lambda^d}{(t + K_\sigma\Lambda^\sigma)^2}\delta l$$

so that the new coarse grained parameters are

$$\begin{cases} \tilde{t} = t + 4u\dfrac{(n+2)K_d\Lambda^d}{t+K_\sigma\Lambda^\sigma}\delta l - u^2 C_t \\ \tilde{K}_2 = K_2 - u^2 C_{K_2} \\ \tilde{K}_\sigma = K_\sigma \\ \tilde{u} = u - 4u^2\dfrac{(n+8)K_d\Lambda^d}{(t+K_\sigma\Lambda^\sigma)^2}\delta l \end{cases} .$$

Let us introduce the parameter $\epsilon = 2\sigma - d$. After rescaling and renormalizing, we obtain the following recursion relations:

$$\begin{cases} \dfrac{dt}{dl} = \sigma t + 4u\dfrac{(n+2)K_d\Lambda^d}{t+K_\sigma\Lambda^\sigma} - u^2 C_t \\ \dfrac{du}{dl} = \epsilon u - 4u^2\dfrac{(n+8)K_d\Lambda^d}{(t+K_\sigma\Lambda^\sigma)^2} \end{cases} .$$

(f) For $d < 2\sigma$, calculate the critical exponents $\nu$ and $\eta$ to first order in $\epsilon = d - 2\sigma$.

[See M.E. Fisher, S.-K. Ma and B.G. Nickel, Phys. Rev. Lett. **29**, 917 (1972).]

• From the recursion relations, we can now obtain the non-trivial fixed point $(t^*, u^*)$, which satisfies $dt/dl = du/dl = 0$, as

$$\begin{cases} u^* = \dfrac{K_\sigma^2\Lambda^\epsilon}{4(n+8)K_d}\epsilon \approx \dfrac{K_\sigma^2}{4(n+8)\mathrm{K}_{d=2\sigma}}\epsilon \\ t^* = -\dfrac{K_\sigma\Lambda^\sigma(n+2)}{\sigma(n+8)}\epsilon \end{cases} ,$$

where, up to linear order in $\epsilon$, we have set $t = 0$ in the denominators of both $dt/dl$ and $du/dl$, and performed other similar simplifications.

Linearizing the recursion relations in the vicinity of the non-trivial fixed point gives

$$\frac{d}{dl}\begin{pmatrix} \delta t \\ \delta u \end{pmatrix} = \begin{pmatrix} \sigma - \dfrac{(n+2)}{(n+8)}\epsilon & \cdots \\ \mathcal{O}(\epsilon^2) & -\epsilon \end{pmatrix}\begin{pmatrix} \delta t \\ \delta u \end{pmatrix} .$$

Hence the eigenvalues can be read directly from the diagonal elements of the matrix. The first one is positive and controls the critical behavior, whereas the second is negative and consequently irrelevant for $d < 2\sigma$.

The new exponent $\nu$ is then

$$\nu = \frac{1}{y_t} = \left(\sigma - \frac{(n+2)}{(n+8)}\epsilon\right)^{-1} = \frac{1}{\sigma} + \frac{(n+2)}{(n+8)\sigma^2}\epsilon + \mathcal{O}(\epsilon^2),$$

and the exponent $\eta$ remains the same as before, $\eta = 2 - \sigma + \mathcal{O}(\epsilon^2)$, since the parameter $K_\sigma$ has not been renormalized at linear order in $\epsilon$. In fact, it can be shown that the RG procedure only generates analytic corrections, and hence the coefficient of the non-analytic term $K_\sigma$ is unrenormalized to all orders, resulting in $\eta = 2 - \sigma$, *exactly.*

(g) What is the critical behavior if $J(r) \propto \exp(-r/a)$? Explain!

• A short-range interaction term of the form $J(r) \propto \exp(-r/a)$ will modify the actual value of the critical temperature $T_c$, and other non-universal quantities, but it will not change the value of the critical exponents, since its Fourier transform is analytic:

$$\int d^d\mathbf{x} d^d\mathbf{y} \vec{m}(\mathbf{x}) \cdot \vec{m}(\mathbf{y}) \mathrm{e}^{-\frac{|\mathbf{x}-\mathbf{y}|}{a}} = \int^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d} \vec{m}(\mathbf{q}) \cdot \vec{m}(-\mathbf{q}) \int d^d\mathbf{r}\ \mathrm{e}^{-\frac{r}{a}}\ \mathrm{e}^{2i\mathbf{q}\cdot\mathbf{r}}$$
$$= \int^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d}\ C(q)\vec{m}(\mathbf{q}) \cdot \vec{m}(-\mathbf{q}),$$

where

$$C(q) = 2\pi a^d \int_0^\infty dx x^{d-1} \mathrm{e}^{-x} J_o(aqx) = a^d \sum_{n=0}^\infty c_n (qa)^{2n}.$$

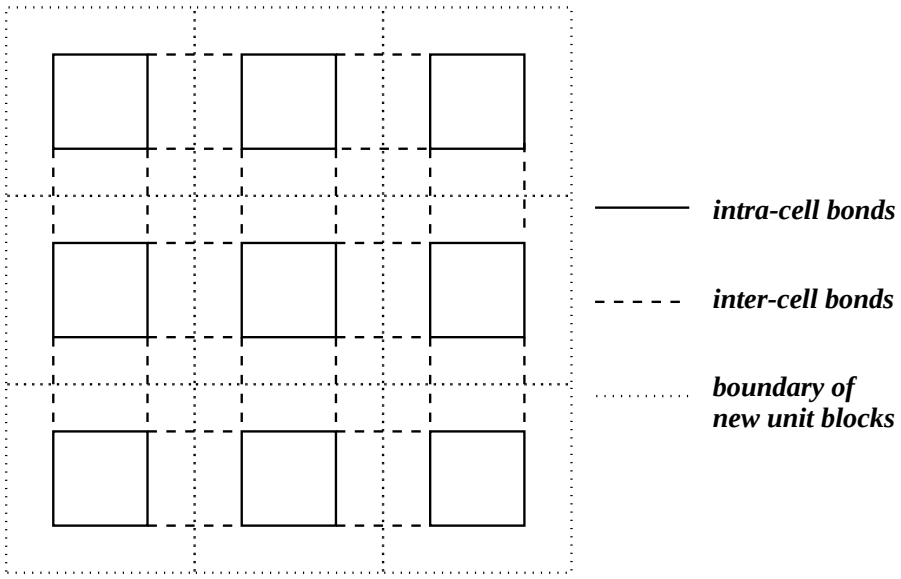
********

**Solutions to problems from chapter 6- Lattice Systems**

**1.** *Cumulant method:* Apply the Niemeijer–van Leeuwen first order cumulant expansion to the Ising model on a *square* lattice with $-\beta\mathcal{H} = K\sum_{<ij>}\sigma_i\sigma_j$, by following these steps:

(a) For an RG with $b = 2$, divide the bonds into *intra–cell* components $\beta\mathcal{H}_0$; and *inter–cell* components $\mathcal{U}$.

• The $N$ sites of the square lattice are partitioned into $N/4$ cells as indicated in the figure below (the *intra-cell* and *inter-cell* bonds are represented by solid and dashed lines respectively).




The renormalized Hamiltonian $\beta\mathcal{H}'[\sigma'_\alpha]$ is calculated from

$$\beta\mathcal{H}'\left[\sigma'_\alpha\right] = -\ln Z_0\left[\sigma'_\alpha\right] + \langle\mathcal{U}\rangle_0 - \frac{1}{2}\left(\langle\mathcal{U}^2\rangle_0 - \langle\mathcal{U}\rangle_0^2\right) + \mathcal{O}\left(\mathcal{U}^3\right),$$

where $\langle\rangle_0$ indicates expectation values calculated with the weight $\exp(-\beta\mathcal{H}_0)$ at fixed $[\sigma'_\alpha]$.

(b) For each cell $\alpha$, define a renormalized spin $\sigma'_\alpha = \mathrm{sign}(\sigma^1_\alpha + \sigma^2_\alpha + \sigma^3_\alpha + \sigma^4_\alpha)$. This choice becomes ambiguous for configurations such that $\sum_{i=1}^4 \sigma^i_\alpha = 0$. Distribute the weight of these configurations equally between $\sigma'_\alpha = +1$ and $-1$ (i.e. put a factor of $1/2$ in addition to the Boltzmann weight). Make a table for all possible configurations of a cell, the internal probability $\exp(-\beta\mathcal{H}_0)$, and the weights contributing to $\sigma'_\alpha = \pm 1$.

| $\sigma'_\alpha = 1$ | Weight | | Weight |
|---|---|---|---|
| + + / + + | $e^{4K}$ | - + / + + | $4 \times 1$ |
| + + / - - | $4 \times 1 \times \frac{1}{2}$ | + - / - + | $2 \times e^{-4K} \times \frac{1}{2}$ |

| $\sigma'_\alpha = -1$ | Weight | | Weight |
|---|---|---|---|
| - - / - - | $e^{4K}$ | + - / - - | $4 \times 1$ |
| + + / - - | $4 \times 1 \times \frac{1}{2}$ | + - / - + | $2 \times e^{-4K} \times \frac{1}{2}$ |

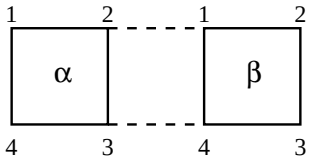
• The possible intracell configurations compatible with a renormalized spin $\sigma'_\alpha = \pm 1$, and their corresponding contributions to the intra-cell probability $\exp(-\beta \mathcal{H}_0)$, are given below, resulting in

$$Z_0 \left[\sigma'_\alpha\right] = \prod_\alpha \left(e^{4K} + 6 + e^{-4K}\right) = \left(e^{4K} + 6 + e^{-4K}\right)^{N/4}.$$

(c) Express $\langle \mathcal{U} \rangle_0$ in terms of the cell spins $\sigma'_\alpha$; and hence obtain the recursion relation $K'(K)$.

• The first cumulant of the interaction term is

$$-\langle \mathcal{U} \rangle_0 = K \sum_{\langle \alpha, \beta \rangle} \langle \sigma_{\alpha 2} \sigma_{\beta 1} + \sigma_{\alpha 3} \sigma_{\beta 4} \rangle_0 = 2K \sum_{\langle \alpha, \beta \rangle} \langle \sigma_{\alpha 2} \rangle_0 \langle \sigma_{\beta 1} \rangle_0,$$



where, for $\sigma'_\alpha = 1$,

$$\langle \sigma_{\alpha i} \rangle_0 = \frac{e^{4K} + (3-1) + 0 + 0}{(\mathrm{e}^{4K} + 6 + e^{-4K})} = \frac{e^{4K} + 2}{(e^{4K} + 6 + e^{-4K})}.$$

Clearly, for $\sigma'_\alpha = -1$ we obtain the same result with a global negative sign, and thus

$$\langle \sigma_{\alpha i} \rangle_0 = \sigma'_\alpha \frac{\mathrm{e}^{4K} + 2}{(e^{4K} + 6 + e^{-4K})}.$$

As a result,

$$-\beta \mathcal{H}' \left[\sigma'_\alpha\right] = \frac{N}{4} \ln \left(e^{4K} + 6 + e^{-4K}\right) + 2K \left(\frac{e^{4K} + 2}{e^{4K} + 6 + e^{-4K}}\right)^2 \sum_{\langle \alpha, \beta \rangle} \sigma'_\alpha \sigma'_\beta,$$

corresponding to the recursion relation $K'(K)$,

$$K' = 2K \left(\frac{e^{4K} + 2}{e^{4K} + 6 + \mathrm{e}^{-4K}}\right)^2.$$

(d) Find the fixed point $K^*$, and the thermal eigenvalue $y_t$.

• To find the fixed point with $K' = K = K^*$, we introduce the variable $x = e^{4K^*}$. Hence, we have to solve the equation

$$\frac{x+2}{x+6+x^{-1}} = \frac{1}{\sqrt{2}}, \qquad \text{or} \qquad \left(\sqrt{2}-1\right) x^2 - \left(6 - 2\sqrt{2}\right) x - 1 = 0,$$

whose only meaningful solution is $x \simeq 7.96$, resulting in $K^* \simeq 0.52$.

To obtain the thermal eigenvalue, let us linearize the recursion relation around this non-trivial fixed point,

$$\left.\frac{\partial K'}{\partial K}\right|_{K^*} = b^{y_t}, \quad \Rightarrow \quad 2^{y_t} = 1 + 8K^* \left[\frac{e^{4K^*}}{\mathrm{e}^{4K^*} + 2} - \frac{e^{4K^*} - e^{-4K^*}}{e^{4K^*} + 6 + \mathrm{e}^{-4K^*}}\right], \quad \Rightarrow \quad y_t \simeq 1.006.$$

(e) In the presence of a small magnetic field $h \sum_i \sigma_i$, find the recursion relation for $h$; and calculate the magnetic eigenvalue $y_h$ at the fixed point.

• In the presence of a small magnetic field, we will have an extra contribution to the Hamiltonian

$$h \sum_{\alpha, i} \langle \sigma_{\alpha, i} \rangle_0 = 4h \frac{e^{4K} + 2}{(\mathrm{e}^{4K} + 6 + e^{-4K})} \sum_\alpha \sigma'_\alpha \,.$$

Therefore,

$$h' = 4h\frac{e^{4K}+2}{(e^{4K}+6+e^{-4K})}.$$

(f) Compare $K^*$, $y_t$, and $y_h$ to their exact values.

• The cumulant method gives a value of $K^* = 0.52$, while the critical point of the Ising model on a square lattice is located at $K_c \approx 0.44$. The exact values of $y_t$ and $y_h$ for the two dimensional Ising model are respectively 1 and 1.875, while the cumulant method yields $y_t \approx 1.006$ and $y_h \approx 1.5$. As in the case of a triangular lattice, $y_h$ is lower than the exact result. Nevertheless, the thermal exponent $y_t$ is fortuitously close to its exact value.

********

**2.** *Migdal–Kadanoff method:* Consider Potts spins $s_i = (1, 2, \cdots, q)$, on sites $i$ of a hypercubic lattice, interacting with their nearest neighbors via a Hamiltonian

$$-\beta\mathcal{H} = K\sum_{<ij>}\delta_{s_i,s_j}\ .$$

(a) In $d = 1$ find the exact recursion relations by a $b = 2$ renormalization/decimation process. Indentify all fixed points and note their stability.

• In $d = 1$, if we average over the $q$ possible values of $s_1$, we obtain

$$\sum_{s_1=1}^{q} e^{K(\delta_{\sigma_1 s_1}+\delta_{s_1\sigma_2})} = \begin{cases} q-1+e^{2K} & \text{if} \quad \sigma_1=\sigma_2 \\ q-2+2e^{K} & \text{if} \quad \sigma_1\neq\sigma_2 \end{cases} \quad = \quad e^{g'+K'\delta_{\sigma_1\sigma_2}},$$

from which we arrive at the exact recursion relations:

$$e^{K'} = \frac{q-1+e^{2K}}{q-2+2e^{K}}, \qquad e^{g'} = q-2+2e^{K}.$$

To find the fixed points we set $K' = K = K^*$. As in the previous problem, let us introduce the variable $x = e^{K^*}$. Hence, we have to solve the equation

$$x = \frac{q-1+x^2}{q-2+2x}, \qquad \text{or} \qquad x^2+(q-2)x-(q-1) = 0,$$

whose only meaningful solution is $x = 1$, resulting in $K^* = 0$. To check its stability, we consider $K \ll 1$, so that

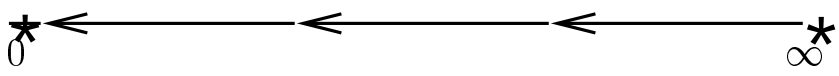
$$K' \simeq \ln\left(\frac{q+2K+2K^2}{q+2K+K^2}\right) \simeq \frac{K^2}{q} \ll K,$$

which indicates that this fixed point is stable.

In addition, $K^* \to \infty$ is also a fixed point. If we consider $K \gg 1$,

$$e^{K'} \simeq \frac{1}{2}e^{K}, \quad \Longrightarrow \quad K' = K - \ln 2 < K,$$

which implies that this fixed point is unstable.



(b) Write down the recursion relation $K'(K)$ in $d$–dimensions for $b = 2$, using the Migdal–Kadanoff bond moving scheme.

• In the Migdal-Kadanoff approximation, moving bonds strengthens the remaining bonds by a factor $2^{d-1}$. Therefore, in the decimated lattice we have

$$e^{K'} = \frac{q - 1 + e^{2\times 2^{d-1}K}}{q - 2 + 2e^{2^{d-1}K}}.$$

(c) By considering the stability of the fixed points at zero and infinite coupling, prove the existence of a non–trivial fixed point at finite $K^*$ for $d > 1$.

• In the vicinity of the fixed point $K^* = 0$, i.e. for $K \ll 1$,

$$K' \simeq \frac{2^{2d-2}K^2}{q} \ll K,$$

and consequently, this point is again stable. However, for $K^* \to \infty$, we have

$$e^{K'} \simeq \frac{1}{2}\exp\left[\left(2^d - 2^{d-1}\right)K\right], \quad \Longrightarrow \quad K' = 2^{d-1}K - \ln 2 \gg K,$$

which implies that this fixed point is now stable provided that $d > 1$.

• As a result, there must be a finite $K^*$ fixed point, which separates the flows to the other fixed points.


(d) For $d = 2$, obtain $K^*$ and $y_t$, for $q = 3$, 1, and 0.

• Let us now discuss a few particular cases in $d = 2$. For instance, if we consider $q = 3$, the non-trivial fixed point is a solution of the equation

$$x = \frac{2 + x^4}{1 + 2x^2}, \qquad \text{or} \qquad x^4 - 2x^3 - x + 2 = (x - 2)(x^3 - 1) = 0,$$

which clearly yields a non-trivial fixed point at $K^* = \ln 2 \simeq 0.69$. The thermal exponent for this point

$$\left.\frac{\partial K'}{\partial K}\right|_{K^*} = 2^{y_t} = 4\left[\frac{e^{4K^*}}{\mathrm{e}^{4K^*} + 2} - \frac{e^{2K^*}}{1 + 2e^{2K^*}}\right] = \frac{16}{9}, \quad \Longrightarrow \quad y_t \simeq 0.83,$$

which can be compared to the exact values, $K^* = 1.005$, and $y_t = 1.2$.

By analytic continuation for $q \to 1$, we obtain

$$e^{K'} = \frac{e^{4K}}{-1 + 2e^{2K}}.$$

The non-trivial fixed point is a solution of the equation

$$x = \frac{x^4}{-1 + 2x^2}, \qquad \text{or} \qquad (x^3 - 2x^2 + 1) = (x-1)(x^2 - x - 1) = 0,$$

whose only non-trivial solution is $x = (1 + \sqrt{5})/2 = 1.62$, resulting in $K^* = 0.48$. The thermal exponent for this point

$$\left.\frac{\partial K'}{\partial K}\right|_{K^*} = 2^{y_t} = 4\left[1 - e^{-K^*}\right], \quad \Longrightarrow \quad y_t \simeq 0.61.$$

As discussed in the next problem set, the Potts model for $q \to 1$ can be mapped onto the problem of *bond percolation*, which despite being a purely geometrical phenomenon, shows many features completely analogous to those of a continuous thermal phase transition.

And finally for $q \to 0$, relevant to *lattice animals* (see PS#9), we obtain

$$e^{K'} = \frac{-1 + e^{4K}}{-2 + 2e^{2K}},$$

for which we have to solve the equation

$$x = \frac{-1 + x^4}{-2 + 2x^2}, \qquad \text{or} \qquad x^4 - 2x^3 + 2x - 1 = (x-1)^3(x+1) = 0,$$

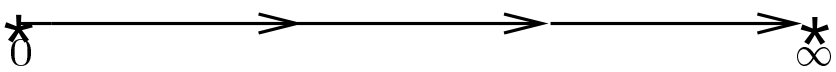
whose only finite solution is the trivial one, $x = 1$. For $q \to 0$, if $K \ll 1$, we obtain

$$K' \simeq K + \frac{K^2}{2} > K,$$

indicating that this fixed point is now unstable. Note that the first correction only indicates marginal stability ($y_t = 0$). Nevertheless, for $K^* \to \infty$, we have

$$e^{K'} \simeq \frac{1}{2}\exp[2K], \quad \Longrightarrow \quad K' = 2K - \ln 2 \gg K,$$

which implies that this fixed point is stable.



********

**3.** *The Potts model:* The *transfer matrix* procedure can be extended to Potts model, where the spin $s_i$ on each site takes $q$ values $s_i = (1, 2, \cdots, q)$; and the Hamiltonian is $-\beta\mathcal{H} = K\sum_{i=1}^{N}\delta_{s_i,s_{i+1}} + K\delta_{s_N,s_1}$.

(a) Write down the transfer matrix and diagonalize it. Note that you do not have to solve a $q^{\text{th}}$ order secular equation as it is easy to guess the eigenvectors from the symmetry of the matrix.

• The partition function is

$$Z = \sum_{\{s_i\}} < s_1|T|s_2 >< s_2|T|s_3 > \cdots < s_{N-1}|T|s_N >< s_N|T|s_1 >= \text{tr}(T^N),$$

where $< s_i|T|s_j >= \exp\left(K\delta_{s_i,s_j}\right)$ is a $q \times q$ transfer matrix. The diagonal elements of the matrix are $e^K$, while the off-diagonal elements are unity. The eigenvectors of the matrix are easily found by inspection. There is one eigenvectors with all elements equal; the corresponding eigenvalue is $\lambda_1 = e^K + q - 1$. There are also $(q-1)$ eigenvectors orthogonal to the first, i.e. the sum of whose elements is zero. This corresponding eigenvalues are degenerate and equal to $e^K - 1$. Thus

$$Z = \sum_{\alpha} \lambda_\alpha^N = \left(e^K + q - 1\right)^N + (q-1)\left(e^K - 1\right)^N.$$

(b) Calculate the free energy per site.

• Since the largest eigenvalue dominates for $N \gg 1$,

$$\frac{\ln Z}{N} = \ln\left(e^K + q - 1\right).$$

(c) Give the expression for the correlation length $\xi$ (you don't need to provide a detailed derivation), and discuss its behavior as $T = 1/K \to 0$.

• Correlations decay as the ratio of the eigenvalues to the power of the separation. Hence the correlation length is

$$\xi = \left[\ln\left(\frac{\lambda_1}{\lambda_2}\right)\right]^{-1} = \left[\ln\left(\frac{e^K + q - 1}{e^K - 1}\right)\right]^{-1}.$$

In the limit of $K \to \infty$, expanding the above result gives

$$\xi \simeq \frac{e^K}{q} = \frac{1}{q} \exp\left(\frac{1}{T}\right).$$

********

---

**4.** *The spin–1 model:* Consider a linear chain where the spin $s_i$ at each site takes on three values $s_i = -1,\ 0,\ +1$. The spins interact via a Hamiltonian

$$-\beta\mathcal{H} = \sum_i K s_i s_{i+1}.$$

(a) Write down the transfer matrix $\langle s|T|s'\rangle = e^{Kss'}$ explicitly.

• The explicit form of the 3x3 transfer matrix is

$$T = \begin{pmatrix} e^K & 1 & e^{-K} \\ 1 & 1 & 1 \\ e^{-K} & 1 & e^K \end{pmatrix}.$$

(b) Use symmetry properties to find the largest eigenvalue of $T$ and hence obtain the expression for the free energy per site $(\ln Z/N)$.

• Because of the symmetry of $T$, we guess an eigenvector of the form $v_1 = \begin{pmatrix} 1 \\ r \\ 1 \end{pmatrix}$. For an eigenvalue $\lambda$, we then obtain the pair of equations

$$\begin{cases} \left(e^K + e^{-K}\right) + r = \lambda \\ 2 + r = r\lambda \end{cases}, \quad \Longrightarrow \quad r = \frac{2}{\lambda - 1} = \lambda - 2\cosh K.$$

This leads to the quadratic equation

$$\lambda^2 - \lambda\left(1 + 2\cosh K\right) - 2\left(1 - \cosh K\right) = 0,$$

which has solutions

$$2\lambda_\pm = 1 + 2\cosh K \pm \sqrt{1 + 4\cosh K + 4\cosh^2 K + 8 - 8\cosh K}.$$

The partition function is related to the largest eigenvalue in the limit of large $N$, and hence

$$\frac{\ln Z}{N} = \ln \lambda_+ = \ln\left[1 + 2\cosh K + \sqrt{1 + 4\cosh K + 4\cosh^2 K + 8 - 8\cosh K}\right] - \ln 2.$$

(c) Obtain the expression for the correlation length $\xi$, and note its behavior as $K \to \infty$.

• The second largest eigenvalue corresponds to the eigenvector $v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, and is equal to

$$\lambda_2 = e^K - e^{-K} = 2 \sinh K.$$

The ratio of eigenvalues is related to the correlation length by

$$\xi = \frac{1}{\ln(\lambda_1/\lambda_2)} = \ln^{-1} \left[ \frac{1 + 2\cosh K + \sqrt{1 + 4\cosh K + 4\cosh^2 K + 8 - 8\cosh K}}{4 \sinh K} \right].$$

In the limit of large $K$,

$$\lambda_1 \simeq e^K \left(1 + 4e^{-2K} + \cdots\right), \qquad \text{and} \qquad \lambda_2 \simeq e^K \left(1 - 2e^{-2K} + \cdots\right),$$

such that the correlation length diverges as

$$\lim_{K \to \infty} \xi(K) \simeq \frac{e^{2K}}{5}.$$

(d) If we try to perform a renormalization group by decimation on the above chain we find that additional interactions are generated. Write down the simplest generalization of $\beta\mathcal{H}$ whose parameter space is closed under such RG.

• The only symmetry present in the Hamiltonian is inversion, $\{s_i\} \to \{-s_i\}$, and the most general spin-1 Hamiltonian consistent with this symmetry is

$$B(s, s') = g + K s s' + \frac{\mu}{2}\left(s^2 + s'^2\right) + \Delta (s s')^2.$$

********

ectionClock model: Each site of the lattice is occupied by a $q$-valued spin $s_i \equiv 1, 2, \cdots, q$, with an underlying translational symmetry modulus $q$, i.e. the system is invariant under $s_i \to (s_i + n)_{\mathrm{mod}q}$. The most general Hamiltonian subject to this symmetry with nearest–neighbor interactions is

$$\beta\mathcal{H}_C = -\sum_{<i,j>} J(|s_i - s_j|_{\mathrm{mod}q}),$$

where $J(n)$ is any function, e.g. $J(n) = J\cos(2\pi n/q)$. *Potts models* are a special case of Clock models with full *permutation symmetry*, and Ising model is obtained in the limit of $q = 2$.

(e) For a closed linear chain of $N$ clock spins subject to the above Hamiltonian show that the partition function $Z = \mathrm{tr}\,[\exp(-\beta\mathcal{H})]$ can be written as

$$Z = \mathrm{tr}\,[\langle s_1|T|s_2\rangle\langle s_2|T|s_3\rangle \cdots \langle s_N|T|s_1\rangle] \ ;$$

where $T \equiv \langle s_i|T|s_j\rangle = \exp\,[J(s_i - s_j)]$ is a $q \times q$ transfer matrix.

• This is a simple generalization of the previous examples, and again the partition function for $N$ sites with period boundary conditions is $Z = \mathrm{tr}\,T^N$, with the $q \times q$ matrix whose elements are $T \equiv \langle s_i|T|s_j\rangle = \exp\,[J(s_i - s_j)]$.

(f) Write down the transfer matrix explicitly and diagonalize it. Note that you do not have to solve a $q^{\mathrm{th}}$ order secular equation; because of the translational symmetry, the eigenvalues are easily obtained by discrete Fourier transformation as

$$\lambda(k) = \sum_{n=1}^{q} \exp\left[J(n) + \frac{2\pi i n k}{q}\right].$$

• Because of the symmetry $T \equiv \langle s_i|T|s_j\rangle = T(|s - s'|_{\mathrm{mod}q})$, we anticipate that eigenvectors of the matrix have the simple form

$$v_k = \frac{1}{\sqrt{q}} \begin{pmatrix} 1 \\ \omega^k \\ \omega^{2k} \\ \vdots \\ \omega^{(q-1)k} \end{pmatrix}, \quad \text{where} \quad \omega = e^{2\pi i/q} \quad \text{is a } q^{th} \text{ root of unity.}$$

It can then be explicitly verified that $Tv_k = \lambda_k v_k$, with

$$\lambda_k = \sum_{n=1}^{q} \exp\left[J(n) + \frac{2\pi i n k}{q}\right].$$

(g) Show that $Z = \sum_{k=1}^{q} \lambda(k)^N \approx \lambda(0)^N$ for $N \to \infty$. Write down the expression for the free energy per site $\beta f = -\ln Z/N$.

• For periodic boundary conditions

$$Z = \mathrm{tr}\, T^N = \lambda_0^N + \lambda_1^N + \cdots = \lambda_0^N \left[ 1 + \sum_{k=1}^{q-1} \left( \frac{\lambda_k}{\lambda_0} \right)^N \right].$$

For $N \to \infty$, since $\lambda_0$ is the largest eigenvalue, we obtain

$$\beta f = -\frac{\ln Z}{N} = -\ln \lambda_0 = -\ln \left[ \sum_{k=0}^{q-1} e^{J(k)} \right].$$

(h) Show that the correlation function can be calculated from

$$\left\langle \delta_{s_i, s_{i+\ell}} \right\rangle = \frac{1}{Z} \sum_{\alpha=1}^{q} \mathrm{tr} \left[ \Pi_\alpha T^\ell \Pi_\alpha T^{N-\ell} \right],$$

where $\Pi_\alpha$ is a projection matrix. Hence show that $\left\langle \delta_{s_i, s_{i+\ell}} \right\rangle_c \sim [\lambda(1)/\lambda(0)]^\ell$. (You do not have to explicitly calculate the constant of proportionality.)

• Consider the matrix $\Pi_\alpha$ that has zero elements everywhere, except for a 1 for its $(\alpha, \alpha)$ element. Then

$$\left\langle \delta_{s_i, s_{i+\ell}} \right\rangle = \frac{1}{Z} \sum_{\alpha=1}^{q} \mathrm{tr} \left[ \Pi_\alpha T^\ell \Pi_\alpha T^{N-\ell} \right].$$

Evaluating the trace in the basis of (Fourier) eigenvectors gives

$$\left\langle \delta_{s_i, s_{i+\ell}} \right\rangle = \frac{1}{Z} \sum_{\alpha=1}^{q} \sum_{k,k'} < k' | \Pi_\alpha | k > \lambda_k^\ell < k | \Pi_\alpha | k' > \lambda_{k'}^{N-\ell}.$$

The main contributions to the sum come from the the two largest eigenvalues, $\lambda_0$ and $\lambda_1$; hence

$$\left\langle \delta_{s_i, s_{i+\ell}} \right\rangle = \frac{1}{\lambda_0^N} \sum_{\alpha=1}^{q} \left[ \lambda_0^N \frac{1}{q^2} + \lambda_0^{N-\ell} \lambda_1^\ell \frac{(\omega \omega^*)^\alpha}{q^2} \right] = \frac{1}{q} \left[ 1 + \left( \frac{\lambda_1}{\lambda_0} \right)^\ell \right] \simeq \frac{1}{q} e^{-\ell/\xi}.$$

(Note that if the second eigenvalue is degenerate, there will also be a degeneracy factor.) The correlation length is given as

$$\xi = \ln^{-1} \left( \frac{\lambda_1}{\lambda_0} \right).$$

********

**5.** *XY model:* Consider two component unit spins $\vec{s_i} = (\cos\theta_i, \sin\theta_i)$ in one dimension, with the nearest neighbor interactions described by $-\beta\mathcal{H} = K\sum_{i=1}^{N} \vec{s_i}\cdot\vec{s_{i+1}}$.

(a) Write down the transfer matrix $\langle\theta|T|\theta'\rangle$, and show that it can be diagonalized with eigenvectors $f_m(\theta) \propto e^{im\theta}$ for integer $m$.

• In the basis of the angles $\{\theta_i\}$,

$$\langle\theta|T|\theta'\rangle = \exp\left[K\cos\left(\theta-\theta'\right)\right].$$

The above transfer 'matrix' is diagonalized in the Fourier basis with $\langle\theta|m\rangle = e^{im\theta}$ (not normalized), as it is easily checked that

$$\langle\theta|T|\theta'\rangle\,\langle\theta'|m\rangle = \int_0^{2\pi}\frac{d\theta'}{2\pi}e^{K\cos(\theta-\theta')+im\theta'} = e^{im\theta}\int_0^{2\pi}\frac{d\phi}{2\pi}e^{K\cos(\phi)+im\phi} = \lambda_m\langle\theta|m\rangle,$$

with

$$\lambda_m = \int_0^{2\pi}\frac{d\phi}{2\pi}e^{K\cos(\phi)+im\phi}.$$

(b) Calulate the free energy per site, and comment on the behavior of the heat capacity as $T \propto K^{-1} \to 0$.

• For large $K$ the eigenvalues can be approximated by the saddle point method as

$$\lambda_m \simeq \frac{e^K}{2\pi}\int_{-\infty}^{\infty} e^{-K\theta^2/2+im\theta} = \frac{e^K}{\sqrt{2\pi}K}e^{-\frac{m^2}{2K}}.$$

They are ordered with $\lambda_0$ being the largest, and thus

$$\frac{\ln Z}{N} \simeq \ln\lambda_0 \approx K - \frac{1}{2}\ln(2\pi K) + \cdots.$$

The dimensionless parameter $K$ can be related to a bond energy $J$ by $K = \beta J$, where $\beta = 1/(k_B T)$. Thus the energy of the chain in the limit of low temperatures is given by

$$E = -\frac{\partial \ln Z}{\partial\beta} = J\frac{\partial \ln Z}{\partial K} = -NJ\left(1 - \frac{1}{2K} + \cdot\right) = -NJ + N\frac{k_B T}{2} + \cdots.$$

The first term is just the energy of the perfectly ordered chain. The second is the energy of excitations into the harmonic modes. Since there are $N$ such quadratic modes, the potential energy of excitations is $N \times (k_B T)/2$. Note that since quantum effects are ignored, there is no quenching of the oscillator energies, and a finite heat capacity of $Nk_B/2$ at zero temperature.

(c) Find the correlation length $\xi$, and note its behavior as $K \to \infty$.

• As usual, the correlation length is related to the ratio of the two largest eigenvalues as

$$\xi = -\ln^{-1}\left(\frac{\lambda_1}{\lambda_0}\right) \approx_{(K\to\infty)} 2K \propto \frac{1}{T}.$$

Note that the divergence of the correlation length as $T \to 0$ is very different for cases where there is a gap in the excitation energy (above the ground state), and when there is none.

********

**6.** *One dimensional gas:* The transfer matrix method can also be applied to a one dimensional gas of particles with short-range interactions, as described in this problem.

(a) Show that for a potential with a hard core that screens the interactions from further neighbors, the Hamiltonian for $N$ particles can be written as

$$\mathcal{H} = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i=2}^{N} \mathcal{V}(x_i - x_{i-1}).$$

The (indistinguishable) particles are labeled with coordinates $\{x_i\}$ such that

$$0 \le x_1 \le x_2 \le \cdots \le x_N \le L,$$

where $L$ is the length of the box confining the particles.

• Each particle $i$ interacts only with adjacent particles $i-1$ and $i+1$, as the hard cores from these nearest neighbors screen the interactions with any other particle. Thus we need only consider nearest neighbor interactions, and, included the kinetic energies, the Hamiltonian is given by

$$\mathcal{H} = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i=2}^{N} \mathcal{V}(x_i - x_{i-1}), \qquad \text{for} \qquad 0 \le x_1 \le x_2 \le \cdots x_N \le L.$$

(b) Write the expression for the partition function $Z(T, N, L)$. Change variables to $\delta_1 = x_1$, $\delta_2 = x_2 - x_1$, $\cdots$, $\delta_N = x_N - x_{N-1}$, and carefully indicate the allowed ranges of integration and the constraints.

- The partition function is

$$Z(T,N,L) = \frac{1}{h^N}\int_0^L dx_1 \int_{x_1}^L dx_2 \cdots \int_{x_{N-1}}^L dx_N \exp\left[-\beta \sum_{i=2}^N \mathcal{V}(x_i - x_{i-1})\right] \cdot \int_{-\infty}^{\infty} dp_1 \cdots \int_{-\infty}^{\infty} dp_N \exp\left[-\beta \sum_{i=1} N \frac{p_i^2}{2m}\right]$$
$$= \frac{1}{\lambda^N}\int_0^L dx_1 \int_{x_1}^L dx_2 \cdots \int_{x_{N-1}}^L dx_N \exp\left[-\beta \sum_{i=2}^N \mathcal{V}(x_i - x_{i-1})\right],$$

where $\lambda = h/\sqrt{2\pi m k_B T}$. (Note that there is no $N!$ factor, as the ordering of the particles is specified.) Introducing a new set of variables

$$\delta_1 = x_1, \quad \delta_2 = x_2 - x_1, \quad \cdots \quad \delta_n = x_N - x_{N-1},$$

or equivalently

$$x_1 = \delta_1, \quad x_2 = \delta_1 + \delta_2, \quad \cdots \quad x_N = \sum_{i=1}^N \delta_i,$$

the integration becomes

$$Z(T,N,L) = \frac{1}{\lambda^N}\int_0^L d\delta_1 \int_0^{L-\delta_1} d\delta_2 \int_0^{L-(\delta_1+\delta_2)} d\delta_3 \cdots \int_0^{L-\sum_{i=1}^N \delta_i} d\delta_N e^{-\beta \sum_{i=2}^N \mathcal{V}_{(\delta_i)}}.$$

This integration can also be expressed as

$$Z(T,N,L) = \frac{1}{\lambda^N}\left[\int d\delta_1 d\delta_2 \cdots d\delta_N\right]' \exp\left[-\beta \sum_{i=2}^N \mathcal{V}(\delta_i)\right],$$

with the constraint

$$0 \le \sum_{i=1}^N \delta_i \le L.$$

This constraint can be put into the equation explicitly with the use of the step function

$$\Theta(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad x \ge 0 \end{cases},$$

as

$$Z(T,N,L) = \frac{1}{\lambda^N}\int_0^{\infty} d\delta_1 \int_0^{\infty} d\delta_2 \cdots \int_0^{\infty} d\delta_N \exp\left[-\beta \sum_{i=2}^N \mathcal{V}(\delta_i)\right] \Theta\left(L - \sum_{i=1}^N \delta_i\right).$$

(c) Consider the Gibbs partition function obtained from the Laplace transformation

$$\mathcal{Z}(T,N,P) = \int_0^\infty dL \exp(-\beta P L) Z(T,N,L),$$

and by extremizing the integrand find the standard formula for $P$ in the canonical ensemble.

- The Gibbs partition function is

$$\mathcal{Z}(T,N,P) = \int_0^\infty dL \exp(-\beta P L) Z(T,N,L).$$

The saddle point is obtained by extremizing the integrand with respect to $L$,

$$\frac{\partial}{\partial L} \exp(-\beta P L) Z(T,N,L)\Big|_{T,N} = 0,$$

which implies that

$$\beta P = \frac{\partial}{\partial L} \ln Z(T,N,L)\Big|_{T,N}, \qquad \Longrightarrow \qquad P = k_B T \frac{\partial \ln Z}{\partial L}\Big|_{T,N}.$$

From thermodynamics, for a one–dimensional gas we have

$$dF = -S dT - P dL, \quad \Longrightarrow \quad P = -\frac{\partial F}{\partial L}\Big|_{T,N}.$$

Further noting that

$$F = -k_B T \ln Z,$$

again results in

$$P_{\text{canonical}} = k_B T \frac{\partial \ln Z}{\partial L}\Big|_{T,N}.$$

(d) Change variables from $L$ to $\delta_{N+1} = L - \sum_{i=1}^N \delta_i$, and find the expression for $\mathcal{Z}(T,N,P)$ as a product over one-dimensional integrals over each $\delta_i$.

- The expression for the partition function given above is

$$Z(T,N,L) = \frac{1}{\lambda^N} \int_0^\infty d\delta_1 \int_0^\infty d\delta_2 \cdots \int_0^\infty d\delta_N \exp\left[-\beta \sum_{i=2}^N \mathcal{V}(\delta_i)\right] \Theta\left(L - \sum_{i=1}^N \delta_i\right).$$

The Laplace transform of this equation is

$$\begin{aligned}\mathcal{Z}(T,N,P) &= \frac{1}{\lambda^N}\int_0^\infty dL\exp(-\beta PL)\int_0^\infty d\delta_1\int_0^\infty d\delta_2\cdots\int_0^\infty d\delta_N \\ &\qquad\cdot\exp\left[-\beta\sum_{i=2}^{N}\mathcal{V}(\delta_i)\right]\Theta\left(L-\sum_{i=1}^{N}\delta_i\right) \\ &= \frac{1}{\lambda^N\beta P}\int_0^\infty d\delta_1\int_0^\infty d\delta_2\cdots\int_0^\infty d\delta_N\exp\left[-\beta\sum_{i=2}^{N}\mathcal{V}(\delta_i)\right]\exp\left[-\beta P\left(\sum_{i=1}^{N}\delta_1\right)\right] \\ &= \frac{1}{\lambda^N(\beta P)^2}\int_0^\infty d\delta_2\cdots\int_0^\infty d\delta_N\exp\left\{-\sum_{i=2}^{N}\left[\beta\mathcal{V}(\delta_i)+\beta P\delta_i\right]\right\}.\end{aligned}$$

Since the integrals for different $\delta_i's$ are equivalent, we obtain

$$\mathcal{Z}(T,N,P) = \frac{1}{\lambda^N(\beta P)^2}\left\{\int_0^\infty d\delta\exp\left[-\beta\left(\mathcal{V}(\delta)+P\delta\right)\right]\right\}^{N-1}.$$

This expression can also be obtained directly, without use of the step function as follows.

$$\begin{aligned}\mathcal{Z}(T,N,P) &= \frac{1}{\lambda^N}\int_0^L d\delta_1\int_0^{L-\delta_1} d\delta_2\int_0^{L-(\delta_1+\delta_2)} d\delta_3\cdots\int_0^{L-\sum_{i=1}^{N}\delta_i} d\delta_N \\ &\qquad\cdot\int_0^\infty dL\exp\left[-\beta PL-\beta\left(\sum_{i=2}^{N}\mathcal{V}(\delta)\right)\right] \\ &= \frac{1}{\lambda^N}\int_0^L d\delta_1\int_0^{L-\delta_1} d\delta_2\cdots\int_0^{L-\sum_{i=1}^{N}\delta_i} d\delta_N\int_{-\sum_{i=1}^{N}\delta_i}^\infty d\left(L-\sum_{i=1}^{N}\delta_i\right) \\ &\qquad\cdot\exp\left\{-\beta P\left[\sum_{i=1}^{N}\delta_i+\left(L-\sum_{i=1}^{N}\delta_i\right)\right]-\beta\left(\sum_{i=2}^{N}\mathcal{V}(\delta_i)\right)\right\}.\end{aligned}$$

Change variables to $\delta_{N+1}\equiv L-\sum_{i=1}^{N}\delta_i$, and note that each of the $\delta's$ indicates the distance between neighboring particles. The size of the gas $L$, has been extended to any value, hence each $\delta$ can be varied independently from 0 to $\infty$. Thus the Gibbs partition

function is

$$\begin{aligned}\mathcal{Z}(T,N,P) &= \frac{1}{\lambda^N}\int_0^\infty d\delta_1 \int_0^\infty d\delta_2 \cdots \int_0^\infty d\delta_N \int_0^\infty d\delta_{N+1} \\ &\qquad \cdot \exp\left[-\beta P\left(\sum_{i=1}^{N+1}\delta_i\right) - \beta\left(\sum_{i=2}^{N}\mathcal{V}(\delta_i)\right)\right] \\ &= \frac{1}{\lambda^N}\left(\int_0^\infty d\delta \cdot \exp\left[-\beta\mathcal{V}(\delta) - \beta P\delta\right]\right)^{N-1} \int_0^\infty d\delta_1 \exp(-\beta P\delta_1) \\ &\qquad \cdot \int_0^\infty d\delta_{N+1}\exp(-\beta P\delta_{N+1}) \\ &= \frac{1}{\lambda^N(\beta P)^2}\left\{\int_0^\infty d\delta \exp\left[-\beta\left(\mathcal{V}(\delta)+P\delta\right)\right]\right\}^{N-1}.\end{aligned}$$

(e) At a fixed pressure $P$, find expressions for the mean length $L(T,N,P)$, and the density $n = N/L(T,N,P)$ (involving ratios of integrals which should be easy to interpret).

- The mean length is

$$\begin{aligned}L(T,N,P) &= -k_B T\,\frac{\partial}{\partial(\beta P)}\ln\mathcal{Z}(T,N,P)\Big|_{T,N} \\ &= \frac{2}{\beta P} + (N-1)\frac{\int_0^\infty d\delta\cdot\delta\cdot\exp\left[-\beta\mathcal{V}(\delta)-\beta P\delta\right]}{\int_0^\infty d\delta\cdot\exp\left[-\beta\mathcal{V}(\delta)-\beta P\delta\right]},\end{aligned}$$

and the density $n$ is given by

$$n = \frac{N}{L(T,N,P)} = N\left\{\frac{2k_BT}{P} + (N-1)\frac{\int_0^\infty d\delta\cdot\delta\cdot\exp\left[-\beta\left(\mathcal{V}(\delta)-P\delta\right)\right]}{\int_0^\infty d\delta\cdot\exp\left[-\beta\left(\mathcal{V}(\delta)-P\delta\right)\right]}\right\}^{-1}.$$

Note that for an ideal gas $\mathcal{V}_{\text{i.g.}}(\delta) = 0$, and

$$L_{\text{i.g.}}(T,N,P) = \frac{(N+1)k_BT}{P},$$

leading to

$$n(p)_{\text{i.g.}} = \frac{N}{N+1}\frac{P}{k_BT}.$$

Since the expression for $n(T,P)$ in part (e) is continuous and non-singular for any choice of potential, there is in fact no condensation transition for the one-dimensional gas. By contrast, the approximate van der Waals equation (or the mean-field treatment) incorrectly predicts such a transition.

(f) For a hard sphere gas, with minimum separation $a$ between particles, calculate the equation of state $P(T, n)$.

• For a hard sphere gas

$$\delta_i \geq a, \qquad \text{for} \qquad i = 2, 3, \cdots, N,$$

the Gibbs partition function is

$$\begin{aligned}\mathcal{Z}(T, N, P) &= \frac{1}{\lambda^N (\beta P)^2} \left[ \int_a^\infty d\delta \exp\left(-\beta \mathcal{V}(\delta) - \beta P \delta\right) \right]^{N-1} \\ &= \frac{1}{\lambda^N (\beta P)^2} \left[ \int_a^\infty d\delta \exp\left(-\beta P \delta\right) \right]^{N-1} \\ &= \frac{1}{\lambda^N} \left( \frac{1}{\beta P} \right)^{N+1} \exp\left(-\beta P a\right)^{N-1}.\end{aligned}$$

From the partition function, we can calculate the mean length

$$L = -k_B T \left. \frac{\partial \ln \mathcal{Z}}{\partial P} \right|_{T,N} = \frac{(N+1) k_B T}{P} + (N-1)a,$$

which after rearrangement yields

$$\beta P = \frac{(N+1)}{L - (N-1)a} = \frac{n + 1/L}{1 - (n - 1/L)a} \approx (n + 1/l)(1 + (n - 1/L)a + (n - 1/L)^2 a^2 + \cdots).$$

For $N \gg 1$, $n \gg 1/L$, and

$$\beta P \approx n(1 + na + n^2 a^2 + \cdots) = n + an^2 + a^2 n^3 + \cdots,$$

which gives the virial coefficients

$$B_\ell(T) = a^{\ell - 1}.$$

The value of $B_3 = a^2$ agrees with the result obtained in an earlier problem. Also note that the exact 'excluded volume' is $(N-1)a$, as opposed to the estimate of $Na/2$ used in deriving the van der Waals equation.

********

**7.** *Potts chain (RG):* Consider a one-dimensional array of $N$ Potts spins $s_i = 1, 2, \cdots, q$, subject to the Hamiltonian $-\beta\mathcal{H} = J \sum_i \delta_{s_i, s_{i+1}}$.

(a) Using the transfer matrix method (or otherwise) calculate the partition function $Z$, and the correlation length $\xi$.

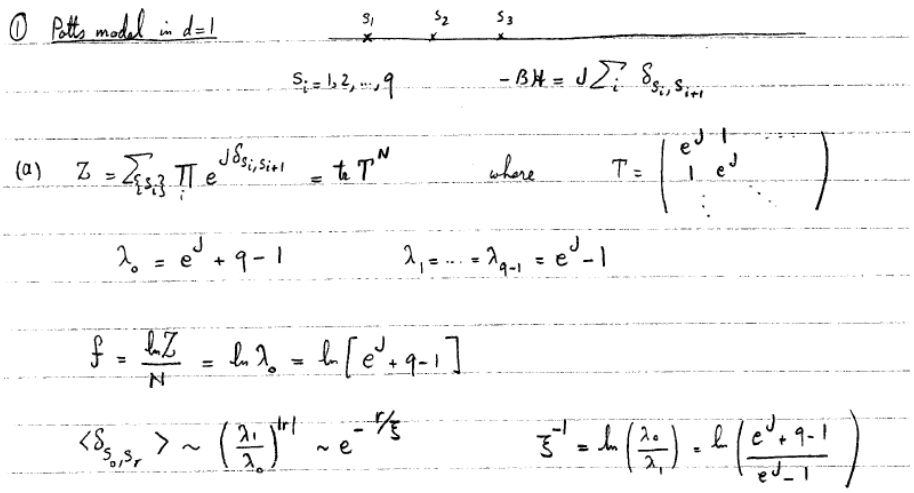
•

(b) Is the system critical at zero temperature for antiferromagnetic couplings $J < 0$?

•

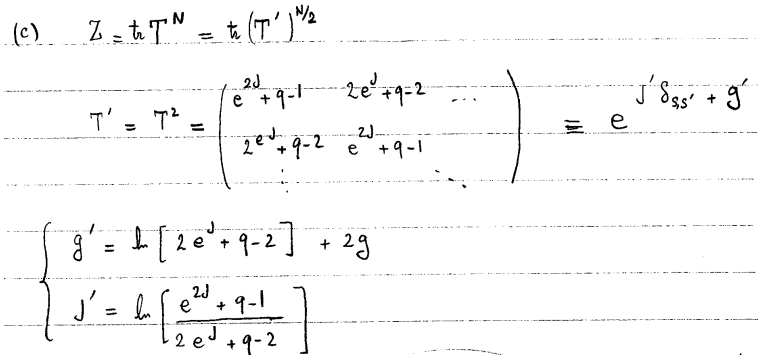
(b) $\lim_{J \to +\infty} \frac{\lambda_1}{\lambda_0} = 1 \qquad \xi \to \infty$

$\lim_{J \to -\infty} \frac{\lambda_1}{\lambda_0} = \left(-\frac{1}{q-1}\right) \qquad \xi^{-1} = \ln(q-1)$ is finite



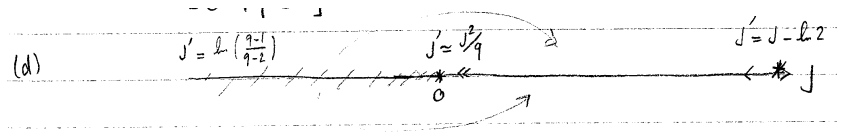

(c) Construct a renormalization group (RG) treatment by eliminating every other spin. Write down the recursion relations for the coupling $J$, and the additive constant $g$.

•




(d) Discuss the fixed points, and their stability.

•




(e) Write the expression for $\ln Z$ in terms of the additive constants of successive rescalings.

•

(f) Show that the recursion relations are simplified when written in terms of $t(J) \equiv e^{-1/\xi(J)}$.

•

(e) $\ln Z_N(J) = \frac{N}{2} g'(J) + \ln Z_{N/2}(J'(J))$

$= \frac{N}{2} g'(J_0) + \frac{N}{4} g'(J_1) + \ln Z_{N/4}(J_2)$

$= \frac{N}{2} g'(J_0) + \frac{N}{4} g'(J_1) + \frac{N}{8} g'(J_2) + \cdots$

$f = \frac{\ln Z}{N} = \sum_{\alpha=0}^{\infty} \frac{1}{2^{\alpha+1}} g'(J_\alpha)$

(f) $T' = T^2 \qquad \lambda_i' = \lambda_i^2 \qquad t \equiv \frac{\lambda_1}{\lambda_0} = e^{-1/T}$

$t' = t^2 \qquad t_n = t^{2^n}$

$t = \frac{e^J - 1}{e^J + q - 1} \qquad e^J t + (q-1)t = e^J - 1 \qquad e^J = \frac{1 + (q-1)t}{1-t}$

$g'(J) = \ln\left[2e^J + q - 2\right] = \ln\left[\frac{2 + (2q-2)t + (q-2)(1-t)}{1-t}\right] = \ln\left[q\,\frac{1+t}{1-t}\right]$

(g) Use the result in (f) to express the series in (e) in terms of $t$. Show that the answer can be reduced to that obtained in part (a), upon using the result

$$\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \ln\left(\frac{1+t^{2^n}}{1-t^{2^n}}\right) = -\ln(1-t).$$

•

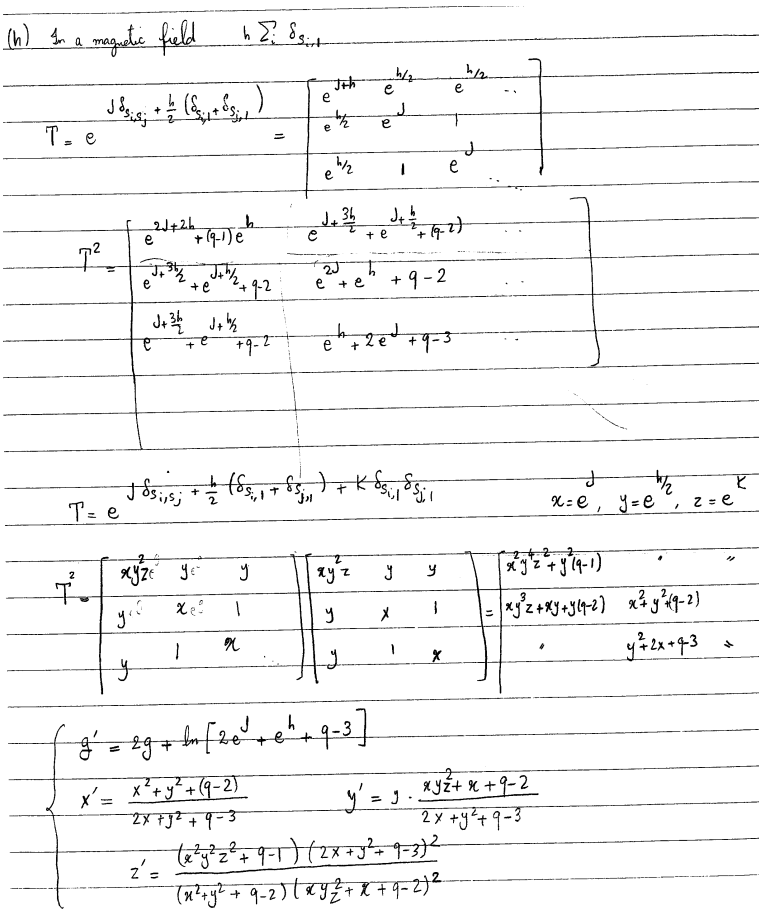
(g) $f = \frac{\ln Z}{N} = \sum_{\alpha=0}^{\infty} \frac{1}{2^{\alpha+1}} \left[\ln q + \ln\left(\frac{1+t^{2^\alpha}}{1-t^{2^\alpha}}\right)\right]$

$= \ln q - \ln(1-t)$

$= \ln q - \ln\left[1 - \frac{e^J - 1}{e^J + q - 1}\right] = \ln q - \ln \frac{q}{e^J + q - 1}$

$= \ln(e^J + q - 1)$ ✓

(h) Repeat the RG calculation of part (c), when a small symmetry breaking term $h \sum_i \delta_{s_i,1}$ is added to $-\beta\mathcal{H}$. You will find that an additional coupling term $K \sum_i \delta_{s_i,1}\delta_{s_{i+1},1}$ is generated under RG. Calculate the recursion relations in the three parameter space $(J, K, h)$.



•

(i) Find the magnetic eigenvalues at the zero temperature fixed point where $J \to \infty$, and obtain the form of the correlation length close to zero temperature.

•

********

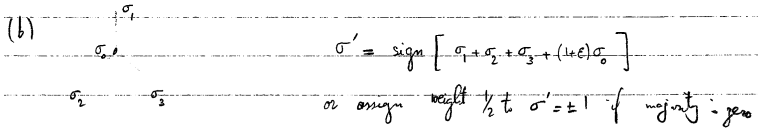
**8.** *Cluster RG:* Consider Ising spins on a *hexagonal lattice* with nearest neighbor interactions $J$.

(a) Group the sites into clusters of four in preparation for a position space renormalization group with $b = 2$.

•

(b) How can the majority rule be modified to define the renormalized spin of each cluster.

•



(i) $x \to \infty \qquad x' = \frac{x}{2}$

$$\begin{cases} y' = y\,\dfrac{(y^2 z + 1)}{2} \\ z' = \dfrac{y^2 z^2\, 4}{(y^2 z + 1)^2} \end{cases} \qquad h, K \to 0$$

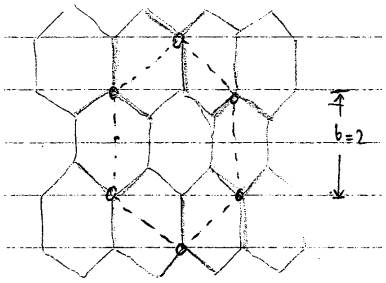
$$\begin{cases} 1 + \frac{h'}{2} = \left(1 + \frac{h}{2}\right)\frac{1}{2}\left[1 + h + K + 1\right] = \left(1 + \frac{h}{2}\right)\left(1 + \frac{h}{2} + \frac{K}{2}\right) = 1 + h + \frac{K}{2} \\ 1 + K' = \dfrac{4\,(1 + h + 2K)}{(1 + h + K)^2} = (1 + h + 2K)\left(1 - \frac{h}{2} - \frac{K}{2}\right)^2 \end{cases}$$

$$\begin{cases} h' = 2h + K \\ K' = K \end{cases} \qquad \begin{pmatrix} h' \\ K' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} h \\ K \end{pmatrix} \qquad y_h = 2 \quad y_K = 1$$

$$\Xi\left(e^J, h, K\right) = 2\,\Xi\left(\frac{e^J}{2}, 2(h + K), K\right)$$

$$\approx 2^n\,\Xi\left(\frac{e^J}{2^n}, 2^n(h + K), K\right) \qquad 2^n = e^J$$

$$= e^J\, f\left(K, e^J(h + K)\right)$$




(c) For a scheme in which the central site is chosen as the tie–breaker, make a table of all possible configurations of site–spins for a given value of the cluster–spin.

•

(d) Focus on a pair of neighboring clusters. Indicate the contributions of intra–cluster and inter–cluster bonds to the total energy.

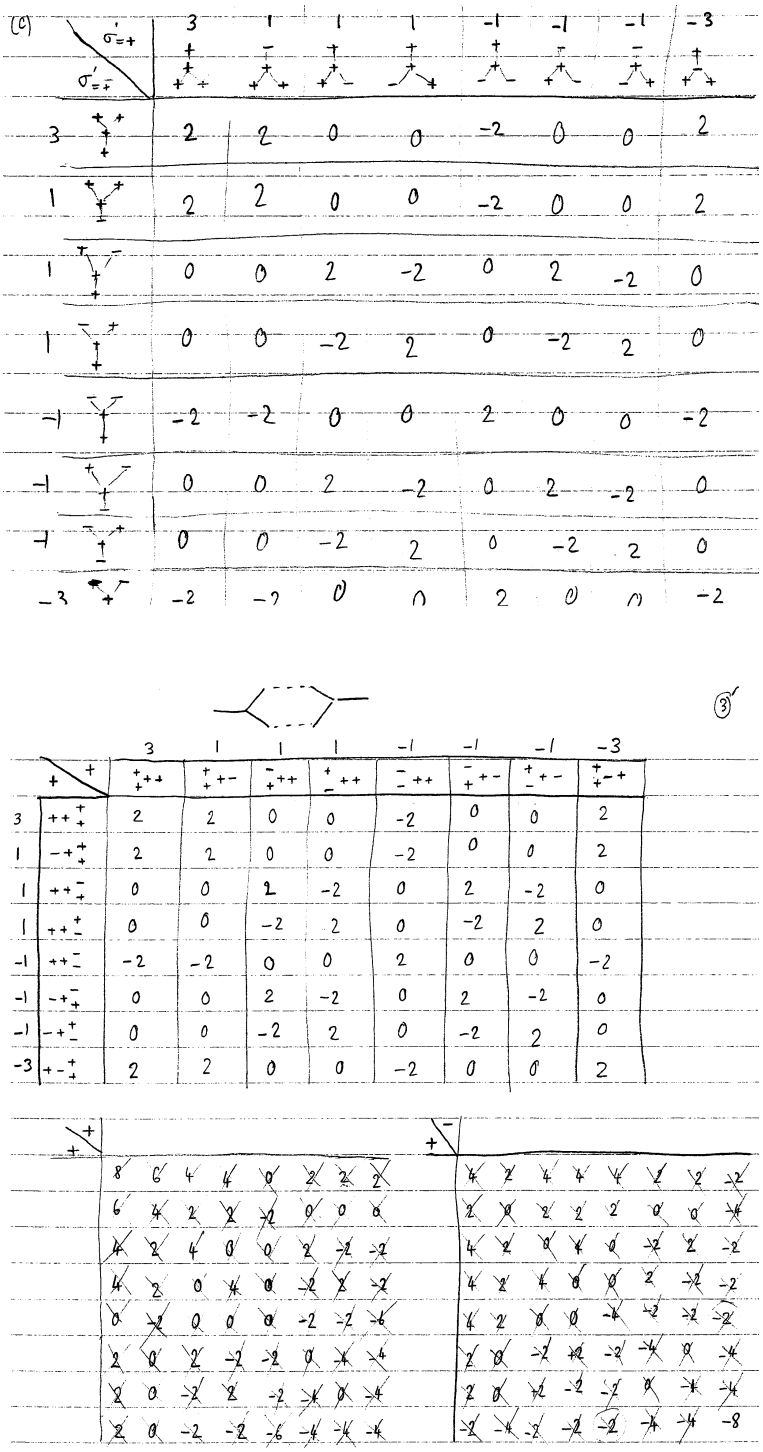
•

(e) Show that in zero magnetic field, the Boltzmann weights of parallel and anti-parallel clusters are given by

$$R(+,+) = x^8 + 2x^6 + 7x^4 + 14x^2 + 17 + 14x^{-2} + 7x^{-4} + 2x^{-6},$$

and

$$R(+,-) = 9x^4 + 16x^2 + 13 + 16x^{-2} + 9x^{-4} + x^{-8},$$



where $x = e^J$.

• Summing over the contributions in the previous tables gives the answer quoted above.

(f) Find the expression for the resulting recursion relation $J'(J)$.

•

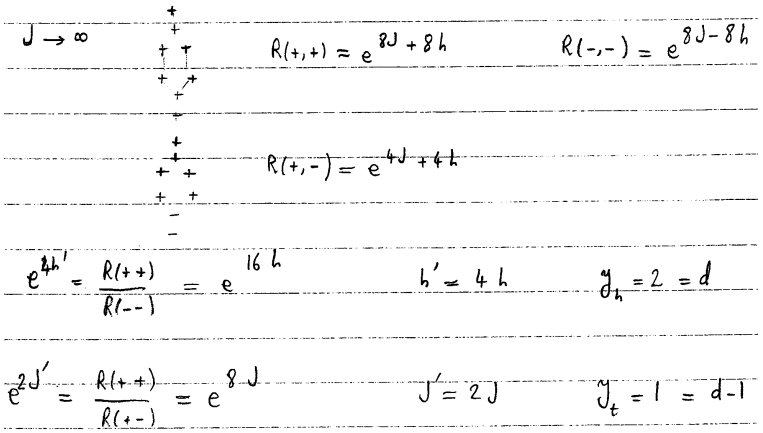
$$\begin{cases} R(+,+) = e^{J'+g'} \\ R(+,-) = e^{-J'+g'} \end{cases} \qquad J' = \frac{1}{2}\ln\left(\frac{R(+,+)}{R(+,-)}\right)$$

(g) Estimate the critical *ferromagnetic* coupling $J_c$, and the exponent $\nu$ obtained from this RG scheme, and compare with the exact values.

• The non-trivial fixed point occurs at $J^* \approx 1.05$, while the exact value of the hexagonal lattice critical coupling is $J_c \approx 0.66$. The value of the eigenvalue $y_t$ and hence the exponent $\nu = 1/y_t$ is close to the exact value of 1.

(h) What are the values of the magnetic and thermal exponents $(y_h, y_t)$ at the zero temperature ferromagnetic fixed point?

•




(i) Is the above scheme also applicable for anti-ferromagnetic interactions? What symmetry of the original problem is not respected by this transformation?

• The distinction between the two sublattices of the hexagonal lattice is not respected by this RG scheme, and hence it does not reproduce the anti-ferromagnetic phase transition.

********

**9.** *Transition probability matrix:* Consider a system of two Ising spins with a coupling $K$, which can thus be in one of four states.

(a) Explicitly write the $4 \times 4$ transition matrix corresponding to single spin flips for a Metropolis algorithm. Verify that the equilibrium weights are indeed a left eigenvector of this matrix.

(b) Repeat the above exercise if both single spin and double spin flips are allowed. The two types of moves are chosen randomly with probabilities $p$ and $q = 1 - p$.

********

**Solutions to problems from chapter 7 - Series Expansions**

**1.** *Continuous spins:* In the standard $\mathcal{O}(n)$ model, $n$ component unit vectors are placed on the sites of a lattice. The nearest neighbor spins are then connected by a bond $J\vec{s}_i \cdot \vec{s}_j$. In fact, if we are only interested in universal properties, any generalized interaction $f(\vec{s}_i \cdot \vec{s}_j)$ leads to the same critical behavior. By analogy with the Ising model, a suitable choice is

$$\exp\left[f(\vec{s}_i \cdot \vec{s}_j)\right] = 1 + (nt)\vec{s}_i \cdot \vec{s}_j,$$

resulting in the so called *loop model*.

(a) Construct a high temperature expansion of the loop model (for the partition function $Z$) in the parameter $t$, on a two-dimensional *hexagonal* (honeycomb) lattice.

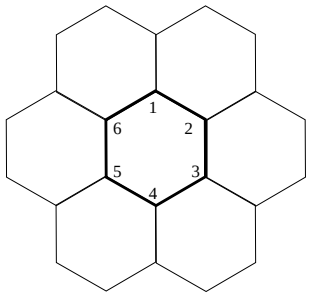
• The partition function for the loop model has the form

$$Z = \int \{\mathcal{D}\mathbf{s}_i\} \prod_{\langle ij \rangle} \left[1 + (nt)\mathbf{s}_i \cdot \mathbf{s}_j\right],$$

that we can expand in powers of the parameter $t$. If the total number of nearest neighbor bonds on the lattice is $N_B$, the above product generates $2^{N_B}$ possible terms. Each term may be represented by a graph on the lattice, in which a bond joining spins $i$ and $j$ is included if the factor $\mathbf{s}_i \cdot \mathbf{s}_j$ appears in the term considered. Moreover, each included bond carries a factor of $nt$. As in the Ising model, the integral over the variables $\{\mathbf{s}_i\}$ leaves only graphs with an even number of bonds emanating from each site, because

$$\int d\mathbf{s}\ s_\alpha = \int d\mathbf{s}\ s_\alpha s_\beta s_\gamma = \cdots = 0.$$

In a honeycomb lattice, as plotted below, there are only 1, 2, or 3 bonds emerging from each site. Thus the only contributing graphs are those with two bonds at each site, which, as any bond can only appear once, are closed *self-avoiding loops*.



While the honeycomb lattice has the advantage of not allowing intersections of loops at a site, the universal results are equally applicable to other lattices.

We shall rescale all integrals over spin by the $n$-dimensional solid angle, such that $\int d\mathbf{s} = 1$. Since $s_\alpha s_\alpha = 1$, it immediately follows that

$$\int d\mathbf{s}\ s_\alpha s_\beta = \frac{\delta_{\alpha\beta}}{n},$$

resulting in

$$\int d\mathbf{s}'\ (s_\alpha s'_\alpha)(s'_\beta s''_\beta) = \frac{1}{n} s_\alpha s''_\alpha.$$

A sequence of such integrals forces the components of the spins around any loop to be the same, and there is a factor $n$ when integrating over the last spin in the loop, for instance

$$\int \{\mathcal{D}\mathbf{s}_i\}\ (s_{1\alpha}s_{2\alpha})(s_{2\beta}s_{3\beta})(s_{3\gamma}s_{4\gamma})(s_{4\delta}s_{5\delta})(s_{5\eta}s_{6\eta})(s_{6\nu}s_{1\nu}) = \frac{\delta_{\alpha\beta}\delta_{\beta\gamma}\delta_{\gamma\delta}\delta_{\delta\eta}\delta_{\eta\nu}\delta_{\alpha\nu}}{n^6} = \frac{n}{n^6}.$$

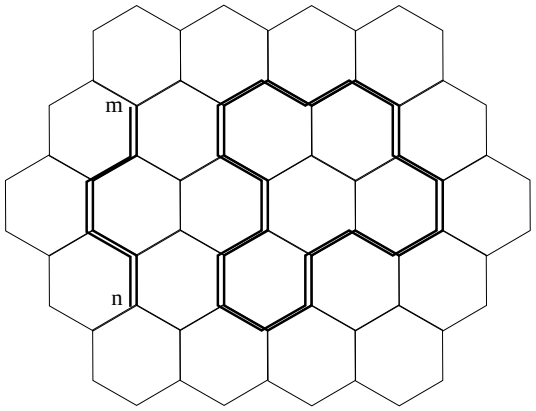
Since each bond carrier a factor of $nt$, each loop finally contributes a factor $n \times t^\ell$, where $\ell$ is the number of bonds in the loop. The partition function may then be written as

$$Z = \sum_{\text{self}-\text{avoiding loops}} n^{N_\ell} t^{N_b},$$

where the sum runs over distinct disconnected or self-avoiding loops collections with a bond fugacity $t$, and $N_\ell$, $N_b$ are the number of loops, and the number of bonds in the graph, respectively. Note that, as we are only interested in the critical behavior of the model, any global analytic prefactor is unimportant.

(b) Show that the limit $n \to 0$ describes the configurations of a single self–avoiding polymer on the lattice.

• While $Z = 1$, at exactly $n = 0$, one may obtain non-trivial information by considering the limit $n \to 0$. The leading term ($\mathcal{O}(n^1)$) when $n \to 0$ picks out just those configurations with a single self-avoiding loop, i.e. $N_\ell = 1$.



The correlation function can also be calculated graphically from

$$G_{\alpha\beta}(n-m) = \langle s_{n\alpha} s_{m\beta} \rangle = \frac{1}{Z} \int \{\mathcal{D}\mathbf{s}_i\} s_{n\alpha} s_{m\beta} \prod_{\langle ij \rangle} \left[1 + (nt)\mathbf{s}_i \cdot \mathbf{s}_j\right].$$

After disregarding any global prefactor, and taking the limit $n \to 0$, the only surviving graph consists of a single line going from $n$ to $m$, and the index of all the spins along the line is fixed to be the same. All other possible graphs disappear in the limit $n \to 0$. Therefore, we are left with a sum over self-avoiding walks that go from $n$ to $m$, each carrying a factor $t^\ell$, where $\ell$ indicates the length of the walk. If we denote by $W_\ell(R)$ the number of self-avoiding walks of length $\ell$ whose end-to-end distance is $R$, we can write that

$$\sum_\ell W_\ell(R) t^\ell = \lim_{n \to 0} G(R).$$

As in the case of phantom random walks, we expect that for small $t$, small paths dominate the behavior of the correlation function. As $t$ increases, larger paths dominate the sum, and, ultimately, we will find a singularity at a particular $t_c$, at which arbitrarily long paths become possible.

Although we presented the mapping of self-avoiding walks to the $n \to 0$ limit of the $\mathcal{O}(n)$ model for a honeycomb lattice, the critical behavior should be universal, and therefore independent of this lattice choice. What is more, various scaling properties of self-avoiding walks can be deduced from the $\mathcal{O}(n)$ model with $n \to 0$. Let us, for instance, characterize the mean square end-to-end distance of a self-avoiding walk, defined as

$$\left\langle R^2 \right\rangle = \frac{1}{W_\ell} \sum_R R^2 W_\ell(R),$$

where $W_\ell = \sum_R W_\ell(R)$ is the total number of self-avoiding walks of length $\ell$.

The singular part of the correlation function decays with separation $R$ as $G \propto |R|^{-(d-2+\eta)}$, up to the correlation length $\xi$, which diverges as $\xi \propto (t_c - t)^{-\nu}$. Hence,

$$\sum_R R^2 G(R) \propto \xi^{d+2-(d-2+\eta)} = (t_c - t)^{-\nu(4-\eta)} = (t_c - t)^{-\gamma-2\nu}.$$

We noted above that $G(t, R)$ is the generating function of $W_\ell(R)$, in the sense that $\sum_\ell W_\ell(R) t^\ell = G(t, R)$. Similarly $\sum_\ell W_\ell t^\ell$ is the generating function of $W_\ell$, and is related to the susceptibility $\chi$, by

$$\sum_\ell W_\ell t^\ell = \sum_R G(R) = \chi \propto (t_c - t)^{-\gamma}.$$

To obtain the singular behavior of $W_\ell$ from its generating function, we perform a Taylor expansion of $(t_c - t)^{-\gamma}$, as

$$\sum_\ell W_\ell t^l = t_c^{-\gamma} \left(1 - \frac{t}{t_c}\right)^{-\gamma} = t_c^{-\gamma} \sum_\ell \frac{\Gamma(1-\gamma)}{\Gamma(1+\ell)\Gamma(1-\gamma-\ell)} \left(\frac{t}{t_c}\right)^\ell ,$$

which results in

$$W_\ell = \frac{\Gamma(1-\gamma)}{\Gamma(1+\ell)\Gamma(1-\gamma-\ell)}\ t_c^{-\ell-\gamma}.$$

After using that $\Gamma(p)\Gamma(1-p) = \pi/\sin p\pi$, considering $\ell \to \infty$, and the asymptotic expression of the gamma function, we obtain

$$W_\ell \propto \frac{\Gamma(\gamma+\ell)}{\Gamma(1+\ell)}\ t_c^{-\ell} \propto \ell^{\gamma-1} t_c^{-\ell},$$

and, similarly one can estimate $\sum_R R^2 W_\ell(R)$ from $\sum_R R^2 G(R)$, yielding

$$\left\langle R^2 \right\rangle \propto \frac{\ell^{2\nu+\gamma-1} t_c^{-\ell}}{\ell^{\gamma-1} t_c^{-\ell}} = \ell^{2\nu}.$$

Setting $n = 0$ in the results of the $\epsilon$-expansion for the $\mathcal{O}(n)$ model, for instance, gives the exponent $\nu = 1/2 + \epsilon/16 + \mathcal{O}(\epsilon^2)$, characterizing the mean square end-to-end distance of a self-avoiding polymer as a function of its length $\ell$, rather than $\nu_0 = 1/2$ which describes the scaling of phantom random walks. Because of self-avoidance, the (polymeric) walk is swollen, giving a larger exponent $\nu$. The results of the first order expansion for $\epsilon = 1$, 2, and 3, in $d = 3$, 2, and 1 are 0.56, 0.625, and 0.69, to be compared to 0.59, 3/4 (exact), and 1 (exact).

********

**2.** *Potts model I:* Consider Potts spins $s_i = (1, 2, \cdots, q)$, interacting via the Hamiltonian $-\beta\mathcal{H} = K \sum_{<ij>} \delta_{s_i,s_j}$ .

(a) To treat this problem graphically at high temperatures, the Boltzmann weight for each bond is written as

$$\exp\left(K\delta_{s_i,s_j}\right) = C(K)\left[1 + T(K) g(s_i, s_j)\right],$$

with $g(s, s') = q\delta_{s,s'} - 1$. Find $C(K)$ and $T(K)$.

• To determine the two unknowns $C(K)$ and $T(K)$, we can use the expressions

$$\begin{cases} e^K = C\left[1 + T(q-1)\right] & \text{if} \quad s_i = s_j \\ 1 = C\left[1 - T\right] & \text{if} \quad s_i \neq s_j \end{cases},$$

from which we obtain

$$T(K) = \frac{e^K - 1}{e^K + q - 1}, \qquad \text{and} \qquad C(K) = \frac{e^K + q - 1}{q}.$$

(b) Show that

$$\sum_{s=1}^{q} g(s, s') = 0 \ , \ \sum_{s=1}^{q} g(s_1, s) g(s, s_2) = q g(s_1, s_2) \ , \ \text{and} \ \sum_{s,s'}^{q} g(s, s') g(s', s) = q^2 (q - 1).$$

• Moreover, it is easy to check that

$$\sum_{s=1}^{q} g(s, s') = q - 1 - (q - 1) = 0,$$

$$\sum_{s=1}^{q} g(s_1, s) g(s, s_2) = \sum_{s=1}^{q} \left[ q^2 \delta_{s_1 s} \delta_{s_2 s} - q(\delta_{s_1 s} + \delta_{s_2 s}) + 1 \right] = q \left( q \delta_{s_1 s_2} - 1 \right) = q g(s_1, s_2),$$

$$\sum_{s,s'=1}^{q} g(s, s') g(s, s') = \sum_{s,s'=1}^{q} \left[ q^2 \delta_{ss'} \delta_{ss'} - 2q \delta_{ss'} + 1 \right] = q^3 - 2q^2 + q^2 = q^2 (q - 1).$$

(c) Use the above results to calculate the free energy, and the correlation function $\langle g(s_m, s_n) \rangle$ for a one–dimensional chain.

• The factor $T(K)$ will be our high temperature expansion parameter. Each bond contributes a factor $T g(s_i, s_j)$ and, since $\sum_s g(s, s') = 0$, there can not be only one bond per any site. As in the Ising case considered in lectures, each bond can only be considered once, and the only graphs that survive have no dangling bonds. As a result, for a one-dimensional chain, with for instance open boundary conditions, it is impossible to draw any acceptable graph, and we obtain

$$Z = \sum_{\{s_i\}} \prod_{\langle ij \rangle} C(K) \left[ 1 + T(K) g(s_i, s_j) \right] = C(K)^{N-1} q^N = q \left( e^K + q - 1 \right)^{N-1}.$$

Ignoring the boundary effects, i.e., that there are $N - 1$ bonds in the chain, the free energy per site is obtained as

$$-\frac{\beta F}{N} = \ln \left( e^K + q - 1 \right).$$

With the same method, we can also calculate the correlation function $\langle g(s_n s_m)\rangle$. To get a nonzero contribution, we have to consider a graph that directly connects these two sites. Assuming that $n > m$, this gives

$$\begin{aligned}\langle g(s_n s_m)\rangle &= \frac{C(K)^N}{Z}\sum_{\{s_i\}} g(s_n s_m)\prod_{\langle ij\rangle}\left[1+T(K)g(s_i,s_j)\right]\\ &= \frac{C(K)^N}{Z}T(K)^{n-m}\sum_{\{s_i\}} g(s_n s_m)g(s_m,s_{m+1})\cdots g(s_{n-1},s_n)\\ &= \frac{C(K)^N}{Z}T(K)^{n-m}q^{n-m+1}(q-1)q^{N-(n-m)-1} = T^{n-m}(q-1)\end{aligned}$$

where we have used the relationships obtained in (b).

(d) Calculate the partition function on the square lattice to order of $T^4$. Also calculate the first term in the low–temperature expansion of this problem.

• The first term in the high temperature series for a square lattice comes from a square of 4 bonds. There are a total of $N$ such squares. Therefore,

$$Z = \sum_{\{s_i\}}\prod_{\langle ij\rangle} C(K)\left[1+T(K)g(s_i,s_j)\right] = C(K)^{2N}q^N\left[1+NT(K)^4(q-1)+\cdots\right].$$

Note that any closed loop involving $\ell$ bonds without intersections contributes $T^\ell q^\ell(q-1)$.

On the other hand, at low temperatures, the energy is minimized by the spins all being in one of the $q$ possible states. The lowest energy excitation is a single spin in a different state, resulting in an energy cost of $K\times 4$ with a degeneracy factor $N\times(q-1)$, resulting in

$$Z = qe^{2NK}\left[1+N(q-1)e^{-4K}+\cdots\right].$$

(e) By comparing the first terms in low- and high–temperature series, find a duality rule for Potts models. Don't worry about higher order graphs, they will work out! Assuming a single transition temperature, find the value of $K_c(q)$.

• Comparing these expansions, we find the following duality condition for the Potts model

$$e^{-\tilde{K}} = T(K) = \frac{e^K-1}{e^K+q-1}.$$

This duality rule maps the low temperature expansion to a high temperature series, or vice versa. It also maps pairs of points, $\tilde{K}\Leftrightarrow K$, since we can rewrite the above relationship in a symmetric way

$$\left(e^{\tilde{K}}-1\right)\left(e^K-1\right) = q,$$

and consequently, if there is a single singular point $K_c$, it must be self-dual point,

$$K_c = \tilde{K}_c, \qquad \Longrightarrow \qquad K_c = \ln\left(\sqrt{q}+1\right).$$

(f) How do the higher order terms in the high–temperature series for the Potts model differ from those of the Ising model? What is the fundamental difference that sets apart the graphs for $q = 2$? (This is ultimately the reason why only the Ising model is solvable.)
• As mentioned in lectures, the Potts model with $q = 2$ can be mapped to the Ising model by noticing that $\delta_{ss'} = (1 + ss')/2$. However, higher order terms in the high-temperature series of the Potts model involve, in general, graphs with *three* or more bonds emanating from each site. These configurations do not correspond to a random walk, not even a constrained one as introduced in class for the 2d Ising model on a square lattice. The quantity

$$\sum_{s_1=1}^{q} g(s_1,s_2)g(s_1,s_3)g(s_1,s_4) = q^3\delta_{s_2s_3}\delta_{s_2s_4} - q^2(\delta_{s_2s_3}+\delta_{s_2s_4}+\delta_{s_3s_4}) + 2q,$$

is always zero when $q = 2$ (as can be easily checked for any possible state of the spins $s_2$, $s_3$ and $s_4$), but is in general different from zero for $q > 2$. This is the fundamental difference that ultimately sets apart the case $q = 2$. Note that the corresponding diagrams in the low temperature expansion involve adjacent regions in 3 (or more) distinct states.

********

**3.** *Potts model II:* An alternative expansion is obtained by starting with

$$\exp\left[K\delta(s_i,s_j)\right] = 1 + v(K)\delta(s_i,s_j),$$

where $v(K) = e^K - 1$. In this case, the sum over spins *does not* remove any graphs, and all choices of distributing bonds at random on the lattice are acceptable.

(a) Including a magnetic field $h\sum_i \delta_{s_i,1}$, show that the partition function takes the form

$$Z(q,K,h) = \sum_{\text{all graphs}} \prod_{\text{clusters c in graph}} \left[v^{n_b^c} \times \left(q - 1 + e^{hn_s^c}\right)\right],$$

where $n_b^c$ and $n_s^c$ are the numbers of bonds and sites in cluster $c$. This is known as the *random cluster expansion.*

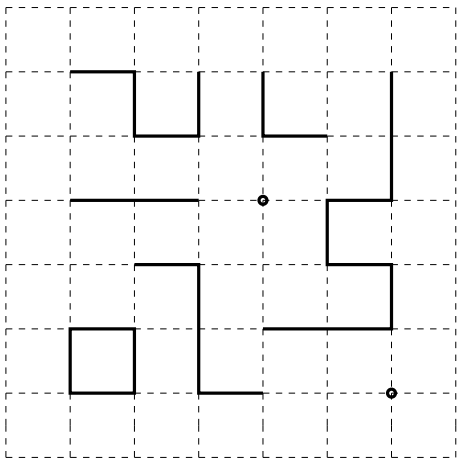
• Including a symmetry breaking field along direction 1, the partition function

$$Z = \sum_{\{s_i\}} \prod_{\langle ij \rangle} [1 + v(K)\delta(s_i, s_j)] \prod_i e^{h\delta_{s_i,1}},$$

can be expanded in powers of $v(K)$ as follows. As usual, if there is a total number $N_B$ of nearest neighbor bonds on the lattice, the product over bonds generates $2^{N_B}$ possible terms. Each term may be represented by a graph on the lattice, in which a bond joining sites $i$ and $j$ is included if the factor $v\delta(s_i, s_j)$ appears in the term considered. Each included bond carries a factor $v(K)$, as well as a delta function enforcing the equality of the spins on the sites which it connects. In general, these bonds form clusters of different sizes and shapes, and within each cluster, the delta functions force the spins at each vertex to be the same. The sum $\sum_{\{s_i\}}$ therefore gives a factor of $(q-1)+e^{hn_s^c}$ for each cluster $c$, where $n_s^c$ is the number of point in the cluster. The partition function may then be written as

$$Z(q, v, h) = \sum_{\text{all graphs}} \prod_{\text{clusters in graph}} \left[ v(K)^{n_b^c} \left( q - 1 + e^{hn_c^s} \right) \right],$$

where $n_b^c$ is the number of bonds in cluster c, and the sum runs over all distinct cluster collections. Note that an isolated site is also included in this definition of a cluster. While the Potts model was originally defined for integer $q$, using this expansion, we can evaluate $Z$ for all values of $q$.



(b) Show that the limit $q \to 1$ describes a *percolation* problem, in which bonds are randomly distributed on the lattice with probability $p = v/(v+1)$. What is the percolation threshold on the square lattice?

• In the problem of *bond percolation*, bonds are independently distributed on the lattice, with a probability $p$ of being present. The weight for a given configuration of occupied and absent bonds bonds is therefore

$$W(\text{graph}) = (1-p)^{zN} \prod_{\text{clusters in graph}} \left(\frac{p}{1-p}\right)^{n_b^c}.$$

The prefactor of $(1-p)^{zN}$ is merely the weight of the configuration with no bonds. The above weights clearly become identical to those appearing in the random cluster expansion of the Potts model for $q = 1$ (and $h = 0$). Clearly, we have to set $p = v/(v+1)$, and neglect an overall factor of $(1+v)^N$, which is analytic in $v$, and does not affect any singular behavior. The partition function itself is trivial in this limit as $Z(1, v, h) = (1+v)^{zN} e^{hN}$. On the other hand, we can obtain information on the number of clusters by considering the limit of $q \to 1$ from

$$\left.\frac{\partial \ln Z(q,v)}{\partial q}\right|_{q=1} = \sum_{\text{all graphs}} [\text{probablility of graph}] \sum_{\text{clusters in graph}} e^{-h n_c^s}.$$

Various properties of interest to percolation can then be calculated from the above generating function. This mapping enables us to extract the scaling laws at the percolation point, which is a continuous geometrical phase transition. The analog of the critical temperature is played by the percolation threshold $p_c$, which we can calculate using the expression obtained in problem 2 as $p_c = 1/2$ (after noting that $v^* = 1$).

An alternative way of obtaining this threshold is to find a duality rule for the percolation problem itself: One can similarly think of the problem in terms of empty bonds with a corresponding probability $q$. As $p$ plays the role of temperature, there is a mapping of low $p$ to high $q$ or vice versa, and such that $q = 1-p$. The self-dual point is then obtained by setting $p^* = 1 - p^*$, resulting in $p^* = 1/2$.

(c) Show that in the limit $q \to 0$, only a single connected cluster contributes to leading order. The enumeration of all such clusters is known as listing *branched lattice animals*.

• The partition function $Z(q, v, h)$ goes to zero at $q = 0$, but again information about geometrical lattice structure can be obtain by taking the limit $q \to 0$ in an appropriate fashion. In particular, if we set $v = q^a x$, then

$$Z(q, v = xq^a, h = 0) = \sum_{\text{all graphs}} x^{N_b} q^{N_c + aN_b},$$

where $N_b$ and $N_c$ are the total number of bonds and clusters. The leading dependence on $q$ as $q \to 0$ comes from graphs with the lowest number of $N_c + aN_b$, and depends on the value of $a$. For $0 < a < 1$, these are the *spanning trees*, which connect all sites of the lattice (hence $N_c = 1$) and that enclose no loops (hence $N_b = N - 1$). Such spanning trees have a power of $x^{a(N-1)} q^{aN-a+1}$, and all other graphs have higher powers of $q$. For $a = 0$ one can add bonds to the spanning cluster (creating loops) without changing the power, as long as all sites remain connected in a single cluster. These have a relation to a problem referred to as *branched lattice animals*.

********

**4.** *Ising model in a field:* Consider the partition function for the Ising model ($\sigma_i = \pm 1$) on a square lattice, *in a magnetic field* $h$; i.e.

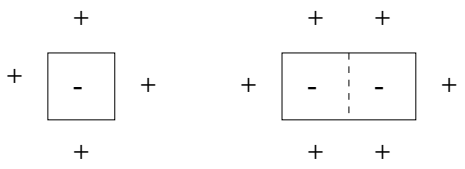
$$Z = \sum_{\{\sigma_i\}} \exp \left[ K \sum_{<ij>} \sigma_i \sigma_j + h \sum_i \sigma_i \right].$$

(a) Find the general behavior of the terms in a low–temperature expansion for $Z$.

• At low temperatures, almost all the spins are oriented in the same direction; low energy excitations correspond to islands of flipped spins. The general behavior of the terms in a low-temperature expansion of the partition function is then of the form

$$e^{-2KL} e^{-2hA},$$

where $L$ is the perimeter of the island, or the number of unsatisfied bonds, and $A$ is the area of the island, or the number of flipped spins.



(b) Think of a model whose high–temperature series reproduces the generic behavior found in (a); and hence obtain the Hamiltonian, and interactions of the dual model.

• To reproduce terms that are proportional to the area, we may consider plaquette-like interactions of the form

$$\tilde{h} \sum_{\text{plaquettes}} \sigma_1\sigma_2\sigma_3\sigma_4,$$

where $\sigma$ is a spin variable defined on each bond of the lattice. Thus, we can define a (lattice gauge) model whose Hamiltonian is given by

$$-\beta\mathcal{H} = \tilde{K} \sum_{\text{bonds}} \sigma + \tilde{h} \sum_{\text{plaquettes}} \sigma_1\sigma_2\sigma_3\sigma_4,$$

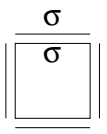
and, as usual, rewrite

$$e^{\tilde{K}\sigma} = \cosh\tilde{K}\left(1+\tanh\tilde{K}\right)\ \sigma,$$
$$e^{\tilde{h}\sigma_1\sigma_2\sigma_3\sigma_4} = \cosh\tilde{h}\ \left(1+\tanh\tilde{h}\right)\ \sigma_1\sigma_2\sigma_3\sigma_4.$$

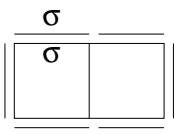
The high temperature series of the partition function will be then of the form,

$$Z \propto \sum_{\text{bonds}} \sum_{\text{plaquettes}} \left(\tanh\tilde{K}\right)^{N_b} \left(\tanh\tilde{h}\right)^{N_p},$$

where $N_b$ and $N_p$, are the number of bonds and plaquettes involved in each term of the expansion. Nevertheless, we will only have non-zero contributions when the bond variables $\sigma$, appear an even number of times, either in a plaquette or as a bond. For example, the lowest order contributions contain 1 plaquette (so that $A = 1$), and 4 extra bonds ($L = 4$), or 2 plaquettes ($A = 2$), and 6 bonds ($L = 6$), etc.




L=4, A=1




L=6, A=2

In general, we obtain terms of the form

$$\left(\tanh \tilde{K}\right)^{L} \left(\tanh \tilde{h}\right)^{A},$$

with $L$ and $A$, the perimeter and the area of the graph considered. Hence, as in the Ising model, we can write the duality relations

$$e^{-2K} = \tanh \tilde{K}, \qquad \text{and} \qquad e^{-2h} = \tanh \tilde{h}.$$

********

**5.** *Potts duality:* Consider Potts spins, $s_i = (1, 2, \cdots, q)$, placed on the sites of a *square lattice* of $N$ sites, interacting with their nearest-neighbors through a Hamiltonian

$$-\beta\mathcal{H} = K \sum_{<ij>} \delta_{s_i, s_j}.$$

(a) By comparing the first terms of high and low temperature series, or by any other method, show that the partition function has the property

$$Z(K) = qe^{2NK}\Xi\left[e^{-K}\right] = q^{-N}\left[e^{K} + q - 1\right]^{2N}\Xi\left[\frac{e^{K}-1}{e^{K}+(q-1)}\right],$$

for some function $\Xi$, and hence locate the critical point $K_c(q)$.

- As discussed in problem 3 of this chapter, the low temperature series takes the form

$$Z = qe^{2NK}\left[1 + N(q-1)e^{-4K} + \cdots\right] \equiv qe^{2NK}\Xi\left[e^{-K}\right],$$

while at high temperatures

$$\begin{aligned} Z &= \left[\frac{e^{K}+q-1}{q}\right]^{2N} q^{N}\left[1 + N(q-1)\left(\frac{e^{K}-1}{e^{K}+q-1}\right)^{4} + \cdots\right] \\ &\equiv q^{-N}\left[e^{K}+q-1\right]^{2N}\Xi\left[\frac{e^{K}-1}{e^{K}+q-1}\right]. \end{aligned}$$

Both of the above series for $\Xi$ are in fact the same, leading to the duality condition

$$e^{-\tilde{K}} = \frac{e^{K}-1}{e^{K}+q-1},$$

and a critical (self-dual) point of

$$K_c = \tilde{K}_c, \qquad \Longrightarrow \qquad K_c = \ln\left(\sqrt{q}+1\right).$$

(b) Starting from the duality expression for $Z(K)$, derive a similar relation for the internal energy $U(K) = \langle \beta\mathcal{H} \rangle = -\partial \ln Z / \partial \ln K$. Use this to calculate the exact value of $U$ at the critical point.

• The duality relation for the partition function gives

$$\ln Z(K) = \ln q + 2NK + \ln \Xi\left[e^{-K}\right] = -N\ln q + 2N \ln\left[e^K + q - 1\right] + \ln \Xi\left[\frac{e^K-1}{e^K+q-1}\right].$$

The internal energy $U(K)$ is then obtained from

$$\begin{aligned} -\frac{U(K)}{K} = \frac{\partial}{\partial K}\ln Z(K) &= 2N - e^{-K}\ln\Xi'\left[e^{-K}\right] \\ &= 2N\frac{e^K}{e^K+q-1} + \frac{qe^K}{\left(e^K+q-1\right)^2}\ln\Xi'\left[\frac{e^K-1}{e^K+q-1}\right]. \end{aligned}$$

$\ln\Xi'$ is the derivative of $\ln\Xi$ with respect to its argument, whose value is not known in general. However, at the critical point $K_c$, the arguments of $\ln\Xi'$ from the high and low temperature forms of the above expression are the same. Substituting $e^{K_c} = 1+\sqrt{q}$, we obtain

$$2N - \frac{\ln\Xi'_c}{1+\sqrt{q}} = \frac{2N}{\sqrt{q}} + \frac{\ln\Xi'_c}{1+\sqrt{q}}, \quad \Longrightarrow \quad \ln\Xi'_c = \frac{q-1}{\sqrt{q}}N,$$

and,

$$-\frac{U(K_c)}{K_c} = N\left(2 - \frac{q-1}{\sqrt{q}+q}\right), \quad \Longrightarrow \quad U(K_c) = NK_c\frac{\sqrt{q}+1}{\sqrt{q}}.$$

********

**6.** *Anisotropic random walks:* Consider the ensemble of all random walks on a square lattice starting at the origin (0,0). Each walk has a weight of ${t_x}^{\ell_x} \times {t_y}^{\ell_y}$, where $\ell_x$ and $\ell_y$ are the number of steps taken along the $x$ and $y$ directions respectively.

(a) Calculate the total weight $W(x,y)$, of all walks terminating at $(x,y)$. Show that $W$ is well defined only for $\overline{t} = (t_x+t_y)/2 < t_c = 1/4$.

• Defining $\langle 0,0|\, W(\ell)\, |x,y\rangle$ to be the weight of all walks of $\ell$ steps terminating at $(x,y)$, we can follow the steps in sec.VI.F of the lecture notes. In the anisotropic case, Eq.(VI.47) (applied $\ell$ times) is trivially recast into

$$\begin{aligned}\langle x,y|\, T^{\ell}\, |q_x,q_y\rangle &= \sum_{x',y'} \langle x,y|\, T^{\ell}\, |x',y'\rangle \, \langle x',y'|\, q_x,q_y\rangle \\ &= \left(2t_x\cos q_x + 2t_y \cos q_y\right)^{\ell} \langle x,y|\, q_x,q_y\rangle\,,\end{aligned}$$

where $\langle x,y|\, q_x,q_y\rangle = e^{iq_x x + iq_y y}/\sqrt{N}$. Since $W(x,y) = \sum_\ell \langle 0,0|\, W(\ell)\, |x,y\rangle$, its Fourier transform is calculated as

$$\begin{aligned}W(q_x,q_y) &= \sum_{\ell}\sum_{x,y} \langle 0,0|\, T^{\ell}\, |x,y\rangle \, \langle x,y|\, q_x,q_y\rangle \\ &= \sum_{\ell} \left(2t_x\cos q_x + 2t_y\cos q_y\right)^{\ell} = \frac{1}{1-\left(2t_x\cos q_x + 2t_y\cos q_y\right)}.\end{aligned}$$

Finally, Fourier transforming back gives

$$W(x,y) = \int_{-\pi}^{\pi} \frac{d^2q}{(2\pi)^2} W(q_x,q_y)\, e^{-iq_x x - iq_y y} = \int_{-\pi}^{\pi} \frac{d^2q}{(2\pi)^2} \frac{e^{-iq_x x - iq_y y}}{1-\left(2t_x\cos q_x + 2t_y\cos q_y\right)}.$$

Note that the summation of the series is legitimate (for all $q$'s) only for $2t_x + 2t_y < 1$, *i.e.* for $\bar{t} = (t_x+t_y)/2 < t_c = 1/4$.

(b) What is the shape of a curve $W(x,y) =$ constant, for large $x$ and $y$, and close to the transition?

• For $x$ and $y$ large, the main contributions to the above integral come from small $q$'s. To second order in $q_x$ and $q_y$, the denominator of the integrand reads

$$1 - 2\left(t_x + t_y\right) + t_x q_x^2 + t_y q_y^2.$$

Then, with $q_i' \equiv \sqrt{t_i}\, q_i$, we have

$$W(x,y) \approx \int_{-\infty}^{\infty} \frac{d^2q'}{(2\pi)^2\sqrt{t_x t_y}} \frac{e^{-i\mathbf{q}'\cdot\mathbf{v}}}{1-2\left(t_x+t_y\right)+\mathbf{q}'^2},$$

where we have extended the limits of integration to infinity, and $\mathbf{v} = \left(\frac{x}{\sqrt{t_x}}, \frac{y}{\sqrt{t_y}}\right)$. As the denominator is rotationally invariant, the integral depends only on the magnitude of the vector $\mathbf{v}$. In other words, $W(x,y)$ is constant along ellipses

$$\frac{x^2}{t_x} + \frac{y^2}{t_y} = \text{constant}.$$

(c) How does the average number of steps, $\langle \ell \rangle = \langle \ell_x + \ell_y \rangle$, diverge as $\bar{t}$ approaches $t_c$?

• The weight of all walks of length $\ell$, irrespective of their end point location, is

$$\sum_{x,y} \langle 0,0| \, W(\ell) \, |x,y\rangle = \langle 0,0| \, T^{\ell} \, |q_x = 0, q_y = 0\rangle = (2t_x + 2t_y)^{\ell} = (4\bar{t})^{\ell}.$$

Therefore,

$$\langle \ell \rangle = \frac{\sum_\ell \ell \, (4\bar{t})^{\ell}}{\sum_\ell (4\bar{t})^{\ell}} = 4\bar{t} \frac{\partial}{\partial (4\bar{t})} \ln \left[ \sum_\ell (4\bar{t})^{\ell} \right] = 4\bar{t} \frac{\partial}{\partial (4\bar{t})} \ln \frac{1}{1-4\bar{t}} = \frac{4\bar{t}}{1-4\bar{t}},$$

*i.e.*

$$\langle \ell \rangle = \frac{\bar{t}}{t_c - \bar{t}},$$

diverges linearly close to the singular value of $\bar{t}$.

********

**7.** *Anisotropic Ising model:* Consider the anisotropic Ising model on a square lattice with a Hamiltonian

$$-\beta \mathcal{H} = \sum_{x,y} \left( K_x \sigma_{x,y} \sigma_{x+1,y} + K_y \sigma_{x,y} \sigma_{x,y+1} \right);$$

i.e. with bonds of different strengths along the $x$ and $y$ directions.

(a) By following the method presented in the text, calculate the free energy for this model. You do not have to write down every step of the derivation. Just sketch the steps that need to be modified due to anisotropy; and calculate the final answer for $\ln Z/N$.

• The Hamiltonian

$$-\beta \mathcal{H} = \sum_{x,y} \left( K_x \sigma_{x,y} \sigma_{x+1,y} + K_y \sigma_{x,y} \sigma_{x,y+1} \right),$$

leads to

$$Z = \sum \left( 2 \cosh K_x \cosh K_y \right)^N t_x^{\ell_x} t_y^{\ell_y},$$

where $t_i = \tanh K_i$, and the sum runs over all closed graphs. By extension of the isotropic case,

$$f = \frac{\ln Z}{N} = \ln \left( 2 \cosh K_x \cosh K_y \right) + \sum_{\ell_x, \ell_y} \frac{t_x^{\ell_x} t_y^{\ell_y}}{\ell_x + \ell_y} \langle 0| \, W^* (\ell_x, \ell_y) \, |0\rangle,$$

where

$$\langle 0|\, W^*\left(\ell_x, \ell_y\right)|0\rangle = \frac{1}{2}\sum{}' (-1)^{\text{number of crossings}},$$

and the primed sum runs over all directed $(\ell_x, \ell_y)$-steps walks from $(0,0)$ to $(0,0)$ with no U-turns. As in the isotropic case, this is evaluated by taking the trace of powers of the $4N \times 4N$ matrix described by Eq.(VI.66), which is block diagonalized by Fourier transformation. However, unlike the isotropic case, in which each element is multiplied by $t$, here they are multiplied by $t_x$ and $t_y$, respectively, resulting in

$$f = \ln\left(2\cosh K_x \cosh K_y\right) + \frac{1}{2}\int \frac{d^2q}{(2\pi)^2} \mathrm{tr}\ln\left[1 - \mathrm{T}\left(\mathbf{q}\right)^*\right],$$

where

$$\begin{aligned}
\mathrm{tr}\ln\left[1 - \mathrm{T}\left(\mathbf{q}\right)^*\right] &= \ln\det\left[1 - T\left(\mathbf{q}\right)^*\right] \\
&= \ln\left[\left(1+t_x^2\right)\left(1+t_y^2\right) - 2t_x\left(1-t_y^2\right)\cos q_x - 2t_y\left(1-t_x^2\right)\cos q_y\right] \\
&= \ln\left[\frac{\cosh 2K_x \cosh 2K_y - \sinh 2K_x \cos q_x - \sinh 2K_y \cos q_y}{\cosh^2 K_x \cosh^2 K_y}\right],
\end{aligned}$$

resulting in

$$f = \ln 2 + \frac{1}{2}\int \frac{d^2q}{(2\pi)^2} \ln\left(\cosh 2K_x \cosh 2K_y - \sinh 2K_x \cos q_x - \sinh 2K_y \cos q_y\right).$$

(b) Find the critical boundary in the $(K_x, K_y)$ plane from the singularity of the free energy. Show that it coincides with the condition $K_x = \tilde{K}_y$, where $\tilde{K}$ indicates the standard dual interaction to $K$.

• The argument of the logarithm is minimal at $q_x = q_y = 0$, and equal to

$$\begin{aligned}
&\cosh 2K_x \cosh 2K_y - \sinh 2K_x - \sinh 2K_y \\
&= \frac{1}{2}\left(e^{K_x}\sqrt{\cosh 2K_y - 1} - e^{-K_x}\sqrt{\cosh 2K_y + 1}\right)^2.
\end{aligned}$$

Therefore, the critical line is given by

$$e^{2K_x} = \sqrt{\frac{\cosh 2K_y + 1}{\cosh 2K_y - 1}} = \coth K_y.$$

Note that this condition can be rewritten as

$$\sinh 2K_x = \frac{1}{2}\left(\coth K_y - \tanh K_y\right) = \frac{1}{\sinh 2K_y},$$

*i.e.* the critical boundary can be described as $K_x = \tilde{K}_y$, where the dual interactions, $\tilde{K}$ and $K$, are related by $\sinh 2K \sinh 2\tilde{K} = 1$.

(c) Find the singular part of $\ln Z/N$, and comment on how anisotropy affects critical behavior in the exponent and amplitude ratios.

• The singular part of $\ln Z/N$ for the anisotropic case can be written as

$$f_S = \frac{1}{2} \int \frac{d^2q}{(2\pi)^2} \ln \left[ \left( e^{K_x} \sqrt{\cosh 2K_y - 1} - e^{-K_x} \sqrt{\cosh 2K_y + 1} \right)^2 + \sum_{i=x,y} \frac{q_i^2}{2} \sinh 2K_i \right] .$$

In order to rewrite this expression in a form closer to that of the singular part of the free energy in the isotropic case, let

$$q_i = \sqrt{\frac{2}{\sinh 2K_i}} q_i',$$

and

$$\delta t = e^{K_x} \sqrt{\cosh 2K_y - 1} - e^{-K_x} \sqrt{\cosh 2K_y + 1}$$

($\delta t$ goes linearly through zero as $(K_x, K_y)$ follows a curve which intersects the critical boundary). Then

$$f_S = \frac{1}{\sqrt{\sinh 2K_x \sinh 2K_y}} \int \frac{d^2q'}{(2\pi)^2} \ln \left( \delta t^2 + q'^2 \right) .$$

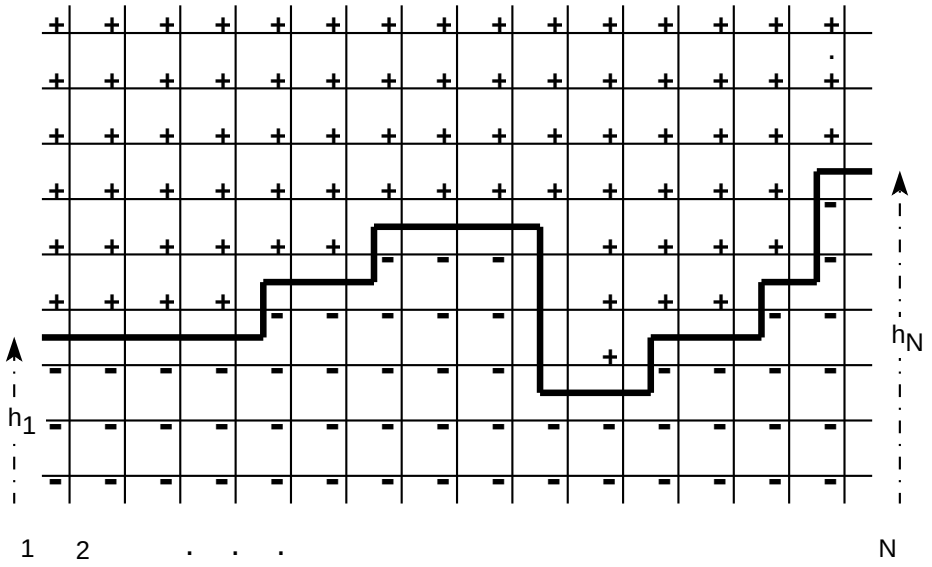
Thus, upon approaching the critical boundary ($\sinh 2K_x \sinh 2K_y = 1$), the singular part of the anisotropic free energy coincides more and more precisely with the isotropic one, and the exponents and amplitude ratios are unchanged by the anisotropy. (The amplitude itself does vary along the critical line.)

********

**8.** *Müller–Hartmann and Zittartz estimate* of the interfacial energy of the $d = 2$ Ising model on a square lattice.

(a) Consider an interface on the square lattice with periodic boundary conditions in one direction. Ignoring islands and overhangs, the configurations can be labeled by heights $h_n$ for $1 \leq n \leq L$. Show that for an anisotropic Ising model of interactions $(K_x, K_Y)$, the energy of an interface along the $x$-direction is

$$-\beta\mathcal{H} = -2K_y L - 2K_x \sum_n |h_{n+1} - h_n| .$$




• For each unsatisfied $(+-)$ bond, the energy is increased by $2K_i$ from the ground state energy, with $i = x$ if the unsatisfied bond is vertical, and $i = y$ if the latter is horizontal. Ignoring islands and overhangs, the number of horizontal bond of the interface is $L$, while the number of vertical bonds is $\sum_n |h_{n+1} - h_n|$, yielding

$$-\beta\mathcal{H} = -2K_y L - 2K_x \sum_{n=1}^{L} |h_{n+1} - h_n| \,.$$

(b) Write down a column–to–column transfer matrix $\langle h|T|h'\rangle$, and diagonalize it.

• We can define

$$\langle h| \, T \, |h'\rangle \equiv \exp\left(-2K_y - 2K_x \, |h' - h|\right),$$

or, in matrix form,

$$T = e^{-2K_y} \begin{pmatrix} 1 & e^{-2K_x} & e^{-4K_x} & \cdots & e^{-HK_x} & e^{-HK_x} & e^{-2\left(\frac{H}{2}-1\right)K_x} & \cdots & e^{-2K_x} \\ e^{-2K_x} & 1 & e^{-2K_x} & \cdots & e^{-2\left(\frac{H}{2}-1\right)K_x} & e^{-2\left(\frac{H}{2}+1\right)K_x} & e^{-HK_x} & \cdots & e^{-4K_x} \\ \cdots & & & & & & & & \end{pmatrix}$$

where $H$ is the vertical size of the lattice. In the $H \to \infty$ limit, $T$ is easily diagonalized since each line can be obtained from the previous line by a single column shift. The eigenvectors of such matrices are composed by the complex roots of unity (this is equivalent to the statement that a translationally invariant system is diagonal in Fourier modes). To the eigenvector

$$\left(e^{i\frac{2\pi}{k}}, e^{i\frac{2\pi}{k}\cdot 2}, e^{i\frac{2\pi}{k}\cdot 3}, \cdots, e^{i\frac{2\pi}{k}\cdot(H+1)}\right),$$

is associated the eigenvalue

$$\lambda_k = e^{-2K_y} \sum_{n=1}^{H+1} T_{1n} e^{i\frac{2\pi}{k}\cdot(n-1)}.$$

Note that there are $H+1$ eigenvectors, corresponding to $k = 1, \cdots, H+1$.

(c) Obtain the interface free energy using the result in (b), or by any other method.

• One way of obtaining the free energy is to evaluate the largest eigenvalue of $T$. Since all elements of $T$ are positive, the eigenvector $(1, 1, \cdots, 1)$ has the largest eigenvalue

$$\begin{aligned}\lambda_1 = e^{-2K_y} \sum_{n=1}^{H+1} T_{1n} &= e^{-2K_y} \left( 1 + 2 \sum_{n=1}^{H/2} e^{-2K_x n} \right) \\ &= e^{-2K_y} \left( 2 \sum_{n=0}^{H/2} e^{-2K_x n} - 1 \right) = e^{-2K_y} \coth K_x,\end{aligned}$$

in the $H \to \infty$ limit. Then, $F = -Lk_B T \ln \lambda_1$.

Alternatively, we can directly sum the partition function, as

$$\begin{aligned}Z &= e^{-2K_y L} \sum_{\{h_n\}} \exp\left( -2K_x \sum_{n=1}^{L} |h_{n+1} - h_n| \right) = e^{-2K_y L} \left[ \sum_d \exp\left( -2K_x |d| \right) \right]^L \\ &= \left[ e^{-2K_y} \left( 2 \sum_{d \geq 0} e^{-2K_x d} - 1 \right) \right]^L = \left( e^{-2K_y} \coth K_x \right)^L,\end{aligned}$$

yielding

$$F = -Lk_B T \left[ \ln\left( \coth K_x \right) - 2K_y \right].$$

(d) Find the condition between $K_x$ and $K_y$ for which the interfacial free energy vanishes. Does this correspond to the critical boundary of the original 2d Ising model as computed in the previous problem?

• The interfacial free energy vanishes for

$$\coth K_x = e^{2K_y},$$

which coincides with the condition found in problem 2. This illustrates that long wavelength fluctuations, such as interfaces, are responsible for destroying order at criticality.

********

**9.** *Anisotropic Landau theory:* Consider an $n$–component magnetization field $\vec{m}(\mathbf{x})$ in $d$–dimensions.

(a) Using the previous problems on anisotropy as a guide, generalize the standard Landau–Ginzburg Hamiltonian to include the effects of spacial anisotropy.

• Requiring different coupling constants in the different spatial directions, along with rotational invariance in spin space, leads to the following leading terms of the Hamiltonian,

$$-\beta\mathcal{H} = \int d^d x \left[ \frac{t}{2}\vec{m}\left(\mathbf{x}\right)^2 + \sum_{i=1}^{d} \frac{K_i}{2} \frac{\partial \vec{m}}{\partial x_i} \cdot \frac{\partial \vec{m}}{\partial x_i} + u\vec{m}\left(\mathbf{x}\right)^4 \right].$$

(b) Are such anisotropies "relevant?"

• Clearly, the apparent anisotropy can be eliminated by the rescaling

$$x_i' = \sqrt{\frac{K}{K_i}} x_i.$$

In terms of the primed space variables, the Hamiltonian is isotropic. In particular, the universal features are identical in the anisotropic and isotropic cases, and the anisotropy is thus "irrelevant" (provided all $K_i$ are non-vanishing).

(c) In $La_2CuO_4$, the Cu atoms are arranged on the sites of a square lattice in planes, and the planes are then stacked together. Each Cu atom carries a spin which we assume to be classical, and can point along any direction in space. There is a very strong antiferromagnetic interaction in each plane. There is also a very weak inter-plane interaction that prefers to align successive layers. Sketch the low–temperature magnetic phase, and indicate to what universality class the order–disorder transition belongs.

• For classical spins, this combination of antiferromagnetic and ferromagnetic couplings is equivalent to a purely ferromagnetic (anisotropic) system, since we can redefine (*e.g.* in the partition function) all the spins on one of the two sublattices with an opposite sign. Therefore, the critical behavior belongs to the $d = 3$, $n = 3$ universality class.

Nevertheless, there is a range of temperatures for which the in-plane correlation length is large compared to the lattice spacing, while the inter-plane correlation length is of the order of the lattice spacing. The behavior of the system is then well described by a $d = 2$, $n = 3$ theory.

********

---

**10.** *Energy by duality:* Consider the Ising model ($\sigma_i = \pm 1$) on a square lattice with $-\beta\mathcal{H} = K \sum_{<ij>} \sigma_i \sigma_j$.

(a) Starting from the duality expression for the free energy, derive a similar relation for the internal energy $U(K) = \langle H \rangle = -\partial \ln Z / \partial \ln K$.

• The low temperature partition function has the form

$$Z(K) = e^{2KN} \sum_{\text{islands}} e^{-2KL},$$

whereas the high temperature expansion looks like

$$Z(K) = \left(2\cosh^2 K\right)^N \sum_{\text{loops}} (\tanh K)^L.$$

What is more, we found that we can write a duality relation of the form

$$\tanh K = e^{-2\tilde{K}}, \qquad \text{or} \qquad \sinh 2K \sinh 2\tilde{K} = 1,$$

which we can use to rewrite the partition function as

$$Z(K) = \left(2\cosh^2 K\right)^N e^{-2\tilde{K}N} Z(\tilde{K}) = (\sinh 2K)^N Z(\tilde{K}),$$

yielding

$$-\beta F(K) = N \ln(2 \sinh 2K) - \beta F(\tilde{K}).$$

The internal energy $U(K)$ is then obtained from

$$U(K) = -K \frac{\partial}{\partial K} \ln Z(K) = -NK \frac{2\cosh 2K}{\sinh 2K} - K \left( \frac{\partial}{\partial \tilde{K}} \ln Z(\tilde{K}) \right) \frac{\partial \tilde{K}}{\partial K}.$$

After taking the logarithm of the relation $\sinh 2K \sinh 2\tilde{K} = 1$, and differentiating it, we obtain

$$\frac{\partial \tilde{K}}{\partial K} = -\frac{\tanh 2\tilde{K}}{\tanh 2K},$$

and, as a result

$$U(K) = -2NK \coth 2K - \frac{K}{\tilde{K}} U(\tilde{K}) \frac{\tanh 2\tilde{K}}{\tanh 2K}$$

or, equivalently

$$U(K) \frac{\tanh 2K}{K} + U(\tilde{K}) \frac{\tanh 2\tilde{K}}{\tilde{K}} = -2N.$$

(b) Using (a), calculate the exact value of $U$ at the critical point $K_c$.

• Using the last result, we can calculate the exact value of $U$ at the critical point, $K^* = K = \tilde{K}$,

$$U(K^*) = -\frac{NK^*}{\tanh 2K^*} = -\frac{N\sqrt{2}}{2}\ln(\sqrt{2}+1),$$

where we have used $K^* = \ln(\sqrt{2}+1)/2$, and $\sinh 2K^* = 1$.

********

**11.** *Clock model duality:* Consider spins $s_i = (1, 2, \cdots, q)$ placed on the sites of a square lattice, interacting via the clock model Hamiltonian

$$\beta\mathcal{H}_C = -\sum_{<i,j>} J\left(|(s_i - s_j)_{\mathrm{mod}q}|\right),$$

(a) Change from the $N$ site variables to the $2N$ bond variables $b_{ij} = s_i - s_j$. Show that the difference in the number of variables can be accounted for by the constraint that around each plaquette (elementary square) the sum of the four bond variables must be zero modulus $q$.

• Around any closed loop, e.g. for the four bonds making up an elementary square,

$$S_p \equiv \sum b_{ij}^p = \sum (s_i - s_j) = 0,$$

as any $s_i$ appears once with a positive and once with a negative sign. We may insist that all bond variables are positive, using $b_{ij} = (s_i - s_j)_{\mathrm{mod}q}$, in which case the above equality is also valid modulus $q$.

(b) The constraints can be implemented by adding "delta–functions"

$$\delta\left[S_p\right]_{\mathrm{mod}q} = \frac{1}{q}\sum_{n_p=1}^{q} \exp\left[\frac{2\pi i n_p S_p}{q}\right],$$

for each plaquette. Show that after summing over the bond variables, the partition function can be written in terms of the dual variables, as

$$Z = q^{-N}\sum_{\{n_p\}}\prod_{\langle p,p'\rangle}\lambda\left(n_p - n_{p'}\right) \equiv \sum_{\{n_p\}}\exp\left[\sum_{\langle p,p'\rangle}\tilde{J}\left(n_p - n_{p'}\right)\right],$$

where $\lambda(k)$ is the discrete Fourier transform of $e^{J(n)}$.

• In terms of the constrained bond variables, the partition function is

$$Z = \sum_{\{b_{ij}\}} \prod_p \delta\left[S_p\right]_{\mathrm{mod}q} \prod_{<ij>} e^{J(b_{ij})} = \sum_{\{b_{ij}\}} \prod_p \sum_{n_p} \frac{1}{q} \exp\left[\frac{2\pi i n_p}{q} \sum b_{ij}^p\right] \prod_{<ij>} e^{J(b_{ij})}.$$

Now we can do the sum over each bond variable $b_{ij}$ independently. Note, however, that a given $b_{ij}$ appears twice in the constraints appropriate to its adjacent plaquettes, say $p$ and $p'$, and thus contributes

$$\sum_{b_{ij}} \exp\left[\frac{2\pi i b_{ij}}{q}(n_p - n_{p'}) + J\left(b_{ij}\right)\right] \equiv \lambda(n_p - n_{p'}) \equiv e^{\tilde{J}\left(n_p - n_{p'}\right)}.$$

The difference in the exponent is again meaningful modulus $q$, and thus

$$Z = q^{-N} \sum_{\{n_p\}} \prod_{\langle p,p'\rangle} e^{\tilde{J}\left(n_p - n_{p'}\right)} = \sum_{\{n_p\}} \exp\left[\sum_{\langle p,p'\rangle} \tilde{J}\left(n_p - n_{p'}\right)\right].$$

(c) Calculate the dual interaction parameter of a Potts model, and hence locate the critical point $J_c(q)$.

• For the Potts model, the discrete Fourier transform has only two values. If $n_p = n_{p'}$, then

$$\lambda(n_p = n_{p'}) = e^J + q - 1,$$

and otherwise

$$\lambda(n_p \neq n_{p'}) = e^J - 1,$$

since the sum of all roots of unity (excluding one itself) is minus one. The ratio of these two cases gives the dual Potts interaction as

$$e^{\tilde{J}} = \frac{e^J + q - 1}{e^J - 1}.$$

Assuming that there is only one critical point (which is correct), it has to be at the self-dual point $j_c(q) = \tilde{J}_c(q)$. This leads to a quadratic equation for $x = e^{J_c(q)}$,

$$x^2 - 2x - (q-1) = 0, \quad \Longrightarrow \quad x = 1 + \sqrt{q-1}, \quad \Longrightarrow \quad J_c(q) = \ln\left[1 + \sqrt{q-1}\right].$$

(d) Construct the dual of the anisotropic Potts model, with

$$-\beta\mathcal{H} = \sum_{x,y} \left( J_x \delta_{s_{x,y},s_{x+1,y}} + J_y \delta_{s_{x,y},s_{x,y+1}} \right);$$

i.e. with bonds of different strengths along the $x$ and $y$ directions. Find the line of self–dual interactions in the plane $(J_x, J_y)$.

- The dual of the anisotropic square lattice is another anisotropic square lattice with

$$\tilde{J}_x = \ln \frac{e^{J_y} + q - 1}{e^{J_y} - 1}, \quad \text{and} \quad \tilde{J}_y = \ln \frac{e^{J_x} + q - 1}{e^{J_x} - 1}.$$

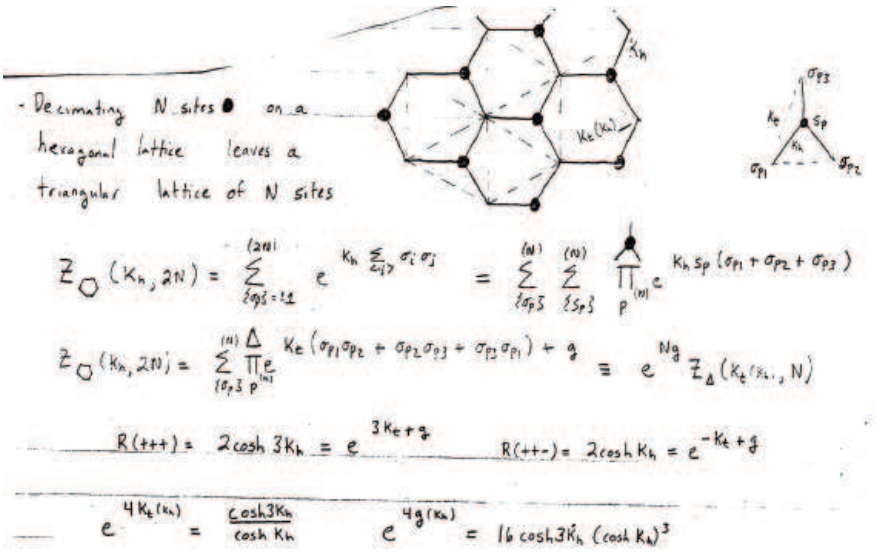
It is easy to check that the $\tilde{\tilde{J}} = J$, and hence $J_y = \tilde{J}_x$ is a line that maps onto itself under duality. It turns out that this is also the critical line of the anisotropic Potts model on the square lattice.

********

**12.** *Triangular/hexagonal lattice Ising model:* For any *planar* network of bonds, one can define a geometrical dual by connecting the centers of neighboring plaquettes. Each bond of the dual lattice crosses a bond of the original lattice, allowing for a *local* mapping. Clearly, the dual of a triangular lattice is a hexagonal (or honeycomb) lattice, and vice versa.

(a) Consider the Ising models on a hexagonal lattice with nearest neighbor interaction strength $K_h$. Note that the hexagonal lattice is *bipartite*, i.e. can be separated into two sublattices. In the partition function, do a partial sum over all spins in one sublattice. Show that the remaining spins form a triangular lattice with nearest neighbor interaction $K_t(K_h)$. (This is called the *star–triangle transformation.*)

-



(b) Show that the dual of a triangular Ising model is a hexagonal Ising model with the usual duality relation $\tilde{K}(K)$.

•

b) $Z_\Delta(K_t, N) = \sum_{\{\sigma_i\}=\pm1}^{(N)} e^{\sum_{\langle ij\rangle} K_t \sigma_i \sigma_j} = \sum_{\{t_{ij}\}=\pm1}^{(3N)} e^{\sum_{\langle ij\rangle} K_t t_{ij}} \left( \prod_p^{\Delta}{}_{(2N)} \left( \frac{1 + t_{12}^p t_{23}^p t_{31}^p}{2} \right) \right)$

$t_{ij} \equiv \sigma_i \sigma_j$ We can associate a variable $n_p = 0, 1$ with each plaquette

$Z_\Delta(K_t, N) = \frac{1}{2^{2N}} \sum_{\{n_p\}}^{(2N)} \sum_{\{t_{ij}\}}^{(3N)} \left( \prod_{\langle ij\rangle}^{(3N)} e^{K_t t_{ij}} \right) \prod_p^{\Delta}{}_{(2N)} \left( t_{12}^p t_{23}^p t_{31}^p \right)^{n_p}$

Now associate every bond on the triangular lattice with its perpendicular bisector, or dual. These dual bonds form a hexagonal lattice. For each bond variable $\tilde{t}_{ij}$ there is a factor $e^{K\tilde{t}_{ij}} \tilde{t}_{ij}^{\,n_i + n_j}$ where $n_i$ and $n_j$ are associated with the plaquettes that share $t_{ij}$ and form the sides of the dual hexagonal lattice.

$Z_\Delta(K_t, N) = \frac{1}{2^{2N}} \sum_{\{n_i\}}^{(2N)} \sum_{\tilde{t}_{ij}}^{(3N)} \prod_{\langle ij\rangle}^{\hexagon}{}_{(3N)} e^{K_t \tilde{t}_{ij}} \tilde{t}_{ij}^{\,n_i + n_j}$

$Z_\Delta(K_t, N) = \frac{1}{2^{2N}} \sum_{\{n_i\}}^{(2N)} \prod_{\langle ij\rangle}^{\hexagon}{}_{(3N)} \left( e^{K_t} + (-1)^{n_i + n_j} e^{-K_t} \right)$ let $\sigma_i \equiv 2n_i - 1$

$Z_\Delta(K_t, N) = \frac{1}{2^{2N}} \sum_{\{\sigma_i\}}^{(2N)} \prod_{\langle ij\rangle}^{\hexagon}{}_{(3N)} \left( e^{K_t} - (-1)^{\frac{\sigma_i + \sigma_j}{2}} e^{-K_t} \right) = \frac{1}{2^{2N}} \sum_{\{\sigma_i\}}^{(2N)} \prod_{\langle ij\rangle}^{\hexagon}{}_{(3N)} e^{\tilde{K}_h(K_t)\sigma_i\sigma_j + \tilde{g}}$

$Z_\Delta(K_t, N) = \frac{e^{3N\tilde{g}}}{2^{2N}} Z_{\hexagon}\left(\tilde{K}_h(K_t), 2N\right)$

$R(++) = e^{K_t} + e^{-K_t} = e^{\tilde{K}_h(K_t) + \tilde{g}}$ $\qquad e^{2\tilde{g}} = 2\sinh 2K_t$

$R(+-) = e^{K_t} - e^{-K_t} = e^{-\tilde{K}_h(K_t) + \tilde{g}}$ $\qquad e^{2\tilde{K}_h(K_t)} = \coth K_t$

The duality result is: $Z_\Delta(K_t, N) = \left( \frac{\sinh^3 2K_t}{2} \right)^{\frac{N}{2}} Z_{\hexagon}\left(\tilde{K}_h(K_t), 2N\right)$

$(\sinh 2K_t)\left(\sinh 2\tilde{K}_h(K_t)\right) = 1$

(c) By combining the previous results, obtain the critical couplings $K_t^*$ and $K_h^*$ of triangular and hexagonal lattices.

•

Combining the results of parts a and b

$$Z_0(K_h, 2N) = \left(\frac{e^{2g}\sinh^3 2K_t}{2}\right)^{\frac{N}{2}} Z_0\left(\tilde{K}_h(K_t(K_h)), 2N\right)$$

The self dual point occurs when $\tilde{K}_h^*(K_t^*(K_h^*)) = K_h^*$

this is equivalent to $e^{2g^*}\sinh^3 2K_t^* = 2$

using results from part a

$$\therefore e^{4g^*} = 16\cosh 3K_h^*(\cosh K_h^*)^3 = 16\coth^2 K_h^*\cosh^4 K_h^*$$

$$2\coth K_h^*\cosh^2 K_h^*\sinh^3 2K_t^* = 1$$

$$\frac{4\cosh^4 K_h^*}{\sinh^4 2K_h} = 1 \qquad \sinh^2 K_h^* = \frac{1}{2}$$

$$\sqrt{3} = \sinh 2K_h^* = (\sinh 2K_t^*)^{-1} \qquad K_h^* = 0.658$$

$$K_t^* \simeq 0.275$$

********

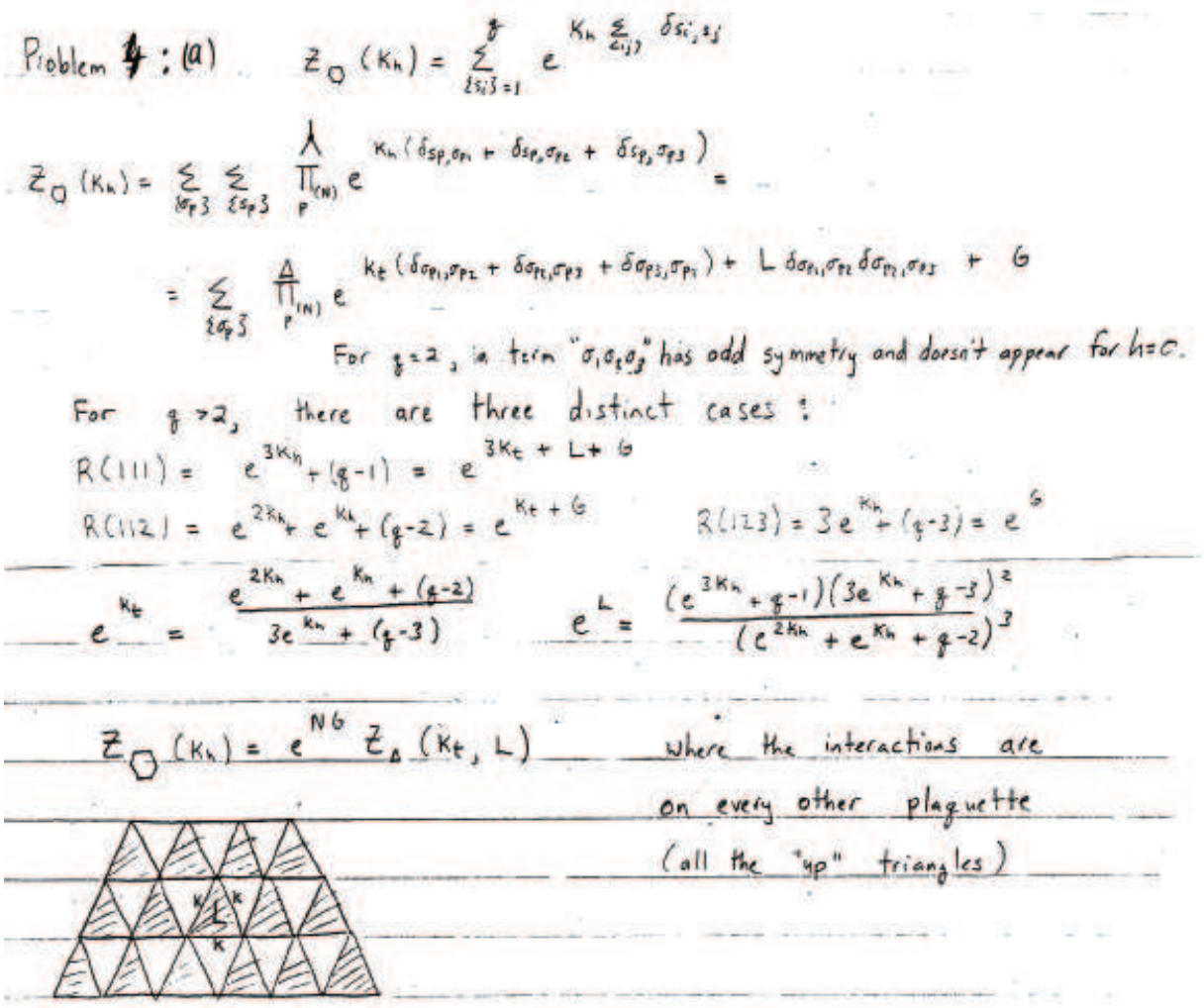
**13.** *Triangular/hexagonal lattice Potts model:* The steps of the previous problem can be repeated for a general Potts model.

(a) Consider Potts spins ($s_i = 1, 2, \cdots, q$) on a hexagonal lattice with nearest neighbor interaction $K_h\delta_{s_i,s_j}$. Perform the star-triangle decimation to show that the remaining spins form a triangular lattice with nearest neighbor interaction $K_t(K_h)$, *and* a three spin interaction $L(K_h)$. Why is $L$ absent in the Ising model?

•

(b) What is the dual of the Potts model on the triangular lattice?

•




(c) Clearly, the model is not self–dual due to the additional interaction. Nonetheless, obtain the critical value such that $\tilde{K}_t(K_c) = K_c$. Then check that $L(K_c) = 0$, i.e. while the model in general is not self–dual, it is self–dual right at criticality, leading to the exact value of $K_c(q)$!

•

b) Disregarding, for the moment, the three point interaction $L$, we can use a duality transformation similar to problem 2.

$$Z_\Delta(K_t) = \sum_{\{s_i\}} \prod_{\langle ij\rangle} e^{K_t \delta_{s_i, s_j}}$$

The Potts interaction depends only on the "distance" between states. This symmetry suggest that the appropriate bond variable is

$$t_{ij} \equiv s_i - s_j \pmod q$$

$$Z_\Delta = \left(\frac{1}{q}\right)^{2N} \sum_{\{t_{ij}\}} \sum_{\{\tau_i\}} \prod_{\langle ij\rangle}^{\Delta} e^{K_t \delta_{t_{ij},0}} \prod_{p}^{(2N)} e^{\frac{2\pi i}{q} \tau_p (t_{12}^p + t_{23}^p + t_{31}^p)}$$

$$Z_\Delta = \left(\frac{1}{q}\right)^{2N} \sum_{\{\tau_i\}} \prod_{\langle ij\rangle}^{\hexagon} \left( \sum_{t_{ij}=0}^{q-1} e^{K_t \delta_{t_{ij},0}} e^{\frac{2\pi i}{q} t_{ij} (\tau_i + \tau_j)} \right)$$

let $\tau_j = q - \tau_j$ for every other hexagonal lattice site

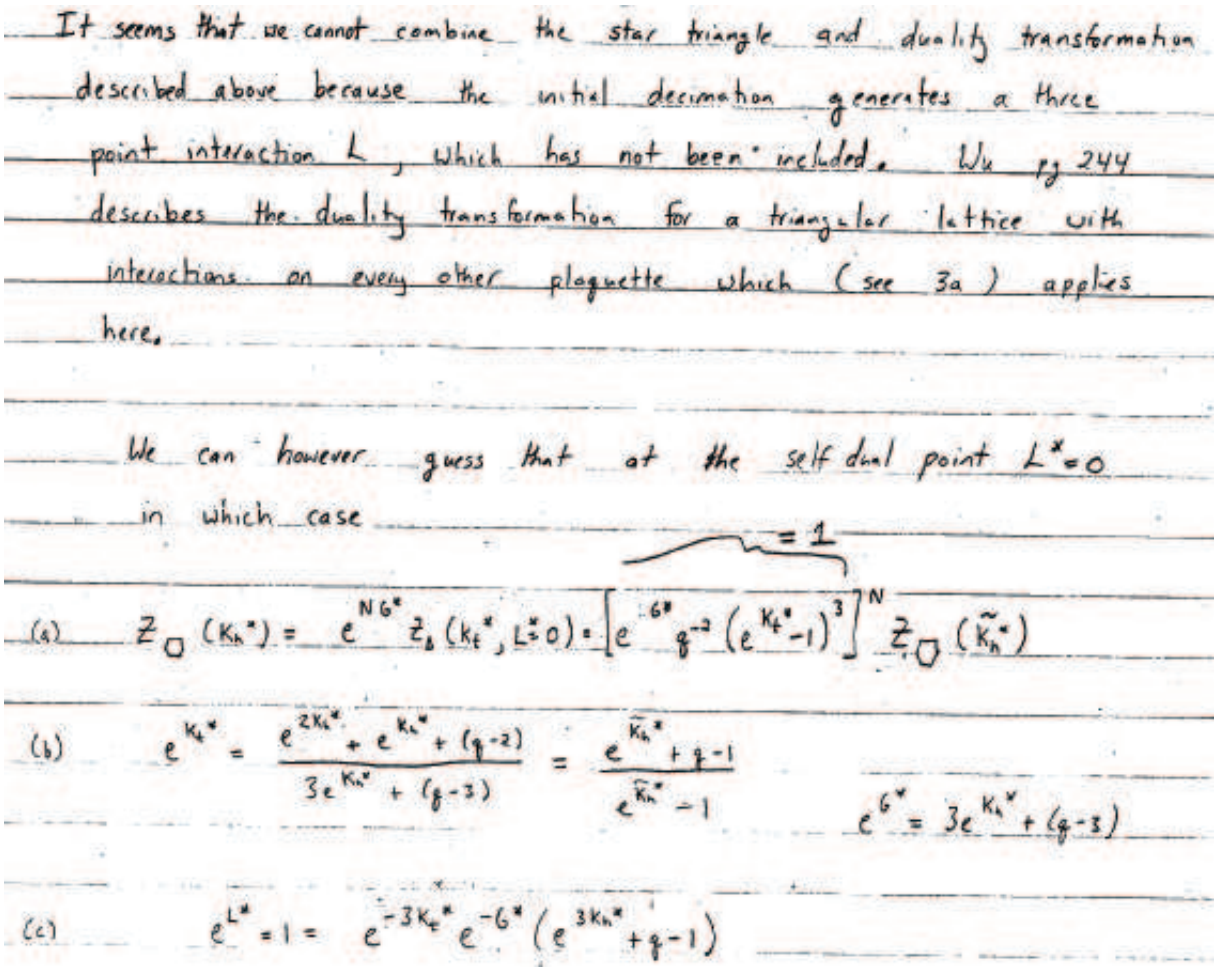
$$Z_\Delta(K_t) = \left(\frac{1}{q}\right)^{2N} \sum_{\{\tau_i\}}^{q} \prod_{\langle ij\rangle}^{\hexagon} e^{K_t + [q \delta_{\tau_i - \tau_j \pmod q, 0}]} - 1 = \left(\frac{1}{q}\right)^{2N} \sum_{\{\tau_i\}}^{q} e^{\sum_{\langle ij\rangle}^{\hexagon} \tilde{K}_h \delta_{\tau_i, \tau_j} + \tilde{G}}$$

$$Z_\Delta(K_t) = \left[\frac{1}{q}\left(e^{K_t} - 1\right)^{3/2}\right]^{2N} Z_0(\tilde{K}_h) \qquad e^{\tilde{K}_h} = \frac{e^{K_t} + q - 1}{e^{K_t} - 1}$$

********

**14.** *Cubic lattice:* The geometric concept of duality can be extended to general dimensions $d$. However, the dual of a geometric element of dimension $D$ is an entity of dimension $d-D$. For example, the dual of a bond ($D = 1$) in $d = 3$ is a plaquette ($D = 2$), as demonstrated in this problem.

(a) Consider a clock model on a cubic lattice of $N$ points. Change to the $3N$ bond variables $b_{ij} = s_i - s_j$. (Note that one must make a convention about the positive directions on the three axes.) Show that there are now $2N$ constraints associated with the plaquettes of this lattice.




(b) Implement the constraints through discrete delta-functions by associating an auxiliary variable $n_p$ with each plaquette. It is useful to imagine $n_p$ as defined on a bond of the dual lattice, perpendicular to the plaquette $p$.

(c) By summing over the bond variables in $Z$, obtain the dual Hamiltonian

$$\beta\tilde{\mathcal{H}} = \sum_p \tilde{J}\left(n_{12}^p - n_{23}^p + n_{34}^p - n_{41}^p\right),$$

where the sum is over all plaquettes $p$ of the dual lattice, with $\{n_{ij}^p\}$ indicating the four bonds around plaquette $p$.

(d) Note that $\beta\tilde{\mathcal{H}}$ is left invariant if all the six bonds going out of any site are simultaneously increased by the same integer. Thus unlike the original model which only had a *global translation symmetry*, the dual model has a *local*, i.e. *gauge symmetry*.

(e) Consider a Potts gauge theory defined on the plaquettes of a four dimensional hypercubic lattice. Find its critical coupling $J_c(q)$.

Equations (a), (b), and (c) must be satisfied for $K_h^* = \tilde{K}_h^* \equiv \ln x$, $K_t^*$ For this analysis to be correct.

Substituting (b) into (a):

$$\left(3(x-1)+g\right)\left(\frac{x^2-2x+1}{3(x-1)+g}\right)^3 = g^2$$

$$(x-1)^3 = g\left(3(x-1)+g\right) \qquad y = x-1$$

(d) $\boxed{y^3 - 3gy - g^2 = 0}$ $\qquad e^{K_h^*} = y+1 \qquad e^{K_t^*} = \frac{y+g}{y}$

At this point it is worthwhile to check that the solution to (d) satisfies (c)

$$e^{-L^*} = \frac{\left(y^3 + 3gy^2 + 3g^2y + g^3\right)(3y+g)}{y^3\left(x^3+g-1\right)} \stackrel{?}{=} 1 \qquad \text{substituting } y^3 = 3gy + g^2$$

$$x^3 + g - 1 \stackrel{?}{=} 3y^2 + 3(g+1)y + g(g+1)$$

$$y^3 - 3gy - g^2 \stackrel{?}{=} 0$$

This problem was first solved in

D. Kim and R.I. Joseph J. Phys. C Vol 7, 1974 (pg L167)

The solution is:

$$y = \begin{cases} 2g^{1/2}\cos\left[\frac{1}{3}\tan^{-1}\left(\frac{4}{g}-1\right)^{1/2}\right], & g \le 4 \\ 2g^{1/2}\cosh\left[\frac{1}{3}\tanh^{-1}\left(1-4/g\right)^{1/2}\right], & g \ge 4 \end{cases}$$

********

## Percolation

Fluids do not pass through a solid with a small concentration of holes. However, beyond a threshold concentration, the holes overlap, and the fluid can *percolate* through

a connected channel in the material. *Percolation* is a classical *geometric* phase transition, and has been used as a model of many breakdown or failure processes. The loss of rigidity in an elastic network, conductivity in resistor nets, magnetization in diluted magnets are but a few examples.

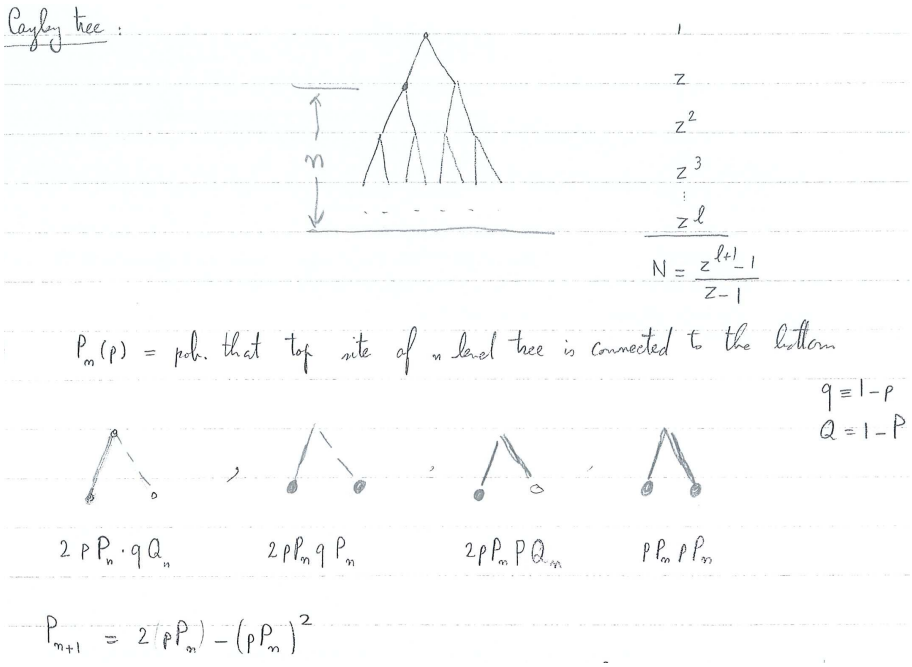
In simple models of percolation, elements of a lattice (sites or bonds) are independently occupied with a probability $p$. A cluster is defined as a connected (by neighboring bonds) set of these occupied elements. At small $p$, only small clusters exist, and the probability that two sites, separated by a distance $r$, are connected to each other decays as $\exp(-r/\xi)$. The correlation length $\xi(p)$ grows with increasing $p$, diverging at the *percolation threshold* $p_c$ as $\xi(p) \sim |p_c - p|^{-\nu}$. An infinite cluster first appears at this threshold, and percolates through the (infinite) system for all $p > p_c$. The analog of the order parameter is the probability $P(p)$ that a site belongs to this infinite cluster. On approaching $p_c$ from above, it vanishes as $P(p) \sim |p_c - p|^{\beta}$. While the value of $p_c$ depends on the details of the model, the exponents $\beta$ and $\nu$ are *universal*, varying only with the spatial dimension $d$.

In the following problems we shall focus on *bond* percolation, i.e. $p$ denotes the probability that a bond on the lattice is occupied.

**15.** *Cayley trees:* Consider a hierarchical lattice in which each site at one level is connected to $z$ sites at the level below. Thus the $n$-th level of the tree has $z^n$ sites.

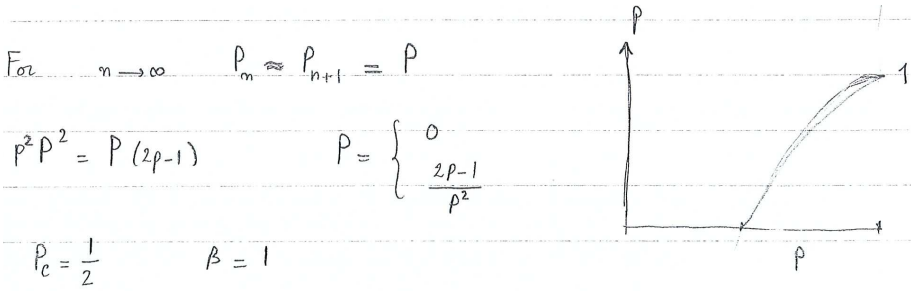
(a) For $z = 2$, obtain a recursion relation for the probability $P_n(p)$ that the top site of a tree of $n$ levels is connected to some site at the bottom level.

•



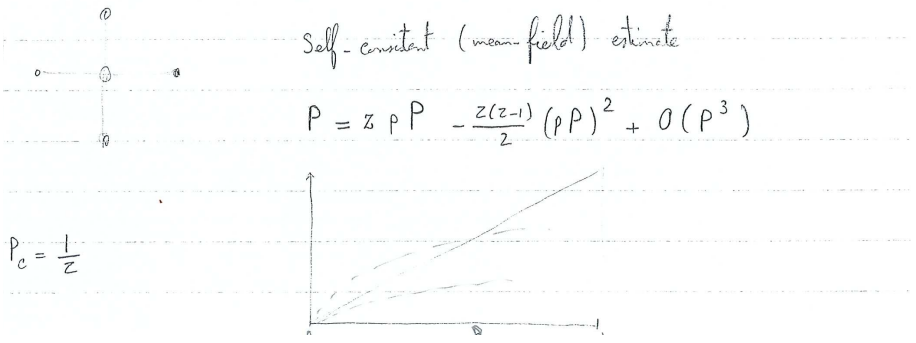
(b) Find the limiting behavior of $P(p)$ for infinitely many levels, and give the exponent $\beta$ for this tree.

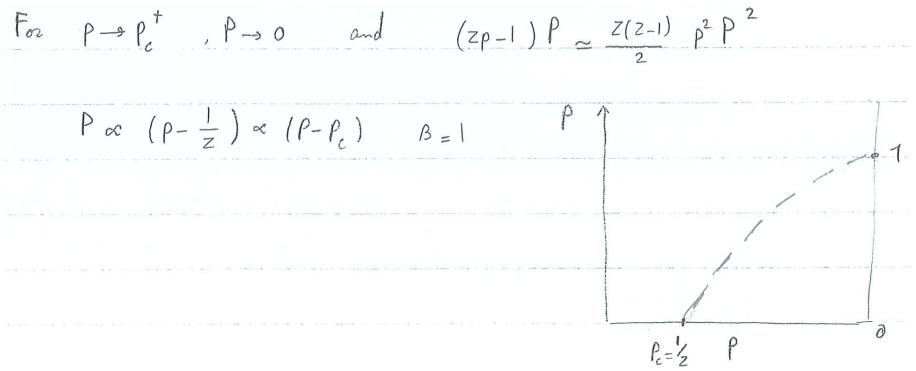
•

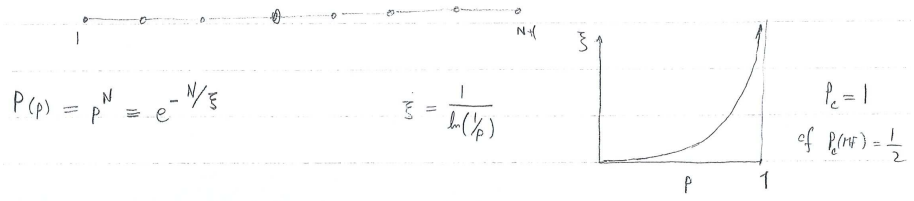



(c) In a *mean-field* approximation, $P(p)$ for a lattice of coordination number $z$ is calculated self-consistently, in a manner similar to the above. Give the mean-field estimates for $p_c$ and the exponent $\beta$.

•






(d) The one-dimensional chain corresponds to $z = 1$. Find the probability that end-points of an open chain of $N + 1$ sites are connected, and hence deduce the correlation length $\xi$.

•

********

**16.** *Duality* has a very natural interpretation in percolation: If a bond is occupied, its dual is empty, and vice versa. Thus the occupation probability for dual bonds is $\tilde{p} = 1 - p \equiv q$. Since, by construction, the original and dual elements do not intersect, one or the other percolates through the system.

(a) The dual of a chain in which $N$ bonds are connected in *series*, has $N$ bonds connected in *parallel*. What is the corresponding (non-) percolation probability?

• Consider $N$ bonds forming a circle (1d chain with periodic boundary conditions). The dual system has one point inside the circle and one point outside. While on the original chain neighboring points are connected in series with probability $p$, on the dual lattice, the inside and outside points are connected by $N$ dual bonds in parallel with probability $q = 1 - p$, i.e. a connected path on the original chain blocks a possible dual path. The probability that the original chain is connected is $p^N$. The probability that the two dual points *are not connected* is also $p^N$.

(b) The bond percolation problem on a square lattice is self-dual. What is its threshold $p_c$?

• The dual to the square lattice is another square lattice whose sites are the centers of the squares of the original lattice. If a bond is not connected on the original lattice, a path exists between the centers of neighboring squares. Thus $\tilde{p} = 1 - p = q$. The self-dual point, which is the bond percolation threshold occurs at $p_c = 1/2$.

(c) Bond percolation in three dimensions is dual to plaquette percolation. Is it possible to percolate while maintaining solid integrity in $d = 3$?

• Yes, there is an intermediate range of values between $p_c$ and $\tilde{p}_c$ in which both bonds and plaquettes percolate.

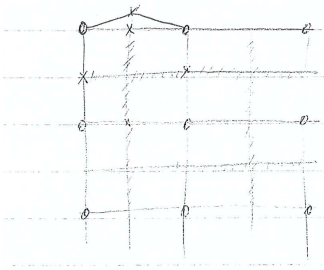
(d) The triangular and hexagonal lattices are dual to each other. Combined with the star–triangle transformation that also maps the two lattices, this leads to exact values of $p_c$ for these lattices. However, this calculation is not trivial, and it is best to follow the steps for calculations of the critical couplings of $q$-state Potts models in an earlier problem.

********

**17.** *Position–space renormalization group (PSRG):* In the Migdal–Kadanoff approximation, a decimation is performed after bonds are moved to connect the remaining sites in a one-dimensional geometry. For a PSRG rescaling factor $b$ on a $d$–dimensional hypercubic lattice, the retained sites are connected by $b^{d-1}$ strands in parallel, each strand having $b$ bonds in series. This scheme is naturally exact for $d = 1$, and becomes progressively worse in higher dimensions.

(a) Apply the above scheme to $b = d = 2$, and compare the resulting estimates of $p_c$ and $y_t = 1/\nu$ to the exact values for the square lattice ($p_c = 1/2$ and $y_t = 3/4$).

•



$\bar{p} = p^2$

$q' = \bar{q}^2 = (1-\bar{p})^2 = (1-p^2)^2 = 1-p'$

$p' = 1-(1-p^2)^2 = 2p^2 - p^4$

Fixed points $p^* = 0$ $p' = 2p^2$ stable

$p^* = 1$ $\frac{\partial p'}{\partial p} = 4p - 4p^3 = 0$

$q' = (1-(1-q)^2)^2 = (2q-q^2)^2 \simeq 4q^2$

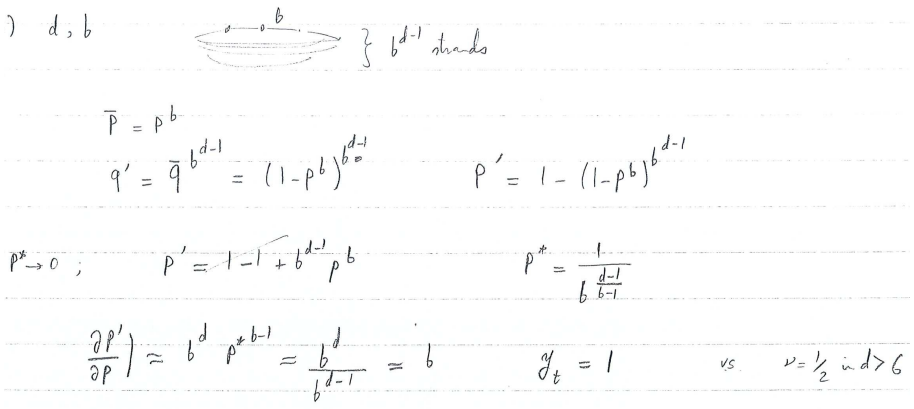
$0 = p^{*3} - 2p^* + 1 = (p^*-1)(p^{*2}+p^*-1)$

$p^* = \frac{1}{2}(-1+\sqrt{5}) \simeq 0.62$

$\left.\frac{\partial p'}{\partial p}\right|_{p^*} = 4p^* - 4p^{*3} \quad (2p^*-1)$

$= 4p^* - 8p^{*2} + 4 = -4p^* + 4 = +2\cdot 2\sqrt{5} + 4 = 6 - 2\sqrt{5} \simeq 1.53 = b^{y_t}$

$y_t \simeq 0.61$ cf. $y_t = 3/4$



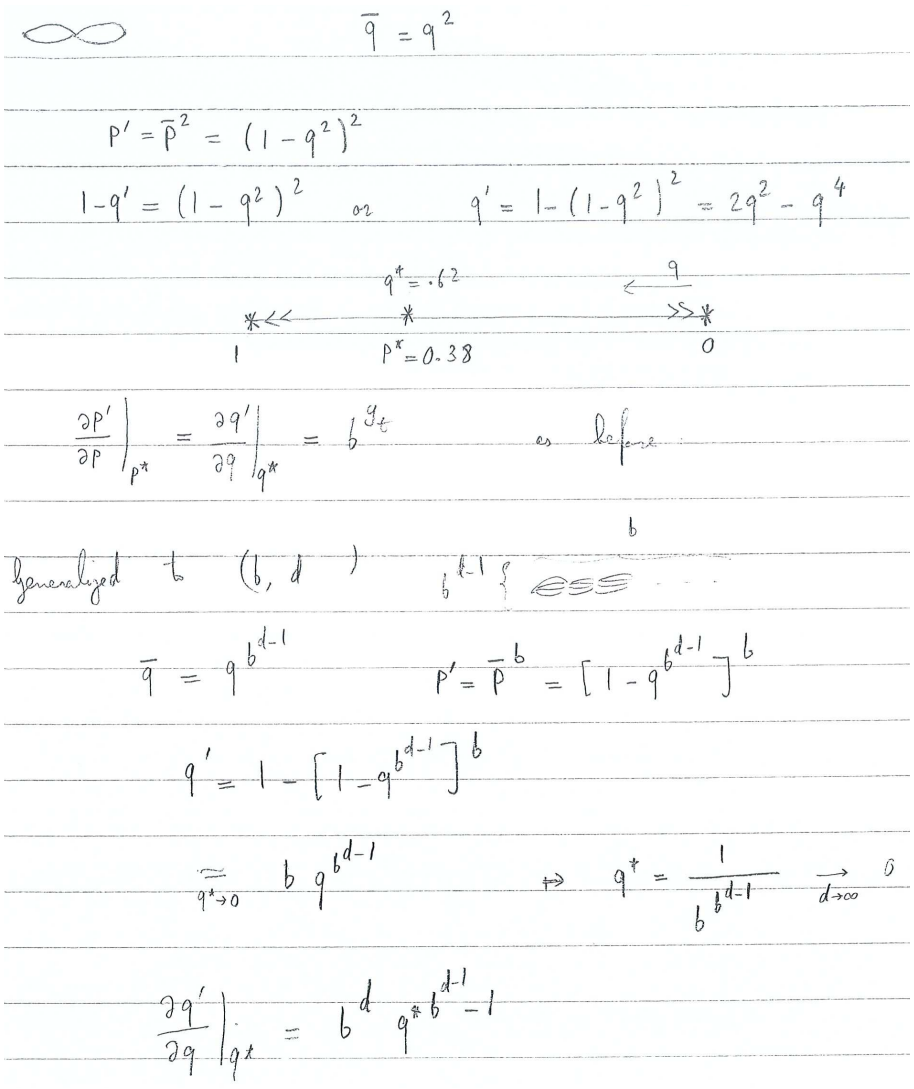

(b) Find the limiting values of $p_c$ and $y_t$ for large $d$. (It can be shown that $\nu = 1/2$ *exactly*, above an upper critical dimension $d_u = 6$.)

•

(c) In an alternative PSRG scheme, first $b^{d-1}$ bonds are connected in parallel, the resulting groups are then joined in series. Repeat the above calculations with this scheme.

•

********




**Solutions to problems from chapter 8 - Beyond Spin Waves**

**1.** *Anisotropic nonlinear $\sigma$ model:* Consider unit $n$-component spins, $\vec{s}(\mathbf{x}) = (s_1, \cdots, s_n)$ with $\sum_\alpha s_\alpha^2 = 1$, subject to a Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x} \left[ \frac{1}{2T} (\nabla \vec{s})^2 + g s_1^2 \right].$$

For $g = 0$, renormalization group equations are obtained through rescaling distances by a factor $b = e^\ell$, and spins by a factor $\zeta = b^{y_s}$ with $y_s = -\frac{(n-1)}{4\pi} T$, and lead to the flow equation

$$\frac{dT}{d\ell} = -\epsilon T + \frac{(n-2)}{2\pi} T^2 + \mathcal{O}(T^3),$$

where $\epsilon = d - 2$.

(a) Find the fixed point, and the thermal eigenvalue $y_T$.

• Setting $dT/d\ell$ to zero, the fixed point is obtained as

$$T^* = \frac{2\pi\epsilon}{n-2} + \mathcal{O}(\epsilon^2).$$

Linearizing the recursion relation gives

$$y_T = -\epsilon + \frac{(n-2)}{\pi} T^* = +\epsilon + \mathcal{O}(\epsilon^2).$$

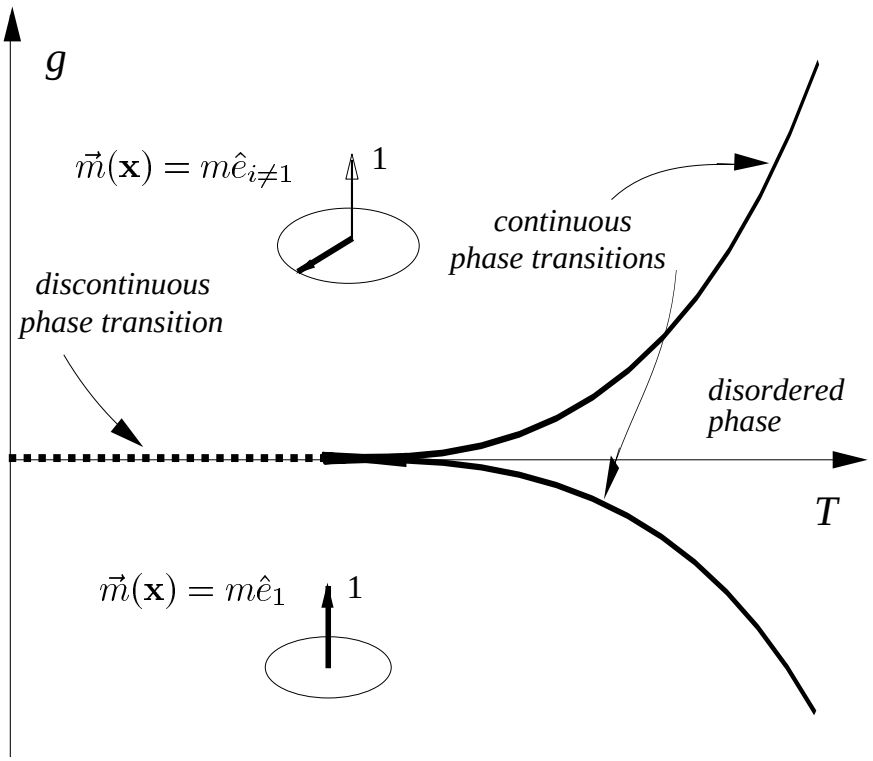
(b) Write the renormalization group equation for $g$ in the vicinity of the above fixed point, and obtain the corresponding eigenvalue $y_g$.

• Rescalings $x \to b\mathbf{x}'$ and $\vec{s} \to \zeta\vec{s}'$, lead to $g \to g' = b^d\zeta^2 g$, and hence

$$y_g = d + 2y_s = d - \frac{n-1}{2\pi} T^* = 2 + \epsilon - \frac{n-1}{n-2}\epsilon = 2 - \frac{1}{n-2}\epsilon + \mathcal{O}(\epsilon^2).$$

(c) Sketch the phase diagram as a function of $T$ and $g$, indicating the phases, and paying careful attention to the shape of the phase boundary as $g \to 0$.

•



The term proportional to $g$ removes full rotational symmetry and leads to a bicritical phase diagram as discussed in recitations. The phase for $g < 0$ has order along direction 1, while $g > 0$ favors ordering along any one of the $(n-1)$ directions orthogonal to 1. The phase boundaries as $g \to 0$ behave as $g \propto (\delta T)^{\phi}$, with $\phi = y_g/y_t \approx 2/\epsilon + \mathcal{O}(1)$.

********

**2.** *Matrix models:* In some situations, the order parameter is a matrix rather than a vector. For example, in triangular (Heisenberg) antiferromagnets each triplet of spins aligns at 120°, locally defining a plane. The variations of this plane across the system are described by a $3 \times 3$ rotation matrix. We can construct a nonlinear $\sigma$ model to describe a generalization of this problem as follows. Consider the Hamiltonian

$$\beta\mathcal{H} = \frac{K}{4} \int d^d\mathbf{x} \ \mathrm{tr} \left[ \nabla M(\mathbf{x}) \cdot \nabla M^T(\mathbf{x}) \right] \quad ,$$

where $M$ is a *real,* $N \times N$ *orthogonal* matrix, and 'tr' denotes the trace operation. The condition of orthogonality is that $MM^T = M^T M = I$, where $I$ is the $N \times N$ identity matrix, and $M^T$ is the transposed matrix, $M^T_{ij} = M_{ji}$. The partition function is obtained by summing over all matrix functionals, as

$$Z = \int \mathcal{D}M(\mathbf{x}) \delta \left( M(\mathbf{x}) M^T(\mathbf{x}) - I \right) e^{-\beta \mathcal{H}[M(\mathbf{x})]} \quad .$$

(a) Rewrite the Hamiltonian and the orthogonality constraint in terms of the matrix elements $M_{ij}$ $(i, j = 1, \cdots, N)$. Describe the ground state of the system.

• In terms of the matrix elements, the Hamiltonian reads

$$\beta\mathcal{H} = \frac{K}{4} \int d^d x \sum_{i,j} \nabla M_{ij} \cdot \nabla M_{ij},$$

and the orthogonality condition becomes

$$\sum_k M_{ik} M_{jk} = \delta_{ij}.$$

Since $\nabla M_{ij} \cdot \nabla M_{ij} \geq 0$, any constant (spatially uniform) orthogonal matrix realizes a ground state.

(b) Define the symmetric and anti-symmetric matrices

$$\begin{cases} \sigma = \frac{1}{2}\left(M + M^T\right) = \sigma^T \\ \pi = \frac{1}{2}\left(M - M^T\right) = -\pi^T \end{cases} .$$

Express $\beta\mathcal{H}$ and the orthogonality constraint in terms of the matrices $\sigma$ and $\pi$.

• As $M = \sigma + \pi$ and $M^T = \sigma - \pi$,

$$\beta\mathcal{H} = \frac{K}{4}\int d^d x \,\mathrm{tr}\left[\nabla\left(\sigma+\pi\right)\cdot\nabla\left(\sigma-\pi\right)\right] = \frac{K}{4}\int d^d x \,\mathrm{tr}\left[\left(\nabla\sigma\right)^2 - \left(\nabla\pi\right)^2\right],$$

where we have used the (easily checked) fact that the trace of the cummutator of matrices $\nabla\sigma$ and $\nabla\pi$ is zero. Similarly, the orthogonality condition is written as

$$\sigma^2 - \pi^2 = I,$$

where $I$ is the unit matrix.

(c) Consider *small fluctuations* about the ordered state $M(\mathbf{x}) = I$. Show that $\sigma$ can be expanded in powers of $\pi$ as

$$\sigma = I - \frac{1}{2}\pi\pi^T + \cdots.$$

Use the orthogonality constraint to integrate out $\sigma$, and obtain an expression for $\beta H$ to fourth order in $\pi$. Note that there are two distinct types of fourth order terms. *Do not include* terms generated by the argument of the delta function. As shown for the nonlinear $\sigma$ model in the text, these terms do not effect the results at lowest order.

• Taking the square root of

$$\sigma^2 = I + \pi^2 = I - \pi\pi^T,$$

results in

$$\sigma = I - \frac{1}{2}\,\pi\pi^T + \mathcal{O}\left(\pi^4\right),$$

(as can easily be checked by calculating the square of $I - \pi\pi^T/2$). We now integrate out $\sigma$, to obtain

$$Z = \int \mathcal{D}\pi\left(\mathbf{x}\right)\exp\left\{\frac{K}{4}\int d^d x\,\mathrm{tr}\left[\left(\nabla\pi\right)^2 - \frac{1}{4}\left(\nabla\left(\pi\pi^T\right)\right)^2\right]\right\},$$

where $\mathcal{D}\pi\left(\mathbf{x}\right) = \prod_{j>i}\mathcal{D}\pi_{ij}\left(\mathbf{x}\right)$, and $\pi$ is a matrix with zeros along the diagonal, and elements below the diagonal given by $\pi_{ij} = -\pi_{ji}$. Note that we have not included the

terms generated by the argument of the delta function. Such term, which ensure that the measure of integration over $\pi$ is symmetric, do not contribute to the renormalization of $K$ at the lowest order. Note also that the fourth order terms are of two distinct types, due to the non-commutativity of $\pi$ and $\nabla\pi$. Indeed,

$$\begin{aligned}\left[\nabla\left(\pi\pi^T\right)\right]^2 &= \left[\nabla\left(\pi^2\right)\right]^2 = \left[\left(\nabla\pi\right)\pi + \pi\nabla\pi\right]^2 \\ &= \left(\nabla\pi\right)\pi\cdot\left(\nabla\pi\right)\pi + \left(\nabla\pi\right)\pi^2\cdot\nabla\pi + \pi\left(\nabla\pi\right)^2\pi + \pi\left(\nabla\pi\right)\pi\cdot\nabla\pi,\end{aligned}$$

and, since the trace is unchanged by cyclic permutations,

$$\mathrm{tr}\left[\nabla\left(\pi\pi^T\right)\right]^2 = 2\,\mathrm{tr}\left[\left(\pi\nabla\pi\right)^2 + \pi^2\left(\nabla\pi\right)^2\right].$$

(d) For an $N$-vector order parameter there are $N-1$ Goldstone modes. Show that an orthogonal $N\times N$ order parameter leads to $N(N-1)/2$ such modes.

• The anti-symmetry of $\pi$ imposes $N\left(N+1\right)/2$ conditions on the $N\times N$ matrix elements, and thus there are $N^2 - N\left(N+1\right)/2 = N\left(N-1\right)/2$ independent components (Goldstone modes) for the matrix. Alternatively, the orthogonality of $M$ similarly imposes $N\left(N+1\right)/2$ constraints, leading to $N\left(N-1\right)/2$ degrees of freedom. [Note that in the analogous calculation for the $\mathcal{O}\left(n\right)$ model, there is one condition constraining the magnitude of the spins to unity; and the remaining $n-1$ angular components are Goldstone modes.]

(e) Consider the quadratic piece of $\beta\mathcal{H}$. Show that the two point correlation function in Fourier space is

$$\langle\pi_{ij}(\mathbf{q})\pi_{kl}(\mathbf{q}')\rangle = \frac{(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')}{Kq^2}\left[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right].$$

• In terms of the Fourier components $\pi_{ij}(\mathbf{q})$, the quadratic part of the Hamiltonian in (c) has the form

$$\beta\mathcal{H}_0 = \frac{K}{2}\sum_{i<j}\int\frac{d^d\mathbf{q}}{(2\pi)^d}q^2|\pi_{ij}(\mathbf{q})|^2,$$

leading to the bare expectation values

$$\left\langle\pi_{ij}\left(\mathbf{q}\right)\pi_{ij}\left(\mathbf{q}'\right)\right\rangle_0 = \frac{\left(2\pi\right)^d\delta^d\left(\mathbf{q}+\mathbf{q}'\right)}{Kq^2},$$

and

$$\langle \pi_{ij}(\mathbf{q})\,\pi_{kl}(\mathbf{q}')\rangle_0 = 0, \text{ if the pairs } (ij) \text{ and } (kl) \text{ are different.}$$

Furthermore, since $\pi$ is anti-symmetric,

$$\langle \pi_{ij}(\mathbf{q})\,\pi_{ji}(\mathbf{q}')\rangle_0 = -\langle \pi_{ij}(\mathbf{q})\,\pi_{ij}(\mathbf{q}')\rangle_0 ,$$

and in particular $\langle \pi_{ii}(\mathbf{q})\,\pi_{jj}(\mathbf{q}')\rangle_0 = 0$. These results can be summarized by

$$\langle \pi_{ij}(\mathbf{q})\,\pi_{kl}(\mathbf{q}')\rangle_0 = \frac{(2\pi)^d\,\delta^d(\mathbf{q}+\mathbf{q}')}{Kq^2}\left(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}\right).$$

We shall now construct a renormalization group by removing Fourier modes $M^{>}(\mathbf{q})$, with $\mathbf{q}$ in the shell $\Lambda/b < |\mathbf{q}| < \Lambda$.

(f) Calculate the coarse grained expectation value for $\langle \mathrm{tr}(\sigma)\rangle_0^{>}$ at low temperatures after removing these modes. Identify the scaling factor, $M'(\mathbf{x}') = M^{<}(\mathbf{x})/\zeta$, that restores $\mathrm{tr}(M') = \mathrm{tr}(\sigma') = N$.

• As a result of fluctuations of short wavelength modes, $\mathrm{tr}\,\sigma$ is reduced to

$$\begin{aligned}
\langle \mathrm{tr}\,\sigma\rangle_0^{>} &= \left\langle \mathrm{tr}\left(I+\frac{\pi^2}{2}+\cdots\right)\right\rangle_0^{>} \approx N+\frac{1}{2}\left\langle \mathrm{tr}\,\pi^2\right\rangle_0^{>} \\
&= N+\frac{1}{2}\left\langle \sum_{i\neq j}\pi_{ij}\pi_{ji}\right\rangle_0^{>} = N-\frac{1}{2}\left\langle \sum_{i\neq j}\pi_{ij}^2\right\rangle_0^{>} = N-\frac{1}{2}\left(N^2-N\right)\left\langle \pi_{ij}^2\right\rangle_0^{>} \\
&= N\left(1-\frac{N-1}{2}\int_{\Lambda/b}^{\Lambda}\frac{d^dq}{(2\pi)^d}\frac{1}{Kq^2}\right) = N\left[1-\frac{N-1}{2K}I_d(b)\right].
\end{aligned}$$

To restore $\mathrm{tr}\,M' = \mathrm{tr}\,\sigma' = N$, we rescale all components of the matrix by

$$\zeta = 1-\frac{N-1}{2K}I_d(b).$$

**NOTE**: An orthogonal matrix $M$ is invertible ($M^{-1} = M^T$), and therefore diagonalizable. In diagonal form, the transposed matrix is equal to the matrix itself, and so its square is the identity, implying that each eigenvalue is either $+1$ or $-1$. Thus, if $M$ is chosen to be very close to the identity, all eigenvalues are $+1$, and $\mathrm{tr}\,M = N$ (as the trace is independent of the coordinate basis).

(g) Use perturbation theory to calculate the coarse grained coupling constant $\tilde{K}$. Evaluate only the two diagrams that directly renormalize the $(\nabla\pi_{ij})^2$ term in $\beta\mathcal{H}$, and show that

$$\tilde{K} = K+\frac{N}{2}\int_{\Lambda/b}^{\Lambda}\frac{d^d\mathbf{q}}{(2\pi)^d}\frac{1}{q^2} \quad .$$

• Distinguishing between the greater and lesser modes, we write the partition function as

$$Z = \int \mathcal{D}\pi^{<}\, \mathcal{D}\pi^{>}\, e^{-\beta\mathcal{H}_0^{<} - \beta\mathcal{H}_0^{>} + U\left[\pi^{<},\pi^{>}\right]} = \int \mathcal{D}\pi^{<}\, e^{-\delta f_b^0 - \beta\mathcal{H}_0^{<}}\, \left\langle e^{U} \right\rangle_0^{>},$$

where $\mathcal{H}_0$ denotes the quadratic part, and

$$\begin{aligned} U &= -\frac{K}{8} \sum_{i,j,k,l} \int d^d x \left[ \left(\nabla \pi_{ij}\right) \pi_{jk} \cdot \left(\nabla \pi_{kl}\right) \pi_{li} + \pi_{ij} \left(\nabla \pi_{jk}\right) \cdot \left(\nabla \pi_{kl}\right) \pi_{li} \right] \\ &= \frac{K}{8} \sum_{i,j,k,l} \int \frac{d^d q_1 d^d q_2 d^d q_3}{(2\pi)^{3d}} \left[ \left(\mathbf{q}_1 \cdot \mathbf{q}_3 + \mathbf{q}_2 \cdot \mathbf{q}_3\right) \cdot \right. \\ &\qquad \left. \cdot \pi_{ij}\left(\mathbf{q}_1\right) \pi_{jk}\left(\mathbf{q}_2\right) \pi_{kl}\left(\mathbf{q}_3\right) \pi_{li}\left(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3\right) \right]. \end{aligned}$$

To first order in $U$, the following two averages contribute to the renormalization of $K$:

$$\begin{aligned} (i) \qquad & \frac{K}{8} \sum_{i,j,k,l} \int \frac{d^d q_1 d^d q_2 d^d q_3}{(2\pi)^{3d}} \left\langle \pi_{jk}^{>}\left(\mathbf{q}_2\right) \pi_{li}^{>}\left(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3\right) \right\rangle_0^{>} \left(\mathbf{q}_1 \cdot \mathbf{q}_3\right) \pi_{ij}^{<}\left(\mathbf{q}_1\right) \pi_{kl}^{<}\left(\mathbf{q}_3\right) \\ &= \frac{K}{8} \left( \int_{\Lambda/b}^{\Lambda} \frac{d^d q'}{(2\pi)^d} \frac{1}{K q'^2} \right) \left( \int_0^{\Lambda/b} \frac{d^d q}{(2\pi)^d} q^2 \sum_{i,j} \pi_{ij}^{<}\left(\mathbf{q}\right) \pi_{ji}^{<}\left(-\mathbf{q}\right) \right), \end{aligned}$$

and

$$\begin{aligned} (ii)\ & \frac{K}{8} \sum_{j,k,l} \int \frac{d^d q_1 d^d q_2 d^d q_3}{(2\pi)^{3d}} \left\langle \sum_{i \neq j,l} \pi_{ij}^{>}\left(\mathbf{q}_2\right) \pi_{li}^{>}\left(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3\right) \right\rangle_0^{>} \left(\mathbf{q}_2 \cdot \mathbf{q}_3\right) \pi_{jk}^{<}\left(\mathbf{q}_2\right) \pi_{kl}^{<}\left(\mathbf{q}_3\right) \\ &= \frac{K}{8} \left[ (N-1) \int_{\Lambda/b}^{\Lambda} \frac{d^d q'}{(2\pi)^d} \frac{1}{K q'^2} \right] \left( \int_0^{\Lambda/b} \frac{d^d q}{(2\pi)^d} q^2 \sum_{j,k} \pi_{jk}^{<}\left(\mathbf{q}\right) \pi_{kj}^{<}\left(-\mathbf{q}\right) \right). \end{aligned}$$

Adding up the two contributions results in an effective coupling

$$\frac{\tilde{K}}{4} = \frac{K}{4} + \frac{K}{8} N \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{K q^2}, \qquad i.e. \qquad \tilde{K} = K + \frac{N}{2} I_d\left(b\right).$$

(h) Using the result from part (f), show that after matrix rescaling, the RG equation for $K'$ is given by:

$$K' = b^{d-2} \left[ K - \frac{N-2}{2} \int_{\Lambda/b}^{\Lambda} \frac{d^d \mathbf{q}}{(2\pi)^d} \frac{1}{q^2} \right] \quad .$$

• After coarse-graining, renormalizing the fields, and rescaling,

$$K' = b^{d-2}\zeta^2\tilde{K} = b^{d-2}\left[1 - \frac{N-1}{K} I_d\left(b\right)\right] K \left[1 + \frac{N}{2K} I_d\left(b\right)\right]$$
$$= b^{d-2}\left[K - \frac{N-2}{2} I_d\left(b\right) + \mathcal{O}\left(1/K\right)\right],$$

*i.e.*, to lowest non-trivial order,

$$K' = b^{d-2}\left[K - \frac{N-2}{2}\int_{\Lambda/b}^{\Lambda} \frac{d^d q}{\left(2\pi\right)^d}\frac{1}{q^2}\right].$$

(i) Obtain the *differential* RG equation for $T = 1/K$, by considering $b = 1 + \delta\ell$. Sketch the flows for $d < 2$ and $d = 2$. For $d = 2 + \epsilon$, compute $T_c$ and the critical exponent $\nu$.

• Differential recursion relations are obtained for infinitesimal $b = 1 + \delta\ell$, as

$$K' = K + \frac{dK}{d\ell}\delta\ell = \left[1 + \left(d-2\right)\delta\ell\right]\left[K - \frac{N-2}{2} K_d \Lambda^{d-2}\delta\ell\right],$$

leading to

$$\frac{dK}{d\ell} = \left(d-2\right)K - \frac{N-2}{2} K_d \Lambda^{d-2}.$$

To obtain the corresponding equation for $T = 1/K$, we divide the above relation by $-K^2$, to get

$$\frac{dT}{d\ell} = \left(2-d\right)T + \frac{N-2}{2} K_d \Lambda^{d-2} T^2.$$

For $d < 2$, we have the two usual trivial fixed points: 0 (unstable) and $\infty$ (stable). The system is mapped unto higher temperatures by coarse-graining. The same applies for the case $d = 2$ and $N > 2$.

For $d > 2$, both 0 and $\infty$ are stable, and a non-trivial unstable fixed point appears at a finite temperature given by $dT/d\ell = 0$, *i.e.*

$$T^* = \frac{2\left(d-2\right)}{\left(N-2\right)K_d\Lambda^{d-2}} = \frac{4\pi\epsilon}{N-2} + \mathcal{O}\left(\epsilon^2\right).$$

In the vicinity of the fixed point, the flows are described by

$$\delta T' = \left[1 + \frac{d}{dT}\left(\frac{dT}{d\ell}\right)\Big|_{T^*}\delta\ell\right]\delta T = \left\{1 + \left[\left(2-d\right) + \left(N-2\right)K_d\Lambda^{d-2}T^*\right]\delta\ell\right\}\delta T$$
$$= \left(1 + \epsilon\delta\ell\right)\delta T.$$

Thus, from

$$\delta T' = b^{y_T}\delta T = (1 + y_T \delta\ell)\,\delta T,$$

we get $y_T = \epsilon$, and

$$\nu = \frac{1}{\epsilon}.$$

(j) Consider a small symmetry breaking term $-h\int d^d\mathbf{x}\ \mathrm{tr}(M)$, added to the Hamiltonian. Find the renormalization of $h$, and identify the corresponding exponent $y_h$.

• As usual, $h$ renormalizes according to

$$\begin{aligned} h' = b^d\zeta h &= (1 + d\delta\ell)\left(K - \frac{N-1}{2K}K_d\Lambda^{d-2}\delta\ell\right)h \\ &= \left[1 + \left(d - \frac{N-1}{2K}K_d\Lambda^{d-2}\right)\delta\ell + \mathcal{O}\left(\delta\ell^2\right)\right]h. \end{aligned}$$

From $h' = b^{y_h}h = (1 + y_h\delta\ell)\,h$, we obtain

$$y_h = d - \frac{N-1}{2K^*}K_d\Lambda^{d-2} = d - \frac{N-1}{N-2}\,(d-2) = 2 - \frac{\epsilon}{N-2} + \mathcal{O}\left(\epsilon^2\right).$$

Combining RG and symmetry arguments, it can be shown that the $3\times 3$ matrix model is perturbatively equivalent to the $N = 4$ vector model at all orders. This would suggest that stacked triangular antiferromagnets provide a realization of the $\mathcal{O}(4)$ universality class; see P. Azaria, B. Delamotte, and T. Jolicoeur, J. Appl. Phys. **69**, 6170 (1991). However, non-perturbative (topological aspects) appear to remove this equivalence as discussed in S.V. Isakov, T. Senthil, Y.B. Kim, Phys. Rev. B **72**, 174417 (2005).

********

**3.** *The roughening transition:* In an earlier problem we examined a continuum interface model which in $d = 3$ is described by the Hamiltonian

$$\beta\mathcal{H}_0 = -\frac{K}{2}\int d^2\mathbf{x}\,(\nabla h)^2 \quad ,$$

where $h(\mathbf{x})$ is the interface height at location $\mathbf{x}$. For a crystalline facet, the allowed values of $h$ are multiples of the lattice spacing. In the continuum, this tendency for integer $h$ can be mimicked by adding a term

$$-\beta U = y_0\int d^2\mathbf{x}\,\cos\left(2\pi h\right),$$

to the Hamiltonian. Treat $-\beta U$ as a perturbation, and proceed to construct a renormalization group as follows:

(a) Show that

$$\left\langle \exp\left[ i \sum_\alpha q_\alpha h(\mathbf{x}_\alpha) \right] \right\rangle_0 = \exp\left[ \frac{1}{K} \sum_{\alpha<\beta} q_\alpha q_\beta C(\mathbf{x}_\alpha - \mathbf{x}_\beta) \right]$$

for $\sum_\alpha q_\alpha = 0$, and zero otherwise. ($C(\mathbf{x}) = \ln|\mathbf{x}|/2\pi$ is the Coulomb interaction in two dimensions.)

• The translational invariance of the Hamiltonian constrains $\langle \exp[i \sum_\alpha q_\alpha h(\mathbf{x}_\alpha)] \rangle_0$ to vanish unless $\sum_\alpha q_\alpha = 0$, as implied by the following relation

$$\begin{aligned} \exp\left( i\delta \sum_\alpha q_\alpha \right) \left\langle \exp\left[ i \sum_\alpha q_\alpha h(\mathbf{x}_\alpha) \right] \right\rangle_0 &= \left\langle \exp\left\{ i \sum_\alpha q_\alpha [h(\mathbf{x}_\alpha) + \delta] \right\} \right\rangle_0 \\ &= \left\langle \exp\left[ i \sum_\alpha q_\alpha h(\mathbf{x}_\alpha) \right] \right\rangle_0 . \end{aligned}$$

The last equality follows from the symmetry $\mathcal{H}[h(\mathbf{x}) + \delta] = \mathcal{H}[h(\mathbf{x})]$. Using general properties of Gaussian averages, we can set

$$\begin{aligned} \left\langle \exp\left[ i \sum_\alpha q_\alpha h(\mathbf{x}_\alpha) \right] \right\rangle_0 &= \exp\left[ -\frac{1}{2} \sum_{\alpha\beta} q_\alpha q_\beta \langle h(\mathbf{x}_\alpha) h(\mathbf{x}_\beta) \rangle_0 \right] \\ = \exp\left[ \frac{1}{4} \sum_{\alpha\beta} q_\alpha q_\beta \left\langle (h(\mathbf{x}_\alpha) - h(\mathbf{x}_\beta))^2 \right\rangle_0 \right] . \end{aligned}$$

Note that the quantity $\langle h(\mathbf{x}_\alpha) h(\mathbf{x}_\beta) \rangle_0$ is ambiguous because of the symmetry $h(\mathbf{x}) \to h(\mathbf{x}) + \delta$. When $\sum_\alpha q_\alpha = 0$, we can replace this quantity in the above sum with the height difference $\left\langle (h(\mathbf{x}_\alpha) - h(\mathbf{x}_\beta))^2 \right\rangle_0$ which is independent of this symmetry. (The ambiguity, or symmetry, results from the kernel of the quadratic form having a zero eigenvalue, which means that inverting it requires care.) We can now proceed as usual, and

$$\begin{aligned} \left\langle \exp\left[ i \sum_\alpha q_\alpha h(\mathbf{x}_\alpha) \right] \right\rangle_0 &= \exp\left[ \sum_{\alpha,\beta} \frac{q_\alpha q_\beta}{4} \int \frac{d^2q}{(2\pi)^2} \frac{\left(e^{i\mathbf{q}\cdot\mathbf{x}_\alpha} - e^{i\mathbf{q}\cdot\mathbf{x}_\beta}\right)\left(e^{-i\mathbf{q}\cdot\mathbf{x}_\alpha} - e^{-i\mathbf{q}\cdot\mathbf{x}_\beta}\right)}{Kq^2} \right] \\ &= \exp\left[ \sum_{\alpha<\beta} q_\alpha q_\beta \int \frac{d^2q}{(2\pi)^2} \frac{1 - \cos(\mathbf{q}\cdot(\mathbf{x}_\alpha - \mathbf{x}_\beta))}{Kq^2} \right] \\ &= \exp\left[ \frac{1}{K} \sum_{\alpha<\beta} q_\alpha q_\beta C(\mathbf{x}_\alpha - \mathbf{x}_\beta) \right] , \end{aligned}$$

where

$$C(\mathbf{x}) = \int \frac{d^2q}{(2\pi)^2} \frac{1-\cos(\mathbf{q}\cdot\mathbf{x})}{q^2} = \frac{1}{2\pi}\ln\frac{|\mathbf{x}|}{a},$$

is the Coulomb interaction in two dimensions, with a short distance cutoff $a$.

(b) Prove that

$$\left\langle |h(\mathbf{x})-h(\mathbf{y})|^2 \right\rangle = -\left.\frac{d^2}{dk^2} G_k(\mathbf{x}-\mathbf{y})\right|_{k=0},$$

where $G_k(\mathbf{x}-\mathbf{y}) = \left\langle \exp\left[ik\big(h(\mathbf{x})-h(\mathbf{y})\big)\right]\right\rangle$.

• From the definition of $G_k(\mathbf{x}-\mathbf{y})$,

$$\frac{d^2}{dk^2} G_k(\mathbf{x}-\mathbf{y}) = -\left\langle [h(\mathbf{x})-h(\mathbf{y})]^2 \exp\left[ik\left(h(\mathbf{x})-h(\mathbf{y})\right)\right]\right\rangle.$$

Setting $k$ to zero results in the identity

$$\left\langle [h(\mathbf{x})-h(\mathbf{y})]^2 \right\rangle = -\left.\frac{d^2}{dk^2} G_k(\mathbf{x}-\mathbf{y})\right|_{k=0}.$$

(c) Use the results in (a) to calculate $G_k(\mathbf{x}-\mathbf{y})$ in perturbation theory to order of $y_0^2$. (Hint: Set $\cos(2\pi h) = \left(e^{2i\pi h}+e^{-2i\pi h}\right)/2$. The first order terms vanish according to the result in (a), while the second order contribution is identical in structure to that of the Coulomb gas described in this chapter.)

• Following the hint, we write the perturbation as

$$-U = y_0 \int d^2x \cos(2\pi h) = \frac{y_0}{2}\int d^2x \left[e^{2i\pi h}+e^{-2i\pi h}\right].$$

The perturbation expansion for $G_k(\mathbf{x}-\mathbf{y}) = \langle \exp[ik(h(\mathbf{x})-h(\mathbf{y}))]\rangle \equiv \langle \mathcal{G}_k(\mathbf{x}-\mathbf{y})\rangle$ is calculated as

$$\begin{aligned}\langle \mathcal{G}_k\rangle = &\langle \mathcal{G}_k\rangle_0 - \left(\langle \mathcal{G}_k U\rangle_0 - \langle \mathcal{G}_k\rangle_0 \langle U\rangle_0\right) \\ &+ \frac{1}{2}\left(\left\langle \mathcal{G}_k U^2\right\rangle_0 - 2\langle \mathcal{G}_k U\rangle_0 \langle U\rangle_0 + 2\langle \mathcal{G}_k\rangle_0 \langle U\rangle_0^2 - \langle \mathcal{G}_k\rangle_0 \left\langle U^2\right\rangle_0\right) + \mathcal{O}\left(U^3\right).\end{aligned}$$

From part (a),

$$\langle U\rangle_0 = \langle \mathcal{G}_k U\rangle_0 = 0,$$

and

$$\langle \mathcal{G}_k\rangle_0 = \exp\left[-\frac{k^2}{K}C(\mathbf{x}-\mathbf{y})\right] = \left(\frac{|\mathbf{x}-\mathbf{y}|}{a}\right)^{-\frac{k^2}{2\pi K}}.$$

Furthermore,

$$\begin{aligned}\left\langle U^2\right\rangle_0 &= \frac{y_0^2}{2}\int d^2\mathbf{x}'d^2\mathbf{x}''\left\langle \exp\left[2i\pi\left(h\left(\mathbf{x}'\right)-h\left(\mathbf{x}''\right)\right)\right]\right\rangle_0 \\ &= \frac{y_0^2}{2}\int d^2\mathbf{x}'d^2\mathbf{x}''\left\langle \mathcal{G}_{2\pi}\left(\mathbf{x}'-\mathbf{x}''\right)\right\rangle_0 = \frac{y_0^2}{2}\int d^2\mathbf{x}'d^2\mathbf{x}''\exp\left[-\frac{\left(2\pi\right)^2}{K}C\left(\mathbf{x}'-\mathbf{x}''\right)\right],\end{aligned}$$

and similarly,

$$\begin{aligned}&\left\langle \exp\left[ik\left(h\left(\mathbf{x}\right)-h\left(\mathbf{y}\right)\right)\right]U^2\right\rangle_0 = \\ &= \frac{y_0^2}{2}\int d^2\mathbf{x}'d^2\mathbf{x}''\exp\left\{-\frac{k^2}{K}C\left(\mathbf{x}-\mathbf{y}\right)-\frac{\left(2\pi\right)^2}{K}C\left(\mathbf{x}'-\mathbf{x}''\right)\right. \\ &\left.+\frac{2\pi k}{K}\left[C\left(\mathbf{x}-\mathbf{x}'\right)+C\left(\mathbf{y}-\mathbf{x}''\right)\right]-\frac{2\pi k}{K}\left[C\left(\mathbf{x}-\mathbf{x}''\right)+C\left(\mathbf{y}-\mathbf{x}'\right)\right]\right\}.\end{aligned}$$

Thus, the second order part of $G_k\left(\mathbf{x}-\mathbf{y}\right)$ is

$$\begin{aligned}&\frac{y_0^2}{4}\exp\left[-\frac{k^2}{K}C\left(\mathbf{x}-\mathbf{y}\right)\right]\int d^2\mathbf{x}'d^2\mathbf{x}''\exp\left[-\frac{4\pi^2}{K}C\left(\mathbf{x}'-\mathbf{x}''\right)\right]\cdot \\ &\cdot\left\{\exp\left[\frac{2\pi k}{K}\left(C\left(\mathbf{x}-\mathbf{x}'\right)+C\left(\mathbf{y}-\mathbf{x}''\right)-C\left(\mathbf{x}-\mathbf{x}''\right)-C\left(\mathbf{y}-\mathbf{x}'\right)\right)\right]-1\right\},\end{aligned}$$

and

$$G_k\left(\mathbf{x}-\mathbf{y}\right)=e^{-\frac{k^2}{K}C\left(\mathbf{x}-\mathbf{y}\right)}\left\{1+\frac{y_0^2}{4}\int d^2\mathbf{x}'d^2\mathbf{x}''e^{-\frac{4\pi^2}{K}C\left(\mathbf{x}'-\mathbf{x}''\right)}\left(e^{\frac{2\pi k}{K}\mathcal{D}}-1\right)+\mathcal{O}\left(y_0^4\right)\right\},$$

where

$$\mathcal{D}=C\left(\mathbf{x}-\mathbf{x}'\right)+C\left(\mathbf{y}-\mathbf{x}''\right)-C\left(\mathbf{x}-\mathbf{x}''\right)-C\left(\mathbf{y}-\mathbf{x}'\right).$$

(d) Write the perturbation result in terms of an effective interaction $K$, and show that perturbation theory fails for $K$ larger than a critical $K_c$.

• The above expression for $G_k\left(\mathbf{x}-\mathbf{y}\right)$ is very similar to that of obtained in dealing with the renormalization of the Coulomb gas of vortices in the XY model. Following the steps in the lecture notes, without further calculations, we find

$$\begin{aligned}G_k\left(\mathbf{x}-\mathbf{y}\right) &= e^{-\frac{k^2}{K}C\left(\mathbf{x}-\mathbf{y}\right)}\left\{1+\frac{y_0^2}{4}\times\frac{1}{2}\left(\frac{2\pi k}{K}\right)^2\times C\left(\mathbf{x}-\mathbf{y}\right)\times 2\pi\int drr^3e^{-\frac{2\pi\ln(r/a)}{K}}\right\} \\ &= e^{-\frac{k^2}{K}C\left(\mathbf{x}-\mathbf{y}\right)}\left\{1+\frac{\pi^3k^2}{K^2}y_0^2C\left(\mathbf{x}-\mathbf{y}\right)\int drr^3e^{-\frac{2\pi\ln(r/a)}{K}}\right\}.\end{aligned}$$

The second order term can be exponentiated to contribute to an effective coupling constant $K_{\rm eff}$, according to

$$\frac{1}{K_{\rm eff}} = \frac{1}{K} - \frac{\pi^3}{K^2} a^{2\pi/K} y_0^2 \int_a^\infty dr r^{3-2\pi/K}.$$

Clearly, the perturbation theory is inconsistent if the above integral diverges, *i.e.* if

$$K > \frac{\pi}{2} \equiv K_c.$$

(e) Recast the perturbation result in part (d) into renormalization group equations for $K$ and $y_0$, by changing the "lattice spacing" from $a$ to $ae^\ell$.

• After dividing the integral into two parts, from $a$ to $ab$ and from $ab$ to $\infty$, respectively, and rescaling the variable of integration in the second part, in order to retrieve the usual limits of integration, we have

$$\frac{1}{K_{\rm eff}} = \frac{1}{K} - \frac{\pi^3}{K^2} a^{2\pi/K} y_0^2 \int_a^{ab} dr r^{3-2\pi/K} - \frac{\pi^3}{K^2} a^{2\pi/K} \times y_0^2 b^{4-2\pi/K} \times \int_a^\infty dr r^{3-2\pi/K}.$$

(To order $y_0^2$, we can indifferently write $K$ or $K'$ (defined below) in the last term.) In other words, the coarse-grained system is described by an interaction identical in form, but parameterized by the renormalized quantities

$$\frac{1}{K'} = \frac{1}{K} - \frac{\pi^3}{K^2} a^{2\pi/K} y_0^2 \int_a^{ab} dr r^{3-2\pi/K},$$

and

$$y_0'^{\,2} = b^{4-2\pi/K} y_0^2.$$

With $b = e^\ell \approx 1 + \ell$, these RG relations are written as the following differential equations, which describe the renormalization group flows

$$\begin{cases} \dfrac{dK}{d\ell} = \pi^3 a^4 y_0^2 + \mathcal{O}\left(y_0^4\right) \\ \dfrac{dy_0}{d\ell} = \left(2 - \dfrac{\pi}{K}\right) y_0 + \mathcal{O}\left(y_0^3\right) \end{cases}.$$

(f) Using the recursion relations, discuss the phase diagram and phases of this model.

• These RG equations are similar to those of the XY model, with $K$ (here) playing the role of $T$ in the Coulomb gas. For non-vanishing $y_0$, $K$ is relevant, and thus flows to larger and larger values (outside of the perturbative domain) if $y_0$ is also relevant ($K > \pi/2$), suggesting a smooth phase at low temperatures ($T \sim K^{-1}$). At small values of $K$, $y_0$ is

irrelevant, and the flows terminate on a fixed line with $y_0 = 0$ and $K \leq \pi/2$, corresponding to a rough phase at high temperatures.

(g) For large separations $|\mathbf{x} - \mathbf{y}|$, find the magnitude of the discontinuous jump in $\left\langle |h(\mathbf{x}) - h(\mathbf{y})|^2 \right\rangle$ at the transition.

• We want to calculate the long distance correlations in the vicinity of the transition. Equivalently, we can compute the coarse-grained correlations. If the system is prepared at $K = \pi/2^-$ and $y_0 \approx 0$, under coarse-graining, $K \to \pi/2^-$ and $y_0 \to 0$, resulting in

$$G_k(\mathbf{x} - \mathbf{y}) \to \langle \mathcal{G}_k \rangle_0 = \exp\left[ -\frac{2k^2}{\pi} C(\mathbf{x} - \mathbf{y}) \right].$$

From part (b),

$$\left\langle [h(\mathbf{x}) - h(\mathbf{y})]^2 \right\rangle = -\left. \frac{d^2}{dk^2} G_k(\mathbf{x} - \mathbf{y}) \right|_{k=0} = \frac{4}{\pi} C(\mathbf{x} - \mathbf{y}) = \frac{2}{\pi^2} \ln|\mathbf{x} - \mathbf{y}|.$$

On the other hand, if the system is prepared at $K = \pi/2^+$, then $K \to \infty$ under the RG (assuming that the relevance of $K$ holds also away from the perturbative regime), and

$$\left\langle [h(\mathbf{x}) - h(\mathbf{y})]^2 \right\rangle \to 0.$$

Thus, the magnitude of the jump in $\left\langle [h(\mathbf{x}) - h(\mathbf{y})]^2 \right\rangle$ at the transition is

$$\frac{2}{\pi^2} \ln|\mathbf{x} - \mathbf{y}|.$$

********

**4.** *Roughening and duality:* Consider a discretized version of the Hamiltonian in the previous problem, in which for each site $i$ of a square lattice there is an integer valued height $h_i$. The Hamiltonian is

$$\beta\mathcal{H} = \frac{K}{2} \sum_{<i,j>} |h_i - h_j|^{\infty} \quad ,$$

where the "$\infty$" power means that there is no energy cost for $\Delta h = 0$; an energy cost of $K/2$ for $\Delta h = \pm 1$; and $\Delta h = \pm 2$ or higher *are not allowed* for neighboring sites. (This is known as the restricted solid on solid (RSOS) model.)

(a) Construct the dual model either diagrammatically, or by following these steps:

**(i)** Change from the $N$ site variables $h_i$, to the $2N$ bond variables $n_{ij} = h_i - h_j$. Show that the sum of $n_{ij}$ around any plaquette is constrained to be zero.

**(ii)** Impose the constraints by using the identity $\int_0^{2\pi} d\theta e^{i\theta n}/2\pi = \delta_{n,0}$, for integer $n$.

**(iii)** After imposing the constraints, you can sum freely over the bond variables $n_{ij}$ to obtain a dual interaction $\tilde{v}(\theta_i - \theta_j)$ between dual variables $\theta_i$ on neighboring plaquettes.

• (i) In terms of bond variables $n_{ij} = h_i - h_j$, the Hamiltonian is written as

$$-\beta\mathcal{H} = -\frac{K}{2}\sum_{\langle ij\rangle} |n_{ij}|^{\infty} .$$

Clearly,

$$\sum_{\text{any closed loop}} n_{ij} = h_{i_1} - h_{i_2} + h_{i_2} - h_{i_3} + \cdots + h_{i_{n-1}} - h_{i_n} = 0,$$

since $h_{i_1} = h_{i_n}$ for a closed path.

(ii) This constraint, applied to the $N$ plaquettes, reduces the number of degrees of freedom from an apparent $2N$ (bonds), to the correct figure $N$, and the partition function becomes

$$Z = \sum_{\{n_{ij}\}} e^{-\beta\mathcal{H}} \prod_{\alpha} \delta_{\sum_{\langle ij\rangle} n_{ij}^{\alpha},0},$$

where the index $\alpha$ labels the $N$ plaquettes, and $n_{ij}^{\alpha}$ is non-zero and equal to $n_{ij}$ only if the bond $\langle ij\rangle$ belongs to plaquette $\alpha$. Expressing the Kronecker delta in its exponential representation, we get

$$Z = \sum_{\{n_{ij}\}} e^{-\frac{K}{2}\sum_{\langle ij\rangle}|n_{ij}|^{\infty}} \prod_{\alpha} \left( \int_0^{2\pi} \frac{d\theta_{\alpha}}{2\pi} e^{i\theta_{\alpha}\sum_{\langle ij\rangle} n_{ij}^{\alpha}} \right).$$

(iii) As each bond belongs to two neighboring plaquettes, we can label the bonds by $\alpha\beta$ rather than $ij$, leading to

$$\begin{aligned} Z &= \left( \prod_{\gamma} \int_0^{2\pi} \frac{d\theta_{\gamma}}{2\pi} \right) \sum_{\{n_{\alpha\beta}\}} \exp\left( \sum_{\langle\alpha\beta\rangle} \left\{ -\frac{K}{2} |n_{\alpha\beta}|^{\infty} + i\left(\theta_{\alpha} - \theta_{\beta}\right) n_{\alpha\beta} \right\} \right) \\ &= \left( \prod_{\gamma} \int_0^{2\pi} \frac{d\theta_{\gamma}}{2\pi} \right) \prod_{\langle\alpha\beta\rangle} \sum_{n_{\alpha\beta}} \exp\left( \left\{ -\frac{K}{2} |n_{\alpha\beta}|^{\infty} + i\left(\theta_{\alpha} - \theta_{\beta}\right) n_{\alpha\beta} \right\} \right). \end{aligned}$$

Note that if all plaquettes are traversed in the same sense, the variable $n_{\alpha\beta}$ occurs in opposite senses (with opposite signs) for the constraint variables $\theta_\alpha$ and $\theta_\beta$ on neighboring plaquettes. We can now sum freely over the bond variables, and since

$$\sum_{n=0,+1,-1} \exp\left(-\frac{K}{2}|n| + i\left(\theta_\alpha - \theta_\beta\right)n\right) = 1 + 2e^{-\frac{K}{2}}\cos\left(\theta_\alpha - \theta_\beta\right),$$

we obtain

$$Z = \left(\prod_\gamma \int_0^{2\pi} \frac{d\theta_\gamma}{2\pi}\right) \exp\left(\sum_{\langle\alpha\beta\rangle} \ln\left[1 + 2e^{-\frac{K}{2}}\cos\left(\theta_\alpha - \theta_\beta\right)\right]\right).$$

(b) Show that for large $K$, the dual problem is just the XY model. Is this conclusion consistent with the renormalization group results of the previous problem? (Also note the connection with the loop model considered in the problems of the previous chapter.)

• This is the loop gas model introduced earlier. For $K$ large,

$$\ln\left[1 + 2e^{-\frac{K}{2}}\cos\left(\theta_\alpha - \theta_\beta\right)\right] \approx 2e^{-\frac{K}{2}}\cos\left(\theta_\alpha - \theta_\beta\right),$$

and

$$Z = \left(\prod_\gamma \int_0^{2\pi} \frac{d\theta_\gamma}{2\pi}\right) e^{\sum_{\langle\alpha\beta\rangle} 2e^{-\frac{K}{2}}\cos(\theta_\alpha - \theta_\beta)}.$$

This is none other than the partition function for the XY model, if we identify

$$K_{\mathrm{XY}} = 4e^{-\frac{K}{2}},$$

consistent with the results of problem 1, in which we found that the low temperature behavior in the roughening problem corresponds to the high temperature phase in the XY model, and vice versa.

(c) Does the one dimensional version of this Hamiltonian, i.e. a 2d interface with

$$-\beta\mathcal{H} = -\frac{K}{2}\sum_i |h_i - h_{i+1}|^\infty \quad ,$$

have a roughening transition?

• In one dimension, we can directly sum the partition function, as

$$\begin{aligned} Z &= \sum_{\{h_i\}} \exp\left(-\frac{K}{2}\sum_i |h_i - h_{i+1}|^\infty\right) = \sum_{\{n_i\}} \exp\left(-\frac{K}{2}\sum_i |n_i|^\infty\right) \\ &= \prod_i \sum_{n_i} \exp\left(-\frac{K}{2}|n_i|^\infty\right) = \prod_i \left(1 + 2e^{-K/2}\right) = \left(1 + 2e^{-K/2}\right)^N, \end{aligned}$$

($n_i = h_i - h_{i+1}$). The expression thus obtained is an analytic function of $K$ (for $0 < K < \infty$), in the $N \to \infty$ limit, and there is therefore no phase transition at a finite non-zero temperature.

********

---

**5.** *Nonlinear $\sigma$ model with long–range interactions:* Consider unit $n$-component spins, $\vec{s}(\mathbf{x}) = (s_1, s_2, \cdots, s_n)$ with $|\vec{s}(\mathbf{x})|^2 = \sum_i s_i(\mathbf{x})^2 = 1$, interacting via a Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x} \int d^d\mathbf{y}\, K(|\mathbf{x}-\mathbf{y}|)\, \vec{s}(\mathbf{x}) \cdot \vec{s}(\mathbf{y}) \quad .$$

(a) The long-range interaction, $K(x)$, is the Fourier transform of $Kq^\omega/2$ with $\omega < 2$. What kind of asymptotic decay of interactions at long distances is consistent with such decay? (Dimensional analysis is sufficient for the answer, and no explicit integrations are required.)

• The dependence of the interaction strength on the separation $x$ between spins in real space is obtained by Fourier transformation, as

$$K(x) = \int \frac{d^dq}{(2\pi)^d}\, e^{i\mathbf{q}\cdot\mathbf{x}} \left(\frac{Kq^\omega}{2}\right) \propto \frac{K}{|x|^{d+\omega}} \quad \text{for } |x| \to \infty,$$

where the asymptotic dependence on $|x|$ is obtained by dimensional analysis.

(b) Close to zero temperature we can set $\vec{s}\,(\mathbf{x}) = (\vec{\pi}\,(\mathbf{x}), \sigma(\mathbf{x}))$, where $\vec{\pi}\,(\mathbf{x})$ is an $n-1$ component vector representing *small fluctuations* around the ground state. Find the effective Hamiltonian for $\vec{\pi}\,(\mathbf{x})$ after integrating out $\{\sigma(\mathbf{x})\}$.

• Employing the constraint of unit vectors, each integration over $\vec{s}$ can be decomposed as

$$\int d^n\vec{s}\; \delta\left(s^2-1\right) \to \int d^{n-1}\vec{\pi}\;\, d\sigma\; \delta\left(\pi^2+\sigma^2-1\right) \to$$
$$\int d^{n-1}\vec{\pi}\;\, d\sigma\; \delta\left[\left(\sigma-\sqrt{1-\pi^2}\right)\left(\sigma+\sqrt{1-\pi^2}\right)\right] \to \int \frac{d^{n-1}\vec{\pi}\;\, d\sigma}{2\sqrt{1-\pi^2}}\; \delta\left(\sigma-\sqrt{1-\pi^2}\right).$$

The $\delta$-function can now be used to replace the field $\sigma$ with $\sqrt{1-\pi^2}$. This leads to a final partition function

$$Z = \int \mathcal{D}\vec{\pi}\,(\mathbf{x}) \exp\left[-\beta\mathcal{H}_{\rm eff}(\vec{\pi}\,)\right],$$

with the effective Hamiltonian

$$\beta\mathcal{H}_{\rm eff} = \int d^d\mathbf{x}\, d^d\mathbf{y}\, K(|\mathbf{x}-\mathbf{y}|) \left[\vec{\pi}\,(\mathbf{x}) \cdot \vec{\pi}\,(\mathbf{y}) + \sqrt{1-\pi^2(\mathbf{x})}\sqrt{1-\pi^2(\mathbf{y})}\right] +$$
$$\frac{\rho}{2} \int d^d\mathbf{x}\, \ln\left(\sqrt{1-\pi^2(\mathbf{x})}\right),$$

where $\rho = N/V$ is the density of degrees of freedom.

(c) Fourier transform the *quadratic part* of the above Hamiltonian focusing only on terms proportional to $K$, and hence calculate the expectation value $\langle \pi_i(\mathbf{q})\pi_j(\mathbf{q}')\rangle_0$.

• To leading order in temperature ($K^{-1}$), the unperturbed Hamiltonian (after Fourier transformation) is

$$\beta\mathcal{H}_0 = \int \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{Kq^\omega}{2}\ \vec{\pi}\,(\mathbf{q})\cdot\vec{\pi}\,(-\mathbf{q}).$$

From this Gaussian form, we can read

$$\langle \pi_i(\mathbf{q})\pi_j(\mathbf{q}')\rangle_0 = \frac{\delta_{ij}(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')}{Kq^\omega}.$$

We shall now construct a renormalization group by removing Fourier modes, $\vec{\pi}^{\,>}(\mathbf{q})$, with $\mathbf{q}$ in the shell $\Lambda/b < |\mathbf{q}| < \Lambda$.

(d) Calculate the coarse grained expectation value for $\langle\sigma\rangle_0^>$ to order of $\pi^2$ after removing these modes. Identify the scaling factor, $\vec{s}\,'(\mathbf{x}') = \vec{s}^{\,<}(\mathbf{x})/\zeta$, that restores $\vec{s}\,'$ to unit length.

• Due to thermal fluctuations of the modes $\vec{\pi}^{\,>}(\mathbf{q})$, the length of the spin is reduced to

$$\langle\sigma\rangle = 1 - \frac{1}{2}\langle\vec{\pi}\cdot\vec{\pi}\,\rangle = 1 - \frac{n-1}{2}\int_{\Lambda/b}^{\Lambda}\frac{d^dq}{(2\pi)^d}\ \frac{1}{Kq^\omega} = 1 - \frac{n-1}{2}\frac{K_d\Lambda^{d-\omega}}{K}\ \frac{1-b^{\omega-d}}{d-\omega}.$$

The renormalized spins must be rescaled by this factor to remain unit vectors. In particular, for an infinitesimal rescaling $b = 1+\delta\ell$,

$$\zeta = 1 - \frac{n-1}{2}\frac{K_d\Lambda^{d-\omega}}{K}\delta\ell.$$

(e) A simplifying feature of long–range interactions is that the coarse grained coupling constant is not modified by the perturbation, i.e. $\tilde{K} = K$ to all orders in a perturbative calculation. Use this information, along simple with dimensional analysis, to express the renormalized interaction, $K'(b)$, in terms of $K$, $b$, and $\zeta$.

• The parameters of the Hamiltonian are changed as $\mathbf{x} \to b\mathbf{x}'$ and $\vec{s} \to \zeta\vec{s}\,'$, resulting in a renormalization of the interaction constant to

$$K'(b) = Kb^{2d}\frac{1}{b^{d+\omega}}\zeta^2 = \zeta^2 b^{d-\omega}\ K.$$

(f) Obtain the *differential* RG equation for $T = 1/K$ by considering $b = 1 + \delta\ell$.

• Setting $b = 1 + \delta\ell$, and expanding to lowest order in $\delta\ell$ gives

$$K + \delta\ell \frac{dK}{d\ell} = [1 + (d-\omega)\delta\ell] \left[1 - \frac{n-1}{2} \frac{K_d \Lambda^{d-\omega}}{K} \delta\ell\right]^2 K.$$

The differential recursion relations are now obtained as

$$\frac{dK}{d\ell} = (d-\omega)K - (n-1)K_d\Lambda^{d-\omega}, \quad \Rightarrow \quad \frac{dT}{d\ell} = -(d-\omega)T + (n-1)K_d\Lambda^{d-\omega}T^2.$$

(g) For $d = \omega + \epsilon$, compute $T_c$ and the critical exponent $\nu$ to lowest order in $\epsilon$.

• Setting $dT/d\ell = 0$ gives

$$T^* = T_c = \frac{d-\omega}{(n-1)K_d\Lambda^{d-\omega}} = \frac{\epsilon}{(n-1)K_\omega} + \mathcal{O}(\epsilon^2).$$

Linearizing around this fixed point gives

$$\frac{d\delta T}{d\ell} = -(d-\omega)\delta T + 2(n-1)K_d\Lambda^{d-\omega}T^*\delta T = +\epsilon\delta T.$$

This corresponds to $y_t = \epsilon$ to lowest order, and hence $\nu = 1/y_t \approx 1/\epsilon$.

(h) Add a small symmetry breaking term, $-\vec{h} \cdot \int d^d\mathbf{x}\, \vec{s}(\mathbf{x})$, to the Hamiltonian. Find the renormalization of $h$ and identify the corresponding exponent $y_h$.

• Under the rescalings $\mathbf{x} \to b\mathbf{x}'$ and $\vec{s} \to \zeta\vec{s}'$, the magnetic field changes (in magnitude) to $h'(b) = b^d\zeta\ h$, and $b = 1 + \delta\ell$, we get

$$\frac{dh}{d\ell} = \left(d - \frac{n-1}{2} \frac{K_d\Lambda^{d-\omega}}{K}\right) h.$$

Evaluated at the fixed point $1/K^* = T^*$, this gives

$$y_h = d - \frac{d-\omega}{2} = \frac{d+\omega}{2}.$$

********

**6.** *The XY model in $2 + \epsilon$ dimensions:* The recursion relations of the XY model in two dimensions can be generalized to $d = 2 + \epsilon$ dimensions, and take the form:

$$\begin{cases} \dfrac{dT}{d\ell} = -\epsilon T + 4\pi^3 y^2 \\ \dfrac{dy}{d\ell} = \left(2 - \dfrac{\pi}{T}\right) y \end{cases}.$$

(a) Calculate the position of the fixed point for the finite temperature phase transition.

• There is a trivial fixed point at $T^* = y^* = 0$ which is the sink for flows inside the ordered phase. Flows in the high temperature phase go towards $y \to \infty$ and $T \to \infty$. In between there is a fixed point at finite $T^*$ and $y^*$, given by

$$T^* = \frac{\pi}{2}, \qquad \text{and} \qquad y^* = \sqrt{\frac{\epsilon}{8\pi^2}}.$$

[Note that the location of the fixed point is accurate only to order of $\sqrt{\epsilon}$.

(b) Obtain the eigenvalues at this fixed point to *lowest* contributing order in $\epsilon$.

• Linearizing the recursion relations in the vicinity of this fixed point gives

$$\frac{d}{d\ell}\begin{pmatrix}\delta T\\ \delta y\end{pmatrix} = \begin{pmatrix}-\epsilon & 8\pi^3 y^*\\ \dfrac{\pi y^*}{T^{*2}} & 2-\dfrac{\pi}{T^*}\end{pmatrix}\begin{pmatrix}\delta T\\ \delta y\end{pmatrix} = \begin{pmatrix}-\epsilon & 2\pi^2\sqrt{2\epsilon}\\ \dfrac{\sqrt{2\epsilon}}{\pi^2} & 0\end{pmatrix}\begin{pmatrix}\delta T\\ \delta y\end{pmatrix}.$$

The eigenvalues satisfy the equation $y^2 + \epsilon y - 4\epsilon = 0$. To lowest order, the eigenvalues are $\pm 2\sqrt{\epsilon}$. There is one irrelevant eigenvalue, the positive (relevant) eigevalue is identified with $y_T$.

(c) Estimate the exponents $\nu$ and $\alpha$ for the superfluid transition in $d = 3$ from these results. [Be careful in keeping track of only the lowest nontrivial power of $\epsilon$ in your expressions.]

• The divergence of the correlation length is controlled by the exponent $\nu = 1/y_T = 1/(2\sqrt{\epsilon})$, which evaluates to 1/2 for $\epsilon = 1$ in $d = 3$. The heat capacity is governed by the exponent $\alpha = 2 - d\nu$, whose correct expansion to order of $\sqrt{\epsilon}$ gives

$$\alpha = 2 - \frac{2+\epsilon}{y_T} = -\frac{1}{\sqrt{\epsilon}} + \mathcal{O}(1),$$

which evaluates to $\alpha = -1$ for $\epsilon = 1$ in $d = 3$.

********

**7.** *Symmetry breaking fields:* Let us investigate adding a term

$$-\beta\mathcal{H}_p = h_p \int d^2\mathbf{x} \cos\left(p\theta(\mathbf{x})\right),$$

to the XY model. There are a number of possible causes for such a symmetry breaking field: $p = 1$ is the usual 'magnetic field,' $p = 2$, 3, 4, and 6 could be due to couplings to an underlying lattice of rectangular, hexagonal, square, or triangular symmetry respectively.

As $h_p \to \infty$, the spin becomes discrete, taking one of $p$ possible values, and the model becomes equivalent to the previously discussed clock models.

(a) Assume that we are in the low temperature phase so that vortices are absent, i.e. the vortex fugacity $y$ is zero (in the RG sense). In this case, we can ignore the angular nature of $\theta$ and replace it with a scalar field $\phi$, leading to the partition function

$$Z = \int D\phi(\mathbf{x}) \exp\left\{ - \int d^2\mathbf{x} \left[ \frac{K}{2} (\nabla\phi)^2 + h_p \cos(p\phi) \right] \right\}.$$

This is known as the sine–Gordon model, and is equivalent to the roughening transition. Obtain the recursion relations for $h_p$ and $K$.

• The spin waves are represented by the angle $\phi(\mathbf{x})$, which ignoring its angular character can be integrated over all values. Forgetting the angular character is equivalent to leaving out vortices by setting their fugacity to $y = 0$. This leads to the above Sine-Gordon model. Following the steps of problem 3 of chapter 8, on the roughening transition, we then arrive to recursion relations

$$\begin{cases} \dfrac{dh_p}{d\ell} = \left( 2 - \dfrac{p^2}{4\pi K} \right) h_p, \\ \dfrac{dK^{-1}}{d\ell} = -\dfrac{\pi p^2 h_p^2}{2} K^{-2}. \end{cases}$$

(b) Show that once vortices are included, the recursion relations are

$$\begin{cases} \dfrac{dh_p}{d\ell} = \left( 2 - \dfrac{p^2}{4\pi K} \right) h_p, \\ \dfrac{dK^{-1}}{d\ell} = -\dfrac{\pi p^2 h_p^2}{2} K^{-2} + 4\pi^3 y^2, \\ \dfrac{dy}{d\ell} = (2 - \pi K)\, y. \end{cases}$$

• The usual recursion relation for the XY model (without the symmetry breaking field) are

$$\begin{cases} \dfrac{dK^{-1}}{d\ell} = 4\pi^3 y^2, \\ \dfrac{dy}{d\ell} = (2 - \pi K)\, y. \end{cases}$$

When both $h_p$ and $y$ are small, we can add the two sets of recursion relations to obtain the three equations above.

(c) Show that the above RG equations are only valid for $\frac{8\pi}{p^2} < K^{-1} < \frac{\pi}{2}$, and thus only apply for $p > 4$. Sketch possible phase diagrams for $p > 4$ and $p < 4$. In fact $p = 4$ is rather special as there is a marginal operator $h_4$, and the transition to the 4-fold phase (cubic anisotropy) has continuously varying critical exponents!

• The above perturbative recursion relations remain valid only if both $h_p$ and $y$ asymptotically go to zero under RG. Hence, we must require

$$\frac{8\pi}{p^2} < K^{-1} < \frac{\pi}{2}, \qquad \text{or} \qquad \frac{4}{p^2} < \eta = \frac{1}{2\pi K} < \frac{1}{4}.$$

Depending on the value of $p$, different types of behavior are possible:

*(1) For* $p > 4$ (e.g. $p = 6$), there is a finite range of $K^{-1}$ over which the above inequalities hold. In this range, it is possible to have a phase which is controlled by spin-waves. The spin-correlations in this phase decay algebraically with an exponent that ranges from $4/p^2$ to $1/4$. At higher temperatures vortices appear and the resulting disordered phase has algebraic decay of correlations. At low-enough temperatures, $h_p$ becomes relevant, fixing the spins along one of $p$ discrete directions. This phase (like the Ising model) has no Goldstone modes.

*(2) For* $p < 4$ (e.g. the Ising model at $p = 2$) there can be no intermediate phase with algebraic order. We expect the usual transition from ordered to disordered states (in the Ising universality for $p = 2$, and the 3-state Potts universality for $p = 4$.

*(3) The case* $p = 4$ is rather special. The field $h_4$ is a marginal operator and the exponents vary continuously at the transition between ordered and disordered phases.

********

**8.** *Inverse-square interactions:* Consider a scalar field $s(x)$ in one-dimension, subject to an energy

$$-\beta\mathcal{H}_s = \frac{K}{2}\int dxdy \frac{s(x)s(y)}{|x-y|^2} + \int dx \Phi[s(x)].$$

The local energy $\Phi[s]$ strongly favors $s(x) = \pm 1$ (e.g. $\Phi[s] = g(s^2-1)^2$, with $g \gg 1$).

(a) With $K > 0$, the ground state is ferromagnetic. Estimate the energy cost of a single domain wall in a chain of length $L$. You may assume that the transition from $s = +1$ to $s = -1$ occurs over a short distance cutoff $a$.

• Let us consider a domain wall at $x = 0$, separating $s = +1$ on the left, and $s = -1$ on the right. Spins at $-x$ and $+y$ are now connected by a dissatisfied bond at the energy cost of $2K/|x+y|^2$. Note that there are two factors of 2 involved: the first occurs because each

pair of points is considered once, rather than twice in $(K/2)\int dxdy$, the second accounts for the factor of 2 in the difference of energies of a satisfied and unsatisfied bond. Adding up all such energies leads to an energy cost of

$$\beta E_1 = 2K \int_{a/2}^{L/2} dxdy \frac{1}{|x+y|^2} \approx 2K \ln\left(\frac{L}{a}\right),$$

where we have ignored subleading corrections.

(b) From the probability of the formation of a single kink, obtain a lower bound for the critical coupling $K_c$, separating ordered and disordered phases.

• Since the kink can be placed in anyone of approximately $(L/a)$ locations along the chain, its weight scales as

$$W_1 \propto \left(\frac{L}{a}\right) \exp\left[-2K \ln\left(\frac{L}{a}\right)\right] \propto \left(\frac{L}{a}\right)^{1-2K}.$$

Even if all other excitations are suppressed, a single kink configuration will spontaneously appear for $K < 1/2$ due to its large weight for $(L/a) \to \infty$. Since other excitations are also possible, this is only a lower bound on the critical value of $K$.

(c) Show that the energy of a dilute set of domain walls located at positions $\{x_i\}$ is given by

$$-\beta \mathcal{H}_Q = 4K \sum_{i<j} q_i q_j \ln\left(\frac{|x_i - x_j|}{a}\right) + \ln y_0 \sum_i |q_i|,$$

where $q_i = \pm 1$ depending on whether $s(x)$ increases or decreases at the domain wall. (Hints: Perform integrations by part, and coarse-grain to size $a$. The function $\Phi[s]$ only contributes to the core energy of the domain wall, which results in the fugacity $y_0$.)

• Performing two integrations by part on the nonlocal part of the Hamiltonian gives

$$\frac{K}{2} \int dxdy \frac{s(x)s(y)}{|x-y|^2} = \frac{K}{2} \int dxdy \partial_x s(x) \partial_y s(y) \ln\left(\frac{|x-y|}{a}\right).$$

(The additional factor of $\ln a$ makes no difference if $s(x)$ is the same at the edges.) The function $s(x)$ is mostly constant at low temperatures, except at the domain walls where it rapidly switches between -1 and +1 over a distance $a$. Assuming that such domain walls are well-separated, we can integrate out fluctuations over scales shorter than $a$. In the vicinity of a kink at $x_i$, integration gives $\int dx \partial_x s(x) = 2q_i$, where $q_i = \pm 1$. The nonlocal part then results in an interaction

$$\frac{K}{2} \sum_{i,j} (2q_i)(2q_j) \ln\left(\frac{|x_i - x_j|}{a}\right) = 4K \sum_{i<j} q_i q_j \ln\left(\frac{|x_i - x_j|}{a}\right) + \text{core contributions}.$$

There are also contributions to the core energy from the local part $\Phi[s]$. The core energies are simply proportional to the number of domain walls (irrespective of their sign), resulting in

$$-\beta\mathcal{H}_Q = 4K\sum_{i<j} q_i q_j \ln\left(\frac{|x_i - x_j|}{a}\right) + \ln y_0 \sum_i |q_i|.$$

(d) The logarithmic interaction between two opposite domain walls at a large distance $L$, is reduced due to screening by other domain walls in between. This interaction can be calculated perturbatively in $y_0$, and to lowest order is described by an effective coupling (see later)

$$K \to K_{eff} = K - 2Ky_0^2 \int_a^\infty dr r \left(\frac{a}{r}\right)^{4K} + \mathcal{O}(y_0^4). \tag{1}$$

By changing the cutoff from $a$ to $ba = (1+\delta\ell)a$, construct differential recursion relations for the parameters $K$ and $y_0$.

• Let us break the above integration into two parts, to get

$$\begin{aligned} K_{eff} &= K - 2Ky_0^2 \int_a^{ba} dr r \left(\frac{a}{r}\right)^{4K} - 2Ky_0^2 \int_{ba}^\infty dr r \left(\frac{a}{r}\right)^{4K} + \mathcal{O}(y_0^4) \\ &= K' - 2K' y_0^2 b^{2-4K} \int_a^\infty dr' r' \left(\frac{a}{r'}\right)^{4K} + \mathcal{O}(y_0^4), \end{aligned}$$

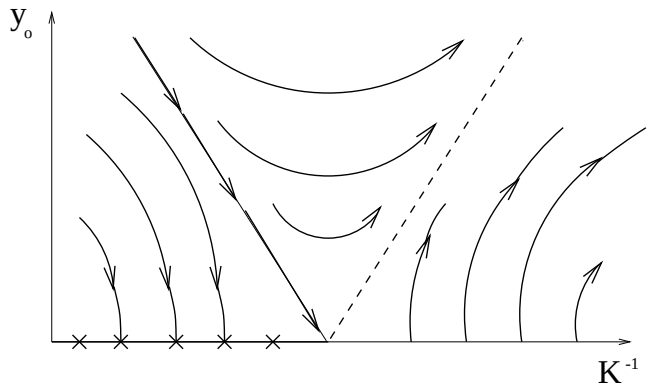
where $r' = br$, and we can identify the renormalized parameters

$$\begin{cases} K' = K - 2Ky_0^2 \displaystyle\int_a^{ba} dr r \left(\frac{a}{r}\right)^{4K} \approx K - 2Ky_0^2 a^2 \delta\ell \\ y_o' = y_0 b^{1-2K} \approx y_0 + (1-2K) y_0 \delta\ell \end{cases}.$$

The last two equations are obtained by setting $b = 1 + \delta\ell$, and result in the differential recursion relations

$$\begin{cases} \dfrac{dK}{d\ell} = -2Ky_0^2 a^2 \\ \dfrac{dy_0}{d\ell} = (1-2K) y_0 \end{cases}.$$

(e) Sketch the renormalization group flows as a function of $T = K^{-1}$ and $y_0$, and discuss the phases of the model.




•

The trajectories at low temperature with $y_0 \to 0$, correspond to the ferromagnetic phase, where there are asymptotically no domain walls. (Note that because of the long-range interactions, it is possible to have an ordered phase even in $d = 1$.) The high temperature phase, where $y_0$ becomes large, is disordered.

(f) Derive the effective interaction given above as Eq.(1). (Hint: This is somewhat easier than the corresponding calculation for the two-dimensional Coulomb gas, as the charges along the chain must alternate.)

• Consider two domain walls separated by a distance $L$. The direct interaction between them can be partly screened by another pair of domain walls. Indicating the distances of the screening pair of domain walls from the left-most one by $R + r/2$ and $R - r/2$, the effective interaction is calculated as

$$\begin{aligned} e^{-\beta V(L)} &= e^{-4K\ln(L/a)} \times \frac{\left[1 + y_0^2 \int dR dr\, e^{-4K\ln(r/a)+4KD(R,r)} + \mathcal{O}(y_0^4)\right]}{\left[1 + y_0^2 \int dR dr\, e^{-4K\ln(r/a)} + \mathcal{O}(y_0^4)\right]} \\ &= e^{-4K\ln(L/a)} \left[1 + y_0^2 \int_a^L dR dr\, e^{-4K\ln(r/a)} \left(e^{4KD(R,r)} - 1\right) + \mathcal{O}(y_0^4)\right], \end{aligned}$$

with the dipole-dipole interaction

$$\begin{aligned} D(R,r) &= \ln\left(R + \frac{r}{2}\right) - \ln\left(R - \frac{r}{2}\right) - \ln\left(L - R + \frac{r}{2}\right) + \ln\left(L - R - \frac{r}{2}\right) \\ &= r\frac{d}{dR}\left[\ln(R) + \ln(L-R)\right] + \mathcal{O}(r^3). \end{aligned}$$

We next set $\exp(4KD) - 1 = 4KD + \mathcal{O}(D^2)$, and use the above form for $D$ to obtain

$$e^{-\beta V(L)} = e^{-4K\ln(L/a)} \left\{ \left[1 + y_0^2 \int_a^L dr dR\, e^{-4K\ln(r/a)} \times \left(4Kr\frac{d}{dR}\left[\ln(R) + \ln(L-R)\right]\right) + \mathcal{O}(r^3) + \mathcal{O}(y_0^4)\right] \right\}.$$

The integration over $R$ is now easily performed to give

$$e^{-\beta V(L)} = e^{-4K\ln(L/a)}\left[1 + 8Ky_0^2 \ln\left(\frac{L}{a}\right)\int_a^L drre^{-4K\ln(r/a)} + \mathcal{O}(y_0^4)\right]$$
$$\approx \exp\left\{-4K\ln\left(\frac{r}{a}\right)\left[1 - 2y_0^2\int_a^L drr\left(\frac{a}{r}\right)^{4K} + \mathcal{O}(y_0^4)\right]\right\},$$

from which we can read off the expression for the effective coupling $K_{eff}$ given earlier.

********

**9.** *Melting:* The elastic energy cost of a deformation $u_i(\mathbf{x})$, of an isotropic lattice is

$$-\beta\mathcal{H} = \frac{1}{2}\int d^d\mathbf{x}\left[2\mu u_{ij}u_{ij} + \lambda u_{ii}u_{jj}\right],$$

where $u_{ij}(\mathbf{x}) = \left(\partial_i u_j + \partial_j u_i\right)/2$, is the strain tensor.

(a) Express the energy in terms of the Fourier transforms $u_i(\mathbf{q})$, and find the normal modes of vibrations.

• In terms of Fourier modes $u_{ij}(\mathbf{q}) = i[q_i u_j(\mathbf{q}) + q_j u_i(\mathbf{q})]/2$, and

$$-\beta\mathcal{H} = \frac{1}{2}\int\frac{d^d\mathbf{q}}{(2\pi)^d}\left[2\mu u_{ij}(\mathbf{q})u_{ij}(-\mathbf{q}) + \lambda u_{ii}(\mathbf{q})u_{jj}(-\mathbf{q})\right]$$
$$= \frac{1}{2}\int\frac{d^d\mathbf{q}}{(2\pi)^d}\left[\frac{\mu}{2}\left(q_i u_j(\mathbf{q}) + q_j u_i(\mathbf{q})\right)\left(q_i u_j^*(\mathbf{q}) + q_j u_i^*(\mathbf{q})\right) + \lambda q_i u_i(\mathbf{q}) q_j u_j^*(\mathbf{q})\right]$$
$$= \frac{1}{2}\int\frac{d^d\mathbf{q}}{(2\pi)^d}\left[\mu q^2\delta_{ij} + (\mu+\lambda)q_i q_j\right]u_i(\mathbf{q})u_j^*(\mathbf{q}) \equiv \frac{1}{2}\int\frac{d^d\mathbf{q}}{(2\pi)^d}M_{ij}(\mathbf{q})u_i(\mathbf{q})u_j^*(\mathbf{q}).$$

The normal modes are obtained by making the matrix $M_{ij}(\mathbf{q})$ diagonal. The *longitudinal phonons* have $\vec{q} \parallel \vec{u}_\ell$, while for *transverse phonons* have $\vec{q} \perp \vec{u}_t$, resulting in

$$-\beta\mathcal{H} = \frac{1}{2}\int\frac{d^d\mathbf{q}}{(2\pi)^d}\left[(2\mu+\lambda)q^2|\vec{u}_\ell(\mathbf{q})|^2 + \mu q^2|\vec{u}_t(\mathbf{q})|^2\right].$$

(b) Calculate the expectation value $\langle u_i(\mathbf{q})u_j(\mathbf{q}')\rangle$.

• From the Gaussian form of the interaction we conclude

$$\langle u_i(\mathbf{q})u_j(\mathbf{q}')\rangle = (2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')M_{ij}^{-1}(\mathbf{q}).$$

By symmetry, the inverse matrix must also have the form $M^{-1}_{ij}(\mathbf{q}) = A(\mathbf{q})\delta_{ij} + B(\mathbf{q})q_iq_j$, with the functions $A(\mathbf{q})$ and $B(\mathbf{q})$ obtained from

$$\begin{aligned}\delta_{ik} &= \sum_j M^{-1}_{ij}(\mathbf{q})M_{jk}(\mathbf{q}) = \left[A(\mathbf{q})\delta_{ij} + B(\mathbf{q})q_iq_j\right]\left[\mu q^2\delta_{jk} + (\mu+\lambda)q_jq_k\right] \\ &= \mu q^2 A\delta_{ik} + \left[(\mu+\lambda)A + \mu q^2 B + (\mu+\lambda)q^2 B\right] q_iq_k.\end{aligned}$$

Equating the two sides of the above equation gives

$$A(\mathbf{q}) = \frac{1}{\mu q^2}, \quad \text{and} \quad B(\mathbf{q}) = -\frac{(\mu+\lambda)A}{(2\mu+\lambda)q^2} = -\frac{\mu+\lambda}{\mu(2\mu+\lambda)q^4},$$

resulting in

$$M^{-1}_{ij}(\mathbf{q}) = \frac{1}{\mu q^2}\left[\delta_{ij} - \frac{\mu+\lambda}{2\mu+\lambda}\frac{q_iq_j}{q^2}\right].$$

(c) Assuming a short-distance cutoff of the order of the lattice spacing $a$, calculate $U^2(\mathbf{x}) \equiv \left\langle (\vec{u}(\mathbf{x}) - \vec{u}(\mathbf{0}))^2 \right\rangle$.

• Using the above expectation value of Fourier components, we obtain

$$\begin{aligned}U^2(\mathbf{x}) &= \sum_i \langle (u_i(\mathbf{x}) - u_i(\mathbf{0}))\,(u_i(\mathbf{x}) - u_i(\mathbf{0}))\rangle \\ &= \int \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{2-2\cos(\mathbf{q}\cdot\mathbf{x})}{\mu q^2}\sum_i\left(\delta_{ii} - \frac{\mu+\lambda}{2\mu+\lambda}\frac{q_iq_i}{q^2}\right) \\ &= \frac{2}{\mu}\left(d - \frac{\mu+\lambda}{2\mu+\lambda}\right)C_d(\mathbf{x}).\end{aligned}$$

The Coulomb potential $C_d(\mathbf{x})$ satisfies $\nabla^2 C_d(\mathbf{x}) = \delta^d(\mathbf{x})$, and is given by $C_d(\mathbf{x}) = |\mathbf{x}|^{2-d}/[S_d(2-d)] + c_0$. Choosing the constant of integration such that $U^2(a) = 0$, where $a$ is the short distance cutoff, leads to

$$U^2(\mathbf{x}) = \frac{2}{\mu}\left(d - \frac{\mu+\lambda}{2\mu+\lambda}\right)\frac{x^{2-d} - a^{2-d}}{S_d(2-d)}.$$

(d) The (heuristic) *Lindemann criterion* states that the lattice melts when deformations grow to a fraction of the lattice spacing, i.e. for $\lim_{|\mathbf{x}|\to\infty} U(\mathbf{x}) = c_L a$. Assuming that $\mu = \hat{\mu}/(k_BT)$ and $\lambda = \hat{\lambda}/(k_BT)$, use the above criterion to calculate the melting temperature $T_m$. Comment on the behavior of $T_m$ as a function of dimension $d$.

• In dimensions $d > 2$, the maximum extent of fluctuations is finite, and related to temperature $T$ by

$$\lim_{|\mathbf{x}|\to\infty} U^2(\mathbf{x}) = \frac{2k_BT}{\hat{\mu}} \left( d - \frac{\hat{\mu}+\hat{\lambda}}{2\hat{\mu}+\hat{\lambda}} \right) \frac{a^{2-d}}{S_d(d-2)}.$$

According to the Lindemann criterion, the melting temperature is obtained by equating the above expression to $(c_L a)^2$, leading to

$$T_m = \frac{(d-2)S_d c_L^2}{2k_B} \frac{\hat{\mu}a^d}{\left[d - (\hat{\mu}+\hat{\lambda})/(2\hat{\mu}+\hat{\lambda})\right]}.$$

Note that the melting temperature vanishes at $d = 2$. This is again a consequence of the divergence of fluctuations in $d \leq 2$ due to proliferation of (phonon) Goldstone modes. The Lindemann criterion is too crude to account for the quasi-long-range translational order in $d = 2$, and its destruction by dislocations.

********