CHAPTER V

Eigenfunctions and Eigen Values

In this chapter we shall develop Fourier's methods to their greatest generality and thereby open up the boundary value problems of physics to mathematical treatment. The most striking demonstration of the power of these methods was given in 1926 when Erwin Schrödinger recognized the quantum numbers as eigen values of his wave equation and thereby put the tools of modern analysis at the service of atomic physics. It was fortunate that he had the aid of his Zürich colleague Hermann Weyl who had, as the greatest pupil and later the successor of Hilbert in Göttingen, an essential part in the development of the theory of integral equations. However we should note that, while the viewpoint of *integral equations* is important for the rigorous mathematical foundation, in particular for the existence proofs for the eigenfunctions and their eigen values, the older viewpoint of *partial differential equations* leads to the same concepts in a natural manner. We shall start by demonstrating this with an example which was known long before integral equations.

§ 25. Eigen Values and Eigenfunctions of the Vibrating Membrane

The subject of the following consideration is a membrane without proper elasticity (see p. 33) which is clamped into a frame whose resistance to distortion is entirely due to the stresses working on its edge. We consider these stresses as perpendicular to the edge in the plane of the membrane. For the deformed membrane this results in a pressure N which acts perpendicular to the surface and is equal to T times the mean curvature of the membrane, and hence is equal to $T\,\Delta u$ for a small deformation u. The wave equation (7.4) for a pure harmonic oscillation of frequency ω then yields

$$-\sigma\,\omega^2\,u = T\,\Delta u, \qquad \sigma = \text{surface density.}$$

This we rewrite in the customary form

$$\Delta u + k^2\,u = 0, \qquad k^2 = \frac{\sigma\,\omega^2}{T}. \tag{1}$$

If we do not consider k^2 as constant but as an arbitrary function $F(x,y)$,

then according to (10.6) this is the general linear *self-adjoint* elliptic differential equation of second order in two variables in its normal form.

The non-trivial solutions of (1) which satisfy the boundary condition $u = 0$ are called *eigenfunctions* and the corresponding k are called the *eigen values* of the problem. If k^2 or $F(x,y)$ were negative then no eigen values would exist, as we saw in the introduction to exercise II.2. The fact that eigen values do exist for positive k^2, namely, an *infinite number*, can be shown first for the simplest examples.

a) The rectangle $0 \leq x \leq a$, $0 \leq y \leq b$. The boundary conditions are satisfied by

$$u = u_{nm} = \sin n\pi \frac{x}{a} \sin m\pi \frac{y}{b}, \quad \begin{cases} n = 1, 2, \cdots \infty, \\ m = 1, 2, \cdots \infty. \end{cases} \tag{2}$$

From the differential equation we then have

$$k = k_{nm} = \pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}. \tag{2a}$$

We shall ignore the constant factor by which the solutions can be multiplied. We first assume a and b to be *incommensurable*. Then all the k_{nm} are different and only *one* eigenfunction u corresponds to each k. The number of eigen values is *infinite*.

b) Circle, circular ring, circular sector. For the *full circle* $0 \leq r \leq a$ we can write:

$$u = I_m(kr)\, e^{\pm i m\varphi}, \quad m = 0, 1, 2, \ldots \infty, \tag{3}$$

where k satisfies the boundary condition

$$I_m(ka) = 0. \tag{3a}$$

Since this equation has infinitely many roots (see Fig. 21) there are again *infinitely many eigen values* $k = k_{nm}$. The roots of (3a) are all different, but for $m > 0$ there are two eigenfunctions for each eigen value corresponding to the different signs in (3), or, in other words, corresponding to the double possibility $\begin{matrix}\cos \\ \sin\end{matrix} m\varphi$. We say that the problem is *degenerate* for $m > 0$; in our case it is *simply degenerate*. According to (20.4b) the (non-degenerate) basic tone of the circular membrane is $k_{10} = 2.40/a$.

For the *circular ring* $b \leq r \leq a$ we write

$$u = [I_m(kr) + c\, N_m(kr)]\, e^{\pm i m\varphi}. \tag{4}$$

Here we need both particular solutions I and N of the Bessel differential

equation (we could, of course, consider H^1 and H^2 instead) in order to be able to satisfy the two boundary conditions:

$$\begin{aligned} I_m(k\,a) + c\,N_m(k\,a) &= 0, \\ I_m(k\,b) + c\,N_m(k\,b) &= 0. \end{aligned} \tag{4a}$$

Here too there exists an infinite number of different k_{nm} with their associated c_{nm}. This problem, too, is simply degenerate when $m > 0$, since, according to (4), there are then two different u_{nm} for each k_{nm}.

For the *circular sector* $0 \leq r \leq a$, $0 \leq \varphi \leq \alpha$ we set

$$u = I_\mu(k\,r)\,\sin\,\mu\varphi, \qquad \mu = m\,\frac{\pi}{\alpha}, \tag{5}$$

where the k are determined by the condition $I_\mu(ka) = 0$. Infinitely many eigen values $k = k_{nm}$ exist; the problem is not degenerate.

The most general region which can be treated in this manner is the *circular ring sector* $b \leq r \leq a$, $0 \leq \varphi \leq \alpha$, which is bounded by two circular arcs and two radii.

c) Ellipse and elliptic-hyperbolic curvilinear quadrangle. The wave equation (1) written in elliptic coordinates ξ, η can be separated (see v.II exercise IV.3) and leads to a so-called Mathieu equation in each coordinate. The solution $\xi = \text{const}$ yields the ellipses which belong to the family of curves; $\eta = \text{const}$ yields the hyperbolas of the family. For the full ellipse we have, in addition to the boundary condition $u = 0$, a condition of continuity for $\xi = 0$ (focal line) and the condition of periodicity for $\eta = \pm\pi$. The determination of the eigen values leads to complicated transcendental equations which we cannot discuss here. The most general region of this kind is the curvilinear quadrangle whose boundary consists of two elliptic arcs and two arcs of hyperbolas which are confocal with the former.

The simple examples which we considered here are special cases of the *fundamental theorem* of the theory of oscillating systems with infinitely many degrees of freedom and their eigenfunctions: *For an arbitrary region an infinite sequence of eigen values k exists for which there is a solution of the corresponding differential equation $\Delta u + k^2 u = 0$ which is continuous in the interior of the region and satisfies the boundary condition $u = 0$* (or any of the other boundary conditions on p. 63).[1] The problem of finding a rigorous proof for this theorem has repeatedly challenged the ingenuity of mathematicians, starting with Poincaré's great work (*Rendic. Circ. Math. di Palermo*, 1894) and culminating in the

[1] The same theorem holds for the eigen value λ of the general self-adjoint differential equation $\Delta u + \lambda F(x, y)\,u = 0$, $F > 0$.

Fredholm-Hilbert theory of integral equations. Here we must be satisfied with proving the related theorem for mechanical systems with a finite number of degrees of freedom: *A system with f degrees of freedom which is in stable equilibrium, can have exactly f linearly independent small (or more precisely, infinitely small) sine-like oscillations about this state.*

We write the kinetic energy for the neighborhood of a state of equilibrium $q_1 = q_2 = \ldots = q_f = 0$ in the form:

$$T = \frac{1}{2} \sum \sum a_{nm} \dot{q}_n \dot{q}_m .$$

Because q is so small we consider the a_{nm} as constants. At the same time the potential energy V becomes a quadratic form in the q_n with constant coefficients since the linear terms $\partial V/\partial q_n$ vanish in the expansion of V in terms of the q_i around the state of equilibrium

$$V - V_0 = \frac{1}{2} \sum \sum b_{nm} q_n q_m .$$

Now it is always possible to transform *both* the above quadratic forms simultaneously into sums of squares by a linear transformation (transformation to principal axes of quadratic surfaces). Performing this transformation we obtain:

$$T = \frac{1}{2} \sum a_n x_n^2 , \qquad V - V_0 = \frac{1}{2} \sum b_n x_n^2 .$$

The new coordinates x_n are called *normal coordinates* of the system. According to the Lagrange equation we then have

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{x}_n} = - \frac{\partial V}{\partial x_n} , \quad \text{hence} \quad a_n \ddot{x}_n = - b_n x_n .$$

T is a positive definite quadratic form and so is $V - V_0$ for a stable equilibrium; hence the a_n and b_n are positive. Thus for every normal coordinate we obtain a stable oscillation

$$x_n = c_n e^{i \omega_n t} \quad \text{with} \quad \omega_n^2 = \frac{b_n}{a_n} > 0 ,$$

which gives as many oscillations as there are degrees of freedom. In the limit $f \to \infty$ there corresponds an eigen value k_n to every ω_n, and to the totality of $q_1, \ldots, q_n$ that belong to the individual x_n there now corresponds the eigenfunction u_f. The k and the ω_n are both *real.*

We point out that the fact that the k are real can also be proved directly from the differential equation without passing to the limit. If a k were complex then the corresponding u would be complex and the conjugate function u^* would have to satisfy the conjugate differential equa-

tion $\Delta u^* + k^{*2} u^* = 0$ with the boundary condition $u^* = 0$. From Green's theorem

$$\int (u\,\Delta u^* - u^*\,\Delta u)\,d\sigma = \int \left(u\frac{\partial u^*}{\partial n} - u^*\frac{\partial u}{\partial n}\right) ds, \tag{6}$$

where the right side vanishes due to the boundary conditions; it follows that

$$(k^2 - k^{*2}) \int u\,u^*\,d\sigma = 0.$$

But uu^* is always $\geqq 0$; hence the integral cannot vanish and hence we must have $k = k^*$, and k must be real. The physical meaning of the real character of the k is that under our conditions the oscillation process is always free from absorption.

Up to now we have assumed our problem to be *non-degenerate*. However, for the perturbation theory of wave mechanics the degenerate cases are of special interest. We return to our example of the rectangle and no longer assume the sides a,b, to be incommensurable. This is certainly the case for the *square* $a = b$. Then we obtain from (2a)

$$k_{nm} = \frac{\pi}{a}\sqrt{n^2 + m^2}, \quad \text{hence} \quad k_{nm} = k_{mn};$$

but according to (2) we have $u_{nm} \neq u_{mn}$, unless $n = m$, namely,

$$u_{n\,m} = \sin n\,\pi\,\frac{x}{a}\,\sin m\,\pi\,\frac{y}{a},$$

but

$$u_{m\,n} = \sin m\,\pi\,\frac{x}{a}\,\sin n\,\pi\,\frac{y}{a},$$

All oscillations with $n \neq m$ are therefore (at least) *simply degenerate*, since two different types of oscillations u_{nm} and u_{mn} correspond to the same k_{mn}. Only the basic oscillation k_{11} and its overtones $k_{nn} = n\,k_{11}$ (which in this special case are harmonic) are *non-degenerate*.

Let us examine somewhat more closely the cases $n = 1$, $m = 2$ and $n = 2, m = 1$ (hence $k_{12} = k_{21} = \sqrt{5}\,\pi/a$). In Figs. 23 and 24 we characterize the corresponding eigenfunctions by their nodal lines. These are the lines $u = 0$ in which powder strewn on the membrane would collect. Together with u_{12} and u_{21} we have, belonging to $k_{12} = k_{21}$, the eigenfunctions

$$u = u_{12} + \lambda\,u_{21}, \tag{7}$$

where λ is an arbitrary constant. By a continuous

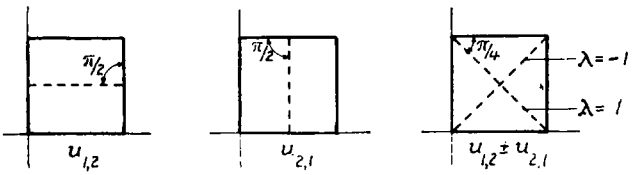

Fig. 23. Simple degeneration in the case of a quadratic membrane for $n = 1$. $m = 2$ or $m = 1$, $n = 2$. The diagonals are the nodal lines for $\lambda = \pm 1$.

deformation of λ the form of the nodal lines within the family (7) is continuously deformed. We compute the linear combinations with $\lambda = \pm 1$:

$$u = \sin \pi \frac{x}{a} \sin 2\pi \frac{y}{a} \pm \sin 2\pi \frac{x}{a} \sin \pi \frac{y}{a}$$

$$= 2 \sin \pi \frac{x}{a} \sin \pi \frac{y}{a} \left(\cos \pi \frac{y}{a} \pm \cos \pi \frac{x}{a} \right).$$

From the last expression here we see that the diagonal $y = x$ is a nodal line of $\lambda = -1$ while the other diagonal $y = a - x$ belongs to $\lambda = +1$. Fig. 24 shows the behavior of the lines for arbitrary values of the parameter λ.

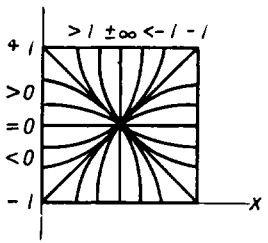

Fig. 24. Total picture of the possible nodal lines for the quadratic membrane. The numbers on the left and on top are the values of the parameter λ in (7).

Under certain conditions higher degenerations occur in the case of the quadratic membrane. For example, if we have

$$n_1^2 + m_1^2 = n_2^2 + m_2^2;$$

then for the eigen value

$$k = \frac{\pi}{a} \sqrt{n_1^2 + m_1^2} = \frac{\pi}{a} \sqrt{n_2^2 + m_2^2}$$

we have four linearly independent eigenfunctions

$$u_{n_1 m_1},\ u_{n_2 m_2},\ u_{m_1 n_1},\ u_{m_2 n_2}.$$

Hence we have a case of triple degeneration. The higher degeneration here depends on whether or not a number can be expressed as the sum of two squares in more than one way, as for example

$$65 = 1^2 + 8^2 = 4^2 + 7^2.$$

According to Gauss' *Disquisitiones Arithmeticae* this is the case for every sum of two squares among whose prime factors there are at least two

different ones of the form $4n + 1$. Such primes permit the complex decomposition

$$4n + 1 = (a + bi)(a - bi)$$

with integral a,b; and the different groupings of the complex factors lead to different representations as a sum of squares. In our example $65 = 5.13$ we have

$$5 = (1 + 2i)(1 - 2i) \text{ and } 13 = (2 + 3i)(2 - 3i)$$

and hence

$$65 = \begin{cases} (1+2i)\ (2+3i) \cdot (1-2i)\ (2-3i) = (-4+7i)\ (-4-7i) \\ \qquad\qquad = 4^2 + 7^2, \\ (1+2i)\ (2-3i) \cdot (1-2i)\ (2+3i) = (8+i) \cdot (8-i) \\ \qquad\qquad = 8^2 + 1^2. \end{cases}$$

For any two eigenfunctions u,u' with $k \neq k'$ we have the *orthogonality theorem*

$$\int u\, u'\, d\sigma = 0 \tag{8}$$

as a result of Green's theorem. The proof is the same as in (6) if we replace u^* by u'. But this deduction fails if u and u' belong to the same degenerate state, in other words if $k = k'$.

In order to avoid cumbersome considerations of special cases, it is desirable to force orthogonality also in the degenerate cases. It will prove convenient to introduce the abbreviation of Courant-Hilbert[2] for the integral in (8):

$$\int u\, u'\, d\sigma = (u, u'). \tag{8a}$$

In §26 we shall return to a discussion of the connection between this expression and the scalar product in ordinary vector analysis. We call the integral in (8a) the "scalar product" of u and u'.

We first prove the theorem that n continuous, real, mutually orthogonal but otherwise arbitrary functions $u_1, u_2, \ldots, u_\nu, \ldots, u_n$ are *linearly independent*. For if there existed an equation of the form

$$\sum_{\nu=0}^{n} c_\nu u_\nu = 0 \quad \text{with} \quad (u_\mu, u_\nu) = 0 \quad \text{for all} \quad \mu \neq \nu\ ,$$

then by the "scalar multiplication" of the equation by u_μ we would obtain

[2] Courant-Hilbert, Methoden der mathematischen Physik, 2nd ed., Springer, Berlin 1931, Chapter II.

$$c_\mu (u_\mu, u_\mu) = 0\,, \text{ hence } \quad c_\mu = 0 \quad \text{for all } \mu\,,$$

which contradicts the assumption of linear dependence.

We now proceed step by step and first treat the case of *simple degeneracy*. Let u_1,u_2 be continuous, real, not necessarily orthogonal functions belonging to the same eigen value. We consider the family

$$u = c_1 u_1 + c_2 u_2$$

and consider the member which is orthogonal to u_1. This member is given by the condition

$$0 = (u_1, u) = c_1 (u_1, u_1) + c_2 (u_1, u_2)\,.$$

We satisfy this condition by setting

$$c_1 = -(u_1, u_2)\,, \; c_2 = (u_1, u_1) \tag{9}$$

where $c_2 \neq 0$ and hence $u \neq 0$. In u_1 and u we have two mutually orthogonal eigenfunctions of the family, which we choose as the representatives of the family instead of u_1,u_2. We now can normalize u by multiplication with a constant factor such that

$$(u,u) = (u_1,u_1). \tag{9a}$$

For *twofold degeneracy* let u_1,u_2 be two functions that are normalized according to (9) and (9a), and let u_3 be a function of the same eigen value that is not necessarily orthogonal to the first two. We consider the family

$$u = c_1 u_1 + c_2 u_2 + c_3 u_3$$

and select the member of the family that is orthogonal to both u_1 and u_2, thus obtaining the conditions

$$0 = (u_1, u) = c_1 (u_1, u_1) + c_3 (u_1, u_3)\,,$$
$$0 = (u_2, u) = c_2 (u_2, u_2) + c_3 (u_2, u_3)\,.$$

We satisfy both conditions by setting

$$c_1 = -(u_1, u_3),\; c_2 = -(u_2, u_3),\; c_3 = (u_1, u_1) = (u_2, u_2). \tag{10}$$

The functions u_1,u_2, u are mutually orthogonal and hence linearly independent; furthermore we can normalize u so as to obtain:

$$(u,u) = (u_1,u_1) = (u_2,u_2). \tag{10a}$$

We thereby obtain the desired orthogonalization for twofold degeneracy.

This process can obviously be continued in the case of higher

degeneracy. The degenerate eigenfunctions are thus made mutually orthogonal; due to (8) they are already orthogonal to the eigenfunctions which belong to different k.

To the orthogonality condition (8) we add the *normality condition*

$$(u,\ u) = \int u^2\, d\sigma = 1 \tag{11}$$

This "normalization to 1" leads to a certain simplification of the orthogonalization process above (see, e.g. (10a)). We shall see in §26 that (11) also has its vector-analytic analog. We still must mention that for complex u equation (11) must be amended to read

$$(u\, u^*) = \int u\, u^*\, d\sigma = 1 \tag{11a}$$

and that in separable problems the normalization is best carried out for each individual factor. Thus in (2) we have to multiply the sine functions by the factors

$$\sqrt{\frac{2}{a}} \qquad \text{and} \qquad \sqrt{\frac{2}{b}} \qquad \text{respectively} \tag{12}$$

and in (3) we have to multiply the exponential function and the Bessel function by the factors

$$\frac{1}{\sqrt{2\pi}} \qquad \text{and} \qquad \frac{\sqrt{2}}{a\, I'_m(ka)} \qquad \text{respectively} \tag{12a}$$

(the latter is due to (20.9a)). Thus our solutions in (2) and (4) at the beginning of this section are determined also with respect to their amplitudes.

From the above-mentioned examples we deduce two theorems concerning nodal lines, which we shall prove now for membranes with arbitrary boundaries:

1. If several nodal lines intersect at a point then they intersect at equal angles (isogonally): for two such lines the angle is $\pi/2$, for ν lines it is π/ν.

2. The larger the eigen value k, the finer the subdivision of the membrane into regions of alternating signs; for $k \to \infty$ the nodal lines become everywhere dense.

In order to demonstrate that theorem 1 holds for our special examples we refer to Figs. 23 and 24, where the boundary itself must be considered a nodal line and the angles are $\pi/2$ and $\pi/4$ as shown. In the case of the full circle we see from (3) that there are m radial lines intersecting at its center at an angle of π/m. In order to show that theorem 2 is satisfied in our cases it suffices to note that for the case of the rectangle and the eigen value k_{nm} the rectangle is subdivided into sub-

rectangles of sides a/n and b/m, so that for $k \to \infty$ at least one side approaches zero.

For the proof of theorem 1 we develop u in the neighborhood of the point O in a Fourier series. We use an r, φ - coordinate system whose origin is at O. For any shape of the membrane we obtain the expansion

$$u = \sum_n I_n(kr)\,(a_n \cos n\varphi + b_n \sin n\varphi) \tag{13}$$

which converges in a certain neighborhood of O, where the a,b are determined coefficients which can be computed from the given u. The fact that the radial functions in the Fourier expansion must be the Bessel functions I_n follows from differential equation (1) and the regularity of u at O. Now if there is to exist *at least* one nodal line through the point O ($r = 0$), then according to (13)

$$0 = I_0(0)\,a_0, \quad \text{and hence } a_0 = 0.$$

Then if a_1 and b_1 are not both zero there is *only* one line through O whose direction is determined by the equation:

$$0 = I_1(kr)\,(a_1 \cos \varphi + b_1 \sin \varphi).$$

Hence for $r > 0$

$$\tan \varphi = -\frac{a_1}{b_1}.$$

This determines the direction of the nodal line *uniquely*.

If there is to be more than one nodal line through O then we must have $a_1 = b_1 = 0$. If we do not at the same time have $a_2 = b_2 = 0$ then according to (13) we have

$$0 = I_2(kr)\,(a_2 \cos 2\varphi + b_2 \sin 2\varphi)$$

or, if there are to be ν nodal lines through O, and hence all a,b up to but not including a_ν, b_ν vanish, then we have

$$0 = I_\nu(kr)\,(a_\nu \cos \nu\varphi + b_\nu \sin \nu\varphi).$$

In the latter case we have for $r > 0$

$$\tan \nu\varphi = -\frac{a_\nu}{b_\nu}. \tag{13a}$$

The right side of this equation is given by our Fourier expansion and shall be denoted by $\tan \alpha$. The general solution of (13a) is then:

$$\varphi = \alpha, \quad \alpha + \frac{\pi}{\nu}, \quad \alpha + \frac{2\pi}{\nu}, \ \ldots, \ \alpha + \frac{(\nu-1)\pi}{\nu}. \tag{13b}$$

These angles differ by the constant amount π/ν, which proves the isogonality.

Passing to the proof of theorem 2, we consider two functions u,v where u is a solution of (1) that satisfies the given boundary condition and v is the special solution

$$v = I_0(kr).$$

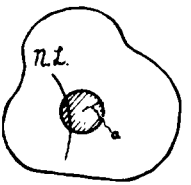

Fig. 25. With increasing k the nodal lines become denser and denser regardless of the shape of the membrane. The proof is given by considering a small disc anywhere on the membrane whose radius a decreases to zero for increasing k. N.L. stands for a nodal line which intersects the disc.

The value of k which is common to u and v is assumed to be large. With the help of this large k we define a small length a by setting $k\,a = \varrho_1$ where ϱ_1 is the first root of the equation $I_0(\varrho) = 0$. We consider a circular disc of radius a situated anywhere on the nodal line pattern of the eigenfunction u (see Fig. 25). With this disc as our domain of integration we apply Green's theorem:

$$\int (u\,\Delta v - v\,\Delta u)\,d\sigma = \int \left(u\,\frac{\partial v}{\partial n} - v\,\frac{\partial u}{\partial n}\right) ds. \tag{14}$$

The left side vanishes since both u and v satisfy the differential equation (1) with the same k. On the right side we have for $r = a$

$$v = 0 \text{ and } \frac{\partial v}{\partial n} = k\,I_0'(\varrho_1) \neq 0.$$

If we set $ds = a\,d\varphi$ then equation (14) becomes

$$\varrho_1\,I_0'(\varrho_1) \int_0^{2\pi} u\,d\varphi = 0,$$

and hence

$$\int_0^{2\pi} u\,d\varphi = 0. \tag{15}$$

According to this u assumes both positive and negative values on the circumference of the disc. Hence there must be at least two zeros of u on the circumference; that is, our disc must be intersected by at least one nodal line. The disc becomes smaller as k becomes larger and hence for increasing eigen value k the nodal lines become *arbitrarily dense*. This holds for every part of the nodal line pattern.

§ 26. General Remarks Concerning the Boundary Value Problems of Acoustics and of Heat Conduction

The eigenfunctions of the oscillating membrane can be adapted directly to the spatial case. Here we do not think of an oscillating rigid body, but (in order to avoid all complications involving vectors and tensors) rather of an oscillating air mass in the interior of a closed rigid hull of finite extension. Just as on p.166, we interpret the scalar function u as the velocity potential of the air oscillations and we again set the boundary condition $\partial u/\partial n = 0$.

For the rectangular solid with side lengths a,b,c we have, in analogy to (25.2),

$$u = u_{n\,m\,l} = \cos n\pi \frac{x}{a} \cos m\pi \frac{y}{b} \cos l\pi \frac{z}{c} \tag{1}$$

with eigen value

$$k = k_{n\,m\,l} = \pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{l^2}{c^2}}. \tag{1a}$$

This state is non-degenerate if a,b,c are incommensurable.

For a sphere of radius a we obtain the general eigenfunction in analogy to (25.3):

$$u = u_{n\,m\,l} = \psi_n (k_{n\,l}\, r)\, P_n^m (\cos \vartheta)\, e^{i m \varphi} \tag{2}$$

Under our boundary condition the eigen value is given by

$$\psi_n' (k_{n\,l}\, a) = 0, \tag{2a}$$

where k_n is the l-th root of this equation. This state is $2n$-fold degenerate, since $k_{n\,l}$ is independent of m and the different states P_n^m for upper index $-n \leq m \leq +n$ belong to the same $k_{n\,l}$.

Also in this category are the eigenfunctions of the circular cylinder $(0 < r < a,\ 0 < z < h)$ which we derived in §20 C. With the boundary condition $\partial u/\partial n = 0$ they are given by

$$u_{n\,l\,m} = I_n (\lambda\, r)\, e^{\pm i n \varphi} \cos m\pi \frac{z}{h}; \tag{3}$$

the corresponding eigen value is determined from the equation $I_n' (\lambda\, a) = 0$, the l-th solution of which we denote by $\lambda_{n\,l}$. Therefore

$$k_{n\,l\,m}^2 = \lambda_{n\,l}^2 + m^2\pi^2/h^2. \tag{3a}$$

Due to the factor $\exp(\pm i n \varphi)$ in (3) this state is simply degenerate for $n > 0$.

We now consider these eigenfunctions "normalized to 1" where we have to keep in mind the remarks on pp. 173,174 . Then for example in (1) we have to replace $\cos\, n\,\pi\, x/a$ by

$$\cos n\,\pi\,\frac{x}{a}\,\sqrt{\frac{2}{a}}$$

and according to (22.31b) we have to replace P_n^m in (2) by

$$\Pi_n^m = P_n^m \qquad \sqrt{(n+\tfrac{1}{2})\,\frac{(n-m)!}{(n+m)!}},$$

etc. (see exercise V.1).

We now generalize the fundamental theorem on p. 169 and its (mathematically non-rigorous) proof to the case of an arbitrary spatial region S. The theorem now reads: *There exists an infinite system of eigenfunctions*

$$u_1, u_2, \ldots, u_n, \ldots,$$

whose elements are regular in the interior of S and satisfy the differential equation

$$\Delta u_n + k_n^2\, u_n = 0,$$

as well as a homogeneous boundary condition. The corresponding eigen values

$$k_1, k_2, \ldots, k_n, \ldots,$$

ordered in an increasing sequence, are infinite in number and increase to infinity; if S is bounded then they form a "discrete spectrum" and they are real since the differential equation was assumed free from absorption.

This system of eigenfunctions satisfies the conditions of *orthogonality* and of *normality:*

$$\int u_n\, u_m\, d\tau = \delta_{n\,m}, \tag{4}$$

which according to (25.8a) can be written as

$$(u_n,\, u_m) = \delta_{n\,m} \tag{4a}$$

or for complex eigenfunctions

$$(u_n,\, u_m^*) = \delta_{n\,m}. \tag{4b}$$

If the system of u_n is *complete* (see p. 5) then we claim that any continuous point function f given on S can be expanded in the u_n:

$$f = \sum A_n\, u_n. \tag{5}$$

If this expansion is possible then, according to (4b), we obtain from (5) through termwise integration

$$A_n = \int f u_n^* \, d\tau . \tag{5a}$$

That this expansion is possible is postulated by the *Ohm-Rayleigh principle*, which we shall assume in the following discussion without presenting its mathematical proof. In connection with the name of this principle we remark: Georg Simon Ohm was not only the discoverer of the basic law of Galvanic conduction, but also did profound research in acoustics. He found that the differences in the tone-color of different musical instruments are the result of differences in the mixture of basic tone and overtones. Since, according to (25.1), the overtones ω_n are related to the k_n, and since they are harmonic with the basic tone only for strings and organ pipes, so the construction of an arbitrary tone-color means the construction of an arbitrary function from the (in general anharmonic) eigen values. In Lord Rayleigh's classic book, *Theory of Sound*, this principle is generalized in the sense of equation (5) and is applied in many directions.

We shall now make some remarks about so-called *Hilbert space*, not only to justify the notation (u_n,u_m) of (4a,b) which is reminiscent of vector analysis, but also to give the Ohm-Rayleigh principle an elegant geometric interpretation, which in the hands of the Hilbert school has even been worked out as a means of proving this principle.

In accord with Courant-Hilbert (see p. 172) we define, in a space of N dimensions, the basis vectors $\mathfrak{e}_1$, $\mathfrak{e}_2, \ldots, \mathfrak{e}_N$ (corresponding to the i,j,k of three-dimensional vector analysis) which lie in the coordinate directions $x_1,x_2, \ldots ,x_N$ and whose scalar product is to satisfy the condition

$$(\mathfrak{e}_n \mathfrak{e}_m) = \delta_{nm} \tag{6}$$

We further consider a vector which forms the angles α_1, $\alpha_2, \ldots$ with the coordinate axes

$$\mathfrak{a} = \cos \alpha_1 \, \mathfrak{e}_1 + \cos \alpha_2 \, \mathfrak{e}_2 + \cdots + \cos \alpha_N \, \mathfrak{e}_N \tag{7}$$

and we call it a *unit vector* if the scalar product of $\mathfrak{a}$ with itself has the value 1:

$$(\mathfrak{a},\mathfrak{a}) = \sum_{n=1}^{N} \cos^2 \alpha_n = 1 . \tag{7a}$$

A second unit vector $\mathfrak{b}$ with direction angles β_n is called *orthogonal* to

$\mathfrak{a}$ if the scalar product of $\mathfrak{a}$ and $\mathfrak{b}$ vanishes:

$$(7b) \qquad (\mathfrak{a},\mathfrak{b}) = \sum_{n=1}^{N} \cos \alpha_n \cos \beta_n = 0.$$

Equations (7a,b) are seen to be generalizations of well-known formulas from three-dimensional analytic geometry.

In the limit $N \to \infty$ we now obtain Hilbert space. Here we note a formal analogy between the basis vectors $\mathfrak{e}_n$ and the elements u_n of our system of eigenfunctions. The relations between the latter as written in the form (4a) are formally the same as the relations (6) between the $\mathfrak{e}_n$. The system u_n, if it is complete, can serve as substitute for the basis $\mathfrak{e}_n$. The same is true for the u_n^* in the case of complex u_n. Every other system of functions that is orthogonalized and normalized to 1 can be composed from the u_n in the sense of equation (7) and can be visualized as a *vector* in Hilbert space. Two such vectors can be transformed into each other by a *rotation* of Hilbert space. But according to (5) any function f is composed of the u_n. With the system of coordinates which is formed by the u_n the function f is associated by (5) to a certain *point* of Hilbert space. The coordinates of this point as measured in the system u_n are the expansion coefficients A_n. Hilbert space thus becomes a *function space*. The association between the arbitrary functions and the points of the space of infinitely many dimensions is one-to-one. If we join the point which represents the function f to the origin of the coordinate system of the u_n, then this infinite dimensional vector represents the function f. According to (5a), which we can write in the form $A_n = (f, u_n^*)$, the coordinates of the representative point are the projections of the representative vector on the axes of the system of u_n^*.

From these highly abstract generalizations we return to the physical applications. For the time being we restrict ourselves to the simple problems of acoustics and heat conduction in their historical form. We defer the questions of wave mechanics to the end of this chapter.

The general problem of *acoustics* for the interior of an arbitrary shell S is the following: the wave equation

$$(8) \qquad \frac{\partial^2 v}{\partial t^2} = c^2 \Delta v, \qquad c = \text{speed of sound},$$

is to be solved with the boundary condition $\partial v/\partial n = 0$ so that for $t = 0$ the functions v and $\partial v/\partial t$ become equal to arbitrary prescribed functions v_0 and v_1 in S. This problem is solved by:

$$(9) \qquad v = \sum A_n u_n \cos \omega_n t + \sum B_n u_n \sin \omega_n t,$$

where the A_n and B_n are to be determined so that

$$v_0 = \sum A_n u_n \quad \text{and} \quad v_1 = \sum B_n \omega_n u_n \tag{9a}$$

Due to the relation of the ω_n with the eigen values k_n (namely $c = \omega_n/k_n$) the second equation can be rewritten as:

$$\frac{v_1}{c} = \sum B_n k_n u_n. \tag{9b}$$

From this we obtain as in (5a)

$$A_n = \int v_0 u_n^* \, d\tau, \quad B_n = \frac{1}{k_n c} \int v_1, \, u_n^* \, d\tau. \tag{9c}$$

We see that this is an *initial value problem;* the *boundary value problem* has been shifted to the u_n.

The general *heat conduction problem* can be solved in the same manner. The difference is that now *one* arbitrary function v_0 suffices to describe the initial state, the initial temperature variation $\partial v/\partial t$ being determined by the differential equation of heat conduction. As a boundary condition we may use any one of the conditions a),b),c) on p. 63, to which we then also subject the eigenfunctions u_n.

We now set

$$v = \sum A_n u_n e^{-\varkappa k^2 n t} \tag{10}$$

where $\varkappa$ stands for temperature (not heat) conductivity. The coefficients A_n are again determined by the initial condition $v = v_0$:

$$A_n = \int v_0 u_n^* \, d\tau. \tag{10a}$$

In addition to this initial condition the function v satisfies the differential equation (25.1) and the boundary condition to which the u_n are subjected.

The potential equation $\Delta u = 0$ has *no* eigenfunctions, or rather every solution which is regular in the interior of S and which satisfies the boundary condition $u = 0$ or $\partial u/\partial n = 0$ must be zero or constant in the interior of S. Hence there can be here no closed "nodal surfaces" $u = 0$ or $\partial u/\partial n = 0$. However, in the next section we shall construct a solution of the general potential boundary value problem (given values $u = U$ on the boundary) from the eigenfunctions of the wave equation.

A solution of the potential equation which is regular in S can also have no maximum or minimum in the interior of S. Extremal values of u can be assumed only on the boundary of S. This follows from Gauss' theorem on the *arithmetic mean* which can be deduced from Green's theorem (see exercise V.2).

Also, *no* eigenfunctions exist for the differential equation $\Delta u - k^2 u = 0$ or the more general $\Delta u - F u = 0$ for positive $F(x,y,z)$ (see exercise II.2).

§ 27. Free and Forced Oscillations. Green's Function for the Wave Equation

The eigenfunctions correspond to free oscillations; in a non-absorbing medium they need no energy supply. We now wish to consider *forced* oscillations, which must be stimulated in the rhythm of their period in order to be able to continue in their purely periodic state. Just as the free oscillations, they are to satisfy a homogeneous surface condition, e.g., $u = 0$; the region S will be assumed to be bounded in the discussions in this section. The measure of stimulation shall, for the time being, be assumed to be a continuous point function in the interior of S, and in analogy to the Poisson equation of potential theory we denote it by ϱ .[3] Correspondingly we write the differential equation of forced oscillations as:

$$\Delta u + k^2 u = \varrho. \tag{1}$$

Here $k = \omega/c$, as we remarked in (26.9a), where ω is the circular frequency of the stimulation and c is the speed of sound. We assume

$$k \neq k_n, \tag{2}$$

i.e., k is different from every eigen value of the region S for the same boundary condition. The case of "resonance" $k = k_n$ will be treated at the end of this section.

According to the Ohm-Rayleigh principle we can expand ϱ in terms of the normalized u_n as in (26.5) and (26.5a):

$$\varrho = \sum A_n u_n, \qquad A_n = \int \varrho\, u_n^*\, d\tau, \tag{3}$$

we also write the solution u of (1) in the same form:

$$u = \sum B_n u_n. \tag{3a}$$

Substituting these expansions in (1) and considering the differential equations $\Delta u_n + k_n^2 u_n = 0$, which differ from (1) and which are satisfied by the eigenfunctions u_n, by equating the coefficients of u_n on both sides we obtain

[3] The function ϱ does not represent charge density as in potential theory, but is of dimension sec^{-1} if u stands for an acoustic velocity potential.

(4) $$B_n = \frac{A_n}{k^2 - k_n^2}, \qquad u = \sum \frac{A_n u_n}{k^2 - k_n^2}.$$

We now consider the special case in which ϱ is a δ-function[4] and hence the stimulation is limited to a simple source point Q of yield 1 (see §10 C). We then have

$$\int_Q \varrho \, d\tau = 1,$$

for a domain of integration which contains the point Q, and

$$\int \varrho \, d\tau = 0.$$

for a domain of integration which does not contain Q. Hence we obtain from (3)

(4a) $$A_n = u_n^*(Q) \int_Q \varrho \, d\tau = u_n^*(Q)$$

and from (4)

(5) $$G(P, Q) = \sum \frac{u_n(P)\, u_n^*(Q)}{k^2 - k_n^2}.$$

where the u of (4) is now denoted by the more suggestive $G(P,Q)$. Indeed this solution is *Green's function of our differential equation* (1) *for arbitrary positions of the action point P and the source point Q and an arbitrary region S.* We assume only the complete system of eigenfunctions and eigen values for the region S. It should be noted that the Ohm-Rayleigh principle has not been applied to the singular δ-function, but only to the continuous function ϱ of (3), which may, e.g., be taken as a regular Gauss error function. Hence in our derivation we do not need the expansion in terms of the u_n of an arbitrary function but only of certain special everywhere regular functions. In the same manner the termwise differentiation which was needed in the derivation of (4) has been carried out on the regular function (3a) before passage to the limit and not on the limit (5).

Green's function is also the solution of an *integral equation.* In order to demonstrate this we recall equation (10.13a), which holds for every self-adjoint differential expression $L(u)$ and hence in particular for the wave equation $\Delta u + k^2 u$. For the three-dimensional case and the boundary value $u = 0$ it reads:

(6) $$u_Q = \int \varrho(P)\, G(P, Q)\, d\tau_P.$$

[4] We have dropped the name "peak function" ("Zackenfunktion") which was introduced by the author (see *Jahresber. Deutschen Math. Vereinigung* **21**, 312, 1912) in favor of Dirac's notation " δ-function."

The function $G(P,Q)$ is called the "*kernel*" of the integral equation. Corresponding to the reciprocity theorem d) on p. 1 , which, for complex G, has to be rewritten as

$$G(P,Q) = G^*(Q,P), \tag{6a}$$

we call G a "symmetric kernel." From the structure of (5) we see directly that (6a) is satisfied.

The convergence of the series in (5) is absolute only in the one-dimensional case; in the two- or more dimensional case the convergence is conditioned by the alternation of signs of the eigenfunctions for a suitable arrangement of the series. This is the reason equation (5) does not appear explicitly in Hilbert's theory of integral equations, but in an integrated form in which it converges absolutely. In the one-dimensional case equation (5) has been rigorously proven by Erhardt Schmidt.[5]

The non-absolute convergence of (5) becomes apparent if we try to show by termwise differentiation that the differential equation (1) is satisfied. For then we obtain from the n-th term

$$\Delta u_n + k^2 u_n = \Delta u_n + k_n^2 u_n + (k^2 - k_n^2) u_n = (k^2 - k_n^2) u_n$$

and cancelling the factor $k^2 - k_n^2$ with the denominator and summing with respect to n we obtain

$$\Delta G + k^2 G = \sum u_n(P)\, u_n^*(Q). \tag{6b}$$

For $P = Q$ the sum on the right side consists of positive terms and diverges, as it should; the fact that it converges for $P \neq Q$ and vanishes throughout is caused by the alternating signs and cannot be proven from this representation. The order of increase for $P \to Q$ can be deduced directly from the differential equation (1) as follows. We consider a sphere with small radius r and center Q, and integrate (1) over its interior. Due to the δ-character of ϱ the right side becomes equal to 1. According to Gauss' theorem the first term on the left side becomes

$$\int \frac{\partial G}{\partial r}\, d\sigma = 4\pi r^2 \frac{\partial G}{\partial r},$$

while the second term vanishes. Hence we have

$$\frac{\partial G}{\partial r} = \frac{1}{4\pi r^2}, \qquad G = -\frac{1}{4\pi r} + \text{Const} \quad \text{for } r \to 0. \tag{7}$$

This expresses the fact that $G(P,Q)$ has a *unit source* in the point $P = Q$.

The above formulas can be interpreted best in Hilbert space (see

[5] In his famous dissertation, Göttingen, 1905.

p. 179). Namely, equation (6b) states that $\Delta G + k^2 G$ is the scalar product of the two unit vectors $u(P)$ and $u^*(Q)$. Hence these unit vectors are orthogonal if $u(P)$ and $u(Q)$ are different ($P \neq Q$); if $u(P)$ and $u(Q)$ are equal ($P = Q$) then orthogonality is of course excluded; instead the product becomes infinite. The expression (5) is constructed from the individual terms of the same product with the "resonance denominator" $k^2 - k_n^2$ as weighting factor.

Despite its poor convergence equation (5) has frequently been found useful in wave mechanical computations (see §30). For the time being we apply it in order to close a gap in the theory of spherical harmonics. But first we make a few preparatory remarks:

1. If the system of eigenfunctions is *separable* then the summation in (5) decomposes into three summations corresponding to the three coordinates. For the rectangular solid we should have:

$$\Sigma = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \tag{8}$$

where n,m,l are as in (26.1).

2. Green's function depends only on the position of the points P,Q relative to the boundary surface σ and on their distance R. It is independent of the orientation of the coordinates in space. A transformation of the coordinate system which transforms the surface σ into itself and leaves R fixed leaves $G(P,Q)$ *invariant.*

3. If σ is the surface of a sphere then the condition of invariance is satisfied for every rotation of the *spherical polar system* r, ϑ, φ with $r = 0$ as the center of the sphere. The coordinates r, ϑ, φ shall be those of P, $r_0, \vartheta_0, \varphi_0$ those of Q.

4. In the latter case we face the additional fact that the system of eigenfunctions (26.2) is *degenerate,* since the eigen value k_n as defined by (26.2a) is independent of m. Writing G as a triple sum in analogy to (8), we can take the denominator $k^2 - k_n^2$ and the radial part of the eigenfunctions in front of the summation over m. Hence we have

$$G(P,Q) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{\Psi_n(k_{nl} r)\, \Psi_n(k_{nl} r_0)}{k^2 - k_{nl}^2}\, Y_n , \tag{9}$$

$$Y_n = \sum_{m=-n}^{+n} \Pi_n^m(\cos\vartheta)\, \Pi_n^m(\cos\vartheta_0)\, e^{im(\varphi - \varphi_0)} . \tag{9a}$$

where Ψ_n stands for the function ψ_n of (26.2) normalized to 1, and Π_n for the spherical harmonic P_n nomalized in the same manner. The function Y_n is a surface spherical harmonic. In (9a) we have used the fact that, due to the real character of Ψ_n and Π_n^m the *conjugate*

complex of the eigenfunction

$$\Psi_n(k_{nl}\, r)\, \Pi_n^m(\cos\vartheta)\, e^{im\varphi}$$

for the argument $Q = (\, r_0, \vartheta_0, \varphi_0 \,)$ can be written as

$$\Psi_n(k_{nl}\, r_0)\, \Pi_n^m(\cos\vartheta_0)\, e^{-im\varphi_0},$$

for all values of m between $-n$ and $+n$.

From remark 2 concerning the invariance of G, and from representation (9), we now see that the surface spherical harmonic (9a) has an invariant meaning which is independent of the rotation of the polar coordinate system. But this is the very theorem which we assumed as an axiom for the proof of the addition theorem of spherical harmonics on p. 133. That proof is now completed.

Up to now we have assumed that stimulation of forced oscillation takes place in the *interior* of the region S. We now wish to assume that stimulation takes place from the *surface*. This is the case if, instead of the homogeneous boundary condition $u = 0$, we prescribe the *inhomogeneous* boundary condition

$$u = U. \tag{10}$$

The surface is then held in pulsation with the rhythm ω of the forced oscillation and with the amplitude U which may vary from point to point, while in the interior of S the differential equation (1) holds throughout with $\varrho = 0$. From (10.12) we know that this boundary value problem can be solved with the help of Green's function by the formula

$$u_Q = \int U \frac{\partial G}{\partial \nu_P}\, d\sigma_P, \tag{11}$$

where the variable of integration on the right side is P and the domain of integration is the surface of S ($d\sigma_P =$ element of surface, $d\nu_P =$ element of normal at the point P). According to (5) equation (11) becomes

$$u_Q = \sum \frac{u_n^*(Q)}{k^2 - k_n^2} \int U \frac{\partial u_n(P)}{\partial \nu_P}\, d\sigma_P. \tag{11a}$$

This formula contains the general solution of the famous *Dirichlet problem of potential theory*, for by setting $k = 0$ we obtain

$$u_Q = -\sum \frac{u_n^*(Q)}{k_n^2} \int U \frac{\partial u_n(P)}{\partial \nu_P}\, d\sigma_P. \tag{12}$$

The remarkable fact about this solution is that it is not expanded in

particular solutions of the differential equation $\Delta u = 0$ concerned, but rather in the eigenfunctions of the *wave equation* (there are no eigenfunctions of the potential equation). Equation (12) remains valid if instead of the boundary condition (10) we prescribe the more general condition

$$\frac{\partial u}{\partial n} + h\,u = U$$

except that in this case we must subject the eigenfunctions u_n to the corresponding homogeneous condition

$$\frac{\partial u}{\partial n} + h\,u = 0$$

In the special case of a sphere of radius a we obtain from (12) and the boundary condition (10)

$$\frac{2\pi}{a^2}\,u\,(r_0, \vartheta_0, \varphi_0) = -\sum_n \sum_l \sum_m \frac{A_{nm}}{k_{nl}}\,\Psi_n\,(k_{nl}\,r_0)\,\Psi'_n\,(k_{nl}\,a)\,\Pi_n^m\,(\cos\vartheta_0)\,e^{-im\varphi_0}, \tag{13}$$

$$A_{nm} = \iint U\,\Pi_n^m\,(\cos\vartheta)\,e^{im\varphi}\sin\vartheta\,d\vartheta\,d\varphi. \tag{13a}$$

where Ψ_n and Π_n have the same meaning as before. The extra factor 2π on the left side of (13) is due to the fact that, as with the Bessel functions and the spherical harmonics, we have to normalize the two functions $\exp(-im\varphi_0)$ and $\exp(im\varphi)$ to 1.

Written in terms of the same variables $Q = (r_0, \vartheta_0, \varphi_0)$ and expanded in terms of particular solutions of $\Delta u = 0$ our solution reads:

$$2\pi\,u\,(r_0, \vartheta_0, \varphi_0) = \sum_n \sum_m A_{nm} \left(\frac{r_0}{a}\right)^n \Pi_n^m\,(\cos\vartheta_0)\,e^{-im\varphi_0}. \tag{14}$$

By comparing these solutions we obtain remarkable summation formulas (see exercise V.3).

Finally, we must consider the exceptional case $k = k_m$. From the mechanics and the electrodynamics of oscillating systems we know the "resonance catastrophe": if the rhythm of the stimulating force equals a proper frequency of the system the oscillations increase to infinity. The condition for this event is $\omega = \omega_m$, and hence $k = k_m$. Equation (1) then assumes the form:

$$\Delta u + k_m^2\,u = \varrho. \tag{15}$$

Here we have an inhomogeneous equation whose left side coincides with the homogeneous equation of a free oscillation.

For simplicity we first consider the two-dimensional case of the membrane of §25, which now, however, is subjected to a periodically changing transversal pressure[6] $\varrho = \varrho(x, y)$ with an arbitrary distribution over the membrane. Do pressure distributions exist for which the resonance catastrophe is avoided, that is, for which equation (15) has continuous solutions throughout (for the boundary condition $u = 0$)? The answer to this question is physically evident: for such a solution the pressure on the membrane may do *no work*. Hence we must have:

$$\int \varrho\, u_m\, d\sigma = 0. \tag{16}$$

The pressure distribution must be orthogonal to the eigenfunction $u = u_m$ with which it is in resonance, e.g., it may have equal magnitude in oppositely oscillating sectors of the membrane; *in particular the pressure along a nodal line may be of arbitrary strength.*

This orthogonality theorem is a corner stone in the theory of integral equations and has important applications in the perturbation theory of wave mechanics. Here we must be content with uncovering its physical basis.

The orthogonality theorem can be adapted directly to the three-dimensional case if in (16) we replace the surface integral with respect to $d\sigma$ by a volume integral with respect to $d\tau$. Then we see that the expansion coefficients A_n and B_n in (3) and (4) vanish for $n = m$. By passing from the continuous distribution ϱ to a δ-function we obtain information about Green's function in the case of resonance. From $A_m = 0$ and equation (4a) we have $u_m^*(Q) = 0$. In other words: *The singularity of Green's function must lie on a nodal surface of the critical proper oscillation u_m.*

For this position and only for this position of the source point Q an everywhere regular Green's function exists. The special form of Green's function for the case of resonance is obtained from the general form (5) by omitting the term involving k_m; it therefore reads:

$$G(P, Q) = \sum_{n \neq m} \frac{u_n(P)\, u_n^*(Q)}{k_m^2 - k_n^2}. \tag{17}$$

§ 28. Infinite Domains and Continuous Spectra of Eigen Values. The Condition of Radiation

With increasing domain the eigen values become closer and closer; for an infinite domain they are dense everywhere; we then deal with a *continuous spectrum of eigen values.*

[6] More precisely: pressure divided by surface tension T (see equation (25.1)). The dimension of ϱ is not that of pressure dyn/cm.², but $\frac{\text{dyn}}{\text{cm}^2} / \frac{\text{dyn}}{\text{cm}} = \text{cm}^{-1}$.

Let us consider, e.g., the interior of a sphere of radius a for vanishing boundary values. For the case of purely radial oscillations its eigen values are given by the equation

$$\psi_0(k_\nu a) = 0, \qquad \psi_0(\varrho) = \frac{\sin \varrho}{\varrho}. \tag{1}$$

Hence $k_\nu a = \nu\pi$ and the difference of successive eigen values is

$$\Delta k_\nu = \frac{\pi}{a} \to 0 \quad \text{for} \quad a \to \infty.$$

We may therefore consider the function $\psi_0(kr)$ which is everywhere regular and vanishes at infinity as an *eigenfunction of infinite space.* Thus, if we have an acoustical or an optical problem in which the prescribed sources are in the finite domain (with a discrete or a continuous distribution), and which is to be solved for a given wave number k, then we can always add the function ψ_0 to the solution. Hence oscillation problems (in contrast to potential problems) are *not* determined *uniquely* by their prescribed sources in the finite domain. This paradoxical result shows that the condition of *vanishing* at infinity is not sufficient, and that we have to replace it by a stronger condition at infinity. We call it the *condition of radiation:* the sources must be *sources,* not *sinks,* of energy. The energy which is radiated from the sources must scatter to infinity; *no energy may be radiated from infinity into the prescribed singularities of the field* (plane waves are excluded since for them even the condition $u = 0$ fails to hold at infinity).

For our special eigenfunctions

$$\psi_0(kr) = \frac{1}{2i}\left(\frac{e^{ikr}}{r} - \frac{e^{-ikr}}{r}\right)$$

the state of affairs is simple: for the time dependence $\exp(-i\omega t)$ e^{ikr}/r is a *radiated,* e^{-ikr}/r an *absorbed,* $\psi_0(kr)$ a *standing* wave (nodal surfaces $kr = \nu\pi$). By excluding absorption from infinity we exclude the addition of the eigenfunction $\psi_0(kr)$. Hence the permissible singularities are restricted to the form

$$u = C\frac{e^{ikr}}{r} \tag{1a}$$

For these singularities we have the condition

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0, \tag{2}$$

It is called the general *condition of radiation* and we shall apply it to all

acoustic and electrodynamic oscillation problems that are generated by sources in the finite domain.

In fact, condition (2) holds not only for the spherical wave (1a) which radiates from $r = 0$, but it also holds for a stimulation which acts at the point $x = x_0$, $y = y_0$, $z = z_0$

$$u = C\frac{e^{ikR}}{R}, \quad R^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$$

Hence, for a continuous stimulation of the spatial density $\varrho = \varrho(x_0, y_0, z_0)$ we have:

$$u = \int \varrho \frac{e^{ikR}}{R}\, dx_0\, dy_0\, dz_0 .$$

This holds not only for unbounded space, but also in the case where there are bounded surfaces σ on which arbitrary linear boundary conditions are prescribed, whether homogeneous, e.g., $u = 0$, or inhomogeneous, e.g., $u = U$. In the former case we have scattered or reflected radiation emanating from the surface σ, whereas in the latter case we have radiation that is stimulated by the pulsating surface σ itself (see p. 186).

As counterpart to the radiation condition (2) we have what may be called the "absorption condition":

$$\text{(2a)} \qquad \lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} + iku\right) = 0 .$$

We demonstrate the general validity of the radiation condition by showing that it guarantees the *uniqueness* of solution of the above general oscillation problem. We may then be convinced that the unique solution of the *mathematical* problem is identical with the *solution that is realized in nature*. Our problem is the following:

a) In the exterior of a surface σ, which may consist of several surfaces $\sigma_1, \sigma_2, \ldots$, the function u is to satisfy the differential equation

$$\Delta u + k^2 u = \varrho$$

The function ϱ measures the yield of the sources which may be continuously distributed or concentrated in single points. The function ϱ is given and must vanish at infinity with sufficient rapidity.

b) On σ the function u is to satisfy $u = U$, where U is a given point function on σ. The surface σ lies entirely in the finite domain.

c) In the finite domain u satisfies the condition (2). The quantity r in (2) stands for the distance from any fixed finite point $r = 0$. Around

this point we draw a sphere Σ of radius $r \to \infty$, which does not intersect the surface σ. The surface element on the sphere is $d\Sigma = r^2\, d\omega$, where $d\omega$ is the solid angle seen from $r = 0$. The region between Σ and σ is called S.

d) Except at possible prescribed sources the function u is to satisfy those conditions of continuity which we prescribed in the derivation of the differential equation.

We assume that two solutions of this problem u_1 and u_2 exist and, as usual, form

$$w = u_1 - u_2, \tag{3}$$

as well as the conjugate function w^*. These functions satisfy the conditions a) to d) with $\varrho = 0$ and $U = 0$. Then in Green's theorem

$$\int_S (w \Delta w^* - w^* \Delta w)\, d\tau = \left\{ \int_\sigma d\sigma + \int_\Sigma r^2 d\omega \right\} \left(w \frac{\partial w^*}{\partial n} - w^* \frac{\partial w}{\partial n} \right) \tag{4}$$

the integral on the left and the first integral on the right vanish. *Hence, the integral over the sphere Σ must also vanish.*

For the further discussion we write:

$$w = \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{f_n(\vartheta, \varphi)}{r^n}, \tag{5}$$

which is shown to be sufficiently general by the following: we consider w expanded in surface spherical harmonics $Y_n(\vartheta, \varphi)$. According to §24 A the coefficients must be of the form

$$C_n\, \zeta_n^1(k\, r) + D_n\, \zeta_n^2(k\, r)$$

where ζ is connected with the half-index Hankel functions by equation (21.15). But here we must have $D_n = 0$, because of the behavior of ζ_n^2 for large values of the argument (see §21 D, p. 117). At the same place we learned that the ζ_n^1 are composed of a finite number of terms of the form $e^{ikr}/(k\, r)^m$, $m < n$. By arranging this expansion in spherical harmonics according to powers of r^{-n} we obtain (5), where the $f_n(\vartheta, \varphi)$ turn out to be finite sums of surface spherical harmonics.

The f_n satisfy a simple *recursion formula.* According to (22.4) the differential equation $\Delta w + k^2 w = 0$ written in terms of r, ϑ, φ, yields the equation

$$\frac{\partial^2 (r\, w)}{\partial r^2} + \frac{1}{r^2} D(r w) + k^2\, r\, w = 0. \tag{6}$$

where D is the differential symbol of (23.15b) in the coordinates ϑ, φ. Applying (6) to (5) we obtain

$$e^{ikr} \sum_{n=0}^{\infty} \left(-\frac{2ikn}{r^{n+1}} + \frac{n(n+1)}{r^{n+2}} + \frac{D}{r^{n+2}} \right) f_n = 0 .$$

Replacing the index of summation n in the first term of the parentheses by $n + 1$ we obtain

$$e^{ikr} \sum_{n=0}^{\infty} \frac{1}{r^{n+2}} [-2ik(n+1) f_{n+1} + \{n(n+1) + D\} f_n] = 0$$

and hence the recursion formula:

$$2ik(n+1) f_{n+1} = \{n(n+1) + D\} f_n . \tag{6a}$$

Hence: *if* $f_0 = 0$ *then all* $f_1 = f_2 = \ldots = 0$.

We now investigate the remaining integral in (4). Since we are interested in the limit for $r \to \infty$ we can replace w by the first term of its expansion (5), ignoring the higher powers of $1/r$, whence:

$$w = \frac{e^{ikr}}{r} f_0 , \qquad w^* = \frac{e^{-ikr}}{r} f_0^* ;$$

$$\frac{\partial w}{\partial n} = ik \frac{e^{ikr}}{r} f_0 , \qquad \frac{\partial w^*}{\partial n} = -ik \frac{e^{-ikr}}{r} f_0^* .$$

Thus we obtain:

$$\int r^2 d\omega \left(w \frac{\partial w^*}{\partial n} - w^* \frac{\partial w}{\partial n} \right) = -2ik \int f_0 f_0^* d\omega .$$

The integrand is *positive* as long as $f_0 \neq 0$. But we saw in (4) that this integral must vanish. Hence

$$f_0 = 0, \text{ and due to (6a) } f_1 = f_2 = \ldots = 0.$$

Therefore

$$w = 0 \text{ and } u_2 = u_1.$$

The author's original proof[7] of this uniqueness theorem assumed, in addition to the conditions a),b),c) for u, the existence of Green's function for the exterior of the surface and an additional "finality condition." The fact that the latter is superfluous has been rigorously proven by F. Rellich[8] even for the case of an arbitrary number of dimensions h

[7] See footnote on p. 183 and Frank-Mises II, chap. XIX, §5. The form of the proof given in the text is essentially F. Sauter's.

[8] *Jahresber Deutschen Math. Vereinigung* **53**, 57 (1943), which also treats the case in which the surface σ stretches to infinity.

where the radiation condition reads

$$\text{(7)} \qquad \lim_{r\to\infty} r^{\frac{h-1}{2}}\left(\frac{\partial u}{\partial r} - i\,k\,u\right) = 0.$$

In the two-dimensional case $h = 2$, where, as we know, the spherical wave e^{ikr}/r is replaced by the cylindrical wave $H_0^1(kr)$, equation (7) becomes

$$\text{(7a)} \qquad \lim_{r\to\infty} r^{\frac{1}{2}}\left(\frac{\partial u}{\partial r} - i\,k\,u\right) = 0,$$

which actually is satisfied by $u = H_0^1(kr)$. In the one-dimensional case, where the radiating wave is given by $\exp(i\,k\,|x|)$, equation (7) becomes

$$\text{(7b)} \qquad \lim_{|x|\to\infty}\left(\frac{\partial u}{\partial |x|} - i\,k\,u\right) = 0.$$

Following Rellich, we stress the fact that no radiating solution u of the wave equation can exist which, in every direction, approaches zero more rapidly than $1/r$. For such a function u we would have $f_0 = 0$ in (5) and, as we have seen, this causes u to vanish identically. In this respect the wave equation differs from the potential equation. For the latter solutions exist which, for increasing r, decrease more rapidly than $1/r$, the so-called dipole, quadrupole, and octupole fields of §24 C. For the wave equation such an r-dependence, which implies a pole of higher order than $1/r$ at $r = 0$, can happen only in the so-called "near zone" ($r < \lambda$, $\lambda =$ wavelength); in the "far zone" ($r > \lambda$) every solution of the wave equation behaves like the spherical wave e^{ikr}/r. Potential theory is the limiting case $\lambda = \infty$, as for this case, the near zone reaches to infinity, so to speak.

We now come to the problem of *Green's function for a continuous spectrum*. We first consider in detail the very simplest one-dimensional example ($-\infty < x < +\infty$), in which the radiation condition is the only boundary condition prescribed. Green's function is then identical with the "principal solution" introduced on p. 47, and therefore has a "unit source" at an arbitrary prescribed point $x = x_0$ (see exercise II.3). It must satisfy the conditions:

a) $$\frac{d^2G}{dx^2} + k^2\,G = 0 \quad \text{for } x \neq x_0$$

b) $$\left(\frac{dG}{dx}\right)_{x_0+0} - \left(\frac{dG}{dx}\right)_{x_0-0} = 1, \quad \text{(definition of unit source)}$$

c) $$\frac{dG}{d|x|} - i\,k\,G = 0 \quad \text{for } x = \pm\infty.$$

The solution is seen to be

$$(8)\qquad G=\begin{cases}\dfrac{1}{2ik}\,e^{ik(x-x_0)} & \text{for } x>x_0\,,\\[2ex] \dfrac{1}{2ik}\,e^{-ik(x-x_0)} & \text{for } x<x_0\,.\end{cases}$$

We compare this to the representation (27.5) first for the finite region $-l<x<+l$, but with the usual boundary conditions replaced by the radiation condition. In preparation for a continuous spectrum we change the name k_n of the eigen values to λ ; the eigenfunction $u=u_\lambda$ which belongs to λ is then defined by

$$\text{a)}\qquad \frac{d^2u}{dx^2}+\lambda^2 u=0 \qquad -l<x<+l\,,$$

$$\text{b)}\qquad \frac{du}{d|x|}-iku=0 \qquad |x|=l\,.$$

If we write the solution of a) as:

$$(9)\qquad u=A\,e^{i\lambda x}+B\,e^{-i\lambda x}\,,$$

then according to b) we must have

$$A(\pm\lambda-k)\,e^{\pm i\lambda l}+B(\mp\lambda-k)\,e^{\mp i\lambda l}=0\,,$$
$$\frac{A}{B}=\frac{\lambda+k}{\lambda-k}\,e^{-2i\lambda l}=\frac{\lambda-k}{\lambda+k}\,e^{+2i\lambda l}.$$

From this we obtain the equation for λ:

$$\left(\frac{\lambda-k}{\lambda+k}\right)^2 e^{4i\lambda l}=1\,.$$

This equation splits into the equations

$$(9a)\qquad \frac{\lambda-k}{\lambda+k}\,e^{2i\lambda l}=+1,\quad B=A,\qquad u=2A\cos\lambda x,$$

$$(9b)\qquad \frac{\lambda-k}{\lambda+k}\,e^{2i\lambda l}=-1,\quad B=-A,\qquad u=2iA\sin\lambda x.$$

From (9a) we obtain as first and second approximation

$$(10a)\qquad \begin{aligned}&\text{for } \lambda\gg k,\ \lambda=\frac{\pi}{l}m \qquad \text{and } =\frac{\pi}{l}m\left(1-\frac{ikl}{\pi^2m^2}\right),\quad m\to\infty\,,\\ &\text{for } \lambda\ll k,\ \lambda=\frac{\pi}{l}\left(m+\tfrac{1}{2}\right) \text{ and } =\frac{\pi}{l}\left(m+\tfrac{1}{2}\right)\left(1-\frac{i}{kl}\right),\ m=0,1,2,..\end{aligned}$$

where m is an integer. In the same manner we obtain from (9b)

$$\text{(10b)}\qquad \begin{aligned} &\text{for } \lambda \gg k,\ \lambda = \frac{\pi}{l}(m+\tfrac{1}{2}) \text{ and } = \frac{\pi}{l}(m+\tfrac{1}{2})\left[1-\frac{ikl}{\pi^2(m+\tfrac{1}{2})^2}\right], m\to\infty, \\ &\text{for } \lambda \ll k,\ \lambda = \frac{\pi}{l}m \qquad\quad \text{and } = \frac{\pi}{l}m\left(1-\frac{i}{kl}\right),\ m = 0, 1, 2, \ldots \end{aligned}$$

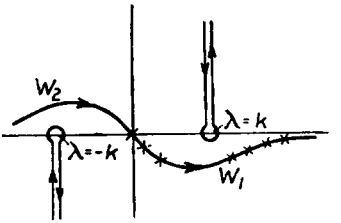

Fig. 26. The path of integration W_1 is completed by the path W_2 to an infinite closed path $W = W_1 + W_2$. For positive $x - x_0$ this path can be deformed so that it runs in the positive imaginary λ- half plane.

We see that the values of λ as calculated from (9a) and (9b) form a sequence (marked by x in Fig. 26) that, starting with $\lambda = 0$, first descends linearly into the negative imaginary λ- half plane[9] and finally for large λ (large m) osculates the real λ- axis from below. According to (9a,b) the successive points alternatingly belong to cosine and sine eigenfunctions. After normalization to 1 these eigenfunctions are

$$\text{(11)}\qquad u = \begin{cases} \dfrac{1}{\sqrt{l(1+A)}} \cos \lambda x\,, \\ \dfrac{1}{\sqrt{l(1-A)}} \sin \lambda x\,, \end{cases} \qquad A = \frac{\sin \lambda l \cos \lambda l}{\lambda l}\,.$$

In the limit $l \to \infty$ the λ- points of Fig. 26 will be everywhere dense on the right half curve denoted by W_1. The difference between two successive points of the sequence (10a) or (10b) then always becomes

$$\text{(11a)}\qquad d\lambda = \frac{\pi}{l} \to 0\,.$$

We now return to the representation (27.5) of Green's function. For $u(P)$ and $u(Q)$ we substitute their expression (11) in the variables x and x_0 respectively, and combine the pairs of successive cosine and sine terms, i.e., the terms which belong to successive eigen values λ. The numerator of (27.5) then becomes

$$u(P)u^*(Q) = \frac{\cos \lambda x \cos \lambda x_0}{l(1+A)} + \frac{\sin \lambda x \sin \lambda x_0}{l(1-A)}\,.$$

According to (11) and (11a) we have for $l \to \infty$

[9] The fact that the eigen values are complex in contrast to the theorem on p. 169 is due to the fact that our present boundary condition is of a complex nature.

$$A = 0, \qquad \frac{1}{l} = \frac{d\lambda}{\pi}.$$

Hence the numerator in (27.5) becomes $\cos \lambda (x - x_0)\, d\lambda/\pi$, while the denominator in our present notation is $k^2 - \lambda^2$. Equation (27.5) then becomes

$$G = \frac{1}{\pi} \int_{W_1} \frac{\cos \lambda (x - x_0)}{k^2 - \lambda^2}\, d\lambda = \frac{1}{2\pi} \int_{W} \frac{e^{i\lambda (x - x_0)}}{k^2 - \lambda^2}\, d\lambda. \tag{12}$$

where W in the last term is the path $W_1 + W_2$ of Fig. 26. The fact that the integration over W_1 is equal to one half the integral over the whole path W follows from the fact that in the integral over W_1 both the numerator and the denominator are even functions of λ. The fact that in the last term we can replace the cosine by the exponential function follows from fact that the sine part of the exponential function is odd in λ and hence vanishes upon integration. The path W is much more convenient than W_1 since it can be deformed away from the origin by the methods of complex integration.

The manner in which this deformation should be performed can be seen from Fig. 26. For positive $x - x_0$ the path W can be drawn over into the positive imaginary λ half plane, for negative $x - x_0$ it can be drawn into the negative imaginary half plane. In the one case the path can not be transformed across the pole $\lambda = +k$ of the integrand in (12), in the other case it can not be transformed across the pole $\lambda = -k$. Forming the residues and combining the two cases we obtain from (12):

$$G = \frac{1}{2ik} e^{ik|x - x_0|}. \tag{13}$$

This is exactly the same as (8).

Hence we see: *The general representation (27.5) of Green's function remains valid for a continuous eigen value spectrum if, in accordance with the radiation condition, we consider a complex path of integration.* If instead we have the "*absorption condition*" (i replaced by $-i$ in (1a) and (2)), then instead of W we have to consider its reflected image on the real λ-axis; we then obtain equation (13) with i replaced by $-i$.

If instead of the one-dimensional case we consider the two- or three-dimensional case and correspondingly replace the coordinate x by the polar coordinates r, φ and r, ϑ, φ, then the spectrum of the eigen values becomes *continuous only in the r-coordinate* but remains *discontinuous* in the angle coordinates. For example in the case of unbounded

three-dimensional space we start from the following representation of Green's function

$$(14)\quad 2\pi G(P,Q) = \sum_{n}\sum_{m} \Pi_n^m(\cos\vartheta)\,\Pi_n^m(\cos\vartheta_0)\,e^{im(\varphi-\varphi_0)} \int_{W_1} \frac{F\,d\lambda}{k^2-\lambda^2},$$

$$(14\,a)\qquad F = \Psi_n(\lambda r)\,\Psi_n(\lambda r_0).$$

Here, as in the preceding section, Π and Ψ stand for the spherical harmonic and Bessel functions normalized to 1; and in the following the Z^1, Z^2 correspond to the Hankel functions ζ^1, ζ^2. The factor 2π on the left side is due to the normalization of the functions $\exp\{im\varphi\}$ and $\exp\{-im\varphi_0\}$. As in Fig. 26 the path W_1 lies in the complex λ-plane from $\lambda = 0$ to $\lambda = \infty$. and again avoids the pole $\lambda = k$ We first give a brief discussion of the way in which this representation can be treated in analogy to the one-dimensional case. This will yield a representation of spherical and cylindrical waves which we have met before.

In order to transform W_1 into the path W of Fig. 26, we write

$$\Psi_n(\lambda r) = \tfrac{1}{2}(Z_n^1(\lambda r) + Z_n^2(\lambda r)),$$

For the convergence problems which arise in connection with the normalization we refer the reader to Appendix I. Due to the properties of Hankel functions (see exercise IV.2, in particular equation (12), and also the discussion in connection with equation (32.13)), we can transform the integral over W_1, which involves the function F of (14a), into the integral over W involving

$$(14\,b)\qquad F_1 = \tfrac{1}{2} Z_n^1(\lambda r)\,\Psi_n(\lambda r_0) \qquad r > r_0,$$

and

$$(14\,c)\qquad F_2 = \tfrac{1}{2}\Psi_n(\lambda r)\,Z_n^1(\lambda r_0) \qquad r < r_0$$

Since the integrand $\tfrac{1}{2}F_{1,2}/(k^2-\lambda^2)$ vanishes at infinity in the positive imaginary λ-plane for both cases $r \gtrless r_0$, the integral of (14) reduces to the residue at the pole $\lambda = k$:

$$(15)\qquad \int F_{1,2}\frac{d\lambda}{k^2-\lambda^2} = \frac{\pi}{2ik}\begin{cases} Z_n^1(kr)\,\Psi_n(kr_0) & r > r_0, \\ \Psi_n(kr)\,Z_n^1(kr_0) & r < r_0. \end{cases}$$

Applying the addition theorem of spherical harmonics (22.34) we obtain from (14):

$$(16)\qquad G(P,Q)=\frac{1}{4ik}\sum_{n=0}^{\infty}\Pi_n(\cos\Theta)\,\Pi_n(1)\begin{cases}Z_n^1(kr)\,\Psi_n(kr_0) & r>r_0,\\ \Psi_n(kr)\,Z_n^1(kr_0) & r<r_0.\end{cases}$$

For reasons of symmetry $G(P,Q)$ in unbounded space is a pure function of the distance

$$R=\sqrt{r^2+r_0^2-2\,r\,r_0\cos\Theta},$$

between P and Q; namely, due to the definition of the unit source on p. 47 we have

$$(16\,a)\qquad G(P,Q)=-\frac{1}{4\pi}\frac{e^{ikR}}{R}=\frac{1}{4\pi i}\zeta_0^1(kR),$$

where ζ_0 is given by (21.15a). If, on the right side of (16), we pass from Π, Ψ, Z to P, ψ, ζ (see Appendix I equation (9a)), we obtain the addition theorem (24.9a).

The corresponding series for the two-dimensional case are contained in (21.3).

More important than the derivation of these known formulas is the generalization to the case in which space is not unbounded but is bounded by a finite closed surface σ (or, in the two-dimensional case, by a curve s) with prescribed boundary conditions. We are then dealing with the *proper problem of Green's function:* to find a function $G(P,Q)$ having a unit source in Q, satisfying the radiation condition at infinity and the given boundary condition on σ (or s).

We choose the special case in which the surface σ is a sphere $r=a$, and the boundary condition is

$$(17)\qquad u=0.$$

The point Q is to lie on the ray

The eigenfunction which belongs to the eigen value λ is no longer $\psi_n(\lambda r)$, but can be written in the (non-normalized) form

$$(18)\qquad u_n(\lambda,r)=\psi_n(\lambda r)+A\,\zeta_n^1(\lambda r)$$

Due to (17) the function A becomes[10]

$$(18a)\qquad A=-\frac{\psi_n(\lambda a)}{\zeta_n^1(\lambda a)}.$$

For the construction of Green's function we shall not follow the general method of equation (14). Instead we shall use a shorter though

[10] The fact that A depends on λ made it necessary to write $u(\lambda,r)$, instead of $u(\lambda\ r)$.

less systematic approach based on equation (24.9) for unbounded space:

$$\left.\begin{matrix}(19a)\\(19b)\end{matrix}\right\} \quad \frac{e^{ikR}}{ikR} = \sum_{n=0}^{\infty} (2n+1)\, P_n(\cos\vartheta) \begin{cases} \psi_n(kr_0)\,\zeta_n^1(kr) & r > r_0, \\ \zeta_n^1(kr_0)\,\psi_n(kr) & r < r_0. \end{cases}$$

Here (19b) will not yet satisfy condition (17) for $r = a$; in order to satisfy (17) we complete the right side of (19b) by adding

$$-\sum_{n=0}^{\infty} (2n+1)\, P_n(\cos\vartheta)\, \zeta_n^1(kr_0)\, \psi_n(ka)\, \frac{\zeta_n^1(kr)}{\zeta_n^1(ka)},$$

Due to (18) the right side of (19b) becomes

$$(20) \qquad \sum_{n=0}^{\infty} (2n+1)\, P_n(\cos\vartheta)\, \zeta_n^1(kr_0)\, u_n(k, r).$$

If we make the same adjunction to (19a) then the continuous passage from (19a) to (19b) for $r = r_0$ is preserved, as is the radiation condition for $r \to \infty$. The right side of (19a) becomes

$$(21) \qquad \sum_{n=0}^{\infty} (2n+1)\, P_n(\cos\vartheta)\, \zeta_n^1(kr)\, u_n(k, r_0).$$

From (20) and (21) we obtain Green's function by adjoining the factor $k/4\pi i$ which, as in (16a), is due to the condition of a unit source. We then have:

$$(22) \quad G(P, Q) = \frac{k}{4\pi i} \sum_{n=0}^{\infty} (2n+1)\, P_n(\cos\vartheta) \begin{cases} \zeta_n^1(kr)\, u_n(k, r_0) & r > r_0, \\ \zeta_n^1(kr_0)\, u_n(k, r) & r < r_0. \end{cases}$$

This way of writing reveals the connection with our general method in (14). The function F of (14a) is now represented by

$$F = u_n(\lambda, r)\, u_n(\lambda, r_0);$$

except for a constant normalizing factor. The corresponding functions F_1, F_2 of (14b,c) become

$$F_{1,2} = \frac{1}{2} \begin{cases} \zeta_n^1(\lambda r)\, u_n(\lambda, r_0) & r > r_0, \\ \zeta_n^1(\lambda r_0)\, u_n(\lambda, r) & r < r_0 \end{cases}$$

By forming the residues for $\lambda = k$ we then obtain equation (22).

In Appendix II of this chapter we shall introduce a novel method of constructing Green's function, which not only improves the convergence

of the series in the most important cases, but also reveals new aspects of the method's applicability.

In the appendix to the following chapter we shall further show that this method would solve the problem of wireless telegraphy on a spherical earth (for infinitely conductive soil and a vertical "dipole antenna") if it were not for the decisive role of the ionosphere.

Finally we remark: a representation of the form (14) remains valid if as the surface σ we choose an ellipsoid instead of a sphere. Instead of the r, ϑ, φ we then have to use the coordinate system of confocal ellipsoids and hyperboloids. The spectrum of eigen values for the exterior of the ellipsoid will then remain discrete in the parameters of the one piece and two piece hyperboloids, but becomes continuous in the parameter of the ellipsoids. By integration over this last parameter we would obtain a simplification similar to that of (22). Even in the most general case where there are no separating coordinates, in which the eigenfunctions can be decomposed into products, we can still use equation (27.5) as a starting point for the representation of Green's function.

§ 29. The Eigen Value Spectrum of Wave Mechanics. Balmer's Term

The Schrödinger equation of wave mechanics for the simple case of the hydrogen atom reads

$$\Delta\psi + \frac{2m}{\hbar^2}(W - V)\,\psi = 0\,. \tag{1}$$

This is our equation (7.15), with the difference that the symbol of energy W has been replaced by the difference of the total energy W and the potential energy V or, mechanically speaking, by the kinetic energy. The Rutherford model for the H-atom consists of a nucleus, the proton with a $+e$ charge, and of an electron with a $-e$ charge that moves in the proton field. Its potential (Coulomb) energy measured in electrostatic units is

$$V = -\frac{e^2}{r}, \tag{2}$$

where r is the distance from the proton and V is normalized so that at infinity we have $V = 0$. The mass energy m_0c^2 of the electron at rest is not to be counted in the total energy. In the following we may consider the proton at rest at the point $r = 0$.

Equation (1) differs from the wave equation we have treated so far because the constant k^2 has been replaced by a *point function* which

becomes *singular* at the point $r = 0$. Whereas we have used k to denote eigen value, we shall now use W as an eigenparameter. Hence, we shall seek those values of W for which (1) has a solution which is continuous in the entire space. These solutions are the eigenfunctions of our "Kepler problem," where the nucleus plays the role of the sun and the electron the role of the planets. Since the electron may move in unbounded space, the spectrum of eigen values will be continuous in the r-coordinate as in equation (28.14). More important for us is the fact that the spectrum also has *discrete* components.

The spectral apparatus gives us the discrete spectrum by measuring the *line spectrum*, which, in the case of hydrogen, is given in the visible range by the Balmer series H_α, H_β, H_γ,H_δ, . . . The lines of this spectrum cumulate at the limit given by the Rydberg constant R. The adjoining *continuum* lies in the near ultraviolet range. Both the discrete and the continuous spectrum are given by the Schrödinger equation. This equation reduces to a simple mathematical formula the enigma of the spectral lines, with their finite cumulation point, the behavior of which differs so fundamentally from that of all mechanical systems.

Niels Bohr gave a general explanation of the Balmer series and its limiting frequencies twelve years before Schrödinger, by endowing the Rutherford model with certain quantum theoretical traits. However the concept of orbits he used lead to diverse contradictions and had to be abandoned in favor of the analytic model of equation (1). The fact that (1) is also based on quantum theory is indicated by the entrance of Planck's constant $\hbar = h/2\pi$.

What is the physical meaning of the eigenfunction ψ ? The answer to this question shows the complete revolution in the concept of nature that quantum theory has brought about: $|\psi|^2 dx\, dy\, dz$ stands for the *probability* with which we may expect to find the hydrogen electron at the point (x,y,z) within an error of dx,dy,dz. Hence, in wave mechanics the concept of probability takes the place of the concept of strict determinism which rules in classical mechanics. The measure of indeterminacy in the atomic range is Planck's h (Heisenberg).

The "normalization of the eigenfunctions to 1," which so far had been introduced only for mathematical simplicity, thereby acquires a fundamental meaning. Namely, the equation

$$\int |\psi|^2 \, d\tau = 1 \tag{3}$$

asserts the *certainty* that the electron is somewhere in space; this condition is necessary from the point of view of wave mechanics. Equation (3) holds for a discrete spectrum; for the continuum it must be modified according to the prescription of Appendix I to this chapter.

We now turn to the integration of (1), introducing the coordinates r, ϑ, φ. If we write the wave equation in the form (22.4) and let[11]

$$\psi = \chi(r)\, P_l^m(\cos\vartheta)\, e^{im\varphi} \tag{4}$$

then according to the differential equation (22.13) we obtain

$$\frac{d^2\chi}{dr^2} + \frac{2}{r}\frac{d\chi}{dr} + \left\{\frac{2m}{\hbar^2}\left(W + \frac{e^2}{r}\right) - \frac{l(l+1)}{r^2}\right\}\chi = 0. \tag{5}$$

We first consider the case in which the electron is *tied* to the nucleus. Then W must be negative since the energy of the electron at rest at infinity is normalized to zero. If it is absorbed by the nucleus and stably tied there then its energy is decreased. If, on the other hand, $W > 0$ then even for an infinite distance from the nucleus the electron has positive kinetic energy and, mechanically speaking, has a hyperbolic orbit.

The asymptotic behavior of χ for $r \to \infty$ is obtained from (5) by neglecting all terms with $1/r$ and $1/r^2$:

$$\frac{d^2\chi}{dr^2} + \frac{2m}{\hbar^2} W\chi = 0.$$

For negative W we write

$$\frac{d^2\chi}{d\varrho^2} = \frac{1}{4}\chi, \quad \chi = e^{-\varrho/2}, \quad \varrho = \frac{2r}{\hbar}\sqrt{-2mW}. \tag{5a}$$

The other solution of (5a), namely, $\chi = \exp(+\varrho/2)$, must be neglected since χ is to be finite everywhere.

In order to obtain an exact solution of (5) we write

$$\chi = e^{-\varrho/2}\, v(\varrho) \tag{6}$$

and obtain as the differential equation for v

$$v'' + \left(\frac{2}{\varrho} - 1\right)v' + \left[\frac{n-1}{\varrho} - \frac{l(l+1)}{\varrho^2}\right]v = 0 \tag{6 a}$$

with the abbreviation

$$n = \frac{m\, e^2/\hbar}{\sqrt{-2mW}}. \tag{6 b}$$

[11] Here we denote the lower index of P by l instead of n, corresponding to wave mechanical usage: l = azimuthal quantum number, n_r = radial quantum number, $n = n_r + l + 1$ = total quantum number, n = magnetic quantum number, where we now have $-l \leq m \leq +l$.

In order to discuss (6a) we use the method of equation (19.36). We write

$$v = \varrho^{\lambda} w, \qquad w = a_0 + a_1 \varrho + \tag{7}$$

and in analogy with (19.37) we obtain:

$$\lambda(\lambda + 1) = l(l + 1), \text{ and hence } \lambda = +l. \tag{7a}$$

The other root of (7a) $\lambda = -l - 1$ must be excluded, since v as well as χ must remain finite for $\varrho = 0$. The recursion formula for the a_k is obtained in analogy with (19.37a) by equating to zero the coefficients of $\varrho^{\lambda + k - 1}$ in the power series obtained from (6a) and (7). Thus we find:

$$a_{k+1}\,[(\lambda + k + 1)(\lambda + k) + 2(\lambda + k + 1) - l(l + 1)] + a_k\,[n - 1 - \lambda - k] = 0\,. \tag{7b}$$

If in this equation we make the coefficient of a_k equal to zero by setting

$$n = k + \lambda + 1, \tag{8}$$

then a_{k+1} vanishes and so do all the subsequent terms in w: *the series breaks off, that is, w becomes a polynomial* of degree k, whose further properties we shall treat later. For the time being we shall stress only the following facts: 1. Due to the factor $\exp(-\varrho/2)$ in (6), we see that as $r \to \infty$ the function χ tends to zero with sufficient rapidity to make possible the normalization of ψ according to (3), no matter what the degree of the polynomial w. 2. If the series did not break off then from (7b) we should obtain an asymptotic behavior of a_k which would make w become infinite to the order $\exp(+\varrho)$ for $\varrho \to \infty$, and the normalization of ψ would be impossible. Hence the requirement that the series for w break off is a wave mechanical *necessity*.

We now consider equation (8). We denote the value of k there by n_r (radial quantum number) and for λ we substitute its value from (7b) (azimuthal quantum number). Hence, according to (8) n is *integral:*

$$n = n_r + l + 1\,. \tag{8a}$$

This number n is called the "total quantum number." From equation (6b) we obtain:

$$W = W_n = -\frac{m e^4}{2\hbar^2 n^2}\,. \tag{8b}$$

Setting W equal to the energy quantum $h\nu$ we obtain

(9) $$\nu = \frac{m e^4}{2 h \hbar^2 n^2} = \frac{R}{n^2}$$

where

(9a) $$R = \frac{2 \pi^2 m e^4}{h^3} .$$

This R is the above mentioned "Rydberg frequency"; it can be measured spectroscopically with extraordinary precision and hence can lead to an improvement of our knowledge of the fundamental constants e,m,h. The number ν of (9) is called the *Balmer term*.

The observable frequency of a spectral line is obtained by the passage of the atom from an initial state 1 to a final state 2 and is computed as the difference of the associated terms ν_2 and ν_1. Hence for the hydrogen spectrum we have

(10) $$\nu = \nu_2 - \nu_1 = R \left(\frac{1}{n_2^2} - \frac{1}{n_1^2} \right).$$

The Balmer series corresponds to the passage into the final state $n_2 = 2$; the Lyman series which lies in the ultraviolet range corresponds to the passage into the fundamental state of the hydrogen atom $n_2 = 1$; in both cases the passage is from an arbitrary initial state $n_1 > n_2$. Hence we have

(10a) $$\nu = R \left(\frac{1}{2^2} - \frac{1}{n^2} \right), \qquad n = 3, 4, 5, \ldots \qquad \text{Balmer series,}$$

(10b) $$\nu = R \left(\frac{1}{1^2} - \frac{1}{n^2} \right), \qquad n = 2, 3, 4, \ldots \qquad \text{Lyman series.}$$

The series with $n_2 = 3$, $n_2 = 4$, ... lie in the infrared domain.

After having learned about the *eigen values* of the H-atom we wish to consider the analytic character of its *eigenfunctions*. With the use of (7), (7a) and (8a) we obtain from (6a)

(11) $$\varrho\, w'' + [2\,(l+1) - \varrho]\, w' + (n - l - 1)\, w = 0.$$

This equation is obtained through $(2l + 1)$-fold differentiation from the simpler differential equation

(12) $$\varrho L'' + (1 - \varrho)\, L' + \mu L = 0 \quad \text{with} \quad \mu = n + l.$$

For every integer μ this equation has one and only one polynomial solution of degree μ. With a suitable normalization we obtain the solutions:

$$\begin{aligned}
&\mu = 0, \quad L = 1,\\
&\mu = 1, \quad L = -\varrho + 1,\\
&\mu = 2, \quad L = \varrho^2 - 4\varrho + 2,\\
&\mu = 3, \quad L = -\varrho^3 + 9\varrho^2 - 18\varrho + 6.\\
&\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\end{aligned}$$

These are precisely the expressions we denoted in exercise I.6 as Laguerre polynomials; equation (12) is the Laguerre differential equation, as indicated by the choice of the letter L. This differential equation coincides with the differential equation (24.29) of the confluent hypergeometric function for the parameters $\alpha = -\mu = -n - l$. Hence we have

$$L = F(-n-l, 1, \varrho) \qquad \text{and} \qquad w = \frac{d^{2l+1}L}{d\varrho^{2l+1}}. \tag{13}$$

Hence from (4), (6), (7) and (7a) we obtain the representation of the hydrogen eigenfunction

$$\psi = N\varrho^l e^{-\varrho/2} \frac{d^{2l+1}L}{d\varrho^{2l+1}} P_l^m(\cos\vartheta)\, e^{im\varphi}. \tag{14}$$

where N is a normalization factor due to (3). From (5a) and (8b) we obtain ϱ :

$$\varrho = \frac{2r}{na}, \qquad a = \frac{\hbar^2}{me^2} \sim \frac{1}{2}\,10^{-8}\,\text{cm}. \tag{14a}$$

where, as is customary, a denotes the "hydrogen radius."

In order to justify this notation, and as a single special application of the above, we compute the "probability density" in the "fundamental state" $n = 1$ of the H-atom. For $n = 1$ we have according to (8a) $l = m = 0$, $n_r = 0$ and hence from (14)

$$\psi = -N_1 e^{-\varrho/2} = -N_1 e^{-r/a}, \qquad |\psi|^2 = N_1^2 e^{-2r/a},$$

where from (3) we obtain $N_1 = (\pi a^3)^{-\frac{1}{2}}$. Hence, the probability of finding the electron is distributed spherically over the nucleus. For $r = 0$ this probability assumes its maximum N_1^2, for $r = a$ its value is only $(N_2/e)^2$, but it only vanishes at infinity. The charge density is proportional to this probability. From the point of view of wave mechanical statistics we do not have an electron which is concentrated at a point, but instead we have a *charge cloud* whose principal part is in the interior of a sphere of radius a.

From the older point of view of orbits we must ascribe a *disc-like*

form to the H-atom. The fundamental state (circular orbit of radius a) then corresponds to a circular disc. In a magnetic field all the circular discs of an H-atom gas would have to be parallel to each other and perpendicular to the magnetic force lines; a light ray passing through this gas would have to show "magnetic double-diffraction." Precise measurements by Schütz, though performed not on an H-atom gas but on the analogous Na-vapor, showed no trace of this phenomenon. This is one of the contradictions which have been cleared up by wave mechanics.

A behavior similar to that of the fundamental state of the H-atom is obtained for all states with $l = 0$, the so-called "s-terms" of spectroscopy. For $l = 0$ we obtain from (14)

$$\psi = N_n e^{-\varrho/2} L_n'(\varrho), \qquad \varrho = \frac{2r}{n a}, \qquad n = n_r + 1,$$

which again means spherical symmetry. Such s-terms are the fundamental states of the alkali atoms Li, Na, K, ... The same holds for all completed shells, e.g., the so-called eight-shells of rare gases. The proof is based on the addition theorem of spherical harmonics. This spherical symmetry of the closed shells is obviously of great importance for all chemical applications.

We have to add a few remarks about the continuous spectrum of hydrogen, that is, about the states $W > 0$ (the "hyperbolic orbits" of the older theory). The electron is then no longer tied to the nucleus but is still in the field of the proton.

According to (5a) and (6b) ϱ and n become *purely* imaginary for $W > 0$. In the asymptotic solution (5a) the two signs of ϱ are equivalent; both solutions $\exp\{\pm \varrho/2\}$ can be used. It is unnecessary, and due to the imaginary character of ϱ it is also impossible, to make the series (7) break off. Hence every value of W is permissible. *The W-spectrum becomes continuous* and reaches from $W = 0$ to $W = \infty$. Since, according to (8b) $W = 0$ corresponds to the limit $n = \infty$ of the discrete spectra, we see that to each of these spectra there adjoins a continuous spectrum in the short wave direction. The analytic form of the representation (14) remains valid; but L is now no longer a Laguerre polynomial but a *confluent hypergeometric series which does not break off*, since the parameter $\alpha = -n - l$ in (13) is no longer negative integral but general complex.

§ 30. Green's Function for the Wave Mechanical Scattering Problem. The Rutherford Formula of Nuclear Physics

Nuclear physics originated with Rutherford's experiments on the scattering of α-rays by heavy atoms. Since the electron shells of the

atom are immaterial for the case of α-rays, we may treat the scattering problem in terms of the continuous H-spectrum. We are dealing, in fact, with a *two-body problem:* a *nucleus* (of charge Ze, where Z is the atomic number, $Z = 1$ for the H-spectrum) and a particle interacting with it (in this case an α-particle with mass m_α and charge $Z'e$ where $Z' = 2$; in the preceding case an electron of mass m and charge $-e$ corresponding to the charge number $Z' = -1$). First we want to find that point of the continuous spectrum that corresponds to the energy constant W_α of the incoming α-rays. For an infinite distance between the α-particle and the nucleus the kinetic energy of the α-particle is

$$W_\alpha = \frac{m_\alpha}{2} v^2, \qquad \text{hence} \qquad 2 m_\alpha W_\alpha = (m_\alpha v)^2 = p^2,$$

where $\vec{p} = m_\alpha \vec{v}$ is the kinetic momentum of the α-particle.

If we now pass from the *corpuscular interpretation* of the α-rays to the "complementary" *wave interpretation*, then $p/\hbar$ is, at the same time, the *wave number*[12] k_α of the α-rays.
Hence we have

$$k_\alpha = \frac{m_\alpha v}{\hbar} = \sqrt{\frac{2 m_\alpha W_\alpha}{\hbar^2}}. \tag{1}$$

We can, therefore, rewrite the variable ϱ of (29.5a) in the form

$$\varrho = 2 i k_\alpha r. \tag{2}$$

For an arbitrary point of the continuous spectrum (i.e., for an arbitrary value W different from W_α) we replace k_α by λ as in §28. Equations (1) and (2) then generalize to

$$\lambda = \sqrt{\frac{2 m_\alpha W}{\hbar^2}}, \qquad \varrho = 2 i \lambda r. \tag{2a}$$

If, as before, we assume the nucleus at rest then the wave equation (29.1) becomes

$$\Delta\psi + \frac{2 m_\alpha}{\hbar^2}\left(W - \frac{Z Z' e^2}{r}\right)\psi = 0. \tag{3}$$

For the time being we replace (3) by:

[12] In fact the formula $k_\alpha = p/\hbar$ is the equation of L. de Broglie: "h times the reciprocal of the wavelength equals the momentum," which in turn is the relativistic completion of Planck's equation: "h times the reciprocal of the time of oscillation equals the energy."

(3a) $$\Delta\psi + K^2\psi = 0, \qquad K^2 = \lambda^2 - \frac{2\,m_\alpha\,ZZ'e^2}{\hbar^2 r};$$

for the point $\lambda = k_\alpha$ of the spectrum we then have

(3b) $$K_\alpha^2 = k_\alpha^2 - \frac{2\,m_\alpha\,ZZ'e^2}{\hbar^2 r}.$$

We note the important fact that in the difference $K^2 - K_\alpha^2$ the potential term, which is a function of position, is eliminated, so that this difference becomes *independent of position:*

(4) $$K^2 - K_\alpha^2 = \lambda^2 - k_\alpha^2.$$

The reader should convince himself that all our previous deductions from Green's theorem, such as the orthogonality of the eigenfunctions in §26 or the representation of Green's function for *constant* k^2 in §27, remain valid for our generalized wave equation $\Delta\psi + K^2\psi = 0$ with K^2 a *function of position* given by (3a,b).

We now return to the Rutherford scattering experiment. If we consider the source of the α-rays (the radium particle) to be point-like and in the finite domain, then we have a *spherical wave* of corpuscular rays, which is modified by the presence of the nucleus in the manner prescribed by the wave equation (3). However if we remove the source to infinity, which is more natural and at the same time simpler, then we have to treat the same problem for the *plane wave.* In both cases the solution is given by Green's function of §28; in the first case for a general position of the source point Q, in the second case for the limit $Q \to \infty$. Since Green's function was to be summed over the complete system of eigenfunctions, we have to consider the discrete as well as the continuous eigen value spectrum for a finite Q. However in the limit $Q \to \infty$ $u(Q)$ vanishes for all eigen values of the discrete spectrum; hence, in this case we have to carry out the integration over only the continuous spectrum. We may retain the expression (29.14) for the eigenfunctions $u(P)$ in question, if we replace ϱ by $2i\lambda r$ in accordance with (2a). If in addition we let $\vartheta = 0$ be in the direction of the line which joins Q to the position O of the nucleus, then the scattering problem becomes symmetric with respect to the axis $\vartheta = 0$, and hence independent of φ, so that the eigenfunctions $u(P)$ must be independent of φ. Therefore according to (29.14) we have

(5) $$u(P) = \chi_l(\varrho)\,P_l(\cos\vartheta), \quad \chi_l = N\varrho^l e^{-\varrho/2}\frac{d^{2l+1}L}{d\varrho^{2l+1}}.$$

The corresponding expression for $u(Q)$ is obtained from (5) by replacing $P_l(\cos\vartheta)$ by $P_l(\cos\vartheta_0) = P_l(-1) = (-1)^l$, and $\chi_l(\varrho)$ by $\chi_l(\varrho_0)$

and then passing to the limit $\varrho_0 \to \infty$. We then obtain the representation of the plane wave from (28.14). By performing the integration over the path W of Fig. 26 and forming the residue at the pole $\lambda = k_\alpha$ we obtain a series representation of the form

$$\sum C_l \chi_l(\varrho) P_l(\cos\vartheta), \quad \varrho = 2ik_\alpha r, \tag{6}$$

where the coefficients C_l are determined in a somewhat cumbersome fashion in terms of the normalizing factors of the χ and P and of the asymptotic behavior of $\chi(\varrho_0)$. This representation was first derived by W. Gordon.[13]

A much simpler representation is obtained if we replace the polar coordinates r, ϑ by the *parabolic coordinates* ξ, η. We thus obtain as the wave function of the scattering process (see Appendix III of this chapter):

$$\psi = e^{ikz} L_n(ik\eta), \quad \eta = r - x = r(1 - \cos\vartheta). \tag{7}$$

Here k is the wave number of (1)

$$k = k_\alpha = \frac{m_\alpha v}{\hbar}; \tag{7a}$$

and n is the total quantum number, which, becomes purely imaginary, for the continuous spectrum as mentioned at the end of §29. This total quantum number is computed from (29.13) where, as in equation (3), we have to replace e^2 by $-ZZ'e^2$:

$$n = \frac{ie^2 ZZ'}{\hbar v}. \tag{7b}$$

The function L_n is the confluent hypergeometric function of (29.13) for $l = 0$:

$$L_n(\varrho) = F(-n, 1, \varrho). \tag{7c}$$

The variable η is the parabolic coordinate defined in (7), the other coordinate is $\xi = r + x = r(1 + \cos\vartheta)$. In the following ϑ will be called the "scattering angle."

From (7) we obtain the asymptotic value for $r \to \infty$

$$\psi = C_1 e^{ikz} + C_2 \frac{e^{ikr}}{r} \tag{8}$$

with the abbreviations

[13] *Z. Physik* (1928). See also the excellent book by Mott and Massey, The Theory of Atomic Collisions, Oxford, 1933, chapter III.

(8a) $$C_1 = \frac{(-i k \eta)^n}{\Gamma(n+1)}, \qquad C_2 = C_1^* \frac{i n/k}{1-\cos\vartheta}.$$

The first term on the right side of (8) represents the incoming *plane wave*, the second term represents the *spherical wave* scattered from the nucleus. The quantities C_1, C_2 are not constants but depend on η ; however, since n is purely imaginary only their phases depend on η. We are interested only in the absolute value of the ratio C_2/C_1, which is independent of η and hence of r, and depends only on the scattering angle ϑ . Namely, from (8a) and (7a,b) we obtain

(9) $$\left|\frac{C_2}{C_1}\right| = \frac{|in|}{k(1-\cos\vartheta)} = \frac{e^2 Z Z'}{m_\alpha v^2 (1-\cos\vartheta)} = \frac{e^2 Z Z'}{2 m_\alpha v^2 \sin^2\vartheta/2}.$$

According to the wave mechanical definition (29.3) of probability density, the square of this quantity is the *ratio of the number of scattered particles per unit of spatial scattering angle and the number of incoming particles per unit of area on a surface perpendicular to the incoming direction.* This law was deduced by Rutherford through geometric consideration of the classical hyperbolic orbits without the help of quantum theory. This was possible owing to the fact that the constant $\hbar$ canceled in (9). Rutherford's law holds not only for α- rays but also for any other particles (protons, electrons, . . .) which are in Coulomb interaction with the nucleus, of course with a correspondingly different meaning of Z' and m_α . The interesting "exchange effect" that occurs for the equality of scattering and the scattered particle will not be discussed here. For very high velocities of the scattering particle we should have to use relativity theory.

Appendix I

Normalization of the Eigenfunctions in the Infinite Domain

In passing from a bounded to an unbounded domain we encounter certain difficulties in convergence which can be removed only by a change in the normalization process. This modified process was introduced by H. Weyl in the theory of integral equations and has been adapted to wave mechanics.

As an example we choose the function $I_\nu(k r)$, where ν is arbitrary and k is a root of the equation $I_\nu(k a) = 0$. According to (20.19) its normalizing integral would be:

(1) $$N = \int_0^a [I_\nu(k r)]^2 \, r \, dr = \frac{a^2}{2} [I'_\nu(k a)]^2;$$

Due to the asymptotic behavior of $a I_\nu'$ this integral becomes divergent in the limit $a \to \infty$ (undetermined between the limits $\pm \infty$). In order to obtain a normalization of I_ν for $a \to \infty$ we start from the more general integral

$$N' = \lim_{a \to \infty} \int_0^a I_\nu(k r)\, I_\nu(k' r)\, r\, dr \tag{1a}$$

According to (21.9a) N' behaves like the function $\delta(k|k')$:[14] namely, it vanishes for $k \neq k'$ and becomes infinite for $k = k'$ so that

$$\int_\Delta N'\, k'\, dk' = 1 \tag{2}$$

where Δ is an arbitrary interval containing the critical point $k' = k$. Since in particular Δ can be chosen arbitrarily small, so that k' can be considered constant in Δ, we may replace (2) by

$$\int_\Delta N'\, dk' = \frac{1}{k}, \tag{2a}$$

and similarly

$$\int_\Delta N' \sqrt{k'}\, dk' = \frac{1}{\sqrt{k}}. \tag{2b}$$

We now change the normalizing integral (1) in the limit $a \to \infty$ to

$$N = \int_0^\infty r\, dr\, I_\nu(k r) \int_\Delta dk'\, I_\nu(k' r), \tag{3}$$

that is, in (1) we replace one[15] of the two proper oscillations I_ν by the group of neighboring proper oscillations

$$\int_\Delta dk'\, I_\nu(k' r) \tag{3a}$$

and thus eliminate the above indeterminacy by averaging, so to speak, by interference within the wave group. From (1a) and (2a) we then obtain

$$N = \int_\Delta N'\, dk' = \frac{1}{k}. \tag{4}$$

In order to make clear the mathematical essence of the above

[14] The symbols σ, r, s of (21.9a) have been replaced here by r, k', k.

[15] According to the original method of Weyl we could also replace *both* eigenfunctions I by "*eigendifferentials*" of the form (3a). Instead of the expressions "group of proper oscillations" or "wave group" used in the text, the less attractive term "wave bundle" is customarily used in wave mechanics.

process, we were somewhat careless in the interchanging of the limiting processes $a \to \infty$ and $\Delta \to 0$ (or in other words $k' \to k$). It would therefore be desirable to deduce equation (4) once more on the basis of Green's theorem. Writing

$$u = I_\nu(kr)\, e^{i\nu\varphi}, \quad v = I_\nu(k'r)\, e^{-i\nu\varphi}, \quad d\sigma = r\, dr\, d\varphi, \quad ds = a\, d\varphi \tag{5}$$

we compute[16] $\int (u\, \Delta v - v\, \Delta u)\, d\sigma$ in the known manner both as a surface integral and by Green's theorem as a contour integral. We then obtain:

$$(k^2 - k'^2) \int_0^a u\, v\, r\, dr = a\left(u \frac{\partial v}{\partial r} - v \frac{\partial u}{\partial r}\right)_{r=a}. \tag{5a}$$

If we divide by $k^2 - k'^2$ and integrate with respect to k' (under the integral sign) from $k' = k - \Delta/2$ to $k' = k + \Delta/2$, and then pass to the limit $a \to \infty$, the left side becomes the normalizing integral N of (3). On the right side we choose a so large that we can compute the I_ν asymptotically. If we choose half the sum of (19.55) and (19.56), the constants $\mp (\nu + \frac{1}{2})\pi/2$ in the exponents partly cancel and partly are of no consequence in the following consideration, and so may be omitted. Then we obtain as the right side of (5a):

$$\frac{1}{2\pi} \int_{k-\Delta/2}^{k+\Delta/2} \frac{1}{\sqrt{k\,k'}} \frac{dk'}{k^2 - k'^2} \{(e^{ika} + e^{-ika})\, ik' (e^{ik'a} - e^{-ik'a}) - (e^{ik'a} + e^{-ik'a})\, ik\, (e^{ika} - e^{-ika})\}. \tag{5b}$$

After multiplying out and collecting terms we obtain

$$\frac{1}{\pi} \int_{k-\Delta/2}^{k+\Delta/2} \frac{dk'}{\sqrt{k\,k'}} \left\{ \frac{\sin(k-k')\,a}{k-k'} + \frac{\sin(k+k')\,a}{k+k'} \right\}. \tag{6}$$

For sufficiently small Δ we can replace $\sqrt{k\,k'}$ by k and put it in front of the integral. If in the first term of the integral we make the substitution $x = (k - k')a$, then it becomes

$$-\int_{x_1}^{x_2} \sin x\, \frac{dx}{x} \quad \text{with} \quad \begin{cases} x_1 = a\,\Delta/2, \\ x_2 = -a\,\Delta/2, \end{cases}$$

In the limit $a \to \infty$ this assumes the value π (see exercise I.5). In the second term of the integral (6) we make the substitution $y = (k + k')a$.

[16] The domain of integration of Green's theorem is not the complete circle of radius a, but the domain of periodicity of u and v as in equation (25.5), namely a circular sector of angle $\alpha = 2\pi/\nu$. In the integration with respect to φ we obtain a factor α on both sides of (5a) that, in the text, has already been canceled.

The limits of integration then become

$$\left.\begin{aligned} y_1 &= (2k - \Delta/2)\,a \\ y_2 &= (2k + \Delta/2)\,a \end{aligned}\right\} \to \infty \quad \text{for} \quad a \to \infty .$$

It is easy to see that this second term vanishes. Hence (6) becomes $1/k$ and (5a) becomes

$$N = \frac{1}{k} \tag{7}$$

which coincides with (4). From this it follows that the Bessel function which is normalized to 1 in the above manner is given by

$$\frac{1}{\sqrt{N}}\, I_\nu(kr) = \sqrt{k}\, I_\nu(kr). \tag{7a}$$

From the relation (21.11)

$$I_{n+\frac{1}{2}}(kr) = \sqrt{\frac{2kr}{\pi}}\, \psi_n(kr) \tag{8}$$

we further see that the function ψ_n normalized to 1, which we denoted by Ψ_n on p. 197, is related to ψ_n by

$$\Psi_n(kr) = \sqrt{\frac{2}{\pi}}\, k\psi_n(kr). \tag{8a}$$

Indeed, from (7a) and (8) we have

$$1 = \int_0^\infty r\,dr \sqrt{k}\, I_{n+\frac{1}{2}}(kr) \int_\Delta dk' \sqrt{k'}\, I_{n+\frac{1}{2}}(k'r) = \frac{2}{\pi} \int_0^\infty r^2\,dr\, k\psi_n(kr) \int_\Delta dk'\, k'\psi_n(k'r)$$

substituting this in (8a) we obtain

$$1 = \int_0^\infty r^2\,dr\ \Psi_n(kr) \int_\Delta dk'\, \Psi_n(k'r), \tag{9}$$

which according to (3) means normalization to 1.

Equation (8) relates not only $I_{n+\frac{1}{2}}$ and ψ_n, but also the associated functions $H^{1,2}_{n+\frac{1}{2}}$ and $\zeta^{1,2}_n$; hence equation (8a) holds also for the ζ_n normalized to 1 which we denoted by Z_n on p. 197.

For a general three-dimensional eigenfunction equation (3) holds in the form

$$N = \int d\tau\, u_n \int_\Delta dk'\, u_n', \tag{10}$$

where u_n and u_n' are the eigenfunctions belonging to the eigen values k and k'. The eigenfunctions normalized to 1 are then

$$U_n = u_n/\sqrt{N}\,. \tag{10a}$$

Appendix II

A New Method for the Solution of the Exterior Boundary Value Problem of the Wave Equation Presented for the Special Case of the Sphere

The "exterior boundary value problem" consists of the construction of a solution of the wave equation $\Delta u + k^2 u = 0$, which is continuous throughout the exterior of the given bounded surface σ, assumes arbitrarily prescribed boundary values $u = U$ on σ, and satisfies the radiation condition at infinity. We know that this solution is best represented by Green's function which vanishes on σ, satisfies the radiation condition at infinity, and at a prescribed point Q has a discontinuity of the character of a unit source.

In the case of a sphere of radius a and a source point Q with the coordinates $r = r_0$, $\vartheta = 0$, we constructed G in the form of equation (28.22):

$$(1)\quad G = \frac{k}{4\pi i}\begin{cases}\sum C_n\, u_n(k, r_0)\, \zeta_n(k\, r)\, P_n(\cos\vartheta) & \text{for } r > r_0,\\ \sum C_n\, \zeta_n(k\, r_0)\, u_n(k, r)\, P_n(\cos\vartheta) & \text{for } r < r_0,\end{cases} \qquad C_n = 2n+1.$$

The radiation condition is satisfied by the factor $\zeta_n(k\,r)$, or more precisely $\zeta_n^1(k\,r)$, in the first line, the boundary condition for $r = a$ is satisfied by the factor in the second line

$$(1b)\qquad u_n(k, r) = \psi_n(k\,r) + A\,\zeta_n(k\,r), \qquad A = -\frac{\psi_n(k\,a)}{\zeta_n(k\,a)}.$$

where n is a positive integral and hence $P_n(\cos\vartheta)$ is continuous for all values $0 \leq \vartheta \leq \pi$.

We now attempt to solve this problem in a more economical manner, by subjecting $\zeta_\nu(k\,r)$ not only to the radiation condition, but also to the boundary condition

$$(2)\qquad \zeta_\nu(k\,a) = 0$$

Then ν must be integral, since the roots of $\zeta_\nu(\varrho) = 0$ coincide with those of $H^1_{\nu+\frac{1}{2}}(\varrho) = 0$. According to (21.41) these roots are non-integral and *complex* (of large absolute value). We denote the consecutive roots which lie in the first quadrant of the complex ν-plane by $\nu_1, \nu_2, \cdots$ and the general root by ν_m. We use $\sum$ to denote summation over the complete system of these roots, and write

$$(3)\qquad G = \sum D_\nu\, \zeta_\nu(k\,r)\, P_\nu(-\cos\vartheta).$$

The function P_ν here is *not a Legendre polynomial* but the *hyper-*

geometric series of (24.24a)

$$P_\nu(-\cos\vartheta) = F\left(-\nu, \nu+1, 1, \frac{1+\cos\vartheta}{2}\right). \tag{3a}$$

The fact that in (3) we used $P_\nu(-\cos\vartheta)$ instead of $P_\nu(+\cos\vartheta)$ is due to the fact that G is to be regular on the ray $\vartheta = \pi$ whereas the ray $\vartheta = 0$ is to contain the singular point Q. According to (3a) the function $P_\nu(-\cos\vartheta)$ for $\vartheta = \pi$ has the value $F(-\nu, \nu+1, 1, 0) = 1$; for $\vartheta = 0$ we obtain from (24.32), after replacing ζ by $-\zeta = -\cos\vartheta$, the value

$$F(-\nu, \nu+1, 1, 1) \to \frac{\sin \nu\pi}{\pi} \log \vartheta^2. \tag{4}$$

We now turn to the determination of the coefficients D_ν, to which F. Sauter has made such valuable contributions. We remark: *the functions* ζ_ν *are mutually orthogonal, that is, we have*

$$\int_{ka}^{\infty} \zeta_\nu(\varrho)\, \zeta_\mu(\varrho)\, d\varrho = 0, \quad \mu \neq \nu \tag{5}$$

This is a consequence of the differential equation for ζ_ν, which in analogy to (21.11) reads:

$$\varrho \frac{d^2\varrho\zeta_\nu}{d\varrho^2} + [\varrho^2 - \nu(\nu+1)]\, \zeta_\nu = 0.$$

If we also write the same equation for ζ_μ and multiply these equations by ζ_μ and ζ_ν respectively, then by integrating the difference of these equations over the fundamental domain $ka \leqq \varrho < \infty$ we obtain:

$$\{\nu(\nu+1) - \mu(\mu+1)\} \int_{ka}^{\infty} \zeta_\mu\, \zeta_\nu\, d\varrho = \varrho\left(\zeta_\mu \frac{d\varrho\,\zeta_\nu}{d\varrho} - \zeta_\nu \frac{d\varrho\,\zeta_\mu}{d\varrho}\right)\Big|_{ka}^{\infty}. \tag{5a}$$

The right side of (5a) vanishes at the lower limit on account of (2), at the upper limit on account of the asymptotic behavior of the ζ according to (19.55). Thus (5) is proven. At the same time (5a) yields the normalizing integral

$$N = \int_{ka}^{\infty} [\zeta_\nu(\varrho)]^2\, d\varrho = \operatorname*{Lim}_{\mu\to\nu} \varrho\, \frac{\zeta_\mu \frac{d\varrho\,\zeta_\nu}{d\varrho} - \zeta_\nu \frac{d\varrho\,\zeta_\mu}{d\varrho}}{\nu(\nu+1) - \mu(\mu+1)}\Bigg|_{ka}^{\infty}$$

Differentiating the numerator and denominator with respect to μ and considering (2) we obtain

$$N_\nu = \frac{(ka)^2}{2\nu+1}\left(\frac{\partial\zeta_\nu}{\partial\nu}\frac{d\zeta_\nu}{d\varrho}\right)_{\varrho=ka}, \tag{6}$$

Introducing the abbreviations

$$\eta_\nu(\varrho) = \frac{\partial \zeta_\nu(\varrho)}{\partial \nu}, \qquad \zeta'_\nu(\varrho) = \frac{d\,\zeta_\nu(\varrho)}{d\varrho}. \tag{6a}$$

we can rewrite (6) in the shorter form

$$N_\nu = \frac{(k\,a)^2}{2\,\nu+1}\,\eta_\nu(k\,a)\,\zeta'_\nu(k\,a). \tag{6b}$$

If we now multiply (3) by $\zeta_{\bar\nu}(k\,r)\,k\,dr$ and integrate from $r = a$ to $r = \infty$ then, on account of the orthogonality and the normalization, we obtain:

$$D_{\bar\nu}\,N_{\bar\nu}\,P_{\bar\nu}(-\cos\vartheta) = \int_a^\infty G\,\zeta_{\bar\nu}(k\,r)\,k\,dr. \tag{7}$$

where $\bar\nu$ is any one of the roots ν.

However this determination of the D is not yet satisfactory since we do not know G. We therefore perform the following limit process: we divide (7) by $P_\nu(-\cos\vartheta)$ (we now omit the bars over the ν) and then let ϑ approach zero. We also split off the source point singularity from G by writing

$$-4\pi G = \frac{e^{ikR}}{R} + g, \qquad R^2 = r^2 + r_0^2 - 2\,r\,r_0\cos\vartheta. \tag{7a}$$

Then equation (7) becomes

$$-4\pi\,D_\nu\,N_\nu = \operatorname*{Lim}_{\vartheta\to 0}\int_a^\infty \left(\frac{e^{ikR}}{R} + g\right)\zeta_\nu(k\,r)\,k\,dr \Big/ P_\nu(-\cos\vartheta). \tag{7b}$$

Now g remains finite whenever $a < r < \infty$ and ϑ is arbitrary, whereas $P_\nu(-\cos\vartheta)$ becomes infinite as $\vartheta \to 0$; hence the contribution of g on the right side of (7b) vanishes. For the same reason we may restrict the contribution of e^{ikR}/R to an integration over the immediate neighborhood of the source point coordinate r_0 by writing

$$r = r_0(1+\eta), \qquad -\varepsilon < \eta < +\varepsilon, \qquad dr = r_0\,d\eta,$$
$$\zeta_\nu(k\,r) = \zeta_\nu(k\,r_0), \qquad e^{ikR} = 1,$$

while the denominator R is approximated by:

$$R = r_0\sqrt{(1+\eta)^2 + 1 - 2\,(1+\eta)\left(1-\frac{\vartheta^2}{2}\right)} \sim r_0\sqrt{\eta^2+\vartheta^2}.$$

Thus (7b) becomes

$$(7c) \qquad 4\pi D_\nu N_\nu = -k\,\zeta_\nu(k r_0) \operatorname*{Lim}_{\vartheta \to 0} \int_{-\varepsilon}^{+\varepsilon} \frac{d\eta}{\sqrt{\eta^2+\vartheta^2}} \Big/ P_\nu(-\cos\vartheta).$$

The limit on the right side is known from (24.31) to be $\pi/\sin \nu\pi$. Hence with the help of (6) we obtain from (7c) the completely determined value

$$(8) \qquad D_\nu = -\frac{1}{4\,k\,a^2} \frac{2\nu+1}{\sin\nu\pi} \frac{\zeta_\nu(k r_0)}{\eta_\nu(k a)\,\zeta'_\nu(k a)}.$$

which no longer depends on ε. Equation (3) for Green's function then assumes the form

$$(9) \qquad G = -\frac{1}{4\,k\,a^2} \sum_\nu \frac{2\nu+1}{\sin\nu\pi} \frac{\zeta_\nu(k r_0)\,\zeta_\nu(k r)}{\eta_\nu(k a)\,\zeta'_\nu(k a)} P_\nu(-\cos\vartheta).$$

This formula becomes considerably simpler if, instead of the ζ, we introduce the normalized eigenfunctions

$$Z_\nu(k r) = \frac{\zeta_\nu(k r)}{\sqrt{N_\nu}}, \qquad Z_\nu(k r_0) = \frac{\zeta_\nu(k r_0)}{\sqrt{N_\nu}}$$

Indeed, with the help of (6b) we can then rewrite (9) as

$$(9a) \qquad G = -\frac{k}{4} \sum_\nu Z_\nu(k r_0)\, Z_\nu(k r) \frac{P_\nu(-\cos\vartheta)}{\sin\nu\pi}.$$

Equation (9a) will prove useful later on; for the time being we shall use equation (9), which has the following advantages and disadvantages:

1. In (9) Green's function is represented by the *same* formula both for $r > r_0$ and for $r < r_0$, not by two *different* formulas as in (1).

2. The general requirement of the reciprocity of Green's function is satisfied owing to the fact that (9) is symmetric in r and r_0. On the other hand in equation (1) the reciprocity of G was expressed by the fact that by interchanging r and r_0 we interchange the two representations for $r > r_0$ and $r < r_0$. The reciprocity of G with respect to the angles ϑ and ϑ_0 (we considered the case in which the latter is zero), can be expressed both in (9) and in (1) by replacing $\cos\vartheta$ by $\cos\Theta$ which is symmetric in ϑ and ϑ_0.

3. The orthogonality relation (5) is essentially different from our previous formulations: in (5) ζ_ν is multiplied by ζ_μ and not by ζ_μ^* as in (25.11a); also, in (5) $\zeta_\nu\,\zeta_\mu$ is multiplied by the one-dimensional interval dr not by $r^2 dr$ as in the application of Green's theorem in exercise V.1b.

4. It is remarkable that our representation (9) seems to break

down for $\vartheta = 0$, since then according to (4) every term of the series becomes infinite like $\log \vartheta^2$, whereas the function it represents is to be regular for $\vartheta = 0$ and $r \neq r_0$. Hence the *whole* ray $\vartheta = 0$(not only the *point* $\vartheta = 0, r = r_0$ on it) must be considered a *singularity* of the representation (9). Hence in the neighborhood of this singularity, that is to say in a more or less narrow cone around this ray, our representation will become more or less useless. The question of the completion of our representation for the interior of such a cone shall be postponed to p. 221.

5. On the other hand equation (9) simplifies for the neighborhood of $\vartheta = \pi$, say for $\vartheta = \pi - \delta$, where in the original form (3) it reads:

$$G = \sum D_\nu \zeta_\nu (k r) P_\nu (\cos \delta). \tag{10}$$

We now let r_0 increase to ∞, that is to say we pass from the primary *spherical* wave G to a *plane* wave u coming in the direction $\vartheta = 0$. Then for $H_{\nu+1/2} (k r_0)$ we may use Hankel's approximation (19.55) no matter what the index ν since now the argument is large compared to the index. Then according to (21.15) we may write:

$$\zeta_\nu (k r_0) = \frac{1}{k r_0} e^{i(k r_0 - (\nu+1)\pi/2)}. \tag{10a}$$

By combining all the factors which are independent of ν into the amplitude A we obtain from (8),

$$D_\nu = A \frac{2\nu+1}{\sin \nu \pi} e^{-i\nu\pi/2} \Big/ \eta_\nu (k a)\, \zeta_\nu' (k a). \tag{10b}$$

Substituting this value in (10) we obtain the diffraction field of the plane wave u in the rear of a sphere of radius a under the angle of diffraction δ. For the time being we set the boundary condition $u = 0$ for $r = a$; later on we shall discuss a boundary condition that is adapted to electromagnetic optics.

6. The great advantage of (9) as compared to (1) lies in its *rapid convergence for large values of ka*. In order to test this convergence we compute the factors in the denominator of (8) for large ka and ν. If as in (21.30a) we set

$$\cos \alpha = \frac{\nu}{\varrho}, \tag{11}$$

then we have according to (21.39)

$$\zeta_\nu (\varrho) = \frac{i}{\varrho \sqrt{\sin \alpha}} \sin z; \tag{11a}$$

where

$$z = \varrho\,(\sin\alpha - \alpha\cos\alpha) - \pi/4. \tag{11b}$$

The roots of $\zeta_\nu = 0$ are given by

$$\sin z = 0,\quad z = z_m = -m\pi,\quad \cos z_m = (-1)^m,\quad \cos^2 z_m = 1. \tag{11c}$$

Hence according to (11a) we obtain for $z = z_m$, if we ignore the slowly varying factor $\sin\alpha$,

$$\frac{\partial\zeta_\nu}{\partial\alpha} = \frac{i}{\varrho\sqrt{\sin\alpha}}\left(\cos z\,\frac{dz}{d\alpha}\right)_{z=z_m} = i\,\alpha\sqrt{\sin\alpha}\,\cos z_m,$$

therefore by (11)

$$\eta_\nu = \frac{\partial\zeta_\nu}{\partial\nu} = \frac{\partial\zeta_\nu}{\partial\alpha}\frac{d\alpha}{d\nu} = \frac{-i\,\alpha}{\varrho\sqrt{\sin\alpha}}\cos z_m. \tag{11d}$$

On the other hand, for $z = z_m$ we obtain from (11a,b), if we remember that $\sin z_m = 0$,

$$\begin{aligned}\zeta'_\nu = \frac{d\zeta_\nu}{d\varrho} &= \frac{\partial\zeta_\nu}{\partial\varrho} + \frac{\partial\zeta_\nu}{\partial\alpha}\frac{d\alpha}{d\varrho} = \frac{i}{\varrho\sqrt{\sin\alpha}}\cos z_m\left(\frac{\partial z}{\partial\varrho} + \frac{\partial z}{\partial\alpha}\frac{d\alpha}{d\varrho}\right)_{z=z_m}\\ &= \frac{i}{\varrho}\sqrt{\sin\alpha}\,\cos z_m.\end{aligned} \tag{11e}$$

Finally we obtain from (11d,e) for $\varrho = ka$

$$\eta_\nu(ka)\;\zeta'_\nu(ka) = \alpha\left(\frac{\cos z_m}{ka}\right)^2 = \frac{\alpha}{(ka)^2}. \tag{12}$$

If we substitute this in (8) we obtain

$$D_\nu = -\frac{k}{4\,\alpha}\,\frac{2\nu+1}{\sin\nu\pi}\,\zeta_\nu(kr_0). \tag{13}$$

According to (21.41) ν in the first approximation is equal to ka, but with increasing m it increases in the positive imaginary direction. Hence $\sin\nu\pi$ increases exponentially in the sequence $\nu_1, \nu_2, \ldots$ and hence D_ν decreases exponentially due to the denominator $\sin\nu\pi$.

The same thing occurs in (13) due to the factor $\zeta_\nu(kr_0)$. This latter factor is to be computed according to Debye's formula (21.32) (the higher saddle point is the determining one) and not according to (11a) (where the saddle points are approximately of the same height). Hence the auxiliary angle now has a meaning different from that in (11): $\cos\alpha$ is equal to ν/kr_0 and not to ν/ka as before. Hence, $\zeta_\nu(kr_0)$

also decreases exponentially on the ν- sequence. The same thing holds for the additional factor $\zeta_\nu(k\,r)$ in (9).

In the special case of wireless telegraphy (appendix to Chapter VI) we would need about 1000 terms for a representation of the type (1), whereas, as we shall see, we need only one or two terms of the corresponding series of the type (9). In this appendix we shall also discuss how one type is obtained from the other by a *purely mathematical transformation* (in the complex plane of the index ν) .

7. The *structure* of Green's function and its *singular behavior for* $\vartheta = 0$ becomes particularly clear in our representation (9a). In order to obtain a rough estimate for the behavior of Green's function for small ϑ we use the approximation formula (24.32)

$$P_\nu(-\cos\vartheta) \to \frac{\sin\,\nu\pi}{\pi} \log\vartheta^2 \quad \text{for} \quad \vartheta \to 0$$

and then obtain from (9a)

$$G = -\frac{k}{4\,\pi} \log\vartheta^2 \sum_\nu Z_\nu(k\,r_0)\, Z_\nu(k\,r)\,. \tag{14}$$

Here $\sum_\nu$ has a " δ- like character." Namely, if we expand a function in the fundamental interval $a < r < \infty$ in an arbitrary normalized orthogonal system of functions $Z_\nu\,(k\,r)$:

$$\delta(r, r_0) = \left\{\begin{matrix} 0 \ldots\ r \neq r_0 \\ \infty \ldots\ r = r_0 \end{matrix}\right\} = \sum A_\nu\, Z_\nu(k\,r) \ldots \text{with} \int_{r_0-\varepsilon}^{r_0+\varepsilon} \delta(r, r_0)\, dr = 1\,,$$

then we obtain formally:

$$A_\nu = \int_0^\infty \delta(r, r_0)\, Z_\nu(k\,r)\, dr = Z_\nu(k\,r_0) \int_{r_0-\varepsilon}^{r_0+\varepsilon} \delta(r, r_0)\, dr = Z_\nu(k\,r_0)\,,$$

but as yet we know nothing about the convergence of this general δ- series

$$\delta(r, r_0) = \sum Z_\nu(k\,r_0)\, Z_\nu(k\,r) \qquad \text{for} \quad r \neq r_0 \tag{14a}$$

Only the divergence for $r = r_0$ (all terms positive) is apparent. The "representation of zero" for $r \neq r_0$ is obtained by an infinitely rapid "oscillation around zero." Hence (14a) is not suitable for the actual computation of G for $\vartheta \to 0$.

We obtain this representation from the defining equation (7a) of the function g, which is continuous throughout, and which for $r = a$ assumes the boundary values

(14b) $$g = g_a = -\frac{e^{ikR_a}}{R_a}, \quad R_a^2 = a^2 + r_0^2 - 2ar_0\cos\vartheta_a;$$

here ϑ_a is the polar distance on the sphere $r = a$. In addition, g must satisfy the differential equation $\Delta g + k^2 g = 0$ in the exterior of the sphere $a < r < \infty$, and the radiation condition at infinity. Hence g can be computed as a solution of the exterior boundary value problem on p. 214, which can be represented in terms of Green's function by

$$g = \int g_a \frac{\partial G}{\partial n} d\sigma_a, \qquad \frac{\partial G}{\partial n} = -\left(\frac{\partial G}{\partial r}\right)_{r=a}.$$

Using the representation (9) of G we obtain for g a series summed over ν, which, on the ray $\vartheta = 0$, $a < r < \infty$, reads:

(14c) $$g(r, \vartheta = 0) = -\frac{\pi}{2}\sum \frac{2\nu+1}{\sin \nu\pi} C_\nu \frac{\zeta_\nu(kr)}{\eta_\nu(ka)},$$

(14d) $$C_\nu = \int_0^\pi g_a P_\nu(-\cos\vartheta_a)\sin\vartheta_a \, d\vartheta_a.$$

Since the singularity $\vartheta = 0$ of $P_\nu(-\cos\vartheta)$ now occurs only under the integral in (14d) and is only of logarithmic order, the coefficients C_ν are all finite; however their explicit computation[17] does not seem to be easy.

In this appendix we have introduced an entirely new kind of "*singular eigenfunctions*," which are essentially different from the "*regular eigenfunctions*" that we have used so far in this chapter. Our singular eigenfunctions

$$\zeta_\nu(kr)\, P_\nu(-\cos\vartheta)$$

are not *free* oscillations, but require a *stimulation* along the ray $\vartheta = 0$. On the other hand each of the particular solutions

$$u_n(k, r)\, P_n(\cos\vartheta) \quad \text{and} \quad \zeta_n(kr)\, P_n(\cos\vartheta)$$

in (1) is *source free* throughout the physical domain $a \leq r < \infty$, $0 \leq \vartheta \leq \pi$. Their stimulation, if we should speak of one, takes place in the *exterior* of this domain, namely in the center $r = 0$, and for u_n at infinity.

In the author's 1912 paper, quoted on p. 183, our "regular" eigenfunctions were called "*improper*" and our "singular" eigenfunctions were called "*proper*." The following discussion may serve to justify this apparently paradoxical notation.

[17] See the discussion of "Whittaker's integral," which is a limiting case of our C_ν, in the textbook by Watson, pp. 239–240.

We start from the fact that for all oscillation problems, whether free or forced, periodic or damped, we have the relation

$$c = \frac{\omega}{k} \tag{15}$$

where we may assume c (the velocity of sound or light) to be real and the carrying medium to be absorption free. Heretofore we have assumed ω to be real and the time dependence to be in the form $\exp\{-i\omega t\}$. The equation

$$a\cos\alpha = \frac{\nu}{k} \tag{15a}$$

which follows from (11), and our condition $\zeta_\nu(ka) = 0$, then yielded a *complex* value of ν *with positive imaginary part*, so that for real k the quantity $a\cos\alpha$ also had the same character and hence was of the form

$$a\cos\alpha = A + iB \text{ with } B > 0. \tag{15b}$$

Now however, while preserving the relations (15), (15a), (15b), we set ν equal to a *positive integer*, say $= n$. Then from these relations we obtain *complex* values for k and ω with *negative imaginary parts:*

$$k = k_1 - i k_2 = \frac{n}{A + iB} = n\frac{A - iB}{A^2 + B^2}, \quad \omega = \omega_1 - i\omega_2 = c(k_1 - i k_2).$$

The boundary condition $\zeta_\nu(ka) = 0$ now becomes

$$\zeta_n\{(k_1 - i k_2)a\} = 0 \tag{15c}$$

and the above purely periodic time dependence factor $\exp\{-i\omega t\}$ becomes

$$\exp\{-i\omega_1 t\}\, e^{-\omega_2 t} = \exp\{-ick_1 t\}\, e^{-ck_2 t}.$$

Appending this time factor and considering the regular character of $P_n(-\cos\vartheta)$ for integral n we obtain from our singular eigenfunction (14) the *damped oscillation*

$$\zeta_n[(k_1 - i k_2)r]\, P_n(-\cos\vartheta)\exp\{-ick_1 t\}\, e^{-ck_2 t}, \tag{16}$$

which is *regular* throughout the region $r > a$. The amplitude is infinite at $t = -\infty$, decreases at a constant rate, and vanishes at $t = +\infty$, whereas the frequency remains constant. (The only exception is the surface $r = a$, since there we have $\zeta_n(ka) = 0$ throughout.) There exist ∞^2 such oscillations of zonal character. Their parameters are the integer n and the complex roots of the transcendental equation $\zeta_n(ka) = 0$

which are infinite in number. (For tesseral spherical harmonics the number of possible oscillations would increase to ∞^3.) These damped oscillations are obvioualy the physically simplest particular solutions of our sphere problem for the boundary condition $u = 0$ and hence deserve the name "proper eigenfunctions." Their close connection with our singular eigenfunctions (16) explain how the latter led to the simplest solution of the boundary value problem.

In the following discussion we apply these eigenfunctions, which so far have been developed only for scalar fields, to the case of the electromagnetic-optic field. We shall see in Chapter VI that this can be done without difficulty. We have to keep in mind the following facts:

1. The boundary condition $u = 0$ must be replaced in the electromagnetic case by

$$\frac{\partial}{\partial r}(r\,u) = 0, \tag{17}$$

as will be shown in Chapter VI.

2. In the transcendental equation $\zeta_\nu(ka) = 0$, or, as we wrote in (15c), $\zeta_n(ka) = 0$, we have to replace the function $\zeta_n(\varrho)$ by

$$\xi_n(\varrho) = \frac{\partial}{\partial r}\{r\,\zeta_n(\varrho)\},$$

which, due to $\varrho = k\,r$, is the same as

$$\xi_n(\varrho) = \zeta_n(\varrho) + \varrho\,\zeta_n'(\varrho). \tag{17 a}$$

In the same manner we have to replace the function η in (6a) by

$$\eta_n(\varrho) = \frac{\partial \xi_n(\varrho)}{\partial n}. \tag{17 b}$$

3. While in the scalar case we had *one* field function u, in the electromagnetic case we have *two* such functions u and v. The function v satisfies the same differential equation as u and a similar boundary condition.

Our damped electromagnetic eigen-oscillations have long been known in the literature. In the case of the *sphere* they were investigated by J. J. Thomson[18] in 1884 as the simplest case of the *Hertz oscillator* which was then the center of interest. They were generalized by M. Abraham[19] to the case of the elongated *ellipsoid of revolution* (rod-like oscillator) and the *Paraboloid of revolution* (wire with free ends). Indeed,

[18] *London Math. Soc. Proc.* **15**, 197, and the textbook Recent Researches in Electricity and Magnetism, Oxford 1893, the so-called "third volume of Maxwell."

[19] *Ann. Physik* **66**, 435 (1898); **67**, 834 (1899); *Math. Ann.* **52**, 81 (1899). For further literature see *Enzykl. d. Math. Wiss.* v. V. 2, Abraham's article, p. 508.

our entire development can, with the introduction of elliptic coordinates, be adapted without fundamental change from the cylindrical and spherical harmonics of the representation (16) to the domain of the Lamé wave functions. We can use this method in order to construct Green's function for the exterior of an ellipsoid or paraboloid and thus obtain general solutions to the associated boundary value problems.

We finally indicate some problems for which the method of this appendix is helpful.

a) Dispersion by colloidal particles. In a 1908 paper G. Mie deduced the impressive color phenomena seen in the ultramicroscope from the dielectricity constant and the conductivity of the individual scattering particles. The particle was assumed to be spherical with a diameter *small compared to the wavelength*, i.e., $ka \ll 1$. In this case the series of type (1) converge sufficiently rapidly. In the opposite case $ka \gg 1$ the use of geometrical optics suffices; but the intermediate case gives rise to difficulties. In this intermediate case we have to use series of the type (9) as specialized for a source at infinity. The fact that in our case the sphere was assumed to be infinitely conductive, while in Mie's case it was assumed to be an arbitrary dispersive medium, does not make an important difference. We must merely replace the *boundary condition* (equation (17)) for the complete conductor by a *transition condition* between the interior and the exterior. The convergence of the series will be the better the nearer we are to the limiting case of geometrical optics.

b) The reflection of a plane wave on the surface of a completely conductive sphere. The diffraction field in the *rear* of a sphere was discussed schematically (i.e., with the simplified boundary condition $u = 0$ and for a scalar field) under 5 above and was represented by the equations (10), (10a) for $ka \gg 1$. On the *front* of the sphere, especially for $\vartheta = 0$, we know from experience of strange interference phenomena, which so far have not been amenable to the usual treatment by series of type (1). The analytical difficulties which arise here are expressed by the singularity of the ray $\vartheta = 0$ in series of the type (9). However, we claim that this problem can be treated in the manner indicated on p. 221, if we take into consideration the actual conditions of the reflection problem.

c) The rainbow. With this classic problem we return to the starting point of Debye's asymptotic investigations (see p. 117) and all subsequent advances in the domain of short waves ($ka \gg 1$). The rainbow problem has since been brought to a beautiful conclusion by B. Van der Pol and H. Bremmer.[20] However, from the viewpoint of method there remains a gap between the wave-optical and geometric-optical method.

It was the task of this appendix to bridge such gaps mathematically.

[20] *Phil. Mag.* **24**, 141, 825 (1937).

Appendix III

The Wave Mechanical Eigenfunctions of the Scattering Problem in Parabolic Coordinates

In the following discussion we outline the steps which lead to the representation (30.7). For details the reader is referred to textbooks on wave mechanics.[21]

The parabolic coordinates $\xi = r + x$, $\eta = r - x$ define, in a plane which passes through the x-axis, a double system of confocal parabolas which have the point $r = 0$ as a common focus. The degenerate parabolas $\xi = 0$, $\eta = 0$ coincide with the negative and positive x-axis respectively; the parabolas $\xi = \infty$, $\eta = \infty$ limit the plane in the direction of large positive and negative x respectively.

If we rotate the plane around the x-axis then ξ, η together with the rotation angle φ form a spatial coordinate system which bears the following relation to the Cartesian coordinates x,y,z:

$$x = \frac{1}{2}(\xi - \eta), \qquad y = \sqrt{\xi\,\eta}\cos\varphi, \qquad z = \sqrt{\xi\,\eta}\sin\varphi.$$

From this we obtain the line element

$$ds^2 = \frac{1}{4}(\xi + \eta)\left(\frac{d\xi^2}{\xi} + \frac{d\eta^2}{\eta}\right) + \xi\,\eta\,d\varphi^2. \tag{1}$$

With its help $\Delta\psi$ is transformed into

$$\Delta\psi = \frac{4}{\xi + \eta}\left(\frac{\partial}{\partial\xi}\,\xi\,\frac{\partial\psi}{\partial\xi} + \frac{\partial}{\partial\eta}\,\eta\,\frac{\partial\psi}{\partial\eta}\right) + \frac{1}{\xi\,\eta}\,\frac{\partial^2\psi}{\partial\varphi^2}.$$

The wave equation (29.1) for an interaction energy

$$V = \frac{Z\,Z'\,e^2}{r} = \frac{2\,Z\,Z'\,e^2}{\xi + \eta}$$

and for independence from φ, becomes

$$\frac{\partial}{\partial\xi}\,\xi\,\frac{\partial\psi}{\partial\xi} + \frac{\partial}{\partial\eta}\,\eta\,\frac{\partial\psi}{\partial\eta} + \frac{m_\alpha}{2\,\hbar^2}[(\xi + \eta)\,W - 2\,Z\,Z'\,e^2)]\,\psi = 0. \tag{2}$$

This can be separated by setting $\psi = \psi_1(\xi)\,\psi_2(\eta)$; with β as the separation constant we then obtain:

[21] For example, the author's Atombau und Spektrallinien, v. II, Chapter V. §6 and Chapter 11, §9. There, in addition, the asymptotic representation (30.8) is derived with the help of a complex integral representation of L that we cannot discuss here.

$$\frac{d}{d\xi}\,\xi\,\frac{d\psi_1}{d\xi} + \left(\frac{m_\alpha W}{2\hbar^2}\,\xi - \frac{m_\alpha Z Z' e^2}{2\hbar^2} + \beta\right)\psi_1 = 0 \tag{3}$$

$$\frac{d}{d\eta}\,\eta\,\frac{d\psi_2}{d\eta} + \left(\frac{m_\alpha W}{2\hbar^2}\,\eta - \frac{m_\alpha Z Z' e^2}{2\hbar^2} - \beta\right)\psi_2 = 0. \tag{4}$$

The function ψ_1 must satisfy the *radiation condition* (28.2) for $\xi \to \infty$ (large positive x). Written in parabolic coordinates (according to (1) ds_ξ is equal to $\frac{1}{2}\,d\xi$ for large ξ, whence $\partial/\partial r$ in (28.2) becomes $\partial/\partial s_\xi = (d\xi/ds_\xi)\,\partial/\partial\xi \equiv 2\,\partial/\partial\xi$, and we have:

$$\frac{\xi}{2}\left(2\,\frac{d\psi_1}{d\xi} - i\,k\,\psi_1\right) = 0 \text{ with } k = \frac{m_\alpha v}{\hbar}, \text{ as in (30.7a).} \tag{5}$$

Hence we set $\psi_1 = \exp\left(\frac{i\,k}{2}\,\xi\right)$ and get from (3)

$$\left(-\frac{k^2}{4}\,\xi + \frac{i\,k}{2} + \frac{m_\alpha W}{2\hbar^2}\,\xi - \frac{m_\alpha Z Z' e^2}{2\hbar^2} + \beta\right)\psi_1 = 0. \tag{6}$$

The terms with ξ cancel because of the meaning of k and W in equation (30.1). Therefore (6) is satisfied by choosing

$$\beta = \frac{m_\alpha Z Z' e^2}{2\hbar^2} - \frac{i\,k}{2}. \tag{7}$$

We see that for this ψ_1 equation (3) is satisfied not only asymptotically but for all ξ.

Due to (7) equation (4) becomes

$$\frac{d}{d\eta}\,\eta\,\frac{d\psi_2}{d\eta} + \left(\frac{m_\alpha W}{2\hbar^2}\,\eta - \frac{m_\alpha Z Z' e^2}{\hbar^2} + \frac{i\,k}{2}\right)\psi_2 = 0. \tag{8}$$

The function ψ_2 must satisfy the *absorption condition* for $\eta \to \infty$ (large negative x), which, written in analogy to (5), reads:

$$\frac{\eta}{2}\left(2\,\frac{d\psi_2}{d\eta} + i\,k\,\psi_2\right) = 0.$$

Hence for large η we have the first approximation

$$\psi_2 = \exp\left(-i\,k\,\eta/2\right).$$

However, this is not an exact solution of (8). Hence we set the more general

$$\psi_2 = e^{-i\,k\,\eta/2}\,f(\eta).$$

From (8) we obtain the equation for $f(\eta)$

$$\eta\,\frac{d^2 f}{d\eta^2} + (1 - i\,k\,\eta)\,\frac{df}{d\eta} - m_\alpha Z Z'\,\frac{e^2}{\hbar^2}\,f = 0. \tag{9}$$

Equation (9) is the differential equation (29.12) of the Laguerre function $L_\mu(\varrho)$, if we set

$$\varrho = i\,k\,\eta\,, \qquad \mu = -\frac{m_\alpha Z Z' e^2}{i\,k\,\hbar^2}\,.$$

The last value coincides with the imaginary total quantum number n of (30.7b). Hence we have

$$f(\eta) = L_n(i\,k\,\eta)\,, \qquad \psi_2 = e^{-i\,k\,\eta/2}\,L_n(i\,k\,\eta)$$

and finally

$$\psi = \psi_1(\xi)\,\psi_2(\eta) = e^{i\,k\,(\xi-\eta)/2}\,L_n(i\,k\,\eta)\,. \tag{10}$$

Thus we have a quick proof of (30.7.)

Appendix IV

Plane and Spherical Waves in Unlimited Space of an Arbitrary Number of Dimensions

After having treated plane and spherical waves in three-dimensional space and plane and cylinder waves in two-dimensional space, we cannot resist the temptation to adapt these formulas to the many-dimensional case. In this connection we shall encounter remarkable generalizations of the ordinary spherical harmonics, the *Gegenbauer polynomials*, and generalized addition theorems of the *Bessel functions*. A systematic approach to these generalizations is again given by our theorem in §27 about the representation of Green's function in terms of the eigenfunctions for the space in question.

A. Coordinate System and Notations

Let the number of dimensions be $p+2$ so that $p=0$ represents two-dimensional, and $p=1$ represents three-dimensional space. On the one hand we use the Cartesian coordinates $x_1, x_2 \ldots, x_{p+2}$, and on the other hand the polar coordinates $r, \vartheta, \varphi_1, \varphi_2 \ldots \varphi_p$. The connection shall be given by

$$\begin{aligned}
x_1 &= r\cos\vartheta \\
x_2 &= r\sin\vartheta\cos\varphi_1 \\
x_3 &= r\sin\vartheta\sin\varphi_1\cos\varphi_2 \\
&\text{- - - - - - - - - - -} \\
x_{p+1} &= r\sin\vartheta\sin\varphi_1\sin\varphi_2 \ldots \sin\varphi_{p-1}\cos\varphi_p \\
x_{p+2} &= r\sin\vartheta\sin\varphi_1\sin\varphi_2 \ldots \sin\varphi_{p-1}\sin\varphi_p\,.
\end{aligned} \tag{1}$$

In order to cover the whole space $-\infty < x_i < +\infty$ the coordinates $r, \vartheta, \ldots, \varphi_p$ must vary between the limits

$$0 < r < \infty, \quad 0 < \vartheta < \pi, \quad \begin{cases} 0 < \varphi_i < +\pi, & i = 1, 2, \ldots, p-1, \\ -\pi < \varphi_p < +\pi. \end{cases}$$

By forming the sum of the squares in (1) we obtain

$$\sum_{i=1}^{p+2} x_i^2 = r^2. \tag{1a}$$

The definition of the $(p+2)$-dimensional line element is

$$ds^2 = \sum_{i=1}^{p+2} dx_i^2. \tag{1b}$$

If for every direction of the coordinates r, ϑ, φ_i we compute the corresponding ds from (1) then we obtain (in unified form)

$$ds = (dr,\ r\,d\vartheta,\ r\sin\vartheta\,d\varphi_1,\ r\sin\vartheta\sin\varphi_1 d\varphi_2,\ \ldots\ r\sin\vartheta\sin\varphi_1\ldots\sin\varphi_{p-1}\,d\varphi_p). \tag{2}$$

The coefficients of $dr, d\vartheta, d\varphi_1, \ldots, d\varphi_p$ on the right side of (2) will be denoted by $g_1, g_2, \ldots, g_{p+2}$. We then have

$$\begin{aligned} g_1 = 1,\ g_2 = r,\ g_3 = r\sin\vartheta,\ g_4 = r\sin\vartheta\sin\varphi_1, \\ \ldots, g_{p+2} = r\sin\vartheta\sin\varphi_1\ldots\sin\varphi_{p-1}. \end{aligned} \tag{2a}$$

From (2) and (2a) we obtain the $(p+2)$-dimensional volume element

$$d\tau = \bar{g}\,dr\,d\vartheta\,d\varphi_1\ldots d\varphi_p, \tag{2b}$$

$$\bar{g} = \prod_{i=1}^{p+2} g_i = r^{p+1}\sin^p\vartheta\,\sin^{p-1}\varphi_1\sin^{p-2}\varphi_2\ldots\sin\varphi_{p-1}. \tag{2c}$$

We denote the surface element of the unit sphere in $(p+2)$-dimensional space by $d\omega$, its total surface by Ω and set

$$\Omega = \int d\omega = \Omega_\vartheta\,\Omega_\varphi, \tag{2d}$$

where Ω_ϑ and Ω_φ are the components obtained through integration of $d\omega$ with respect to ϑ and $\varphi_1, \varphi_2, \ldots, \varphi_p$ respectively. From (2b,c) we obtain

$$\Omega_\vartheta = \int_0^\pi \sin^p\vartheta\,d\vartheta = \frac{\pi}{2^p}\,\frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+1\right)\Gamma\left(\frac{p}{2}+1\right)} = \frac{\pi}{p}\,\frac{2^{-p+2}\,\Gamma(p)}{\Gamma(p/2)\,\Gamma(p/2)}. \tag{2e}$$

$$\begin{aligned} \Omega_\varphi &= \int_0^\pi \sin^{p-1}\varphi_1\,d\varphi_1 \int_0^\pi \sin^{p-2}\varphi_2\,d\varphi_2 \ldots \int_0^\pi \sin\varphi_{p-1}\,d\varphi_{p-1} \int_{-\pi}^{+\pi} d\varphi_2 \\ &= 2\pi\,\frac{\frac{p+1}{2}}{\Gamma\left(\frac{p+1}{2}\right)}. \end{aligned} \tag{2f}$$

We denote the Laplace operator in our space by Δ_p (thus in three-dimensional space we would denote it by Δ_1) and we write for a function u which depends on r alone

$$\Delta_p u = \frac{1}{\bar{g}} \frac{d}{dr}\left(\frac{\bar{g}}{g_1^2}\frac{du}{dr}\right) = \frac{1}{r^{p+1}}\frac{d}{dr}\left(r^{p+1}\frac{du}{dr}\right). \tag{3}$$

The potential equation $\Delta_p u = 0$ then becomes

$$\frac{d^2u}{dr^2} + \frac{p+1}{r}\frac{du}{dr} = 0. \tag{3a}$$

Except for additive and multiplicative constants of integration we obtain the solution:

$$u = r^{-p}. \tag{4}$$

We generalize this solution to

$$u = R^{-p}, \qquad R^2 = \sum (x_i - y_i)^2. \tag{4a}$$

If we place the second point introduced here on the axis $\vartheta = 0$ and denote its distance from the origin by r_0 then according to (1) we have $y_1 = r_0, y_2 = y_3 = \cdots = y_{p+2} = 0$ and $R^2 = r^2 - 2\,r\,r_0\cos\vartheta + r_0^2$. Hence

$$u = \frac{r_0^p}{R^p} = \frac{1}{\left[1 + \left(\frac{r}{r_0}\right)^2 - 2\,\frac{r}{r_0}\cos\vartheta\right]^{p/2}} \tag{4b}$$

is also a solution of $\Delta_p u = 0$.

As in (22.3) we expand (4b) in ascending (or descending) powers of r/r_0 and call the coefficients *p-dimensional zonal spherical harmonics*

$$P_n(\cos\vartheta\,|\,p)$$

or also *Gegenbauer polynomials.*[22] The *Legendre polynomials* may thus be denoted by

$$P_n(\cos\vartheta\,|\,1)\,.$$

Hence we write[23]

$$\left[1 + \left(\frac{r}{r_0}\right)^2 - 2\,\frac{r}{r_0}\cos\vartheta\right]^{-p/2} = \sum_{n=0}^{\infty}\left(\frac{r}{r_0}\right)^n P_n(\cos\vartheta\,|\,p), \tag{5}$$

and deduce from this

[22] Gegenbauer's original notation (see e.g., *Wien. Akad.* **70** (1875)) is $C_n^{\nu}(\cos\vartheta)$, where $\nu = p/2$.

[23] The defining equation (5) is not limited to integral p; equation (5) breaks down for $p = 0$ since in that case (4b) has to be replaced by the two-dimensional logarithmic potential.

$$
\begin{array}{l|l}
\cos\vartheta = 1 & P_n(1|p) = (-1)^n \binom{-p}{n} = \dfrac{(p+n-1)!}{n!\,(p-1)!} \\
\cos\vartheta = -1 & P_n(-1|p) = (-1)^n P_n(1|p) \\
 & P_{2s+1}(0|p) = 0 \\
\cos\vartheta = 0 & P_{2s}(0|p) = \binom{-p/2}{s} = (-1)^s \dfrac{\Gamma\left(\frac{p}{2}+s\right)}{\Gamma(s+1)\,\Gamma\left(\frac{p}{2}\right)}.
\end{array}
\tag{5a}
$$

For the particular solution of the potential equation $\Delta_p u = 0$ which depends only on r and ϑ

$$u = r^n P_n(\cos\vartheta|p) \tag{5b}$$

we then obtain the differential equation

$$\frac{1}{\bar{g}}\left\{\frac{\partial}{\partial r}\left(\frac{\bar{g}}{g_1^2}\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial\vartheta}\left(\frac{\bar{g}}{g_2^2}\frac{\partial u}{\partial\vartheta}\right)\right\} = 0;$$

and after dividing out the factor $r^{n+p-1}/\bar{g}$ we obtain the ordinary differential equation for $P_n(\cos\vartheta|p)$

$$\left[\frac{d}{d\vartheta}\sin^p\vartheta\frac{d}{d\vartheta} + n(n+p)\sin^p\vartheta\right] P_n(\cos\vartheta|p) = 0. \tag{5c}$$

The reader is asked to check the connection of these and the following formulas with the formulas from the theory of ordinary spherical harmonics.

The Gegenbauer polynomials can be expressed in terms of hypergeometric series in a manner similar to that of the Legendre polynomials in (24.24a); we have

$$P_n(\cos\vartheta|p) = P_n(1|p)\, F\left(-n, n+p, \frac{p+1}{2}, \frac{1-\cos\vartheta}{2}\right). \tag{5d}$$

B. The Eigenfunctions of Unlimited Many-Dimensional Space

From the potential equation we pass to the wave equation. For a function which depends only on r the wave equation is, according to (3),

$$\frac{d^2u}{dr^2} + \frac{p+1}{r}\frac{du}{dr} + k^2 u = 0. \tag{6}$$

If we set $u = r^{-p/2} w$, then we obtain the Bessel differential equation with index $p/2$ for w. Hence (6) is integrated by

$$u = r^{-p/2} I_{p/2}(kr), \tag{6a}$$

and also by

$$u = r^{-p/2} H^1_{p/2}(kr), \tag{6b}$$

$$u = r^{-p/2} H^2_{p/2}(kr). \tag{6c}$$

The function in (6b) behaves asymptotically like

$$C e^{ikr} \Big/ r^{\frac{p+1}{2}}, \qquad C = \sqrt{\frac{2}{k\pi}}\, e^{-\frac{p+1}{2}\frac{i\pi}{2}}$$

and satisfies the radiation condition (28.7)

$$\lim_{r\to\infty} r^{\frac{p+1}{2}} \left(\frac{\partial u}{\partial r} - i k u\right) = 0;$$

In the same manner (6c) satisfies the absorption condition. Hence (6b,c) represent the radiated and absorbed *spherical waves in* $(p+2)$-*dimensional space.* This remains valid for a general position of the source point with the solutions

(7) $$U = R^{-p/2} H_{p/2}(kR), \quad R^2 = r^2 - 2 r r_0 \cos\vartheta + r_0^2.$$

The function in (6a) may be called "eigenfunction of *spherical* symmetry." We now want to find the general eigenfunctions of zonal symmetry. They are of the form

(8) $$u_n(r,\vartheta) = v_n(r)\, P_n(\cos\vartheta|p).$$

From the equation (5c) of P_n we find the differential equation of v_n

$$\left(\frac{d^2}{dr^2} + \frac{p+1}{r}\frac{d}{dr} + k^2 - \frac{n(n+p)}{r^2}\right) v_n = 0.$$

If we treat this equation as we did (6) by setting $v_n = r^{-p/2} w$ then for w we obtain the Bessel differential equation with index $n + p/2$, and hence as the solution which is finite for $r = 0$

$$w = I_{n+p/2}(kr).$$

Hence the eigenfunction becomes

(8a) $$u_n = r^{-p/2} I_{n+p/2}(kr)\, P_n(\cos\vartheta|p).$$

According to §26 any two of these eigenfunctions are mutually *orthogonal*, both in the continuous spectrum $0 < k < \infty$, and in the discrete spectrum $n = 0,1,2,\ldots$.

For two eigenfunction u_n, u_m with equal k but different indices we obtain from (2b,c) and (8):

(9) $$\int u_n u_m\, d\tau = \int_0^\infty I_{n+p/2}(kr)\, I_{m+p/2}(kr)\, r\, dr \int_0^\pi P_n(\cos\vartheta|p)\, P_m(\cos\vartheta|p) \sin^p\vartheta\, d\vartheta\, d\Omega_\varphi.$$

where Ω_φ is as in (2f). Due to the fact that neither Ω_φ nor the integral with respect to r vanish and due to the orthogonality of u_n and u_m we obtain:

(10) $$\int_0^\pi P_n(\cos\vartheta \mid p)\, P_m(\cos\vartheta \mid p) \sin^p\vartheta\, d\vartheta = 0, \quad m \neq n.$$

Note the characteristic factor $\sin^p \vartheta$ in (10), which in the three-dimensional case ($p = 1$) becomes the customary factor $\sin \vartheta$ for the Legendre polynomials. While in the customary analytic derivation of (10) this factor might appear artificial, it follows in our many-dimensional approach directly from the meaning of $d\tau$.

We also note the corresponding normalizing integral for $m = n$

$$(11) \qquad N = \int_0^\pi [P_n(\cos\vartheta \mid p)]^2 \sin^p \vartheta \, d\vartheta = \frac{\Gamma(n+p)}{2^{p-1}(n+p/2)\, n!} \frac{\pi}{\Gamma(p/2)\,\Gamma(p/2)}$$

which is a generalization of the normalizing integral for ordinary zonal spherical harmonics: $N = 1/(n + \frac{1}{2})$ for $p = 1$. The proof of (11) starts from the defining equation (5) of the Gegenbauer polynomials.

With the help of (2e) we can replace (11) by:

$$(11\,a) \qquad N = \frac{p}{2} \frac{\Gamma(n+p)}{(n+p/2)\, n!\, \Gamma(p)} \Omega_\vartheta .$$

C. Spherical Waves and Green's Function in Many-Dimensional Space

The spherical wave of zonal symmetry has been described by equation (7). From this function we obtain Green's function of $(p + 2)$-dimensional unlimited space by adding a factor f such that the source Q of U becomes a unit source. According to §10 C this means

$$(12) \qquad 1 = \int \frac{\partial G}{\partial n} d\sigma = f \int \frac{\partial U}{\partial R} d\sigma = f \int \frac{\partial}{\partial R} [R^{-p/2} H^1_{p/2}(k R)] R^{p+1} d\omega .$$

where the integration is to be taken over a sphere of radius $R \to 0$; $d\sigma$ denotes the surface element on this sphere; $d\omega$, as in (2d), denotes the surface element on the unit sphere. Hence we obtain from (12)

$$(12a) \qquad 1 = f \Omega \lim_{R \to 0} R^{p+1} \frac{\partial}{\partial R} [R^{-p/2} H^1_{p/2}(k R)].$$

For odd p we can use the formula (19.31) for H, which yields

$$\lim_{R \to 0} \cdots = \frac{-i R^{p+1}}{\sin p\pi/2} \frac{\partial}{\partial R} [R^{-p/2} I_{-p/2}(k R)] = \frac{i p}{\sin p\pi/2} \frac{(k/2)^{-p/2}}{\Gamma(-p/2+1)} .$$

Using a well-known Γ- relation we can replace this by

$$\frac{i p}{\pi} \left(\frac{k}{2}\right)^{-p/2} \Gamma\left(\frac{p}{2}\right).$$

For even p we obtain the same value from (19.26) and (19.47). Hence we obtain from (12a)

(12 b) $$f = \frac{\pi}{\Omega}\left(\frac{k}{2}\right)^{p/2} \Big/ i^p \Gamma\left(\frac{p}{2}\right)$$

and from (7) upon multiplication by f

(13) $$G(P,Q) = \frac{\pi}{\Omega}\left(\frac{k}{2}\right)^{p/2}\left(\frac{kR}{2}\right)^{-p/2} H^1_{p/2}(kR) \Big/ i^p \Gamma\left(\frac{p}{2}\right).$$

On the other hand we want to construct $G(P,Q)$ as in §28 from the eigenfunctions $u(P)$ in (8a)

(13 a) $$u(P) = r^{-p/2} I_{n+p/2}(\lambda r)\, P_n(\cos\vartheta|p)$$

and the associated $u(Q)$ for a point Q with the coordinates $r = r_0$, $\vartheta_0 = 0$:

(13 b) $$u(Q) = r_0^{-p/2} I_{n+p/2}(\lambda r_0)\, P_n(1|p).$$

In both representations (13a,b) λ (see equation (28.14)) denotes the variable of integration in the continuous part of the eigen value spectrum. In a similar manner we perform the integration over λ in the complex λ- plane and obtain in analogy to (28.15)

(14) $$\int u(P)\, u(Q)\, \frac{d\lambda}{k^2 - \lambda^2}$$
$$= \frac{\pi i}{2}(r r_0)^{-p/2} \sum_n P_n(\cos\vartheta|p)\, P_n(1|p) \begin{cases} I_{n+p/2}(k r_0)\, H_{n+p/2}(k r) & r > r_0, \\ I_{n+p/2}(k r)\, H_{n+p/2}(k r_0) & r < r_0. \end{cases}$$

In order to be able to apply this formula to Green's function we still must normalize the functions $u(P)$ and $u(Q)$ to one. The general term on the right side of (14) must therefore be: 1) divided by the normalizing factor N of (11a) due to the dependence on ϑ; 2) divided by Ω_p of (2f) due to its independence of the coordinates $\varphi_1, \varphi_2, \ldots, \varphi_p$, and 3) multiplied by k due to the r-dependence according to Appendix I, equation (4). Altogether this yields the factor (see also (5a))

(14 a) $$\frac{k}{\Omega}\frac{2n+p}{p}\frac{n!\,\Gamma(p)}{\Gamma(n+p)} = \frac{k}{\Omega}\frac{2n+p}{p} \Big/ P_n(1|p),$$

which has to be introduced under the Σ- sign of (14). Thus, according to our general theorem of §28 we obtain Green's function of unlimited space. Comparing this with (13) we obtain:

(15) $$\frac{H_{p/2}(kR)}{(kR)^{p/2}} = 2^{p/2}\,\Gamma\left(\frac{p}{2}\right) \sum_{n=0}^{\infty}\left(n+\frac{p}{2}\right) P_n(\cos\vartheta|p) \left\{\ \right\},$$

$$\left\{\ \right\} = \begin{cases} \dfrac{I_{n+p/2}(k r_0)}{(k r_0)^{p/2}} \dfrac{H_{n+p/2}(k r)}{(k r)^{p/2}} & \cdots\ r > r_0, \\ \dfrac{I_{n+p/2}(k r)}{(k r)^{p/2}} \dfrac{H_{n+p/2}(k r_0)}{(k r_0)^{p/2}} & \cdots\ r < r_0. \end{cases}$$

This *general addition theorem of Bessel functions* holds both for $H = H^1$ and for $H = H^2$, and hence for any linear combination of the two, so that in (15) we may replace H on both sides by

$$Z = c_1 H^1 + c_2 H^2,$$

hence, in particular by

$$I = \tfrac{1}{2}(H^1 + H^2),$$

in which latter case the distinction $r \gtrless r_0$ becomes immaterial. The theorem holds, under more general conditions than those assumed in the derivation, if, as in footnote 22, we replace $p/2$ by an arbitrary number, say ν.

D. Passage from the Spherical Wave to the Plane Wave

For $r_0 \to \infty$ we deduce a representation of the plane wave in many-dimensional space from the last line of (15)

First we obtain on the right side, according to Hankel's approximation (19.55),

$$H^1_{n+p/2}(k r_0) = a\, e^{-i n \pi/2}, \qquad a = \sqrt{\frac{2}{\pi k r_0}} \exp\left\{ i \left(k r_0 - \frac{p+1}{2} \frac{\pi}{2} \right) \right\}.$$

Correspondingly, on the left side we obtain

$$H^1_{p/2}(k R) = \sqrt{\frac{2}{\pi k R}} \exp\left\{ i \left(k R - \frac{p+1}{2} \frac{\pi}{2} \right) \right\}.$$

However for $r_0 \to \infty$ we have

$$R = r_0 \left(1 - 2 \frac{r}{r_0} \cos\vartheta + \cdots \right)^{\frac{1}{2}} = r_0 - r \cos\vartheta + \cdots;$$

therefore,

$$H^1_{p/2}(k R) = \sqrt{\frac{2}{\pi k r_0}} \exp\left\{ i \left(k r_0 - k r \cos\vartheta - \frac{p+1}{2} \frac{\pi}{2} \right) \right\}$$

$$= a \exp\{ - i k r \cos\vartheta \},$$

so that the left side of (15), with the corresponding approximation for the denominator, becomes:

$$\frac{a}{(k r_0)^{p/2}} \exp\{ - i k r \cos\vartheta \}.$$

After canceling the common factor on both sides we obtain

$$(16) \qquad e^{-ikr\cos\vartheta} = 2^{p/2}\,\Gamma\left(\frac{p}{2}\right)\sum_{n=0}^{\infty}\left(n+\frac{p}{2}\right)e^{-in\pi/2}\,P_n(\cos\vartheta|p)\,\frac{I_{n+p/2}(kr)}{(kr)^{p/2}}.$$

This represents an incoming wave in the direction of the positive axis $\vartheta = 0$, or, in other words, a wave which proceeds in the negative direction of this axis. The wave which proceeds in the positive direction is obtained from (16) by replacing $+ i$ with $- i$. The reader is asked to verify that this formula coincides for $p = 1$ with the three-dimensional representation (24.7). In the two-dimensional case in which (4b) breaks down (see footnote 22) equation (16) is replaced by the representation (21.2b).

Through a suitable averaging or with the help of an "addition theorem[24] of Gegenbauer polynomials" we obtain from (16) remarkable relations between Bessel functions of integral and of fractional indices.[25]

[24] See the lucid collection of Gegenbauer's results in the book by Magnus and Oberhettinger, Formeln und Sätze über die speziellen Funktionen der Mathematischen Physik, Springer, 1943, particularly p. 77.

[25] G. Bauer, *Sitzungsber. bayr. Akad.*, 1875, p. 247; generalization to higher dimensions, A. Sommerfeld, *Math. Ann.* **119**, 1 (1943).