EXERCISES FOR CHAPTER II

II.1. Elastic rod, open and covered pipe. Compute the *transversal* proper oscillations of a cylindrical rod of length l, which is clamped at $x = 0$ and oscillates freely at $x = l$, and compare them to the proper oscillations of an open and a covered pipe.

II.2. Second form of Green's theorem. Develop the analogue to Green's theorem, see v.II, equation (3.16), for the general elliptic differential expression $L(u)$,

a) where $L(u)$ is brought to the normal form,

b) in the general case.

c) Investigate the conditions under which the boundary value problem becomes unique for a self-adjoint differential expression L.

We may restrict ourselves to the case of two independent variables, for which Green's theorem is

$$\int u\,\Delta v\,d\sigma + \int (\operatorname{grad} u, \operatorname{grad} v)\,d\sigma = \int u\,\frac{\partial v}{\partial n}\,ds\,.$$

II.3. One-dimensional potential theory. Determine the one-dimensional Green's function from the conditions

$$\text{a)}\quad \frac{d^2G}{dx^2} = 0 \qquad \text{for} \qquad \begin{cases} 0 \le x < \xi\,, \\ \xi < x \le l\,, \end{cases}$$

$$\text{b)}\quad G = 0 \qquad \text{for} \quad x = 0 \quad \text{and} \quad x = l,$$

$$\text{c)}\quad \frac{dG_+}{dx} - \frac{dG_-}{dx} = 1 \quad \text{and} \quad G \text{ continuous for } x = \xi\,,$$

and apply it to the (obviously trivial) solution of the boundary value problem:

$$\frac{d^2u}{dx^2} = 0\,, \quad u \text{ continuous for } \ 0 \le x \le l \begin{cases} u = u_0 & \text{for } x = 0, \\ u = u_l & \text{for } x = l. \end{cases}$$

Condition c) means "yield 1" of the source of G which is situated at $x = \xi$; G_+ is the branch $x > \xi$, G_- the branch $x < \xi$ of G.

II.4. Application of Green's method which was developed for heat conduction to the so-called laminar plate flow of an incompressible viscous fluid. We assume the flow to be planar and rectilinear throughout; this means that it is to be *independent* of the third coordinate z and *directed in the direction of the y-axis.* The velocity $\mathbf{v}$ then has the single component $\mathbf{v}_y = v$, which, due to our assumption of incompressibility, is independent not only of z, but also of y, so that the quadratic convection terms ($\mathbf{v}$ grad) $\mathbf{v}$ vanish. The Navier-Stokes equation for v is then according to v.II, equation (16.1)

$$\frac{\partial v}{\partial t} - k\frac{\partial^2 v}{\partial x^2} = -\frac{1}{\varrho}\frac{\partial p}{\partial y}; \tag{1}$$

where k is the kinematic viscosity; the right side is *independent of* x due to the corresponding equation for the vanishing x-component of velocity, hence it is a function of t only, say $a(t)$.

The flow is to be bounded at $x = 0$ by a fixed plate, which is at rest up to the time $t = 0$ and thereafter is in motion with the velocity $v_0(t)$. Due to the adhesion of the fluid to the plate we have for $x = 0$:

$$v = \begin{cases} 0 \ldots t \leq 0, \\ v_0(t) \ldots t > 0. \end{cases} \tag{2}$$

For the linear Couette flow (see v.II, Fig. 19b) we would have to add further boundary conditions on a plane that is at rest at a finite distance from $x = 0$. However, for the sake of simplicity, we shall consider this plate situated at infinity. The limiting case obtained in this manner is known in fluid dynamics as *plate flow*. For this flow we have, in addition to (2), the condition for $x = \infty$:

$$v = 0 \quad \text{and} \quad p = p_0 \text{ (independent of } y). \tag{3}$$

From this it follows that $a(t) = 0$, so that (1) goes over into the equation of heat conduction.

We are thus led to a boundary value problem, which is a specialization of the problem illustrated by Fig. 13 only in that we now have $x_1 = \infty$ and $x_0 = 0$, and is a simplification of that problem because in the initial state in which the plate and hence the fluid are at rest, we have:

$$v = 0 \quad \text{for} \quad t = 0 \quad \text{and} \quad \text{all} \quad x > 0. \tag{4}$$

The solution is obtained as in (12.18), if the principal solution V is replaced by a suitable Green's function. Discuss the resulting velocity profile $v(x)$ for increasing values of t.

EXERCISES FOR CHAPTER III

III.1. Linear conductor with external heat conduction according to Fourier. Let the initial temperature for $x > 0$ be $u(x,0) = f(x)$. How must this function be continued for $x < 0$ so that the condition is