$$\frac{\partial u}{\partial n} + h\,u = 0$$

satisfied for $x = 0$?

III.2. Deduce *the normalization condition in anharmonic analysis* by specialization from Green's theorem.

III.3. Experimental determination of the ratio of outer and inner heat conductivity. A rod is to be kept at the fixed temperatures u_1 and u_3 at its ends $x = 0$ and $x = l$ and is to be in a *stationary* state after the effect of an arbitrary initial state dies out. The flow of temperature would then be *linear* if the lateral surface of the rod were adiabatically closed. Hence, at the middle section of the rod $x = l/2$ we would have temperature $u_2 = (u_1 + u_3)/2$. Hence from the measurement of

$$q = \frac{u_1 + u_3}{2\,u_2}$$

we can determine the ratio of outer and inner heat conduction (essentially our constant h). Deduce the relation between q and h needed for the evaluation of the measurement; according to the above $q = 1$ corresponds to $h = 0$.

III.4. Determination of the ratio of heat conductivity $\varkappa$ to electric conductivity σ. A metal rod is to be heated electrically, where the current i per unit of length gives the rod the Joule heat $i^2/q\sigma$ (q = cross-section of the rod); the rod is to be insulated against lateral heat conduction. Write the differential equation of the stationary state and adapt it to the boundary conditions $u = 0$ for $x = 0$ and $x = l$ that are realized in water baths. The potential difference V at the ends of the rod and the maximal temperature U at the mid-section of the rod are to be measured. From them we are to compute the ratio $\varkappa/\sigma$. For pure metals this ratio has a universal value (Wiedemann-Franz law).

EXERCISES FOR CHAPTER IV

IV.1. Power series expansion of $I_n(\varrho)$. Compute this expansion from the integral representation (19.18)

a) for integral n,

b) for arbitrary n

with the help of a general definition of the Γ- function.

IV.2. Deduce *the so-called circuit relations for H_n^1 and H_n^2* for integral n from the integral representations (19.22).

IV.3. Compute *the logarithmic singularity* of $H_0(\varrho)$ at the origin from the integral representations (19.22).

IV.4. An elementary process for the asymptotic approximation to $H_n^1(\varrho)$. Verify the asymptotic limiting value of $H_n^1(\varrho)$ by successively neglecting $1/\varrho$ and the higher powers of $1/\varrho$ already in the differential equation. This method is of course dubious from a mathematical point of view.

IV.5. Expansion of a function $f(\vartheta, \varphi)$ *on the sphere.*

a) Expand f first in a trigonometric series in φ, and then in spherical harmonics in $\cos\vartheta$. That is, find the expansions

$$f(\vartheta, \varphi) = \sum_{m=-\infty}^{+\infty} C_m e^{im\varphi}, \qquad C_m = \sum_{n=m}^{\infty} A_{mn} P_n^m(\cos\vartheta)$$

and as combination of both these expansions

$$f(\vartheta, \varphi) = \sum_m \sum_n A_{mn} P_n^m(\cos\vartheta) e^{im\varphi} \begin{cases} -\infty < m < +\infty, \\ |m| \le n < \infty. \end{cases} \tag{1}$$

b) Construct f from general surface spherical harmonics Y_n and determine the coefficients from the orthogonality relation for

$$Y_{nm} = P_n^m(\cos\vartheta) e^{-im\varphi},$$

that is, find

$$f(\vartheta, \varphi) = \sum_{n=0}^{\infty} Y_n, \qquad Y_n = \sum_{m=-n}^{+n} A_{nm} P_n^m(\cos\vartheta) e^{im\varphi};$$

and as combination of both expansions:

$$f(\vartheta, \varphi) = \sum_n \sum_m A_{nm} P_n^m(\cos\vartheta) e^{im\varphi} \begin{cases} 0 < n < \infty, \\ -n \le m \le +n. \end{cases} \tag{2}$$

Clarify the apparent dissimilarity in the order of summation in (1) and (2) by a figure (lattice in the m,n-plane) and show that A_{mn} and A_{nm} in (1) and (2) formally have the same meaning (by interchanging the order of summation and integrating).

IV.6. Mapping of the wedge arrangement of Fig. 17 into circular crescents. Transform the 60°-angle wedge of Fig. 17 by reciprocal radii with a suitable position of the center of inversion C (see the text for this figure); the three straight lines 1,-1; 2,-2; 3,-3 then go into three circular arcs and the angles formed by them go into circular crescents. Examine the association of these regions and consider the fact that the Green's function of potential theory can be obtained for each of these regions (spatially speaking they are spherical crescents) by five repeated reflections.

IV.7. Mapping a) *of the plane parallel plate into two tangent spheres,* b) *of two concentric spheres into a plane and a sphere.* Investigate the three-dimensional figure into which the plane parallel plate of p. 74 together with its mirror images are transformed upon inversion. The sphere of inversion is best situated so that it is tangent to the boundary planes of the plate. Show that the plate is thus mapped into the exterior of two tangent spheres; and its mirror images are mapped into the space between two interior tangent spheres. b) Show that two concentric spheres can be inverted into a plane and a sphere. Hence, conversely, we can transform the potential of a sphere towards a plane into the much simpler boundary value problem for two concentric spheres. The same holds for the potential of two arbitrary non-intersecting spheres.

IV.8. Evaluation of two expressions involving Bessel functions. In equation (5) of Appendix I to Chapter IV determine

$$H_n(\varrho)\, I_n'(\varrho) - H_n'(\varrho)\, I_n(\varrho) \tag{I}$$

and in equation (20b) of the same appendix determine

$$\zeta_n(\varrho)\, \psi_n'(\varrho) - \zeta_n'(\varrho)\, \psi_n(\varrho). \tag{II}$$

EXERCISES FOR CHAPTER V

V.1. Normalization questions. Normalize the functions $I_n(\lambda r)$ and $\psi_n(k r)$ of (26.3) and (26.2) to 1 for the basic interval $0 < r < a$ with the boundary conditions $I_n'(\lambda a) = 0$ and $\psi_n'(k a) = 0$ in analogy to equation (20.19).

V.2. The Gauss theorem of arithmetic mean in potential theory. Prove the theorem: The value of a potential function $\overline{u}$ at any point P of its domain of regularity S is equal to the arithmetic mean u of its values on an arbitrary sphere K_a, which has radius a and center P and which lies entirely in S.

V.3. Summation formulas over the roots of Bessel functions. Verify that the coefficients A_{nm} in the expansion (27.13) equal those of (27.14), and derive interesting identities for the Ψ_n from the equality of the coefficients of $\Pi_n^m(\cos\vartheta_0)\, e^{-im\varphi_0}$ in these two expansions. These identities can be rewritten as identities for the ψ_n.