

# Statistical physics of social dynamics

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Statistical physics has proven to be a very fruitful framework to describe phenomena outside the realm of traditional physics. The last years have witnessed the attempt by physicists to study collective phenomena emerging from the interactions of individuals as elementary units in social structures. Here we review the state of the art by focusing on a wide list of topics ranging from opinion, cultural and language dynamics to crowd behavior, hierarchy formation, human dynamics, social spreading. We highlight the connections between these problems and other, more traditional, topics of statistical physics. We also emphasize the comparison of model results with empirical data from social systems.

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## I. INTRODUCTION

The concept that many laws of nature are of statistical origin is so firmly grounded in virtually all fields of modern physics, that statistical physics has acquired the status of a discipline on its own. Given its success and its very general conceptual framework, in recent years there has been a trend toward applications of statistical physics to interdisciplinary fields as diverse as biology, medicine, information technology, computer science, etc.. In this context, physicists have shown a rapidly growing interest for a statistical physical modeling of fields patently very far from their traditional domain of investigations (Chakrabarti *et al.*, 2006; Stauffer *et al.*, 2006b). In social phenomena the basic constituents are not particles but humans and every individual interacts with a limited number of peers, usually negligible compared to the total number of people in the system. In spite of that, human societies are characterized by stunning global regularities (Buchanan, 2007). There are transitions from disorder to order, like the spontaneous formation of a common language/culture or the emergence of consensus about a specific issue. There are examples of scaling and universality. These macroscopic phenomena naturally call for a statistical physics approach to social behavior, i.e., the attempt to understand regularities at large scale as collective effects of the interaction among single individuals, considered as relatively simple entities.

It may be surprising, but the idea of a physical modeling of social phenomena is in some sense older than the idea of statistical modeling of physical phenomena. The discovery of quantitative laws in the collective properties of a large number of people, as revealed for example by birth and death rates or crime statistics, was one of the factors pushing for the development of statistics and led many scientists and philosophers to call for some quantitative understanding (in the sense of physics) on how such precise regularities arise out of the apparently erratic behavior of single individuals. Hobbes, Laplace, Comte, Stuart Mill and many others shared, to a different extent, this line of thought (Ball, 2004). This point of view was well known to Maxwell and Boltzmann and probably played a role when they abandoned the idea of describing the trajectory of single particles and introduced a statistical description for gases, laying the foundations of modern statistical physics. The value of statistical laws for social sciences has been foreseen also by Majorana in his famous tenth article (Majorana, 1942, 2005). But it is only in the past few years that the idea of approaching society within the framework of statistical physics has transformed from a philosophical declaration of principles to a concrete research effort involving a critical mass of physicists. The availability of new large databases as well as the appearance of brand new social phenomena (mostly related to the Internet) and the tendency of social scientists to move toward the formulation of simplified models and their quantitative analysis, have been instrumental for this change.

In this review we mostly discuss several different aspects of a single basic question of social dynamics: how do the interactions between social agents create order out of an initial disordered situation? Order is a translation in the language of physics of what is denoted in social sciences as consensus, agreement, uniformity, while disorder stands for fragmentation or disagreement. It is reasonable to assume that without interactions heterogeneity dominates: left alone, each agent would choose a personal response to a political question, a unique set of cultural features, his own special correspondence between objects and words. Still it is common experience that shared opinions, cultures, languages do exist. The focus of the statistical physics approach to social dynamics is to understand how this comes about. The key factor is that agents interact and this generally tends to make people more similar (although many counterexamples exist). Repeated interactions in time lead to higher degrees of homogeneity, that can be partial or complete depending on the temporal or spatial scales. The investigation of this phenomenon is intrinsically dynamic in nature.

A conceptual difficulty immediately arises when trying to approach social dynamics from the point of view of statistical physics. In usual applications, the elementary components of the systems investigated, atoms and molecules, are relatively simple objects, whose behavior is very well known: the macroscopic phenomena are not due to a complex behavior of single entities, rather to nontrivial collective effects resulting from the interaction of a large number of 'simple' elements.

Humans are exactly the opposite of such simple entities: the detailed behavior of each of them is already the complex outcome of many physiological and psychological processes, still largely unknown. No one knows precisely the dynamics of a single individual, nor the way he interacts with others. Moreover, even if one knew the very nature of such dynamics and such interactions, they would be much more complicated than, say, the forces that atoms exert on each other. It would be impossible to describe them precisely with simple laws and few parameters. Therefore any modeling of social agents inevitably involves a huge and unwarranted simplification of the real problem. It is then clear that **any investigation of models of social dynamics involves two levels of difficulty. The first is in the very definition of sensible and realistic microscopic models; the second is the usual problem of inferring the macroscopic phenomenology out of the microscopic dynamics of such models.** Obtaining useful results out of these models may seem a hopeless task.

The critique that models used by physicists to describe social systems are too simplified to describe any real situation is most of the times very well grounded. This applies also to highly acclaimed models introduced by social scientists, as Schelling's model for urban segregation (Schelling, 1971) and Axelrod's model (Axelrod, 1997) for cultural dissemination. But in this respect, statistical physics brings an important added value. In

most situations qualitative (and even some quantitative) properties of large scale phenomena do not depend on the microscopic details of the process. Only higher level features, as symmetries, dimensionality or conservation laws, are relevant for the global behavior. With this concept of *universality* in mind one can then approach the modelization of social systems, trying to include only the simplest and most important properties of single individuals and looking for qualitative features exhibited by models. A crucial step in this perspective is the comparison with **empirical data** which should be primarily intended as an investigation on whether the trends seen in real data are compatible with plausible microscopic modeling of the individuals, are self-consistent or require additional ingredients.

The statistical physics approach to social dynamics is currently attracting a great deal of interest, as indicated by the large and rapidly increasing number of papers devoted to it. The newcomer can easily feel overwhelmed and get lost in the steadily growing flow of new publications. Even for scholars working in this area it is difficult to keep up on the new results that appear at an impressive pace. In this survey we try to present, in a coherent and structured way, the state of the art in a wide subset of the vast field of social dynamics, pointing out motivations, connections and open problems. Specific review articles already exist for some of the topics we consider and we will mention them where appropriate. We aim at providing an up-to-date and – as much as possible – unified description of the published material. Our hope is that it will be useful both as an introduction to the field and as a reference.

When writing a review on a broad, interdisciplinary and active field, completeness is, needless to say, a goal out of reach. For this reason we spell out explicitly what is in the review and what is not. We focus on some conceptually homogeneous topics, where the common thread is that individuals are viewed as adaptive instead of rational agents, the emphasis being on communication rather than strategy. A large part of the review is devoted to the dynamics of opinions (Sec. III) and to the related field of cultural dissemination (Sec. IV). Another large section describes language dynamics (Sec. V), intended both as the formation and evolution of a language and as the competition between different languages. In addition we discuss some other interesting issues as crowd dynamics (Sec. VI), the emergence of hierarchies (Sec. VII), social spreading phenomena (Sec. IX), coevolution of states and topology (Sec. X) and what is becoming established as ‘human dynamics’ (Sec. VIII). Although it is often very difficult to draw clear borders between disciplines, we have in general neglected works belonging to the field of econophysics as well as to evolutionary theory (Blythe and McKane, 2007) and evolutionary game theory, except for what concerns the problem of language formation. On such topics there are excellent books and reviews (Bouchaud and Potters, 2000; Lux, 2007; Mantegna and Stanley, 1999) to which we refer the

interested reader. **We leave out also the physical investigation of vehicular traffic**, a rather well established and successful field (Chowdhury *et al.*, 2000; Helbing, 2001; Nagatani, 2002), though akin to pedestrian behavior in crowd dynamics. **The hot topic of complex networks** has a big relevance from the social point of view, since many nontrivial topological structures emerge from the self-organization of human agents. Nevertheless, for lack of space, we do not discuss such theme, for which we refer to (Albert and Barabási, 2002; Boccaletti *et al.*, 2006; Dorogovtsev and Mendes, 2002; Newman, 2003a). Networks will be considered but only as substrates where the social dynamics may take place. Similarly, we **do not review the intense recent activity on epidemics** spreading (Anderson and May, 1991; Lloyd and May, 2001) on networks (May, 2006; Pastor-Satorras and Vespignani, 2001), though we devote a section to social spreading phenomena. It is worth remarking that, even if we have done our best to mention relevant social science literature and highlight connections to it, the main focus of this work remains the description of the statistical physics approach to social dynamics.

The reader will realize that, with the exception of some specific sections, there is a striking unbalance between empirical evidence and theoretical modelization, in favor of the latter. This does not correspond to our personal choice: it is a rather objective reflection of a great disproportion in the literature on social dynamics. In this respect things are very different from other related fields like complex networks or econophysics. In those cases the great bursts of activity occurred recently have been essentially data-driven: the availability of unprecedented huge new datasets has spurred first their thorough empirical characterization and, based on that, an intense theoretical activity followed for their modeling, well rooted in the comparison between theory and data. For some of the fields that will be reviewed below, things have gone the other way around. The introduction of a profusion of theoretical models has been mainly justified by vague plausibility arguments, with no direct connection to measurable facts. Very little attention has been paid to a stringent quantitative validation of models and theoretical results. The contribution of physicists in establishing social dynamics as a sound discipline grounded on empirical evidence has been so far insufficient. We hope that the present review will be instrumental in stimulating the statistical physics community towards this goal. The latest developments of the field, especially those related to so-called Social Web, hint at a change of perspective in this direction.

## II. GENERAL FRAMEWORK: CONCEPTS AND TOOLS

Despite their apparent diversity, most research lines we shall review are actually closely connected from the point of view of both the methodologies employed and, more importantly, of the general phenomenology ob-

served. Opinions, cultural and linguistic traits, social status, kinematic features are always modeled in terms of a small set of variables whose dynamics is determined by social interactions. The interpretation of such variables will be different in the various cases: a binary variable will indicate yes/no to a political question in opinion dynamics, two synonyms for a certain object in language evolution, two languages in competition, whether somebody has been reached by a rumor or not, etc.. Other details may differ, but often results obtained in one case can immediately be translated in the context of other sub-fields. In all cases the dynamics tends to reduce the variability of the initial state and this may lead to an ordered state, with all the agents sharing the same features (opinion, cultural or linguistic traits, velocity) or to a fragmented (disordered) state. The way in which those systems evolve can thus be addressed in a unitary way using well known tools and concepts from statistical physics. In this spirit some of the relevant general questions we will consider in the review include: What are the fundamental interaction mechanisms that allow for the emergence of consensus on an issue, a shared culture, a common language, collective motion, a hierarchy? What favors the homogenization process? What hinders it?

Generally speaking the drive toward order is provided by the tendency of interacting agents to become more alike. This effect is often termed 'social influence' in the social science literature (Festinger *et al.*, 1950) and can be seen as a counterpart of ferromagnetic interaction in magnets. Couplings of anti-ferromagnetic type, i.e., pushing people to adopt a state different from the state of their neighbors, are also in some cases important and will be considered.

Any modelization of social agents inevitably neglects a huge number of details. One can often take into account in an effective form such unknown additional ingredients assuming the presence of noise. A time-independent noise in the model parameters often represents the variability in the nature of single individuals. On the other hand a time-dependent noise may generate spontaneous transitions of agents from one state to another. A crucial question has then to do with the stability of the model behavior with respect to such perturbations. Do spontaneous fluctuations slow down or even stop the ordering process? Does diversity of agents' properties strongly affect the model behavior?

An additional relevant feature is the topology of the interaction network. Traditional statistical physics usually deals with structures whose elements are located regularly in space (lattices) or considers the simplifying hypothesis that the interaction pattern is all-to-all, thus guaranteeing that the mean field approximation is correct. This assumption, often also termed homogeneous mixing, generally permits analytical treatment, but it is hardly realistic in a social context. Much more plausible interaction patterns are those denoted as complex networks (see Sec. II.B). The study of the effect of their

nontrivial topological properties on models for social dynamics is a very hot topic.

One concept playing a special role in many social dynamic models and having no equally common counterpart in traditional statistical physics is 'bounded confidence', i.e., the idea that in order to interact two individuals must be not too different. This parallels somewhat the range of interaction in physics: if two particles are too far apart they do not exert any influence on each other. However let us stress that the distance involved in bounded confidence is not spatial, rather being defined in a sort of opinion space. We will discuss in the review several instances of this general principle.

Let us finally clarify some problems with nomenclature. Being a strongly interdisciplinary field, in social dynamics there is a natural tendency towards a rather free (or sloppy) use of terms. This heterogeneity is in some cases very confusing as it happens for some words (like polarization) that have been used with opposite meaning. For the sake of clarity we specify that in this review consensus indicates a configuration where all agents share the same state. When many possible states are possible but only two of them survive in the population we speak of polarization. Fragmentation indicates instead a configuration with more than two surviving states.

## A. Order and disorder: the Ising paradigm

In the previous section we have seen that the common theme of social dynamics is the understanding of the transition from an initial disordered state to a configuration that displays order (at least partially). Such type of transitions abound in traditional statistical physics (Huang, 1987; Kubo *et al.*, 1985). It is worth summarizing some important concepts and tools used in that context, as they are relevant also for the investigation of social dynamics. We will illustrate them using a paradigmatic example of order-disorder transitions in physics, the one exhibited by the Ising model for ferromagnets (Binney *et al.*, 1992). Beyond its relevance as a physics model, the Ising ferromagnet can be seen as a very simple model for opinion dynamics, with agents being influenced by the state of the majority of their interacting partners.

Consider a collection of  $N$  spins (agents)  $s_i$  that can assume two values  $\pm 1$ . Each spin is energetically pushed to be aligned with its nearest neighbors. The total energy is

$$H = -\frac{1}{2} \sum_{\langle i,j \rangle} s_i s_j, \quad (1)$$

where the sum runs on the pairs of nearest-neighbors spins. Among the possible types of dynamics, the most common (Metropolis) (Landau and Binder, 2005) takes as elementary move a single spin flip that is accepted with probability  $\exp(-\Delta E/k_B T)$ , where  $\Delta E$  is the change in

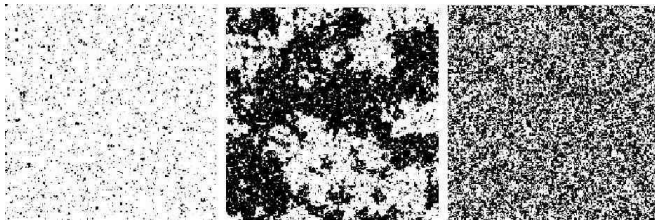


FIG. 1 Snapshots of equilibrium configurations of the Ising model (from left to right) below, at and above  $T_c$ .

energy and  $T$  is the temperature. Ferromagnetic interactions in Eq. (1) drive the system towards one of the two possible ordered states, with all positive or all negative spins. At the same time thermal noise injects fluctuations that tend to destroy order. For low temperature  $T$  the ordering tendency wins and long-range order is established in the system, while above a critical temperature  $T_c$  the system remains macroscopically disordered. The transition point is characterized by the average magnetization  $m = 1/N \sum_i \langle s_i \rangle$  passing from 0 for  $T > T_c$  to a value  $m(T) > 0$  for  $T < T_c$ . The brackets denote the average over different realizations of the dynamics. This kind of transitions is exhibited by a wealth of systems. Let us simply mention, for its similarity with many of the social dynamic models discussed in the review, the Potts model (Wu, 1982), where each spin can assume one out of  $q$  values and equal nearest neighbor values are energetically favored. The Ising model corresponds to the special case  $q = 2$ .

It is important to stress that above  $T_c$  no infinite-range order is established, but on short spatial scales spins are correlated: there are domains of  $+1$  spins (and others of  $-1$  spins) extended over regions of finite size. Below  $T_c$  instead these ordered regions extend to infinity (they span the whole system), although at finite temperature some disordered fluctuations are present on short scales (Fig. 1).

Not only the equilibrium properties just described, that are attained in the long run, are interesting. A much investigated and nontrivial issue (Bray, 1994) is the way the final ordered state at  $T < T_c$  is reached, when the system is initially prepared in a fully disordered state. This ordering dynamics is a prototype for the analogous processes occurring in many models of social dynamics. On short time scales, coexisting ordered domains of small size (both positive and negative) are formed. The subsequent evolution occurs through a *coarsening* process of such domains, which grow larger and larger while their global statistical features remain unchanged over time. This is the dynamic scaling phenomenon: the morphology remains statistically the same if rescaled by the typical domain size, which is the only relevant length in the system and grows over time as a power-law.

Macroscopically, the dynamic driving force towards order is surface tension. Interfaces between domains of opposite magnetization cost in terms of energy and their contribution can be minimized by making them as

straight as possible. This type of ordering is often referred to as curvature-driven and occurs in many of the social systems described in this review. The presence of surface tension is a consequence of the tendency of each spin to become aligned with the majority of its neighbors. When the majority does not play a role, the qualitative features of the ordering process change.

The dynamic aspect of the study of social models requires the monitoring of suitable quantities, able to properly identify the buildup of order. The magnetization of the system is not one of such suitable quantities. It is not sensitive to the size of single ordered domains, while it measures their cumulative extension, which is more or less the same during most of the evolution. The appropriate quantity to monitor the ordering process is the correlation function between pairs of spins at distance  $r$  from each other,  $C(r, t) = \langle s_i(t) s_{i+r}(t) \rangle - \langle s_i(t) \rangle^2$ , where brackets denote averaging over dynamic realizations and an additional average over  $i$  is implicit. The temporal variable  $t$  is measured as the average number of attempted updates per spin. The dynamic scaling property implies that  $C(r, t)$  is a function only of the ratio between the distance and the typical domain size  $L(t)$ :  $C(r, t) = L(t)^d F[r/L(t)]$ .  $L(t)$  grows in time as a power-law  $t^{1/z}$ . The dynamic exponent  $z$  is universal, independent of microscopic details, possibly depending only on qualitative features as conservation of the magnetization or space dimensionality. In the Glauber/Metropolis case  $z = 2$  in any dimension. Another quantity often used is the density of interfaces  $n_a(t) = N_a(t)/N_p$ , where  $N_p$  is the total number of nearest neighbor pairs and  $N_a$  the number of such pairs where the two neighbors are in different states:  $n_a = 1/2$  means that disorder is complete, while  $n_a = 0$  indicates full consensus.

Finally, a word about finite size effects. The very concept of order-disorder phase-transitions is rigorously defined only in the limit of a system with an infinite number of particles (thermodynamic limit), because only in that limit truly singular behavior can arise. Social systems are generally composed by a large number  $N$  of agents, but by far smaller than the number of atoms or molecules in a physical system. The finiteness of  $N$  must play therefore a crucial role in the analysis of models of social dynamics (Toral and Tessone, 2007). Studying what happens when  $N$  changes and even considering the large- $N$  limit is generally very useful, because it helps characterizing well qualitative behaviors, understanding which features are robust, and filtering out non-universal microscopical details.

## B. Role of topology

An important aspect always present in social dynamics is topology, i.e., the structure of the interaction network describing who is interacting with whom, how frequently and with which intensity. Agents are thus supposed to sit on vertices (nodes) of a network, and the edges (links)

define the possible interaction patterns.

The prototype of homogeneous networks is the uncorrelated random graph model proposed by Erdős and Rényi (ER model) (Erdős and Rényi, 1959, 1960), whose construction consists in drawing an (undirected) edge with a fixed probability  $p$  between each possible pair out of  $N$  given vertices. The resulting graph shows a binomial degree distribution, the degree of a node being the number of its connections, with average  $\langle k \rangle \simeq Np$ . The degree distribution converges to a Poissonian for large  $N$ . If  $p$  is sufficiently small (order  $1/N$ ), the graph is sparse and presents locally tree-like structures. In order to account for degree heterogeneity, other constructions have been proposed for random graphs with arbitrary degree distributions (Aiello and Lu, 2001; Catanzaro *et al.*, 2005; Goh *et al.*, 2001; Molloy and Reed, 1995, 1998).

A well-known paradigm, especially for social sciences, is that of “small-world” networks, in which, on the one hand, the average distance between two agents is small (Milgram, 1967), growing only logarithmically with the network size, and, on the other hand, many triangles are present, unlike ER graphs. In order to reconcile both properties, Watts and Strogatz have introduced the small-world network model (Watts and Strogatz, 1998), which allows to interpolate between regular low-dimensional lattices and random networks, by introducing a certain fraction  $p$  of random long-range connections into an initially regular lattice (Newman and Watts, 1999). In (Watts and Strogatz, 1998) two main quantities have been considered: the characteristic path length  $L(p)$ , defined as the number of edges in the shortest path between two vertices, averaged over all pairs of vertices, and the clustering coefficient  $C(p)$ , defined as follows. If a node  $i$  has  $k$  connections, then at most  $k(k-1)/2$  edges can exist between its neighbors (this occurs when every neighbor of  $i$  is connected to every other neighbor). The clustering coefficient  $C(p)$  denotes the fraction of these allowable edges that actually exist, averaged over all nodes. Small-world networks feature high values of  $C(p)$  and low values of  $L(p)$ .

Since many real networks are not static but evolving, with new nodes entering and establishing connections to already existing nodes, many models of growing networks have also been introduced. The Barabási and Albert model (BA) (Barabási and Albert, 1999), has become one of the most famous models for complex heterogeneous networks, and is constructed as follows: starting from a small set of  $m$  fully interconnected nodes, new nodes are introduced one by one. Each new node selects  $m$  older nodes according to the *preferential attachment* rule, i.e., with probability proportional to their degree, and creates links with them. The procedure stops when the required network size  $N$  is reached. The obtained network has average degree  $\langle k \rangle = 2m$ , small clustering coefficient (of order  $1/N$ ) and a power law degree distribution  $P(k) \sim k^{-\gamma}$ , with  $\gamma = 3$ . Graphs with power law degree distributions with  $\gamma \leq 3$  are referred to as *scale-free networks*.

An extensive analysis of the existing network models is out of the scope of the present review and we refer the reader to the huge literature on the so-called complex networks (Albert and Barabási, 2002; Boccaletti *et al.*, 2006; Caldarelli, 2007; Dorogovtsev and Mendes, 2003; Newman, 2003a; Pastor-Satorras and Vespignani, 2004). It is nevertheless important to mention that real networks often differ in many respects from artificial networks. People have used the social network metaphor for over a century to represent complex sets of relationships between members of social systems at all scales, from interpersonal to international. A huge amount of work has been carried out about the so-called social network analysis (SNA), especially in the social science literature (Freeman, 2004; Granovetter, 1973, 1983; Moreno, 1934; Scott, 2000; Wasserman and Faust, 1994). Recently the interest of physicists triggered the investigation of many different networks: from the network of scientific collaborations (Barabási *et al.*, 2002; Newman, 2001a,b, 2004) to that of sexual contacts (Liljeros *et al.*, 2001) and the ongoing social relationships (Holme, 2003), from email exchanges networks (Ebel *et al.*, 2002; Eckmann *et al.*, 2004; Newman *et al.*, 2002) to the dating community network (Holme *et al.*, 2004) and to mobile communication networks (Onnela *et al.*, 2007; Palla *et al.*, 2007), just to quote a few examples. From this experimental work a set of features characterizing social networks have been identified. It has been shown (Newman and Park, 2003) how social networks differ substantially from other types of networks, namely technological or biological. The origin of the difference is twofold. On the one hand they exhibit a positive correlation between adjacent vertices (also called assortativity), while most non-social networks (Newman, 2003b; Pastor-Satorras *et al.*, 2001) are disassortative. A network is said to show assortative mixing if nodes with many connections tend to be linked to other nodes with high degree. On the other hand social networks show clustering coefficients well above those of the corresponding random models. These results opened the way to a modeling activity aimed at reproducing in an artificial and controlled way the same features observed in real social networks (Jin *et al.*, 2001). We cannot review here all these attempts but we have quoted some relevant references all along the review when discussing specific modeling schemes. It is important to keep in mind that future investigations on social dynamics will be forced to take into account in a more stringent way structural and dynamic properties of real social networks (Roehner, 2007).

When applying models of social dynamics on specific topologies several non-trivial effects may arise, potentially leading to important biases for the dynamics. For instance, on networks with strongly heterogeneous degree distributions the dynamics of models with binary asymmetric interaction rules, i.e., where the two selected agents have different roles, may be affected by the order in which the interaction partners are selected. This is a consequence of the fact that, in a



network with degree distribution  $P(k)$ , if one picks a random node and then one of its neighbors, it is likely that the second node has a higher degree than the first. More precisely, the degree distribution of the second is  $kP(k)/\langle k \rangle$  (Pastor-Satorras and Vespignani, 2004). For scale-free degree distributions this may have strong effects (as is the case of the voter model, as seen in Sec. III.B, and of the Naming Game, as seen in Sec. V.B).

### C. Dynamical systems approach

One of the early contribution of physicists to the study of social systems has been the introduction of methods and tools coming from the theory of dynamical systems and non-linear dynamics. This development goes under the name of *sociodynamics* (Helbing, 1991; Weidlich, 1991; Weidlich, 2002). The term sociodynamics has been introduced to refer to a systematic approach to mathematical modeling in the framework of social sciences.

Sociodynamics is a branch of synergetics (Haken, 1978) devoted to social systems, featuring a few important differences. In synergetics one typically starts with a large set of microscopic equations for the elementary components and performs a reduction of the degrees of freedom. This is not the case for social systems, for which equations at the microscopic level are not available. In this case one has to identify relevant macro-variables and construct directly equations for them, based on reasonable and realistic social hypotheses, i.e., informed by social driving forces. The typical procedure consists in defining probabilistic transition rates per unit of time for the jumps between different configurations of the system corresponding to different values of the macro-variables. The transition rates are used as building blocks for setting up the equations of motion for the probabilistic evolution of the set of macro-variables. The central evolution equation in sociodynamics is the master equation, a phenomenological first-order differential equation describing the time evolution of the probability  $P(\mathbf{m}, t)$  for a system to occupy each one of a discrete set of states, defined through the set of macro-variables  $\mathbf{m}$ :

$$\frac{dP(\mathbf{m}, t)}{dt} = \sum_{\mathbf{m}'} [W_{\mathbf{m}', \mathbf{m}} P(\mathbf{m}', t) - W_{\mathbf{m}, \mathbf{m}'} P(\mathbf{m}, t)], \quad (2)$$

where  $W_{\mathbf{m}, \mathbf{m}'}$  represents the transition rate from the state  $\mathbf{m}$  to the state  $\mathbf{m}'$ . The master equation is a gain-loss equation for the probability of each state  $\mathbf{m}$ . The first term is the gain due to transitions from other states  $\mathbf{m}'$ , and the second term is the loss due to transitions into other states  $\mathbf{m}'$ .

While it is relatively easy to write down a master equation, it is quite another matter to solve it. It is usually highly non-linear and some clever simplifications are often needed to extract a solution. In general only numerical solutions are available. Moreover, typically the

master equation contains too much information in comparison to available empirical data. For all these reasons it is highly desirable to derive from the master equation simpler equations of motion for simpler variables. One straightforward possibility is to consider the equations of motion for the average values of the macro-variables  $\mathbf{m}$ , defined as:

$$\overline{m}_k(t) = \sum_{\mathbf{m}} m_k P(\mathbf{m}, t). \quad (3)$$

The exact expression for the equations of motion for  $\overline{m}_k(t)$  does not lead to simplifications because one should already know the full probability distribution  $P(\mathbf{m}, t)$ . On the other hand, under the assumption that the distribution remains unimodal and sharply peaked for the period of time under consideration, one has:

$$\overline{P(\mathbf{m}, t)} \simeq P(\overline{\mathbf{m}(t)}), \quad (4)$$

yielding the approximate equations of motions for  $\overline{m}_k(t)$ , which are now a closed system of coupled differential equations. We refer to (Weidlich, 2002) for a complete derivation of these equations as well as for the discussion of several applications. The approach has also been applied to model behavioral changes (Helbing, 1993a,b; Helbing, 1994).

### D. Agent-based modeling

Computer simulations play an important role in the study of social dynamics since they parallel more traditional approaches of theoretical physics aiming at describing a system in terms of a set of equations, to be later solved numerically and/or, whenever possible, analytically. One of the most successful methodologies used in social dynamics is *agent-based* modeling. The idea is to construct the computational devices (known as agents with some properties) and then simulate them in parallel to model the real phenomena. In physics this technique can be traced back to molecular dynamics (Alder and Wainwright, 1957, 1959) and Metropolis and Monte Carlo (Metropolis *et al.*, 1953) simulations. The goal is to address the problem of the emergence from the lower (micro) level of the social system to the higher (macro) level. The origin of agent-based modeling can be traced back to the 1940s, to the introduction by Von Neumann and Ulam of the notion of cellular automaton (Neumann, 1966; Ulam, 1960), e.g., a machine composed of a collection of cells on a grid. Each cell can be found in a discrete set of states and its update occurs on discrete time steps according to the state of the neighboring cells. A well-known example is Conway's Game of Life, defined in terms of simple rules in a virtual world shaped as a 2-dimensional checkerboard. This kind of algorithms became very popular in population biology (Matsuda *et al.*, 1992).

The notion of agent has been very important in the development of the concept of Artificial Intelligence (McCarthy, 1959; Minsky, 1961), which traditionally focuses on the individual and on rule-based paradigms inspired by psychology. In this framework the term *actors* was used to indicate interactive objects characterized by a certain number of internal states, acting in parallel and exchanging messages (Hewitt, 1970). In computer science the notion of actor turned in that of agent and more emphasis has been put on the interaction level instead of autonomous actions.

Agent-based models were primarily used for social systems by Reynolds, who tried to model the reality of living biological agents, known as artificial life, a term coined in (Langton, 1996). Reynolds introduced the notion of individual-based models, in which one investigates the global consequences of local interactions of members of a population (e.g. plants and animals in ecosystems, vehicles in traffic, people in crowds, or autonomous characters in animation and games). In these models individual agents (possibly heterogeneous) interact in a given environment according to procedural rules tuned by characteristic parameters. One thus focuses on the features of each individual instead of looking at some global quantity averaged over the whole population.

The Artificial Life community has been the first in developing agent-based models (Maes, 1991; Meyer and Wilson, 1990; Steels, 1995; Varela and Bourgine, 1992; Weiss, 1999), but since then agent-based simulations have become an important tool in other scientific fields and in particular in the study of social systems (Axelrod, 2006; Conte *et al.*, 1997; Macy and Willer, 2002; Schweitzer, 2003; Wooldridge, 2002). In (Epstein and Axtell, 1996), by focusing on a bottom-up approach, the first large scale agent model, the Sugarscape, has been introduced to simulate and explore the role of social phenomena such as seasonal migrations, pollution, sexual reproduction, combat, trade and transmission of disease and culture. In this context it is worth mentioning the concept of *Brownian agent* (Schweitzer, 2003) which generalizes that of Brownian particle from statistical mechanics. A Brownian agent is an active particle which possesses internal states, can store energy and information and interacts with other agents through the environment. Again the emphasis is on the parsimony in the agent definition as well as on the interactions, rather than on the autonomous actions. Agents interact either directly or in an indirect way through the external environment, which provides a feedback about the activities of the other agents. Direct interactions are typically local in time and ruled by the underlying topology of the interaction network (see also Sec. II.B). Populations can be homogeneous (i.e., all agents being identical) or heterogeneous. Differently from physical systems, the interactions are usually asymmetrical since the role of the interacting agents can be different both for the actions performed and for the rules to change their

internal states. Agent-based simulations have now acquired a central role in modeling complex systems and a large literature has been rapidly developing in the last few years about the internal structure of the agents, their activities and the multi-agent features. An exhaustive discussion of agent-based models is out of the scope of the present review, but we refer to (Schweitzer, 2003) where the role of active particles is thoroughly discussed with many examples of applications, ranging from structure formation in biological systems and pedestrian traffic to the simulation of urban aggregation or opinion formation processes.

### III. OPINION DYNAMICS

#### A. Introduction

Agreement is one of the most important aspects of social group dynamics. Everyday life presents many situations in which it is necessary for a group to reach shared decisions. Agreement makes a position stronger, and amplifies its impact on society.

The dynamics of agreement/disagreement among individuals is complex, because the individuals are. Statistical physicists working on opinion dynamics aim at defining the opinion states of a population, and the elementary processes that determine transitions between such states. The main question is whether this is possible and whether this approach can shed new light on the process of opinion formation.

In any mathematical model, opinion has to be a variable, or a set of variables, i.e., a collection of numbers. This may appear too reductive, thinking about the complexity of a person and of each individual position. Everyday life, on the contrary, indicates that people are sometimes confronted with a limited number of positions on a specific issue, which often are as few as two: right/left, Windows/Linux, buying/selling, etc. If opinions can be represented by numbers, the challenge is to find an adequate set of mathematical rules to describe the mechanisms responsible for the evolution and changes of them.

The development of opinion dynamics so far has been uncoordinated and based on individual attempts, where social mechanisms considered reasonable by the authors turned into mathematical rules, without a general shared framework and often with no reference to real sociological studies. The first opinion dynamics designed by a physicist was a model proposed in (Weidlich, 1971). The model is based on the probabilistic framework of sociodynamics, which was discussed in Sec. II.C. Later on the Ising model made its first appearance in opinion dynamics (Galam *et al.*, 1982; Galam and Moscovici, 1991). The spin-spin coupling represents the pairwise interaction between agents, the magnetic field the cultural majority or propaganda. Moreover individual fields are introduced that determine personal preferences toward



either orientation. Depending on the strength of the individual fields, the system may reach total consensus toward one of the two possible opinions, or a state where both opinions coexist.

In the last decade, physicists have started to work actively in opinion dynamics, and many models have been designed. We focus on the models that have received more attention in the physics literature, pointing out analogies as well as differences between them: the voter model (Sec. III.B), majority rule models (Sec. III.C), models based on social impact theory (Sec. III.D), Sznaid (Sec. III.E) and bounded confidence models (Sec. III.F). In Sec. III.G other models are briefly discussed. Finally, in Sec. III.H, we review recent work that aims at an empirical validation of opinion dynamics from the analysis of data referring to large scale social phenomena.

## B. Voter model

### 1. Regular lattices

The voter model has been named in this way for the very natural interpretation of its rules in terms of opinion dynamics; for its extremely simple definition, however, the model has been thoroughly investigated also in fields quite far from social dynamics, like probability theory and population genetics. Voter dynamics was first considered in (Clifford and Sudbury, 1973) as a model for the competition of species and named “voter model” in (Holley and Liggett, 1975). It has soon become popular because, despite being a rather crude description of any real process, it is one of the very few non-equilibrium stochastic processes that can be solved exactly in any dimension (Redner, 2001). It can also be seen as a model for dimer-dimer heterogeneous catalysis in the reaction controlled limit (Evans and Ray, 1993).

The definition is extremely simple: each agent is endowed with a binary variable  $s = \pm 1$ . At each time step an agent  $i$  is selected along with one of its neighbors  $j$  and  $s_i = s_j$ , i.e., the agent takes the opinion of the neighbor. This update rule implies that agents imitate their neighbors. They feel the pressure of the majority of their peers only in an average sense: the state of the majority does not play a direct role and more fluctuations may be expected with respect to the zero-temperature Glauber dynamics. Bulk noise is absent in the model, so the states with all sites equal (consensus) are absorbing. Starting from a disordered initial condition, voter dynamics tends to increase the order of the system, as in usual coarsening processes (Scheucher and Spohn, 1988). The question is whether full consensus is reached in a system of infinite size. In one-dimensional lattices the dynamics is exactly the same of the zero-temperature Glauber dynamics. A look at the patterns generated in two-dimensional lattices (Fig. 2) indicates that domains grow but interfaces are very rough, at odds with usual coarsening systems (Bray, 1994).

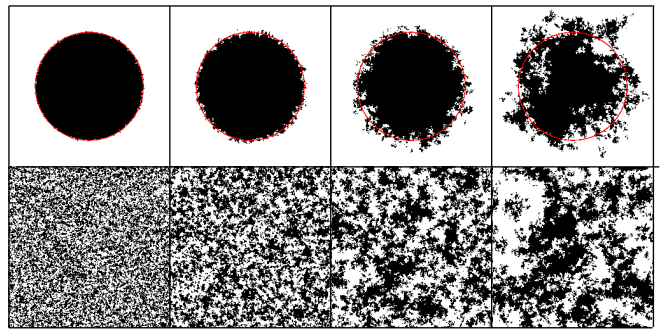


FIG. 2 Evolution of a two-dimensional voter model starting from a droplet (top) or a fully disordered configuration (bottom). From (Dornic *et al.*, 2001).

Early studies, performed by probabilists (Clifford and Sudbury, 1973; Cox and Griffeath, 1986; Holley and Liggett, 1975; Liggett, 1985), have exploited the fact that the model can be exactly mapped on a model of random walkers that coalesce upon encounter. This duality property allows to use the powerful machinery of random walk theory (Liggett, 1985, 1999). We prefer to follow another derivation of the general solution on lattices (Frachebourg and Krapivsky, 1996), based on earlier work (Krapivsky, 1992). Considering a  $d$ -dimensional hypercubic lattice and denoting with  $S = \{s_i\}$  the state of the system, the transition rate for a spin  $k$  to flip is

$$W_k(S) \equiv W(s_k \rightarrow -s_k) = \frac{d}{4} \left( 1 - \frac{1}{2d} s_k \sum_j s_j \right), \quad (5)$$

where  $j$  runs over all  $2d$  nearest neighbors and the prefactor, setting the overall temporal scale, is chosen for convenience. The probability distribution function  $P(S, t)$  obeys the master equation

$$\frac{d}{dt} P(S, t) = \sum_k [W_k(S^k) P(S^k, t) - W_k(S) P(S, t)], \quad (6)$$

where  $S^k$  is equal to  $S$  except for the flipped spin  $s_k$ . The linear structure of the rates (5) has the nice consequence that the equations for correlation functions of any order  $\langle s_k \cdots s_l \rangle \equiv \sum_S P(S, t) s_k \cdots s_l$  can be closed, i.e., they do not depend on higher-order functions and hence can be solved (Scheucher and Spohn, 1988).

The equation for the one-body correlation function is

$$\frac{d}{dt} \langle s_k \rangle = \Delta_k \langle s_k \rangle, \quad (7)$$

where  $\Delta_k$  is the discrete Laplace operator. Summing over  $k$  one sees that the global magnetization  $\langle s \rangle = 1/N \sum_k \langle s_k \rangle$  is conserved. This conservation immediately allows to determine the probability that a finite system will end up with all spins up or down (exit probability), depending on the initial density of up spins

$\rho(0) = (\langle s \rangle + 1)/2$ . This gives  $P_{up}(\rho(0)) = \rho(0)$  in any dimension.

The two-body correlation function obeys

$$\frac{d}{dt} \langle s_k s_l \rangle = (\Delta_k + \Delta_l) \langle s_k s_l \rangle. \quad (8)$$

The structure of this equation, as well as of those for higher-order correlation functions, is similar *in any dimension* to the equations for correlators of the one-dimensional Ising model with zero-temperature Glauber dynamics (Glauber, 1963) and can be solved analogously, via Laplace transform. In this way the asymptotic behavior of the density of active interfaces  $n_a(t) = (1 - \langle s_k s_{k+1} \rangle)/2$  is derived (Frachebourg and Krapivsky, 1996)

$$n_a(t) \sim \begin{cases} t^{-(2-d)/2} & d < 2 \\ 1/\ln(t) & d = 2 \\ a - bt^{-d/2} & d > 2. \end{cases} \quad (9)$$

Eq. (9) shows that for  $d \leq 2$  the voter model undergoes a coarsening process leading to complete consensus. For  $d > 2$  instead, it exhibits asymptotically a finite density of interfaces, i.e., no consensus is reached (in an infinite system) and domains of opposite opinions coexist indefinitely in time. In terms of duality the lack of order in high dimensions is a consequence of the transient nature of random walks in  $d > 2$ : diffusing active interfaces have a finite probability to meet and annihilate. For  $d = 2$  the exact expression of the density of active interfaces for large times is

$$n_a(t) = \frac{\pi}{2 \ln(t) + \ln(256)} + O\left(\frac{\ln t}{t}\right). \quad (10)$$

The large constant value in the denominator of Eq. (10) makes the approach to the asymptotic logarithmic decay very slow, and explains why different laws were hypothesized, based on numerical evidence (Evans and Ray, 1993; Meakin and Scalapino, 1987).

Beyond the expression for the density  $n_a(t)$ , the solution of Eq. (8) allows to write down a scaling form for the correlation function  $C(r, t)$  (Dornic, 1998; Scheucher and Spohn, 1988). In  $d = 2$  the solution violates logarithmically the standard scaling form (see Sec. II.A) holding for usual coarsening phenomena (Bray, 1994). This violation reflects the fact that there is no single domain length in the system. On the contrary, domains of all sizes  $t^\alpha$ , with  $0 < \alpha \leq 1/2$ , are simultaneously present (Cox and Griffeath, 1986; Scheucher and Spohn, 1988).

While in an infinite system consensus is reached only if  $d \leq 2$ , in any dimension consensus is invariably reached asymptotically if the system is finite. The time  $T_N$  needed depends on the system size  $N$  (Cox, 1989):  $T_N \sim N^2$  for  $d = 1$ ,  $T_N \sim N \ln N$  for  $d = 2$ , while  $T_N \sim N$  for  $d > 2$ . It is worth remarking that the way consensus is reached on finite systems has a completely different nature for  $d \leq 2$  (where the system coherently

tends towards order by coarsening) and for  $d > 2$  (where consensus is reached only because of a large random fluctuation).

Expression (5) for the spin-flip rates is rather special. How much of the voter behavior is retained if rates are modified? A natural generalization (Drouffe and Godrèche, 1999; de Oliveira *et al.*, 1993) considers transition rates of the form  $W_k(S) = 1/2[1 - s_k f_k(S)]$ , where  $f_k(S)$  is a local function with  $|f_k(S)| \leq 1$ . A local dynamics that is spatially symmetric and preserves the up-down symmetry requires  $f_k(S)$  to be an odd function of the sum of the nearest neighbors. In a square lattice, the local field can assume five values, but only two of them are independent,  $f(2) = -f(-2) = x$ , and  $f(4) = -f(-4) = y$ . Voter dynamics corresponds to  $x = 1/2$  and  $y = 1$ , while  $x = y$  corresponds to the majority-vote model (Sec. III.C),  $y = 2x/(1 + x^2)$  gives the transition rates of Glauber dynamics, and the case  $y = 2x$  corresponds to the noisy voter model (see below). The significance of the two parameters is straightforward:  $y$  gauges bulk noise, i.e., the possibility that a spin fully surrounded by equal spins flips to the opposite position; The value  $y = 1$  implies absence of such noise. The parameter  $x$  instead measures the amount of interfacial noise. Simulations and a pair approximation treatment show that the phase-diagram of this generalized model is divided in a ferromagnetic region around the  $x = 1, y = 1$  point (zero-temperature Glauber dynamics) and a paramagnetic phase, separated by a line of continuous phase-transitions terminating at the voter model point. Changing the interfacial noise parameter  $x$ , while keeping  $y = 1$ , one finds a jump of the order parameter, indicating a first-order transition. Hence the voter point is critical, sitting exactly at the transition between order and disorder driven by purely interfacial noise.

More physical insight is provided by considering a droplet of up spins surrounded by negative spins (Dornic *et al.*, 2001). The Cahn-Allen theory for curvature-driven coarsening (Bray, 1994) predicts in  $d = 2$  a linear decay in time of the droplet area, the rate being proportional to surface tension. In the voter model instead, the interface of the droplet roughens but its radius remains statistically unchanged (Dall'Asta and Castellano, 2007; Dornic *et al.*, 2001), showing that no surface tension is present (Fig. 2).

The phase diagram of the generalized model (de Oliveira *et al.*, 1993) described above seems to suggest that the voter model is rather peculiar, being only a point in the line of continuous transitions between the ferromagnetic and the paramagnetic phase. However, voter-like behavior (characterized by the absence of surface tension leading to logarithmic ordering in  $d = 2$ ) can be found in other models. It has been argued (Dornic *et al.*, 2001) that voter behavior is generically observed at order-disorder non-equilibrium transitions, driven by interfacial noise, between dynamically symmetric absorbing states. This symmetry may

be enforced either by an up-down symmetry of the local rules or by global conservation of the magnetization. The universal exponents associated to the transition are  $\beta = 0$ , and  $\nu = 1/2$  in all dimensions, while  $\gamma = 1/2$  for  $d = 1$ , and  $\gamma = 1$  for  $d > 2$  with logarithmic corrections at the upper critical dimension  $d = 2$  (Dornic *et al.*, 2001; de Oliveira, 2003).

The original voter dynamics does not include the possibility for a spin to flip spontaneously when equal to all its neighbors. The noisy voter model (Granovsky and Madras, 1995; Scheucher and Spohn, 1988), also called linear Glauber model (de Oliveira, 2003) includes this possibility, via a modification of the rates (5) that keeps the model exactly solvable. The effect of bulk noise is to destroy long-range order: the noisy voter model is always in the paramagnetic phase of the generalized model of (de Oliveira *et al.*, 1993), so that domains form only up to a finite correlation length. As the strength of bulk noise is decreased, the length grows and the voter first-order transition occurs for zero noise.

The investigation of the generalized voter universality class and its connections with other classes of non-equilibrium phase transitions is a complicated and open issue, approached also via field-theoretical methods (Dickman and Tretyakov, 1995; Droz *et al.*, 2003; Hammal *et al.*, 2005).

## 2. Modifications and applications

Being a very simple non-equilibrium dynamics with a nontrivial behavior, the voter model has been investigated with respect to many properties in recent years, including persistence, aging and correlated percolation. Furthermore, many modifications of the original dynamics have been proposed in order to model various types of phenomena or to test the robustness of the voter phenomenology. A natural extension is a voter dynamics for Potts variables (multitype voter model), where many of the results obtained for the Ising case are easily generalizable (Sire and Majumdar, 1995).

One possible modification is the presence of quenched disorder, in the form of one “zealot”, i.e., an individual that does not change its opinion (Mobilia, 2003). This modification breaks the conservation of magnetization: in  $d \leq 2$  the zealot influences all, inducing general consensus with its opinion. In higher dimensions consensus is still not reached, but in the neighborhood of the zealot the stationary state is biased toward his opinion. The case of many zealots has also been addressed (Mobilia and Georgiev, 2005; Mobilia *et al.*, 2007).

Another variant is the constrained voter model (Vazquez *et al.*, 2003), where agents can be in three states (leftists, rightists, or centrists) but interactions involve only centrists, while extremists do not talk to each other. In this way a discrete analogue of

bounded confidence is implemented. Detailed analytical results give the probabilities, as a function of the initial conditions, of ending up with full consensus in one of the three states or with a mixture of the extremists, with little change between  $d = 1$  (Vazquez *et al.*, 2003) and mean field (Vazquez and Redner, 2004). A similar model with three states is the AB-model (Castelló *et al.*, 2006). Here the state of an agent evolves according to the following rules. At each time step one randomly chooses an agent  $i$  and updates its state according to the following transition probabilities:

$$p_{A \rightarrow AB} = 1/2\sigma_B, \quad p_{B \rightarrow AB} = 1/2\sigma_A, \quad (11)$$

$$p_{AB \rightarrow B} = 1/2(1 - \sigma_A), \quad p_{AB \rightarrow A} = 1/2(1 - \sigma_B) \quad (12)$$

where  $\sigma_l$  ( $l=A,B,AB$ ) are the local densities of each state in the neighborhood of  $i$ . The idea here is that, in order to go from A to B one has to pass through the intermediate state AB. At odds with the constrained voter model, however, here extremes do interact, since the rate to go from state A to AB is proportional to the density of neighbors in state B. This implies that consensus on the AB state or a frozen mixture of A and B is not possible, the only two possible absorbing states being those of consensus of A or B type.

In the original voter model the decision on the possible flip of a spin depends on one neighbor. In the “vacillating” voter model (Lambiotte and Redner, 2007) a site checks the state of two of its neighbors and flips if either is different from himself. This leads to a bias towards the zero magnetization state, anti-coarsening behavior and consensus time scaling exponentially with the system size  $N$ . A related one-dimensional model is (Lambiotte and Redner, 2007).

Another modification is the introduction of memory in the form of noise reduction (Dall’Asta and Castellano, 2007). Each spin has associated two counters. When an interaction takes place with a positive (negative) neighbor, instead of modifying the spin the positive (negative) counter is increased by one. The spin is updated only when one of the counters reaches a threshold  $r$ . This change induces an effective surface tension, leading to curvature-driven coarsening dynamics.

A counterintuitive effect is reported in (Stark *et al.*, 2007) where a voter model with “inertia” is presented: the local rate of change of a site decreases linearly with the time since the last flip, up to a finite saturation value. On two-dimensional lattices, Watts-Strogatz and fully connected networks, a sufficiently weak inertia makes global dynamics faster: consensus time is smaller than with fixed rate.

Other variants of the original voter model have been devised for studying ecological problems. Recent publications in the physics literature are the study of diversity in plant communities (voter model with speciation (Zillio *et al.*, 2005)), or the investigation of fixation in the evolution of competing species (biased voter model (Antal *et al.*, 2006)).

### 3. The voter model on networks

Non-regular topologies have nontrivial effects on the ordering dynamics of the voter model.

On a complete graph the Fokker-Planck equation for the probability density of the magnetization has the form of a one-dimensional diffusion equation with a position-dependent diffusion constant, and can be solved analytically (Slanina and Lavička, 2003). The lack of a drift term is the effect of the lack of surface tension in the model dynamics. The average time needed to reach consensus in a finite system can be computed exactly for any value of the initial magnetization and scales as the size of the system  $N$ . The tail of the distribution can also be computed and has an exponential decay  $\exp(-t/N)$ .

When considering disordered topologies different ways of defining the voter dynamics, that are perfectly equivalent on regular lattices, give rise to nonequivalent generalizations of the voter model. When the degree distribution is heterogeneous, the order in which a site and the neighbor to be copied are selected does matter, because high-degree nodes are more easily chosen as neighbors than low-degree vertices.

The most natural generalization (*direct voter model*) is to pick up a site and make it equal to one of its neighbors. In this way one of the fundamental properties of the voter model, conservation of the global magnetization, is violated (Suchecky *et al.*, 2005a; Wu *et al.*, 2004). To restore conservation a *link-update* dynamics must be considered (Suchecky *et al.*, 2005a): a link is selected at random and then one node located at a randomly chosen end is set equal to the other. If instead one chooses first a node and copies its variable to a randomly selected neighbor one obtains the *reverse (or invasion) voter dynamics* (Castellano, 2005).

On highly heterogeneous substrates these different definitions result in different behaviors. The mean consensus time  $T_N$  has been computed in (Sood *et al.*, 2007; Sood and Redner, 2005) for the direct voter dynamics on a generic graph, by exploiting the conservation of a suitably defined degree-weighted density  $\omega$  of up spins,

$$T_N(\omega) = -N \frac{\mu_1^2}{\mu_2} [(1 - \omega) \ln(1 - \omega) + \omega \ln \omega], \quad (13)$$

where  $\mu_k$  is the  $k$ -th moment of the degree-distribution. For networks with scale-free distributed degree (with exponent  $\gamma$ ),  $T_N$  scales then as  $N$  for  $\gamma > 3$  and sublinearly for  $\gamma \leq 3$ , in good agreement with numerical simulations (Castellano *et al.*, 2005; Sood and Redner, 2005; Suchecky *et al.*, 2005a). The same approach gives, for the other versions of voter dynamics on graphs, a linear dependence of the consensus time on  $N$  for link-update dynamics (independent of the degree distribution) and  $T_N \sim N$  for any  $\gamma > 2$  for the reverse-voter dynamics, again in good agreement with simulations (Castellano, 2005; Sood *et al.*, 2007). A general analysis of voter-like dynamics on generic topologies is presented in (Baxter *et al.*, 2008), with particular refer-

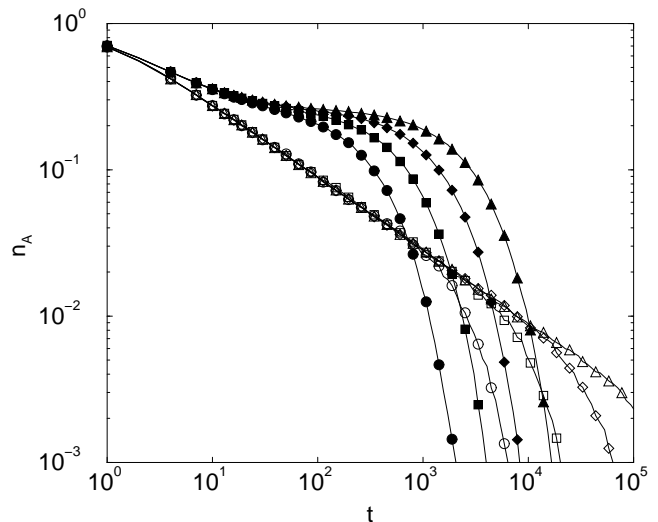


FIG. 3 Log-log plot of the fraction  $n_A$  of active bonds between nodes with different opinions. Empty symbols are for the one-dimensional case ( $p = 0$ ). Filled symbols are for rewiring probability  $p = 0.05$ . Data are for  $N = 200$  (circles),  $N = 400$  (squares),  $N = 800$  (diamonds),  $N = 1600$  (triangles up) and  $N = 3200$  (triangles left). Reprinted figure with permission from (Castellano *et al.*, 2003). Copyright 2003 from EDP Sciences.

ence to applications in population genetics and biodiversity studies.

Another interesting effect of the topology occurs when voter dynamics is considered on small-world networks (Watts and Strogatz, 1998). After an initial regime equal to the one-dimensional behavior, the density of active interfaces forms a plateau (Fig. 3), because shortcuts hinder their diffusive motion. The system remains trapped in a metastable state with coexisting domains of opposite opinions, whose typical length scales as  $1/p$  (Castellano *et al.*, 2003; Vilone and Castellano, 2004),  $p$  being the fraction of long-range connections.

The lifetime of the metastable state scales with the linear system size  $L$  so that for finite systems consensus is eventually reached on a temporal scale shorter than on a regular one-dimensional lattice ( $L^2$ ). For infinite systems instead, the state with coexisting opinions is actually stable, leading to the conclusion that long-range connections prevent the complete ordering of the voter model, in a way similar to what occurs for Glauber dynamics (Boyer and Miramontes, 2003). A general discussion of the interplay between topology and dynamics for the voter model is presented in (Suchecky *et al.*, 2005b). A comparison between the behavior of the voter dynamics and the AB-model on modular networks is in (Castelló *et al.*, 2007).

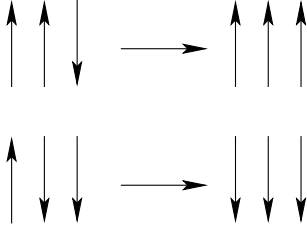


FIG. 4 MR model. The majority opinion inside a discussion group (here of size three) is taken by all agents.

### C. Majority rule model

In a population of  $N$  agents, endowed with binary opinions, a fraction  $p_+$  of agents has opinion  $+1$  while a fraction  $p_- = 1 - p_+$  opinion  $-1$ . For simplicity, suppose that all agents can communicate with each other, so that the social network of contacts is a complete graph. At each iteration, a group of  $r$  agents is selected at random (discussion group): as a consequence of the interaction, all agents take the majority opinion inside the group (Fig. 4). This is the basic principle of the majority rule (MR) model, which was proposed to describe public debates (Galam, 2002).

The group size  $r$  is not fixed, but is selected at each step from a given distribution. If  $r$  is odd, there is always a majority in favor of either opinion. If  $r$  is even, instead, there is the possibility of a tie, i.e., that either opinion is supported by exactly  $r/2$  agents. In this case, one introduces a bias in favor of one of the opinions, say  $+1$ , and that opinion prevails in the group. This prescription is inspired by the principle of social inertia, for which people are reluctant to accept a reform if there is no clear majority in its favor (Friedman and Friedman, 1984). Majority rule with opinion bias was originally applied within a simple model describing hierarchical voting in a society (Galam, 1986, 1990, 1999, 2000).

Defined as  $p_+^0$  the initial fraction of agents with the opinion  $+1$ , the dynamics is characterized by a threshold  $p_c$  such that, for  $p_+^0 > p_c$  ( $p_+^0 < p_c$ ), all agents will have opinion  $+1$  ( $-1$ ) in the long run. The time to reach consensus (in number of updates per spin) scales like  $\log N$  (Tessone *et al.*, 2004). If the group sizes are odd,  $p_c(r) = 1/2$ , due to the symmetry of the two opinions. If there are groups with  $r$  even,  $p_c < 1/2$ , i.e., the favored opinion will eventually be the dominant one, even if it is initially shared by a minority of agents.

The MR model<sup>1</sup> with a fixed group size  $r$  was analytically solved in the mean field limit (Krapivsky and Redner, 2003). The group size  $r$  is odd, to keep the symmetry of the two opinions.

The solution can be derived both for a finite population of  $N$  agents and in the continuum limit of  $N \rightarrow \infty$ . The latter derivation is simpler (Chen and Redner, 2005), and is sketched here.

Let  $s_k = \pm 1$  be the opinion of agent  $k$ ; the average opinion (magnetization) of the system is  $m = 1/N \sum_k s_k = p_+ - p_-$ . The size of each discussion group is 3. At each update step, the number  $N_+$  of agents in state  $+$  increases by one unit if the group state is  $++-$ , while it decreases by one unit if the group state is  $+- -$ . One thus has:

$$dN_+ = 3(p_+^2 p_- - p_+ p_-^2) = -6p_+(p_+ - \frac{1}{2})(p_+ - 1), \quad (14)$$

where the factor of 3 is due to the different permutations of the configurations  $++-$  and  $+- -$ . Eq. (14) can be rewritten as:

$$\frac{dN_+}{N} \frac{N}{3} = \dot{p}_+ = -2p_+(p_+ - \frac{1}{2})(p_+ - 1), \quad (15)$$

with the time increment  $dt = 3/N$ , so that each agent is updated once per unit of time. The fixed points are determined by the condition  $\dot{p}_+ = 0$  and from Eq. (15) we see that this happens when  $p_+ = 0, 1/2$  and  $1$ , respectively. The point  $p_+ = 1/2$  is unstable, whereas the others are stable: starting from any  $p_+ \neq 1/2$ , all agents will converge to the state of initial majority, recovering Galam's result. The integration of Eq. (15) yields that the consensus time grows as  $\log N$ .

On a  $d$ -dimensional lattice, the discussion group is localized around a randomly chosen lattice site. In one dimension, the model is not analytically solvable. Since the average magnetization is not conserved by the MR dynamics, the exit probability, i.e., the probability that the final magnetization is  $+1$ , has a non-trivial dependence on the initial magnetization in the thermodynamic limit and a minority can actually win the contest. Consensus time grows as  $N^2$ . In higher dimensions (Chen and Redner, 2005), the dynamics is characterized by diffusive coarsening. When the initial magnetization is zero, the system may be trapped in metastable states (stripes in  $2d$ , slabs in  $3d$ ), which evolve only very slowly. This leads to the existence of two distinct temporal scales: the most probable consensus time is short but, when metastable states appear, the time needed is exceedingly longer. As a consequence, the average consensus time grows as a power of  $N$ , with a dimension-dependent exponent. When the initial magnetization is non-zero, metastable states quickly disappear. A crude coarse-graining argument reproduces qualitatively the occurrences of metastable configurations for any  $d$ . Numerical simulations show that the MR model in four dimensions does not reproduce the results of the mean field limit, so the upper critical dimension of the MR model is larger than four. The MR dynamics was also investigated on networks with strong degree heterogeneities (Lambiotte, 2007) and on networks with community structure, i.e., graphs consisting of groups of nodes with a comparatively large density

<sup>1</sup> The name Majority Rule model was actually coined in (Krapivsky and Redner, 2003). Since this model is just a special case of the one introduced in (Galam, 2002), we adopt this name since the beginning of the section.

of internal links with respect to the density of connections between different groups (Lambiotte and Ausloos, 2007; Lambiotte *et al.*, 2007a). The MR model was studied on small world lattices (Li *et al.*, 2006) as well.

The MR model has been extended to multi-state opinions and plurality rule (Chen and Redner, 2005). The number of opinion states and the size of the interaction groups are denoted with  $s$  and  $G$ , respectively. In the mean field limit, the system reaches consensus for any choice of  $s$  and  $G$ , in a time that scales like  $\log N$ , as in the 2-state MR model. On a square lattice, if the number of states  $s$  is too large, there are no groups with a majority, so the system does not evolve, otherwise the evolution is based on diffusive coarsening, similarly to that of the 2-state MR model. Again, two different timescales emerge when  $s$  is small, due to the existence of metastable states. When  $s$  and  $G$  approach a threshold, there is only one domain that grows and invades all sites, so there is only one time scale. The plurality rule is a special extension of the MR rule when there are more than two opinion states: in this case, all agents of a group take the opinion with the most representatives in the group. The evolution leads to consensus for any  $s$  and  $G$ , because all interaction groups are active (there is always a relative majority); when the opinions reduce to two, the dynamics becomes identical to that of the 2-state MR model, so there will be metastable states and two different timescales.

Modifications of the MR model include: a model where agents can move in space (Galam *et al.*, 2002; Stauffer, 2002a); a dynamics where each agent interacts with a variable number of neighbors (Tessone *et al.*, 2004); an extension to three opinions (Gekle *et al.*, 2005); the introduction of a probability to favor a particular opinion, that could vary among different individuals and/or social groups (Galam, 2005a); the presence of “contrarians”, i.e., agents that initially take the majority opinion in a group discussion, but that right after the discussion switch to the opposite opinion (Galam, 2004; Stauffer and Sá Martins, 2004); the presence of one-sided contrarians and unsettled agents (Borghesi and Galam, 2006); the presence of inflexible agents, that always stay by their side (Galam and Jacobs, 2007).

We now discuss some variants of the majority rule. In the majority-minority (MM) model (Mobilia and Redner, 2003), one accounts for the possibility that minorities take over: in a discussion group the majority opinion prevails with a probability  $p$ , whereas with a probability  $1 - p$  it is the minority opinion that dominates. For a discussion group of three agents, the magnetization  $m$  changes by an amount  $2p - 4(1 - p)$  at each interaction, which means that, for  $p = p_c = 2/3$ ,  $m$  does not change on average, like in the voter model. In the mean field limit, the model can be solved analytically: the exit probability turns out to be a step function for  $p > p_c$ , (i.e., the system will evolve towards consensus around the initial majority opinion), whereas it equals  $1/2$  for  $p < p_c$ , which means that the system is driven towards zero

magnetization.

Another interesting model based on majority rule is the majority-vote model (Liggett, 1985). At each update step, with a probability  $1 - q$  a spin takes the sign of the majority of its neighbors, with a probability  $q$  it takes the minority spin state. If there is a tie, the spin is flipped with probability  $1/2$ . The parameter  $q$  is the so-called noise parameter. We stress that a single spin is updated at each time step, at variance with the MR model. For  $q = 0$  the model coincides with the Ising model with zero-temperature Glauber kinetics (Glauber, 1963). On a regular lattice, the majority-vote model presents a phase transition from an ordered to a disordered state at a critical value  $q_c$  of the noise parameter (de Oliveira, 1992). The critical exponents of the transition are in the Ising universality class. Recent studies showed that the majority-vote model also generates an order-disorder phase transition on small-world lattices (Campos *et al.*, 2003) and on random graphs (Lima *et al.*, 2008; Pereira and Moreira, 2005).

In a recent model, an agent is convinced if there is at least a fraction  $p$  of its neighbors sharing the same opinion (Klimek *et al.*, 2007). This model interpolates between the rule where majority dominates ( $p = 1/2$ ) and the unanimity rule ( $p = 1$ ), where an agent is influenced by its neighbors only if they all have the same opinion (Lambiotte *et al.*, 2007b). Another model based similar to majority-vote is presented and studied on directed networks in (Sánchez *et al.*, 2002).

## D. Social impact theory

The psychological theory of social impact (Latané, 1981) describes how individuals feel the presence of their peers and how they in turn influence other individuals. The impact of a social group on a subject depends on the number of the individuals in the group, on their convincing power, and on the distance from the subject, where the distance may refer both to spatial proximity or to the closeness in an abstract space of personal relationships. The original cellular automata introduced in (Latané, 1981) and refined in (Nowak *et al.*, 1990) represent a class of dynamic models of statistical mechanics, that are exactly solvable in the mean field limit (Lewenstein *et al.*, 1992).

The starting point is a population of  $N$  individuals. Each individual  $i$  is characterized by an opinion  $\sigma_i = \pm 1$  and by two real-valued parameters, that estimate the strength of its action on the others: persuasiveness  $p_i$  and supportiveness  $s_i$ , that describe the capability to convince someone to change or to keep its opinion, respectively. These parameters are assumed to be random numbers, and introduce a disorder that is responsible for the complex dynamics of the model. The distance of a pair of agents  $i$  and  $j$  is  $d_{ij}$ . In the simplest version, the total impact  $I_i$  that an individual  $i$  experiences from



his/her social environment is

$$I_i = \left[ \sum_{j=1}^N \frac{p_j}{d_{ij}^\alpha} (1 - \sigma_i \sigma_j) \right] - \left[ \sum_{j=1}^N \frac{s_j}{d_{ij}^\alpha} (1 + \sigma_i \sigma_j) \right], \quad (16)$$

where  $\alpha > 2$  expresses how fast the impact decreases with the distance  $d_{ij}$  between two individuals. The first term of Eq. (16) expresses the persuasive impact, i.e. the pressure exerted by the agents with opposite opinions, which tend to enforce an opinion change; the second term instead is the supportive impact, i.e. the pressure exerted by the agents with the same opinion of  $i$ , which favor the status quo. In both cases, the impact of each agent on  $i$  is proportional to its persuasiveness/supportiveness.

The opinion dynamics is expressed by the rule

$$\sigma_i(t+1) = -\text{sgn}[\sigma_i(t)I_i(t) + h_i], \quad (17)$$

where  $h_i$  is a random field representing all sources other than social impact that may affect the opinion (e.g. mass media). According to Eq. (17), a spin flips if the pressure in favor of the opinion change overcomes the pressure to keep the current opinion ( $I_i > 0$  for vanishing  $h_i$ ).

For a system of fully connected agents, and without individual fields, the model presents infinitely many stationary states (Lewenstein *et al.*, 1992). The order parameter of the dynamics is a complex function of one variable, like in spin glasses (Mezard *et al.*, 1987).

In general, in the absence of individual fields, the dynamics leads to the dominance of one opinion on the other, but not to complete consensus. If the initial magnetization is about zero, the system converges to configurations characterized by a large majority of spins in the same opinion state, and by stable domains of spins in the minority opinion state. In the presence of individual fields, these minority domains become metastable: they remain stationary for a very long time, then they suddenly shrink to smaller clusters, which again persist for a very long time, before shrinking again, and so on (“staircase dynamics”).

The dynamics can be modified to account for other processes related to social behavior, such as learning (Kohring, 1996), the response of a population to the simultaneous action of a strong leader and external influence (Holyst *et al.*, 2000; Kacperski and Holyst, 1996, 1997, 1999, 2000) and the mitigation of social impact due to the coexistence of different individuals in a group (Bordogna and Albano, 2007). For a review of statistical mechanical models of social impact, see (Holyst *et al.*, 2001).

Social impact theory neglects a number of realistic features of social interaction: the existence of a memory of the individuals, which reflects the past experience; a finite velocity for the exchange of information between agents; a physical space, where agents have the possibility to move. An important extension of social impact theory that includes those features is based on *active Brownian particles* (Schweitzer, 2003; Schweitzer and Holyst,

2000), that are Brownian particles endowed with some internal energy depot that allows them to move and to perform several tasks as well. The interaction is due to a scalar opinion field, expressing the social impact of all agents/opinions at each point in space; the particles/agents act as sources of the field and are in turn affected by it, both in opinion and in space. Each agent  $i$  is labeled by its opinion  $\sigma_i = \pm 1$  and its personal strength  $s_i$ . The field of opinion  $\sigma$ , at position  $\mathbf{r}$  and time  $t$ , is indicated with  $h_\sigma(\mathbf{r}, t)$ . The transition probability rates  $w(\sigma'|\sigma)$ , for an agent to pass from opinion  $\sigma$  to opinion  $\sigma'$ , with  $\sigma \neq \sigma'$ , are defined as:

$$w(\sigma'|\sigma) = \eta \exp\{[h_{\sigma'}(\mathbf{r}, t) - h_\sigma(\mathbf{r}, t)]/T\}, \quad (18)$$

where  $T$  is a social temperature. The dynamics is expressed by two sets of equations: one set describes the spatio-temporal change of the opinion field

$$\frac{\partial}{\partial t} h_\sigma(\mathbf{r}, t) = \sum_{i=1}^N s_i \delta_{\sigma, \sigma_i} \delta(\mathbf{r} - \mathbf{r}_i) - \gamma h_\sigma(\mathbf{r}, t) + D_h \Delta h_\sigma(\mathbf{r}, t), \quad (19)$$

the other set presents reaction-diffusion equations for the density  $n_\sigma(\mathbf{r}, t)$  of individuals with opinion  $\sigma$ , at position  $\mathbf{r}$  and time  $t$

$$\begin{aligned} \frac{\partial}{\partial t} n_\sigma(\mathbf{r}, t) = & -\nabla[n_\sigma(\mathbf{r}, t)\alpha\nabla h_\sigma(\mathbf{r}, t)] + D_n \Delta n_\sigma(\mathbf{r}, t) \\ & - \sum_{\sigma' \neq \sigma} [w(\sigma'|\sigma)n_\sigma(\mathbf{r}, t) - w(\sigma|\sigma')n_{\sigma'}(\mathbf{r}, t)]. \end{aligned} \quad (20)$$

In the equations above,  $N$  is the number of agents,  $1/\gamma$  is the average lifetime of the field,  $D_h$  is the diffusion constant for information exchange,  $D_n$  the spatial diffusion coefficient of the individuals,  $\alpha$  measures the agents’ response to the field. The three terms on the right-hand side of Eq. (19) represent the microscopic density of the agents’ strength, the relaxation of the field (modeling the memory effects) and its diffusion in space, respectively. The first two terms on the right-hand side of Eq. (20) indicate the change in the agents’ density due to their motion in space, whereas the sum expresses the balance between the gain and loss of individuals due to opinion changes.

Eqs. (19) and (20) are coupled: depending on the local intensity of the field supporting either opinion, an agent can change its opinion, or migrate towards locations where its opinion has a larger support. Opinion changes and migrations have a non-linear feedback on the communication field, which in turn affects the agents, and so on. The model presents three phases, depending on the values of the parameters: a paramagnetic phase, where both opinions have the same probability (1/2) of being selected at every place (high-temperature, high-diffusion), a ferromagnetic phase, with more agents in favor of one opinion over the other (low-temperature, low-diffusion), and a phase in which either opinion prevails in spatially separated domains (segregation).

### E. Sznajd model

In the previous section we have seen that the impact exerted by a social group on an individual increases with the size of the group. We would not pay attention to a single guy staring at a blank wall; instead, if a group of people stares at that wall, we may be tempted to do the same. Convincing somebody is easier for two or more people than for a single individual. This is the basic principle behind the Sznajd model (Stauffer, 2003a; Sznajd-Weron, 2005b). In its original version (Sznajd-Weron and Sznajd, 2000), that we call Sznajd B, agents occupy the sites of a linear chain, and have binary opinions, denoted by Ising spin variables. A pair of neighboring agents  $i$  and  $i + 1$  determines the opinions of their two nearest neighbors  $i - 1$  and  $i + 2$ , according to these rules:

$$\text{if } s_i = s_{i+1}, \text{ then } s_{i-1} = s_i = s_{i+1} = s_{i+2}; \quad (21)$$

$$\text{if } s_i \neq s_{i+1}, \text{ then } s_{i-1} = s_{i+1} \text{ and } s_{i+2} = s_i. \quad (22)$$

So, if the agents of the pair share the same opinion, they successfully impose their opinion on their neighbors. If, instead, the two agents disagree, each agent imposes its opinion on the other agent's neighbor.

Opinions are updated in a random sequential order. Starting from a totally random initial configuration, where both opinions are equally distributed, two types of stationary states are found, corresponding to consensus, with all spins up ( $m = 1$ ) or all spins down ( $m = -1$ ), and to a stalemate, with the same number of up and down spins in antiferromagnetic order ( $m = 0$ ). The latter state is a consequence of rule (22), that favors antiferromagnetic configurations, and has a probability  $1/2$  to be reached. Each of the two (ferromagnetic) consensus states occurs with a probability  $1/4$ . The values of the probability can be easily deduced from the up-down symmetry of the model. The relaxation time of the system into one of the possible attractors has a log-normal distribution (Behera and Schweitzer, 2003). The number of agents that never changed opinion first decays as a power law of time, and then it reaches a constant but finite value, at odds with the Ising model (Stauffer and de Oliveira, 2002). The exit probability has been calculated analytically for both random and correlated initial conditions (Lambiotte and Redner, 2007; Slanina *et al.*, 2007).

Since the very introduction of the Sznajd model, it has been argued that a distinctive feature of its dynamics is the fact that the information flows from the initial pair of agents to their neighbors, at variance with the other opinion dynamics models, in which instead agents are influenced by their neighbors. Because of that the Sznajd model was supposed to describe how opinions spread in a society. On the other hand, in (Behera and Schweitzer, 2003) it has been shown that, in one dimension, the direction of the information flow is actually irrelevant, and

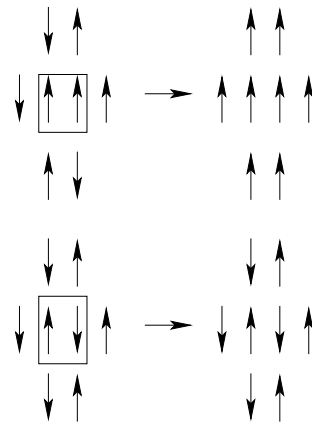


FIG. 5 Sznajd model. In the most common version of the model (Sznajd A), a pair of neighboring agents with the same opinion convince all their neighbors (top), while they have no influence if they disagree (bottom).

that the Sznajd B dynamics is equivalent to a voter dynamics. The only difference with the classic voter model is that an agent is not influenced by its nearest neighbors but by its next-to-nearest neighbors. Indeed, the dynamics of Sznajd B on a linear chain can be summarized by the simple sentence “just follow your next-to-nearest neighbor”. The fact that in Sznajd a pair of agents is updated at a time, whereas in the voter model the dynamics affects a single spin, introduces a factor of two in the average relaxation time of the equivalent voter dynamics; all other features are exactly the same, from the probability to hit the attractors to the distributions of decision and relaxation times. Therefore, Sznajd B does not respect the principle of social validation which motivated its introduction, as each spin is influenced only by a single spin, not by a pair.

Sznajd rule (22) is unrealistic and was soon replaced by alternative recipes in subsequent studies. In the most popular alternative, that we call Sznajd A, only the ferromagnetic rule (21) holds, so the neighbors of a disagreeing agents' pair maintain their opinions. Extensions of the Sznajd model to different substrates usually adopt this prescription and we shall stick to it unless stated otherwise. On the square lattice, for instance, a pair of neighboring agents affect the opinions of their six neighbors only if they agree (Fig. 5). In this case, the exit probability is a step function with threshold at  $m = 0$ : if the initial magnetization  $m < 0$ , the system always attains consensus with  $m = -1$ ; if  $m > 0$  initially, the steady state is consensus with  $m = 1$ . The distribution of the times required to reach complete consensus is broad, but not a log-normal like for Sznajd B in one dimension (Stauffer *et al.*, 2000). We stress that Sznajd B in one dimension has no phase transition in the exit probability, due to the coexistence of ferro- and antiferromagnetic stationary states.

The fixed points of Sznajd A dynamics hold if one changes the size of the pool of persuading agents. The

only exception is represented by the so-called Ochrombel simplification of the Sznajd model (Ochrombel, 2001), in which a single agent imposes its opinion on all its neighbors.

The results mentioned above were derived by computer simulations. In (Slanina and Lavička, 2003) an exact solution for a Sznajd-like dynamics on a complete graph has been given. Here a pair of randomly selected agents  $i$  and  $j$  interacts with a third agent  $k$ , also taken at random. If  $s_i = s_j$ , then  $s_k = s_i = s_j$ , otherwise nothing happens. The evolution equation for the probability density  $P(m, t)$  that the system has magnetization  $m$  at time  $t$  reads:

$$\frac{\partial}{\partial t} P(m, t) = -\frac{\partial}{\partial m} [(1 - m^2)mP(m, t)]. \quad (23)$$

Eq. (23) is derived in the thermodynamic limit and it represents a pure drift of the magnetization. The general solution is:

$$P(m, t) = [(1 - m^2)m]^{-1} f\left(e^{-t} \frac{m}{\sqrt{1 - m^2}}\right), \quad (24)$$

where the function  $f$  depends on the initial conditions. If  $P(m, t = 0) = \delta(m - m_0)$ , i.e., the system starts with a fixed value  $m_0$  of the magnetization,  $P(m, t)$  is a  $\delta$ -function at any moment of the evolution; the center is pushed by the drift towards the extremes  $+1$  if  $m_0 > 0$  or  $-1$  if  $m_0 < 0$ , which are reached asymptotically. So, the initial magnetization  $m_0$  determines the final state of the system, which is consensus, and there is a phase transition when  $m_0$  changes sign. Eq. (23) also allows to derive the behavior of the tail of the distribution of the times to reach the stationary states of the dynamics, which turns out to be exponential.

Some effort has been devoted to find a proper Hamiltonian formulation of Sznajd dynamics (Sznajd-Weron, 2002, 2004, 2005a). It turns out that the rules of the model are equivalent to the minimization of a local function of spin-spin interactions, the so-called *disagreement function*. On a linear chain of spins, the disagreement function for spin  $i$  reads:

$$E_i = -J_1 s_i s_{i+1} - J_2 s_i s_{i+2}, \quad (25)$$

where  $J_1$  and  $J_2$  are coupling constants, whose values determine the type of dynamics, and  $i + 1$ ,  $i + 2$  are the right nearest and next-to-nearest neighbors of  $i$ . Here, spin  $i$  takes the value that minimizes  $E_i$ . The function  $E_i$  and its minimization defines the Two-Component (TC) model (Sznajd-Weron, 2002). We remark that, when  $J_1 J_2 > 0$ , the two terms of  $E_i$  are equivalent, so only one can be kept. Sznajd B dynamics is recovered for  $-J_2 < J_1 < J_2$ ,  $J_2 > 0$ , but the model has a much richer behavior. Based on the values of the pair of parameters  $J_1$  and  $J_2$ , one distinguishes four phases, delimited by the bisectors  $J_1 \pm J_2 = 0$ . Besides the known ferro- and anti-ferromagnetic attractors, a new stationary configuration emerges, with pairs of aligned spins whose signs alternate

(...++--++--...). The TC model has been extended to the square lattice (Sznajd-Weron, 2004), and can be exactly solved in the mean field limit (Sznajd-Weron, 2005a). In general, we stress that the model is not equivalent to a Hamiltonian model at zero temperature, because it is not possible to define a global energy for the system. The sum of the disagreement function  $E_i$  over all spins does not play the role of the energy: the local minimization of  $E_i$  can lead to an increase of its global value (Sznajd-Weron, 2004).

Sznajd dynamics turns out to be a special case of the general sequential probabilistic model (GPM) (Galam, 2005b). Here, opinions are Ising spins: the proportions of both opinions at time  $t$  are  $p(t)$  (+) and  $1 - p(t)$  (-). In the mean field limit, a random group of  $k$  agents is selected, with  $j$  agents with opinion + and  $k - j$  with opinion -. The opinion dynamics of the GPM enforces consensus among the agents of the group, which adopt opinion + with a suitably defined probability  $m_{k,j}$  and opinion - with probability  $1 - m_{k,j}$ . The probability  $p(t + 1)$  to find an agent sharing opinion + after the update is

$$p(t + 1) = \sum_{j=0}^k m_{k,j} p(t)^j [1 - p(t)]^{k-j} \frac{k!}{j!(k-j)!}. \quad (26)$$

The size  $k$  of the random group along with the local probabilities  $\{m_{k,j}\}$  completely define the dynamics of the GPM. A phase diagram can be derived as a function of the local probabilities. Only two different phases are obtained, corresponding to consensus and coexistence of the two opinions in equal proportions. The phase transition occurs at those values of the  $\{m_{k,j}\}$  for which magnetization is on average conserved: here the model has a voter dynamics. With suitable choices of the set  $\{m_{k,j}\}$  the GPM reproduces the MF behavior of all known models with binary opinions: voter, majority rule, Sznajd, the majority-minority model, etc..

We now briefly review the modifications of the Sznajd model. The dynamics has been studied on many different topologies: regular lattices (Chang, 2001; Stauffer *et al.*, 2000), complete graphs (Slanina and Lavička, 2003), random graphs (Rodrigues and da F. Costa, 2005), small-world networks (Elgazzar, 2003; He *et al.*, 2004) and scale-free networks (Bernardes *et al.*, 2002; Bonnekoh, 2003; Rodrigues and da F. Costa, 2005; Sousa, 2005; Sousa and Sánchez, 2006). The Sznajd model on scale-free networks was recently studied (González *et al.*, 2006) within a real space renormalization framework. On any graph, if only Sznajd's ferromagnetic rule (21) holds, the system undergoes a sharp dynamic phase transition from a state with all spins down to a state with all spins up. If the graph is not fixed, but in evolution, like a growing network, the transition becomes a smooth crossover between the two phases (González *et al.*, 2004). The phase transition holds as well if one introduces dilution (Moreira *et al.*, 2001), if the number of opinion states is larger than two (Slanina and Lavička, 2003), if

the influence of the active pair of agents extends beyond their neighborhood (Schulze, 2003b), so it is a very robust feature of the Sznajd model, although it disappears when one includes noise (Stauffer *et al.*, 2000) or antiferromagnetic rules (Chang, 2001; Sznajd-Weron, 2004).

If the random sequential updating so far adopted is replaced by synchronous updating, i.e., if at each iteration all agents of the configurations are paired off and act simultaneously on their neighbors, it may happen that an agent is induced to choose opposite opinions by different neighboring pairs. In this case the agent is “frustrated” and maintains its opinion. Such frustration hinders consensus (Stauffer, 2004; Tu *et al.*, 2005), due to the emergence of stable clusters where both opinions coexist. This problem can be limited if noise is introduced (Sabatelli and Richmond, 2004), or if agents have memory, so that, in case of conflicting advice, they follow the most frequent opinion they had in the past (Sabatelli and Richmond, 2003).

When the possible opinion states are  $q > 2$ , one can introduce bounded confidence, i.e., the realistic principle that only people with similar opinions can have an influence on each other. If we assume that two opinions are similar if their values differ by at most one unit, and that a pair of agents with the same opinion can convince only neighbors of similar opinions, the Sznajd dynamics always leads to complete consensus for  $q = 3$ , whereas for  $q > 3$  it is very likely that at least two opinions survive in the final stationary state (Stauffer, 2002b). Bounded confidence allows for an extension of the Sznajd model to real-valued opinions (Fortunato, 2005b). Other studies focused on the dynamics of clusters of agents with regular opinion patterns, ferromagnetic and/or antiferromagnetic (Schneider and Hirtreiter, 2005b), damage spreading (Klietsch, 2005; Roehner *et al.*, 2004), the combination of Sznajd with other convincing strategies (Sousa and Sánchez, 2006), contrarian behavior (de la Lama *et al.*, 2005; Wio *et al.*, 2006), the effect on the dynamics of agents biased towards the global majority and/or minority opinion (Schneider, 2004; Schneider and Hirtreiter, 2005a). Recently extensions of Sznajd B to higher dimensions have been considered as well (Kondrat and Sznajd-Weron, 2008).

The Sznajd model has found applications in different areas. In politics, it has been used to describe voting behavior in elections (Bernardes *et al.*, 2002; González *et al.*, 2004); we shall discuss this issue in Sec. III.H. Moreover, it was applied to study the interaction of economic and personal attitudes of individuals, which evolve according to different rules but in a coupled manner (Sznajd-Weron and Sznajd, 2005). Sznajd dynamics has also been adopted to model the competition of different products in an open market (Sznajd-Weron and Weron, 2003). The effects of aging, diffusion and a multi-layered society have been considered as well (Schulze, 2003a; Schulze, 2004). Sznajd dynamics has been adapted in a model that describes the spread of opinions among a group

of traders (Sznajd-Weron and Weron, 2002). Finally, Sznajd-like rules have been employed to generate a new class of complex networks (da Fontoura Costa, 2005).

## F. Bounded confidence models

### 1. Continuous opinions

In the models we have so far investigated opinion is a discrete variable, which represents a reasonable description in several instances. However, there are cases in which the position of an individual can vary smoothly from one extreme to the other of the range of possible choices. As an example, one could think of the political orientation of an individual, that is not restricted to the choices of extreme Right/Left, but it includes all the options in between, which may be indicated by the geometric position of the seat of a deputy in the Parliament.

Continuous opinions invalidate some of the concepts adopted in models with discrete choices, like the concepts of majority of an opinion and equality of opinions, so they require a different framework. Indeed, continuous opinion dynamics has historically followed an alternative path. The first studies were carried out by applied mathematicians and aimed at identifying the conditions under which a panel of experts would reach a common decision (Chatterjee and Seneta, 1977; Cohen *et al.*, 1986; Stone, 1961).

The initial state is usually a population of  $N$  agents with randomly assigned opinions, represented by real numbers within some interval. In contrast to discrete opinion dynamics, here all agents usually start with different opinions, and the possible scenarios are more complex, with opinion clusters emerging in the final stationary state. The opinion clusters could be one (consensus), two (polarization) or more (fragmentation). In principle, each agent can interact with every other agent, no matter what their opinions are. In practice, there is a real discussion only if the opinions of the people involved are sufficiently close to each other. This realistic aspect of human communications is called *bounded confidence* (BC); in the literature it is expressed by introducing a real number  $\epsilon$ , the *uncertainty* or *tolerance*, such that an agent, with opinion  $x$ , only interacts with those of its peers whose opinion lies in the interval  $]x - \epsilon, x + \epsilon[$ .

In this section we discuss the most popular BC models, i.e., the Deffuant model (Deffuant *et al.*, 2000) and that of Hegselmann-Krause (Hegselmann and Krause, 2002). BC models have been recently reviewed in (Lorenz, 2007).

### 2. Deffuant model

Let us consider a population of  $N$  agents, represented by the nodes of a graph, where agents may discuss with each other if the corresponding nodes are connected. Each agent  $i$  is initially given an opinion  $x_i$ , randomly

chosen in the interval  $[0, 1]$ . The dynamics is based on random binary encounters, i.e., at each time step, a randomly selected agent discusses with one of its neighbors on the social graph, also chosen at random. Let  $i$  and  $j$  be the pair of interacting agents at time  $t$ , with opinions  $x_i(t)$  and  $x_j(t)$ , respectively. Deffuant dynamics is summarized as follows: if the difference of the opinions  $x_i(t)$  and  $x_j(t)$  exceeds the threshold  $\epsilon$ , nothing happens; if, instead,  $|x_i(t) - x_j(t)| < \epsilon$ , then:

$$x_i(t+1) = x_i(t) + \mu[x_j(t) - x_i(t)], \quad (27)$$

$$x_j(t+1) = x_j(t) + \mu[x_i(t) - x_j(t)]. \quad (28)$$

The parameter  $\mu$  is the so-called convergence parameter, and its value lies in the interval  $[0, 1/2]$ . Deffuant model is based on a compromise strategy: after a constructive debate, the positions of the interacting agents get closer to each other, by the relative amount  $\mu$ . If  $\mu = 1/2$ , the two agents will converge to the average of their opinions before the discussion. For any value of  $\epsilon$  and  $\mu$ , the average opinion of the agents' pair is the same before and after the interaction, so the global average opinion ( $1/2$ ) of the population is an invariant of Deffuant dynamics.

The evolution is due to the instability of the initial uniform configuration near the boundary of the opinion space. Such instability propagates towards the middle of the opinion space, giving rise to patches with an increasing density of agents, that will become the final opinion clusters. Once each cluster is sufficiently far from the others, so that the difference of opinions for agents in distinct clusters exceeds the threshold, only agents inside the same cluster may interact, and the dynamics leads to the convergence of the opinions of all agents in the cluster to the same value. Therefore, the final opinion configuration is a succession of Dirac's delta functions. In general, the number and size of the clusters depend on the threshold  $\epsilon$ , whereas the parameter  $\mu$  affects the convergence time of the dynamics. However, when  $\mu$  is small, the final cluster configuration also depends on  $\mu$  (Laguna *et al.*, 2004; Porfiri *et al.*, 2007).

On complete graphs, regular lattices, random graphs and scale-free networks, for  $\epsilon > \epsilon_c = 1/2$ , all agents share the same opinion  $1/2$ , so there is complete consensus (Fortunato, 2004; Lorenz and Urbig, 2007). This may be a general property of Deffuant model, independently of the underlying social graph. If  $\epsilon$  is small, more clusters emerge (Fig. 6).

Monte Carlo simulations reveal that the number  $n_c$  of clusters in the final configuration can be approximated by the expression  $1/(2\epsilon)$ . This can be understood if we consider that, at stationarity, agents belonging to different opinion clusters cannot interact with each other, which means that the opinion of each cluster must differ by at least  $\epsilon$  from the opinions of its neighboring clusters. In this way, within an interval of length  $2\epsilon$  centered at a cluster, there cannot be other clusters, and the ratio  $1/(2\epsilon)$  is a fair estimate for  $n_c$ .

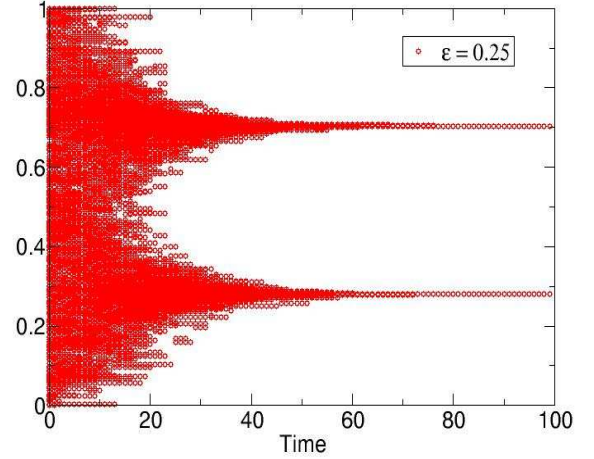


FIG. 6 Deffuant model. Opinion profile of a population of 500 agents during its time evolution, for  $\epsilon = 0.25$ . The population is fully mixed, i.e., everyone may interact with everybody else. The dynamics leads to a polarization of the population in two factions.

Most results on Deffuant dynamics are derived through numerical simulations, as the model is not analytically solvable. However, in the special case of a fully mixed population, where everybody interacts with everybody else, it is possible to write the general rate equation governing the opinion dynamics (Ben-Naim *et al.*, 2003). For this purpose, one neglects individual agents and focuses on the evolution of the opinion population  $P(x, t)$ , where  $P(x, t)dx$  is the probability that an agent has opinion in the interval  $[x, x+dx]$ . The interaction threshold is  $\epsilon = 1$ , but the opinion range is the interval  $[-\Delta, \Delta]$ ; this choice is equivalent to the usual setting of the Deffuant model, if  $\epsilon = 1/2\Delta$ . For simplicity,  $\mu = 1/2$ . The rate equation then reads:

$$\frac{\partial}{\partial t}P(x, t) = \int_{|x_1 - x_2| < 1} dx_1 dx_2 P(x_1, t)P(x_2, t) \times \left[ \delta\left(x - \frac{x_1 + x_2}{2}\right) - \delta(x - x_1) \right]. \quad (29)$$

Eq. (29) conserves the norm  $M_0 = \int_{-\Delta}^{+\Delta} P(x, t)dx$  and the average opinion. The question is to find the asymptotic state  $P_\infty(x) = P(x, t \rightarrow \infty)$ , starting from the flat initial distribution  $P(x, t = 0) = 1$ , for  $x \in [-\Delta, \Delta]$ . If  $\Delta < 1/2$ , all agents interact and Eq. (29) is integrable. In this case, it is possible to show that all agents approach the central opinion 0 and  $P_\infty(x) = M_0\delta(x)$ .

If  $\Delta > 1/2$ , the equation is no longer analytically solvable. The asymptotic distribution is a linear combination of delta functions, i.e.,

$$P_\infty(x) = \sum_{i=1}^p m_i \delta(x - x_i). \quad (30)$$

The cluster masses  $m_i$  must obey the conditions  $\sum_i m_i = M_0$  and  $\sum_i m_i x_i = 0$ ; the latter comes from the conservation of the average opinion. Numerical solutions of Eq. (29) reveal that there are only three types of clusters: major (mass  $> 1$ ), minor (mass  $< 10^{-2}$ ) and a central cluster located at  $x = 0$ . These clusters are generated by a periodic sequence of bifurcations, consisting in the nucleation and annihilation of clusters.

On any graph topology, Deffuant dynamics is always characterized by a stationary state with one or more opinion clusters. However, as the interaction range of an agent is restricted to its topological neighborhood, more opinion clusters emerge for low values of the uncertainty. Opinion homogenization involves only agents in the same cluster: in this way, if two clusters are geometrically separated, there will be no communication between the corresponding agents and the final opinions will be in general different in each cluster, even if their opinions are compatible, so that they would converge to the same opinion on a complete graph. The result is an increased fragmentation of the agents' population. On scale-free networks, the number of surviving opinions in the stationary state is proportional to the number of agents of the population, for fixed  $\epsilon$  (Stauffer and Meyer-Ortmanns, 2004). In particular, nodes with few connections have a sizeable probability to be excluded from the dynamics and to keep their opinion forever (Weisbuch, 2004). The result holds for both static and evolving networks (Sousa, 2004).

Deffuant model can be defined as well if opinions are not continuous but discretized (Stauffer *et al.*, 2004). Here the opinion  $s$  of any agent can take one of  $Q$  values,  $s = 1, 2, \dots, Q$ . Opinions  $s_i$  and  $s_j \neq s_i$  are compatible if  $|s_i - s_j| \leq L$ , where  $L$  is an integer. The rules are the same as in Eqs. (27) and (28), still with a real-valued convergence parameter  $\mu$ , but the shift of the opinions is rounded to the nearest integer. In the limit  $L \rightarrow \infty$  and  $Q \rightarrow \infty$ , with the ratio  $\epsilon = L/Q$  kept constant, one recovers the results of the original model with continuous opinions. If  $L = 1$ , on a complete graph consensus is the only stationary state if  $Q = 2$ , i.e., for binary opinions<sup>2</sup>. Instead, for  $Q > 2$ , complete consensus is never attained, but there are at least two opinion clusters in the final configuration. On scale-free networks the number of surviving opinions in the stationary state is a scaling function of  $Q$  and the population size  $N$ .

Simple modifications of Deffuant model yield rich dynamics. If agents have individual values of  $\epsilon$  (Lorenz, 2008b; Weisbuch *et al.*, 2002), the dynamics is dominated by the agents with large uncertainties. In several papers (Aldashev and Carletti, 2007; Amblard and Deffuant, 2004; Deffuant, 2006; Deffuant *et al.*, 2004; Deffuant *et al.*, 2002;

Weisbuch *et al.*, 2005), the uncertainties are also affected by the dynamics. In addition, they are also coupled to the opinions, based on the principle that a small uncertainty also implies more confidence and a higher probability to affect other opinions. These models are able to explain how extremal positions, initially shared by a minority of people, may eventually dominate in society. In (Ben-Naim, 2005) a model in which Deffuant compromise strategy is combined with spontaneous changes of the agents' opinions, has been studied. The latter phenomenon is described as a diffusion process in the opinion space, which affects cluster formation and evolution, with large clusters steadily overtaking small ones. Other extensions of Deffuant dynamics include the introduction of an external periodic perturbation affecting all agents at once, to simulate the effect of propaganda (Carletti *et al.*, 2006), and the study of a more complex opinion dynamics where the interaction of pairs of agents depends not only on the compatibility of their opinions, but also on the coevolving mutual affinity of the agents (Bagnoli *et al.*, 2007; Carletti *et al.*, 2007). This coupling provides a natural and endogenous way of determining the number of opinion clusters and their positions.

### 3. Hegselmann-Krause model

The model proposed in (Hegselmann and Krause, 2002) (HK) is quite similar to that of Deffuant. Opinions take real values in an interval, say  $[0, 1]$ , and an agent  $i$ , with opinion  $x_i$ , interacts with neighboring agents whose opinions lie in the range  $[x_i - \epsilon, x_i + \epsilon]$ , where  $\epsilon$  is the uncertainty. The difference is given by the update rule: agent  $i$  does not interact with one of its compatible neighbors, like in Deffuant, but with all its compatible neighbors at once. Deffuant's prescription is suitable to describe the opinion dynamics of large populations, where people meet in small groups, like pairs. In contrast, HK rule is intended to describe formal meetings, where there is an effective interaction involving many people at the same time.

On a generic graph, HK update rule for the opinion of agent  $i$  at time  $t$  reads:

$$x_i(t+1) = \frac{\sum_{j: |x_i(t) - x_j(t)| < \epsilon} a_{ij} x_j(t)}{\sum_{j: |x_i(t) - x_j(t)| < \epsilon} a_{ij}}, \quad (31)$$

where  $a_{ij}$  is the adjacency matrix of the graph. So, agent  $i$  takes the average opinion of its compatible neighbors. The model is fully determined by the uncertainty  $\epsilon$ , unlike Deffuant dynamics, for which one needs to specify as well the convergence parameter  $\mu$ . The need to calculate opinion averages of groups of agents that may be rather large makes computer simulations of the HK model rather lengthy as compared to Deffuant's. This may explain why the HK model has not been studied by many authors.

<sup>2</sup> We remark that, for  $L = 1$ , it is impossible for the two interacting opinions to shift towards each other, as only integer opinion values are allowed; so, as a result of the discussion, one agent takes the opinion of the other.



The dynamics develops just like in Deffuant, and leads to the same pattern of stationary states, with the number of final opinion clusters decreasing if  $\epsilon$  increases. In particular, for  $\epsilon$  above some threshold  $\epsilon_c$ , there can only be one cluster. On a complete graph, the final configurations are symmetric with respect to the central opinion  $1/2$ , because the average opinion of the system is conserved by the dynamics (Fortunato *et al.*, 2005), as in Deffuant. The time to reach the stationary state diverges in correspondence to the bifurcation thresholds of opinion clusters, due to the presence of isolated agents lying between consecutive clusters (Fortunato *et al.*, 2005).

The threshold for complete consensus  $\epsilon_c$  can only take one of two values, depending on the behavior of the average degree  $\langle k \rangle$  of the underlying social graph when the number of nodes  $N$  grows large (Fortunato, 2005a). If  $\langle k \rangle$  is constant in the limit of large  $N$ , as for example in lattices,  $\epsilon_c = \epsilon_1 = 1/2$ . Instead, if  $\langle k \rangle \rightarrow \infty$  when  $N \rightarrow \infty$ , as for example in complete graphs,  $\epsilon_c = \epsilon_2 \sim 0.2$ . We have seen instead that, for Deffuant,  $\epsilon_c = 1/2$  on any graph.

The extension of HK to discretized opinions (Fortunato, 2004) is essentially a voter model with bounded confidence: an agent picks at random the opinion of a compatible neighbor. For three opinion values and uncertainty one, the model reduces to the constrained voter model (Vazquez *et al.*, 2003).

Other developments include: the use of alternative recipes to average the opinions in Eq. (31) (Hegselmann and Krause, 2005); an analysis of damage spreading (Fortunato, 2005); the introduction of a general framework where the size of the groups of interacting agents varies from 2 (Deffuant) to  $N$  (HK) (Urbig and Lorenz, 2004); the reformulation of Deffuant and HK dynamics as interactive Markov chains (Lorenz, 2005b; Lorenz, 2006); analytical results on the stability of BC opinion dynamics (Lorenz, 2005a) and their ability to preserve the relative ordering of the opinions (Hendrickx, 2007).

## G. Other models

The opinion dynamics models described so far are based on elementary mechanisms, which explain their success and the many investigations they have stimulated. Such models, however, do not exhaust the wide field of opinion dynamics. The last years witnessed a real explosion of new models, based on similar concepts as the classical models or on entirely new principles. Here we briefly survey these alternative models.

The basic models we have seen are essentially deterministic, i.e., the final state of the system after an interaction is always well defined. Randomness can be introduced, in the form of a social temperature or pure noise, but it is not a fundamental feature. Most models of last generation, instead, focus on the importance of randomness in the process of opinion formation. Ran-

domness is a necessary ingredient of social interactions: both our individual attitudes and the social influence of our peers may vary in a non-predictable way. Besides, the influence of external factors like mass media, propaganda, etc., is also hardly predictable. In this respect, opinion dynamics is a stochastic process.

An interesting variant of Ising dynamics has been recently explored in (Jiang *et al.*, 2007, 2008). Here an agent surrounded by a majority of neighbors with equal opinion will flip its opinion with the usual Metropolis probability  $\exp(-\Delta E/T)$ , where  $\Delta E$  is the increase of the Ising energy due to the flip and  $T$  the (social) temperature. When the majority of the neighbors disagrees with the agent's opinion, the latter flips with a probability  $q$ , which accounts for the possibility that agents keep their opinion in spite of social pressure ("inflexible agents"). For  $q = 1$  one recovers the usual Ising dynamics. For  $q < 1$  instead, the model obeys a non-equilibrium dynamics, since detailed balance is violated. Simulations of the model on the lattice reveal a non-trivial phase diagram: the ferromagnetic transition is continuous for  $q$  larger than a critical value  $q_c$ , and discontinuous for  $q < q_c$ . The critical exponents of the continuous transitions differ both from the mean field and the Ising exponents. On small-world lattices with a large density of shortcuts there is no magnetization transition if  $q$  is sufficiently low (Jiang *et al.*, 2008).

In (Bartolozzi *et al.*, 2005) a model with binary opinions, evolving according to a heat bath dynamics, is proposed. The opinion field acting on a spin is given by a linear combination with random weights of a term proportional to the average opinion of its nearest neighbors on the social network, with a term proportional to the average opinion of the whole network. When the stochastic noise exceeds a threshold, the time evolution of the average opinion of the system is characterized by large intermittent fluctuations; a comparison with the time series of the Dow-Jones index at New York's Stock Exchange reveals striking similarities.

In a recent model (Kuperman and Zanette, 2002), opinions are affected by three processes: social imitation, occurring via majority rule, fashion, expressed by an external modulation acting on all agents, and individual uncertainty, expressed by random noise. Stochastic resonance (Gammaitoni *et al.*, 1998) was observed: a suitable amount of noise leads to a strong amplification of the response of the system to the external modulation. The phenomenon occurs as well if one varies the size of the system for a fixed amount of noise (Tessone and Toral, 2005): here the best response to the external solicitation is achieved for an optimal population size (system size stochastic resonance).

Kinetic models of opinion dynamics were proposed in (Toscani, 2006). Interactions are binary, and the opinions of the interacting pair of agents vary according to a compromise strategy à la Deffuant, combined with the possibility of opinion diffusion, following the original idea (Ben-Naim, 2005) discussed in Sec. III.F.2. The

importance of diffusion in the process is expressed by a random weight. The dynamics can be easily reformulated in terms of Fokker-Planck equations, from which it is possible to deduce the asymptotic opinion configurations of the model. Fokker-Planck equations have also been employed to study a dynamics similar to that of the constrained voter model (Vazquez *et al.*, 2003), but in the presence of a social temperature, inducing spontaneous opinion changes (de la Lama *et al.*, 2006).

In (Martins, 2007, 2008) binary and continuous opinions are combined. The model is based on the simple idea that declared opinions, or actions, are only a projection of the actual opinion of the agents. Two persons can express the same preference but their individual certitudes towards that preference may be different. So one has to distinguish between the internal opinion, which is expressed by a probability, and the external opinion, or action, which is a binary variable. The internal opinion is a measure of how extreme a position is. Agents vary both their actions and their internal opinions based on the observation of the actions of their peers. In this way one can monitor how the convictions of individuals are coupled to their actions. Clusters of agents with the same external opinions display a characteristic pattern, where the agents are very convinced of their choices (extremists) if they are well inside the cluster, whereas they are more open-minded if they lie at the boundary.

Synchronization has also been used to explain consensus formation. A variant of the Kuramoto model (Kuramoto, 1975), where the phases of the oscillators are replaced by unbounded real numbers, representing the opinions, displays a phase transition from an incoherent phase (anarchy), to a synchronized phase (consensus) (Pluchino *et al.*, 2006; Pluchino *et al.*, 2005). In (Di Mare and Latora, 2007) it was shown that several opinion dynamics models can be reformulated in the context of strategic game theory.

Some models focus on specific aspects of opinion dynamics. In (Indekeu, 2004) it has been pointed out that the influence of network hubs in opinion dynamics is overestimated, because it is unlikely that a hub-agent devotes much time to all its social contacts. If each agent puts the same time in its social relationships, this time will be distributed among all its social contacts; so the effective strength of the interaction between two neighboring agents will be the smaller, the larger the degrees of the agents. If the spin-spin couplings are renormalized according to this principle, the Ising model on scale-free networks always has a ferromagnetic threshold, whereas it is known that, with uniform couplings, networks with infinite degree variance are magnetized at any temperature (Aleksiejuk *et al.*, 2002; Leone *et al.*, 2002). The issue of how opinion dynamics is influenced by the hierarchical structure in societies/organizations has also been investigated (Grabowski and Kosiński, 2006b; Laguna *et al.*, 2005). Other authors investigated fashion (Nakayama and Nakamura, 2004), the interplay between opinions and personal taste (Bagnoli *et al.*, 2004)

and the effect of opinion surveys on the outcome of elections (Alves *et al.*, 2002).

It is worth mentioning how the close formal similarities between the fields of opinion and language dynamics leads to the idea that models proposed in the framework of language dynamics could suitably apply also in modeling opinion formation. One example is represented by a variant of the naming game (Baronchelli *et al.*, 2007), as defined in Sec. V.

## H. Empirical data

One of the main contributions of the physical approach to opinion dynamics should be to focus on the quantitative aspects of the phenomenon of consensus formation, besides addressing the mere qualitative question of when and how people agree/disagree. What is needed is then a quantitative phenomenology of opinion dynamics, to define the phenomenon in a more objective way, posing severe constraints on models. Sociological investigations have been so far strongly limited by the impossibility of studying processes involving large groups of individuals. However, the current availability of large datasets and of computers able to handle them makes for the first time such empirical analysis possible.

Elections are among the largest scale phenomena involving people and their opinions. The number of voters is of the order of millions for most countries, and it can easily reach hundreds of millions in countries like Brazil, India and the USA. A great deal of data is nowadays publicly available in electronic form. The first empirical investigations carried out by physicists concerned Brazilian elections (Costa Filho *et al.*, 1999). The study focused on the distribution of the fraction  $\nu$  of votes received by a candidate. Datasets referring to the federal elections in 1998 revealed the existence of a characteristic pattern for the histogram  $P(\nu)$ , with a central portion following the hyperbolic decay  $1/\nu$ , and an exponential cutoff for large values of  $\nu$ . Interestingly, datasets corresponding to candidates to the office of state deputy in several Brazilian states revealed an analogous pattern. A successive analysis on data referring to state and federal elections in 2002 confirmed the results for the elections in 1998, in spite of a change in the political rules that constrained alliances between parties (Filho *et al.*, 2003). Indian data displayed a similar pattern for  $P(\nu)$  across different states, although discrepancies were also found (González *et al.*, 2004). Data on Indonesian elections are consistent with a power law decay of  $P(\nu)$ , with exponent close to one, but are too noisy to be reliable (Situngkir, 2004). Claims that Mexican elections also obey a similar pattern are not clearly supported by the data (Morales-Matamoros *et al.*, 2006).

The peculiar pattern of  $P(\nu)$  was interpreted as the result of a multiplicative process, which yields a log-normal distribution for  $\nu$ , due to the Central Limit Theorem (Costa Filho *et al.*, 1999). The  $1/\nu$  behavior can

indeed be reproduced by a log-normal function, in the limit where the latter has a large variance. A microscopic model based on Sznajd opinion dynamics was proposed in (Bernardes *et al.*, 2002). Here, the graph of personal contacts between voters is a scale-free network *à la* Barabási-Albert; candidates are initially the only nodes of the network in a definite opinion state, a suitably modified Sznajd dynamics spreads the candidates' opinions to all nodes of the network. The model reproduces the empirical curve  $P(\nu)$  derived from Brazilian elections. The same mechanism yields, on different social graphs, like pseudo-fractal networks (González *et al.*, 2004) and a modified Barabási-Albert network with high clustering (Sousa, 2005), a good agreement with empirical data. The weakness of this model, however, is that a non-trivial distribution is only a transient in the evolution of the system. For long times the population will always converge to the only stable state of Sznajd dynamics, where every voter picks the same candidate, and the corresponding distribution is a  $\delta$ -function. All studies stopped the modified Sznajd dynamics after a certain, carefully chosen, time. A recent model based on simple opinion spreading yields a distribution similar to the Brazilian curve, if the underlying social graph is an Erdős-Rényi network, whereas on scale-free networks the same dynamics fails to reproduce the data (Travieso and da Fontoura Costa, 2006).

The power law decay in the central region of  $P(\nu)$ , observed in data sets relative to different countries and years, could suggest that this pattern is a universal feature of the distribution. But this is unlikely because candidates' scores strongly depend on the performance of their parties, which is determined by a much more complex dynamics. Indeed, municipal election data display a different pattern (Lyra *et al.*, 2003). Instead, the performances of candidates of the same party can be objectively compared. This can be done in proportional elections with open lists (Fortunato and Castellano, 2007). In this case the country is divided into constituencies, and each party presents a list of candidates in each constituency. There are three relevant variables: the number of votes  $v$  received by a candidate, the number  $Q$  of candidates presented by the party in the corresponding list and the total number  $N$  of votes received by the party list. Therefore, the distribution of the number of votes received by a candidate should be a function of three variables,  $P(v, Q, N)$ . It turns out instead that  $P(v, Q, N)$  is a scaling function of the single variable  $vQ/N$ , with a log-normal shape, and, remarkably, this function is the same in different countries and years (Fig. 7). This finding justifies a simple microscopic description of voting behavior, using the tools and methods of statistical physics. A model based on word-of-mouth spreading, similar to that of (Travieso and da Fontoura Costa, 2006), is able to reproduce the data.

Other studies disclose a correlation between the scores of a party and the number of its members in German elections (Schneider and Hirtreiter, 2005c) and a polar-

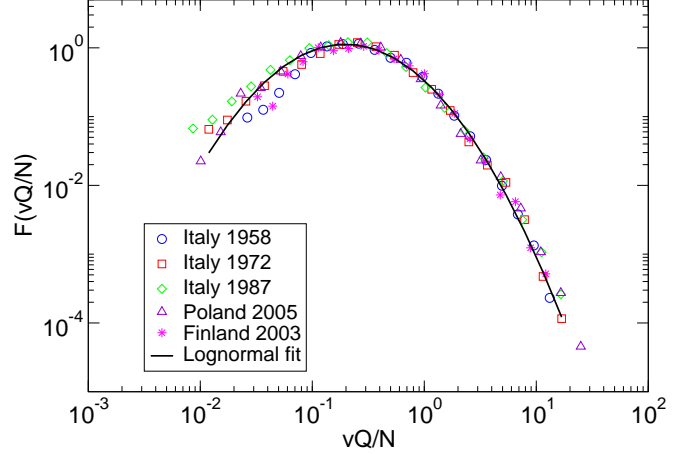


FIG. 7 Distribution of electoral performance for candidates in proportional elections held in Italy, Poland and Finland. The remarkable overlap shows that the curve is a universal feature of the voting process. From (Fortunato and Castellano, 2007).

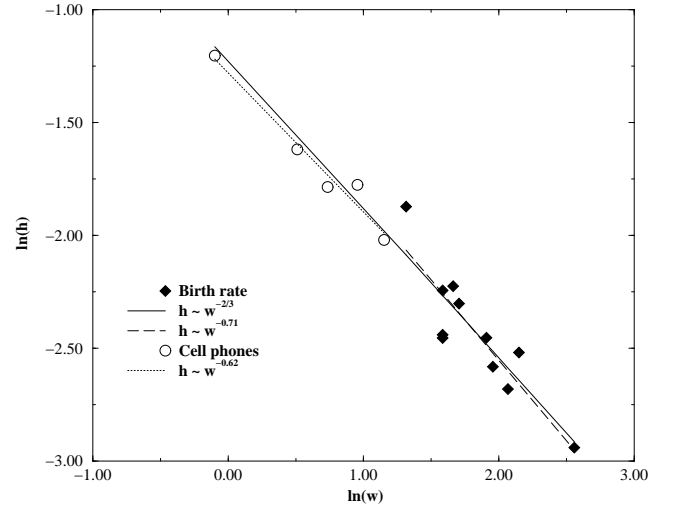


FIG. 8 Relation between the maximal speed of change and the duration of the change for birth rates and the number of mobile phones in several European countries. The linear regression of the data points in double logarithmic scale is consistent with the universal behavior predicted by the Random Field Ising Model at zero temperature. From (Michard and Bouchaud, 2005).

ization of the distribution of votes around two main candidates in Brazilian elections for mayor (Araripe *et al.*, 2006). An empirical investigation of the relation between party size and temporal correlations in electoral results is described in (Andresen *et al.*, 2007).

In (Michard and Bouchaud, 2005) it was suggested that extreme events like booms of products/fashions, financial crashes, crowd panic, etc., are determined by a combination of effects, including the personal atti-

tude of the agents, the public information, which affects all agents, and social pressure, represented by the mutual interaction between the agents. This can be formally described within the framework of the Random Field Ising Model at zero temperature, which successfully describes hysteresis in random magnets and other physical phenomena, like the occurrence of crackling noise (Sethna *et al.*, 2001). Here, opinions are binary, attitudes are real-valued numbers in  $] - \infty, +\infty[$ , corresponding to the random fields, the public information is a global field  $F(t)$ , slowly increasing with the time  $t$ , and the interaction term is the sum of Ising-like couplings between pairs of agents. The order parameter  $O$  of the system is the average opinion of the population. By increasing the field  $F$ ,  $O$  displays a sharp variation, due to large groups of agents that simultaneously switch opinion. The evolution of the speed of change  $dO/dF$  as a function of  $F$  follows a universal bell-shaped curve in the transition region, with a characteristic relation between the height  $h$  of the peak and its width  $w$ :  $h \sim w^{-2/3}$ . This relation was indeed observed in empirical data on extreme events, such as the dramatic drop of birth rates in different European countries in the last decades, the rapid diffusion of mobile phones in Europe in the late '90s, and the decrease of the clapping intensity at the end of applauses (Fig. 8).

For the future, more data are needed. Several phenomena of consensus formation could be empirically analyzed, for instance spreading of fads and innovations, sales dynamics, etc..

#### IV. CULTURAL DYNAMICS

In the previous section we have reviewed the active field of opinion dynamics. In parallel, there has been in recent years a growing interest for the related field of cultural dynamics. The border between the two fields is not sharp and the distinction is not clear-cut. The general attitude is to consider opinion as a scalar variable, while the more faceted culture of an individual is modeled as a vector of variables, whose dynamics is inextricably coupled. This definition is largely arbitrary, but we will adopt it in the review.

The typical questions asked with respect to cultural influence are similar to those related to the dynamics of opinions: what are the microscopic mechanisms that drive the formation of cultural domains? What is the ultimate fate of diversity? Is it bound to persist or all differences eventually disappear in the long run? What is the role of the social network structure?

##### A. Axelrod model

A prominent role in the investigation of cultural dynamics has been played by a model introduced by Axelrod in (Axelrod, 1997), that has attracted a lot of interest

from both social scientists and physicists.

The origin of its success among social scientists is in the inclusion of two mechanisms that are believed to be fundamental in the understanding of the dynamics of cultural assimilation (and diversity): social influence and homophily. The first is the tendency of individuals to become more similar when they interact. The second is the tendency of likes to attract each other, so that they interact more frequently. These two ingredients were generally expected by social scientists to generate a self-reinforcing dynamics leading to a global convergence to a single culture. It turns out instead that the model predicts in some cases the persistence of diversity.

From the point of view of statistical physicists, the Axelrod model is a simple and natural “vectorial” generalization of models of opinion dynamics that gives rise to a very rich and nontrivial phenomenology, with some genuinely novel behavior. The model is defined as follows. Individuals are located on the nodes of a network (or on the sites of a regular lattice) and are endowed with  $F$  integer variables  $(\sigma_1, \dots, \sigma_F)$  that can assume  $q$  values,  $\sigma_f = 0, 1, \dots, q-1$ . The variables are called cultural *features* and  $q$  is the number of the possible *traits* allowed per feature. They are supposed to model the different “beliefs, attitudes and behavior” of individuals. In an elementary dynamic step, an individual  $i$  and one of his neighbors  $j$  are selected and the overlap between them

$$\omega_{i,j} = \frac{1}{F} \sum_{f=1}^F \delta_{\sigma_f(i), \sigma_f(j)}, \quad (32)$$

is computed, where  $\delta_{i,j}$  is Kronecker’s delta. With probability  $\omega_{i,j}$  the interaction takes place: one of the features for which traits are different ( $\sigma_f(i) \neq \sigma_f(j)$ ) is selected and the trait of the neighbor is set equal to  $\sigma_f(i)$ . Otherwise nothing happens. It is immediately clear that the dynamics tends to make interacting individuals more similar, but the interaction is more likely for neighbors already sharing many traits (homophily) and it becomes impossible when no trait is the same. There are two stable configurations for a pair of neighbors: when they are exactly equal, so that they belong to the same cultural region or when they are completely different, i.e., they sit at the border between cultural regions.

Starting from a disordered initial condition (for example with uniform random distribution of the traits) the evolution on any finite system leads unavoidably to one of the many absorbing states, which belong to two classes: the  $q^F$  ordered states, in which all individuals have the same set of variables, or the other, more numerous, frozen states with coexistence of different cultural regions.

It turns out that which of the two classes is reached depends on the number of possible traits  $q$  in the initial condition (Castellano *et al.*, 2000). For small  $q$  individuals share many traits with their neighbors, interactions are possible and quickly full consensus is achieved. For large  $q$  instead, very few individuals share traits. Few interactions occur, leading to the formation of small cul-

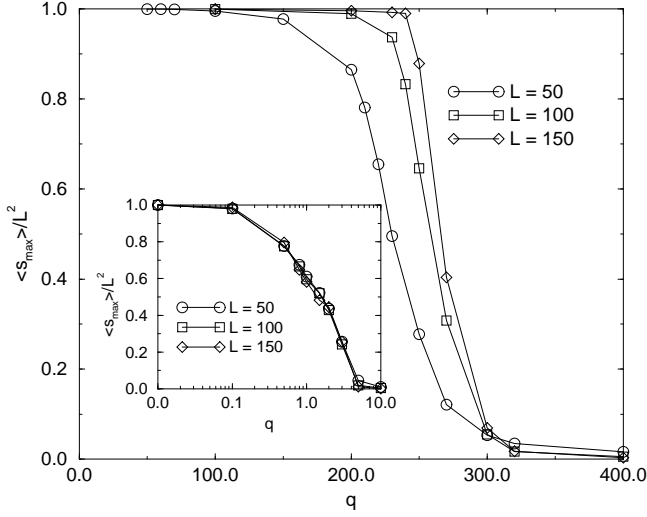


FIG. 9 Axelrod model. Behavior of the order parameter  $\langle S_{max} \rangle / L^2$  vs.  $q$  for three different system sizes and  $F = 10$ . In the inset the same quantity is reported for  $F = 2$ . From (Castellano *et al.*, 2000).

tural domains that are not able to grow: a disordered frozen state. On regular lattices, the two regimes are separated by a phase transition at a critical value  $q_c$ , depending on  $F$  (Fig. 9).

Several order parameters can be defined to characterize the transition. One of them is the average fraction  $\langle S_{max} \rangle / N$  of the system occupied by the largest cultural region.  $N$  is the number of individuals in the system. In the ordered phase this fraction is finite (in the limit  $N \rightarrow \infty$ ), while in the disordered phase cultural domains are of finite size, so that  $\langle S_{max} \rangle / N \sim 1/N$ . Another (dis)order parameter often used (González-Avella *et al.*, 2005) is  $g = \langle N_g \rangle / N$ , where  $N_g$  is the number of different domains in the final state. In the ordered phase  $g \rightarrow 0$ , while it is finite in the disordered phase.

In two dimensions the nature of the transition depends on the value of  $F$ . For  $F = 2$  there is a continuous change in the order parameter at  $q_c$ , while for  $F > 2$  the transition is discontinuous (Fig. 9)<sup>3</sup>. Correspondingly the distribution of the size  $s$  of cultural domains at the transition is a power law with  $P(s) \sim s^{-\tau}$  exponent smaller than 2 ( $\tau \approx 1.6$ ) for  $F = 2$  while the exponent is larger than 2 ( $\tau \approx 2.6$ ) for any  $F > 2$ . In one-dimensional systems instead (Klemm *et al.*, 2003c), the transition is continuous for all values of  $F$ .

It is worth remarking that, upon interaction, the over-

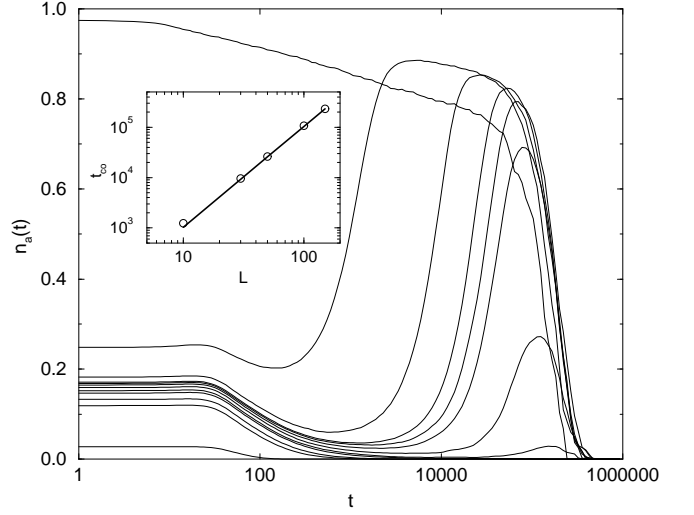


FIG. 10 Plot of the density of active links  $n_a(t)$  for  $F = 10$ ,  $L = 150$  and (top to bottom)  $q = 1, 100, 200, 230, 240, 250, 270, 300, 320, 400, 500, 10000$ . The inset reports the dependence of the freezing time  $t_{co}$  on  $L$  for  $F = 10$  and  $q = 100 < q_c$ . The bold line has slope 2. From (Castellano *et al.*, 2000).

lap between two neighbors always increases by  $1/F$ , but the change of a trait in an individual can make it more dissimilar with respect to his other neighbors. Hence, when the number of neighbors is larger than 2, each interaction can, somewhat paradoxically, result in an increase of the general level of disorder in the system. This competition is at the origin of the nontrivial temporal behavior of the model in  $d = 2$ , illustrated in Fig. 10: below the transition but close to it ( $q \lesssim q_c$ ) the density of active links (connecting sites with overlap different from 0 and 1) has a highly non monotonic behavior.

Most investigations of the Axelrod model are based on numerical simulations of the model dynamics. Analytical approaches are just a few. A simple mean field treatment (Castellano *et al.*, 2000; Vazquez and Redner, 2007; Vilone *et al.*, 2002) consists in writing down rate equations for the densities  $P_m$  of bonds of type  $m$ , i.e., connecting individuals with  $m$  equal and  $F - m$  different features. The natural order parameter in this case is the steady state number of active links  $n_a = \sum_{m=1}^{F-1} P_m$ , that is zero in the disordered phase, while it is finite in the ordered phase. This approach gives a discontinuous transition for any  $F$ . In the particular case of  $F = 2$  the mean field equations can be studied analytically in detail (Vazquez and Redner, 2007), providing insight into the non-monotonic dynamic behavior for  $q \lesssim q_c$  and showing that the approach to the steady state is governed by a timescale diverging as  $|q - q_c|^{-1/2}$ . Some information about the behavior of the Axelrod model for  $F = 2$  and  $q = 2$  is obtained also by a mapping to the

<sup>3</sup> Since  $q$  is discrete calling the transition “continuous” is a slight abuse of language. We will adopt it because the transition is associated with the divergence of a length, as in usual transitions.

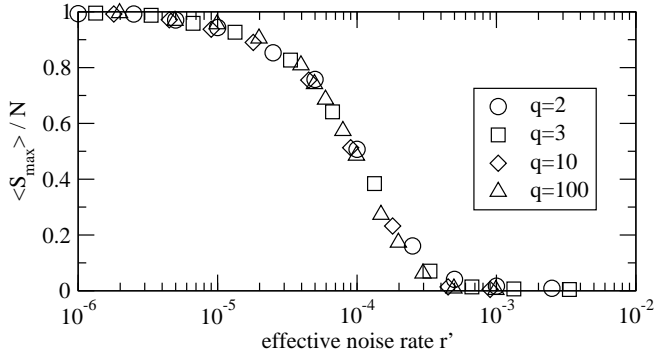


FIG. 11 Order parameter  $\langle S_{\max} \rangle / N$  as a function of the effective noise rate  $r' = r(1 - 1/q)$  for different values of  $q$ . Simulations have been run in systems of size  $N = 50^2$  with  $F = 10$ . From (Klemm *et al.*, 2003a).

constrained voter model (Vazquez and Redner, 2004) discussed in Sec. III.B.

### B. Variants of Axelrod model

In his seminal paper, Axelrod himself mentioned many possible variants of his model, to be studied in order to investigate the effect of additional ingredients as the topology of the interactions, random noise, the effect of mass media and many others. Over the years this program has been followed by many researchers.

The possibility of one individual to change spontaneously one of his traits, independently of his neighborhood, is denoted as “cultural drift” in social science and corresponds to the addition of flipping events driven by random noise. In (Klemm *et al.*, 2003a) it is demonstrated that the inclusion of noise at rate  $r$  has a profound influence on the model, resulting in a noise-induced order-disorder transition, practically independent of the value of the parameter  $q$  (Fig. 11).

For small noise the state of the system is monocultural for any  $q$ , because disordered configurations are unstable with respect to the perturbation introduced by the noise: the random variation of a trait unfreezes in some cases the boundary between two domains leading to the disappearance of one of them in favor of the other. However, when the noise rate is large, the disappearance of domains is compensated by the rapid formation of new ones, so that the steady state is disordered. The threshold between the two behaviors is set by the inverse of the average relaxation time for a perturbation  $T(N)$ , so that the transition occurs for  $r_c T(N) = O(1)$ . An approximate evaluation of the relaxation in  $d = 2$  gives  $T = N \ln(N)$ , in good agreement with simulations, while  $T \sim N^2$  on one dimension (Klemm *et al.*, 2005). Since  $T(N)$  diverges with  $N$ , the conclusion is that, no matter how small the rate of cultural drift is, in the thermodynamic limit the system remains always disordered for any  $q$ .

The discovery of the fragility of the Axelrod model with respect to the presence of noise immediately raises the question “What is the simplest modification of the original model that preserves the existence of a transition in the presence of noise?”. In (Kuperman, 2006) two modified Axelrod-like dynamics have been introduced, where the interaction between individuals is also influenced by which trait is adopted by the majority of agents in the local neighborhood. Similar ingredients are present in two other variants of the Axelrod model recently proposed (Flache and Macy, 2007). A convincing illustration that these modifications lead to a robust phenomenology with respect to the addition of (at least weak) noise is still lacking.

Another variant of the original definition of the model is the introduction of a threshold such that, if the overlap is smaller than a certain value  $\theta$ , no interaction takes place (Flache and Macy, 2007). Unsurprisingly no qualitative change occurs, except for a reduction of the ordered region of the phase diagram (De Sanctis and Galla, 2007). Another possibility named “interaction noise”, is that for  $\omega$  smaller than the threshold, the interaction takes place with probability  $\delta$ . This kind of noise favors ordering but again does not lead to drastic changes of the model behavior (De Sanctis and Galla, 2007).

In order to understand the effect of complex interaction topologies on its behavior, the Axelrod model has been studied on small-world and scale-free graphs (Klemm *et al.*, 2003b). In the first case, the transition between consensus and a disordered multicultural phase is still observed, for values of the control parameter  $q_c$  that grow as a function of the rewiring parameter  $p$ . Since the WS network for  $p = 1$  is a random network (and then practically an infinite-dimensional system) this is consistent with the observation of the transition also in the mean field approaches (Castellano *et al.*, 2000; Vazquez and Redner, 2007). The scale-free nature of the BA network dramatically changes the picture. For a given network of size  $N$  a somewhat smeared-out transition is found for a value  $q_c$ , with bistability of the order parameter, the signature of a first-order transition. However numerical simulations show that the transition threshold grows with  $N$  as  $q_c \sim N^{0.39}$ , so that in the thermodynamic limit the transition disappears and only ordered states are possible. This is similar to what occurs for the Ising model on scale-free networks, where the transition temperature diverges with system size (Leone *et al.*, 2002).

Another natural modification of the original Axelrod model concerns the effect of media, represented by some external field or global coupling in the system. One possible way to implement an external field consists in defining a mass media cultural message as a set of fixed variables  $M = (\mu_1, \mu_2, \dots, \mu_F)$  (González-Avella *et al.*, 2005). With probability  $B$  the selected individual interacts with the external field  $M$  exactly as if it were a neighbor. With probability  $1 - B$  the individual selects instead one of his actual neighbors. Rather unexpected-



edly the external field turns out to favor the multicultural phase, in agreement with early findings (Shibanai *et al.*, 2001). The order-disorder transition point is shifted to smaller values of the control parameter  $q_c(B)$ . For  $B$  larger than a threshold such that  $q_c(B^*) = 0$  only the disordered phase is present: a strong external field favors the alignment of some individuals with it, but it simultaneously induces a decoupling from individuals too far from it.

Similar conclusions are drawn when a global coupling or a local non-uniform coupling are considered (González-Avella *et al.*, 2006). In all cases the ordered region of the phase diagram is reduced with respect to the case of zero field and it shrinks to zero beyond a certain field strength  $B^*$ . Interestingly, for  $q > q_c(B = 0)$  a vanishing field has the opposite effect, leading to an ordered monocultural state. The limit  $B \rightarrow 0$  is therefore discontinuous. The same type of behavior is also found for indirect mass-media feedback, i.e., when sites accept the change of a trait only with probability  $R$ , if the new value of the trait is not the same of the majority (González-Avella *et al.*, 2007).

In the Axelrod model the numerical value of traits is just a label: nothing changes if two neighbors have traits that differ by 1 or by  $q - 1$ . In order to model situations where this difference actually matters, it has been proposed (Flache and Macy, 2006) to consider some features to be “metric”, i.e., such that the contribution to the overlap of a given feature is  $[1 - \Delta\sigma_f/(q - 1)]/F$ , where  $\Delta\sigma_f$  is the difference between the trait values. In this way the Axelrod model becomes similar to the vectorial version of the Deffuant model. Although a systematic investigation has not been performed, it is clear that this variation favors the reaching of consensus, because only maximal trait difference ( $\Delta\sigma_f = q - 1$ ) totally forbids the interaction. A related variation with “metric” features is described in (De Sanctis and Galla, 2007).

Other recent works deal with a version of the Axelrod model with both an external field and noise (Mazzitello *et al.*, 2007) and one where individuals above a fixed threshold do not interact (Parravano *et al.*, 2006).

### C. Other multidimensional models

At odds with the detailed exploration of the behavior of the Axelrod model, much less attention has been paid to other types of dynamics for vectors of opinions.

In the original paper on the Deffuant model (Deffuant *et al.*, 2000), a generalization to vectorial opinions is introduced, considering in this case binary variables instead of continuous ones. This gives a model similar to the Axelrod model with  $q = 2$  traits per feature, with the difference that the probability of interaction between two agents as a function of their overlap is a step function at a bounded confidence threshold  $d$ . In mean field a transition between full

consensus for large threshold and fragmentation for small  $d$  is found.

A similar model is studied in (Laguna *et al.*, 2003). In this case, when two agents are sufficiently close to interact, each pair of different variables may become equal with a probability  $\mu$ . Again a transition between consensus and fragmentation is found as a function of the bounded confidence threshold, but its properties change depending on whether  $\mu = 1$  or  $\mu < 1$ .

A generalization of continuous opinions (the HK model) to the vectorial (2-dimensional) case is reported in (Fortunato *et al.*, 2005) for a square opinion space, with both opinions ranging between 0 and 1, and square or circular confidence ranges. Assuming homogeneous mixing and solving the rate equations, it turns out that no drastic change occurs with respect to the ordinary HK model. The consensus threshold is practically the same. When there is no consensus the position of coexisting clusters is determined by the shape of the opinion space. An extension of Deffuant and HK models to vectorial opinions has been proposed in (Lorenz, 2006, 2008a). Here opinions sit on a hypercubic space or on a simplex, i.e., the components of the opinion vectors sum up to one. It turns out that consensus is easier to attain if the opinion space is a simplex rather than hypercubic.

Other vectorial models are considered in Section X on the coevolution of networks and states

## V. LANGUAGE DYNAMICS

Language dynamics is an emerging field (Wichmann, 2008) that focuses on all processes related to emergence, change, evolution, interactions and extinction of languages. In this section we shall review some of the main directions in which this field is evolving.

Models for language dynamics and evolution can be roughly divided in two main categories: *sociobiological* and *sociocultural* approaches. This distinction somehow parallels the debate *nature versus nurture* (Galton, 1874; Ridley, 2003) which concerns the relative importance of an individual’s innate qualities (“nature”) with respect to personal experiences (“nurture”) in determining or causing individual differences in physical and behavioral traits.

The sociobiological approach (Hurford, 1989; Pinker and Bloom, 1990) postulates that successful communicators, enjoying a selective advantage, are more likely to reproduce than worse communicators. Successful communication contributes thus to biological fitness: i.e., good communicators leave more offspring. The most developed branch of research in this area is represented by the evolutionary approaches. Here the main hypothesis is that communication strategies (which are model-dependent) are innate, in the spirit of the nativist approach (Chomsky, 1965), and transmitted genetically across generations. Thus if one of them is better than the others, in an evolutionary time span it

will displace all rivals, possibly becoming the unique strategy of the population. The term strategy acquires a precise meaning in the context of each particular model. For instance, it can be a strategy for acquiring the lexicon of a language, i.e., a function from samplings of observed behaviors to acquired communicative behavior patterns (Hurford, 1989; Nowak *et al.*, 1999b; Oliphant, 1997; Oliphant and Batali, 1996), or it can simply coincide with the lexicon of the parents (Nowak and Krakauer, 1999), but other possibilities exist (Steels, 2005).

On the other hand, in sociocultural approaches language is seen as a complex dynamical system that evolves and self-organizes, continuously shaped and reshaped by its users (Steels and Baillie, 2000). Here good strategies do not provide higher reproductive success but only better communication abilities. Agents can select better strategies exploiting cultural choice and direct feedback in communications. Moreover, innovations can be introduced due to the inventing ability of the agents. Thus, the study of the self-organization and evolution of language and meaning has led to the idea that a community of language users can be seen as a complex dynamical system which collectively solves the problem of developing a shared communication system. In this perspective, which has been adopted by the novel field of semiotic dynamics, the theoretical tools developed in statistical physics and complex systems science acquire a central role for the study of the self-generated structures of language systems.

## A. Evolutionary approaches

According to the sociobiological approach (Hurford, 1989; Nowak, 2006; Nowak *et al.*, 1999b; Oliphant, 1997; Oliphant and Batali, 1996), evolution is the main responsible both for the origin and the emergence of natural language in humans (Pinker and Bloom, 1990). Consequently, natural selection is the fundamental driving force to be introduced in models. Evolutionary game theory (Smith, 1982) was formulated with the aim of adapting classical game theory (von Neumann and Morgenstern, 1947; Osborne and Rubinstein, 1994) to deal with evolutionary issues, such as the possibility for agents to adapt, learn and evolve. The approach is phenotypic, and the fitness of a certain phenotype is, roughly speaking, proportional to its diffusion in the population. Strategies of classical game theory are substituted by traits (genetic or cultural), that are inherited, possibly with mutations. The search for Nash equilibria (Nash, 1950) becomes the quest for evolutionary stable strategies. A strategy is stable if a group adopting it cannot be invaded by another group adopting a different strategy. Finally, a fundamental assumption is that the payoff from a game is interpreted as the fitness of the agents involved in the game. The Evolutionary Language Game (ELG)

(Nowak and Krakauer, 1999; Nowak *et al.*, 1999b) aims at modeling the emergence of language resorting to evolutionary game theory and to the concept of language game (Wittgenstein, 1953a,b). For a recent experimental paper we refer to (Lieberman *et al.*, 2007).

### 1. Evolutionary language game

In this section we analyze in some detail how the problem of the evolution of a common vocabulary [or more generally a common set of conventions (Lewis, 1969), syntactic or grammatical rules] is addressed in the framework of evolutionary game theory. The formalism we use is mutated by (Nowak *et al.*, 1999b), but the basic structure of the game was already included in the seminal paper (Hurford, 1989) about the evolution of Saussurean signs (de Saussure, 1916).

A population of agents (possibly early hominids) lives in an environment with  $n$  objects. Each individual is able of produce a repertoire of  $m$  words (sounds or signals, in the original terminology) to be associated with objects. Individuals are characterized by two matrices  $P$  and  $Q$ , which together form a language  $L$ . The *production matrix*  $P$  is a  $n \times m$  matrix whose entry,  $p_{ij}$ , denotes the probability of using word  $j$  when seeing object  $i$ , while the *comprehension matrix*  $Q$  is a  $m \times n$  matrix, whose entry,  $q_{ji}$ , denotes the probability for a hearer to associate sound  $j$  with object  $i$ , with the following normalization conditions on the rows  $\sum_{j=1}^m p_{ij} = 1$  and  $\sum_{i=1}^n q_{ji} = 1$ .

A pair of matrices  $P$  and  $Q$  identifies a language  $L$ . Imagine then two individuals  $I_1$  and  $I_2$  speaking languages  $L_1$  (defined by  $P_1$  and  $Q_1$ ) and  $L_2$  (defined by  $P_2$  and  $Q_2$ ). The typical communication between the two involves the speaker, say  $I_1$ , associating the signal  $j$  to the object  $i$  with probability  $p_{ij}$ . The hearer  $I_2$  infers object  $i$  with probability  $\sum_{j=1}^m p_{ij}^{(1)} q_{ji}^{(2)}$ . If one sums over all the possible objects, one gets a measure of the ability, for  $I_1$ , to convey information to  $I_2$ :  $\sum_{i=1}^n \sum_{j=1}^m p_{ij}^{(1)} q_{ji}^{(2)}$ . A symmetrized form of this expression defines the so-called payoff function, i.e., the reward obtained by two individuals speaking languages  $L_1$  and  $L_2$  when they communicate:

$$F(L_1, L_2) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^m (p_{ij}^{(1)} q_{ji}^{(2)} + p_{ij}^{(2)} q_{ji}^{(1)}). \quad (33)$$

From the definition of the payoff it is evident that each agent is treated once as hearer and once as speaker and they both receive a reward for successful communication.

The crucial point of the model is the definition of the matrices  $P$  and  $Q$  which have to be initialized in some way at the beginning of the simulation. In principle there is no reason why  $P$  and  $Q$  should be correlated. On the other hand the best possible payoff is obtained by choosing  $P$  as a binary matrix having at least one 1 in every column (if  $n \geq m$ ) or in every row (if  $n \leq m$ ) and  $Q$  as a

binary matrix with  $q_{ji} = 1$ , if  $p_{ij}$  is the largest entry in a column of  $P$ . If  $n = m$  the maximum payoff is obtained for  $P$  having one 1 in every row and column and  $Q$  being the transposed matrix of  $P$ . In general the maximum payoff is given by  $F_{max} = \min\{m, n\}$ . It is also worth noting that the presence of two completely uncorrelated matrices for the production,  $P$ , and comprehension,  $Q$ , modes, already present in (Hurford, 1989; Oliphant, 1997; Oliphant and Batali, 1996), could lead to pathological situations as remarked in (Komarova and Niyogi, 2004), where a single matrix is adopted for both tasks.

In a typical situation one simulates a population of  $N$  individuals speaking  $N$  different languages  $L_1$  to  $L_N$  (by randomly choosing the matrices  $P_k$  and  $Q_k$ , for  $k = 1, \dots, N$ ). In each round of the game, every individual communicates with every other individual, and the accumulated payoffs are summed up, e.g. the payoff received by individual  $k$  is given by  $F_k = \sum_{l=1}^N F(L_k, L_l)$ , with  $l \neq k$ . As already mentioned, the payoff is interpreted as fitness. In a parental learning scheme each individual will produce an offspring (without sexual reproduction) with the probability  $f_k = F_k / \sum_l F_l$ . In this way each individual gives rise on average to one offspring for the next generation and the population size remains constant. The individuals of the new generation learn the language of their parents by constructing an association matrix  $A$ , whose element  $a_{ij}$  records how many times the individual has observed its parent associating object  $i$  and signal  $j$  in  $K$  different samplings. The production and comprehension matrices  $P$  and  $Q$  are easily derived from the association matrix  $A$  as:

$$p_{ij} = a_{ij} / \sum_{l=1}^m a_{il} \quad q_{ji} = a_{ji} / \sum_{l=1}^m a_{lj}. \quad (34)$$

The form of the matrix  $A$  clearly depends on  $K$ . In the limit  $K \rightarrow \infty$  the offspring reproduces the production matrix of its parent and  $A = P$ . For finite values of  $K$ , learning occurs with incomplete information and this triggers mutations occurring in the reproduction process.

An important observation is in order. In such a scheme the language of an individual, i.e., the pair  $(P, Q)$ , determines its fitness and, as a consequence, the reproduction rate. On the other hand what is inherited is not directly the language but a mechanism to learn the language which is language specific, i.e., a language acquisition device in the spirit of the nativist approach (Chomsky, 1965). Therefore the traits transmitted to the progeny can be different from the language itself.

This evolutionary scheme leads the population to converge to a common language, i.e., a pair of  $(P, Q)$  matrices shared by all individuals. The common language is not necessarily optimal and the system can often get stuck in sub-optimal absorbing states where *synonymy* (two or more signals associated to the same object) or *homonymy* (the same signal used for two or more objects) are present. The convergence properties to an absorbing state depend on the population

size  $N$  as well as on  $K$ , but no systematic analysis has been performed in this direction. Another interesting direction is related to the underlying topology of the game. What described so far corresponds to a fully connected topology, where each agent interacts with the whole population. It is certainly of interest exploring different topological structures, more closely related to the structure of social networks, as discussed in (Hauert *et al.*, 2005; Szabó and Fáth, 2007) (see also the recent (Tavares *et al.*, 2007) on the model proposed in (Komarova *et al.*, 2001)).

The model can then be enriched by adding a probability of errors in perception (Nowak and Krakauer, 1999), i.e., by introducing a probability  $u_{ij}$  of misinterpreting signal  $i$  as signal  $j$ . The terms  $u_{ij}$  are possibly defined in terms of similarities between signals. The maximum payoff for two individuals speaking the same language is now reduced, hence the maximum capacity of information transfer. This result is referred to as *linguistic error limit* (Nowak and Krakauer, 1999; Nowak *et al.*, 1999a): the number of distinguishable signals in a protolanguage, and therefore the number of objects that can be accurately described by this language, is limited. Increasing the number of signals would not increase the capacity of information transfer [it is worth mentioning here the interesting parallel between the formalism of evolutionary language game with that of information theory (Plotkin and Nowak, 2000)]. A possible way out is that of combining signals into words (Smith *et al.*, 2003), opening the way to a potentially unlimited number of objects to refer to. In this case it is shown that the fitness function can overcome the error limit, increasing exponentially with the length of words (Nowak and Krakauer, 1999; Nowak *et al.*, 1999a). This is considered one of the possible ways in which evolution has selected higher order structures in language, e.g. syntax and grammar. We refer to (Nowak, 2006; Nowak and Krakauer, 1999) for details about the higher stages in the evolution of language.

## 2. Quasispecies-like approach

The model described in the previous section can be cast in the framework of a deterministic dynamical system (see for a recent discussion (Traulsen *et al.*, 2005) and references therein). We consider again the association matrix  $A$ , a  $n \times m$  matrix whose entries,  $a_{ij}$ , are non-zero if there is an association between the object  $i$  and the signal  $j$ . In this case we consider a binary matrix with entries taking either the value 0 or 1. The possible number of matrices  $A$  is then  $M = 2^{nm}$ . This matrix is also denoted as the lexical matrix (Komarova and Nowak, 2001). In a population of  $N$  individuals denote now with  $x_k$  the fraction of individuals with the association matrix  $A_k$ , with  $\sum_{k=1}^M x_k = 1$ . One can define the evolution of  $x_k$  as given by the following equation:

$$\dot{x}_k = \sum_l f_l x_l Q_{lk} - \phi x_k, \quad l = 1, \dots, M = 2^{nm}, \quad (35)$$

where  $f_l$  is the fitness of individuals with the association matrix  $A_l$  (from now on individual  $l$ ),  $f_l = \sum_k F(A_l, A_k) x_k$ , with the assumption that  $x_k$  is the probability to speak with an individual  $k$ ;  $\phi$  defines the average fitness of the population,  $\phi = \sum_l f_l x_l$ , while  $Q_{lk}$  denotes the probability that someone learning from an individual with  $A_l$  will end up with  $A_k$ . The second term on the right-hand side keeps the population size constant.

Eqs. (35) represent a particular case of the quasispecies equations (Eigen, 1971; Eigen and Schuster, 1979). The quasispecies model is a description of the process of Darwinian evolution of self-replicating entities within the framework of physical chemistry. These equations provide a qualitative understanding of the evolutionary processes of self-replicating macromolecules such as RNA or DNA or simple asexual organisms such as bacteria or viruses. Quantitative predictions based on this model are difficult, because the parameters that serve as input are hard to obtain from actual biological systems. In the specific case in which mutation is absent, i.e.,  $Q_{ij} = 0$  if  $i \neq j$ , one recovers the so-called *replicator equations* of evolutionary game theory (Smith, 1982), which, it is worth recalling, are equivalent to the Lotka-Volterra equations in  $M - 1$  dimensions (Hofbauer and Sigmund, 1998).

## B. Semiotic Dynamics approach

Semiotic dynamics looks at language as an evolving system where new words and grammatical constructions may be invented or acquired, new meanings may arise, the relation between language and meaning may shift (e.g. if a word adopts a new meaning), the relation between meanings and the world may shift (e.g. if new perceptually grounded categories are introduced).

### 1. The Naming Game

The *Naming Game* (NG) possibly represents the simplest example of the complex processes leading progressively to the establishment of human-like languages. It was expressly conceived to explore the role of self-organization in the evolution of language (Steels, 1995, 1996) and it has acquired, since then, a paradigmatic role in the whole field of semiotic dynamics. The original paper (Steels, 1995), focuses mainly on the formation of vocabularies, i.e., a set of mappings between words and meanings (for instance physical objects). In this context, each agent develops its own vocabulary in a random private fashion. But agents are forced to align their vocabularies, through successive conversation, in order to obtain the benefit of cooperating through communication. Thus, a globally shared vocabulary emerges, or

should emerge, as the result of local adjustments of individual word-meaning association. The communication evolves through successive conversations, i.e., events that involve a certain number of agents (two, in practical implementations) and meanings. It is worth remarking that here conversations are particular cases of language games, which, as already pointed out in (Wittgenstein, 1953a,b), can be used to describe linguistic behavior, even if they can include also non-linguistic behavior, such as pointing.

This original seminal idea triggered a series of contributions along the same lines and many variants have been proposed over the years. It is particularly interesting to mention the work proposed in (Ke *et al.*, 2002), that focuses on an imitation model which simulates how a common vocabulary is formed by agents imitating each other, either using a mere random strategy or a strategy in which imitation follows the majority (which implies non-local information for the agents). A further contribution of this paper is the introduction of an interaction model which uses a probabilistic representation of the vocabulary. The probabilistic scheme is formally similar to the framework of evolutionary game theory seen in Sec. V.A.1, since to each agent a *production* and a *comprehension* matrices are associated. Differently from the approach of ELG, here the matrices are dynamically transformed according to the social learning process and the cultural transmission rule. A similar approach has been proposed in (Lenaerts *et al.*, 2005).

In the next section we shall present a *minimal* version of the NG which results in a drastic simplification of the model definition, while keeping the same overall phenomenology. This version of the NG is suitable for massive numerical simulations and analytical approaches. Moreover the extreme simplicity allows for a direct comparison with other models introduced in other frameworks of statistical physics as well as in other disciplines.

*a. The Minimal Naming Game* The simplest version of the NG (Baronchelli *et al.*, 2006b) is played by a population of  $N$  agents, on a fully connected network, trying to bootstrap a common name for a given object. Each player is characterized by an inventory of word-object associations he/she knows. All agents have empty inventories at time  $t = 0$ . At each time step ( $t = 1, 2, \dots$ ), two players are picked at random and one of them plays as speaker and the other as hearer. Their interaction obeys the rules described in Fig. 12.

*b. Macroscopic analysis* The first property of interest is the time evolution of the total number of words owned by the population  $N_w(t)$ , of the number of different words  $N_d(t)$ , and of the success rate  $S(t)$  (Fig. 13).

After a transient period, the system undergoes spontaneously a disorder-order transition to an asymptotic state where global consensus emerges, i.e., every agent has the same word for the same object. It is remark-

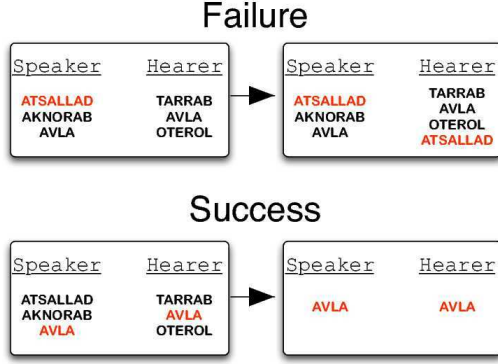


FIG. 12 Naming Game. Examples of the dynamics of the inventories in a failed (top) and a successful (bottom) game. The speaker selects the word highlighted. If the hearer does not possess that word he includes it in his inventory (top). Otherwise both agents erase their inventories only keeping the winning word (bottom).

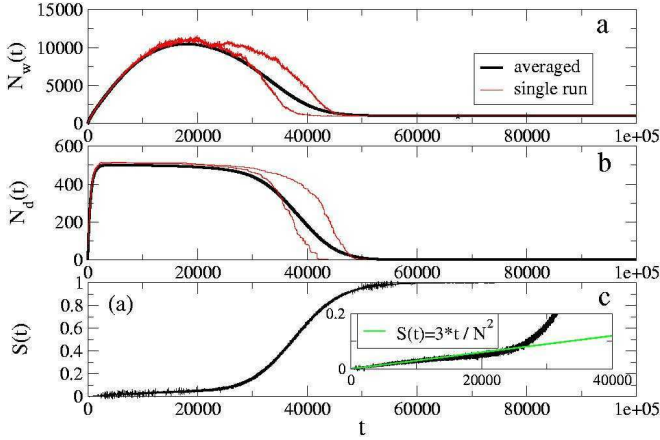


FIG. 13 Naming Game. a) Total number of words present in the system,  $N_w(t)$ ; b) Number of different words,  $N_d(t)$ ; c) Success rate  $S(t)$ , i.e., probability of observing a successful interaction at time  $t$ . The inset shows the linear behavior of  $S(t)$  at small times. The system reaches the final absorbing state, described by  $N_w(t) = N$ ,  $N_d(t) = 1$  and  $S(t) = 1$ , in which a global agreement has been reached. From (Baronchelli *et al.*, 2006b).

able that this happens starting from completely empty inventories for each agent. The asymptotic state is one where a word invented during the time evolution takes over with respect to the other competing words and imposes itself as the leading word. In this sense the system spontaneously selects one of the many possible coherent asymptotic states and the transition can thus be seen as a symmetry breaking transition. The dynamics of the Naming Game is characterized by the following scaling behavior for the convergence time  $t_{conv}$ , the time and the height of the peak of  $N_w(t)$ , namely  $t_{max}$  and  $N_w^{max} = N_w(t_{max})$ . It turns out that all these quantities follow power law behaviors:  $t_{max} \sim N^\alpha$ ,  $t_{conv} \sim N^\beta$ ,

$N_{max} \sim N^\gamma$  and  $t_{diff} = (t_{conv} - t_{max}) \sim N^\delta$ , with exponents  $\alpha = \beta = \gamma = \delta = 1.5$  (with a subtle feature around the disorder-order transition where an additional timescale emerges). The values of those exponents can be understood through simple scaling arguments (Baronchelli *et al.*, 2006b)<sup>4</sup>.

## 2. Symmetry breaking: a controlled case

Consider now a simpler case in which there are only two words at the beginning of the process, say A and B, so that the population can be divided into three classes: the fraction of agents with only A,  $n_A$ , the fraction of those with only the word B,  $n_B$ , and finally the fraction of agents with both words,  $n_{AB}$ . Describing the mean-field time evolution of the three species is straightforward:

$$\begin{aligned} \dot{n}_A &= -n_A n_B + n_{AB}^2 + n_A n_{AB} \\ \dot{n}_B &= -n_A n_B + n_{AB}^2 + n_B n_{AB} \\ \dot{n}_{AB} &= +2n_A n_B - 2n_{AB}^2 - (n_A + n_B)n_{AB}. \end{aligned} \quad (36)$$

The system of differential equations (36) is deterministic. It presents three fixed points in which the system can collapse depending on the initial conditions. If  $n_A(t=0) > n_B(t=0)$  [ $n_B(t=0) > n_A(t=0)$ ], at the end of the evolution we will have the stable fixed point  $n_A = 1$  [ $n_B = 1$ ] and, consequently  $n_B = n_{AB} = 0$  [ $n_A = n_{AB} = 0$ ]. If, on the other hand, we start from  $n_A(t=0) = n_B(t=0)$ , the equations lead to  $n_A = n_B = 2n_{AB} = 0.4$ . The latter situation is clearly unstable, since any external perturbation would make the system fall in one of the two stable fixed points.

Eqs. (36) however, are not only a useful example to clarify the nature of the symmetry breaking process. In fact, they also describe the interaction among two different populations that converged separately to two distinct conventions. In this perspective, Eqs. (36) predict that the larger population will impose its conventions. In the absence of fluctuations, this is true even if the difference is very small: B will dominate if  $n_B(t=0) = 0.5 + \epsilon$  and  $n_A(t=0) = 0.5 - \epsilon$ , for any  $0 < \epsilon \leq 0.5$  and  $n_{AB}(t=0) = 0$ . Data from simulations show that the probability of success of the convention of the minority group  $n_A$  decreases as the system size increases, going to zero in the thermodynamic limit ( $N \rightarrow \infty$ ). A similar approach has been proposed to model the competition between two languages (Abrams and Strogatz, 2003). We discuss this point in Sec. V.D. Here it is worth remarking the formal similarities between modeling the competition between synonyms in a NG framework and the competition between languages: in both cases a synonym or a language are represented by a single feature, e.g. the characters A or B, for instance,

<sup>4</sup> Here the time is the number of binary interactions

in Eqs. (36). The similarity has been made more evident by the subsequent variants of the model introduced in (Abrams and Strogatz, 2003) to include explicitly the possibility of bilingual individuals. In particular in (Minett and Wang, 2008; Wang and Minett, 2005) deterministic models for the competition of two languages have been proposed, which include bilingual individuals. In (Castelló *et al.*, 2006) a modified version of the voter model (see Sec. III.B) including bilingual individuals has been proposed, the so-called AB-model. In a fully connected network and in the limit of infinite population size, the AB-model can be described by coupled differential equations for the fractions of individuals speaking language  $A$ ,  $B$  or  $AB$ , that are, up to a constant normalization factor in the timescale, identical to Eqs. (36).

### 3. The role of the interaction topology

As already mentioned in Sec. II.B, social networks play an important role in determining the dynamics and outcome of language change. The first investigation of the role of topology was proposed in 2004, at the 5th Conference on Language evolution, Leipzig (Ke *et al.*, 2008). Since then many approaches focused on adapting known models on topologies of increasing complexity: regular lattices, random graphs, scale-free graphs, etc.

The NG, as described above, is not unambiguously defined on general networks. As already observed in Sec. II.B and in Sec. III.B, when the degree distribution is heterogeneous, the order in which an agent and one of its neighbors are selected does matter, because high-degree nodes are more easily chosen as neighbors than low-degree nodes. Several variants of the NG on generic networks can be defined. In the *direct NG* (*reverse NG*) a randomly chosen speaker (hearer) selects (again randomly) a hearer (speaker) among its neighbors. In a *neutral* strategy one selects an edge and assigns the role of speaker and hearer with equal probability to the two nodes (Dall'Asta *et al.*, 2006b).

On low-dimensional lattices consensus is reached through a coarsening phenomenon (Baronchelli *et al.*, 2006a) with a competition among the homogeneous clusters corresponding to different conventions, driven by the curvature of the interfaces (Bray, 1994). A scaling of the convergence time as  $\mathcal{O}(N^{1+1/d})$  has been conjectured, where  $d \leq 4$  is the lattice dimension. Low-dimensional lattices require more time to reach consensus compared to a fully connected graph, but a lower use of memory. A similar analysis has been performed for the AB-model (Castelló *et al.*, 2006). The effect of a small-world topology has been investigated in (Dall'Asta *et al.*, 2006a) in the framework of the NG and in (Castelló *et al.*, 2006) for the AB-model. Two different regimes are observed. For times shorter than a crossover time,  $t_{cross} = \mathcal{O}(N/p^2)$ , one observes the usual coarsening phenomena as long as the clusters are one-dimensional, i.e., as long as the typical cluster size

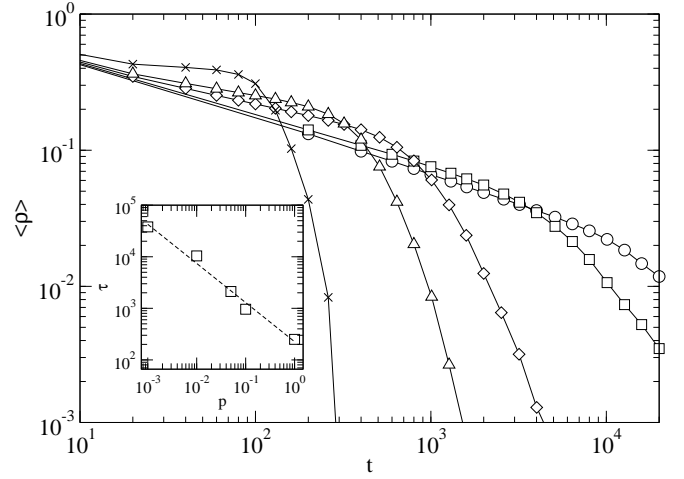


FIG. 14 AB-model. Time evolution of the average density  $\langle \rho \rangle$  of bilingual individuals in small-world networks for different values of the rewiring parameter  $p$ . From left to right:  $p = 1.0, 0.1, 0.05, 0.01, 0.0$ . The inset shows the dependence of the characteristic lifetime  $\tau$  on the rewiring parameter  $p$ . The dashed line corresponds to a power law fit  $\tau \sim p^{-0.76}$ . From (Castelló *et al.*, 2006).

is smaller than  $1/p$ . For times much larger than  $t_{cross}$ , the dynamics is dominated by the existence of short-cuts and enters a mean field like behavior. The convergence time is thus expected to scale as  $N^{3/2}$  and not as  $N^3$  (as in  $d = 1$ ). Small-world topology allows to combine advantages from both finite-dimensional lattices and fully connected networks: on the one hand, only a finite memory per node is needed, unlike the  $\mathcal{O}(N^{1/2})$  in fully connected graphs; on the other hand the convergence time is expected to be much shorter than in finite dimensions. In (Castelló *et al.*, 2006), the dynamics of the AB-model on a two-dimensional small world network has been studied. Also in this case a dynamic stage of coarsening is observed, followed by a fast decay to the  $A$  or  $B$  absorbing states caused by a finite size fluctuation (Fig. 14). The NG has been studied on complex networks as well. Here the convergence time  $t_{conv}$  scales as  $N^\beta$ , with  $\beta \simeq 1.4 \pm 0.1$ , for both Erdős-Rényi (ER) (Erdős and Rényi, 1959, 1960) and Barabási-Albert (BA) (Barabási and Albert, 1999) networks. The scaling laws observed for the convergence time are general robust features not affected by further topological details (Dall'Asta and Baronchelli, 2006; Dall'Asta *et al.*, 2006b). Finally it is worth mentioning how the Naming Games with local broadcasts on random geometric graphs have been investigated in (Lu *et al.*, 2008) as a model for agreement dynamics in large-scale autonomously operating wireless sensor networks.



### C. Other models

A variant of the NG has been introduced with the aim of mimicking the mechanisms leading to opinion and convention formation in a population of individuals (Baronchelli *et al.*, 2007). In particular a new parameter,  $\beta$ , has been added mimicking an *irresolute attitude* of the agents in making decisions ( $\beta = 1$  corresponds to the NG). The parameter  $\beta$  is simply the probability that, in a successful interaction, both the speaker and the hearer update their memories erasing all opinions except the one involved in the interaction (see Fig. 12). This negotiation process displays a non-equilibrium phase transition from an absorbing state in which all agents reach a consensus to an active (not frozen as in the Axelrod model (Axelrod, 1997)) stationary state characterized either by polarization or fragmentation in clusters of agents with different opinions. Interestingly, the model displays the non-equilibrium phase transition also on heterogeneous networks, in contrast with other opinion dynamics models, like for instance the Axelrod model (Klemm *et al.*, 2003b), for which the transition disappears for heterogeneous networks in the thermodynamic limit.

A hybrid approach, combining vertical and horizontal transmission of cultural traits, has been proposed in (Ke *et al.*, 2002), while an evolutionary version of the Naming Game has been introduced in (Lipowski and Lipowska, 2008).

A very interesting approach to language change has been proposed in (Baxter *et al.*, 2006; Croft, 2000), based on an utterance selection model. Another interesting approach to language dynamics, the so-called Iterated Learning Model (ILM) (Kirby, 2001), focuses on cultural evolution and learning (Niyogi, 2006) and explores how the mappings from meanings to signals are transmitted from generation to generation. In this framework several results have been obtained concerning the emergence of a linguistic structure, e.g. compositionality (Smith *et al.*, 2003).

### D. Language competition

Models of language evolution usually focus on a single population, which is supposed to be isolated from the rest of the world. However, real populations are not isolated, but they keep interacting with each other. These steady interactions between peoples play a big role in the evolution of languages.

At present, there are about 6,500 languages in the world, with a very uneven geographic distribution. Most of these languages have very few speakers, and are threatened by extinction (Sutherland, 2003). Indeed, it is plausible that in the future increasing numbers of people will be pushed to adopt a common language by socio-economic factors, leading to the survival of a few major linguistic groups, and to the extinction of all other

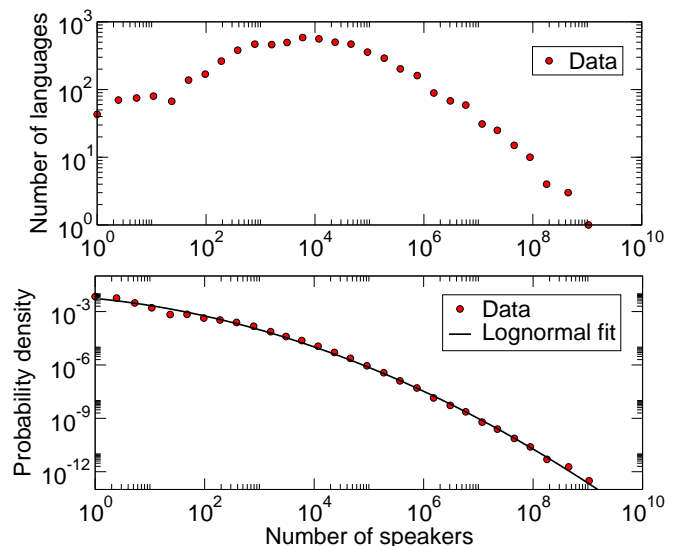


FIG. 15 Distribution of language sizes. The  $x$ -axis represents the number of individuals speaking a language. In the upper diagram, on the  $y$  axis the number of languages spoken by  $x$  individuals is reported. In the lower diagram the number of languages is divided by the size of the corresponding bins, obtaining the probability distribution, well fitted by a log-normal function (continuous line). The total number of languages sampled is 6142. (Data from <http://www.ethnologue.com/>).

languages. According to some estimates, up to 90% of present languages might disappear by the end of the 21st century (Krauss, 1992). The histogram of the language sizes, shown in Fig. 15, has a regular shape, which closely resembles a log-normal distribution<sup>5</sup>. Several models of language competition have been proposed with the aim of reproducing such distribution. However, we stress that the observed distribution of language sizes may not be a stable feature of language diversity, as there is no reason to believe that it has kept its shape over the past centuries and that it will keep it in the future. Moreover, it has been suggested that the current histogram may be a consequence of pure demographic growth (Zanette, 2007b).

Modeling language competition means studying the interaction between languages spoken by adults. Language evolution shares several features with the evolution of biological species. Like species, a language can split into several languages, it can mutate, by modifying words/expressions over time, it can face extinction. Such similarities have fostered the application of models used to describe biological evolution in a language competition context. The models can be divided in two categories: macroscopic models, where only average properties of the system are considered, are based on differential

<sup>5</sup> Similarly, in (Ausloos and Petroni, 2007), the distribution of the number of adherents to religions has been investigated.

equations; microscopic models, where the state of each individual is monitored in time, are based on computer simulations.

### 1. Macroscopic models

The first macroscopic model of language competition was a dynamic model proposed in (Abrams and Strogatz, 2003) (AS), describing how two languages,  $A$  and  $B$ , compete for speakers. The languages do not evolve in time; the attractiveness of each language increases with its number of speakers and perceived status, which expresses the social and economic benefits deriving from speaking that language. We indicate with  $x$  and with  $0 \leq s \leq 1$  the fraction of speakers and the status of  $A$ , respectively. Accordingly, language  $B$  has a fraction  $y = 1 - x$  of speakers, and its status is  $1 - s$ . The dynamics is given by the simple rate equation

$$\frac{dx}{dt} = c(1 - x)x^a s - cx(1 - x)^a(1 - s), \quad (37)$$

where  $a$  and  $c$  are parameters which, along with  $s$ , fix the model dynamics<sup>6</sup>. Eq. (37) expresses the balance between the rates of people switching from language  $B$  to  $A$  and from  $A$  to  $B$ . The dynamics has only two stable fixed points, corresponding to  $x = 0$  and  $x = 1$ . There is a third fixed point, corresponding to  $x = 1/2$ ,  $s = 1/2$ , when the two languages are equivalent, but it is unstable, as confirmed by numerical simulations of a microscopic version of the AS model on different graph topologies (Stauffer *et al.*, 2007). Therefore, the AS model predicts the dominance of one of the two languages and the consequent extinction of the other. The dominant language is the one with the initial majority of speakers and/or higher status. Comparisons with empirical data reveal that the model is able to reproduce the decrease in time of the number of speakers of various endangered languages (Fig. 16). The AS model is minimal and neglects important aspects of sociolinguistic interaction. In actual situations of language competition, the interaction between two languages  $A$  and  $B$  often occurs through speakers who are proficient in both languages. In (Mira and Paredes, 2005) bilingual speakers were introduced in the AS model. A parameter  $k$  expresses the similarity of the two competing languages  $A$  and  $B$  and is related to the probability for monolingual speakers to turn bilingual. For each choice of the AS parameters  $a$ ,  $s$ , there is a critical value  $k_{min}(a, s)$  such that, for  $k > k_{min}(a, s)$ , the system reaches a steady state characterized by the coexistence of one group of monolingual speakers with a group of bilinguals. Monolingual speakers of the endangered language are bound to disappear,

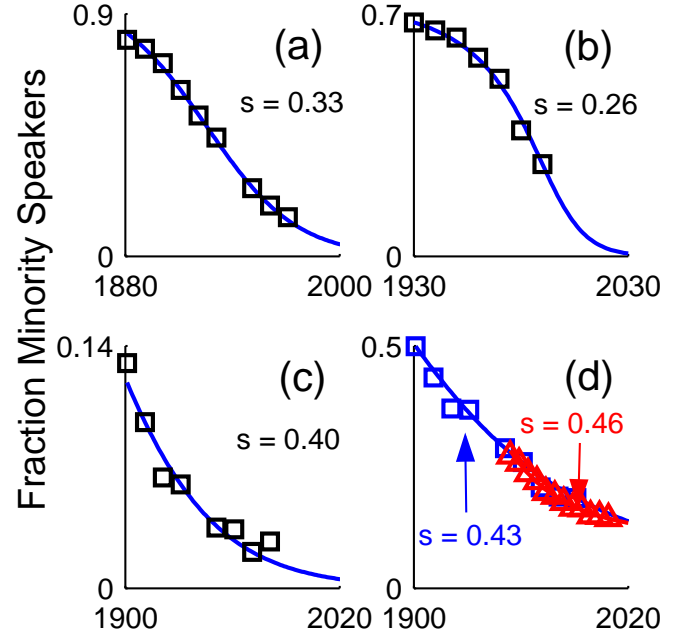


FIG. 16 . Dynamics of language extinction according to the model of Abrams and Strogatz. The four panels show the comparison of the model with real data on the proportion of speakers over time for (a) Scottish Gaelic in Sutherland, Scotland, (b) Quechua in Huanuco, Peru, (c) Welsh in Monmouthshire, Wales and (d) Welsh in all of Wales. In (d) both historical data (squares) and the results of a recent census (triangles) are plotted. From (Abrams and Strogatz, 2003).

but the survival of the language is ensured by bilingualism, provided  $A$  and  $B$  are similar enough. The model describes well historical data on the time evolution of the proportions of speakers of Galician and Castilian Spanish in Galicia. In (Minett and Wang, 2008) a more complex modification of the AS model, incorporating bilingualism and language transmission between adults and from adults to children, was proposed. The model predicts the same extinction scenario of the AS model, unless special strategies of intervention are adopted when the number of speakers of the endangered language decreases below a threshold. Effective intervention strategies are the enhancement of the status of the endangered language and the enforcement of monolingual education of the children.

In (Patriarca and Leppänen, 2004) the effect of population density is introduced, by turning the rate equation of the AS model into a reaction-diffusion equation. Here people can move on a plane, divided in two regions; in each region one language has a higher status than the other one. The system converges to a stable configuration where both languages survive, although they are mostly concentrated in the zones where they are favored. In a recent work (Pinasco and Romanelli, 2006) it was shown that language coexistence in the same region is possible, if one accounts for the population dynamics of the two linguistic communities, instead of considering the whole

<sup>6</sup> We remark that the parameter  $c$  is an overall multiplicative constant of the right-hand side of Eq. (37) and can be absorbed in the time unit, without affecting the dynamics.

population fixed, like in the AS model. The dynamics is now ruled by a set of generalized Lotka-Volterra equations, and presents a non-trivial fixed point when the rate of growth of the population of speakers of the endangered language compensates the rate of conversion to the dominant language.

## 2. Microscopic models

Many microscopic models of language competition represent language as a set of  $F$  independent features ( $F$  usually goes from 8 to 64), with each feature taking one out of  $Q$  values. This is also the representation of culture in the Axelrod model (see Sec. IV.A); indeed, language diversity is an aspect of cultural diversity. If  $Q = 2$ , language is a bit-string, a representation used for biological species (Eigen, 1971). For a recent review of language competition simulations see (Schulze *et al.*, 2008).

In the Schulze model (Schulze and Stauffer, 2005a), the language of each individual evolves according to three mechanisms, corresponding to random changes, transfer of words from one language to another and learning of a new language. There are three parameters:  $p$ ,  $q$  and  $r$ . With probability  $p$ , a randomly chosen feature of an agent's language is modified: with probability  $q$ , the new value is that of the corresponding feature of a randomly picked individual, otherwise a value taken at random. Finally, there is a probability  $(1 - x)^2 r$  that an agent switches to a language spoken by a fraction  $x$  of the population. If agents are the nodes of a network, the language of an individual can only be affected by its neighbors. Simulations show that there is a sharp transition between a phase in which most people speak the same language (low  $p$ ), and a phase in which no language dominates (high  $p$ ) and the distribution of language sizes is roughly log-normal, like the empirical distribution (Fig. 15). The agreement with the data improves by sampling the evolving model distribution over a large time interval (Stauffer *et al.*, 2006c). An analytical formulation of the Schulze model has been recently proposed (Zanette, 2007a).

We notice that here languages have no intrinsic fitness, i.e., they are all equivalent for the dynamics, at variance with biological species and with the macroscopic models of the previous section, where the different status of languages is responsible for their survival/extinction. The possible dominance of one language is determined by initial fluctuations, that make a linguistic community slightly larger than the others and more likely to capture speakers fleeing from other communities.

Several modifications of the Schulze model have been proposed. Agents can age, reproduce and die (Schulze and Stauffer, 2005a); they can move on the sites of a lattice, forming linguistic communities that are spatially localized (Schulze and Stauffer, 2005b); they can be bilingual (Schulze *et al.*, 2008). To avoid the dominance of a single language, it is enough to

stop the flight from an endangered language when the number of its speakers decreases below a threshold (Schulze and Stauffer, 2006). A linguistic taxonomy can be introduced, by classifying languages into families based on similarities of the corresponding bit strings (Wichmann *et al.*, 2006). This enables to control the dynamics of both individual languages and of their families.

The model in (de Oliveira *et al.*, 2006b) describes the colonization of a territory by a population that eventually splits into different linguistic communities. Language is represented by a number, so it has no internal structure. The expansion starts from the central site of a square lattice, with some initial population size. Free sites are occupied by a neighboring population with a probability proportional to the number of people speaking that language, which is a measure of the fitness of that population. The language of a group conquering a new site mutates with a probability that is inversely proportional to its fitness. The simulation stops when all lattice sites have been occupied. The resulting linguistic diversity displays similar features as those observed in real linguistic diversity, like the distribution of language sizes (Fig. 15). The agreement improves by introducing an upper bound for the fitness of a population (de Oliveira *et al.*, 2006a), or by representing languages as bit strings (de Oliveira *et al.*, 2007).

Social Impact Theory (see Sec. III.D) was applied to model language change (Nettle, 1999a,b). Here, there are two languages and agents are induced to join the linguistic majority because it exerts a great social pressure. Language mixing, for which a new language may originate from the merging of two languages, was implemented in (Kosmidis *et al.*, 2005). In this model, the biological fitness of the agents may increase if they learn words of the other language. The model accounts for the emergence of bilingualism in a community where people initially speak only one of two languages. In (Schwämmle, 2005) there are two languages and agents move on a lattice, are subjected to biological aging and can reproduce. People may grow bilingual; bilinguals may forget one of the two languages, if it is minoritarian in their spatial surroundings. As a result, if the two linguistic communities are spatially separated, they can coexist for a long time, before the dynamics will lead to the dominance of one of them. Bilingual agents are also present in the modified version of the voter model proposed in (Castelló *et al.*, 2006), discussed in Sec. III.B.

## VI. CROWD BEHAVIOR

Collective motion is very common in nature. Flocks of birds, fish schools, swarms of insects are among the most spectacular manifestations (Parrish and Hamner, 1997). Humans display similar behavior in many instances: pedestrian motion, panic, vehicular traffic, etc..

The origin of collective motion has represented a puzzle

zle for many years. One has the impression that each individual knows exactly what all its peers are doing in the group and acts accordingly. It is plausible instead that an individual has a clear perception of what happens in its neighborhood, ignoring what most of its peers are doing. We are then faced again with a phenomenon where local interactions determine the emergence of a global property of the system, in this case collective motion. Therefore it is not surprising that in the last years physicists have worked in this field. In this section we shall give a brief account of the most important results on crowd behavior. For a review of the studies on vehicular traffic we refer to (Helbing, 2001; Kerner, 2004; Nagatani, 2002).

### A. Flocking

To study the collective motion of large groups of organisms, the concept of *self-propelled particles* (SPP) has been introduced (Czirók and Vicsek, 1999; Czirók and Vicsek, 2000; Vicsek *et al.*, 1995). SPP are particles driven by an intrinsic force, produced by an energy depot which is internal to the particles, just like it happens in real organisms. In the original model (Vicsek *et al.*, 1995),  $N$  particles move on a squared surface with periodic boundary conditions. A particle  $i$  is characterized by its position  $\mathbf{x}_i$  and velocity  $\mathbf{v}_i$ . For simplicity, it is assumed that the velocities of the particles equal in modulus the constant  $v_0$  at any moment of the system's evolution. This is where the self-propelled feature of the particles sets in. The interaction is expressed by a simple rule: at each time step a particle  $i$  assumes the average direction of motion of the particles lying within a local neighborhood  $S(i)$ , with a random perturbation, i.e.

$$\theta_i(t + \Delta t) = \langle \theta(t) \rangle_{S(i)} + \xi, \quad (38)$$

where  $\theta_i$  is the angle indicating the velocity direction of particle  $i$ ,  $\Delta t$  the time step and the noise  $\xi$  is a random variable with a uniform distribution in the range  $[-\eta/2, \eta/2]$ . Noise is a realistic ingredient of the model, relaxing the strict prescription of alignment to the average direction of motion of the neighboring peers.

The model resembles the classical ferromagnetic  $XY$  model, where the velocities of the particles play the role of the spins and the noise that of temperature (Binney *et al.*, 1992). As in the  $XY$  model, spins tend to align. In the limit  $v_0 \rightarrow 0$  the SPP model resembles Monte Carlo simulations of diluted  $XY$  ferromagnets. However, when  $v_0 \neq 0$ , it is a non-equilibrium model involving transport, as particles continuously move on the plane.

Initially, the directions of the velocity vectors are randomly assigned, so that there is no organized flow of particles in the system. After some relaxation time, the system reaches a steady state. The order parameter is

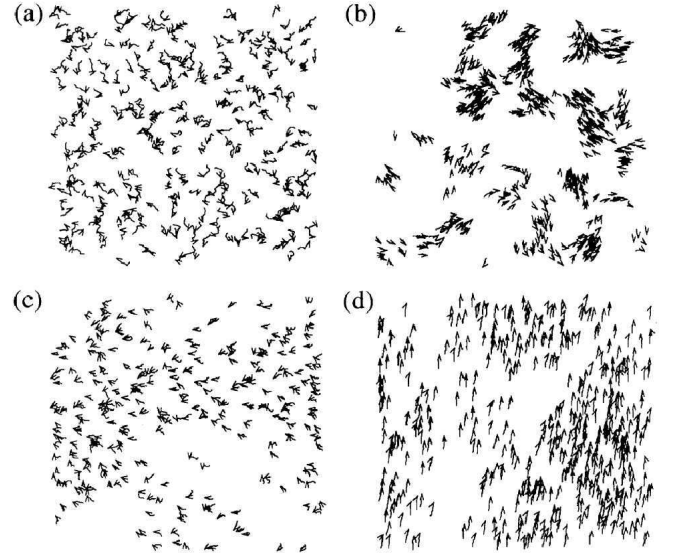


FIG. 17 Velocity fields of particles in the model introduced in (Vicsek *et al.*, 1995). A phase transition from a disordered (a) to an ordered (d) state of collective motion is observed by decreasing the amount of noise in the system. From (Vicsek *et al.*, 1995).

the magnitude  $\Phi$  of the average velocity of the system

$$\Phi = \frac{1}{N} \left| \sum_j \mathbf{v}_j \right|. \quad (39)$$

Collective motion arises when  $\Phi > 0$ . The level of noise  $\eta$  is the control parameter, just like temperature in spin models. Numerical simulations indicate that, in the thermodynamic limit of infinite system size,  $\Phi$  varies with  $\eta$  as follows

$$\Phi(\eta) \sim \begin{cases} \left[ \frac{\eta_c(\rho) - \eta}{\eta_c(\rho)} \right]^\beta & \text{for } \eta < \eta_c(\rho) \\ 0 & \text{for } \eta > \eta_c(\rho) \end{cases} \quad (40)$$

where  $\rho$  is the density of the particles. Eq. (40) indicates that the system undergoes a continuous kinetic transition from a phase, where the initial rotational symmetry of the system is preserved, to a phase where it is broken due to the emergence of collective motion (Fig. 17). The flow is generated for values of the noise lower than a threshold  $\eta_c(\rho)$ , which depends on the particle density. Like in equilibrium phase transitions, the fluctuations of the order parameter diverge at the critical threshold. Numerical estimates yield  $\beta = 0.42 \pm 0.03$ , different from the mean field value  $1/2$ . The critical line  $\eta_c(\rho) \sim \rho^\kappa$ , with  $\kappa = 0.45 \pm 0.05$ , and differs from the corresponding line for diluted ferromagnets: here, when  $\eta \rightarrow 0$ , the critical density vanishes; for diluted ferromagnets, instead, where the absence of noise corresponds to a vanishing temperature, the critical density approaches the finite percolation threshold.

These results hold for any non-vanishing value of the velocity  $v_0$ . It is remarkable that the model has an

ordering phase transition in two dimensions, as it is known that the equilibrium  $XY$  model does not order, due to the formation of Kosterlitz-Thouless vortices (Kosterlitz and Thouless, 1973). Simulations show that, for  $v_0 \approx 0$ , vortices indeed form in the first steps of the evolution, but they are unstable and disappear in the long run (Czirók *et al.*, 1997).

Similar SPP models were also introduced in one and three dimensions. In one dimension (Czirók *et al.*, 1999) the interaction rule has to be modified: since there is only one direction of motion, imposing a constant particle's velocity would yield a trivial dynamics, without contact to real situations. In a realistic dynamics, particles change direction of motion after slowing down. This can be accomplished with a simple modification of rule (38). The resulting model displays the same kinetic transition as in two dimensions, but with different exponents ( $\beta = 0.60 \pm 0.05$ ,  $\kappa \approx 1/4$ ). Collective motion emerges in three dimensions as well (Czirók *et al.*, 1999). Here the kinetic phase transition is less surprising, as in three dimensions even equilibrium continuous spin models have a ferromagnetic phase transition. The critical exponents are found to be consistent with the mean field exponents in equilibrium systems. In the limit  $v_0 \rightarrow 0$  the model becomes analogous to the three-dimensional diluted Heisenberg model (Binney *et al.*, 1992). However, for 3D SPP collective motion can be generated for any value of the particle density  $\rho$ , whereas the static model cannot order for densities lower than the percolation threshold  $\rho_{perc} \approx 1$ .

Equilibrium systems cannot display an ordering phase transition in 2D because of the Mermin-Wagner theorem (Mermin and Wagner, 1966), so the kinetic transition observed in the SPP model is due to its intrinsic dynamical, non-equilibrium character. To explain the peculiar features of the model continuum theories have been proposed (Czirók *et al.*, 1999; Toner and Tu, 1995, 1998), where the relevant variables are the velocity and density fields  $\mathbf{v}(\mathbf{x}, t)$  and  $\rho(\mathbf{x}, t)$ .

In a first approach, continuum hydrodynamic equations of motion were used (Toner and Tu, 1995, 1998). The equation terms are dictated by the strict requirement that rotational symmetry must be preserved

$$\begin{aligned} \partial_t \mathbf{v} + (\mathbf{v} \nabla) \mathbf{v} &= \alpha \mathbf{v} - \beta |\mathbf{v}|^2 \mathbf{v} - \nabla P + D_L \nabla (\nabla \mathbf{v}) \\ &\quad + D_1 \nabla^2 \mathbf{v} + D_2 (\mathbf{v} \nabla)^2 \mathbf{v} + \xi, \\ \partial_t \rho + \nabla (\rho \mathbf{v}) &= 0, \end{aligned} \quad (41)$$

where  $P$  is the analogue of pressure,  $\alpha$ ,  $\beta$  are coefficients ruling the symmetry breaking, the  $D_{L,1,2}$  are diffusion constants and  $\xi$  is uncorrelated Gaussian random noise. The first equation contains some additional terms with respect to standard Navier-Stokes equations for a simple fluid, because the SPP model does not conserve the momentum by construction, so Galilean invariance is broken. The second equation expresses the conservation of the mass. An analysis of these equations based on the dynamic renormalization group (Forster *et al.*, 1977) reveals that the system orders in 2D, and that in more than

four dimensions it behaves just like the corresponding equilibrium model, with equal exponents. On the other hand, there should be no phase transition in one dimension, in contrast to numerical findings. For this reason, a different continuum theory was derived directly from the master equation of the 1D SPP model (Czirók *et al.*, 1999). Its equations read

$$\begin{aligned} \partial_t U &= f(U) + \mu^2 \partial_x^2 U + \alpha \frac{(\partial_x U)(\partial_x \rho)}{\rho} + \zeta \\ \partial_t \rho &= -v_0 \partial_x (\rho U) + D \partial_x^2 \rho, \end{aligned} \quad (42)$$

where  $U(x, t)$  is the velocity field,  $f(U)$  a self-propulsion term,  $\zeta$  the noise. The crucial feature of Eqs. (42) is the existence of the non-linear term  $(\partial_x U)(\partial_x \rho)/\rho$ . A linear stability analysis of the equations reveals that there is an ordered phase if the noise is sufficiently low, as the domain walls separating groups of particles travelling in opposite directions are unstable. Numerical solutions of Eqs. (42) show that the continuum theory and the discrete 1D SPP model belong to the same universality class.

## B. Pedestrian behavior

Pedestrian behavior has been empirically studied since the 1950s (Hankin and Wright, 1958). The first physical modeling was proposed in (Henderson, 1971) where it was conjectured that pedestrian flows are similar to gases or fluids and measurements of pedestrian flows were compared with Navier-Stokes equations. However, realistic macroscopic models should account for effects like maneuvers to avoid collisions, for which energy and momentum are not conserved. Moreover, they should consider the “granular” structure of pedestrian flows, as each pedestrian occupies a volume that cannot be penetrated by others. Therefore, microscopic models have recently attracted much attention (Galea, 2003; Schreckenberg and Sharma, 2001). One distinguishes two main approaches: models based on cellular automata (CA) and the social force model.

In CA models of pedestrian dynamics (Blue and Adler, 1998, 2000; Burstedde *et al.*, 2001; Fukui and Ishibashi, 1999; Maniccam, 2003; Muramatsu *et al.*, 1999; Muramatsu and Nagatani, 2000), time and space are discretized. The pedestrian area is divided into cells, which can be either empty or occupied by a single agent or an obstacle. A pedestrian can move to an empty neighboring cell at each time step. The motion of a single pedestrian is a biased random walk, where the bias is represented by a field residing on the space cells, which determines the transition rates of the agent towards neighboring cells, much like it happens in chemotaxis. CA models are computationally very efficient, but they do not describe well the complex phenomenology observed in real pedestrian dynamics, mostly because of space discretization, which constrains the directions of traffic flows. Therefore, models where agents can move in continuous space are more likely

to be successful. Among them, the social force model introduced by Helbing and coworkers (Helbing, 1994; Helbing *et al.*, 2002, 2000a; Helbing and Molnár, 1995) had a big impact: the main reason is that the actual forces between agents are computed, which allows more quantitative predictions as compared to CA models.

The social force model is based on the concept that behavioral changes of individuals are driven by an external *social force*, which affects the motivation of the individual and determines its actions. Pedestrians have a particular destination and a preferred walking speed. The motion of a pedestrian is determined by its tendency to maintain its speed and direction of motion and the perturbations due to the presence of other pedestrians and physical barriers (walls).

The general equation of motion for a pedestrian  $i$  is

$$m_i \frac{d\mathbf{v}_i}{dt} = \mathbf{F}_i^{(pers)} + \mathbf{F}_i^{(soc)} + \xi_i(t), \quad (43)$$

where  $m_i$  is the mass of  $i$  and  $\mathbf{v}_i$  its velocity. The tendency to maintain the desired walking speed  $\mathbf{v}_i^{(0)}$  is expressed by the first force term  $\mathbf{F}_i^{(pers)}$ , which reads

$$\mathbf{F}_i^{(pers)} = m_i \frac{\mathbf{v}_i^{(0)} - \mathbf{v}_i}{\tau_i}, \quad (44)$$

where  $\tau_i$  is the reaction time of  $i$ . The social force  $\mathbf{F}_i^{(soc)}$ , which describes the influence of the environment, is a superposition of two terms, expressing the interaction with other pedestrians and with barriers, respectively. The interaction with other pedestrians is described by a repulsive potential, expressing the need to avoid collisions, and by an attractive potential, expressing the tendency to come closer to persons/objects that the pedestrian finds interesting. A typical choice for the repulsive force  $\mathbf{F}_{ij}^{(rep)}$  between two pedestrians  $i$  and  $j$  is

$$\mathbf{F}_{ij}^{(rep)} = A_i \exp\left(\frac{r_{ij} - d_{ij}}{B_i}\right) \mathbf{n}_{ij}, \quad (45)$$

where  $A_i$  and  $B_i$  are constants, indicating the strength and the range of the interaction, respectively,  $r_{ij}$  is the sum of the radii of the pedestrians, which are modelled as disks,  $d_{ij}$  the distance between their centers of mass and  $\mathbf{n}_{ij}$  the normalized vector pointing from  $j$  to  $i$ . Pedestrians try to keep a certain distance from walls/obstacles as well, and the repulsion is often described by the same term as in Eq. (45), where one considers the distance of  $i$  from the nearest point of the walls. The noise  $\xi_i(t)$  is added to account for non-predictable individual behavior.

This simple framework predicts realistic scenarios, like the formation of ordered lanes of pedestrians who intend to walk in the same direction and the alternation of streams of pedestrians that try to pass a narrow door into opposite directions. The existence of lanes reduces the risk of collisions and represents a more efficient configuration for the system. On the other hand, this is a spontaneously emerging property of the system, as the agents

are not explicitly instructed by the model to behave this way. The repulsion between pedestrians moving towards each other implies that the pedestrians shift a little aside to avoid the collision: in this way small groups of people moving in the same direction are formed. These groups are stable, due to the minimal interactions between people of each group, and attract other pedestrians who are moving in the same direction.

In (Helbing *et al.*, 2000b) it is shown that a nontrivial non-equilibrium phase transition is induced by noise: a system of particles driven in opposite directions inside a two-dimensional periodic strip can get jammed in crystallized configurations if the level of noise exceeds some critical threshold (*freezing by heating*), in contrast to the expectation that more noise corresponds to more disorder in the system. This can explain how jams can arise in situations of great collective excitation, like panic. Surprisingly, the crystallized state has a higher energy than the disordered state corresponding to particles flowing along the corridor, so it is metastable.

The model introduced in (Helbing *et al.*, 2000b) has been adapted to simulate situations in which people inside a room are reached by a sudden alarming information (e.g. fire alarm) and try to run away through one of the exits (escape panic) (Helbing *et al.*, 2000a). Additional force terms are considered to account for realistic features of panicking crowds, like the impossibility of excessive body compression and of tangential motion of people about each other and along the walls. The model describes phenomena observed in real panic situations: for example, people attempting to leave a room through a single narrow exit, generate intermittent clogging of the exit, so that people are unable to flow continuously out of the room, but groups of individuals of various sizes escape in an irregular succession (Fig. 18a). This bursty behavior was actually observed in an empirical study on mice attempting to exit out of a water pool (Saloma *et al.*, 2003). Moreover, due to the friction of people in contact, the time to empty the room is minimal in correspondence to some optimal value of the individual speed: for higher speeds, the total escape time increases (*faster is slower effect*). Placing columns near the exits improves the situation, even if it seems against intuition. Another situation deals with people trying to escape from a smoky room, i.e., a room whose exits are not visible unless one happens to stand close to them (Fig. 18b). In this case, the agents do not have a preferential direction of motion, as they have first to find the exits. The question is whether it is more effective for the individuals to act on their own or to rely on the action of the people close to them. The process is modeled by introducing a panic parameter, that expresses the relative importance of independent action and herding behavior [where herding is simulated by a term analogous to the alignment rule in (Vicsek *et al.*, 1995)]. It turns out that the optimal chances of survival are attained when each individual adopts a mixed strategy, based both on personal initiative and on herding. In fact, through individ-



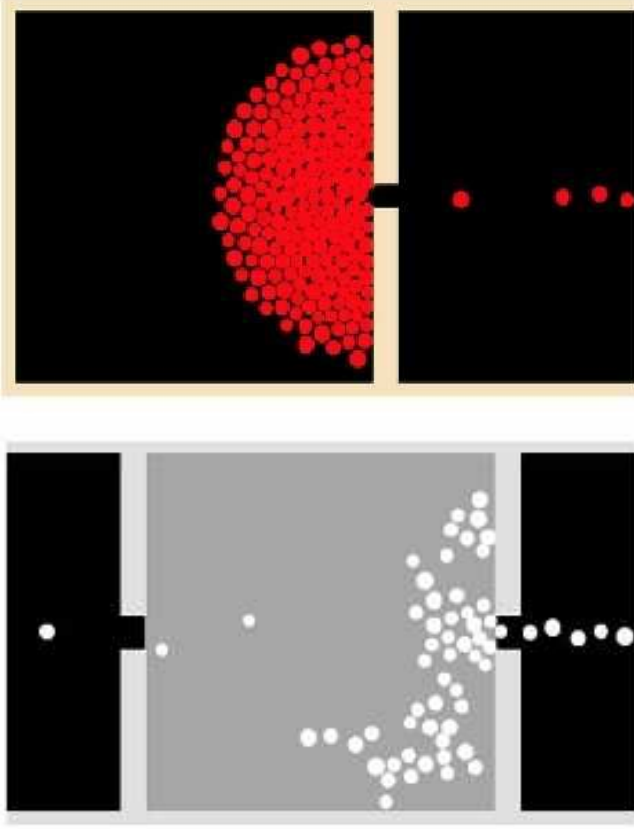


FIG. 18 Panic behavior. (Top) Escape from a room with a single exit. The exit is clogged by the people, who can leave the room only from time to time in bunches of individuals. (Bottom) Escape from a smoky room. The time to empty the room is minimal if people maintain their self-control and look at what the others are doing. Adapted from (Helbing *et al.*, 2000a).

ualistic behavior some lucky ones find quickly the exits and are followed by the others because of imitation. A detailed account of evacuation dynamics can be found in the recent review (Schadschneider *et al.*, 2008).

Other studies concern the statistical features of mobility patterns of individuals in physical space. In (Brockmann *et al.*, 2006) the scaling properties of human travels have been investigated by tracking the worldwide dispersal of bank notes through bill-tracking websites. It turns out that the distribution of traveling distances decays algebraically, and is well reproduced within a two-parameter continuous time random walk model. Moreover also in this case a power-law distribution for the interevent times between two consecutive sighting of the banknotes has been reported. These studies highlight the importance of the web as a platform for social oriented experiments (see also Sect. VIII). Very recently the mobility patterns of individuals have been investigated by tracking the geographical location of mobile phone users (Gonzalez *et al.*, 2008). Also in this case

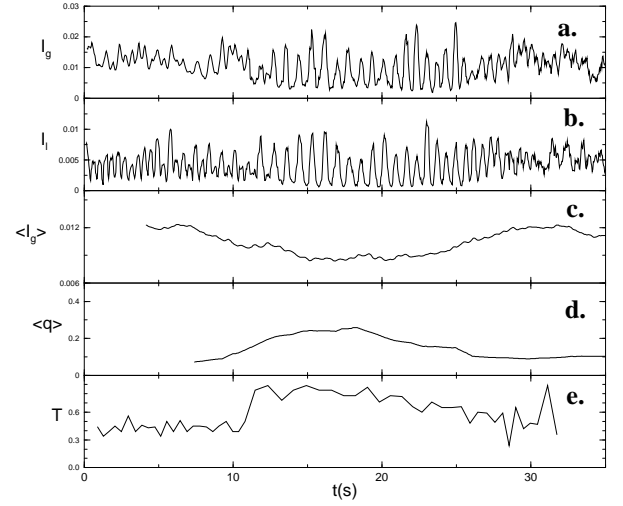


FIG. 19 Applause dynamics. Time evolution of the sound intensity of applauses in theaters and concert halls. The rhythmic applause occurs in the time window between 12s and 25s. a) Global noise intensity, measured by a microphone (averaged over a narrow time window of 0.2s). b) Local noise intensity. c) Average noise intensity over a moving time window of 3s. d) Order parameter  $q$ . e) Clapping period. Adapted from (Néda *et al.*, 2000).

the distribution of displacements over all users is well approximated by a truncated power-law and analyzed in terms of truncated Lévi flights, i.e. random walks with power-law distributed step sizes.

### C. Applause dynamics

Applause represents another remarkable example of social self-organization. At the end of a good performance the audience, after an initial uncoordinated phase, often produces a synchronized clapping, where everybody claps at the same time and with the same frequency. Synchronization occurs in many biological and sociological processes (Strogatz and Stewart, 1993; Strogatz, 1994), from the flashing of Southeast Asian fireflies to the chirping of crickets, from oscillating chemical reactions to menstrual cycles of women living together for long times.

Rhythmic applauses have been explored in detail, both empirically and theoretically. In the first pioneering investigations (Néda *et al.*, 2000; Néda *et al.*, 2000) applauses were recorded after several good theater and opera performances, with microphones placed both at some distance from the audience and close to randomly selected spectators. The intensity of noise, heard at some distance from the audience or by one of the spectators (Fig. 19a, b) shows that the signal becomes periodic during the phase of the rhythmic applause, and that the average intensity of the sound decreases (Fig. 19c). Spectators usually start with a high frequency of clapping, which reduces to approximately a half in the synchro-



nized phase (Fig. 19e). Moreover, an order parameter for synchronization was introduced and computed, i.e. the maximum of the normalized correlation between the signal  $s(t)$  and a harmonic function

$$r(t) = \max_{\{T, \Phi\}} \left[ \frac{\int_{t-T}^{t+T} s(t) \sin(2\pi/T + \Phi) dt}{\int_{t-T}^{t+T} s(t)} \right], \quad (46)$$

where  $\Phi$  and  $T$  vary in  $[0, 2\pi]$  and between two convenient values  $T_{min}$  and  $T_{max}$ , respectively. The time evolution of the order parameter  $q(t)$  clearly indicates the emergence of synchronization (Fig. 19d).

The dynamics of rhythmic applause has been explained in the framework of the Kuramoto model (Kuramoto, 1975). The audience is described by a system of non-identical  $N$  rotators. Each rotator  $i$  is characterized by its phase  $\phi_i$  and natural frequency  $\omega_i$ , which is assigned according to a probability distribution  $g(\omega)$ . The dynamics is expressed by a system of  $N$  coupled differential equations

$$\frac{d\phi_i}{dt} = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\phi_i - \phi_j), \quad (47)$$

where  $K$  is the interaction coupling. Each rotator interacts with all its peers, trying to minimize the phase difference with the other rotators. In the thermodynamic limit  $N \rightarrow \infty$  there is a critical coupling  $K_c$  such that, if  $K > K_c$ , there is a synchronized phase in which the rotators are partially aligned. If the distribution of natural frequencies  $g(\omega)$  is Gaussian, with dispersion  $D$ , one finds that  $K_c = D\sqrt{2/\pi^3}$ , i.e. the critical coupling is proportional to the dispersion of the natural frequencies. So, if  $D$  is small, synchronization is likely to occur. The clapping frequency of the rhythmic applause is indeed small, and so is its dispersion, as confirmed by experiments performed on individual spectators (Néda *et al.*, 2000). On the other hand, the frequencies of the enthusiastic clapping at the beginning of the applause have a much higher dispersion, which hinders synchronization.

From Fig. 19 we see that the phase of rhythmic applause is not stable, as it disappears after a while, although it may be reached again at a later time. This is due to the fact that the noise intensity decreases during the rhythmic applause, because of the lower clapping frequency (Fig. 19c). Spectators may find that the reduced intensity of the sound does not adequately represent their appreciation of the performance and decide to increase the clapping frequency in order to produce more noise. However, this brings the system again in the phase of enthusiastic clapping, and synchronization is lost.

In a more realistic model, spectators are represented as two-mode stochastic oscillators, and are only driven by the goal of producing some desired global level of noise, with or without synchronization (Néda *et al.*, 2003).

#### D. Mexican wave

We conclude with another striking example of coherent collective motion, i.e., the Mexican wave, also called La Ola, which is the wave created by spectators in football stadia when they rapidly leap from the seats with their arms up and successively sit down while a neighboring section of people starts the same sequence. In (Farkas *et al.*, 2002; Farkas and Vicsek, 2006) an empirical study of this peculiar phenomenon has been reported and simple models to describe it proposed. These models were inspired by the literature on excitable media (Bub *et al.*, 2002; Greenberg and Hastings, 1978), where each unit of the system can switch from an inactive to an active state if the density of active units in their neighborhood exceeds a critical threshold. The influence of a neighbor on an excitable subject decreases with its distance from the subject and is higher if the neighbor sits on the side where the wave comes from. The total influence of the neighbors is compared with the activation threshold of the spectator, which is uniformly distributed in some range of values. It turns out that a group of spectators must exceed a critical mass in order to initiate the process. The models are able to reproduce size, form, velocity and stability of real waves.

### VII. FORMATION OF HIERARCHIES

Hierarchical organization is a peculiar feature of many animal species, from insects to fishes, from birds to mammals, including humans (Allee, 1942; Chase, 1980; Guhl, 1968; Wilson, 1971). Individuals usually have a well defined rank inside their group, and the rank essentially determines their role in the community. Highly-ranked individuals have easier access to resources, they have better chances to reproduce, etc.. Hierarchies also allow for an efficient distribution of different tasks within a society, leading to a specialization of the individuals.

The origin of hierarchical structures in animal and human societies is still an open issue and has stimulated a lot of activity in the past decades. The problem is to understand why and how from individuals with initial identical status, inequalities emerge. For instance, one wonders how, in human societies, a strongly elitarian wealth distribution could arise starting from a society where people initially own an equal share of resources. A possible explanation is that hierarchies are produced by intrinsic attributes of the individuals, e.g. differences in weight or size (for animals), and talent or charisma (for humans). However, already in 1951 (Landau, 1951a,b), it was pointed out that intrinsic factors alone could not be responsible for the hierarchies observed in animal communities, and that the interactions between individuals play a crucial role in the establishment of dominance relationships. The hypothesis that hierarchy formation is a self-organization phenomenon due to social dynamics has meanwhile become the most widespread (Chase, 1982;

Chase *et al.*, 2002; Francis, 1988). Here we discuss the most important results in this area.

### A. The Bonabeau model

Dominance relationships seem to be determined by the outcome of fights between individuals. Laboratory experiments on various species hint at the existence of a positive feedback mechanism (Chase *et al.*, 1994; Hogeweg and Hesper, 1983; Theraulaz *et al.*, 1995), according to which individuals who won more fights have an enhanced probability to win future fights as compared to those who were less successful (winner/loser effects). Memory effects are also important: observations on cockroaches show that insects removed from a group and reinserted after some time do not regain immediately their original rank (Dugatkin *et al.*, 1994). Based on these empirical findings, Bonabeau *et al.* proposed a simple model to explain the emergence of hierarchies from an initial egalitarian society (Bonabeau *et al.*, 1995).

In the Bonabeau model, agents occupy the sites of a two-dimensional square lattice with linear dimension  $L$ . Each site can host only one agent and the density of the agents on the lattice is  $\rho$ , which is the control parameter of the system. Every agent performs a random walk on the lattice, moving to a randomly selected neighboring site at each iteration. If the site is free, the agent occupies it. If the site is hosting another agent, a fight arouses between the two, and the winner gets the right to occupy the site. In this way, if the winner is the attacking agent, the two competitors switch their positions at the end of the fight, otherwise they keep their original positions.

The outcome of the fight depends on the relative strength of the two opponents. The strength  $F$  of an agent grows with the number of fights it wins. Agent  $i$  is stronger than agent  $j$  if  $F_i > F_j$ . The fight is a stochastic process, in which the stronger agent has better chances to prevail, but it is not bound to win. The probability  $Q_{ij}$  that agent  $i$  defeats agent  $j$  is expressed by a Fermi function:

$$Q_{ij} = \frac{1}{1 + \exp\{-\eta[F_i - F_j]\}}, \quad (48)$$

where  $\eta$  is a sort of inverse temperature, measuring the level of randomness in the dynamics: for  $\eta \rightarrow \infty$ , the stronger agent always wins, for  $\eta \rightarrow 0$  both agents win with the same probability  $1/2$ . When an agent wins/loses a fight, its strength is increased/decreased by one unit. Memory effects are expressed by a relaxation process of the strengths of all agents, which decrease by a constant fraction  $\mu$  at each time step.

In an egalitarian society, all agents have equal strength. A broad distribution of strength would indicate the existence of hierarchies in the system and is reflected in the distribution of the winning probabilities  $Q_{ij}$  in the stationary state. So, the variance

$$\sigma = \langle Q_{ij}^2 \rangle - \langle Q_{ij} \rangle^2 \quad (49)$$

can be used as order parameter for the system (Sousa and Stauffer, 2000). For an egalitarian society,  $\sigma = 0$ ; a hierarchical society is characterized by a strictly positive value of the variance  $\sigma$ .

In simulations of the Bonabeau model, agents are initially distributed at random on the lattice, the strengths of all agents are usually initialized to zero (egalitarian society) and one iteration consists of one sweep over all agents, with each agent performing a diffusion/fighting step. The main result is that there is a critical density  $\rho_c(\eta, \mu)$  for the agents such that, for  $\rho < \rho_c(\eta, \mu)$ , society is egalitarian, whereas for  $\rho > \rho_c(\eta, \mu)$  a hierarchical organization is created (Bonabeau *et al.*, 1995). This is due to the interplay between competition and relaxation: at low densities, fights are rare, and the dynamics is dominated by relaxation, which keeps the individual strengths around zero; at large densities, instead, the grow of the strengths of some individuals is not counterbalanced by relaxation, and social differences emerge.

#### 1. Mean field solution

The model can be analytically solved in the mean field limit, in which spatial correlations are absent and the density  $\rho$  expresses the probability for two agents to meet and fight. The evolution equation for the strength of an agent  $F_i$  is

$$\frac{dF_i}{dt} = H_i(\{F_j\}) = -\mu F_i + \frac{\rho}{N} \sum_{j=1}^N \frac{\sinh \eta(F_i - F_j)}{1 + \cosh \eta(F_i - F_j)}, \quad (50)$$

where  $N$  is the number of agents. From Eq. (50) one derives that the mean strength  $\langle F \rangle$  decays exponentially to zero, so the stationary states are all characterized by  $\langle F \rangle = 0$ . To check for the stability of the special solution  $F_i = 0, \forall i$ , corresponding to an egalitarian society, one computes the eigenvalues of the Jacobian matrix

$$H_{ij} = \frac{\partial}{\partial F_j} H_i(0, 0, \dots, 0) = \left(-\mu + \frac{\rho\eta}{2}\right) \delta_{ij} - \frac{\rho\eta}{2N}. \quad (51)$$

It turns out that there are only two different eigenvalues, namely  $-\mu < 0$  and  $\rho\eta/2 - \mu$  (in the large- $N$  limit), which is negative only when  $\rho < \rho_c(\eta, \mu) = 2\mu/\eta$ . We conclude that, for  $\rho > \rho_c(\eta, \mu)$ , the egalitarian solution is linearly unstable, and a hierarchy emerges. For  $\eta < 2$  the hierarchy transition is discontinuous, because a subcritical bifurcation occurs at  $\rho_c(\eta, \mu)$ . In this case there are several metastable states, along with the egalitarian one, so it is possible to observe hierarchical structure even for  $\rho < \rho_c(\eta, \mu)$ . Indeed, simulations reveal that the stationary state is very sensitive to the choice of the initial conditions (Bonabeau *et al.*, 1995). For  $\eta \geq 2$ , the bifurcation is critical, so the transition is continuous and characterized by critical exponents and critical slowing down.

## 2. Modifications of the Bonabeau model

The most popular modification of the Bonabeau model (Stauffer, 2003b) consists in replacing  $\eta$  in Eq. (48) with the standard deviation  $\sigma$  of Eq. (49). By doing so, the probabilities  $\{Q_{ij}\}$  are calculated using the variance of their distribution, which changes in time, so there is a feedback mechanism between the running hierarchical structure of society and the dominance relationships between agents.

For this model, analytical work in the mean field limit (Lacasa and Luque, 2006) showed that the egalitarian fixed point is stable at all densities, at odds with simulation results, that support the existence of a phase transition to a hierarchical system (Malarz *et al.*, 2006). The apparent discrepancy is due to the fact that, above a critical density, a saddle-node bifurcation takes place. Both the egalitarian and the hierarchical fixed points are stable, and represent possible endpoints of the dynamics, depending on the initial conditions. The model has been simulated on regular lattices (Stauffer, 2003b), complete graphs (Malarz *et al.*, 2006) and scale-free networks (Gallos, 2005). The phase transition holds in every case, although on scale-free networks the critical density may tend to zero in the thermodynamic limit of infinite agents. On the lattice, the model yields a society equally divided into leaders and followers, which is not realistic. If the variation of the strength is larger for a losing agent than for a winning agent, instead, the fraction of agents that turn into leaders decreases rapidly with the amount of the asymmetry (Stauffer and Sá Martins, 2003).

Some authors proposed modifications of the moving rule for the agents. In (Odagaki and Tsujiguchi, 2006; Tsujiguchi and Odagaki, 2007) two particular situations have been investigated, corresponding to what is called a timid and a challenging society, respectively. Timid agents do not look for fights, but try to move to a free neighboring site. If there are none, they pick a fight with the weakest neighbor. Two phase transitions were observed by increasing the population density: a continuous one, corresponding to the emergence of a middle class of agents, who are fairly successful, and a discontinuous one, corresponding to the birth of a class of winners, who win most of their fights. In a challenging society, agents look for fights, and choose the strongest neighbor as opponent. Hierarchies already emerge at low values of the population density; in addition, since strong agents have a big attractiveness, spatial correlations arise with the formation of small domains of agents at low and intermediate densities.

### B. The Advancement-Divide model

A simple model based on the interplay between advancement and decline, similar to the Bonabeau model, has been proposed in (Ben-Naim and Redner, 2005). Agents have an integer-valued strength, and interact

pairwise. The advancement dynamics is deterministic: the strength of the stronger competitor increases by one unit. If both agents have equal strength, both advance. The memory effect of the Bonabeau model now consists of a declining process, in that the strength of each individual decreases by one unit at rate  $r$ , as long as it is positive. The parameter  $r$  fixes the balance of advancement and decline.

The model is solvable in the mean field limit. We call  $f_j(t)$  the fraction of agents with strength  $j$  at time  $t$ , and  $F_k = \sum_{j=0}^k f_j$ . The dynamics of the cumulative distribution  $F_k$  is described by the non-linear master equation

$$\frac{dF_k}{dt} = r(F_{k+1} - F_k) + F_k(F_{k-1} - F_k), \quad (52)$$

where the first two terms express the contribution of the decline process, the other two that of advancement. At the beginning all agents have zero strength, i.e.  $f_0(0) = 1$ , corresponding to  $F_k(0) = 1, \forall k \geq 0$  (the reflecting wall at  $k = 0$  comes from imposing  $F_{-1}(t) = 0, \forall t$ ).

In the limit of continuous strength, Eq. (52) becomes a partial differential equation,

$$\frac{\partial F(k, t)}{\partial t} = [r - F(k, t)] \frac{\partial F(k, t)}{\partial k}. \quad (53)$$

In the limit of  $t \rightarrow \infty$  and finite strength, and since  $F(k, \infty) \leq 1$  by definition, Eq. (53) leaves only two possibilities:

$$\begin{aligned} F(k, \infty) &= r, & \text{for } r < 1; \\ \partial F(k, \infty)/\partial k &= 0, & \text{for } r \geq 1. \end{aligned} \quad (54)$$

In this way, for  $r \geq 1$ , all agents have finite strength in the stationary state. For  $r < 1$ , instead, we have a hierarchical society, with a fraction  $r$  of agents having finite strength (lower class), coexisting with a fraction  $1 - r$  of agents whose strength diverges with  $t$  (middle class).

A dimensional analysis of Eq. (53) reveals that the strength of the middle class agents increases linearly with  $t$ . One can then check for solutions of the form  $F_k \sim \Phi(x)$ , with  $x = k/t$  and the boundary condition  $\Phi(\infty) = 1$ . Eq. (53) becomes

$$x\Phi'(x) = (\Phi(x) - r)\Phi'(x), \quad (55)$$

whose solution is

$$\Phi(x) \sim \begin{cases} x + r, & x < 1 - r; \\ 1, & x \geq 1 - r. \end{cases} \quad (56)$$

From Eq. (56) one deduces that the maximum strength of the system is  $(1 - r)t$ , i.e. it grows linearly with time, as expected. In the limit of finite strength ( $x \rightarrow 0$ ) one has  $\Phi(0) = r$ , which recovers the result on the fraction of lower class agents.

In a later paper (Ben-Naim *et al.*, 2006b), the model has been generalized by introducing a stochastic advancement dynamics, in which the stronger competitor

of a pair of interacting agents wins with a probability  $p$ . This model yields a richer phase diagram. In some region of the parameter space, a new egalitarian class emerges, in which the strength distribution of the agents is strongly peaked and moves with constant velocity, like a traveling wave. The model has been successfully applied to describe the dynamics of sport competitions (Ben-Naim *et al.*, 2007). Moreover, it has inspired a generalization to competitive games involving more than two players at the same time (Ben-Naim *et al.*, 2006a).

## VIII. HUMAN DYNAMICS

### A. Empirical data

One of the key questions in social dynamics concerns the behavior of single individuals, namely how an individual chooses a convention, takes a decision, schedules his tasks and more generally decides to perform a given action. Most of these questions are obviously very difficult to address, due to the psychological and social factors involved. Nevertheless in the last few years several studies have tried to quantitatively address these questions, mainly relying on the availability of data through the web. A first valuable source of data have been the logs of email exchanges. In particular the structure of email networks has been first studied in (Ebel *et al.*, 2002; Newman *et al.*, 2002), focusing on the spreading of informatic viruses. The emergence of coherent, self-organized, structures in email traffic has been reported in (Eckmann *et al.*, 2004), using an information-theoretic approach based on the analysis of synchronization among triplets of users. It has been highlighted how nontrivial dynamic structures emerge as a consequence of time correlations when users act in a synchronized manner. The observed probability distribution of the response time  $\tau$  until a message is answered, features a broad distribution, roughly approximated by a  $1/\tau$  power law behavior (Fig. 20). The same kind of data have been analyzed in (Johansen, 2004) and a generalized response time distribution  $\sim 1/\tau$  for human behavior in the absence of deadlines has been suggested.

A different scaling, as far as the distribution of response times is concerned, has been observed in mail correspondence patterns (Oliveira and Barabási, 2005; Vázquez *et al.*, 2006). In the correspondence of Einstein, Darwin and Freud it has been found that the probability that a letter will be replied in  $\tau$  days is well approximated by a power-law  $\tau^{-\alpha}$ , with  $\alpha = 3/2$ .

From all these empirical data a picture emerges where human dynamics is characterized by bursts of events separated by short interevent times followed by long periods of inactivity. This burstiness (Kleinberg, 2002) can be quantified by looking for instance at the distribution of interevent times that typically displays heavy-tailed non-Poissonian statistics. A recent study (Alfi *et al.*, 2007) investigated how people react to deadlines for conference

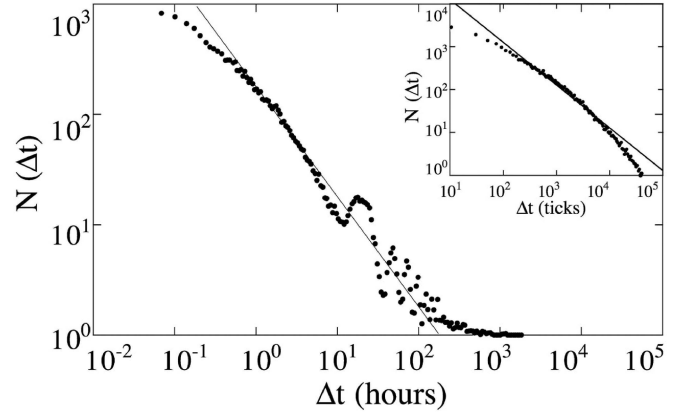


FIG. 20 Distribution of the response time until an email message is answered. (Inset) The same distribution is measured in ticks, i.e., units of messages sent in the system. Binning is logarithmic. The solid lines follow  $\Delta t^{-1}$  and are meant as guides for the eye. From (Eckmann *et al.*, 2004)

registration.

### B. Queuing models

Let us now focus on the explanation of the above mentioned phenomenology. The very same database collected and used in (Eckmann *et al.*, 2004) has been analyzed in (Barabási, 2005), where the origin of bursts and heavy tails in the probability distribution of the response time to an email has been explained as a consequence of a decision-based queuing process. The model is defined as follows. Each human agent has a list with  $L$  tasks, each task being assigned with an *a priori* priority parameter  $x_i$  (for  $i \in [1, \dots, L]$ ), chosen from a distribution  $\rho(x)$ . At each time step the agent selects the task with the highest priority with probability  $p$  and executes it, while with probability  $1 - p$  a randomly selected task is performed. The executed task is then removed from the list and replaced with another one with priority again randomly extracted from  $\rho(x)$ . Computer simulations of the model showed that for the deterministic protocol ( $p \rightarrow 1$ ) the probability distribution of the times spent by the tasks on the list features a power-law tail with exponent  $\alpha = 1$ .

The exact solution of the Barabási model for two tasks ( $L = 2$ ) (Vázquez, 2005) confirmed the  $1/\tau$  behavior with an exponential cut-off over a characteristic time  $(\ln(2/(1+p)))^{-1}$ , which diverges for  $p \rightarrow 1$ . The limit  $p \rightarrow 1$  is subtle because not only the characteristic time diverges but the prefactor vanishes.

This issue has been addressed in (Gabrielli and Caldarelli, 2007), where an exact probabilistic description of the Barabási model for  $L = 2$  has been given in the extremal limit,  $p = 1$ , i.e., when only the most recently added task is executed. In this limit the Barabási model can be exactly mapped into Invasion Percolation (IP) (Wilkinson and Willemsen,

1983) in  $d = 1$ . Using the so-called run time statistics (RTS) (Marsili, 1994), originally introduced to study the IP problem in  $d = 2$ , it has been found that the exact waiting time distribution for a task scales as  $\tau^{-2}$ , unlike the result  $1/\tau$  found in (Vázquez, 2005), which is valid for the stationary state when  $0 < p < 1$ . This behavior disappears in the limit  $p \rightarrow 1$ , since the prefactor vanishes. In summary, for  $0 < p < 1$  finite time deviations from stationarity relax exponentially fast and the dynamics is well described by the stationary state. On the other hand, for  $p \rightarrow 1$  the stationary state becomes trivial and finite time deviations relax so slowly that the task list dynamics has to be described as an intrinsically nonstationary dynamics.

In (Vázquez *et al.*, 2006) the case where limitations on the number of tasks an individual can handle at any time is discussed (Cobham, 1954). Here tasks are, for instance, letters to be replied. One assumes that letters arrive at rate  $\lambda$  following a Poisson process with exponential arrival time distribution, while responses are written at rate  $\mu$ . Each letter is assigned a discrete priority parameter  $x = 1, 2, \dots, r$  upon arrival and the unanswered letter with highest priority will be always chosen for a reply. This model predicts a power law behavior for the waiting time distribution of the individual tasks, with an exponent equal to  $3/2$ , with an exponential cut-off characterized by a characteristic time  $\tau_0$  given by  $\tau_0 = 1/(\mu(1 - \sqrt{\rho})^2)$ , where  $\rho = \lambda/\mu$ . The ratio  $\rho$  is called traffic intensity and it represents the natural control parameter for the model. For  $\rho = 1$  one has a critical regime where the queue length performs a  $1 - d$  random walk with a lower bound at zero. This allows to explain the origin of the exponent  $3/2$  in terms of the exponent of the return time probability for a random walker.

Conditions for the emergence of scaling in the inter-event time distribution have been addressed in (Hidalgo, 2006). A generalization of the queuing model introducing a time-dependent priority for the tasks in the list (aging) has been discussed in (Blanchard and Hongler, 2006). A further generalization of the queue model with continuous valued priorities has been introduced and studied in (Grinstein and Linsker, 2006). In this case a new task is added to the queue with probability  $\lambda$  and assigned a priority  $x$  ( $0 \leq x \leq 1$ ) chosen from a probability distribution  $\rho(x)$ . With probability  $\mu$  the highest priority task is executed. Two asymptotic waiting time distributions have been found analytically. For  $\lambda = \mu < 1$  one obtains a power law (with exponent  $3/2$ ) for the waiting time distribution. For  $\lambda < \mu < 1$  a characteristic time  $\tau_0 = (\mu - \lambda)^2/(4\mu(1 - \lambda)) > 0$  emerges such that, for  $\tau \ll \tau_0$ , the waiting time distribution goes as  $P(\tau) \sim e^{-\tau/\tau_0} \tau^{-5/2}$ , while for  $1 \ll \tau \ll \tau_0$   $P(\tau) \sim \tau^{-3/2}$ . The asymptotic behaviour of  $P(\tau)$  changes with the introduction of a cost for switching between different classes of tasks (Grinstein and Linsker, 2006).

Finally, in (Bedogne and Rodgers, 2007) a continuous model has been introduced, which adopts an infinite queuing list where each element is assigned a real

non-negative number representing its priority of execution. The priorities are drawn from a given probability distribution function  $\rho(x)$ . At each time step a task is selected with a selection probability function, which is an increasing function of the priority:

$$\Pi(x, t) = \frac{x^\gamma P(x, t)}{\int_0^\infty x^\gamma P(x, t) dx}, \quad (57)$$

where  $\gamma$  is a non-negative real number and  $P(x, t)$  is the so-called waiting room priority distribution. The resulting waiting time distribution  $P(\tau)$  can be analytically found and it depends explicitly on the priority distribution density function  $\rho(x)$ . The scaling  $P(\tau) \sim 1/\tau$  is recovered when  $\rho(x)$  is exponentially distributed.

### C. The social web

Recently, a new paradigm where human dynamics plays an important role has been quickly gaining ground on the World Wide Web: Collaborative Tagging (Cattuto *et al.*, 2007; Golder and Huberman, 2006). In web applications like *del.icio.us* (<http://del.icio.us>), *Flickr* (<http://www.flickr.com>), *CiteULike* (<http://www.citeulike.org>), users manage, share and browse collections of online resources by enriching them with semantically meaningful information in the form of freely chosen text labels (*tags*). The paradigm of collaborative tagging has been successfully deployed in web applications designed to organize and share diverse online resources such as bookmarks, digital photographs, academic papers, music and more. Web users interact with a collaborative tagging system by posting content (*resources*) into the system, and associating text strings (*tags*) with that content. Fig. 21 reports the frequency-rank distributions for the tags co-occurring with a few selected ones. The high-rank tail of the experimental curves displays a power-law behavior, signature of an emergent hierarchical structure, corresponding to a generalized Zipf's law (Zipf, 1949) with an exponent between 1 and 2. Since power laws are the standard signature of self-organization and of human activity (Barabási, 2005; Mitzenmacher, 2003; Newman, 2005), the presence of a power-law tail is not surprising. The observed value of the exponent, however, deserves further investigation, because the mechanisms usually invoked to explain Zipf's law and its generalizations (Zanette and Montemurro, 2005) do not look very realistic for the case at hand, and a mechanism grounded on experimental data should be sought. Moreover, the low-rank part of the frequency-rank curves exhibits a flattening typically not observed in systems strictly obeying Zipf's law. Several aspects of the underlying complex dynamics may be responsible for this feature: on the one hand this behavior points to the existence of semantically equivalent and possibly

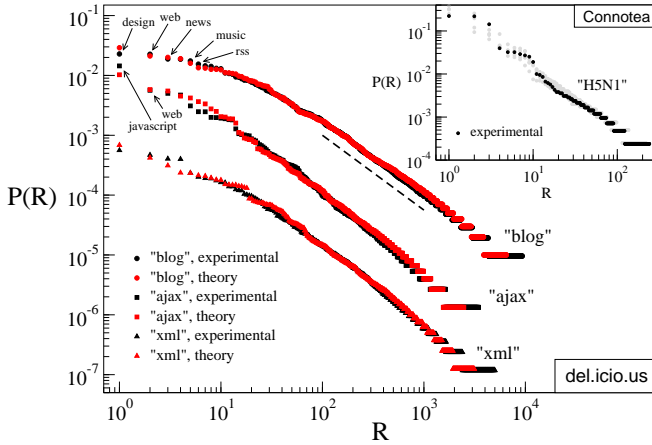


FIG. 21 Frequency-rank plots for tags co-occurring with a selected tag. Experimental data (black symbols) are shown for *del.icio.us* (circles for tags co-occurring with the popular tag *blog*, squares for *ajax* and triangles for *xml*). For the sake of clarity, the curves for *ajax* and *xml* are shifted down by one and two decades, respectively. All curves exhibit a power-law decay for high ranks (a dashed line corresponding to the power law  $R^{-5/4}$  is provided as an aid for eye) and a shallower behavior for low ranks. Gray (red online) symbols are theoretical data obtained by computer simulations of the stochastic process described in (Cattuto *et al.*, 2007, 2006). (Inset) the same graph for the much younger system *Connotea*. From (Cattuto *et al.*, 2007).

competing high-frequency tags (e.g. *blog* and *blogs*). More importantly, this flattening behavior may be ascribed to an underlying hierarchical organization of tags co-occurring with the one we single out: more general tags (semantically speaking) will tend to co-occur with a larger number of other tags.

At the global level the set of tags, though determined with no explicit coordination, evolves in time and leads towards patterns of terminology usage. Hence one observes the emergence of a loose categorization system that can be effectively used to navigate through a large and heterogeneous body of resources. It is interesting to investigate the way in which users interact with those systems. Also for this system a hyperbolic law for the user access to the system has been observed (Cattuto *et al.*, 2007). In particular if one looks at the temporal auto-correlation function for the sequence of tags co-occurring with a given tag (e.g. *blog*), one observes a  $1/(t + \tau)$  behavior, which suggests a heavy-tailed access to the past state of the system, i.e., a memory kernel for the user access to the system. On this basis a stochastic model of user behavior (Cattuto *et al.*, 2007) has been proposed, embodying two main aspects of collaborative tagging: (i) a frequency-bias mechanism related to the idea that users are exposed to each other's tagging activity; (ii) a notion of memory (or aging of resources) in the form of a heavy-tailed access to the past state of the system. Remarkably, this simple scheme is able to account quantitatively for the observed experimental features, with a surprisingly

high accuracy (see Fig. 21). This points to the direction of a universal behavior of users, who, despite the complexity of their own cognitive processes and the uncoordinated and selfish nature of their tagging activity, appear to follow simple activity patterns.

The dynamics of information access on the web represents another source of data and several experiments have been performed in the last few years. In (Johansen, 2001; Johansen and Sornette, 2000) the dynamic response of the *internauts* to a point-like perturbation as the announcement of a web interview on stock market crashes has been investigated. In (Chessa and Murre, 2004, 2006) a cognitive model, based on the mathematical theory of point processes, has been proposed, which extends the results of (Johansen, 2001; Johansen and Sornette, 2000) to download relaxation dynamics. In (Dezsö *et al.*, 2006) the visitation patterns of news documents on a web portal have been considered. The problem of collective attention has been recently addressed in (Wu and Huberman, 2007) in the framework of the website *Digg.com*. *Digg* allows users to submit news stories they discover from the Internet. A new submission immediately appears on a repository webpage called *Upcoming Stories*, where other members can find the story and, if they like it, add a *digg* to it. A so-called *digg* number is shown next to each story's headline, which simply counts how many users have *digged* the story in the past. The distribution of *diggs* accumulated by the news stories has been found to be well approximated by a log-normal curve. This behaviour has been explained in terms of a simple stochastic model where the number of *diggs* acquired by a news story at time  $t$  can be expressed as:

$$N_t = (1 + r_t X_t) N_{t-1}, \quad (58)$$

where  $X_1, X_2, \dots, X_t$  are positive i.i.d. random variables with a given mean and variance. The parameter  $r_t$  is a time-dependent novelty factor consisting of a series of decreasing positive numbers with the property that  $r_1 = 1$  and  $r_t \rightarrow 0$  as  $t \rightarrow \infty$ . Experimentally it has been found that  $r_t$  follows a stretched-exponential relation with a characteristic time of about one hour.

## IX. SOCIAL SPREADING PHENOMENA

Opinion dynamics deals with the competition between different possible responses to the same political question/issue. A key feature is that the alternatives have the same or at least comparable levels of plausibility, so that in the interaction between two agents each of them can in principle influence the other. In phenomena like the propagation of rumors or news, the interaction is intrinsically asymmetric: possible states are very different in nature. The flow is only from those who know to those who do not. The propagation of rumors or news

is an instance of the vast class of social spreading phenomena, that includes the diffusion of fads, the adoption of technological innovations, the success of consumer products mediated by word-of-mouth. The investigation of such types of dynamics has a long tradition in sociology and economics (Bass, 1969; Bikhchandani *et al.*, 1992; Rogers, 2003). Work along this line has been performed recently by statistical physicists on the diffusion of innovations (Guardiola *et al.*, 2002; Llas *et al.*, 2003), the occurrence of information cascades in social and economic systems (Centola *et al.*, 2007; Watts, 2002), disaster spreading in infrastructures (Buzna *et al.*, 2006), the emergence of “hits” in markets (Hidalgo *et al.*, 2006; Sinha and Pan, 2006). Many of the models introduced for these phenomena assume that a local threshold in the fraction of active neighbors must be overcome for the spreading process to occur. Here we will, somewhat arbitrarily, review only the activity on the problem of rumor spreading. This phenomenon is modeled without a threshold: hearing a rumor from a single neighbor is generally enough for a single agent to start spreading it. It is then clear that rumor spreading bears a lot of resemblance with the evolution of an epidemics, with informed people playing the role of infected agents and uninformed of susceptible ones (Goffman and Newell, 1964; Rapoport, 1953). Obviously there are crucial qualitative differences: rumor/news spreading is intentional; it usually involves an (at least perceived) advantage for the receiver, etc. However most of such differences lie in the interpretation of parameters; the analogy is strong and the field is usually seen as closer to epidemiology than to opinion dynamics.

When considering rumor spreading some of the relevant questions to address are similar to those for epidemiology: How many people will eventually be reached by the news? Is there an ‘epidemic threshold’ for the rate of spreading, separating a regime where a finite fraction of people will be informed from one with the info remaining confined to a small neighborhood? What is the detailed temporal evolution? Other issues, more connected to technological applications, deal with the cost of the spreading process and its efficiency. Rumor dynamics has also appealing connections with the search for robust scalable communication protocols in large distributed systems (Kermarrec *et al.*, 2003; Vogels *et al.*, 2003) and “viral” strategies in marketing (Leskovec *et al.*, 2006).

Detailed applications of the common models for epidemics to the investigation of empirical data on the dissemination of ideas exist (Bettencourt *et al.*, 2006; Goffman, 1966), but the most popular model for rumor spreading, introduced in (Daley and Kendall, 1964) (DK), has an important difference. As in the SIR model for epidemiology (Anderson and May, 1991), agents are divided in three classes: ignorants, spreaders and stiflers, i.e., those who have lost interest in diffusing the information/rumor. Their role is exactly the same of the susceptible, infected, recovered agents of the SIR model. The only difference is that while for an epidemics infected (I)

people become recovered or removed (R) *spontaneously* with a certain rate, typically people stop propagating a rumor when they realize that those they want to inform are already informed. Hence the transition to state R is proportional to the density of spreaders  $s(t)$  in the SIR model, while it is proportional to  $s(t)[s(t) + r(t)]$  in the DK model, where  $r(t)$  is the density of stiflers.

The DK model has been studied analytically in the case of homogeneous mixing, revealing that there is no threshold: for any rate  $\lambda$  of the spreading process a finite fraction  $r_\infty$  of people would be informed (Sudbury, 1985), given by the solution of

$$r_\infty = 1 - e^{-(1+\lambda/\alpha)r_\infty}, \quad (59)$$

where  $\alpha$  is the proportionality constant of the transition rate to state R. The same result holds for the very similar Maki-Thompson model (Maki and Thompson, 1973). Hence the nonlinear transition rate removes the threshold of the SIR model. Clearly when both mechanisms for the damping of the propagation are present (self-recovery and the nonlinear DK mechanism) a threshold is recovered, since the linear term prevails for small  $s$  (Nekovee *et al.*, 2007).

In the context of statistical physics the focus has been on the behavior of the DK model on complex networks (Liu *et al.*, 2003; Moreno *et al.*, 2004a,b). When scale-free networks are considered, the fraction  $r_\infty$  of people reached decreases compared to homogeneous nets. This occurs because hubs tend to become stiflers soon and hence hamper the propagation. However, if one considers the efficiency  $E$  of the spreading process, defined as the ratio between  $r_\infty$  and the total traffic  $L$  generated, it is found that for any value of the parameters scale-free networks are more efficient than homogeneous ones, and in a broad range of parameters they are more efficient than the trivial broadcast spreading mechanism (i.e., each node transmits the message to all its neighbors).

A remarkable phenomenon occurs when the DK dynamics takes place on the small-world Watts-Strogatz (WS) network (Zanette, 2001, 2002). In this case there is an ‘epidemic’ transition depending on the rewiring parameter  $p$ . For  $p > p_c$  the rumor propagates to a finite fraction  $r_\infty$  of the network sites. For  $p < p_c$  instead the rumor remains localized around its origin, so  $r_\infty$  vanishes in the thermodynamic limit. Notice that  $p_c$  is finite for  $N \rightarrow \infty$ , at odds with the geometric threshold characterizing the small-world properties of WS networks, that vanishes in the limit of infinite network. Hence the transition is dynamic in nature and cannot be ascribed to a pure geometric effect.

The diffusion of corruption has also been modeled as an epidemic-like process, with people accepting or practicing a corrupt behavior corresponding to infected individuals. The main difference with respect to usual epidemiological models is that the chance of an individual to become corrupt is a strongly nonlinear function of the number of corrupt neighbors. Other modifications include global



coupling terms, modeling the process of people getting corrupt because of a perceived high prevalence of corruption in the society and the response of the society as a whole, which is proportional to the fraction of non-corrupt people. The resulting phenomenology is quite rich (Blanchard *et al.*, 2005).

It is also worth mentioning the model of Dodds and Watts for social contagion (Dodds and Watts, 2004), that introduces memory in the basic SIR model thus taking into account the effect of repeated exposure to a rumor. Also information spreading in a population of diffusing agents (Agliari *et al.*, 2006) has been studied.

Finally, some activity has been devoted to the related problem of gossip spreading. While rumors are about some topic of general interest so that they may potentially extend to all, gossip is the spreading of a rumor about some person and hence it is by definition a local phenomenon; it may concern only people close to the subject in the social network. If only nearest neighbors of the subject can spread, the fraction of them reached by the gossip exhibits a minimum as a function of the degree  $k$  for some empirical and model social networks. Hence there is an ideal number of connections to minimize the gossip propagation (Lind *et al.*, 2007a,b).

## X. COEVOLUTION OF STATES AND TOPOLOGY

All models considered in the previous sections are defined on static substrates: the interaction pattern is fixed and only opinions, not connections, are allowed to change. The opposite case is often considered in many studies of network formation: vertices are endowed with quenched attributes and links are formed or removed depending on such fixed node properties.

In fact, real systems are mostly in between these two extreme cases: both intrinsic properties of nodes (like opinions) and connections among them vary in time over comparable temporal scales. The interplay of the two evolutions is then a natural issue to be investigated. More interestingly, in many cases the two evolutions are explicitly coupled: if an agent finds that one of his contacts is too different, he tends to sever the connection and look for other interaction partners more akin to his own properties.

The investigation of the coevolution of networks and states has recently attracted interest in several contexts, including self-organization in Boolean networks (Bornholdt and Rohlf, 2000), synchronization (Ito and Kaneko, 2001, 2003) and game-theoretic approaches (Eguíluz and Zimmermann, 2000; Ehrhardt *et al.*, 2006; Zimmermann *et al.*, 2001; Zimmermann *et al.*, 2004). A review about these developments is (Gross and Blasius, 2008), that also contains a useful attempt to classify generic patterns of behavior in the field. For opinion and cultural dynamics the study of adaptive networks is still at the beginning, but it promises to be a very active field in the next years.

In coevolving systems, three ingredients are to be specified. The dynamics taking place on nodes usually belongs to standard classes of opinion or cultural dynamics (voter-like, majority-rule, continuous opinion, etc.). Link dynamics can be of many types. Examples exist of prescriptions allowing only link deletion or independent link addition or removal. A more common, simple yet quite realistic, assumption is that links are rewired: an agent unhappy about one connection cuts it and forms a new link with another agent. In this way the average connectivity  $\langle k \rangle$  is conserved. The third ingredient is the relative rate of link or node updates, generally modeled by a probability  $\phi$ : a node is chosen at random; if it is equal to one of its neighbors (or all, depending on the node dynamics) nothing happens. Otherwise with probability  $\phi$  it rewires one of its connections; with probability  $1 - \phi$  it updates its state leaving the topology unchanged.

If node dynamics leads toward consensus, it is reasonable to expect that for small  $\phi$  there will be asymptotically a single network with all nodes in the same state. In the opposite limit  $\phi \rightarrow 1$  instead, rewiring will quickly split the network in disconnected components that will separately reach a different consensus state. These two limit behaviors are hence separated by a topological phase transition for some critical value  $\phi_c$ . Such type of transition was first reported in (Holme and Newman, 2006), where Potts variables assuming  $G$  different values are defined on the nodes of a network.  $G$  is proportional to the number of vertices, so that  $\gamma = G/N$  is constant. At each time step a node and a neighbor are selected and with probability  $1 - \phi$  the node picks the opinion of the neighbor. With probability  $\phi$  instead, the node rewires the link to a new vertex chosen randomly among those having its same opinion. Dynamics continues up to complete separation in components, within which there is full consensus. For large  $\phi$  only rewiring is allowed and the resulting topological clusters trivially coincide with the sets of initial holders of individual opinions. The distribution of their sizes is multinomial. For small  $\phi$  practically only opinion changes are allowed and the final clusters are the components of the initial graph. The phase transition occurring at  $\phi_c$  is characterized by a power-law distribution of cluster sizes, with an exponent about  $-3.5$  (Fig. 22), which differs from the one at the threshold of the giant component formation in random graphs (mean field percolation class). Finite size scaling gives additional exponents that are universal (do not depend on  $\langle k \rangle$  and  $\gamma$ ) and different from percolation. Critical slowing down is also observed.

A similar dynamics, but for a binary state variable, is investigated in (Vazquez *et al.*, 2008). A master equation approach gives an absorbing phase-transition between an active phase for  $\phi < \phi_c = (\langle k \rangle - 2)/(\langle k \rangle - 1)$ , and a frozen phase with zero active links for  $\phi > \phi_c$ . In finite systems, numerical simulations show that in the absorbing phase the network is composed of two disconnected components, one with all spins up and the other with all spins down, while the active phase corresponds to all

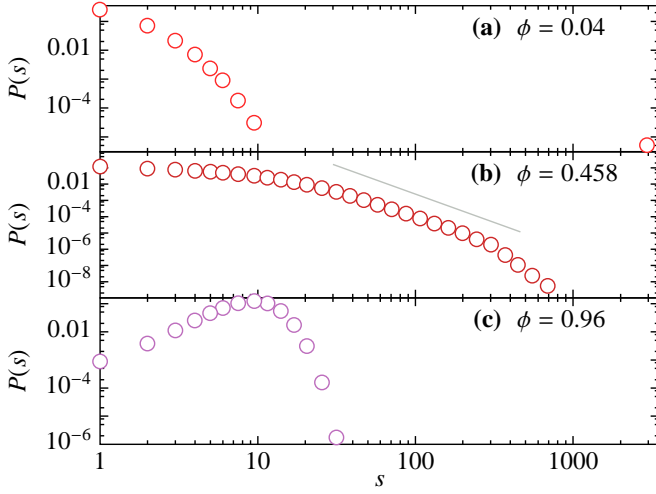


FIG. 22 Coevolution of opinions and networks. Histograms of cluster sizes in the consensus state for values of  $\phi$  above, at and below the critical point in panels (a), (b) and (c), respectively. In panel (b) the distribution appears to follow a power law for part of its range with exponent  $-3.5 \pm 0.3$ , as indicated by the solid line. From (Holme and Newman, 2006).

nodes belonging to the same giant component.

Nardini et al. (Nardini *et al.*, 2007) investigate a very similar model, with only the slight difference that links are rewired to nodes selected completely at random, with no preference for similar vertices holding the same opinion. They focus on the approach to consensus in the limit of small rewiring rate, finding opposite behaviors if the selected node picks the opinion of the neighbor (direct voter dynamics) or the neighbor is convinced by the node (reverse voter). In the first case the consensus time becomes logarithmic in the system size  $N$ ; in the second it diverges exponentially with  $N$ . A mean field approach clarifies the origin of the difference and also allows to understand why the consensus time remains proportional to  $N$  for the AB-model, independently of the direct or reverse update prescription.

Differently from the models just discussed, in the first papers on opinion dynamics in adaptive networks (Gil and Zanette, 2006; Zanette and Gil, 2006) links can only be deleted. In the initial configuration, binary opinions are distributed randomly on the nodes of a fully connected network. If two neighbors disagree, one of them is set equal to the other with probability  $p_1$  (voter dynamics). With probability  $(1 - p_1)p_2$  instead they get disconnected. Iterating this dynamics a stationary state is reached depending only on the combination  $q = p_1/[p_1 + (1 - p_1)p_2]$ . For small  $q$  the system breaks down in two communities of similar size and opposite opinions, with a large fraction of internal connections. For large  $q$  there are two possibilities: a single community with the same opinion or one well connected community with a set of poorly connected smaller communities. In correspondence to an intermediate value  $q_c$ , the total

density of links exhibits a minimum  $r_c(q_c)$ ; both  $r_c$  and  $q_c$  vanish for large system size.

Starting from the usual Axelrod model (see Sec. IV.A) in (Centola *et al.*, 2007) a further step is added: if the overlap  $\omega$  between two nodes is exactly zero, the link is removed and one of the agents connects to another randomly chosen vertex. In this way the transition  $q_c$  between a monocultural state and fragmentation is moved to much larger values: coevolution favors consensus. At  $q_c$  both a cultural and a topological transition take place: the system becomes separated in cultural groups that also form topologically disconnected network subsets. For even higher values of the variability of the initial state  $q = q^* > q_c$  another transition occurs, involving only the network structure. For  $q > q^*$  the system remains culturally disordered but a giant component is formed again. In this regime it is likely that each vertex is completely different from its neighbors, therefore it continuously breaks links and looks (unsuccessfully) for new more similar partners. The transition occurring at  $q_c$  can be explained (Vazquez *et al.*, 2007) in terms of the competition between the temporal scales of cultural and topological evolution. The topological transition occurring at  $q^*$  can be instead seen as the value of  $q$  such that the temporal scale for reaching a topologically stationary state is maximum. Another model of adaptive network coupled to vectorial opinions is introduced in (Grabowski and Kosiński, 2006a).

Network rewiring has also been considered for the dynamics of the Deffuant model (Kozma and Barrat, 2008). At each time step with probability  $1 - w$  a step of the usual opinion dynamics is performed with confidence bound  $d$ , otherwise one agent breaks one link and reconnects it to a randomly chosen other node. By changing  $w$  it is then possible to go from pure opinion dynamics in a static environment to fast topological evolution in a quenched opinion state. Coevolution has opposite effects on the two transitions exhibited by the model on static ER networks. The confidence bound threshold  $d_1$ , above which consensus is found, grows with  $w$ . The threshold  $d_2$  of the transition between a polarized state (for  $d_2 < d < d_1$ ) and a fragmented one with no macroscopic domains (for  $d < d_2$ ) goes instead to zero:  $d_2(w > 0) = 0$ . The fragmented state disappears because, even for small  $d$ , a node can rewire its connections and find other agents with whom to reach an agreement. Another coevolving generalization of Deffuant model is (Stauffer *et al.*, 2006a).

Finally, a Glauber zero-temperature spin dynamics has been recently considered in (Benczik *et al.*, 2007), together with a particular link update rule, different from all previous cases: at each time step two nodes carrying equal spins are connected with probability  $p$ , while nodes with opposite values of the spin variable are connected with probability  $1 - p$ . No topological fragmentation occurs in this case. Consensus is always reached on finite systems but exceedingly long disordered metastable states exist.

## XI. OUTLOOK

With this review we made a first attempt to summarize the many activities in the field of the so-called social dynamics. Our point of view has been that of reviewing what has been done so far in this young but rapidly evolving area, placing the main emphasis on the statistical physics approach, i.e., on the contributions the physics community has been giving to social oriented studies.

Though it is generally very difficult to isolate the contribution of a given community to an intrinsically interdisciplinary endeavor, it is nevertheless useful to identify the contribution the physics community has been giving and the role it could play in the future. In this perspective it is clear how the statistical physics role in social dynamics has been mainly focused on modeling, either by introducing brand new models to capture the main features of a given phenomenology or performing detailed analysis of already existing models, e.g., focusing for instance on the existence of phase transitions, universality, etc.. An inspiring principle has been provided by the quest for simplicity. This has several advantages. It allows for discovering underlying universalities, i.e., realizing that behind the details of the different models there could be a level where the mathematical structure is similar. This implies, in its turn, the possibility to perform mapping with other known models and exploit the background of the already acquired knowledge for those models. In addition physicists placed a great emphasis on the role of scales (system sizes, timescales, etc.) as well as on the topology (i.e., the network of interactions) underlying the observed phenomenology.

Closely related to modeling is the data analysis activity, both considering synthetic data coming from simulations and empirical data gathered from observations of real systems or collected in the framework of newly devised experiments. Data analysis is very important not only for the identification of new phenomenologies or surprising features, but also for the validation of the models against empirical data. In this way a positive feedback mechanism could be triggered between the theoretical and the experimental activities in order to make the results robust, well understood and concrete.

Methodologically we can identify several important directions the research in this area should possibly follow. It would be crucial fostering the interactions across disciplines by promoting scientific activities with concrete mutual exchanges among social scientists, physicists, mathematicians and computer scientists. This would help both in identifying the problems and sharpening the focus, as well as in devising the most suitable theoretical concepts and tools to approach the research.

As for the modeling activity it would highly desirable to identify general classes of behavior, not based on microscopic definitions, but rather on large-scale universal characteristics, in order to converge to a shared theoretical framework based on few fundamental paradigms. The identification of which phenomena are actually described

by the theoretical models must become a priority. For instance, the celebrated Axelrod model has not yet been shown to describe at least semi-quantitatively any concrete situation. Without applications the intense activity on modeling risks to be only a conceptual exercise.

In this perspective a crucial factor will be most likely represented by the availability of large sets of empirical quantitative data. The research carried out so far only rarely relied on empirical datasets, often insufficient to discriminate among different modeling schemes. The joint interdisciplinary activity should then include systematic campaigns of data gathering as well as the devising of new experimental setups for a continuous monitoring of social activities. From this point of view the web may be of great help, both as a platform to perform controlled online social experiments, and as a repository of empirical data on large-scale phenomena, like elections and consumer behavior. It is only in this way that a virtuous circle involving data collection, data analysis, modeling and predictions could be triggered, giving rise to an ever more rigorous and focused approach to socially motivated problems. A successful example in this perspective is the study of traffic and pedestrian behaviors, that in the last few years has attained a high level of maturity, leading to reliable quantitative predictions and control (Helbing *et al.*, 2007) (see also Sec. VI).

We conclude this review by highlighting a few interesting directions that could possibly have a boosting effect on the research in the area of social dynamics.

### 1. Information dynamics and the Social Web

Though only a few years old, the growth of the World Wide Web and its effect on the society have been astonishing, spreading from the research in high-energy physics into other scientific disciplines, academe in general, commerce, entertainment, politics and almost anywhere where communication serves a purpose. Innovation has widened the possibilities for communication. Blogs, wikis and social bookmark tools allow the immediacy of conversation, while the potential of multimedia and interactivity is vast. The reason for this immediate success is the fact that no specific skills are needed for participating. In the so-called Web 2.0 (O'Reilly, 2005) users acquire a completely new role: not only information seekers and consumers, but information architects, cooperate in shaping the way in which knowledge is structured and organized, driven by the notion of meaning and semantics. In this perspective the web is acquiring the status of a platform for *social computing*, able to coordinate and exploit the cognitive abilities of the users for a given task. One striking example is given by a series of *web games* (von Ahn and Dabbish, 2004), where pairs of players are required to coordinate the assignment of shared labels to pictures. As a side effect these games provide a categorization of the images content, an extraordinary difficult task for artificial vision

systems. More generally, the idea that the individual, selfish activity of users on the web can possess very useful side effects, is far more general than the example cited. The techniques to profit from such an unprecedented opportunity are, however, far from trivial. Specific technical and theoretical tools need to be developed in order to take advantage of such a huge quantity of data and to extract from this noisy source solid and usable information (Arrow, 2003; Huberman and Adamic, 2004). Such tools should explicitly consider how users interact on the web, how they manage the continuous flow of data they receive (see Sec. VIII), and, ultimately, what are the basic mechanisms involved in their brain activity. In this sense, it is likely that the new social platforms appearing on the web, could rapidly become a very interesting laboratory for social sciences. In particular we expect the web to have a strong impact on the studies of opinion formation, political and cultural trends, globalization patterns, consumers behavior, marketing strategies.

## 2. Language and communication systems

Language dynamics is a promising field which encompasses a broader range of applications with respect to what described in Sec. V (Loreto and Steels, 2007). In many biological, technological and social systems, a crucial problem is that of the communication among the different components, i.e., the elementary units of the systems. The agents interact among themselves and with the environment in a sensorial and non-symbolic way, their communication system not being predetermined nor fixed from a global entity. The communication system emerges spontaneously as a result of the interactions of the agents and it could change continuously due to the mutations occurring in the agents, in their objectives as well as in the environment. An important question concerns how conventions are established, how communication arises, what kind of communication systems are possible and what are the prerequisites for such an emergence to occur. In this perspective the emergence of a common vocabulary only represents a first stage while it is interesting to investigate the emergence of higher forms of agreement, e.g., compositionality, categories, syntactic or grammatical structures. It is clear how important it would be to cast a theoretical framework where all these problems could be defined, formalized and solved. That would be a major input for the comprehension of many social phenomena as well as for devising new technological instruments.

## 3. Evolution of social networks

As real and online social systems grow ever larger, their analysis becomes more complicated, due to their intrinsic dynamic nature, the heterogeneity of the in-

dividuals, their interests, behavior etc.. In this perspective, the discovery of communities, i.e., the identification of more homogeneous groups of individuals, is a major challenge. In this context, one has to distinguish the communities as typically intended in social network analysis (SNA) (Freeman, 2004; Scott, 2000; Wasserman and Faust, 1994) from a broader definition of communities. In SNA one defines communities over a communication relationship between the users, e.g. if they regularly exchange e-mails or talk to each other. In a more general context, for e.g. providing recommendation strategies, one is more interested in finding communities of users with homogeneous interests and behavior. Such homogeneity is independent of contacts between the users although in most cases there will be at least a partial overlap between communities defined by the user contacts and those by common interests and behavior. Two important areas of research can be identified. On the one hand, there is the question of which observable features in the available data structures are best suited for inferring relationships between individuals or users. Selecting a feature affects the method used to detect communities (Girvan and Newman, 2002), which may be different if one operates in the context of recommendation systems or in the context of semantic networks. On the other hand important advances are foreseeable in the domain of coevolution of dynamics and the underlying social substrates. This topic is still in its infancy, despite the strong interdependence of dynamics and networks in virtually all real phenomena. Empirical data on these processes are becoming available: it is now possible to monitor in detail the evolution of large scale social systems (Palla *et al.*, 2007).

## Acknowledgments

We wish to thank A. Baldassarri, A. Baronchelli, A. Barrat, R. Blythe, E. Caglioti, C. Cattuto, L. Dall'Asta, I. Dornic, J.P. Eckmann, M. Felici, S. Galam, G. Gosti, C. Hidalgo, P. Holme, N.F. Johnson, N.L. Komarova, R. Lambiotte, J. Lorenz, A. McKane, M. Marsili, J. Minett, M. Nowak, J.P. Onnela, F. Radicchi, J.J. Ramasco, S. Redner, M. San Miguel, V.D.P. Servedio, D. Stauffer, L. Steels, D. Urbig, A. Vespignani, D. Vilone, T. Vicsek, W.S.-Y. Wang. This work was partially supported by the EU under contract IST-1940 (ECAgents) and IST-34721 (TAGora). The ECAgents and TAGora projects are funded by the Future and Emerging Technologies program (IST-FET) of the European Commission.

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