HINTS FOR SOLVING THE EXERCISES

I.1. The equation which determines the position of the extrema is

$$\cos x + \cos 3x + \cdots + \cos(2n-1)x = 0. \tag{1}$$

If we write this in the form

$$\operatorname{Re}(e^{ix} + e^{3ix} + \cdots + e^{(2n-1)ix}) = 0, \tag{2}$$

then we can sum this geometric series. We obtain finally

$$\text{(3)} \quad \frac{\sin 2nx}{2\sin x} = 0, \qquad \text{hence} \qquad \text{(4)} \quad x = \frac{\pi}{2n}, \frac{2\pi}{2n}, \ldots, \frac{(2n-1)\pi}{2n}.$$

Due to the denominator in (3) we do not count the points $x = 0$ and $x = \pi$. Equation (4) proves the statement in the exercise.

I.2. By integrating from 0 to x we obtain a sine series from the cosine series (2.17); and if in the sine series we set $x = \pi/2$ then we obtained the analogue to the Leibniz series which follows (2.14). Integrating this sine series we obtain a series in terms of $1 - \cos x$, $1 - \cos 3x, \ldots$ and setting $x = \pi/2$ here we obtain the next analogue to (2.16), from which we deduce the value of Σ_6. This process can be continued indefinitely, but it does not seem to lead to a transparent law for the successive analogues.

I.3. The two processes mentioned in the exercise lead to the series

$$\sin x = \frac{2}{\pi}\left(1 - \frac{2}{1\cdot 3}\cos 2x - \frac{2}{3\cdot 5}\cos 4x \ldots\right)$$

from which, by setting, e.g., $x = 0$, we obtain a representation of $\frac{1}{2}$ as a series in the reciprocals of odd integers.

I.4. Consider case a). If in (4.8) we replace x by t and perform the integration with respect to ξ we obtain

$$f(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{d\omega}{\omega}\sin\frac{\omega\tau}{2}\, e^{i\omega t} = \int_0^{\infty} a(\omega)\cos\omega t\, d\omega. \tag{1}$$

where $|a(\omega)|$ stands for the amplitude of the spectrum of $f(t)$ with the frequency ω and, according to (1),

$$a(\omega) = \frac{1}{\pi}\frac{\sin\omega\tau/2}{\omega/2}. \tag{2}$$

This function has its principal maximum of altitude τ/π at $\omega = 0$, followed by secondary maxima of decreasing altitudes at intervals with lengths asymptotically equal to $\Delta\omega = 2\pi/\tau$.

In case b), where according to the figure we are dealing with a function f which is odd in t, we obtain in the same manner:

$$
\begin{aligned}
f(t) &= \frac{i}{\pi}\int_{-\infty}^{+\infty} \frac{d\omega}{\omega}\left(1 - \cos\frac{\omega\tau}{2}\right) e^{i\omega t} = \int_0^\infty b(\omega)\sin\omega t\, d\omega\,; \\
b(\omega) &= -\frac{1}{\pi}\,\frac{1-\cos\omega\tau/2}{\omega/2} = -\frac{1}{\pi}\,\frac{\sin^2\omega\tau/4}{\omega/4}.
\end{aligned} \tag{3}
$$

We now have $b(\omega) = 0$ for $\omega = 0$; the first maximum lies at $\omega \sim 4.7/\tau$; as before, the subsequent secondary maxima successively decrease in altitude.

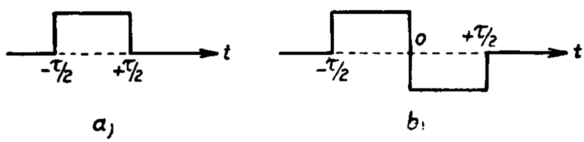

Fig. 33.

a) $f(t) = 0$ for $|t| > \tau/2$,
$f(t) = 1$ for $|t| < \tau/2$.

b) $f(t) = 0$ for $|t| > \tau/2$,
$f(t) = 1$ for $-\tau/2 < t < 0$,
$= -1$ for $0 < t < \tau/2$.

In both cases a) and b) we are dealing with a "grooved spectrum" that extends to infinity.

In the beginning of the theory of x-rays an attempt was made to interpret them as ether impulses of the type a) or b). From a spectral point of view, which is the only one that is physically justified, this is not a departure from the wave interpretation, but merely an (arbitrary special) assumption about the nature of the x-ray spectrum.

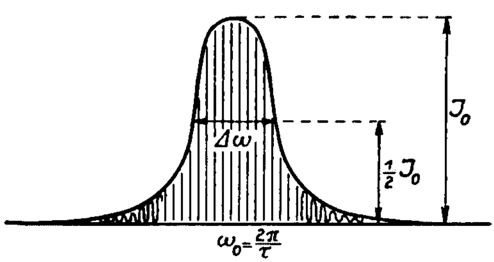

Fig. 33c. Schematic representation of the intensity in the spectrum of a wave process of frequency $\omega_0 = \dfrac{2\pi}{\tau}$, which breaks off on both sides. Here $\Delta\omega$ is the half-value width of the corresponding spectral line. The ruled middle portion of the spectrum consists of $\sin^2$-like oscillations, just like the unruled part.

For the wave process of length $2T = 2n\tau$ which breaks off on both sides (Fig. 33c) we start most conveniently from equation (4.11b) and find

$$(4) \qquad f(t) = \int_0^\infty b(\omega) \sin \omega t \, d\omega, \qquad b(\omega) = \frac{4}{\tau} \frac{\sin \omega T}{\omega^2 - \left(\frac{2\pi}{\tau}\right)^2}.$$

The principal maximum of b lies, as expected, at the frequency $\omega_0 = 2\pi/\tau$ and has the altitude $b_0 = n\tau/\pi$ corresponding to the intensity $I_0 = \left(\frac{n\tau}{\pi}\right)^2$, and hence increases with increasing length. This principal maximum is flanked on both sides by secondary maxima of successively decreasing altitudes at intervals with length asymptotically approaching $\pi n \tau$; for all these maxima we have $\sin \omega T \sim 1$. We seek the two maxima for which $I = \frac{1}{2} I_0$, that is, according to (4), those maxima for which

$$(5) \qquad \omega^2 - \omega_0^2 = \pm \frac{4\sqrt{2}\,\pi}{n\tau^2} = \pm \frac{\sqrt{2}}{\pi n} \omega_0^2, \quad \omega^2 = \omega_0^2 \left(1 \pm \frac{\sqrt{2}}{\pi n}\right).$$

The difference of their frequencies is the so-called half-value width of the corresponding spectral line. If we assume $n \gg 1$ then this frequency difference is, according to (5),

$$(6) \qquad \Delta\omega = \frac{\sqrt{2}}{\pi n} \omega_0 = \frac{2\sqrt{2}}{T}.$$

Hence the half value width decreases with increasing T as stated in the exercise. Only for $T \to \infty$ do we obtain an absolutely sharp spectral line.

I.5. The function $f(t)$ of Fig. 33a) for $\tau/2 = 1$ is known in the mathematical literature as the *Dirichlet discontinuous factor:*

$$(1) \qquad D = \frac{2}{\pi} \int_0^\infty \sin \omega \cos \omega t \, \frac{d\omega}{\omega} = \begin{cases} 1 & |t| < 1, \\ \frac{1}{2} & |t| = 1, \\ 0 & |t| > 1. \end{cases}$$

If we set $t = 0$ here then we obtain the fundamental integral

$$(2) \qquad \int_0^\infty \sin \omega \frac{d\omega}{\omega} = \frac{\pi}{2} \qquad \text{or} \qquad (2\text{a}) \quad \int_{-\infty}^{+\infty} \sin \omega \frac{d\omega}{\omega} = \pi.$$

which was used in connection with Fig. 4, p. 11. This can be verified directly through complex integration: since $\sin \omega/\omega$. is analytic on the real axis and in its neighborhood, we can avoid the point $\omega = 0$ (e.g., as in Fig. 34a, p. 301) below the real axis, and we then can decompose (2a) into the difference of the integrals

$$(3) \qquad \mathrm{I} = \frac{1}{2i}\int e^{i\omega}\frac{d\omega}{\omega}, \qquad \mathrm{II} = \frac{1}{2i}\int e^{-i\omega}\frac{d\omega}{\omega},$$

both integrals being taken over the heavy line of the figure. The path of integration in II can be deformed into the infinite part of the *lower* half plane, where the integral vanishes. The path of integration of I must be deformed into the infinite part of the upper half plane since $\exp(i\omega)$ vanishes only there; however, it cannot be deformed across the pole $\omega = 0$. The residue at the pole is $2\pi i$. This proves (2a) and hence (2).

Finally, we easily verify that those parts of the path of integration which in the figure are indicated by short arrows and their dotted continuations also make no contribution to I and II.

The method of complex integration also serves to extend the statements of the preceding exercise for the wave which is bounded on both sides. Such a wave can be considered as the superposition of two waves, which are bounded on one side, of opposite phase, one ranging from $t = -T$ to $t = \infty$, the other from $t = +T$ to $t = \infty$. However these processes cannot be represented individually in the Fourier manner, due to the divergences at $t = \infty$. For this purpose it is necessary to transfer the path of integration in equation (4) of the preceding exercise from the real axis into the complex domain (as shown in Fig. 34b), and then to perform the decomposition. This is explained by the following transformations which start from equation (4) on p. 299:

$$f(t) = \frac{4}{\tau}\int_0^\infty \sin\omega t \sin\omega T \frac{d\omega}{\omega^2 - \left(\frac{2\pi}{\tau}\right)^2}$$

$$= \frac{2}{\tau}\int_0^\infty \left\{\cos\omega(t-T) - \cos\omega(t+T)\right\}\frac{d\omega}{\omega^2 - \left(\frac{2\pi}{\tau}\right)^2}$$

$$= \frac{1}{\tau}\int_{-\infty}^{+\infty} \left(e^{i\omega(t-T)} - e^{i\omega(t+T)}\right)\frac{d\omega}{\omega^2 - \left(\frac{2\pi}{\tau}\right)^2} = \mathrm{I}\text{–}\mathrm{II},$$

$$\left.\begin{matrix}\mathrm{I}\\ \mathrm{II}\end{matrix}\right\} = -\frac{1}{\tau}\int e^{i\omega(t\pm T)}\frac{d\omega}{\omega^2 - \left(\frac{2\pi}{\tau}\right)^2},$$

where the integral signs without upper and lower limits are to be taken over the complex path of Fig. 34b.

We claim that I represents the wave process starting at $t = -T$

and II represents that starting at $t = +T$, both continuing to $t = \infty$.

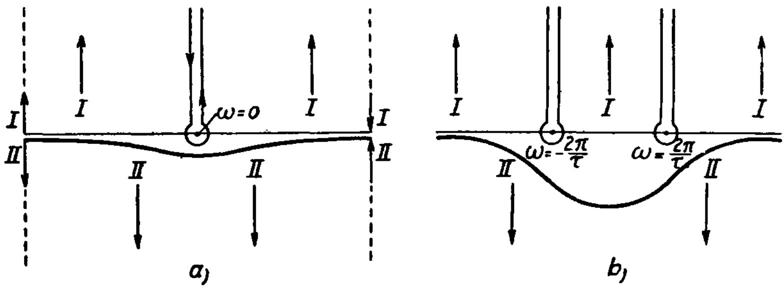

Fig. 34. a) Dirichlet discontinuous factor. We prove equation (3) by deforming the path of integration of II downward, that of I upward. b) Wave process which breaks off on both sides. The complex integrals I and II represent wave processes which are bounded on one side.

In order to prove this we set $T = 0$ for simplicity, and show that

$$f_0(t) = -\frac{1}{\tau}\int e^{i\,\omega\,t}\,\frac{d\omega}{\omega^2 - \left(\frac{2\pi}{\tau}\right)^2} \tag{4}$$

is a sine wave which starts at $t = 0$ and continues to $t = \infty$. The proof is given in a manner similar to that of (3): for $t < 0$ the path of integration can be drawn into the infinite part of the lower half plane and the integral vanishes there. For $t > 0$ the path of integration must be drawn into the upper half plane. Due to the poles $\omega = \pm 2\pi/\tau$ we obtain the residues

$$2\pi i\,\frac{e^{2\pi i t/\tau}}{4\pi/\tau} \qquad \text{and} \qquad 2\pi i\,\frac{e^{-2\pi i t/\tau}}{-4\pi/\tau};$$

so that

$$f_0(t) = -\frac{i}{2}\left(e^{2\pi i t/\tau} - e^{-2\pi i t/\tau}\right) = \sin\frac{2\pi t}{\tau}.$$

This completes the proof.

If instead of starting from (4) we start from

$$f_0(t) = -\frac{1}{2\pi}\,\mathrm{Re}\int e^{i\,\omega\,t}\,\frac{d\omega}{\omega - 2\pi/\tau} \tag{5}$$

then we see that for the same choice of the path of integration and the same deformation of this path we obtain the same result as before:

$$f_0(t) = \begin{cases} 0 & t < 0, \\ \sin\dfrac{2\pi t}{\tau} & t > 0. \end{cases} \tag{6}$$

The interest in this representation lies in the optic theory of dispersion. We imagine that perpendicularly to the plane $x = 0$ the wave (6) enters a medium filling the half space $x > 0$, and decompose it according to (5) into partial waves of the form $a(\omega)\, e^{i\omega t}$. Each of these waves propagates in the direction of increasing x independently of all other waves and we represent it here by $a(\omega) \exp[i(kz - \omega t)]$. The wave number k in a dispersion-free medium would be $k_0 = \omega/c$; due to the induced oscillation of the electrons (numbering N per cm.^{3}) we have

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{N e^2/m}{\omega^2 - \omega_e^2}\right), \tag{7}$$

where ω_e is the proper frequency of the oscillating electrons (for the sake of simplicity we neglect the damping of these oscillations). Hence in the infinite part of the ω-plane we have:

$$k = k_0, \qquad k x - \omega t = k_0 x - \omega t = \frac{\omega}{c}(x - c t).$$

Thus the question of whether the path of integration is to be deformed in the direction of the positive or negative half plane is determined by the sign of $x - ct$. Since this criterion is independent of ω it is the same for all partial waves, so that $x = ct$ stands for the entire light stimulation at the point x of the dispersive medium. *The head of our light signal therefore propagates with the vacuum velocity $dx/dt = c$, not*, as one might think, *with the phase velocity $V = \omega/k$ which is characteristic for the dispersive medium.* We interpret this in the following way: the electrons are at rest for $t < x/c$ and start plane oscillation for $t = x/c$. The full amplitude corresponding to the incoming oscillation is attained only at a later time that is determined not by the phase velocity V but by the *group velocity $U = d\omega/dk$*. The oscillation processes which precede this time may be called *forerunners of the light signal.*

I.6. a) *Hermite polynomials.* Due to the even character of $g(x) = e^{-x^2}$ and the fact that the interval is $-\infty < x < +\infty$ we see that the Hermite polynomials, like the spherical harmonics P_n, are even or odd functions of x depending on whether n is even or odd. Considering this fact and the customary normalization of H_n (see exercise) write:

$$H_0 = 1, \quad H_1 = 2x, \quad H_2 = 4x^2 + a, \quad H_3 = 8x^3 + bx, \quad H_4 = 16x^4 + cx^2 + d$$

and compute the coefficients a, b, c, d through a repeated application of the orthogonality condition (result:

$$a = -2,\ b = -12,\ c = -48,\ d = +12).$$

b) *Laguerre polynomials.* Due to the weighting factor $g(x) = e^{-x}$ and the fact that the interval is $0 < x < \infty$ the polynomials are no longer even or odd. Considering this fact and the customary normalization (see exercise) write:

$$L_0 = 1\,, \quad L_1 = a\,x + 1\,, \quad L_2 = b\,x^2 + c\,x + 2\,, \quad L_3 = dx^3 + e\,x^2 + fx + 6$$

and compute the coefficients $a, b, \ldots f$ as in a) result: $a = -1$, $b = 1$, $c = -4$, $d = -1$, $e = 9$, $f = -18$).

II.1. The differential equation (7.8) to which this exercise relates is obtained as follows from the theory of *beam bending:* we start from the differential equation:

$$\frac{\partial^4 u}{\partial x^4} = \frac{1}{EI}\frac{\partial^2 M}{\partial x^2}\,. \tag{1}$$

The bending moment M of the exterior static load of the beam is to be replaced by the moment of the dynamic inertia resistances

$$-\varrho\frac{\partial^2 u}{\partial t^2}$$

(ϱ = mass per unit of length of the oscillating beam). Let this beam be clamped at $x = 0$ and let the free end be $x = l$. All the cross-sections $x < \xi < l$ contribute to the bending moment at the cross-section x, each cross-section with the lever-arm $\xi - x$. Hence we have

$$M = -\varrho\int_x^l (\xi - x)\frac{\partial^2 u}{\partial t^2}\,d\xi\,, \quad \frac{\partial M}{\partial x} = \varrho\int_x^l \frac{\partial^2 u}{\partial t^2}\,d\xi\,, \quad \frac{\partial^2 M}{\partial x^2} = -\varrho\frac{\partial^2 u}{\partial t^2}\,. \tag{2}$$

Substituting this in (1) we obtain (7.8) and for the constant c (of dimension cm.^{2}/sec) we obtain

$$c = \sqrt{\frac{E\,I}{\varrho}}\,. \tag{3}$$

According to (2) we have at the free end $M = \dfrac{\partial M}{\partial x} = 0$. According to equation (1) above this means

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^3 u}{\partial x^3} = 0 \qquad x = l\,. \tag{4}$$

On the other hand the clamping implies

$$u = \frac{\partial u}{\partial x} = 0 \quad \text{for } x = 0\,. \tag{5}$$

If we write $u = U\,e^{i\,\omega t}$, $U = e^{\alpha x}$ then (7.8) yields

$$\alpha^4 = \frac{\omega^2}{c^2}\,, \qquad \alpha = \pm k \text{ and } = \pm i\,k\,, \qquad k = \sqrt{\frac{\omega}{c}}\,.$$

Hence there are four particular solutions of the differential equation for U:

$$e^{kx}, \; e^{-kx}, \; e^{ikx}, \; e^{-ikx},$$

which for the following can be combined more conveniently into the forms

$$\sinh kx, \; \cosh kx, \; \sin kx, \; \cos kx.$$

Hence the general solution becomes

$$U = A \sinh kx + B \cosh kx + C \sin kx + D \cos kx.$$

According to (4) and (5) there are four relations among the constants of integration A,B,C,D, from which we obtain through elimination the transcendental equation

$$\cos kl = -\frac{1}{\cosh kl}. \tag{6}$$

The graphical treatment of this equation in the manner of Fig. 7 yields an infinity of roots at intervals which asymptotically become

$$k_{n+1} - k_n = \frac{\pi}{l}, \qquad \omega_{n+1} - \omega_n = 2\,c\,n\,\frac{\pi^2}{l^2} \tag{7}$$

For the basic oscillation we have

$$k = k_1 = 1.875/l, \qquad \omega = \omega_1 = c\,k_1^2. \tag{7a}$$

The differential equation of a *pipe* is the same as that of an oscillating string, that is equation (7.6) where u = longitudinal velocity of air, c = velocity of sound. For the pipe which is open on both ends, or one which is covered at $x = 0$ and open at $x = l$ we have the boundary conditions

$$\frac{\partial u}{\partial x} = 0 \quad \text{for } x = 0 \quad \text{and} \quad x = l \text{ (open pipe)} \tag{8a}$$

or

$$u = 0 \quad \text{for} \quad x = 0, \quad \frac{\partial u}{\partial x} = 0 \quad \text{for} \quad x = l \text{ (covered pipe)} \tag{8b}$$

(due to the hydrodynamic continuity equation $\partial u/\partial x = 0$ means the same as $\partial p/\partial t = 0$, that is $p = p_0$ = atmospheric pressure which we assume to hold approximately). Writing $u = U\,e^{i\omega t}$ for the proper oscillations we obtain from (8a, b)

$$U = A \cos k_n x, \quad k_n = n\,\frac{\pi}{l}, \qquad \omega_n = c\,k_n, \quad k_1 = 3.14/l \tag{9a}$$

$$U = B \sin k_n x, \quad k_n = \left(n + \frac{1}{2}\right)\frac{\pi}{l}, \quad \omega_n = c\,k_n, \quad k_1 = 1.57/l. \tag{9b}$$

The value (7a) of k_1 lies between the values (9b) and (9a). The sequence of ω is harmonic for both the open and covered pipe; for the elastic rod it becomes harmonic only asymptotically for high overtones (see (7)).

II.2. a) Using the identities

$$v\,\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x}\,\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(v\,\frac{\partial u}{\partial x}\right),$$

$$D\,v\,\frac{\partial u}{\partial x} + u\,\frac{\partial D\,v}{\partial x} = \frac{\partial}{\partial x}\,(D\,u\,v),$$

and writing $L(u)$ in the normal form (10.1), we obtain

$$v\,L\,(u) + \varLambda\,(u, v, \ldots) = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \tag{1}$$

with

$$\varLambda\,(u, v, \ldots) = (\operatorname{grad} u,\ \operatorname{grad} v) + u\left(\frac{\partial D\,v}{\partial x} + \frac{\partial E\,v}{\partial y} - F\,v\right), \tag{2}$$

$$X = v\,\frac{\partial u}{\partial x} + Du\,v, \qquad Y = v\,\frac{\partial u}{\partial y} + E\,u\,v. \tag{3}$$

Here $\varLambda$ is a *bilinear form* in the u,v and their first derivatives. If L is self-adjoint then, because $D = E = 0$, equation (10.6), we have:

$$\varLambda = (\operatorname{grad} u,\ \operatorname{grad} v) - F\,u\,v, \tag{4}$$

which is symmetric in u and v.

b) If L has the general form (8.1) then we again have equation (1), but with

$$\begin{aligned}\varLambda\,(u, v \ldots) = {}& A\,\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial x} + B\left(\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\,\frac{\partial v}{\partial x}\right) + C\,\frac{\partial u}{\partial y}\,\frac{\partial v}{\partial y}\\ & + u\left(\frac{\partial D\,v}{\partial x} + \frac{\partial E\,v}{\partial y} - F\,v\right) + v\left(\frac{\partial A}{\partial x}\,\frac{\partial u}{\partial x} + \frac{\partial B}{\partial x}\,\frac{\partial u}{\partial y} + \frac{\partial B}{\partial y}\,\frac{\partial u}{\partial x} + \frac{\partial C}{\partial y}\,\frac{\partial u}{\partial y}\right).\end{aligned} \tag{5}$$

$$X = v\left(A\,\frac{\partial u}{\partial x} + B\,\frac{\partial u}{\partial y} + D\,u\right), \qquad Y = v\left(B\,\frac{\partial u}{\partial x} + C\,\frac{\partial u}{\partial y} + E\,u\right). \tag{6}$$

If L is self-adjoint then, due to (10.6), the expression $\varLambda$ simplifies to

$$\varLambda = A\,\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial x} + B\left(\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\,\frac{\partial v}{\partial x}\right) + C\,\frac{\partial u}{\partial y}\,\frac{\partial v}{\partial y} + \frac{\partial D\,u\,v}{\partial x} + \frac{\partial E\,u\,v}{\partial y} - F\,u\,v,$$

which is again *symmetric in the u and v.* This can be further simplified by taking the terms

$$\frac{\partial D\,u\,v}{\partial x} \quad \text{and} \quad \frac{\partial E\,u\,v}{\partial y}$$

over to the X,Y on the right side of (1). We then obtain:

$$A = A \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + B\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + C\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} - F u v, \tag{7}$$

$$X = v\left(A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y}\right), \qquad Y = v\left(B\frac{\partial u}{\partial x} + C\frac{\partial u}{\partial y}\right). \tag{8}$$

If we now integrate equation (1) over a region S with the boundary curve C we obtain the most general version of the *second form of Green's theorem:*

$$\int v\, L(u)\, d\sigma + \int A\,(u, v, \ldots)\, d\sigma = \int \{X \cos(n, x) + Y \cos(n, y)\}\, ds. \tag{9}$$

c) In order to investigate whether the solution of the differential equation for a given boundary condition is *unique* we proceed as in the case of the equation $\Delta u = 0$:

Assuming that for given values of u on the boundary curve C different solutions u_1, u_2 exist, we set as in (9)

$$u = v = u_1 - u_2$$

Then, because of the linearity of L, the first term on the left side of (9) vanishes. Also, because $v = 0$ on the curve C, we see by (6) that $X = Y = 0$ on C, so that the right side of (9) vanishes. Thus (9) becomes

$$\int A\,(u, u, \ldots)\, d\sigma = 0. \tag{10}$$

If we restrict ourselves to the *self-adjoint* case and introduce the abbreviations $\xi = \partial u/\partial x$, $\eta = \partial u/\partial y$ we obtain

$$A\,(u, u, \ldots) = \begin{cases} \xi^2 + \eta^2 - F u^2 & \text{according to (4)} \\ A\,\xi^2 + 2\,B\,\xi\,\eta + C\,\eta^2 - F u^2 & \text{according to (7)} \end{cases} \tag{11}$$

The upper line of (11) contradicts (10) if $F(x,y)$ is *negative* throughout S; the second line contradicts (10) if the quadratic form $A\xi^2 + 2B\xi\eta + C\eta^2$ is *definite* and $F(x,y)$ has the *opposite* sign throughout S. In both these cases we conclude from (10) that

$$u = 0 \quad \text{and hence} \quad u_1 = u_2 \tag{12}$$

that is the *uniqueness of the boundary value problem.*

This is identical with the *non-existence of "eigenfunctions"* (see Chapter V). In particular, for the self-adjoint differential equation in the normal form $\Delta u + Fu = 0$, where $F = \text{const.} = \pm k^2$, we see that the differential equation

$$\Delta u - k^2 u = 0 \tag{13}$$

has no *eigenfunctions* in contrast to the equation of the oscillating membrane

(13a) $$\Delta u + k^2 u = 0\,,$$

where the eigenfunctions are of basic interest.

The fact that for the self-adjoint differential equation of elliptic type the *uniqueness question for the boundary value problem* or *the question about the existence of eigenfunctions* can be decided quite generally on the basis of the second form of Green's theorem, explains the preferred role of these differential equations in mathematical physics.

In order to stress the physical importance of equation (13) we remark that as the *Yukawa meson equation* it plays the same role in *nuclear physics* as is played by the potential equation in *electron physics.*

II.3. From the conditions a), b), c) of the exercise we obtain

(1) $$\begin{cases} \text{for } 0 < x < \xi \ldots G_- = -\left(1 - \dfrac{\xi}{l}\right) x\,, \\[2ex] \text{for } \xi < x < l \ldots G_+ = -\left(1 - \dfrac{x}{l}\right) \xi\,. \end{cases}$$

On the other hand the solution of the "boundary value problem" for u is obviously

(2) $$u = u_0 + (u_l - u_0)\,\frac{x}{l}\,.$$

Figure 35 represents the lines for G and u. Verify that Green's equation (10.12) for one dimension

(3) $$u_\xi = u_l \left(\frac{\partial G_+}{\partial x}\right)_l - u_0 \left(\frac{\partial G_-}{\partial x}\right)_0$$

II.4. According to equation (2) of the exercise, $v = v_0(t)$ is given for $x = 0$ and $t > 0$; hence the required Green's function G must satisfy the condition:

(1) $$G = 0 \quad \text{for} \quad x = 0.$$

This function is obtained from the principal solution $V(x, t; \xi, \tau)$, equation (12.16), through reflection on the line $x = 0$ (see also §13):

$$\begin{aligned} G &= V(x, t;\ \xi, \tau) - V(x, t; -\xi, \tau) \\ &= \{4\pi k(\tau - t)\}^{-\frac{1}{2}} \left(\exp\left\{\frac{-(x-\xi)^2}{4k(\tau - t)}\right\} - \exp\left\{\frac{-(x+\xi)^2}{4k(\tau - t)}\right\}\right), \end{aligned}$$

and hence for $x = 0$:

(2) $$\frac{\partial G}{\partial x} = \frac{\xi}{2\sqrt{\pi}}\,\{k(\tau - t)\}^{-\frac{3}{2}} \exp\left\{\frac{-\xi^2}{4k(\tau - t)}\right\}.$$

This value is to be substituted in (12.18) for $\partial V/\partial x$, so that we obtain the following simplification. On the right side we have to cancel the first term, since according to equation (4) of the exercise we have

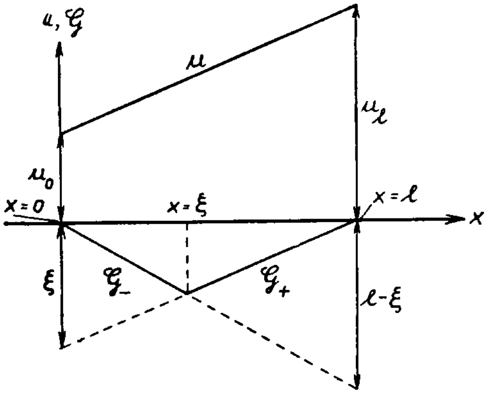

Fig. 35. Behavior of Green's function G with the source point $x = \xi$ and behavior of the potential function u with the boundary values u_0, u_l. The figure at the same time indicates, by the dotted extensions, an elementary construction of the ordinate of G at the point $x = \xi$ coincides with (2) if we use the values of G from (1).

$V = 0$ for $t = 0$. In the second term the part corresponding to $x_1 = \infty$ vanishes, so that only the term corresponding to $x_0 = 0$ remains, which according to (12.18) is to be taken negative; in this term the part multiplied by V vanishes because of (1). Hence, due to equation (2) of the exercise we have:

$$v(\xi, \tau) = \frac{\xi}{2\sqrt{\pi k}} \int_0^\tau v_0(t) \exp\left\{\frac{-\xi^2}{4k(\tau - t)}\right\} \frac{dt}{(\tau - t)^{\frac{3}{2}}}. \tag{3}$$

If instead of t we substitute the variable of integration $p = \xi/\sqrt{4k(\tau - t)}$ then we obtain

$$v(\xi, \tau) = \frac{2}{\sqrt{\pi}} \int_{\xi/\sqrt{4k\tau}}^{\infty} v_0\left(\tau - \frac{\xi^2}{4kp^2}\right) e^{-p^2}\, dp \tag{4}$$

as the final solution of the problem.

In order to discuss (4) we expand

$$v_0\left(\tau - \frac{\xi^2}{4kp^2}\right) = v_0(\tau) - \frac{v_0'(\tau)}{1!} \frac{\xi^2}{4kp^2} + \frac{v_0''(\tau)}{2!} \frac{\xi^4}{(4kp^2)^2} - \cdots. \tag{5}$$

Replacing the variables ξ, τ by x,t then we obtain from (4):

$$v(x,t) = v_0(t)\, I_0(z) - \frac{x^2}{4\,k}\frac{v_0'(t)}{1\,!}\, I_1(z) + \frac{x^4}{16\,k^2}\frac{v_0''(t)}{2\,!}\, I_2(z) - \cdots, \tag{6}$$

with the abbreviations

$$z = \frac{x}{\sqrt{4\,k\,t}}, \qquad I_0(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-p^2}\,dp, \qquad I_n(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-p^2}\frac{dp}{p^{2n}}. \tag{7}$$

Here $I_0(z)$ is essentially the well known and frequently tabulated "error integral." We have for $z \to \infty$

$$I_0(0) = 1, \quad I_0(z) \sim \frac{e^{-z^2}}{\sqrt{\pi}\; z}. \tag{8}$$

The corresponding statements for $I_n(z)$ are

$$I_n(z)\cdot z^{2n} \to 0, \qquad I_n(z)\cdot z^{2n} \sim \frac{e^{-z^2}}{\sqrt{\pi}\; z}. \tag{8a}$$

The expansion (6) is valid for $z < 1$. For $z \gg 1$ we have:

$$v(x,t) \sim \frac{e^{-z^2}}{z} \to 0. \tag{9a}$$

The transition between these limiting laws takes place at $z = 1$, that is, for

$$x \sim \sqrt{4\,k\,t}. \tag{9b}$$

The plate flow investigated here is a useful preparation for recent investigations of the turbulence problem (see v. II, §38).

III.1. According to (13.1) we write

$$u(x,t) = \int_0^\infty \{f(\xi)\, U(x,\xi) + f(-\xi)\, U(x,-\xi)\}\, d\xi. \tag{1}$$

Here $f(\xi)$ is given and we have to find $f(-\xi)$. For $x = 0$ we have

$$(4\,\pi\,k\,t)^{\frac{1}{2}}\, u(0,t) = \int_0^\infty \{\, f(\xi) + f(-\xi)\,\}\, e^{-\frac{\xi^2}{4kt}}\, d\xi, \tag{2}$$

$$(4\,\pi\,k\,t)^{\frac{1}{2}} \left(\frac{\partial u(x,t)}{\partial x}\right)_{x=0} = -\int_0^\infty \{\, f(\xi) - f(-\xi)\,\}\frac{d}{d\xi}\, e^{-\frac{\xi^2}{4kt}}\, d\xi. \tag{3}$$

If we assume that $f(\xi)$ can be made continuous at $\xi = 0$, that is, that

$$\lim_{\xi\to 0} f(-\xi) = \lim_{\xi\to 0} f(+\xi) = f_0, \qquad \xi > 0, \tag{4}$$

and if we transform the integral in (3) by integrating by parts, then the resulting term free of the integral sign vanishes and the right side of (3) becomes

$$\int_0^\infty \frac{d}{d\xi}\{f(\xi)-f(-\xi)\}\, e^{-\frac{\xi^2}{4kt}}\, d\xi. \tag{4a}$$

The condition imposed in the exercise for $x = 0$ is satisfied by the integrands in (2) and (4a). Considering the fact that here $\partial/\partial n$ is the same as $-\partial/\partial x$, we write:

$$\left(\frac{d}{d\xi}+h\right)f(-\xi) = X, \qquad X = \left(\frac{d}{d\xi}-h\right)f(\xi). \tag{5}$$

The differential equation for $f(-\xi)$ obtained in this way becomes integrable if we multiply by $\exp(h\,\xi)$ and yields

$$f(-\xi) = f(\xi) - 2\,h\,e^{-h\xi}\int_0^\xi e^{h\eta}\, f(\eta)\, d\eta. \tag{6}$$

This expression for $f(-\xi)$ is to be substituted in (1). Verify that we obtain the representation of G in (13.15) if in (1) we now specialize f to a δ-function.

III.2. We are dealing with Green's theorem (16.6) from which we are to deduce the normalizing integral (6.3a) by the limit process $\lambda_m \to \lambda_n$. Since this integral assumes the form 0/0 we apply de l'Hospital's rule by first differentiating the numerator and denominator with respect to λ_m and then setting $\lambda_m = \lambda_n$:

$$\frac{\left.\dfrac{du_n}{d\lambda_n}\dfrac{du_n}{dx} - u_n\dfrac{d}{d\lambda_n}\dfrac{du_n}{dx}\right|_{x=0}^{x=l}}{2\,k_n\dfrac{dk_n}{d\lambda_n}}.$$

The numerator must be computed for $x = l$ only, since for $x = 0$ it vanishes even before we pass to the limit.

According to (16.5) and (16.5a) we obtain

$$\int_0^l u_n^2\, dx = \frac{l}{2}\left(\cos^2\lambda_n\pi + \sin^2\lambda_n\pi - \frac{1}{\lambda_n\pi}\sin\lambda_n\pi\cos\lambda_n\pi\right),$$

which coincides with (6.4a) if we specialize our present l to π.

We performed this calculation mainly in order to be able to use it as a model in later cases where the normalizing integral cannot be integrated in an elementary manner.

III.3. In the stationary case equation (16.11) becomes

$$\frac{d^2u}{dx^2} = \lambda^2 u, \qquad \lambda = \sqrt{\frac{2h}{b}}. \tag{1}$$

From this we obtain as the general solution

$$u = A e^{\lambda x} + B e^{-\lambda x};$$

The coefficients A and B are computed from the values

$$u = u_1 \text{ for } x = 0, \; u = u_3 \text{ for } x = l.$$

We obtain

$$u = \frac{u_3 \sinh \lambda x + u_1 \sinh \lambda(l - x)}{\sinh \lambda l} \tag{2}$$

and, setting $x = l/2$,

$$u_2 = \frac{(u_1 + u_3) \sinh \lambda l/2}{\sinh \lambda l} = \frac{u_1 + u_3}{2 \cosh \lambda l/2}.$$

The symbol q, which was introduced in the exercise now becomes:

$$q = \cosh \lambda l/2. \tag{3}$$

From this we obtain a quadratic equation for $\exp(\lambda l/2)$, which yields

$$\frac{\lambda l}{2} = \log \left(q + \sqrt{q^2 - 1}\right). \tag{4}$$

According to (1) we also have h expressed in terms of q. According to (13.5) h stands for the ratio of the exterior heat conductivity (which was denoted there by $4aT_0^3$) to the interior heat conductivity $\varkappa$. For $q = 1$ we obtain from (4) that $h = 0$ as required in the statement of the exercise.

III.4. In the stationary state the energy extracted from the. element of the rod (length dx, cross-section q) through heat conduction must equal the Joule heat generated in the same element. If we express the current i in terms of the potential difference V then we obtain as the differential equation of the stationary state:

$$\frac{d^2u}{dx^2} = -a, \quad a = \frac{V^2}{l^2} \frac{\sigma}{\varkappa}.$$

Due to the boundary conditions at the ends of the rod the integration of this differential equation yields a symmetric parabola as the graph of the temperature process. We determine a in terms of the maximal temperature U and obtain:

$$\frac{\varkappa}{\sigma} = \frac{1}{8} \frac{V^2}{U}.$$

Thus, through measurement of V and U we can verify the empirical law of Wiedemann and Franz which, according to the theory of metal electrons, asserts that

$$\frac{\varkappa}{\sigma} = \frac{\pi^2}{3}\left(\frac{k}{e}\right)^2 T,$$

where T = the absolute temperature, k = the Boltzmann constant, and e = the electron charge.

IV.1. a) For integral n we can expand the function $\exp(i\varrho\cos w)$ of (19.18) in the well known power series. As the coefficients of ϱ^k we then obtain

$$\text{(1)} \qquad a_k = \frac{e^{i(k-n)\pi/2}}{2\pi\, k!\, 2^k} \int_{-\pi}^{+\pi} (e^{iw} + e^{-iw})^k e^{inw}\, dw.$$

If we perform the binomial expansion under the integral sign then only one term remains upon integration, and even that term remains only for $k - n \geqq 0$. This result agrees with (19.34).

b) For non-integral n (ϱ is assumed real) we substitute in (19.14)

$$\text{(2)} \qquad t = \frac{\varrho}{2}\, e^{-i(w-3\pi/2)}, \qquad dw = i\,\frac{dt}{t}.$$

The path W_0, which may be assumed rectangular, is then transformed into the loop of Fig. 37a which starts from $+\infty$, circles the origin in clockwise direction, and returns to $+\infty$ according to the scheme

$$\begin{array}{llllllll} w = i\infty - \frac{\pi}{2}, & -\frac{\pi}{2}, & 0, & +\frac{\pi}{2}, & \pi, & \frac{3\pi}{2}, & i\infty + \frac{3\pi}{2}; \\ t = +\infty \quad, & e^{2\pi i}\frac{\varrho}{2}, & e^{3\pi i/2}\frac{\varrho}{2}, & e^{i\pi}\frac{\varrho}{2}, & e^{i\pi/2}\frac{\varrho}{2}, & \frac{\varrho}{2}, & +\infty \end{array}$$

Equation (19.14) then becomes

$$\text{(3)} \qquad I_n(\varrho) = -\frac{e^{i\pi n}}{2\pi i}\left(\frac{\varrho}{2}\right)^n \int_S e^{-t+\frac{\varrho^2}{4t}}\, t^{-n-1}\, dt.$$

If we now expand in $\exp\left(\frac{\varrho^2}{4t}\right)$ we again obtain the series (19.34), provided we use the following general definition of the Γ-function:

$$\text{(4)} \qquad \frac{1}{\Gamma(x+1)} = \frac{e^{i\pi(x+1)}}{2\pi i}\int_S e^{-t}\, t^{-x-1}\, dt;$$

We can easily verify that this definition coincides with the elementary definition $\Gamma(x+1) = x!$ for integral x by forming residues for $t = 0$.

IV.2. In order to complete the investigation of the real part of $i\varrho\cos w$ (p. 86) we compute for complex $\varrho = |\varrho|\, e^{i\Theta}$ the quantity

(1) $$X = \mathbf{Re}\,(i\,\varrho\cos w) = \frac{|\varrho|}{2}\,[\sin(p-\Theta)\,e^{q} - \sin(p+\Theta)\,e^{-q}].$$

In order that X become negative in the infinite part of the upper half of the w-plane ($q \gg 1$) we must have

(2) $$\sin(p-\Theta) < 0\,.$$

The shaded strip

$$-\pi < p < 0$$

of Figs. 18 and 19 then shifts into the strip,

$$-\pi + \Theta < p < \Theta,$$

that is, the strip is shifted to the right or left in Fig. 18 according as θ increases or decreases. The opposite holds in the lower half of the w-plane, where according to (1) we must replace (2) by

(3) $$\sin(p+\Theta) > 0.$$

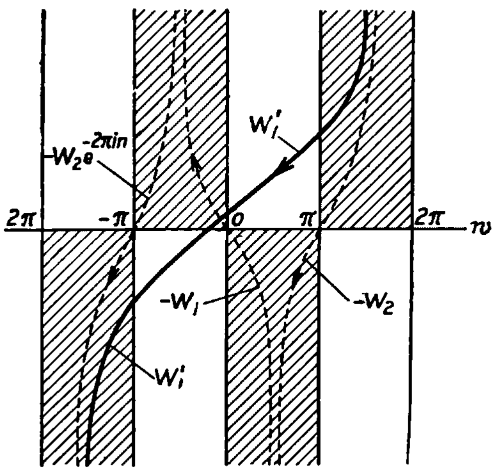

Fig. 36a. Distortion of the path of integration W_1 of H_n^1 into W_1' for a full circuit around the origin.

For $0 < \Theta < \pi$ (upper half of the positive ϱ-plane) the shaded regions of the upper and lower half planes have finite segments of the real axis in common, so that the path W_1 can be situated entirely within the shaded region. From this follows, without the use of the asymptotic formula (19.55), that H^1 vanishes for $\varrho \to \infty$ in the upper half plane. Due to this shift of the shaded regions we also see that for $0 > \Theta > -\pi$ (lower half of the ϱ-plane) the path W_1 must necessarily lead across the non-shaded region so that H^1 becomes infinite as $\varrho \to \infty$. The opposites of both these statements holds for W_2 and H^2.

Figure 36a illustrates the effect of this shift on a full circuit of ϱ around the origin with respect to the path W_1. The beginning of W_1 has been shifted by 2π, the end by -2π; thus W_1 has been distorted into W_1'. However W_1' can be decomposed into three partial paths of the same character as the original W_1, W_2 following the symbolic equation

$$W_1' = -W_2 - W_1 - W_2 e^{-2\pi i n}. \tag{4}$$

Here $-W_1$ is the middle one of the three shaded partial paths of the figure and it differs from W_1 in orientation only; $-W_2$ is the path on the right; the path on the left is obtained from W_2 by replacing w by $w - 2\pi$, thereby changing the factor $\exp(i n w)$ in the integrand by the factor $e^{-2\pi i n}$ Thus, as a result of (4) we obtain:

$$H_n^1(\varrho e^{2\pi i}) = -H_n^1(\varrho) - H_n^2(\varrho)(1 + e^{-2\pi i n}). \tag{5}$$

When n is an integer this becomes

$$H_n^1(\varrho e^{2\pi i}) = -H_n^1(\varrho) - 2 H_n^2(\varrho), \tag{6}$$

which we can rewrite in the form:

$$H_n^1(\varrho e^{2\pi i}) - H_n^1(\varrho) = -2\{H_n^1(\varrho) + H_n^2(\varrho)\} = -4 I_n(\varrho). \tag{6a}$$

This change in H_n^1 of $4I_n$ together with the relation $H^1 = I + iN$ correspond to the change in $\log \varrho$ of $2\pi i$ in the formula (19.47) for N.

Figure 36b represents the correspondingly distorted path W_2' of H^2 for a full positive circuit of ϱ around the origin. This path can be decomposed into five partial paths of the same character as W_1, W_2 following the symbolic equation:

$$W_2' = W_2 + W_1 + W_1 e^{2\pi i n} + W_2 e^{-2\pi i n} + W_2 e^{+2\pi i n}. \tag{7}$$

Here W_2 is the middle one of the five partial paths, W_1 is the path adjacent on the left, $W_1 e^{2\pi i n}$ is the path adjacent on the right, etc. Instead of (5) we now obtain

$$H_n^2(\varrho e^{2\pi i}) = H_n^2(\varrho)(1 + e^{-2\pi i n} + e^{+2\pi i n}) + H_n^1(\varrho)(1 + e^{2\pi i n}). \tag{8}$$

When n is an integer this becomes

$$H_n^2(\varrho e^{2\pi i}) = 3 H_n^2(\varrho) + 2 H_n^1(\varrho), \tag{9}$$

$$H_n^2(\varrho e^{2\pi i}) - H_n^2(\varrho) = 2(H_n^1(\varrho) + H_n^2(\varrho)) = 4 I_n(\varrho). \tag{9a}$$

The change (9a) together with the relation $H^2 = I - iN$ again correspond to the charge in $\log \varrho$ of $2\pi i$ in (19.47).

The equations (5) and (8) are the so-called *circuit relations* of the Hankel functions for the angle increment $\Delta\Theta = 2\pi$. These relations correspond to the "relationes inter contiguas" which were established by Gauss for the hypergeometric functions (see §24). Just as the equations (6a), (9a) for integral n are obtained from (19.47), the general relations (5) and (8) can be derived from the representations (19.31) and (19.30).

The circuit relations can be generalized from the full circuits $\Delta\Theta = 2\pi\nu$ (ν = integer) to half-circuits $\Delta\Theta = \pi\nu$. For a later applica-

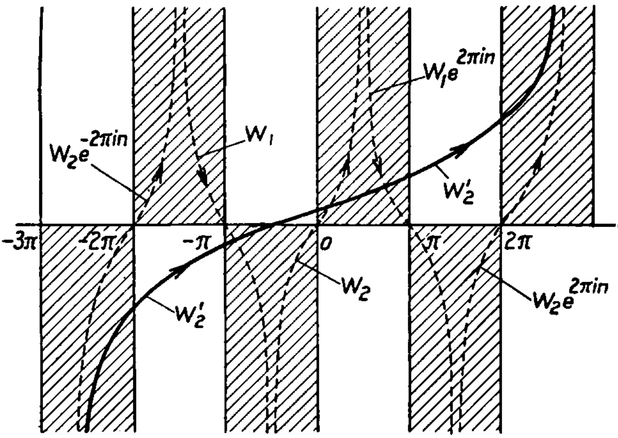

Fig. 36b. Distortion of the path W_2 of H_n^2 into W_2' for a full circuit around the origin.

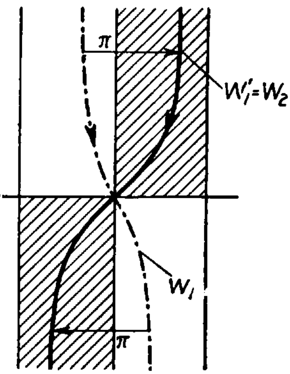

Fig. 36c. The half-circuit relation for H_0^1.

tion in §32 we discuss the half-circuit relation for $\nu = 1$ and $n = 0$. The relation reads

$$H_0^1(\varrho\, e^{i\pi}) = -H_0^2(\varrho). \tag{10}$$

For a proof, a look at Fig. 36c suffices. For real ϱ the path W_1 (indicated by dots and dashes in the figure) leads from the region $-\pi < p < 0$ to the region $0 < p < +\pi$; for the present argument $\varrho e^{i\pi}$ this path has been shifted by $+\pi$ in the upper part and by $-\pi$ in the lower part, as indicated by the arrows in the figure. The path W_1' obtained in this manner is identical with the path W_2 for H^2. But this is the statement of equation (10).

A relation which is analogous to (10) is obtained if we replace ϱ by $\varrho e^{-i\pi}$:

$$H_0^2(\varrho\, e^{-i\pi}) = -H_0^1(\varrho). \tag{10a}$$

Considering the factor exp (inw) of the integrand we can generalize (10) to

$$H_n^1(\varrho\, e^{i\pi}) = -e^{-n\pi i} H_n^2(\varrho), \tag{11}$$

for arbitrary n, or (replacing ϱ by $\varrho^{-i\pi}$)

$$H_n^2(\varrho\, e^{-i\pi}) = -e^{+n\pi i} H_n^1(\varrho). \tag{11a}$$

These half-circuit relations can also be derived directly from the equations (19.30) and (19.31) with the help of the equations

$$(12)\qquad I_n(\varrho\cdot e^{i\pi\nu}) = e^{i\pi n\nu}\, I_n(\varrho), \quad I_{-n}(\varrho\cdot e^{i\pi\nu}) = e^{-i\pi\nu n}\, I_{-n}(\varrho).$$

which follow from (19.34).

These relations become very simple if we write them for $\psi_n(\varrho)$ and $\zeta_n(\varrho)$ in which the corresponding Hankel and Bessel functions of index $n+\frac{1}{2}$ with integral n are multiplied by $\sqrt{2\varrho/\pi}$; namely we have

$$(13)\qquad \zeta_n^{1,2}(\varrho\, e^{\pm i\pi}) = (-1)^{n+1}\,\zeta_n^{2,1}(\varrho), \qquad \psi_n(\varrho\, e^{\pm i\pi}) = (-1)^{n+1}\,\psi_n(\varrho).$$

In the representation (19.22) for $H_0^1(\varrho)$ we substitute

$$x = i\varrho\cos w, \qquad dw = \frac{i\,dx}{\sqrt{x^2+\varrho^2}}.$$

The path W_1, which for convenience is to be taken rectangular, is then transformed into the x-plane according to the scheme

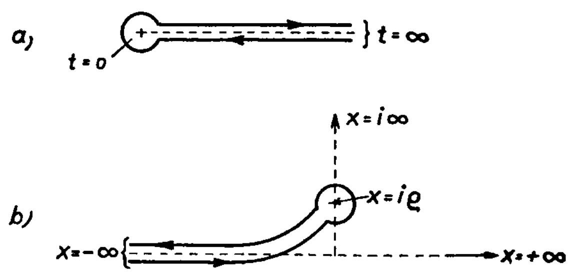

Fig. 37a. The loop integrals for $1/\Gamma(x+1)$. b. The loop integral for $H_0^1(\varrho)$ for small ϱ.

$$\begin{array}{llllll} w = & -\frac{\pi}{2}+i\infty, & -\frac{\pi}{2}, & 0, & +\frac{\pi}{2}, & \frac{\pi}{2}+i\infty, \\ x = & -\infty & , & 0, \quad i\varrho, & 0, & -\infty \quad . \end{array}$$

We thus have the loop integral of Fig. 37b that begins at the negative infinite end of the real x-axis, circles the point $x = i\varrho$ and returns to the negative infinite end of the real axis; the orientation of this loop is controlled by a small displacement of the real branch of W_1. From (19.22) we then obtain:

$$(1)\qquad H_0^1(\varrho) = \frac{2i}{\pi}\int_{-\infty}^{+i\varrho}\frac{e^x\,dx}{\sqrt{x^2+\varrho^2}}.$$

This integral is obtained by the combination of the two branches of Fig. 37b, where the originally negative sign of the returning branch has been reversed through the complete circuit around the branch point

$x = i\varrho$. Through integration by parts with respect to x we obtain

$$(2) \qquad H_0^1(\varrho) = \frac{2i}{\pi}\left\{e^x \log\left(x + \sqrt{x^2+\varrho^2}\right)\right\}_{x=i\varrho} - \frac{2i}{\pi}\int_{-\infty}^{i\varrho} e^x \log\left(x+\sqrt{x^2+\varrho^2}\right) dx\,.$$

Substituting $x = i\varrho$ in the first term and letting $\varrho \to 0$ in both terms we obtain

$$(3) \qquad \operatorname*{Lim}_{\varrho\to 0} H_0^1(\varrho) = \frac{2i}{\pi}\log i\varrho - \frac{2i}{\pi}\int_{-\infty}^{0} e^x \log 2x\, dx\,.$$

With the substitution $x = -t$ the last integral becomes

$$(4) \qquad \log(-2) + \int_0^\infty e^{-t}\log t\, dt = \log(-2) - C = \log\left(\frac{-2}{\gamma}\right).$$

Here C and γ are the quantities defined in (19.41a) (check using the Laplace integral for the Γ-function). Combining (4) and (3) we obtain from (1):

$$(5) \qquad \operatorname*{Lim}_{\varrho\to 0} H_0^1(\varrho) = \frac{2i}{\pi}\left(\log\frac{\gamma\varrho}{2} - \frac{i\pi}{2}\right) = \frac{2i}{\pi}\log\frac{\gamma\varrho}{2} + 1\,.$$

Due to the relation

$$H_0 = I_0 + i\,N_0, \quad \operatorname*{Lim}_{\varrho\to 0} H_0 = 1 + i \operatorname*{Lim}_{\varrho\to 0} N_0$$

equation (5) coincides with the equation (19.48) for N.

IV.4. 1. If we neglect $1/\varrho$ in the differential equation (19.11) we obtain $H_n^1 = A\,e^{i\varrho}$ (A = constant of integration; the solution involving $e^{-i\varrho}$ corresponds to H_n^2).

2. We now consider A not as a constant but as a "slowly varying function of ϱ" such that A'', A'/ϱ and A/ϱ^2 can be neglected. This yields a differential equation for $A(\varrho)$, from which we obtain $A = B/\sqrt{\varrho}$. The normalizing constant B cannot, of course, be determined in this manner.

IV.5. a) According to the equations (1.12), (22.14), (22.31) we obtain as the coefficients C_m and A_{mn} of the exercise:

$$C_m = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(\vartheta_0,\varphi_0)\, e^{-im\varphi_0}\, d\varphi_0,$$

$$A_{mn} = \frac{1}{N_n^m}\int_{-\pi}^{+\pi} C_m(\vartheta_0)\, P_n^m(\cos\vartheta_0)\sin\vartheta_0\, d\vartheta_0$$

$$= \frac{n+\frac{1}{2}}{2\pi}\frac{(n-m)!}{(n+m)!}\int\limits_0^{\pi}\sin\vartheta_0\,d\vartheta_0\int\limits_{-\pi}^{+\pi}d\varphi_0 f(\vartheta_0,\varphi_0)\,P_n^m(\cos\vartheta_0)\,e^{-im\varphi_0}.$$

b) From the scheme for $f(\vartheta,\varphi)$ in the exercise we see that if we multiply $f(\vartheta_0,\varphi_0)$ by

$$Y_{\nu\mu} = P_\nu^\mu(\cos\vartheta_0)\,e^{-i\mu\varphi_0}$$

and integrate with respect to φ_0 then we obtain

$$\int\limits_{-\pi}^{+\pi} f(\vartheta_0,\varphi_0)\,Y_{\nu\mu}\,d\varphi_0 = 2\pi\sum_n A_{n\mu}\,P_n^\mu(\cos\vartheta_0)\,P_\nu^\mu(\cos\vartheta_0),$$

Integrating with respect to $\sin\vartheta_0\,d\vartheta_0$ from (22.14) and (22.31) we obtain the result

$$\int\int f(\vartheta_0,\varphi_0)\,Y_{\nu\mu}\,d\sigma_0 = 2\pi\,N_\nu^\mu\,A_{\nu\mu},$$

After a change in notation (ν, μ instead of n,m) this coincides with the expression for A_{mn} in a) except for the order of summation (see Fig. 38).

In a) the horizontal strips $|m| \leq n < \infty$ are summed in the vertical direction, and in b) the vertical strips $-n \leq m \leq +n$ are summed in the horizontal direction. In both cases the total domain of summation is bounded by the lines $n = \pm m$; thus the sums are the same.

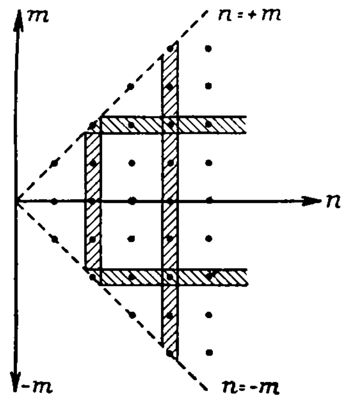

Fig. 38. The double sum in the number lattice of n,m; arranged in horizontal strips for case a) and in vertical strips for case b).

IV.6. Again we draw the 60° wedge of Fig. 17 together with its five reflected images, so that the original wedge lies symmetric with respect to the horizontal plane. For reasons of convenience we situate the center of inversion C on the horizontal line through the vertex O of the wedge and we let the circle of inversion (dotted circle in the figure) pass through O. Since the point at infinity is mapped into C and the points of intersection O and S_1 of the line 1 with the circle of inversion remain fixed, the position of the circle into which the straight line 1, − 1 is transformed is determined by the points O,C,S_1. The arcs of the circle which correspond to the half lines 1 and − 1 are again denoted by 1 and − 1. The same holds for the line 2, − 2 which is mapped into a circle of the same radius passing through the points O,C,S_2. The line 3, − 3 goes into a circle of diameter OC in which the upper and lower semicircles correspond to the half lines − 3 and 3.

Now the wedge 1,2 is mapped into the exterior of the circular diangle $C,S_1,1,O,2,S_2,C$; both regions are indicated by a shading of the boundary. We now seek the images of the reflected wedges. All these images are interiors of certain circular diangles (crescents); e.g., the wedge 2,3 is mapped into the crescent $C,S_2 2,0,3,C$, and the wedge -2, -1 is mapped into the small lens-like region $C,-2,O,-1,C$ in the center of our figure.

Up to now we have described the drawing as a *plane* figure and spoken of straight lines, circles, circular diangles, etc. However there is nothing that prevents us from interpreting the figure as *three-dimensional* and to speak of planes and spheres instead of straight lines and circles. These spheres are then situated with their centers in the plane of the drawing. The original wedge 1,2 then is mapped into the exterior of the two intersecting spheres which belong to the circles 1, $-$ 1 and 2, $-$ 2; in the same manner the reflected wedges correspond to the regions bounded by two of the spheres 1,2,3.

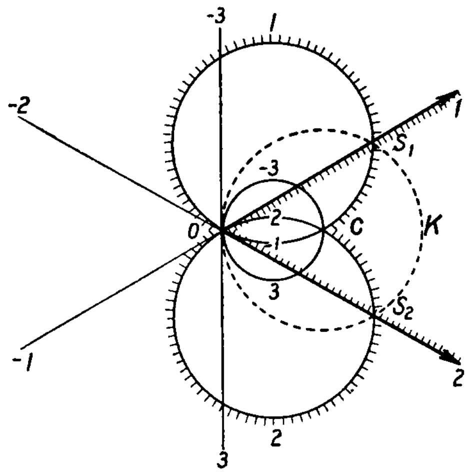

Fig. 39. The wedge 1,2 of the 60° angle is mapped by inversion into the exterior of $C,1,O,2,C$ of the intersecting spheres $1,-1$ and $2,-2$; the reflected wedges 2,3; $3,-1$;. . . are mapped into spherical crescents.

Just as before, we obtained *Green's function* for the wedge from the elementary reflections in Fig. 17, now in the case of the *potential equation* (but only in this case) we obtain Green's function for the corresponding circular or spherical regions by finding the "electric image" of the given pole upon inversion on the boundary circles or spheres 1,2,3 and by giving alternating signs to these poles. For the symmetric structure of our problem it suffices to have five such electric images in order to satisfy the boundary condition $u = 0$ on the boundary of each of the regions under consideration.

IV.7. a) In the inversion in the sphere K (broken line) of Fig. 40 all the infinite points of the reflecting planes ± 1, ± 2, ± 3, . . . are mapped into the center of inversion C; thus the planes $\pm$ 1 go into the spheres $+$ 1 and $-$ 1 which are tangent at C and have diameter equal to the radius a of the sphere of inversion. Here the exterior of the spheres $\pm$ 1 corresponds to the interior of the plate and the interior of these spheres corresponds to the exterior of the plate. The planes $\pm$ 2 are mapped into spheres which are again tangent at C but have a diameter of only $a/3$. The images of the planes $\pm$ 3 are in turn spheres which lie in the interior of the spheres $\pm$ 2, have the diameter $a/5$ and are tangent at C. The region bounded by two consecutive spheres in this sequence corresponds to a reflected image of the original plate.

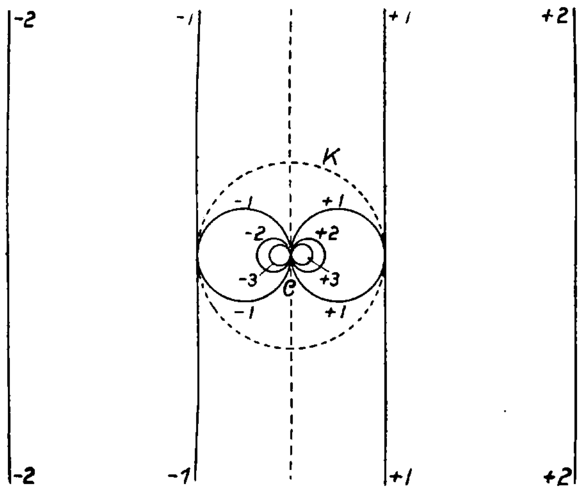

Fig. 40. Inversion of a plane parallel plate and its successive reflected images into a system of spheres tangent at the center of inversion C.

Green's function of potential theory for the exterior of two tangent spheres (e.g., the spheres $\pm$ 1 of our figure) can be deduced by inversion from Green's function for the plane parallel plate. The infinitely many image points of the arbitrarily prescribed pole of Green's function that arise in the inversion are situated in the successive spherical regions mentioned above, and they accumulate at the point C.

b) If the radii of the concentric spheres I and II are a and $2a$ then we may choose the radius of the sphere of inversion equal to a and place its center C on sphere I. Then sphere I is mapped into the plane E_{I}, and II is mapped into a sphere K_{II} of radius $2a/3$; the minimum distance between K_{II} and E_{I} is $a/6$. Conversely, E_{I} and K_{II} are mapped into the concentric spheres I and II.

For an arbitrary position of the non-intersecting plane E and sphere K we can proceed in the following way (kindly communicated to me by Caratheodory): from the center of K we drop the perpendicular L to E; from the foot F of L we draw tangents (of length t) to K and draw the auxiliary sphere H with center F and radius t. As the center of inversion we choose one of the points of intersection S of L and H. Then L is transformed into a straight line, H becomes a plane which is perpendicular to L, and E and K become spheres which are perpendicular to H and L and hence have their center at the point of intersection of H and L, that is, in the center of inversion. The radius of inversion remains arbitrary and determines only the size of the concentric spheres.

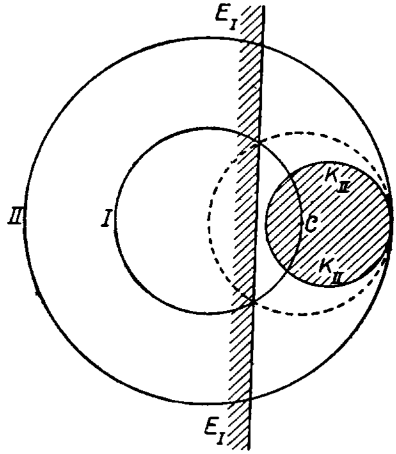

Fig. 41. Two concentric spheres I and II are transformed through inversion into the plane E_I and the sphere K_{II} (which are shaded in the figure).

Instead of E and K we may also consider two arbitrary nonintersecting spheres K_1 and K_2 (see the last statement in the exercise). In order to transform them into two concentric spheres we start from the pencil of spheres $K_1 + \lambda K_2 = 0$. The pencil contains two spheres of radius 0, namely the two poles of the bipolar coordinate system. If we choose one of these poles as the center of inversion then all the spheres of the pencil, including K_1 and K_2 are mapped into *concentric* spheres.

IV.8. Let u_1 and u_2 be linearly independent solutions of the differential equation

$$L(u) = u'' + pu' + qu = 0$$

of second order, where p and q are arbitrary given functions of the independent variable ϱ. Then for $X = u_1u_2' - u_2u_1'$ we have:

$$u_1 L(u_2) - u_2 L(u_1) = \frac{dX}{d\varrho} + p\,X = 0,$$

hence $X = C\,e^{-\int p\,d\varrho}$, $C =$ constant of integration.

a) For the Bessel differential equation (19.11) we have
and hence

$$X = C\,e^{-\log\varrho} = \frac{C}{\varrho}.$$

If we take $u_1 = H_n^1$, $u_2 = H_n^2$ then C is determined most simply from the asymptotic values (19.55), (19.56). We obtain

$$X = \frac{-4i}{\varrho\,\pi}, \qquad C = \frac{-4i}{\pi}.$$

Due to $I_n = \frac{1}{2}(H_n^1 + H_n^2)$, we see that the expression (I) in the exercise, where H is to stand for H^1, equals half the above X, so that

$$(\mathrm{I}) = -\frac{2\,i}{\varrho\,\pi} \quad ;$$

the sign is reversed if H is to stand for H^2.

The determination of C becomes somewhat less simple if we start from $\varrho = 0$ instead of $\varrho = \infty$.

b) In the differential equation (21.11a) we have $p = 2/\varrho$, and hence $X = C/\varrho^2$. If we take $u_1 = \zeta_n^1$, $u_2 = \zeta_n^2$, then according to (21.14) we have, for $\varrho \to \infty$,

$$\zeta_n^{1,2} = \frac{1}{\varrho}\, e^{\pm i[\varrho - (n+1)\pi/2]}, \qquad X = -\frac{2\,i}{\varrho^2}, \qquad C = -2\,i\,.$$

For the expression (II) we then obtain:

$$(\mathrm{II}) = \mp\frac{i}{\varrho^2} \quad ,$$

where the sign depends on whether we set ζ equal to ζ^1 or ζ^2.

V.1. a) Due to $I_n'(\lambda a) = 0$ the limit process of (20.9) yields the normalizing integral

$$N = -a \lim_{\varepsilon \to 0} \frac{I_n(\lambda a)\, I_n'(\lambda a + \varepsilon a)}{2\,\varepsilon} = -\frac{a^2}{2}\, I_n(\lambda a)\, I_n''(\lambda a)$$

instead of (20.19). With the help of the Bessel differential equation we obtain

$$N = \frac{1}{2\,\lambda^2}(\lambda^2 a^2 - n^2)\, I_n^2(\lambda a)\,.$$

Hence if we "normalize $I_n(\lambda r)$ to 1" we obtain

$$\sqrt{\frac{2\,\lambda^2}{\lambda^2 a^2 - n^2}}\;\frac{I_n(\lambda r)}{I_n(\lambda a)}\,.$$

b) From the relation (21.11) between $\psi_n(\varrho)$ and $I_{n+\frac{1}{2}}(\varrho)$ we obtain for the present normalizing integral

$$N = \int_0^a \psi_n^2(k\,r)\, r^2\, dr = \frac{\pi}{2\,k}\int_0^a I_{n+\frac{1}{2}}^2(k\,r)\, r\, dr\,.$$

For the boundary condition

$$\psi_n'(k\,a) = \sqrt{\frac{\pi}{2\,k\,a}}\left(I_{n+\frac{1}{2}}'(k\,a) - \frac{1}{2\,k\,a}\, I_{n+\frac{1}{2}}(k\,a)\right) = 0$$

of the exercise we obtain by a limit process analogous to that of a)

$$N = -\frac{\pi}{2k}\frac{a^2}{2} J_{n+\frac{1}{2}}(ka)\left\{J''_{n+\frac{1}{2}}(ka) + \frac{1}{2ka} J'_{n+\frac{1}{2}}(ka)\right\}$$

with the help of the Bessel differential equation we may therefore write

$$N = \frac{a}{2k^2}\psi_n^2(ka)\{k^2a^2 - n(n+1)\};$$

thus the normalized form of ψ_n is

$$\psi'_n = \sqrt{\frac{2k^2/a}{k^2a^2 - n(n+1)}}\,\frac{\psi_n(kr)}{\psi_n(ka)}.$$

V.2. The proof follows from Green's theorem

$$(1)\qquad \int (u\Delta v - v\Delta u)\,d\tau = \int \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) d\sigma,$$

if we set $v = 1/r$ where r stands for the distance of the point of integration from P. Due to the singularity of v at P we surround P in the customary manner by a sphere K_ϱ of radius $\varrho \to 0$. If we extend the integration on the left side over the region bounded by K_a and K_ϱ, then the left side vanishes and the right side becomes the sum of the surface integrals over K_a and K_ϱ (in both cases n stands for the exterior normal to the region). By letting $\varrho \to 0$ in the integral over K_ϱ we obtain from (1)

$$(2)\qquad 0 = 4\pi u_P - \frac{1}{a^2}\int_{K_a} u\,d\sigma - \frac{1}{a}\int_{K_a}\frac{\partial u}{\partial n}\,d\sigma.$$

The third term on the right here vanishes since throughout the interior of S we have $\Delta u = 0$. Thus equation (2) proves the theorem of the arithmetic mean.

V.3. From equation (27.14) and the condition $u = U$ on the sphere $r_0 = a$ we obtain

$$(1)\qquad 2\pi U = \sum_n \sum_m A_{nm} \Pi_n^m(\cos\vartheta_0)\, e^{-im\varphi_0};$$

Multiplying by $e^{i\mu\varphi_0}$ and integrating with respect to φ_0 from 0 to π we obtain:

$$(2)\qquad \int_0^{2\pi} U e^{i\mu\varphi_0}\,d\varphi_0 = \sum_n A_{n\mu}\Pi_n^\mu(\cos\vartheta_0);$$

multiplying by $\Pi_\nu^\mu(\cos\vartheta_0)\sin\vartheta_0$ and integrating with respect to ϑ_0 from

0 to π we have:

(3) $$\int_0^\pi \int_0^{2\pi} U\, \Pi_\nu^\mu (\cos\vartheta_0)\, e^{i\mu\varphi_0} \sin\vartheta_0\, d\vartheta_0\, d\varphi_0 = A_{\nu\mu}\,,$$

which coincides with (27.13a) except for notation.

Comparing the r-dependence of (27.13) and (27.14) we obtain

(4) $$-a^2 \sum_l \frac{1}{k_{nl}} \Psi_n(k_{nl} r_0)\, \Psi_n'(k_{nl} a) = \left(\frac{r_0}{a}\right)^n,$$

The summation extends over all the roots $k = k_n$ of the equation $\Psi_n(k a) = 0$, which are the same as the roots of the equation $\psi_n(k a) = 0$. In order to determine the Ψ_n in terms of the ψ_n we use equation (21.11)

(5) $$\psi_n(x) = \sqrt{\frac{\pi}{2x}}\, I_{n+\frac{1}{2}}(x)$$

and the relation (20.19) (which holds for non-integral n, too)

(5a) $$\int_0^a [I_{n+\frac{1}{2}}(k r)]^2\, r\, dr = \frac{a^2}{2} [I'_{n+\frac{1}{2}}(k a)]^2,$$

where k is a root of $I_{n+\frac{1}{2}}(k a) = 0$, and hence a root of $\psi_n(k a) = 0$. From (5) and (5a) we obtain:

(6) $$\int_0^a [\psi_n(k r)]^2\, r^2\, dr = \frac{a^3}{2} [\psi_n'(k a)]^2.$$

If we now set $\Psi_n = N \psi_n$ and impose the condition

(7) $$\int_0^a \Psi_n^2(k r)\, r^2\, dr = 1,$$

we obtain

$$N^2 = 2/a^3\, [\psi_n'(k a)]^2.$$

Rewriting the equation (4) in terms of ψ_n, and adopting the notation $\alpha = r_0/a$, we obtain:

(8) $$2 \sum_l \frac{\psi_n(k_{nl}\, \alpha\, a)}{a\, k_{nl}\, \psi_n'(k_{nl} a)} = -\alpha^n.$$

For $n = 0$ we have according to (21.11)

$$\psi_0(x) = \frac{\sin x}{x};$$

therefore

$$\psi_0(k a) = 0 \quad \text{for} \quad k = k_{0l} = \frac{l\pi}{a}, \qquad l = \pm 1, \pm 2, \ldots$$

In this particularly simple case equation (8) becomes

$$2\sum_{l=1}^{+\infty}(-1)^l\frac{\sin l\pi\alpha}{l}=-\pi\alpha. \tag{9}$$

For $\alpha=\frac{1}{2}$ this yields

$$1-\frac{1}{3}+\frac{1}{5}-\cdots=\frac{\pi}{4},$$

which is the Leibniz series (2.8). In general we obtain from (2.9) for $x=2\pi\alpha$ a representation of the "saw-tooth profile"

$$\left.\begin{array}{l} +\frac{1}{2}(\pi-x) \\ \\ -\frac{1}{2}(\pi+x) \end{array}\right\} = \sin x+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\cdots \text{ for } \left\{\begin{array}{l} 0<x<+\pi, \\ \\ -\pi<x<0. \end{array}\right.$$

The reader is asked to verify this as a further exercise for chapter I. It is apparent, however, that equation (8) for $n>0$ contains far deeper and more general analytic relations.

VI.1. a) Due to the fact that Π has the z-direction and depends only on z and $r^2 = x^2+y^2$, we obtain from (31.4)

$$\mathbf{E}_x=\frac{\partial}{\partial x}\frac{\partial\Pi}{\partial z}=\frac{x}{r}\frac{\partial^2\Pi}{\partial r\,\partial z},$$
$$\mathbf{E}_y=\frac{\partial}{\partial y}\frac{\partial\Pi}{\partial z}=\frac{y}{r}\frac{\partial^2\Pi}{\partial r\,\partial z}.$$

For the form a) of the exercise we have

$$\frac{\partial\Pi}{\partial z}=\frac{z-h}{R}\frac{d}{dR}\frac{e^{ikR}}{R}+\frac{z+h}{R'}\frac{d}{dR'}\frac{e^{ikR'}}{R'}.$$

which vanishes when $z=0$ since then $R=R'$. Hence the expressions for $\mathbf{E}_x$ and $\mathbf{E}_y$, which are obtained by differentiation with respect to x,y or r, also vanish for $z=0$.

b) According to (31.4) we now have

$$\mathbf{E}_x=k^2\Pi+\frac{\partial}{\partial x}\frac{\partial\Pi}{\partial x},$$
$$\mathbf{E}_y=\frac{\partial}{\partial y}\frac{\partial\Pi}{\partial x}.$$

But since in the form b) Π vanishes for $z=0$, we also have $\mathbf{E}_x$ and $\mathbf{E}_y$ vanishing for all points (x,y) on the earth's surface.

c) From (35.1) we obtain

$$\mathbf{E}_x=i\mu_0\omega\frac{\partial\Pi_z}{\partial y},\quad \mathbf{E}_y=-i\mu_0\omega\frac{\partial\Pi_z}{\partial x}.$$

These derivatives vanish for $z = 0$, since in the form c) Π_z itself vanishes for $z = 0$.

d) From (35.1) we now have

$$\mathbf{E}_x \equiv 0, \qquad \mathbf{E}_y = i\,\mu_0\,\omega\,\frac{\partial \Pi_x}{\partial z}.$$

But according to the form d) we have

$$\frac{\partial \Pi_x}{\partial z} = \frac{z-h}{R}\frac{d}{dR}\frac{e^{ikR}}{R} + \frac{z+h}{R'}\frac{d}{dR'}\frac{e^{ikR'}}{R'}.$$

which vanishes for $z = 0$ since then $R' = R$.

VI.2. This exercise is instructive not only for the understanding of Zenneck waves, but also for the general knowledge of electromagnetic rotational fields and for their representation using complex operators.

From (32.20) according to the prescription (31.4), we obtain for the *air*

$$\text{(1)} \qquad \begin{aligned} \mathbf{E}_x &= -i\,p\sqrt{p^2-k^2}\,A\,k_E^2\,e^{ipx-\sqrt{p^2-k^2}\,z},\\ \mathbf{E}_z &= p^2\,A\,k_E^2\,e^{ipx-\sqrt{p^2-k^2}\,z}. \end{aligned}$$

These expressions, multiplied by the exponential time factor, represent an elliptic oscillation as known from optics. Due to the complex nature of the right sides of (1) the principal axes of the oscillation ellipse are oblique to the x- and z-direction. If we form the absolute values of $\mathbf{E}_x$ and $\mathbf{E}_z$ together with their negatives, then we obtain the limits between which $\mathbf{E}_x$ and $\mathbf{E}_z$ oscillate, in other words we obtain a rectangle circumscribed about the ellipse. The ratio of the sides of this rectangle is given by the absolute value of

$$\text{(2)} \qquad \frac{\mathbf{E}_x}{\mathbf{E}_z} = \frac{\sqrt{k^2-p^2}}{p} = \frac{1}{n}.$$

The value $1/n$ is obtained from the definitions (32.16a) and (32.2) of p and n. Because $|n| > 1$ the rectangle is tall and narrow (See Fig. 42a).

On the other hand, due to (32.20) and (31.7), in the *earth* we have

$$\text{(3)} \qquad \begin{aligned} \mathbf{E}_x &= +i\,p\sqrt{p^2-p_E^2}\;A\,k^2\,e^{ipx+\sqrt{p^2-k_E^2}\,z},\\ \mathbf{E}_z &= \qquad\qquad\quad p^2\,A\,k^2\,e^{ipx+\sqrt{p^2-k_E^2}\,z}. \end{aligned}$$

Hence again we have an elliptic oscillation that, is this time, situated in a rectangle with the ratio of sides given by the absolute value of

$$\text{(4)} \qquad \frac{\mathbf{E}_x}{\mathbf{E}_z} = \frac{\sqrt{k_E^2-p^2}}{p} = n,$$

where n is again obtained from (32.16a) and (32.2). Because $|n| > 1$ the rectangle is now broad and squat (See Fig. 42b). The present ellipse is traversed in the opposite sense of the former, as is seen from the reciprocity of the values n and $1/n$ in (4) and (2).

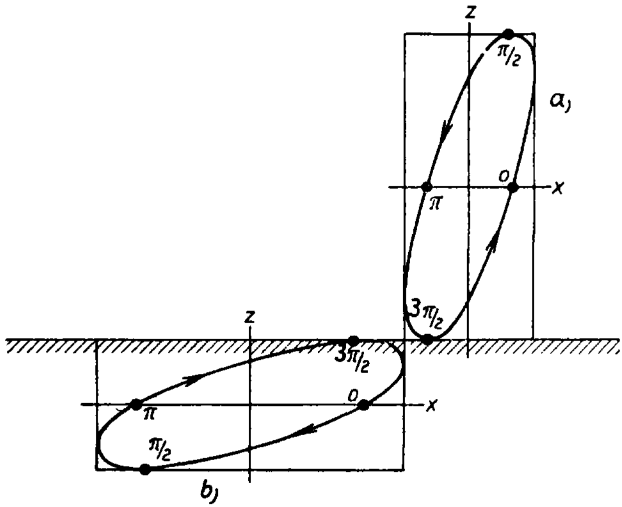

Fig. 42. The rotational field of the Zenneck wave a) in the air, narrow ellipse, b) in the earth, wide ellipse, congruent and of opposite orientation to each other.

If we think of the field in the air as pushing forward with its phase velocity in the positive x-direction, then the field in the earth appears to lag behind against the resistance there.

VI.3. The electric field strength at the antenna and in the direction of the antenna is $\mathbf{E} = \mathrm{Re}\{E\,e^{-i\omega t}\}$. At an antenna which is short compared to the wavelength this field strength does the work $\mathbf{E} j_t dt$ in the time dt. Hence according to equation (36.20) we have as the time average of work

$$W = j\,l \int_0^\tau \mathrm{Re}\{E\,e^{-i\omega t}\}\,\mathrm{Re}\{i\,e^{-i\omega t}\}\,\frac{dt}{\tau}.$$

where τ stands for the time of oscillation. With the method given on p. 271 we obtain

$$W = \frac{j\,l}{4}\int_0^\tau (E\,e^{-i\omega t} + E^*\,e^{+i\omega t})\,(i\,e^{-i\omega t} - i\,e^{+i\omega t})\,\frac{dt}{\tau}$$

$$= \frac{j\,l}{4}(-\,i\,E + i\,E^*).$$

and hence also

(1) $$W = \frac{j\,l}{2}\operatorname{Re}\{-i\,E\}.$$

a) *Vertical antenna.* From (31.4) and the differential equation of Π we obtain for $\Pi = \Pi_z$ and $\mathbf{E} = \mathbf{E}_z$

$$E = k^2 \Pi + \frac{\partial^2 \Pi}{\partial z^2} = -\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Pi}{\partial r}.$$

We express Π with the help of equation (32.9) and both e^{ikR}/R and $e^{ikR'}/R'$ are expressed with the help of (31.14). Since the r-dependence of these three terms is given by $I_0(\lambda r)$, the application of the operator

$$-\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}$$

under the integral sign yields the common factor $+\lambda^2 I_0(\lambda r)$, and hence at the point $r = 0$ of the antenna it yields the factor λ^2. Thus we obtain from (32.9):

(3) $$\operatorname{Re}\{-i\,E\} = \operatorname{Re}\left\{-i\int_0^\infty (e^{-\mu|z-h|} + e^{-\mu(z+h)})\frac{\lambda^3\,d\lambda}{\mu} + 2i\int_0^\infty e^{-\mu(z+h)}\frac{\mu_E}{n^2\mu+\mu_E}\frac{\lambda^3\,d\lambda}{\mu}\right\}.$$

Since μ is real for $\lambda > k$ (see p. 273), the integral over $k < \lambda < \infty$ in the first line does not contribute to the real part and we can pass to $z = h$, that is, to the point of the antenna, without encountering difficulties of convergence. Thus we obtain

(4) $$\operatorname{Re}\{-i\,E\} = \int_0^k -i(1+e^{-2\mu h})\frac{\lambda^3\,d\lambda}{|\mu|} + 2\operatorname{Re}\left\{i\int_0^\infty e^{-2\mu h}\frac{\mu_E}{n^2\mu+\mu_E}\frac{\lambda^3\,d\lambda}{\mu}\right\}.$$

If we take the values of these integrals given by (36.13) to (36.17), substitute (4) in (1), and append the factor (36.22) in order to express the result in terms of our units, then we obtain the value of W from (36.23).

b) *Horizontal antenna.* For $\Pi = (\Pi_x, \Pi_z)$ and $\mathbf{E} = \mathbf{E}_x$ we obtain from (31.4):

(5) $$E = k^2 \Pi_x + \frac{\partial^2 \Pi_x}{\partial x^2} + \frac{\partial^2 \Pi_z}{\partial x\,\partial z}.$$

Now according to (33.12) and (33.15) the x-dependence of Π_x is given by $I_0(\lambda r)$, and the x-dependence of Π_z is given by $\frac{x}{r} I_1(\lambda r)$. Hence for small x,y we have

$$I_0(\lambda r) = 1 - \frac{\lambda^2}{4}(x^2 + y^2) + \cdots, \qquad \frac{x}{r} I_1(\lambda r) = \frac{\lambda}{2} x + \cdots,$$

and for $r = 0$

$$\left(k^2 + \frac{\partial^2}{\partial x^2}\right) I_0 = \frac{1}{2}(2k^2 - \lambda^2), \qquad \frac{\partial}{\partial x}\frac{x}{r} I_1 = \frac{\lambda}{2}. \tag{6}$$

These factors $\frac{1}{2}(2k^2 - \lambda^2)$ and $\lambda/2$ appear under the integral signs of the equations (33.12) and (33.15) in the computation of (5), where for the first two terms on the right side of (33.12) we have to use (31.4) and where in (33.15) we have to perform the differentiation with respect to z, in addition to the differentiation with respect to x (this yields a factor $-\mu$ under the integral sign). Thus, instead of (5) we obtain

$$\begin{aligned} E = {} & \frac{1}{2}\int_0^\infty \frac{2k^2 - \lambda^2}{\mu}\left(e^{-\mu|z-h|} - e^{-\mu(z+h)}\right)\lambda^3\, d\lambda \\ & + \int_0^\infty e^{-\mu(z+h)}\left[\frac{2k^2 - \lambda^2}{\mu + \mu_E} + \frac{\lambda^2}{k^2}\frac{\mu(\mu - \mu_E)}{n^2\mu + \mu_E}\right]\lambda\, d\lambda. \end{aligned} \tag{7}$$

The first term in [ ] is due to the third term in (33.12), the second term is due to (33.15). If we form the common denominator of [ ] and observe that $\mu^2 - \mu_E^2 = k^2(n^2 - 1)$ (see p. 260), then we obtain

$$[\,] = \frac{\lambda^2 - 2\mu\mu_E}{n^2\mu + \mu_E}.$$

Hence for $z = 0$ the second line of (7) becomes identical with the integral for L in (36.17a). If in the first line of (7) we pass to $\mathbf{Re}\{-iE\}$ according to the procedure of (3), then we can again replace the upper limit ∞ by k and carry out the integration as in (36.16a). If we then pass to our system of units we obtain exactly the expression in (36.23a).