CHAPTER III

Boundary Value Problems in Heat Conduction

§13. Heat Conductors Bounded on One Side

In the preceding section we treated the equalization process for a linear heat conductor that is infinite in both directions and represented it by equation (12.10a):

$$(1) \qquad u(x,t) = \int_{-\infty}^{+\infty} f(\xi)\, U\, d\xi, \qquad U = (4\pi k t)^{-\frac{1}{2}} \exp\left\{-\frac{(x-\xi)^2}{4kt}\right\}.$$

By the substitution

$$(1a) \qquad \xi = x + \sqrt{4kt}\, z$$

it goes over into the *Laplace form*

$$(2) \qquad u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} f(x + \sqrt{4kt}\, z)\, e^{-z^2}\, dz\,.$$

It is instructive to compare this with d'Alembert's solution (9.2): in the latter we have two arbitrary functions F_1, F_2, corresponding to the hyperbolic type of the wave equation, whereas in (1) and (2) we have *one* arbitrary function f corresponding to the parabolic type of the equation of heat conduction.

In the case of a heat conductor which is bounded on one side, $0 < x < \infty$, we have to deal first with the boundary condition at $x = 0$:

a) A given temperature $u(0,t)$; in particular the *isothermal* boundary condition

$$(3a) \qquad u = 0.$$

b) A given heat flow $\mathbf{G}\,(0,t)$ (notation as in §7, equations (9) to (12)); in particular the *adiabatic* boundary condition

$$(3b) \qquad \frac{\partial u}{\partial x} = 0.$$

c) A linear combination of both which takes into consideration the so-called *outer heat conduction*, written in the conventional form

$$\frac{\partial u}{\partial n} + hu = 0. \tag{3c}$$

Here n stands for the *outer* normal which in our case is in the direction of the *negative* x-axis. The name "outer heat conduction" summarizes the effect of convection, the radiation into the surrounding medium and the heat conduction into that medium which is usually negligible. We note that (3c) is obtained as an approximation of the radiation law of Stefan-Boltzmann which states: the radiation of heat per unit of time and area of a body of absolute temperature T is proportional to T^4. If we denote the factor of proportionality by a and if the end of the rod[1] is in a neighborhood of temperature T_0 which radiates towards the end of the rod an amount of heat aT_0^4 per unit of time and area, then the energy emitted in the normal direction by a surface element $d\sigma$ of the end of the rod is given by:

$$dQ_n = a(T^4 - T_0^4)d\sigma\, dt.$$

Since usually both temperatures T and T_0 are far from absolute zero, we get

$$dQ_n \sim 4\, a\, T_0^3\, u\, d\sigma\, dt \quad \text{with} \quad u = T - T_0 \ll T \quad \text{and} \quad \ll T_0. \tag{4}$$

This amount of heat dQ_n must be balanced by the heat flow from the interior of the rod which is given by Fourier's law (7.12). Hence we write:

$$dQ_n = -\varkappa \frac{\partial u}{\partial n} d\sigma\, dt. \tag{4a}$$

By comparison of (4) and (4a) we obtain:

$$4\, a\, T_0^3\, u = -\varkappa \frac{\partial u}{\partial n},$$

and hence

$$\frac{\partial u}{\partial n} + h\, u = 0, \qquad h = \frac{4\, a\, T_0^3}{\varkappa}, \tag{5}$$

which corresponds to (3c) and shows h to be a *positive* constant.

We first treat conditions (3a) and (3b). These conditions are satisfied if we develop f, which is given only for $0 < x < \infty$, into a *pure sine or*

[1] We speak here of a "rod" although the linear heat conductor need not have the form of a thin rod but may have an arbitrary cross-section so long as its state depends only on *one* coordinate; for a real rod one must add the adiabatic condition $\partial u/\partial n = 0$ for the lateral surface (see the end of §16).

cosine integral, or in other words, if we continue f in the negative x-axis as an *even* or *odd* function (see equations (4.11a,b)). If as in (12.8) we append the time dependence factor $\exp(-\omega^2 k t)$ which is required by the equation of heat conduction, and integrate with respect to ω, then (1) becomes

$$u(x, t) = \int_0^\infty f(\xi)\, U(\xi)\, d\xi \mp \int_0^\infty f(\xi)\, U(-\xi)\, d\xi\,. \tag{6}$$

The second integral which was originally taken from $-\infty$ to 0 has been converted by a change of sign of the variable of integration into an integral from 0 to $+\infty$. The principal solution $U(\xi)$ is then transformed into

$$U(-\xi) = (4\,\pi\, k\, t)^{-\frac{1}{2}} \exp\left\{-\frac{(x+\xi)^2}{4\,k\,t}\right\} \tag{7}$$

which is the expression for a unit heat pole at $x = -\xi$, $t = 0$. Equation (6) becomes:

$$u(x, t) = \int_0^\infty f(\xi)\, G\, d\xi, \qquad G = U(\xi) \mp U(-\xi)\,. \tag{8}$$

This *Green's function* G satisfies all the conditions of p. 61. It has only one heat pole in the region $0 < x < \infty$, since the additional heat pole at $x = -\xi$ lies outside the region; it also complies with the condition that G satisfy the *adjoint* equation in the variables ξ, τ, since in our case G is independent of τ and a change of sign in τ becomes immaterial.

It would be more intuitive to start from a single heat pole at $x = \xi$ and to reflect it on the boundary $x = 0$, with a negative or positive sign of U depending on whether we impose condition (3a) or (3b). In this manner we would first construct Green's function and then reconstruct the given initial temperature $f(x)$ by the successive superposition of the heat poles of strength $f(\xi)\, d\xi$. From now on we shall use mainly this intuitive process, that is we restrict ourselves to the *construction of Green's function* from which we can write down the solution for arbitrary initial temperature $f(x)$ as in (8). We first use this

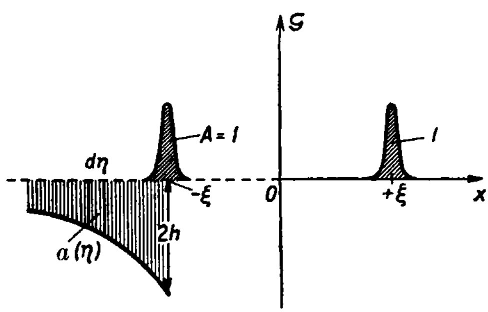


Fig. 15. Green's function for a linear heat conductor which is bounded on one side with outer heat conduction. A heat pole is at $x = +\xi$, its mirror image at $x = -\xi$, with an associated continuous spectrum of heat sources.

process for the somewhat more complicated boundary condition (3c). (In problem III.1 we shall treat the same boundary condition according to Fourier's process.)

We notice at once that an isolated reflected point $x = -\xi$ will not be sufficient, but that we also need a continuous sequence of heat sources which we shall place at all points $\eta < -\xi$. Let A and $a(\eta)\, d\eta$ be the yields of the isolated and the continuous heat sources (see Fig. 15). The corresponding function G is then

$$G = U(\xi) + A\,U(-\xi) + \int_{-\infty}^{-\xi} a(\eta)\, U(\eta)\, d\eta$$

(9)

$$= (4\pi k t)^{-\frac{1}{2}} \left[e^{-\frac{(x-\xi)^2}{4kt}} + A\, e^{-\frac{(x+\xi)^2}{4kt}} + \int_{-\infty}^{-\xi} a(\eta)\, e^{-\frac{(x-\eta)^2}{4kt}}\, d\eta \right]$$

Hence at $x = 0$:

(10) $$(4\pi k t)^{\frac{1}{2}}\, G = (1 + A)\, e^{-\frac{\xi^2}{4kt}} + \int_{-\infty}^{-\xi} a(\eta)\, e^{-\frac{\eta^2}{4kt}}\, d\eta.$$

From (9) we form $\partial G/\partial x$. Then if we replace $\partial/\partial x$ by $-\partial/\partial\eta$ under the integral sign we obtain for $x = 0$

(10a) $$(4\pi k t)^{\frac{1}{2}} \frac{\partial G}{\partial x} = \xi\left(\frac{1}{2kt} - \frac{A}{2kt}\right) e^{-\frac{\xi^2}{4kt}} - \int_{-\infty}^{-\xi} a(\eta)\, \frac{\partial}{\partial\eta} e^{-\frac{\eta^2}{4kt}}\, d\eta.$$

and after integrating by parts:

(11) $$(4\pi k t)^{\frac{1}{2}} \frac{\partial G}{\partial x} = \left[(1 - A)\frac{\xi}{2kt} - a(-\xi)\right] e^{-\frac{\xi^2}{4kt}} + \int_{-\infty}^{-\xi} a'(\eta)\, e^{-\frac{\eta^2}{4kt}}\, d\eta.$$

If we substitute (10) and (11) in condition (3c) with $\partial/\partial n$ replaced by $-\partial/\partial x$, then (3c) must be satisfied identically for all $t > 0$. By setting the terms of different time dependence individually equal to zero we obtain:

(12) $$A - 1 = 0 \quad \ldots \quad A = +1,$$

(13) $$a(-\xi) + h(1 + A) = 0 \quad \ldots \quad a(-\xi) = -2h,$$

and the differential equation:

$$a'(\eta) - k\, a(\eta) = 0 \tag{14}$$

Considering (13) we see that (14) has the solution

$$a(\eta) = b\, e^{h\eta} = -\, 2\, h\, e^{h(\xi + \eta)}$$

This determines the constant A and the function $a(\eta)$. The fact that this determination is unique will be demonstrated in §17.

The result is

$$(4\pi k t)^{\frac{1}{2}}\, G = e^{-\frac{(x-\xi)^2}{4kt}} + e^{-\frac{(x+\xi)^2}{4kt}} - 2\, h\, e^{h\xi} \int\limits_{-\infty}^{-\xi} e^{-\frac{(x-\eta)^2}{4kt}}\, e^{h\eta}\, d\eta\,. \tag{15}$$

For numerical applications this integral can be reduced to the tabulated normal error function.[2]

Only when the given initial values $f(x)$ are particularly simple will it be more convenient to use equations (1) and (2) instead of the more intuitive method of Green's function. We illustrate this by an example which also shows the translation of problems of heat conduction into the language of *diffusion problems.*

Let the bottom section of a cylindrical vessel, $0 < x < H$, be filled with a concentrated solution (say $CuSO_4$); above it let there be a layer of pure solvent (water) to an arbitrary height $x = \infty$. Let the concentration of the solution be u and let the initial concentration be 1. At the base of the cylinder we have $\partial u/\partial x = 0$ at all times, since the dissolved salt molecules cannot penetrate the base.

This condition may also be satisfied by extending the vessel downward and by prescribing the reflected initial distribution as above. (For a finite height of water column one would have to use the somewhat more complicated reflection process of §16.) The initial distribution of u is then:

$$f(x) = \begin{cases} 1 \cdots -H < x < +H\,, \\ 0 \cdots \quad H < |x| < \infty\,. \end{cases}$$

Equation (2) yields

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int\limits_{z_1}^{z_2} e^{-z^2}\, dz\,. \tag{16}$$

[2] E.g., in Jahnke-Emde's tables of functions, 3rd ed., Teubner, Leipzig, 1938, p. 24

The values for the limits of integration z_1 and z_2 are obtained if in (1a) we let $\xi = \pm H$:

$$z_1 = \frac{-H - x}{\sqrt{4\,kt}}, \qquad z_2 = \frac{H - x}{\sqrt{4\,kt}}. \tag{17}$$

Using the customary notation

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int\limits_0^x e^{-z^2}\,dz \tag{18}$$

for the error function, we can write our solution (16) with astonishing simplicity:

$$u(x,\,t) = \frac{1}{2}\{\Phi(z_2) - \Phi(z_1)\}. \tag{19}$$

§14. The Problem of the Earth's Temperature

We treat the surface of the earth as a plane and assume an averaged *purely periodic* temperature $f(t)$ (annual averaged or daily averaged temperature). In order to determine the temperature in the earth's interior[3] we can in general use the method described in Fig. 13, by setting $x_0 = 0$ (surface of the earth), $x_1 = \infty$ (great depth), and $u_0 = f(t)$ for $x = 0$. It is convenient in our case to expand $f(t)$ into a complex Fourier-series:

$$f(t) = \sum_{n=-\infty}^{+\infty} C_n\, e^{2\pi i n t/T}, \qquad T = \text{length of year or day} \tag{1}$$

and to set for the temperature in the interior of the earth at a depth x:

$$u(x,\,t) = \sum_{n=-\infty}^{+\infty} C_n\, u_n(x)\, e^{2\pi i n t/T}. \tag{2}$$

Each individual term of this series must satisfy the basic law of heat conduction. This yields the ordinary differential equation for u_n:

$$\frac{d^2 u_n}{dx^2} = p_n^2\, u_n \quad \text{with} \quad p_n^2 = \frac{2\pi i n}{kT}. \tag{3}$$

In order that (2) go into (1) for $x = 0$, we must have (3a)

$$u_n(0) = 1. \tag{3a}$$

[3] The physical problem of "geothermic depth" (increase of temperature in the interior of the earth due to radioactive or nuclear processes) is of course ignored here.

Depending on whether n is positive or negative we set

$$2\,in = (1 \pm i)^2\,|n|$$

and

(4) $$p_n = (1 \pm i)\,q_n\,, \qquad q_n = \sqrt{\frac{|n|\,\pi}{kT}} > 0.$$

The general solution of (3) is then

(5) $$u_n(x) = A_n\,e^{(1\pm i)\,q_n x} + B_n\,e^{-(1\pm i)\,q_n x}.$$

Here we must have $A_n = 0$, since otherwise the temperature would become infinite for $x \to \infty$, and $B_n = 1$, to satisfy (3a). Substituting this in (2) we get

(6) $$u(x,t) = \sum_{n=-\infty}^{+\infty} C_n\,e^{-(1\pm i)q_n x}\,e^{2\pi i n t/T}\,.$$

In order to translate this into real language we write for $n > 0$

$$C_n = |C_n|\,e^{i\gamma_n}\,.$$

According to (1.13) C_n for $n < 0$ has the same absolute value but the negative phase. Equation (6) then becomes

(7) $$u(x,t) = C_0 + 2\sum_{n=1}^{\infty} |C_n|e^{-q_n x}\cos\left(2\pi n\,\frac{t}{T} + \gamma_n - q_n x\right).$$

We see that the amplitude $|\,C_n\,|$ of the n-th partial wave is *damped exponentially* with increasing depth x, and that this damping increases with increasing n. At the same time the phase of the partial wave is *retarded* increasingly with increasing x and n.

We now consider the numerical values. For an average type of soil we have the approximate temperature conductivity

$$k = 2\cdot 10^{-3}\,\frac{\mathrm{cm}^2}{\mathrm{sec}}.$$

For the period of one year $T = 365 \times 24 \times 60 \times 60 = 3.15 \times 10^7$ sec. and for $x = 1\ m = 100$ cm. we have then

(8) $$q_1 x = 0.7 \sim \frac{\pi}{4}\,, \qquad e^{-q_1 x} \sim \frac{1}{2}\,.$$

At a depth of 4 meters we already have a "phase lag" $q_1 x = \pi$ and an amplitude damping $2^{-4} = 1/_{16}$. Even for the first and principal partial wave of temperature fluctuation *it is winter at a depth of 4 meters when it is summer on the surface; the amplitude is only a fraction of the surface amplitude.* For the higher partial waves $n > 1$ the phase lag and amplitude damping are correspondingly greater owing to the factor $\sqrt{|n|}$ in q_n. We may say that the earth acts as a *harmonic analyzer* (see p. 4) by singling out the principal (though much weakened) wave from among all partial waves.

As a special example we consider the yearly curve of an "extremely continental climate," namely a uniform summer temperature and the same negative winter temperature which we shall set arbitrarily $= \pm 1$ This year-curve is represented graphically by the meander line of Fig. 1. and analytically by (2.2)

$$(9) \qquad u(0,t) = \frac{4}{\pi}\left(\sin\tau + \frac{1}{3}\sin 3\tau + \frac{1}{5}\sin 5\tau + \cdots\right). \qquad \tau = 2\pi\frac{t}{T}\,.$$

In order to obtain the corresponding series $u(x,t)$ we must, according to (2.1a), specialize the coefficients C in (7) as follows:

$$C_{2n} = 0\,, \quad |C_{2n+1}| = \frac{2}{\pi(2n+1)}\,, \qquad \gamma_{2n+1} = -\frac{\pi}{2}\,.$$

However it is somewhat simpler to apply the calculation process used for (2) directly to equation (9). We immediately get:

$$(9a) \qquad u(x,t) = \frac{4}{\pi}\left(e^{-q_1 x}\sin(\tau - q_1 x) + \frac{1}{3}e^{-q_3 x}\sin(3\tau - q_3 x) + \cdots\right).$$

Then, substituting the values of the q we get for $x = 100$ cm.

$$(9b) \qquad \begin{aligned} u(x,t) \approx \frac{4}{\pi}\Bigg[\frac{1}{2}\sin\left(\tau - \frac{\pi}{4}\right) &+ \frac{1}{3\cdot 3.4}\sin\left(3\tau - \frac{\sqrt{3}\pi}{4}\right) \\ &+ \frac{1}{5\cdot 4.8}\sin\left(5\tau - \frac{\sqrt{5}\pi}{4}\right) + \cdots\Bigg] \end{aligned}$$

and for $x = 400$ cm.

$$(9c) \qquad \begin{aligned} u(x,t) \approx \frac{4}{\pi}\Bigg(\frac{1}{16}\sin(\tau - \pi) &+ \frac{10^{-2}}{3\cdot 1.3}\sin(3\tau - \sqrt{3}\pi) \\ &+ \frac{10^{-2}}{5\cdot 5.3}\sin(5\tau - \sqrt{5}\pi) + \cdots\Bigg). \end{aligned}$$

A comparison of (9) with (9b,c) shows clearly the influence of depth on the amplitude and the phase of the temperature process.

This shows the usefulness of a deep cellar. It has not only much smaller temperature fluctuations than the surface of the earth, but is also warmer in winter than in summer (or it would be if there were no air flow).

Our conclusions become even more striking if we pass from the consideration of an averaged yearly temperature to that of an averaged *daily* temperature. The q_n are then increased by the factor $\sqrt{365} \sim 19$. Hence the damping and phase lag which for a yearly period belongs to a depth x occurs now at a depth of $x/19$. The decrease of amplitude to 1/16 for the principal term (see (9c)) and the reversal of the time of day (midnight instead of noon) will now occur at a depth of only $x = 400/19 = 21$ cm. Hence the daily fluctuations of temperature enter into the earth with noticeable intensity for only a few centimeters; the whole process takes place in a *thin surface layer.*

We deal here with an obvious analogue to the *skin effect* of electricity. The fact that in practice it is particularly observable on cylindrical wires makes no difference here; for a conductor bounded by a plane it occurs quantitatively in almost the same manner. Our daily curve corresponds to an alternating current of high frequency, our yearly curve to one that is 365 times slower. We know from §7 that the differential equations are the same in both cases, but for electricity we interpret the coefficient k as the specific resistance of the conductor.

§15. The Problem of a Ring-Shaped Heat Conductor

We now turn to the case of a heat conductor of finite length 1. However, at its two ends $x = \pm 1/2$ we do not prescribe the boundary conditions a),b) or c) of p. 63, but instead the much simpler *condition of periodicity.* By this we mean that not only u but also all its derivatives

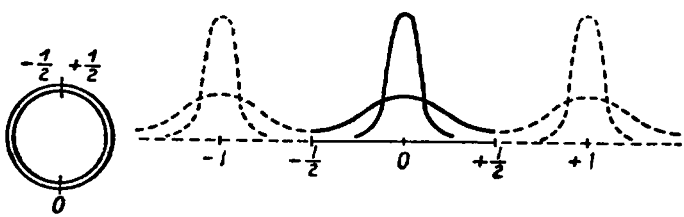


Fig. 16. Heat conduction in a ring. Heat pole at $x = 0$ with periodic repetitions. Temperature distribution for $kt < 1$ (steep curve) and for $kt > 1$ (flat curve).

shall coincide at the ends. We achieve this by bending our rod into a ring so that the two ends coincide. The shape of the ring is of no importance since, as in all cases of linear heat conduction, we must consider the lateral surface of the ring as adiabatically closed. In Fig. 16 we have drawn a circular ring.

As initial temperature we take $f(x)$ which is arbitrary but symmetric with respect to $x = 0$. Its Fourier expansion is a pure cosine series which automatically satisfies the periodicity condition at the ends From (4.1) and (4.2) we get, after setting $a = 1/2$:

$$f(x) = \sum_{n=0}^{\infty} A_n \cos(2\pi n x), \qquad \begin{aligned} A_0 &= \int_{-1/2}^{+1/2} f(z)\, dz, \\ A_n &= 2\int_{-1/2}^{+1/2} f(z) \cos(2\pi n z)\, dz. \end{aligned} \tag{1}$$

In order to obtain the corresponding solution $u(x,t)$ of the equation of heat conduction we merely must multiply the n-th term by

$$e^{-(2\pi n)^2 k t}$$

We then obtain

$$u(x, t) = \sum_{n=0}^{\infty} A_n e^{-4\pi^2 n^2 k t} \cos(2\pi n x). \tag{2}$$

We now consider $f(x)$ to be a " δ function" by writing

$$f(x) = 0 \quad \text{for } x \neq 0, \quad \text{but} \quad \int_{-\varepsilon}^{+\varepsilon} f(x)\, dx = 1.$$

Then in (1) we get $A_0 = 1$, $A_1 = A_2 = \ldots = 2$, and if we replace u by the customary ϑ we get:

$$\vartheta(x, t) = 1 + 2\sum_{n=1}^{\infty} e^{-4\pi^2 n^2 k t} \cos(2\pi n x). \tag{3}$$

The letter ϑ stands for the *theta-function* which was introduced by C. G. J. Jacoby in the theory of elliptic functions and which is of paramount importance in all numerical computations.[4] The fact that it satisfies the equation of heat conduction is frequently used there as an incidental property, whereas we use this property for the definition of ϑ.

We now have to adjust our notation t to the theory of the ϑ function by setting

[4] The reason for its special convergence was mentioned in §3, p. 15: since the ϑ series together with all its derivatives is periodic, and therefore has no jumps at $x = +1/2$ and $x = -1/2$, its terms vanish with increasing n more rapidly than any power of n.

$$\tau = 4\pi i k t. \tag{4}$$

This τ, which of course has nothing to do with the symbol $t-\tau$ of the principal solution U, does not have the dimension of time and is positive-imaginary in our case. (In the theory of elliptic function τ is in general complex with positive imaginary part, namely the ratio of the two periods of these functions). Written in terms of τ (3) becomes:

$$\vartheta(x|\tau) = 1 + 2\sum_{n=1}^{\infty} e^{i\pi\tau n^2}\cos(2\pi n x). \tag{5}$$

This form converges very well for large τ or, in other words, for *large kt.* It represents the *later phases* of the damping out of the unit source exceptionally well, but it does not help us for the beginning of this process. We therefore complete Fourier's process which, in analogy to the reflection process, is based on a periodic repetition of the initial state (see the right half of Fig. 16).

We have rolled off the cut ring on the x-axis both to the right and the left in an infinite sequence. From the heat source $U_0(x,t)$ given in the ring we get, at the points $x = n$ ($n = \pm 1, \pm 2, \ldots$), the identical heat sources:

$$U_n(x, t) = \frac{1}{\sqrt{4\pi k t}}\exp\left\{-\frac{(x-n)^2}{4kt}\right\}. \tag{6}$$

In the series

$$u(x, t) = \sum_{n=-\infty}^{+\infty} U_n(x, t) \tag{7}$$

we have a second representation of the damping out process which converges excellently for *small values of kt.* This is so because for such values we need consider only U_0 and its immediate successors, the subsequent U_n having no effect on account of the factor $\exp(n^2/4\,k\,t)$ of (6). Equation (7) is therefore the desired complement of (5). The figure shows the nature of both representations: the flat curve shows the behavior for large kt according to (5), the steep curve shows the behavior for small kt according to (7).

Oddly enough we can bring (7) into a form very similar to that of the ϑ series in (5). All we must do is put the factor $\exp(-x^2/4\,k\,t)$ outside the summation and combine the terms with $\pm n$. Equation (7) then becomes

$$u(x, t) = (4\pi k t)^{-\frac{1}{2}}\exp\left\{-\frac{x^2}{4\,kt}\right\}\cdot\left[1 + 2\sum_{n=1}^{\infty} e^{-\frac{n^2}{4kt}}\cos\frac{i n x}{2\,kt}\right]. \tag{7a}$$

If we replace t by τ according to (4) then the bracketed term becomes

$$1 + 2\sum_{n=1}^{\infty} e^{-\frac{\pi i n^2}{\tau}} \cos 2\pi n \frac{x}{\tau}.$$

This differs from (5) only in that x/τ has replaced x in the argument of the cosines and that $-1/\tau$ has replaced τ in the exponents. Hence the bracket of (7a) is

$$\vartheta\left(\frac{x}{\tau}\middle| -\frac{1}{\tau}\right).$$

Substituting this in (7) and remembering that both (5) and (7) are solutions of the same problem of heat conduction we obtain

$$\vartheta(x|\tau) = \left(\frac{\tau}{i}\right)^{-\frac{1}{2}} \exp\left\{-\frac{\pi i x^2}{\tau}\right\} \cdot \vartheta\left(\frac{x}{\tau}\middle| -\frac{1}{\tau}\right)$$

or conversely

$$\vartheta\left(\frac{x}{\tau}\middle| -\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}} \exp\left\{\frac{\pi i x^2}{\tau}\right\} \cdot \vartheta(x|\tau). \tag{8}$$

This is the famous *transformation formula of the* ϑ *function.* It is used in the theory of elliptic functions to transform the series $\vartheta(x|\tau)$ which converges slowly for small τ into the very rapidly converging series $\vartheta\left(\frac{x}{\tau}\middle|-\frac{1}{\tau}\right)$. For us it constitutes the passage from Fourier's method to the method of heat poles. In quantum theory formula (8) is of importance for the rotational energy of diatomic molecules and for the calculation of their specific heat for low temperatures.

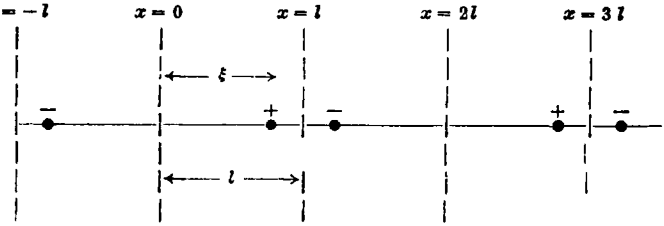
§16. Linear Heat Conductors Bounded on Both Ends

By setting the length of the ring in the preceding section equal to 1, we tacitly introduced a new dimensionless coordinate $x' = x/l$ and wrote x instead of x'. For the case of a rod of length l, which we shall consider now, we must replace x by x/l whenever we apply one of the preceding formulas. The meaning of τ, which has the dimension of x^2, must then be amended in the manner described on the following page.

We first give a table of the problems corresponding to the boundary conditions a) and b) of p. 63 and of their solutions by both Fourier's method and the method of heat poles. The latter leads to an *infinite* sequence of reflections since not only the primary heat pole but also all its images are reflected at both ends of the rod. Let us consider a room with parallel mirrors as an optical example; the chandelier will be reflected in both mirrors not once but in infinite repetition.

a) a)

$$u = 0 \text{ for } \begin{cases} x = 0 \\ x = l \end{cases}$$

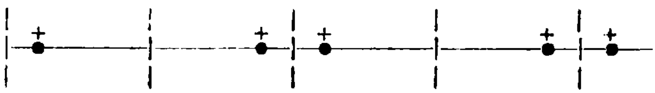


$$f(x) = \sum B_n \sin \pi n \frac{x}{l}, \quad B_n = \frac{2}{l} \int_0^l f(x) \sin \pi n \frac{x}{l} \, dx,$$

$$G = \vartheta\left(\frac{x-\xi}{2l} \,\Big|\, \tau\right) - \vartheta\left(\frac{x+\xi}{2l} \,\Big|\, \tau\right).$$

b) b)

$$\frac{\partial u}{\partial x} = 0 \text{ for } \begin{cases} x = 0 \\ x = l \end{cases}$$

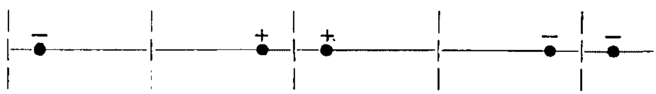


$$f(x) = \sum A_n \cos \pi n \frac{x}{l}, \quad A_n = \frac{2}{l} \int_0^l f(x) \cos \pi n \frac{x}{l} \, dx, \quad A_0 = \frac{1}{l} \int_0^l f(x) \, dx,$$

$$G = \vartheta\left(\frac{x-\xi}{2l} \,\Big|\, \tau\right) + \vartheta\left(\frac{x+\xi}{2l} \,\Big|\, \tau\right).$$

a) b)

$$u = 0 \text{ for } x = 0$$
$$\frac{\partial u}{\partial x} = 0 \quad ,, \quad x = l$$

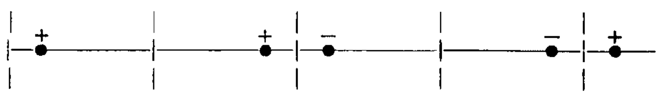


$$f(x) = \sum B_n \sin \pi (n + \tfrac{1}{2}) \frac{x}{l}, \quad B_n = \frac{2}{l} \int_0^l f(x) \sin (n + \tfrac{1}{2}) \frac{x}{l} \, dx,$$

$$G = \vartheta\left(\frac{x-\xi}{4l} \,\Big|\, \tau\right) - \vartheta\left(\frac{x+\xi}{4l} \,\Big|\, \tau\right) + \vartheta\left(\frac{x+\xi-2l}{4l} \,\Big|\, \tau\right) - \vartheta\left(\frac{x-\xi-2l}{4l} \,\Big|\, \tau\right).$$

b) a)

$$\frac{\partial u}{\partial x} = 0 \text{ for } x = 0$$
$$u = 0 \quad ,, \quad x = l$$



$$f(x) = \sum A_n \cos \pi (n + \tfrac{1}{2}) \frac{x}{l}, \quad A_n = \frac{2}{l} \int_0^l f(x) \cos (n + \tfrac{1}{2}) \frac{x}{l} \, dx,$$

$$G = \vartheta\left(\frac{x-\xi}{4l} \,\Big|\, \tau\right) + \vartheta\left(\frac{x+\xi}{4l} \,\Big|\, \tau\right) - \vartheta\left(\frac{x+\xi-2l}{4l} \,\Big|\, \tau\right) - \vartheta\left(\frac{x-\xi-2l}{4l} \,\Big|\, \tau\right).$$

We see immediately that the functions $f(x)$ in this table satisfy the boundary conditions a)a) to b)b); these boundary conditions then hold for the corresponding solutions of the boundary value problems $u(x,t)$, which are obtained from $f(x)$ according to *Fourier's method* by multiplying the series of f termwise by

$$e^{-(\pi n/l)^2 kt} \qquad \text{or} \qquad e^{-[\pi(n+{}^1/_2)/l]^2 kt}$$

The diagrams show the positions and the signs of the heat poles according to the *reflection method.* In the first two cases the heat poles are seen to have the period $2l$, in the last two they have the period $4l$. Their summation yields Green's function $G = \Sigma U$ which is expressed here in terms of the ϑ function of the preceding section. In the formulas of the preceding section, where the period was taken equal to 1 and the heat pole was at $x = 0$, we have to replace x by $\frac{x-\xi_i}{2l}$ for a)a) and b)b) and by $\frac{x-\xi_i}{4l}$ for a)b) and b)a), where ξ_i stands for the position of any heat pole of the sequence which is summed by ϑ. (Due to the periodicity the choice of the heat pole is immaterial.) In our formulas we have chosen for ξ_i the heat pole of the initial region $0 < x < l$ or of one of the adjacent regions. From Green's function we get the solution of the boundary value problem for arbitrary initial values $u(x,0) = f(x)$ according to the general formula

$$u(x, t) = \int_0^l f(\xi)\, G(x, \xi; t)\, d\xi\,. \tag{1}$$

We now wish to treat the *boundary condition* c) of p. 64, where we particularly consider the combination a)c). In order to satisfy condition a) at $x = 0$ we set

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \lambda_n \pi \frac{x}{l}. \tag{2}$$

That is to say, in the solution for a)a) we replace the sequence of integers n by the sequence

$$\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$$

which we wish to determine in such a way that for $x = l$ condition c) is satisfied. This leads to the transcendental equation

$$\lambda_n \frac{\pi}{l} \cos \lambda_n \pi + h \sin \lambda_n \pi = 0\,,$$

or

$$\tan \lambda_n \pi = -\frac{\pi \lambda_n}{h l}. \tag{3}$$

This is equation (6.2a) with $\alpha = -\pi/h\,l$; its solution was illustrated in Fig. 7. Hence we are dealing here with a typical case of *anharmonic Fourier analysis*. The values of the coefficients B_n in (2) can be taken directly from (6.3b). We obtain the final solution of our boundary value problem a)c) if we multiply the terms in the sum of (2) by the required time factors:

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \lambda_n \pi \frac{x}{l} \cdot \exp\left\{-\left(\frac{\lambda_n \pi}{l}\right)^2 k\,t\right\}. \tag{4}$$

We mentioned in §6 that the formal computation of the coefficients B can be replaced by a physically meaningful one. We shall do this now in such a manner that we shall be able to refer to this case in all future expansions in "eigenfunctions."

We consider two arbitrary terms of (2):

$$u_n = \sin \lambda_n \frac{\pi x}{l}, \qquad u_m = \sin \lambda_m \frac{\pi x}{l}; \tag{5}$$

They satisfy the differential equations:

$$\frac{d^2 u_n}{dx^2} + k_n^2\, u_n = 0, \qquad \frac{d^2 u_m}{dx^2} + k_m^2\, u_m = 0, \qquad \begin{cases} k_n = \lambda_n \dfrac{\pi}{l}, \\ k_m = \lambda_m \dfrac{\pi}{l}. \end{cases} \tag{5 a}$$

Hence

$$u_m \frac{d^2 u_n}{dx^2} - u_n \frac{d^2 u_m}{dx^2} = (k_m^2 - k_n^2)\, u_m\, u_n. \tag{5 b}$$

The left side is a total derivative (for the case of Green's theorem, p. 44, we spoke of a "divergence"). The integration of (5b) over the fundamental region $0 < x < l$ reduces to the boundary points on the left (in Green's theorem we said "to a boundary integral"). From this we obtain the value of the right side without further calculation:

$$(k_m^2 - k_n^2) \int_0^l u_m\, u_n\, dx = u_m \frac{du_n}{dx} - u_n \frac{du_m}{dx} \Bigg|_{x=0}^{x=l}. \tag{6}$$

Here the right side vanishes for $x = 0$ since according to (5) we have then $u_n = u_m = 0$; but it also vanishes for $x = l$, since the boundary condition c) holds for every individual term of (2) and hence the du/dx are proportional to the u. Therefore the *condition of orthogonality* (6.3) is satisfied for $k_m \neq k_n$.

We shall show in exercise III.2 that the normalizing integral (6.3b) can be obtained almost without further calculation.

The expressions as well as the mathematical formulations in this section were based on the assumption that the lateral surface of the rod is closed to heat flow, an assumption that is open to reasonable doubt. We shall show now that our formulas can be used also in the case of incomplete closure with respect to heat flow.

Instead of imposing the adiabatic condition b) on an element of area $d\sigma$ of the lateral surface of our rod, which we assume to be a circular cylinder, we impose condition c) which states that an amount of heat

$$-\varkappa \frac{\partial u}{\partial n} d\sigma = \varkappa\, h\, u\, d\sigma$$

passes out through $d\sigma$ per unit of time. We apply this to the case of an element of a cylindrical rod of altitude dx and radius of cross section b so that the lateral area is $2\pi b\, dx$ and the outer normal dn is in the direction of the extended radius. The amount of heat passing out of the lateral surface is

$$dQ_1 = \varkappa\, h\, u \cdot 2\pi\, b\, dx\, dt. \tag{7}$$

The amount of heat flowing out of the bases $x =$ const. and $x + dx$ $=$ const. of this element is

$$dQ_2 = -\varkappa \frac{\partial^2 u}{\partial x^2} \cdot \pi\, b^2\, dx\, dt \tag{8}$$

According to (7.9) the total outflux of heat from the rod element is equal to the product of div G dt with its volume. Hence we have

$$\text{div G} \cdot \pi\, b^2 dx\, dt = dQ_1 + dQ_2. \tag{9}$$

and after substituting (7) and (8)

$$\text{div G} = \frac{2\varkappa h}{b} u - \varkappa \frac{\partial^2 u}{\partial x^2}. \tag{10}$$

According to (7.11) div G is proportional to $-\partial u/\partial t$. Substituting this and dividing by $\varkappa$ we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} + \frac{2h}{b} u. \tag{11}$$

Hence the "outer-heat conduction" through the lateral surface only modifies our differential equation by the additional term of equation (11). Our derivation depended on the assumption that the linear character of the thermal state, i.e., its sole dependence on x, is not affected by the lateral radiation, an assumption which seems plausible for sufficiently small cross section.

The integration of (11) is very simple. We let:

$$u = v\, e^{-\lambda t};$$

After division by $\exp(-\lambda t)$ equation (11) becomes:

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{k}\frac{\partial v}{\partial t} + \left(\frac{2h}{b} - \frac{\lambda}{k}\right) v,$$

which is the ordinary equation of heat conduction if we set:

$$\lambda = \frac{2hk}{b}. \tag{12}$$

Hence all the developments of this chapter are valid for a rod with outer heat conduction if we multiply by the factor $\exp\left(-\frac{2hk}{b}t\right)$

In exercises III.3 and III.4 we shall see an elegant experimental determination of the ratios inner to outer heat conduction and heat conduction to electron conduction in metals.

§ 17. Reflection in the Plane and in Space

We finally leave the case of linear heat conduction and turn to spatial regions which are bounded by planes and which can be treated by the simple *reflection method.* The corresponding plane regions bounded by straight lines will be treated in a very similar manner.

The simplest case is that of a *half space* with the boundary conditions $u = 0$ or $\partial u/\partial n = 0$. Since we know the spatial function of a heat pole from equation (12.14), we can write Green's function for the half space directly. If we take the boundary of the half space to be $z = 0$ and the source point to be (ξ, η, ζ) then we have

$$(4\pi k t)^{\frac{3}{2}} G = e^{-\frac{r^2}{4kt}} \mp e^{-\frac{r'^2}{4kt}} \begin{cases} r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2, \\ r'^2 = (x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2 \end{cases} \tag{1}$$

Since for $z = 0$

$$r^2 = r'^2 \quad \text{and} \quad \frac{\partial r^2}{\partial z} = -\frac{\partial r'^2}{\partial z}$$

we get

$$G = 0 \qquad \text{or} \quad \frac{\partial G}{\partial n} = -\frac{\partial G}{\partial z} = 0 \quad \text{for } z = 0.$$

Even for the boundary condition c) of p. 64 we can transfer the solution (13.15) directly to the spatial case. We have

$$(2) \qquad (4\pi k t)^{\frac{3}{2}} G = e^{-\frac{r^2}{4kt}} + e^{-\frac{r'^2}{4kt}} - 2h e^{h\zeta} e^{-\frac{\varrho^2}{4kt}} \int_{-\infty}^{-\zeta} e^{-\frac{(z-\beta)^2}{4kt}} e^{h\beta}\, d\beta.$$

with

$$\varrho^2 = (x-\xi)^2 + (y-\eta)^2,$$

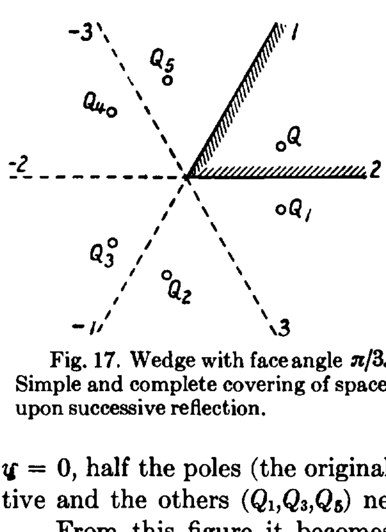


Fig. 17. Wedge with face angle $\pi/3$. Simple and complete covering of space upon successive reflection.

Not all spatial regions bounded by planes can be treated according to the reflection process. It is necessary that, under successive reflections of the original region, space be covered *completely and simply.* We demonstrate this with the example of a *wedge.* If it has an angle of 60° (see Fig. 17) then it is reproduced five times upon successive reflections whereupon the process terminates. Hence Green's function can be represented by a sum of six heat poles where, for the boundary condition $u = 0$, half the poles (the original pole Q and its images Q_2, Q_4) are positive and the others (Q_1, Q_3, Q_5) negative.

From this figure it becomes apparent that the reflection process may be attempted only for those polyhedrons whose *face angles are all submultiples of* π (not merely of 2π). In the case of wedges the angle $2\pi/3 = 120°$ leads to a double covering of space, $3\pi/2$ leads to a triple covering, and every angle which is incommensurable with π leads to an infinite covering. A particularly interesting case is that of space with a half plane removed, a wedge of angle 2π so to speak. Its treatment according to a reflection process requires the study of the principal solution in a "two-sheeted Riemann space" whose branch line is the edge of the half plane.[5]

[5] This solution was given by the author in 1894, *Math. Ann.* **45**, first for the case of heat conduction and soon thereafter for the refraction of light (*ibid.* **47**). For details see Frank-Mises, 2nd ed. (8th ed. of Riemann-Weber), Vieweg, 1935, chapter 20.

Among the polyhedral regions we consider first the interior of a *cube* (the exterior would lead to most complicated ramifications) and as its generalization the *rectangular solid.* The mirror images of the given primary source point form eight superimposed spatial lattices corresponding to the eight combinations of signs $\pm\xi, \pm\eta, \pm\zeta$. Each of these lattices taken separately forms a triply periodic solution of the differential equation, so to speak a higher ϑ function (see below). If the base of a rectangular solid is divided by its diagonals into four isosceles triangles then a rectangular cylinder with one of them as base is also a polyhedron of the required kind. Another example is given by a rectangular cylinder whose base is an equilateral triangle or half an equilateral triangle obtained through bisection by the altitude. A rectangular cylinder whose base is a regular hexagon has face angles of $2\pi/3$ and therefore leads to a double, not a simple, covering of space.

Everything said about the subdivision of rectangular solids is of course true for cubes. In addition, for suitable subdivisions the cube yields permissible tetrahedra: Lamé's "tetrahedra 1/6 and 1/24," of which the former fills out the cube upon six reflections, the latter upon twenty-four, and another tetrahedron which was discovered by Schönflies in his general investigations on crystal structure.[6]

For all these regions we can not only solve the problem of heat conduction but *any physical process of isotropic symmetry,* such as an acoustic, optic, or electric process, by the reflection method. The very word "reflection" reminds us of the optical application.

The set of permissible regions is extended very considerably if we no longer impose *boundary conditions,* but require *periodicity,* as in the case of the ring in the beginning of §15. Then instead of a rectangular solid we can treat an *arbitrary parallelipiped;* all we have to do is to repeat periodically the pole of Green's function in the initial domain in all its images under the translation group. The elliptic ϑ function of the ring is then replaced by a higher ϑ function (hyperelliptic Abelian); however we shall not go into this since there are no immediate physical applications.

Everything said here about spatial regions can be transferred directly to plane regions. Half space is replaced by half plane, rectangular solid by rectangle, the rectangular cylinder whose base is an equilateral triangle by that triangle. In formula (1) for Green's function of half space we have to replace the exponent $\frac{3}{2}$ on the left by the exponent 1,

[6] G. Lamé, Leçons sur la theorie de la chaleur, Paris 1861; he does not use the reflection method but Fourier's method with a suitable continuation of the arbitrary initial distribution. For A. Schönflies' tetrahedron see *Math. Ann.* **34.**

and the three-dimensional square of distances on the right by the corresponding square of distances in the plane.

Unfortunately this method of reflection for problems of heat conduction can not be applied to spherical (or circular) regions (see §23).

§ 18. Uniqueness of Solution for Arbitrarily Shaped Heat Conductors

The physicist may consider such a proof superfluous; we shall consider it, however, on account of its mathematical elegance and the importance of its method.

It will suffice to use Green's theorem of potential theory which we formulated in exercise II.2 as the "second form." The parabolic character of the equation of heat conduction plays no particular role here; it would become important if, as in Fig. 13, we were to impose time dependent boundary conditions, but we shall restrict ourselves here to the boundary conditions a),b),c) of p. 63.

Our heat conductor may have an arbitrary boundary; as part of this boundary we include the boundaries of any possible inner cavities. On this total boundary surface σ there may be given an arbitrary combination of the boundary conditions

$$\text{a) } u = f_1(\sigma), \qquad \text{b) } \frac{\partial u}{\partial n} = f_2(\sigma), \qquad \text{c) } \frac{\partial u}{\partial n} + hu = f_3(\sigma)$$

("non-homogeneous" boundary conditions where f_1, f_2, f_3 are arbitrary point functions on σ, in contrast to the previous "homogeneous" boundary conditions where the right sides are zero). In addition we assume the initial temperature u to be given as an arbitrary point function $f(x,y,z)$.

Let u_1 and u_2 be two different solutions of the equation of heat conduction under these initial and boundary conditions. Their difference $u_1 - u_2 = w$ then satisfies same differential equation as u_1, u_2

$$\Delta w = \frac{1}{k}\frac{\partial w}{\partial t} \tag{1}$$

with a distribution over σ of the "homogeneous" boundary conditions

$$\text{a) } w = 0, \qquad \text{b) } \frac{\partial w}{\partial n} = 0, \qquad \text{c) } \frac{\partial w}{\partial n} + hw = 0 \tag{2}$$

and the initial condition

$$w = 0 \qquad \text{for} \qquad t = 0. \tag{3}$$

Setting both u and v equal to w in Green's theorem of exercise II.2 we get:

$$\int w\,\Delta w\,d\tau + \int (\text{grad } w, \text{grad } w)\, d\tau = \int w \frac{\partial w}{\partial n}\, d\sigma. \tag{4}$$

Considering (1) and (2) this becomes

$$\frac{1}{2\,k}\frac{\partial}{\partial t}\int w^2\, d\tau = -\int D w\; d\tau - \int h w^2\, d\sigma_c. \tag{5}$$

where Dw is the so-called first differential parameter:

$$Dw = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2.$$

The last term of (5) is to be considered only over that part of σ on which the boundary condition c) holds, as indicated by the subscript c attached to $d\sigma$.

Equation (5) contains a contradiction: the right side is *negative* since $h > 0$ as established in (13.5). (We no longer have to assume that h is a constant; h may vary on the surface σ_c depending on the local structure of the surface.) The left side of (5) is certainly *positive* for small t, since w^2 is 0 for $t = 0$ and therefore can only increase for increasing t. In order to make this contradiction even more apparent and to extend it to arbitrary t, we integrate (5) with respect to t:

$$\frac{1}{2\,k}\int w^2\, d\tau = -\int_0^t dt \int D w\, d\tau - \int_0^t dt \int h\, w^2\, d\sigma_c. \tag{6}$$

The only possibility of removing this contradiction is in setting:

$$w = 0, \qquad \text{hence} \qquad u_1 = u\,. \tag{7}$$

This *uniqueness result* can also be expressed as follows: in heat conduction there exist no eigenfunctions for any shape of the conductor (see Chapter V). In this sense heat conduction and all analogous *equalization processes* differ in a characteristic manner from *oscillation processes.*