Introduction to the N-body problem

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Outline of the talk

Dark matter in the galaxies

General relativity and its Newtonian limit

Boltzmann equation and the N-body method

Constructing a galaxy

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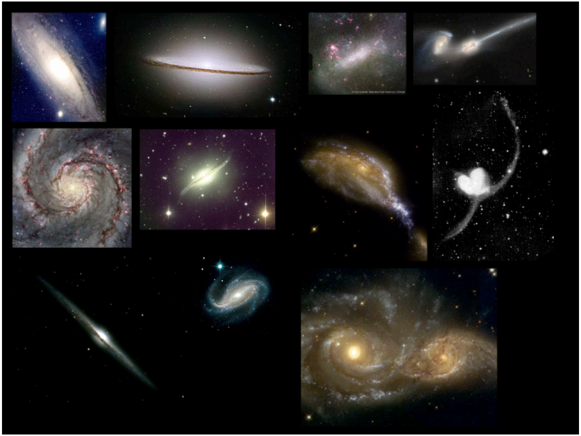
Morphological classification of galaxies

The fork ...



Morphological classification of galaxies

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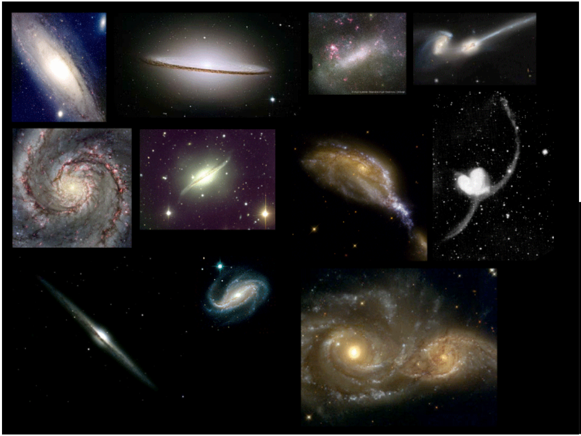

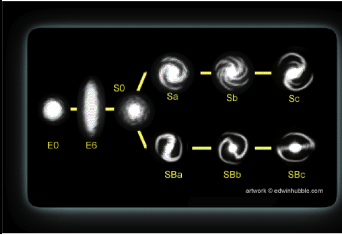

Morphological classification of galaxies

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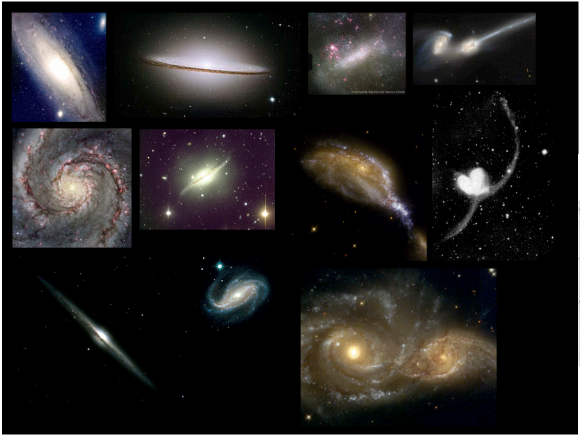

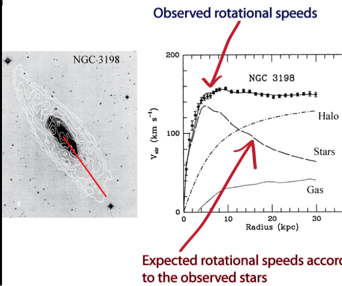

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General relativity theory and field equations

The Lagrangian is

$$\mathcal{L} = -\frac{\sqrt{-g}}{16\pi G} R + \mathcal{L}_M(g_{\mu\nu}) \ ,$$

Here $g_{\mu\nu}$ is the metric, $\mathcal{L}_M(g_{\mu\nu})$ is the matter Lagrangian. When we make the variation of the action with respect to the metric and the scalar field we obtain the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \ = \ 8\pi G T_{\mu\nu} \ ,$$

for the metric $g_{\mu\nu}$. Here $T_{\mu\nu}$ is the energy-momentum.

Newtonian limit of the GR

- Therefore, in the present study, we need to consider the **limit of a static GR**, and then we need to describe the theory in its Newtonian approximation, that is, where **gravity is weak (and time independent) and velocities of dark matter particles are non-relativistic**.
- One defines the perturbations

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $\eta_{\mu\nu}$ is the Minkowski metric.

Newtonian limit of the GR and solutions ...

The Newtonian approximation gives

$$R_{00} = \frac{1}{2}\nabla^2 h_{00} \quad = \quad 4\pi G_N \rho \, , \tag{2.1}$$

Note that Equation (2.1) can be cast as Poisson equation for $\Phi_N \equiv (1/2)h_{00}$

$$\nabla^2 \Phi_N = 4\pi G_N \rho$$

and represents the Newtonian limit of GR.

The next step is to find solutions for this new Newtonian potential given a density profile, that is, to find the so–called potential–density pairs. General solutions to Eq. (2.1) can be found in terms of the corresponding Green function, and the new Newtonian potential is

$$\Phi_N \equiv \frac{1}{2}h_{00} \quad = \quad -G_N \int d\mathbf{r}_s \frac{\rho(\mathbf{r}_s)}{|\mathbf{r} - \mathbf{r}_s|} + \text{B.C.} \tag{2.2}$$

Solutions Newtonian limit of the GR .....

Multipole expansion of Poisson equation

The Poisson's Green function can be expanded in terms of the spherical harmonics, $Y_{ln}(\theta, \varphi)$,

$$\frac{1}{|\mathbf{r} - \mathbf{r}_s|} = 4\pi \sum_{l=0}^{\infty} \sum_{n=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{ln}^*(\theta', \varphi') Y_{ln}(\theta, \varphi),$$

where $r_<$ is the smaller of $|\mathbf{r}|$ and $|\mathbf{r}_s|$, and $r_>$ is the larger of $|\mathbf{r}|$ and $|\mathbf{r}_s|$ and it allows us that the standard gravitational potential due to a distribution of mass $\rho(\mathbf{r})$, without considering the boundary condition, can be written as (Jackson, 1975)

$$\psi(\mathbf{r}) = \psi^{(i)} + \psi^{(e)}$$

where $\psi^{(i)}$ ($\psi^{(e)}$) are the internal (external) multipole expansion of ψ,

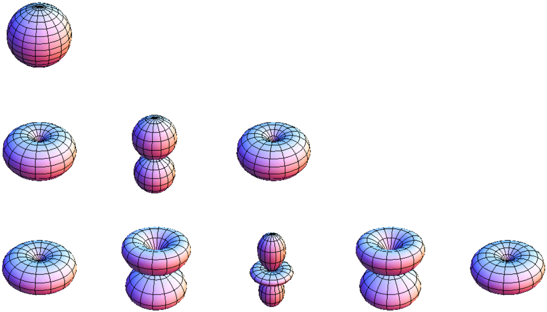
$$\begin{aligned} \psi^{(i)} &= -\sum_{l=0}^{\infty} \sum_{n=-l}^{l} \frac{\sqrt{4\pi}}{2l+1} q_{ln}^{(i)} Y_{ln}(\theta, \varphi) r^l \,, \\ \psi^{(e)} &= -\sum_{l=0}^{\infty} \sum_{n=-l}^{l} \frac{\sqrt{4\pi}}{2l+1} q_{ln}^{(e)} \frac{Y_{ln}(\theta, \varphi)}{r^{l+1}} \,, \end{aligned}$$

Solutions Newtonian limit of GR ......

Multipole expansion of Poisson equation

Here, the coefficients of the expansions $\psi^{(i)}$ and $\psi^{(e)}$, and known as internal and external multipoles, respectively, are given by

$$
\begin{aligned}
q_{ln}^{(i)} &= \sqrt{4\pi} \int_{V(r \leq r')} d\mathbf{r}' \, \frac{1}{r'^{l+1}} Y_{ln}^{*}(\theta', \varphi') \rho(\mathbf{r}') \,, \\
q_{ln}^{(e)} &= \sqrt{4\pi} \int_{V(r > r')} d\mathbf{r}' \, Y_{ln}^{*}(\theta', \varphi') r'^{l} \rho(\mathbf{r}') \,.
\end{aligned}
$$



Solutions Newtonian limit of GR ......

Multipole expansion of Poisson equation : Cartesians (1)

We may write expansions above in cartesian coordinates up to quadrupoles. For the internal multipole expansion we have

$$\psi^{(i)} = -M^{(i)} - \mathbf{r}\cdot\mathbf{p}^{(i)} - \frac{1}{2}\mathbf{r}\cdot\mathbf{Q}^{(i)}\cdot\mathbf{r}\,, \tag{2.3}$$

and its force is

$$\mathbf{F}_{\psi}^{(i)} = \mathbf{p}^{(i)} + \mathbf{Q}^{(i)}\cdot\mathbf{r}\,, \tag{2.4}$$

where

$$M^{(i)} \equiv \int_{V(r\leq r')} d\mathbf{r}'\,\frac{1}{r'}\rho(\mathbf{r}')\,, \tag{2.5}$$

$$p_i^{(i)} \equiv \int_{V(r\leq r')} d\mathbf{r}'\,x_i'\,\frac{1}{r'^3}\rho(\mathbf{r}')\,, \tag{2.6}$$

$$Q_{ij}^{(i)} \equiv \int_{V(r\leq r')} d\mathbf{r}'\,(3x_i'x_j' - r'^2\delta_{ij})\frac{1}{r'^5}\rho(\mathbf{r}')\,. \tag{2.7}$$

Solutions Newtonian limit of GR ......

Multipole expansion of Poisson equation : Cartesians (2)

For the external multipoles we have

$$\psi^{(e)} = -\frac{M^{(e)}}{r} - \frac{\mathbf{r}\cdot\mathbf{p}^{(e)}}{r^3} - \frac{1}{2}\frac{\mathbf{r}\cdot\mathbf{Q}^{(e)}\cdot\mathbf{r}}{r^5}, \tag{2.8}$$

and its force is

$$\mathbf{F}_{\psi}^{(e)} = -\frac{M^{(e)}}{r^3}\mathbf{r} + \frac{\mathbf{p}^{(e)}}{r^3} - 3\frac{\mathbf{p}^{(e)}\cdot\mathbf{r}}{r^5}\mathbf{r} + \frac{\mathbf{Q}^{(e)}\cdot\mathbf{r}}{r^5} - \frac{5}{2}\frac{\mathbf{r}\cdot\mathbf{Q}^{(e)}\cdot\mathbf{r}}{r^7}\mathbf{r}, \tag{2.9}$$

where

$$M^{(e)} \equiv \int_{V(r>r')} d\mathbf{r}'\,\rho(\mathbf{r}'), \tag{2.10}$$

$$p_i^{(e)} \equiv \int_{V(r>r')} d\mathbf{r}'\,x_i'\,\rho(\mathbf{r}'), \tag{2.11}$$

$$Q_{ij}^{(e)} \equiv \int_{V(r>r')} d\mathbf{r}'\,(3x_i'x_j' - r'^2\delta_{ij})\rho(\mathbf{r}'). \tag{2.12}$$

The external multipoles have the usual meaning, i.e., $M^{(e)}$ is the mass, $\mathbf{p}^{(e)}$ is the dipole moment, and $\mathbf{Q}^{(e)}$ is the traceless quadrupole tensor, of the volume $V(r > r')$. We may atach to the internal multipoles similar meaning, i.e., $M^{(i)}$ is the internal "mass", $\mathbf{p}^{(i)}$ is the internal "dipole" moment, and $\mathbf{Q}^{(i)}$ is the traceless internal "quadrupole" tensor, of the volume $V(r \leq r')$.

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Boltzmann equation in GR

The Vlasov-Poisson equations ...

In the Newtonian limit of GR to describe the evolution of the six-dimensional, one-particle distribution function, $f(\mathbf{x}, \mathbf{p}, t)$ we need to solve the Vlasov-Poisson equation. The Vlasov equation is,

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial f}{\partial \mathbf{x}} - m\nabla\Phi_N(\mathbf{x}) \cdot \frac{\partial f}{\partial \mathbf{p}} = 0 \tag{3.1}$$

where $\Phi_N = (1/2)h_{00} = \Phi_N$ with,

$$\nabla^2\Phi_N(\mathbf{x}) = 4\pi G_N \, \rho(\mathbf{x})$$

the Poisson equation. Above equations form the Vlasov-Poisson equations, constitutes a collisionless, mean-field approximation to the evolution of the full N-body distribution in the framework of the Newtonian limit of GR.

Boltzmann equation and N-body method in a STT

An N-body code attempts to solve the Vlasov-Poisson system of equations by representing the one-particle distribution function as

$$f(\mathbf{x},\mathbf{p}) = \sum_{i=1}^{N} \delta(\mathbf{x}-\mathbf{x}_i)\,\delta(\mathbf{p}-\mathbf{p}_i)$$

Substitution of (15) in the Vlasov-Poisson system of equations yields the exact Newton's equations for a system of N gravitating particles

$$\ddot{\mathbf{x}}_i = -G_N \sum_{j\neq i} \frac{m_j(\mathbf{x}_i-\mathbf{x}_j)}{|\mathbf{x}_i-\mathbf{x}_j|^3} \tag{3.2}$$

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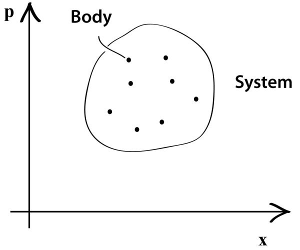
Galactic Dynamics:

Boltzmann and Jean equations

The basic equation to describe the dynamics of galaxies is the Boltzmann equation

$$\frac{\partial f}{\partial t} + \dot{\mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{x}} + \dot{\mathbf{p}} \cdot \frac{\partial f}{\partial \mathbf{p}} = \left(\frac{\partial f}{\partial t}\right)_{coll}$$

Stars are collisionless, gas is collisional.



Jeans' equations:

$$\begin{aligned} \frac{\partial \nu}{\partial t} + \frac{\partial(\nu \bar{v}_i)}{\partial x_i} &= 0 \\ \nu \frac{\partial \bar{v}_j}{\partial t} + \nu \bar{v}_i \frac{\partial \bar{v}_j}{\partial x_i} &= -\nu \frac{\partial \Phi_N}{\partial x_j} - \frac{\partial \nu \sigma_{ij}^2}{\partial x_i} \end{aligned}$$

Definitions:

$$\begin{aligned} \nu(\mathbf{r}, t) &\equiv \int d^3 v f(\mathbf{r}, \mathbf{v}, t) \\ \bar{v}_i(\mathbf{r}, t) &\equiv \frac{1}{\nu} \int d^3 v v_i f(\mathbf{r}, \mathbf{v}, t) \\ \sigma_{ij}^2 &\equiv \overline{v_i v_j} - \bar{v}_i \bar{v}_j \end{aligned}$$

A galaxy model is constructed by looking stationary solutions of the Boltzmann equations with spherical and axisymmetric symmetries.

Galactic Dynamics:

initial conditions ... spatial distributions

The bulge density profile is given by (Hernquist, 1990):

$$\rho_b(r) = \frac{M_b a_b}{2\pi} \frac{1}{r(r+a_b)^3}, \tag{4.1}$$

and for the halo we use a Dehnen density profile with $\gamma = 0$ (Dehnen, 1993):

$$\rho_h(r) = \frac{3M_h}{4\pi} \frac{a_h}{(r+a_h)^4}. \tag{4.2}$$

We assume that disk follows the exponential profile (Freeman, 1970):

$$\rho_d(r,z) = \frac{M_d \alpha^2}{4\pi z_0} e^{-\alpha r} \mathrm{sech}^2\left(\frac{z}{z_0}\right). \tag{4.3}$$

In these equations M_b, a_b and M_h, a_h are the mass and length of bulge and halo respectively, and M_d, α^{-1} y z_0 are the mass, length scale and the thickness length scale of the disk, respectively.

Particles velocities are obtained using the Schwarzschild distribution,

$$f_{B,H}(v_r, v_\phi, v_\theta) \propto \exp\left[-\frac{v_r^2}{2\sigma_r^2} - \frac{v_\phi^2}{2\sigma_\phi^2} - \frac{v_\theta^2}{2\sigma_\theta^2}\right] \tag{4.4}$$

wher σ_r, σ_ϕ, and σ_θ are the dispersion of velocities and in general they are functions of r.
For an isotropic ellipsoid the above velocity distribution is the Maxwell distribution.

Galactic Dynamics:

initial conditions ... velocity distributions ... spherical

For a spherically simmetric mass distribution and without rotation the dispersion of velocities is obtained using Jeans' equation

$$\frac{d}{dr}\left(\rho(r)\sigma_r^2\right) + \frac{\rho(r)}{r}\left[2\sigma_r^2 - (\sigma_\theta^2 + \sigma_\phi^2)\right] = -\rho(r)\frac{d\Phi}{dr} \tag{4.5}$$

If the distribution of velocities is isotropic

$$\sigma_r^2 = \sigma_\theta^2 = \sigma_\phi^2 \tag{4.6}$$

The above equation can be integrated to give a general expression for the dispersion of velocities:

$$\sigma_r^2(r) = \frac{1}{\rho(r)}\int_r^\infty \rho(r')\frac{d\Phi}{dr'}dr' \tag{4.7}$$

Particles velocities can be found by inverting the equation

$$F(v,r) = \frac{4\pi}{(2\pi\sigma^2)^{2/3}}v^2 \exp\left[-\frac{v^2}{2\sigma_r^2}\right] \tag{4.8}$$

In practice it is convenient to cut the Gaussian distribution at some finite value. A natural choice is the scape velocity V_e.

Galactic Dynamics:

initial conditions ... velocity distributions ... axisymmetric

The velocity profiles for the disk are computed using the epiciclic approximation, which consists in assuming that velocity dispersions are small ($\sigma_R, \sigma_z, \sigma_\phi \ll R\,\omega$):

$$f_D(v_R, v_z, v_\phi) \propto \exp\left[-\frac{v_R^2}{2\sigma_R^2} - \frac{v_z^2}{2\sigma_z^2} - \frac{(v_\phi - V_0)^2}{2\sigma_\phi^2}\right] \tag{4.9}$$

Observations in the exterior of disk galaxies suggest that the radial dispersion is proportional to the surface radial density:

$$\sigma_R^2 \propto (-\alpha\, R) \tag{4.10}$$

The vertical dispersion in the isothermal shell approximation is also related to the surface density of the disk:

$$\sigma_z^2 = \pi G\, z_0\, \Sigma(r) \tag{4.11}$$

The ratio σ_R^2/σ_z^2 is constant through the disk and is considered equal to 4, i.e.,

$$\sigma_R^2 = 4\,\sigma_z^2 \tag{4.12}$$

Galactic Dynamics:

initial conditions ... velocity distributions ... axisymmetric ..

The azimuthal dispersion is simply related to radial dispersion through the epiciclic approximation for the Schwarzschild velocity distribution

$$\sigma_\phi^2 = \frac{\kappa^2}{4\omega^2}\sigma_R^2 \tag{4.13}$$

where ω is the angular frequency, computed from the potential

$$\omega = \frac{\partial\Phi(R)}{\partial R} \tag{4.14}$$

and κ is the epiciclic frequency defined by

$$\kappa^2(R) = 4\omega^2(R) + R\frac{d}{dR}\left[\omega^2(R)\right] \tag{4.15}$$

For an exponential surface density profile, the azimuthal drift velocity is given approximately by

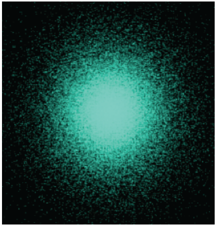
$$V_0^2 = V_c^2 + \sigma_R^2 - \sigma_\phi^2 - 2\alpha R \tag{4.16}$$

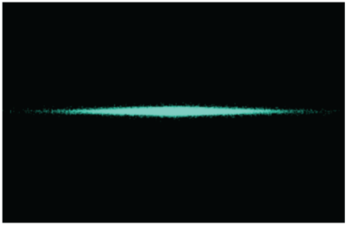
where $V_c^2 = R\,\omega$ is the azimuthal circular velocity of the disk.

Once velocity dispersions are computed, the velocity components of particles in the disk can be found by inverting the above Gaussian distribution which includes the drift velocity V_0.

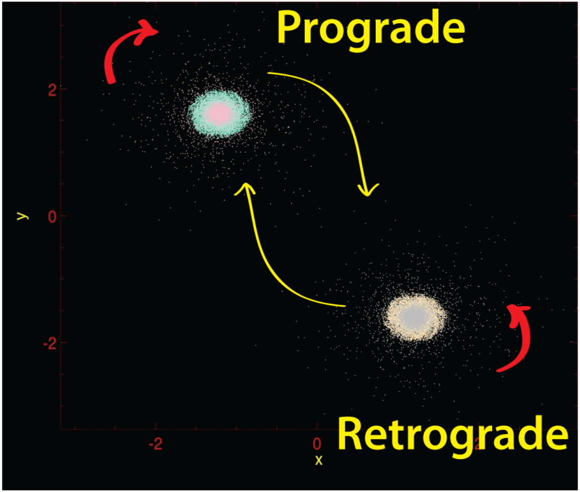
Galactic Dynamics:

Finally we build the galaxy using a Monte Carlo procedure





Galactic Dynamics: geometry of the collision



- $t_p = 0.75$ Gyr, $P = 2$, $R_d = 32$ kpc, $R_p = 160$ kpc

  (Gabbasov, et al., A&A, 2006).

Galactic Dynamics:

parameters of the galaxy model

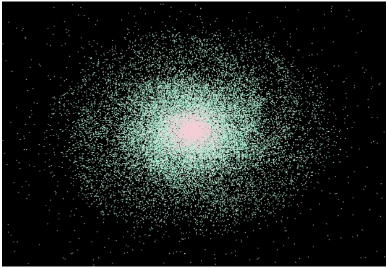


Table: Parameters of the galaxy model. The units of mass and length are $2.2 \cdot 10^{11}$ $M_\odot$ and 40 kpc, respectively.

Component	Mass	Number of particles	Cutoff radius	Scale-length
Bulge	0.0625	$0.05N$	1.5	0.04168
Disc	0.1875	$0.15N$	0.4	0.0833
Halo	1.0	$0.8N$	6.0	0.1

The number of particles in each component are assigned in proportion to their masses:

$M_{\mathrm{b}} : M_{\mathrm{d}} : M_{\mathrm{h}} = 1 : 3 : 16.$