Test Problems in Optimization

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Abstract

Test functions are important to validate new optimization algorithms and to compare the performance of various algorithms. There are many test functions in the literature, but there is no standard list or set of test functions one has to follow. New optimization algorithms should be tested using at least a subset of functions with diverse properties so as to make sure whether or not the tested algorithm can solve certain type of optimization efficiently. Here we provide a selected list of test problems for unconstrained optimization.

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In order to validate any new optimization algorithm, we have to validate it against standard test functions so as to compare its performance with well-established or existing algorithms. There are many test functions, so there is no standard list or set of test functions one has to follow. However, various test functions do exist, so new algorithms should be tested using at least a subset of functions with diverse properties so as to make sure whether or not the tested algorithm can solve certain type of optimization efficiently.

In this appendix, we will provide a subset of commonly used test functions with simple bounds as constraints, though they are often listed as unconstrained problems in literature. We will list the function form $f(\boldsymbol{x})$, its search domain, optimal solutions $\boldsymbol{x}_*$ and/or optimal objective value f_*. Here, we use $\boldsymbol{x} = (x_1, ..., x_n)^T$ where n is the dimension.

Ackley's function:

$$f(\boldsymbol{x}) = -20 \exp\Big[-\frac{1}{5}\sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2}\,\Big] - \exp\Big[\frac{1}{n}\sum_{i=1}^{n}\cos(2\pi x_i)\Big] + 20 + e, \tag{1}$$

where $n = 1, 2, ...,$ and $-32.768 \leq x_i \leq 32.768$ for $i = 1, 2, ..., n$. This function has the global minimum $f_* = 0$ at $\boldsymbol{x}_* = (0, 0, ..., 0)$.

De Jong's functions: The simplest of De Jong's functions is the so-called sphere function

$$f(\boldsymbol{x}) = \sum_{i=1}^{n} x_i^2, \quad -5.12 \leq x_i \leq 5.12, \tag{2}$$

whose global minimum is obviously $f_* = 0$ at $(0, 0, ..., 0)$. This function is unimodal and convex. A related function is the so-called weighted sphere function or hyper-ellipsoid function

$$f(\boldsymbol{x}) = \sum_{i=1}^{n} i x_i^2, \quad -5.12 \leq x_i \leq 5.12, \tag{3}$$

which is also convex and unimodal with a global minimum $f_* = 0$ at $\boldsymbol{x}_* = (0, 0, ..., 0)$. Another related test function is the sum of different power function

$$f(\boldsymbol{x}) = \sum_{i=1}^{n} |x_i|^{i+1}, \quad -1 \leq x_i \leq 1, \tag{4}$$

which has a global minimum $f_* = 0$ at $(0, 0, ..., 0)$.

Easom's function:

$$f(\boldsymbol{x}) = -\cos(x)\cos(y)\exp\Big[-(x-\pi)^2 + (y-\pi)^2\Big], \tag{5}$$

whose global minimum is $f_* = -1$ at $\boldsymbol{x}_* = (\pi, \pi)$ within $-100 \leq x, y \leq 100$. It has many local minima. Xin-She Yang extended in 2008 this function to n dimensions, and we have

$$f(\boldsymbol{x}) = -(-1)^n\Big(\prod_{i=1}^{n}\cos^2(x_i)\Big)\exp\Big[-\sum_{i=1}^{n}(x_i-\pi)^2\Big], \tag{6}$$

whose global minimum $f_* = -1$ occurs at $\boldsymbol{x}_* = (\pi, \pi, ..., \pi)$. Here the domain is $-2\pi \leq x_i \leq 2\pi$ where $i = 1, 2, ..., n$.

Equality-Constrained Function:

$$f(\boldsymbol{x}) = -(\sqrt{n})^n \prod_{i=1}^{n} x_i, \tag{7}$$

subject to an equality constraint (a hyper-sphere)

$$\sum_{i=1}^{n} x_i^2 = 1. \tag{8}$$

The global minimum $f_* = -1$ of $f(\boldsymbol{x})$ occurs at $\boldsymbol{x}_*(1/\sqrt{n}, ..., 1/\sqrt{n})$ within the domain $0 \le x_i \le 1$ for $i = 1, 2, ..., n$.

Griewank's function:

$$f(\boldsymbol{x}) = \frac{1}{4000} \sum_{i=1}^{n} x_i^2 - \prod_{i=1}^{n} \cos(\frac{x_i}{\sqrt{i}}) + 1, \quad -600 \le x_i \le 600, \tag{9}$$

whose global minimum is $f_* = 0$ at $\boldsymbol{x}_* = (0, 0, ..., 0)$. This function is highly multimodal.

Michaelwicz's function:

$$f(\boldsymbol{x}) = -\sum_{i=1}^{n} \sin(x_i) \cdot \Big[\sin(\frac{i x_i^2}{\pi})\Big]^{2m}, \tag{10}$$

where $m = 10$, and $0 \le x_i \le \pi$ for $i = 1, 2, ..., n$. In 2D case, we have

$$f(x, y) = -\sin(x)\sin^{20}(\frac{x^2}{\pi}) - \sin(y)\sin^{20}(\frac{2y^2}{\pi}), \tag{11}$$

where $(x, y) \in [0, 5] \times [0, 5]$. This function has a global minimum $f_* \approx -1.8013$ at $\boldsymbol{x}_* = (x_*, y_*) = (2.20319, 1.57049)$.

Perm Functions:

$$f(\boldsymbol{x}) = \sum_{j=1}^{n} \Big\{ \sum_{i=1}^{n} (i^j + \beta)\Big[(\frac{x_i}{i})^j - 1\Big]\Big\}, \quad (\beta > 0), \tag{12}$$

which has the global minimum $f_* = 0$ at $\boldsymbol{x}_* = (1, 2, ..., n)$ in the search domain $-n \le x_i \le n$ for $i = 1, ..., n$. A related function

$$f(\boldsymbol{x}) = \sum_{j=1}^{n} \Big\{ \sum_{i=1}^{n} (i + \beta)\Big[x_i^j - (\frac{1}{i})^j\Big]\Big\}^2, \tag{13}$$

has the global minimum $f_* = 0$ at $(1, 1/2, 1/3, ..., 1/n)$ within the bounds $-1 \le x_i \le 1$ for all $i = 1, 2, ..., n$. As $\beta > 0$ becomes smaller, the global minimum becomes almost indistinguishable from their local minima. In fact, in the extreme case $\beta = 0$, every solution is also a global minimum.

Rastrigin's function:

$$f(\boldsymbol{x}) = 10n + \sum_{i=1}^{n} \Big[x_i^2 - 10\cos(2\pi x_i)\Big], \quad -5.12 \le x_i \le 5.12, \tag{14}$$

whose global minimum is $f_* = 0$ at $(0, 0, ..., 0)$. This function is highly multimodal.

Rosenbrock's function:

$$f(\boldsymbol{x}) = \sum_{i=1}^{n-1} \Big[(x_i - 1)^2 + 100(x_{i+1} - x_i^2)^2\Big], \tag{15}$$

whose global minimum $f_* = 0$ occurs at $\boldsymbol{x}_* = (1, 1, ..., 1)$ in the domain $-5 \leq x_i \leq 5$ where $i = 1, 2, ..., n$. In the 2D case, it is often written as

$$f(x, y) = (x - 1)^2 + 100(y - x^2)^2, \tag{16}$$

which is often referred to as the banana function.

Schwefel's function:

$$f(\boldsymbol{x}) = -\sum_{i=1}^{n} x_i \sin\Big(\sqrt{|x_i|}\Big), \quad -500 \leq x_i \leq 500, \tag{17}$$

whose global minimum $f_* \approx -418.9829n$ occurs at $x_i = 420.9687$ where $i = 1, 2, ..., n$.

Six-hump camel back function:

$$f(x, y) = (4 - 2.1x^2 + \frac{1}{3}x^4)x^2 + xy + 4(y^2 - 1)y^2, \tag{18}$$

where $-3 \leq x \leq 3$ and $-2 \leq y \leq 2$. This function has two global minima $f_* \approx -1.0316$ at $(x_*, y_*) = (0.0898, -0.7126)$ and $(-0.0898, 0.7126)$.

Shubert's function:

$$f(\boldsymbol{x}) = \Big[\sum_{i=1}^{n} i \cos\Big(i + (i+1)x\Big)\Big] \cdot \Big[\sum_{i=1}^{n} i \cos\Big(i + (i+1)y\Big)\Big], \tag{19}$$

which has 18 global minima $f_* \approx -186.7309$ for $n = 5$ in the search domain $-10 \leq x, y \leq 10$.

Xin-She Yang's functions:

$$f(\boldsymbol{x}) = \Big(\sum_{i=1}^{n} |x_i|\Big) \exp\Big[-\sum_{i=1}^{n} \sin(x_i^2)\Big], \tag{20}$$

which has the global minimum $f_* = 0$ at $\boldsymbol{x}_* = (0, 0, ..., 0)$ in the domain $-2\pi \leq x_i \leq 2\pi$ where $i = 1, 2, ..., n$. This function is not smooth, and its derivatives are not well defined at the optimum $(0, 0, ..., 0)$.

A related function is

$$f(\boldsymbol{x}) = -\Big(\sum_{i=1}^{n} |x_i|\Big) \exp\Big(-\sum_{i=1}^{n} x_i^2\Big), \quad -10 \leq x_i \leq 10, \tag{21}$$

which has multiple global minima. For example, for $n = 2$, we have 4 equal minima $f_* = -1/\sqrt{e} \approx -0.6065$ at $(1/2, 1/2)$, $(1/2, -1/2)$, $(-1/2, 1/2)$ and $(-1/2, -1/2)$.

Yang also designed a standing-wave function with a defect

$$f(\boldsymbol{x}) = \Big[e^{-\sum_{i=1}^{n} (x_i/\beta)^{2m}} - 2 e^{-\sum_{i=1}^{n} x_i^2} \Big] \cdot \prod_{i=1}^{n} \cos^2 x_i, \quad m = 5, \tag{22}$$

which has many local minima and the unique global minimum $f_* = -1$ at $\boldsymbol{x}_* = (0, 0, ..., 0)$ for $\beta = 15$ within the domain $-20 \le x_i \le 20$ for $i = 1, 2, ..., n$.

The location of the defect can easily be shift to other positions. For example,

$$f(\boldsymbol{x}) = \Big[e^{-\sum_{i=1}^{n} (x_i/\beta)^{2m}} - 2 e^{-\sum_{i=1}^{n} (x_i - \pi)^2} \Big] \cdot \prod_{i=1}^{n} \cos^2 x_i, \quad m = 5, \tag{23}$$

has a unique global minimum $f_* = -1$ at $\boldsymbol{x}_* = (\pi, \pi, ..., \pi)$

Yang also proposed another multimodal function

$$f(\boldsymbol{x}) = \Big\{ \Big[\sum_{i=1}^{n} \sin^2(x_i) \Big] - \exp(-\sum_{i=1}^{n} x_i^2) \Big\} \cdot \exp\Big[-\sum_{i=1}^{n} \sin^2 \sqrt{|x_i|} \Big], \tag{24}$$

whose global minimum $f_* = -1$ occurs at $\boldsymbol{x}_* = (0, 0, ..., 0)$ in the domain $-10 \le x_i \le 10$ where $i = 1, 2, ..., n$. In the 2D case, its landscape looks like a wonderful candlestick.

Most test functions are deterministic. Yang designed a test function with stochastic components

$$f(x, y) = -5 e^{-\beta[(x-\pi)^2 + (y-\pi)^2]} - \sum_{j=1}^{K} \sum_{i=1}^{K} \epsilon_{ij} e^{-\alpha[(x-i)^2 + (y-j)^2]}, \tag{25}$$

where $\alpha, \beta > 0$ are scaling parameters, which can often be taken as $\alpha = \beta = 1$. Here ϵ_{ij} are random variables and can be drawn from a uniform distribution $\epsilon_{ij} \sim$ Unif[0,1]. The domain is $0 \le x, y \le K$ and $K = 10$. This function has K^2 local valleys at grid locations and the fixed global minimum at $\boldsymbol{x}_* = (\pi, \pi)$. It is worth pointing that the minimum $f_{\min}$ is random, rather than a fixed value; it may vary from $-(K^2 + 5)$ to -5, depending α and β as well as the random numbers drawn.

Furthermore, he also designed a stochastic function

$$f(\boldsymbol{x}) = \sum_{i=1}^{n} \epsilon_i \left| x_i - \frac{1}{i} \right|, \quad -5 \le x_i \le 5, \tag{26}$$

where ϵ_i $(i = 1, 2, ..., n)$ are random variables which are uniformly distributed in $[0, 1]$. That is, ϵ_i ~Unif$[0, 1]$. This function has the unique minimum $f_* = 0$ at $\boldsymbol{x}_* = (1, 1/2, ..., 1/n)$ which is also singular.

Zakharov's functions:

$$f(\boldsymbol{x}) = \sum_{i=1}^{n} x_i^2 + \Big(\frac{1}{2} \sum_{i=1}^{n} i x_i \Big)^2 + \Big(\frac{1}{2} \sum_{i=1}^{n} i x_i \Big)^4, \tag{27}$$

whose global minimum $f_* = 0$ occurs at $\boldsymbol{x}_* = (0, 0, ..., 0)$. Obviously, we can generalize this function as

$$f(\boldsymbol{x}) = \sum_{i=1}^{n} x_i^2 + \sum_{k=1}^{K} J_n^{2k}, \tag{28}$$

where $K = 1, 2, ..., 20$ and

$$J_n = \frac{1}{2} \sum_{i=1}^{n} i x_i. \tag{29}$$

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