Numerical optimization for large scale problems
Constrained optimization

Problem list, constrained optimization
a.y. 2022/23

Please choose one of the following problems. Whatever problem you choose, I expect you to submit **a single pdf file**. The file should be submitted through the `exercise.polito.it` page from which you downloaded the assignment. If you work in a group (max 3 people) please upload the file **only once** but clearly state in the file name the family names of all team-mates.

The document is expected to report an introductory analysis of the problem, tables and/or figures summarizing your results and comments on your results. Please use **captions** in order to explain what every table and/or figure is reporting, and quote it also in the text (e.g., "In Figure xx we report the plot of....", "In Table yy we compare ...").

In general you are expected to test your solver/solvers on one or more common problems, with different values of some parameters and possibly different starting points. In all the cases you should compare the results obtained, for example in terms of number of iterations and computing time, commenting your results also in view of the values of the parameters used and of the theory.

As an **appendix**, please add the commented scripts/functions you implemented in your favorite programming language. Please make sure to use sensible names for the variables and functions, and to provide enough comments and explanations to render the code readable to a non expert of the specific language.

1. Projected gradient method
   Consider the problem described by equation (3) in [1]. Use your own implementation of the projected gradient method to solve the problem with $n = 10^d$ and $d = 3, 4, 5$, both using exact derivatives and using finite differences to approximate the gradient. Compare the behavior of the two implementations, using the following values for the increment h:

   $$h = 10^{-k}||\hat{x}||, \qquad k = 2, 4, 6, 8, 10, 12$$

   where $\hat{x}$ is the point at which the derivatives have to be approximated.

   Repeat considering the same function but with the following constraints, indicating what is the minimum point in each case:

   (a) $X = [1, 5.12]^n$;
   (b) $X = [-5.12, 5.12] \times [1, 5.12]^{n-1}$;
   (c) $X = [-5.12, 5.12]^{n/2} \times [1, 5.12]^{n/2}$.

2. Equality constrained quadratic programming problem
   Consider the problem

$$
\begin{aligned}
\min_{x\in\mathbb{R}^n} \quad & \textstyle\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n-1} x_i x_{i+1} + \sum_{i=1}^{n} x_i \\
\text{s.t.} \quad & \text{the sum } x_1 + x_{1+K} + x_{1+2K} + \ldots \text{ should be 1} \\
& \text{the sum } x_2 + x_{2+K} + x_{2+2K} + \ldots \text{ should be 1} \\
& \qquad \vdots \\
& \text{the sum } x_K + x_{2K} + x_{3K} + \ldots \text{ should be 1}
\end{aligned}
$$

   Solve the problem solving the KKT conditions, comparing both a full solution of the KKT system and a suitable reduced form. Use a direct solver whenever possible (i.e., no memory fault or to long computations), otherwise use suitable iterative solvers. Solve the problem with $n = 10^4$ and $n = 10^5$; use $K = 100$ and $k = 500$.

3. Interior Point Method applied to a QP problem
   Consider the problem

$$
\begin{aligned}
\min_{x\in\mathbb{R}^n} \quad & \textstyle\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n-1} x_i x_{i+1} + \sum_{i=1}^{n} x_i \\
\text{s.t.} \quad & \text{the sum } x_1 + x_{1+K} + x_{1+2K} + \ldots \text{ should be 1} \\
& \text{the sum } x_2 + x_{2+K} + x_{2+2K} + \ldots \text{ should be 1} \\
& \qquad \vdots \\
& \text{the sum } x_K + x_{2K} + x_{3K} + \ldots \text{ should be 1} \\
& x_i \geq 0 \qquad \forall i
\end{aligned}
$$

   Use your own implementation of the Predictor-Corrector Interior Point Method to solve the problem with $n = 10^4$ and $n = 10^5$; use $K = 100$ and $k = 500$. Compare different strategies to solve the linear system at each iteration.

[1] https://www.researchgate.net/publication/45932888_Test_Problems_in_Optimization