

CT561: Systems Modelling and Simulation

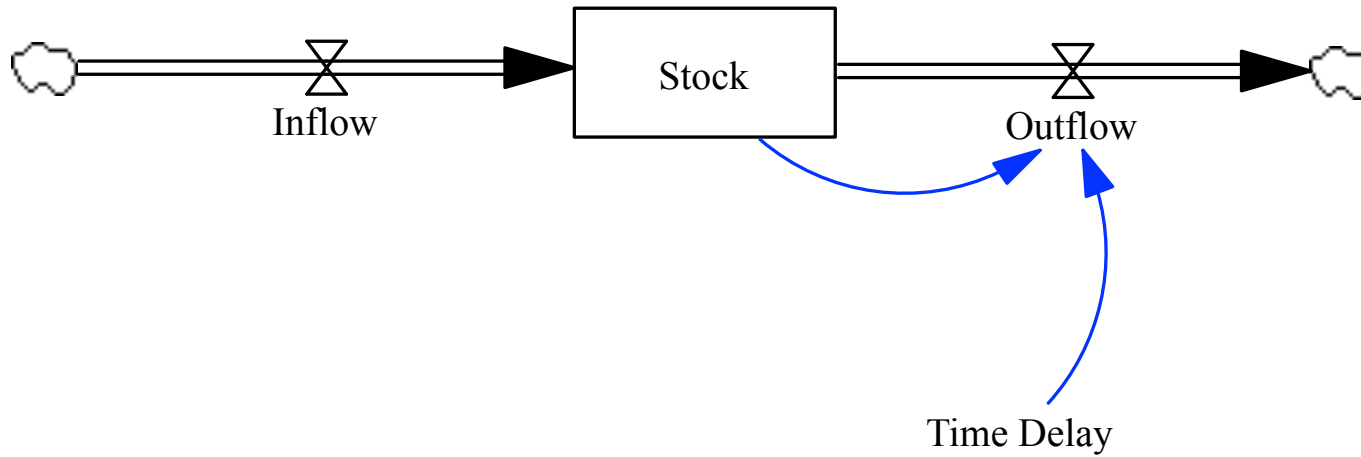
Week 10: Modelling Diffusion and the SIR Model

<https://github.com/JimDuggan/CT561>

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Recap: Tutorial Model (in R)

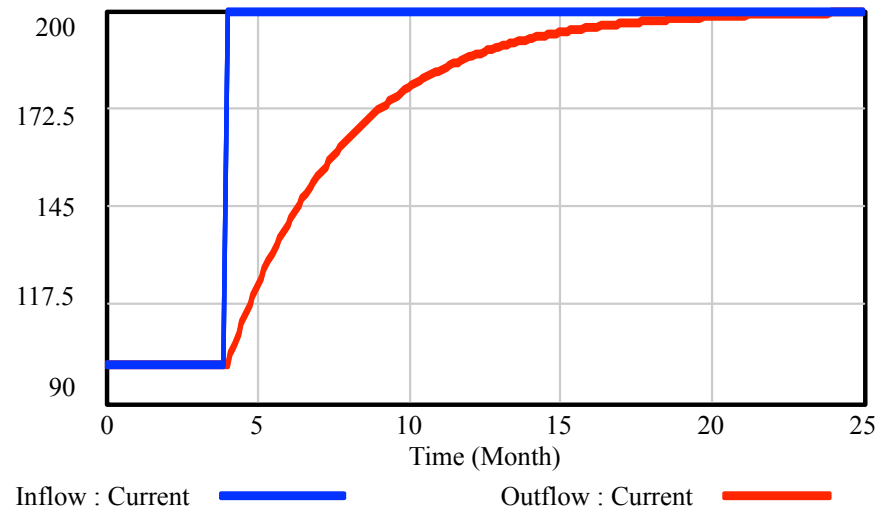


```
Inflow=100+step(100,4)
```

```
Outflow=Stock/Time Delay
```

```
Stock= INTEG (Inflow-Outflow, 400)
```

```
Time Delay=4
```



R Code

```
library(deSolve)
library(ggplot2)

# Model equations (from Vensim)
# INITIAL TIME = 0
# FINAL TIME = 25
# Inflow=100+step(100,4)
# Outflow=Stock/Time Delay
# Stock= INTEG (Inflow-Outflow, 400)
# Time Delay=4
# TIME STEP = 0.125

START<-0; FINISH<-25; STEP<-0.125;
simtime <- seq(START, FINISH, by=STEP)
```

Exploring vector simtime

time = 4 is at vector location 33

```
> simtime
```

[1]	0.000	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.000	1.125	1.250	1.375
[13]	1.500	1.625	1.750	1.875	2.000	2.125	2.250	2.375	2.500	2.625	2.750	2.875
[25]	3.000	3.125	3.250	3.375	3.500	3.625	3.750	3.875	4.000	4.125	4.250	4.375
[37]	4.500	4.625	4.750	4.875	5.000	5.125	5.250	5.375	5.500	5.625	5.750	5.875
[49]	6.000	6.125	6.250	6.375	6.500	6.625	6.750	6.875	7.000	7.125	7.250	7.375
[61]	7.500	7.625	7.750	7.875	8.000	8.125	8.250	8.375	8.500	8.625	8.750	8.875
[73]	9.000	9.125	9.250	9.375	9.500	9.625	9.750	9.875	10.000	10.125	10.250	10.375
[85]	10.500	10.625	10.750	10.875	11.000	11.125	11.250	11.375	11.500	11.625	11.750	11.875
[97]	12.000	12.125	12.250	12.375	12.500	12.625	12.750	12.875	13.000	13.125	13.250	13.375
[109]	13.500	13.625	13.750	13.875	14.000	14.125	14.250	14.375	14.500	14.625	14.750	14.875
[121]	15.000	15.125	15.250	15.375	15.500	15.625	15.750	15.875	16.000	16.125	16.250	16.375
[133]	16.500	16.625	16.750	16.875	17.000	17.125	17.250	17.375	17.500	17.625	17.750	17.875
[145]	18.000	18.125	18.250	18.375	18.500	18.625	18.750	18.875	19.000	19.125	19.250	19.375
[157]	19.500	19.625	19.750	19.875	20.000	20.125	20.250	20.375	20.500	20.625	20.750	20.875
[169]	21.000	21.125	21.250	21.375	21.500	21.625	21.750	21.875	22.000	22.125	22.250	22.375
[181]	22.500	22.625	22.750	22.875	23.000	23.125	23.250	23.375	23.500	23.625	23.750	23.875
[193]	24.000	24.125	24.250	24.375	24.500	24.625	24.750	24.875	25.000			

Setting up a global data frame

```
# Setup the step function in a global data frame
input <- rep(NA,length(simtime))
input[1:(4/STEP)]<-100
input[((4/STEP)+1):length(simtime)]<-200

simData<-data.frame(time=simtime, aInput=input)
```

The input vector (step function)
step happens at vector location 33

```
> input
[1] 100 100 100 100 100 100 100 100 100 100 100 100 100 100 100 100 100
[18] 100 100 100 100 100 100 100 100 100 100 100 100 100 100 100 200 200
[35] 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200
[52] 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200
[69] 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200
[86] 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200
[103] 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200
[120] 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200
[137] 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200
[154] 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200
[171] 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200
[188] 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200 200
```


The simData data frame

```
> simData[1:10,]
```

	time	aInput
1	0.000	100
2	0.125	100
3	0.250	100
4	0.375	100
5	0.500	100
6	0.625	100
7	0.750	100
8	0.875	100
9	1.000	100
10	1.125	100

```
> simData[30:40,]
```

	time	aInput
30	3.625	100
31	3.750	100
32	3.875	100
33	4.000	200
34	4.125	200
35	4.250	200
36	4.375	200
37	4.500	200
38	4.625	200
39	4.750	200
40	4.875	200

```
> simData[191:201,]
```

	time	aInput
191	23.750	200
192	23.875	200
193	24.000	200
194	24.125	200
195	24.250	200
196	24.375	200
197	24.500	200
198	24.625	200
199	24.750	200
200	24.875	200
201	25.000	200

The model function

```
# model equations
model <- function(time, stocks, auxs)
{
  with(as.list(c(stocks, auxs)),{
    index<-which(simData$time==time)

    fInflow<-simData$aInput[index]
    fOutflow<-sStock/aTimeDelay

    dS_dt <- fInflow - fOutflow

    return (list(c(dS_dt),
                  Inflow=fInflow,
                  Outflow=fOutflow))
  })
}
```

Inflow=100+step(100,4)

Outflow=Stock/Time Delay

Stock= INTEG (Inflow-Outflow, 400)

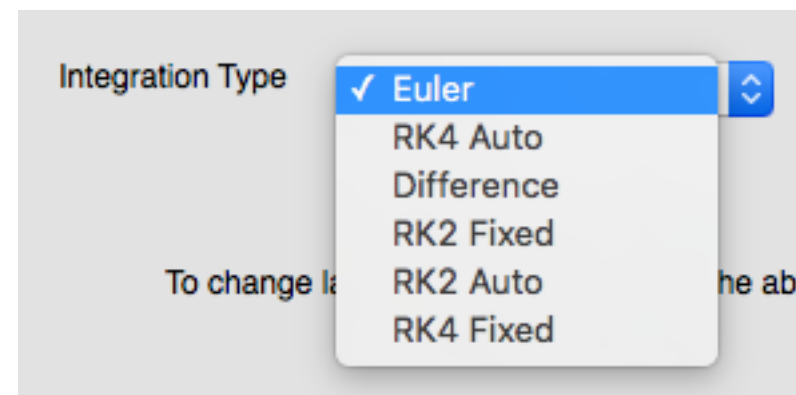
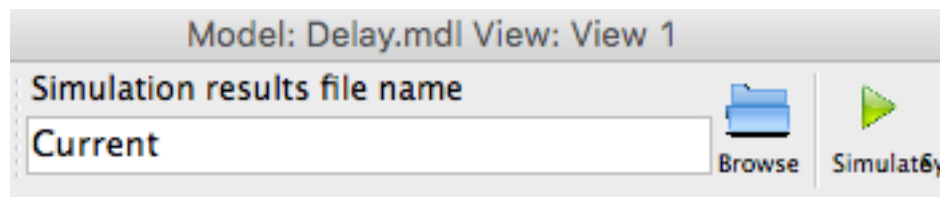
Setup and run

```
stocks <- c(sStock=400)
auxs <- c(aTimeDelay=4)
```

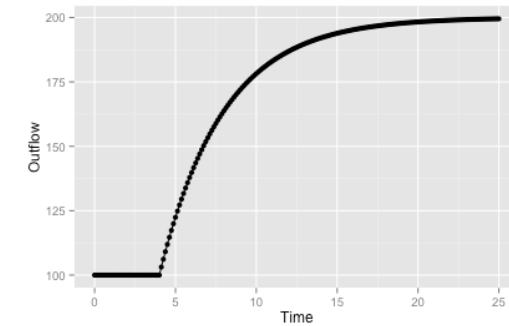
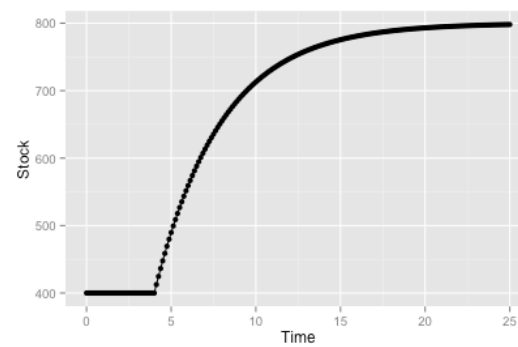
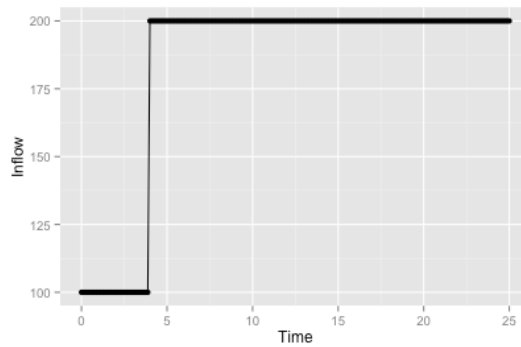
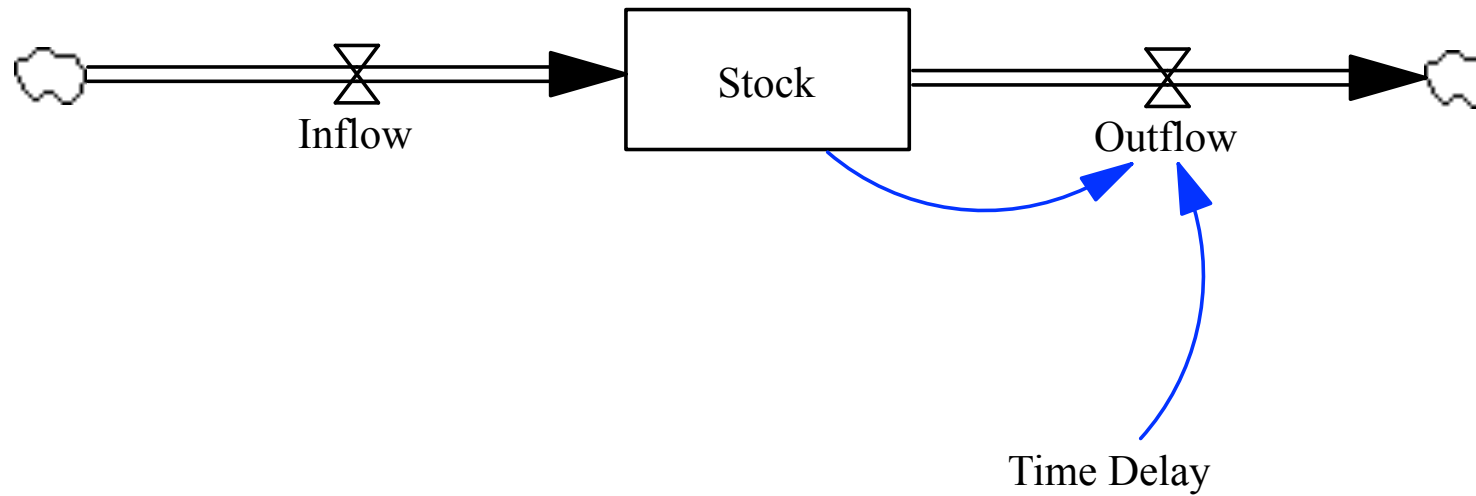
Stock= INTEG (Inflow-Outflow, **400**)

Time Delay=4

```
o<-data.frame(ode(y=stocks, simtime, func = model,
  parms=auxs, method="euler"))
```

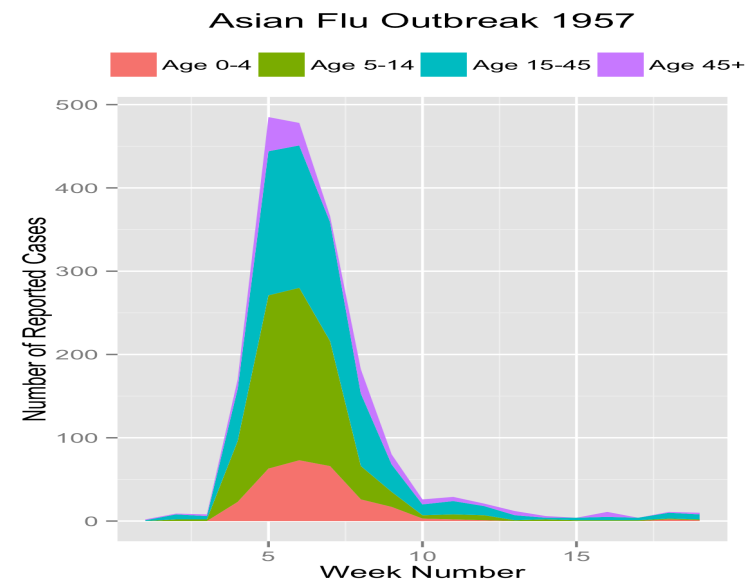
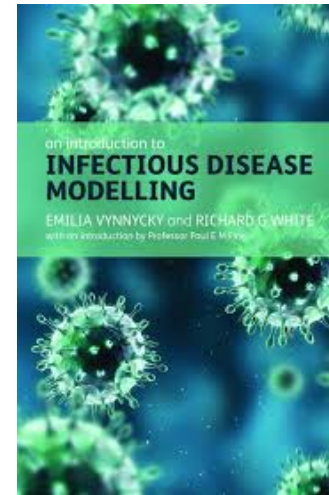


View the results

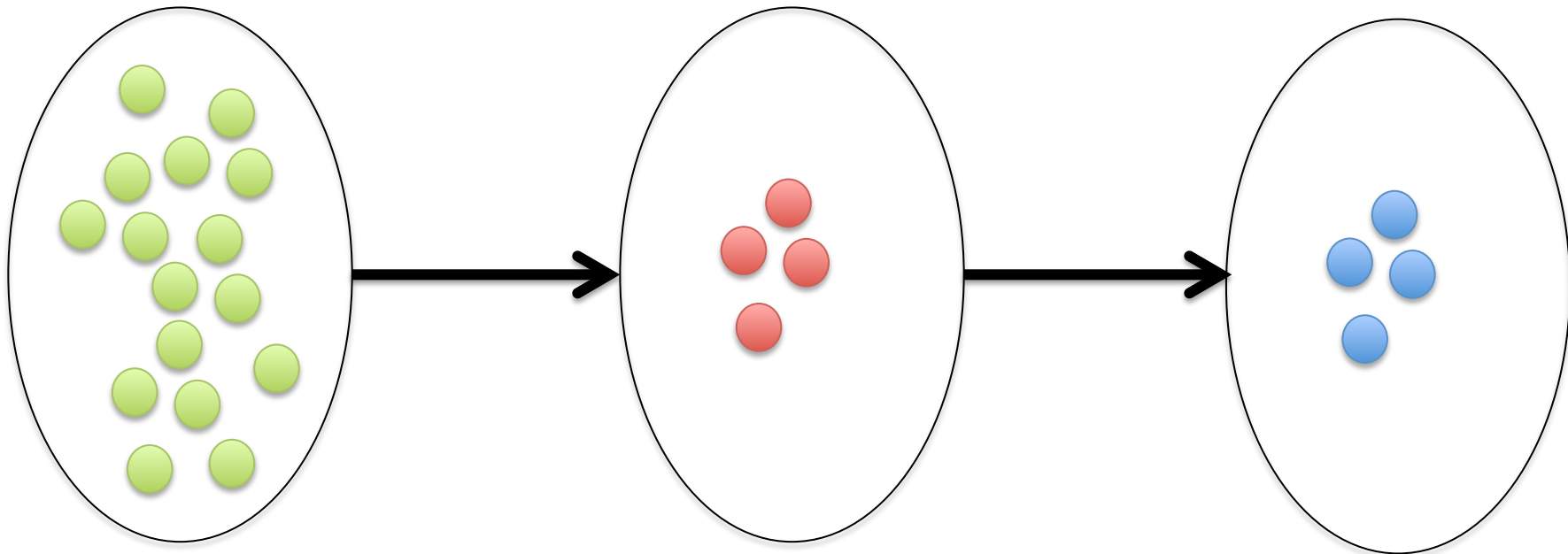
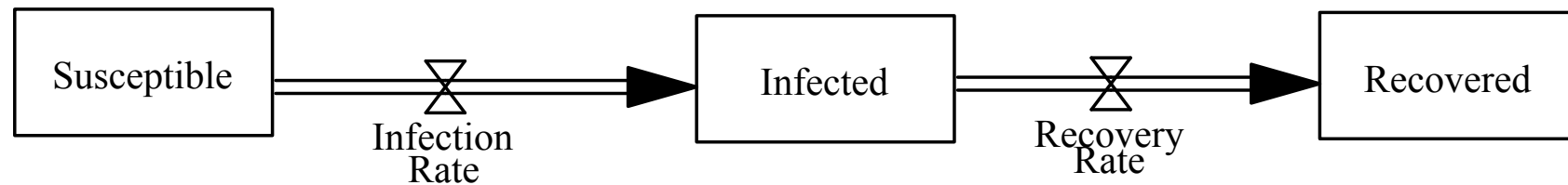


Public Health: Modelling Infectious Disease Outbreaks

- Public Health
- Modelling transmission of infectious diseases
- SIR Model (3 stocks, 2 flows)



Need a model structure (Stocks and Flows)



Stock Equations

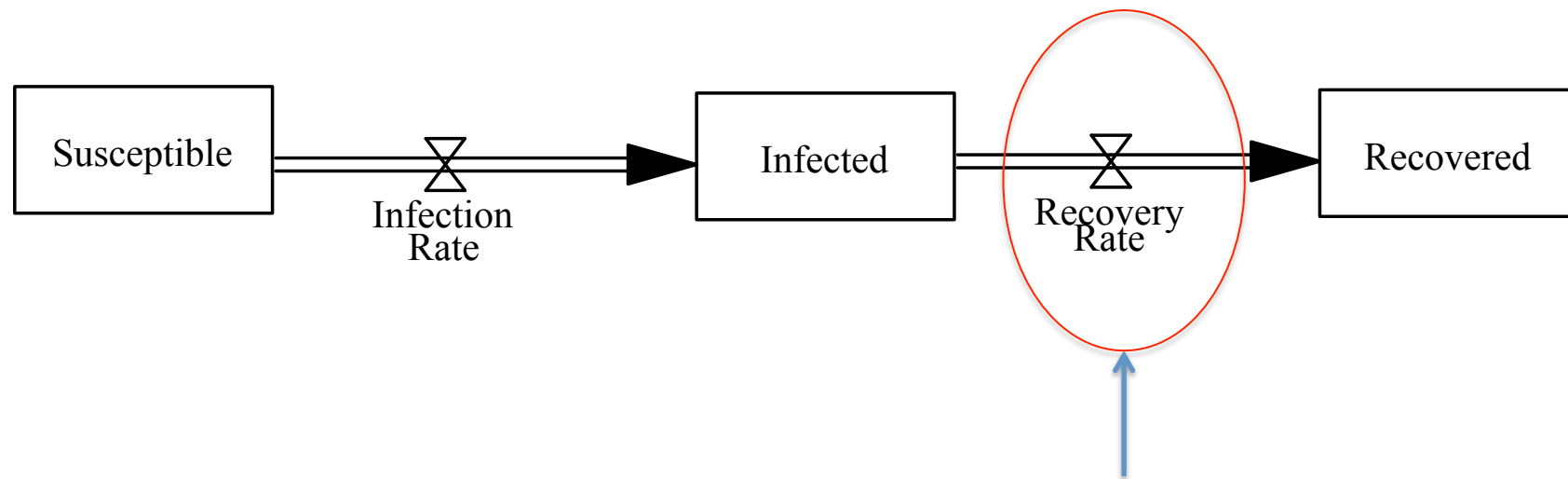


$$Susceptible (S) = INTEGRAL(-IR, 99999)$$

$$Infected (I) = INTEGRAL(IR - RR, 1)$$

$$Recovered(R) = INTEGRAL(RR, 0)$$

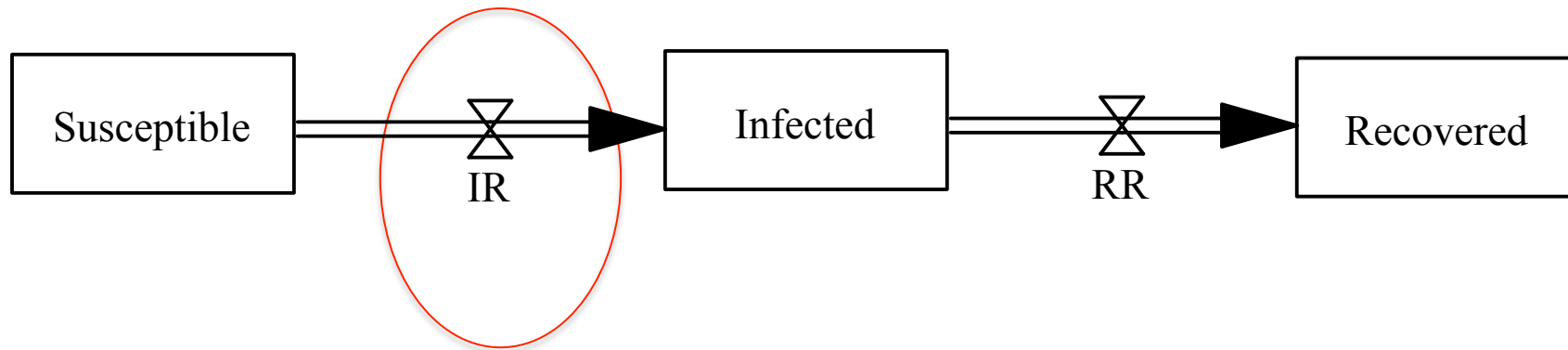
What about the flows?



First Order Delay Structure

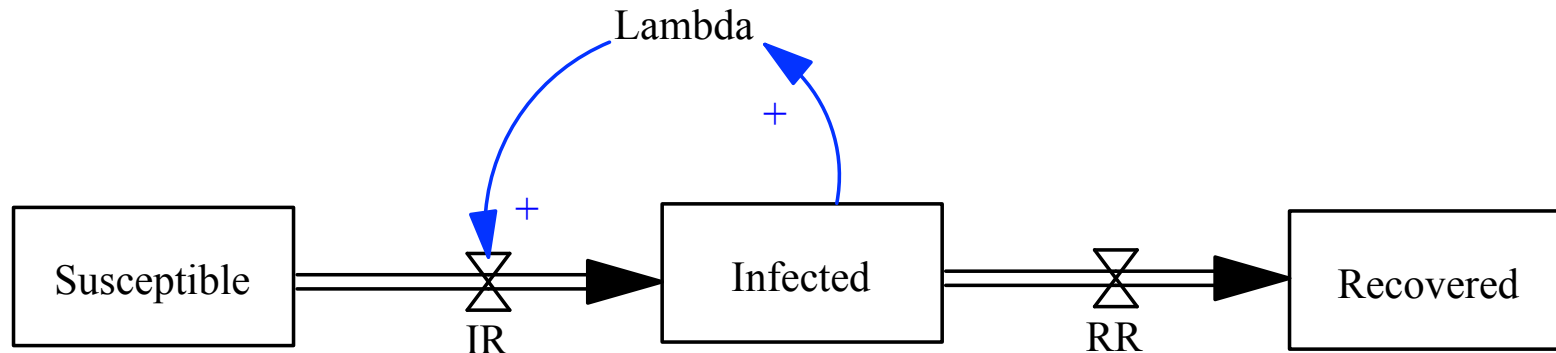
$$RR = \frac{I}{D}$$
$$Delay (D) = 2$$

Infection Rate?



- Infection spreads through contact
- As the number of infected increase, so to does the infection rate
- A positive feedback process

Lambda – Force of Infection (Attack Rate)



↑ Infected
↑ Lambda
↑ IR

→ Lambda
→ IR
→ Infected

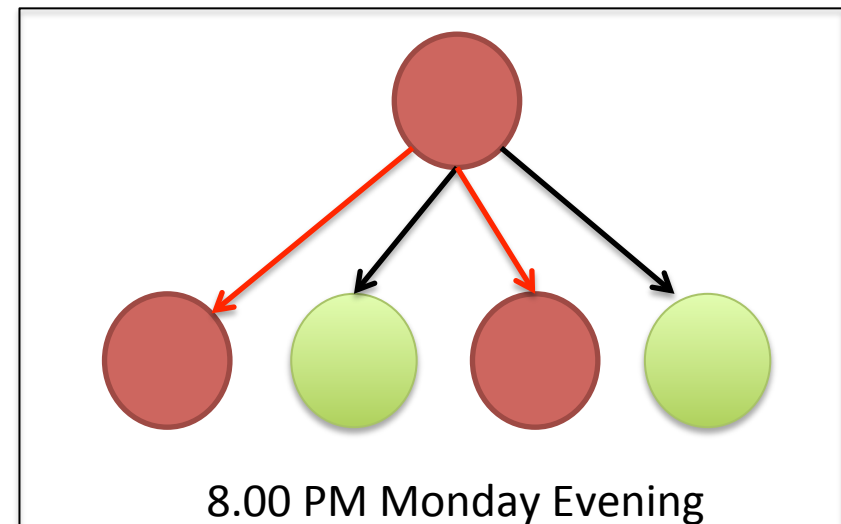
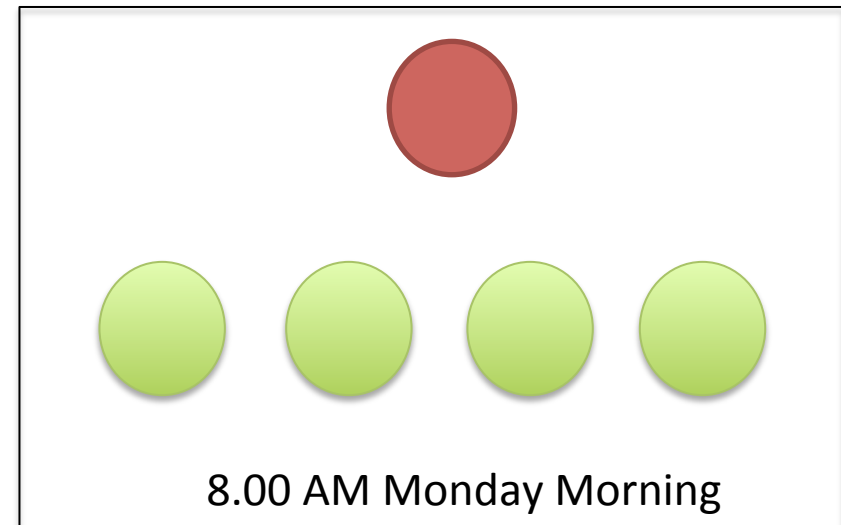
↑
↑
↑

The rate at which susceptible individuals become infected per unit time

Proportional to the number infected

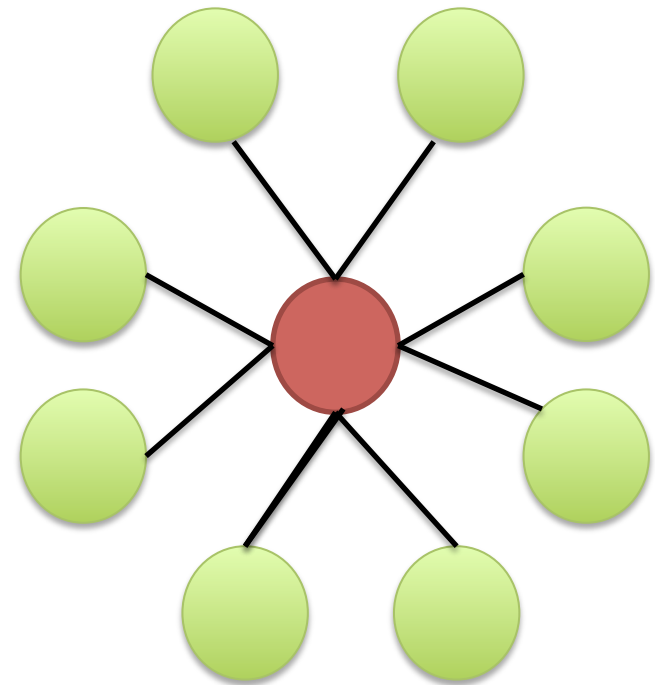
Effective Contact (C_e)

- Defined a one which is sufficient to lead to infection, were it to occur between a **susceptible** and **infectious** individuals
- For example, if $C_e = 2$
 - An infectious person will infect two susceptible people in one day
 - They could meet 4 people, and pass on the virus with probability (0.50)



Beta (β)

- **Per capita rate** at which two specific individuals come into effective contact per unit time
- An important parameter used to model interactions between infectious and susceptible individuals



$$\beta = \frac{c_e}{N}$$

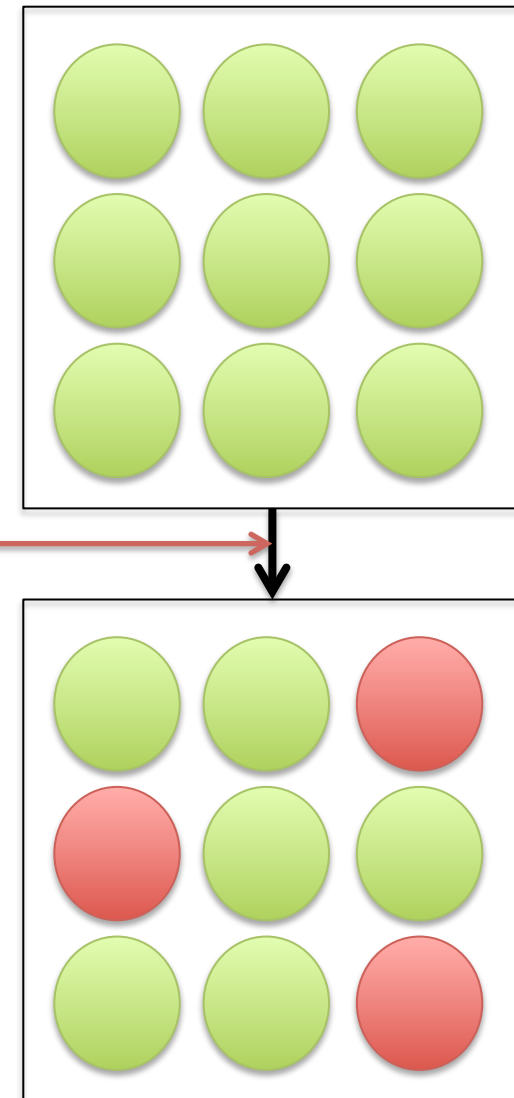
$$\beta = \frac{2}{10000} = 0.0002$$

Lambda – Force of Infection

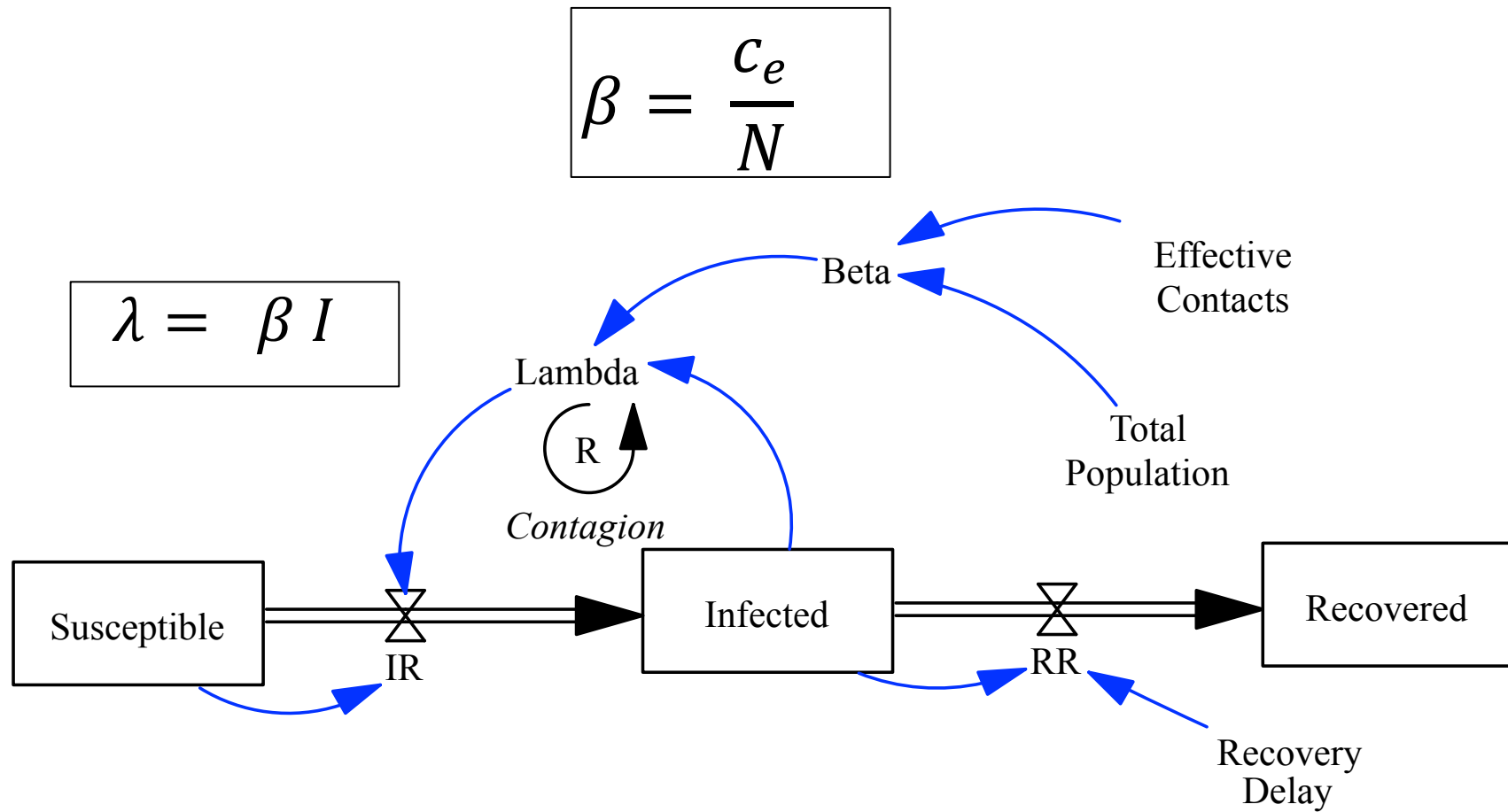
- The rate at which susceptible individuals become infected per unit time
- Also known as the hazard rate or incidence rate

$$\lambda = \beta I$$

$$\lambda = \frac{1}{3}$$



A Stock and Flow Model



Model Equations

Total Population = 10000

Susceptible= INTEG (-IR, 9999)

Effective Contacts=2

Infected= INTEG (IR-RR, 1)

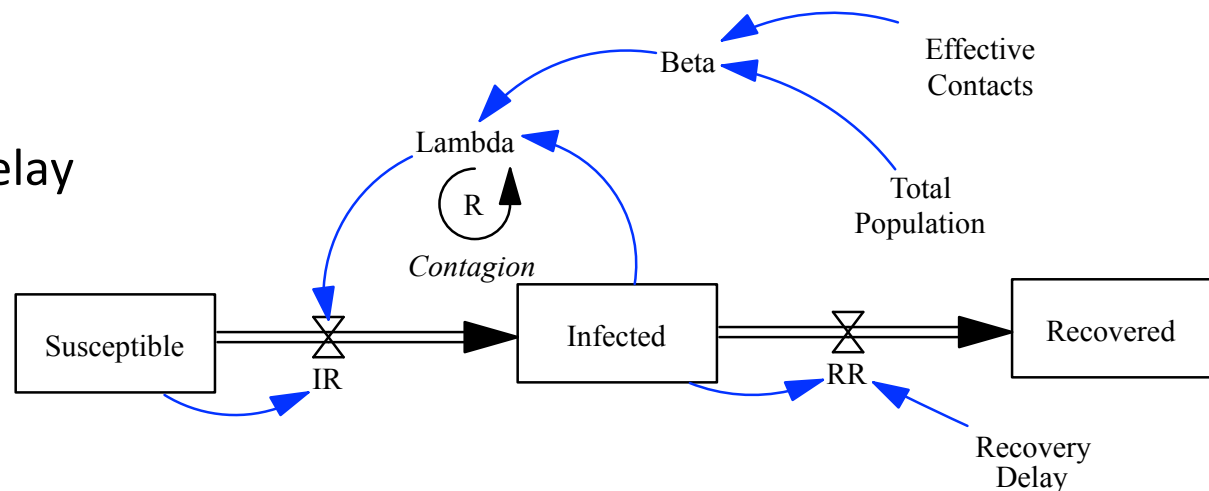
Beta= Effective Contacts/Total Population

Recovered= INTEG (RR, 0)

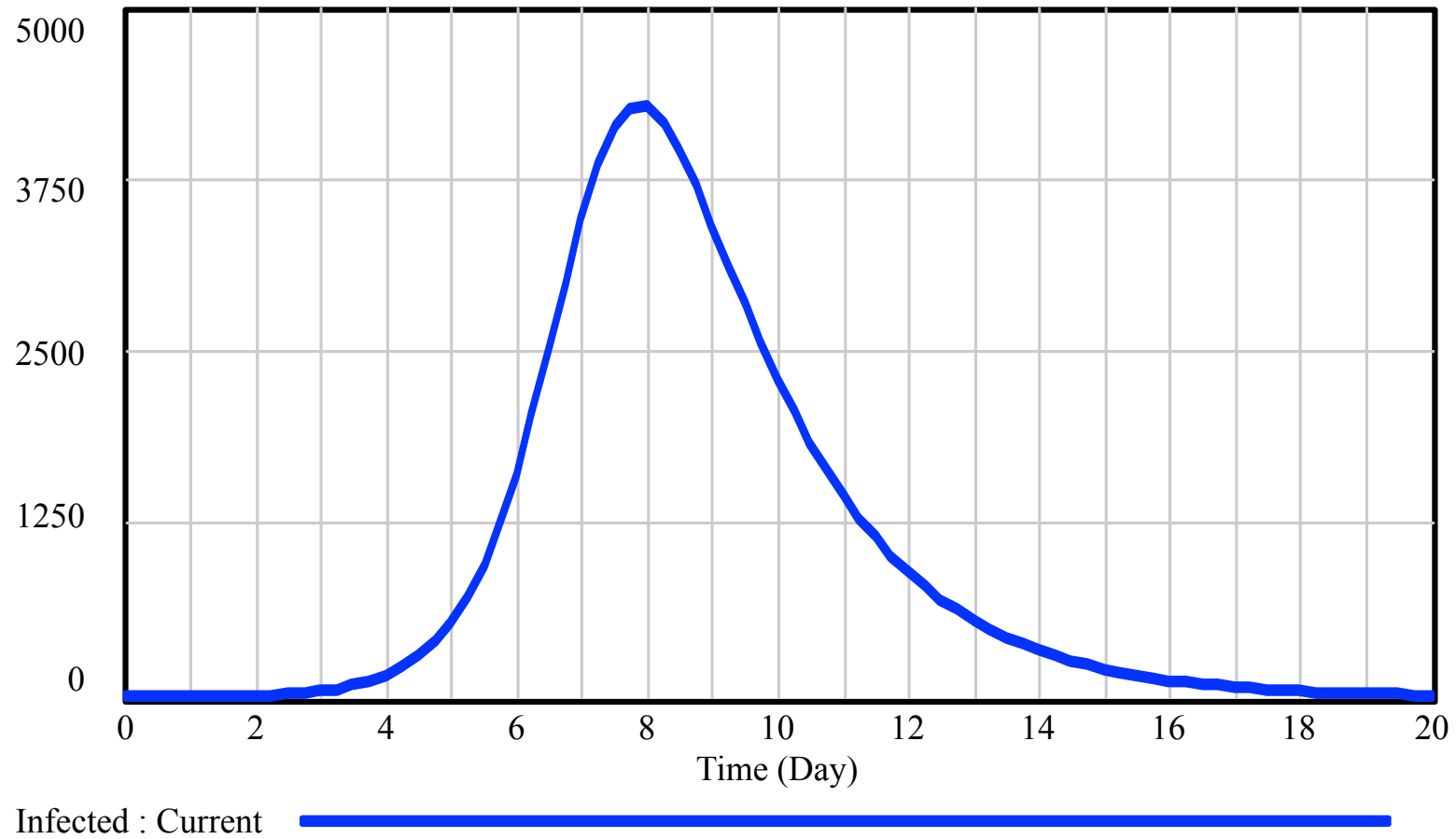
Lambda = Beta*Infected

IR=Lambda*Susceptible

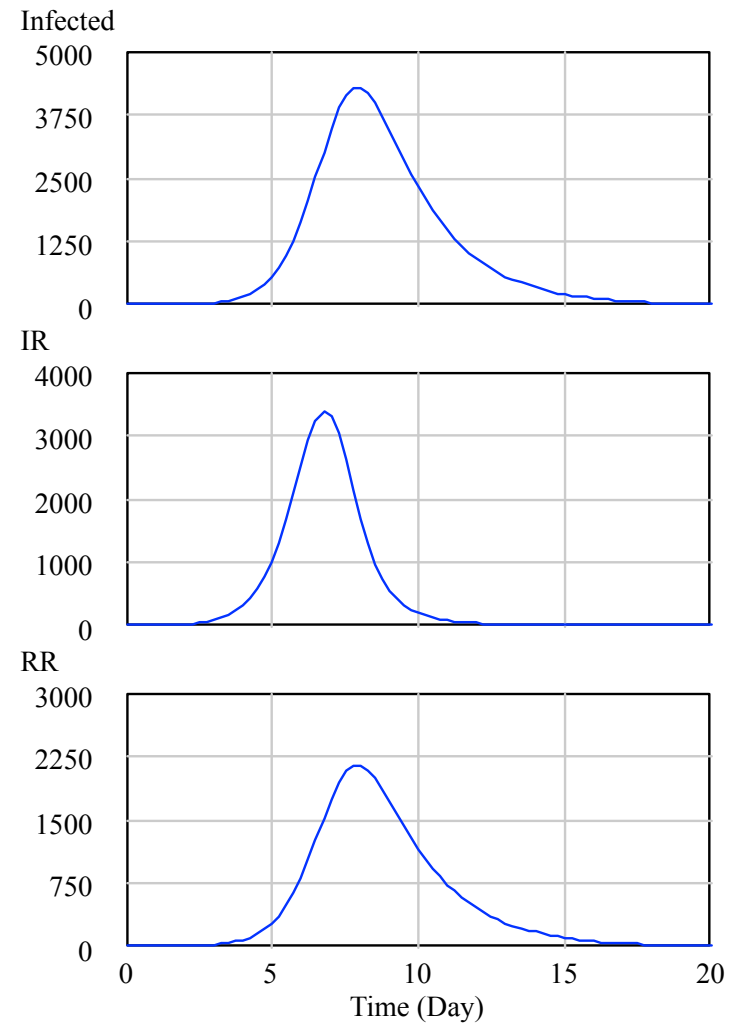
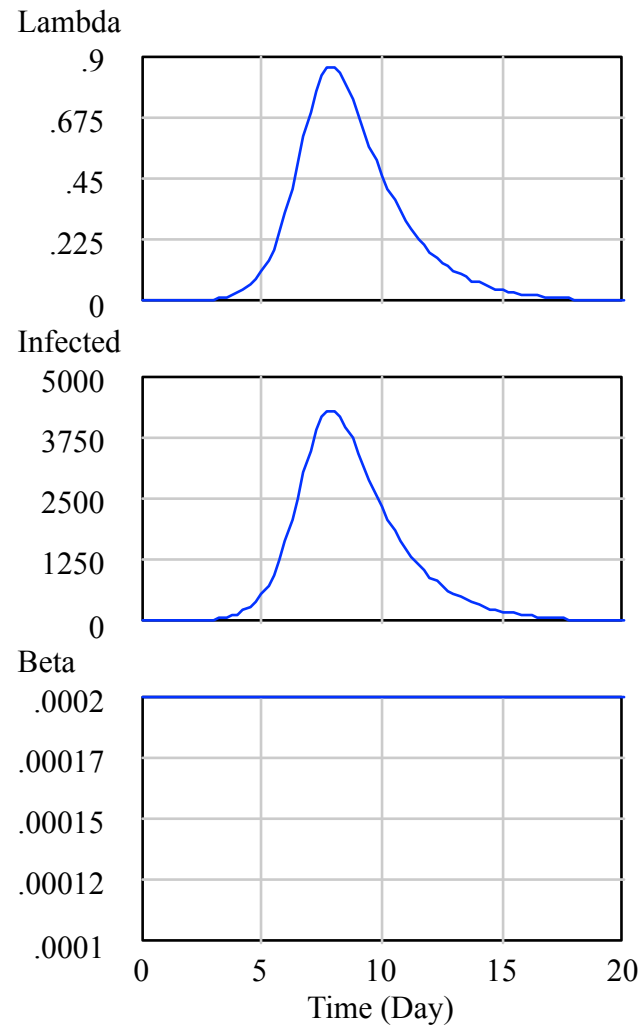
RR=Infected/Recovery Delay



Simulation Output



Exploring Variables...



Challenge 10.1

- Draw the stock and flow model the corresponds to the following equations

$$\frac{dS}{dt} = -\lambda S$$

$$\frac{dI}{dt} = fE - rI$$

$$\frac{dE}{dt} = \lambda S - fE$$

$$\frac{dR}{dt} = rI$$

The rate at which something occurs is $1 / \text{Average time to the event}$
 f and r are rates.

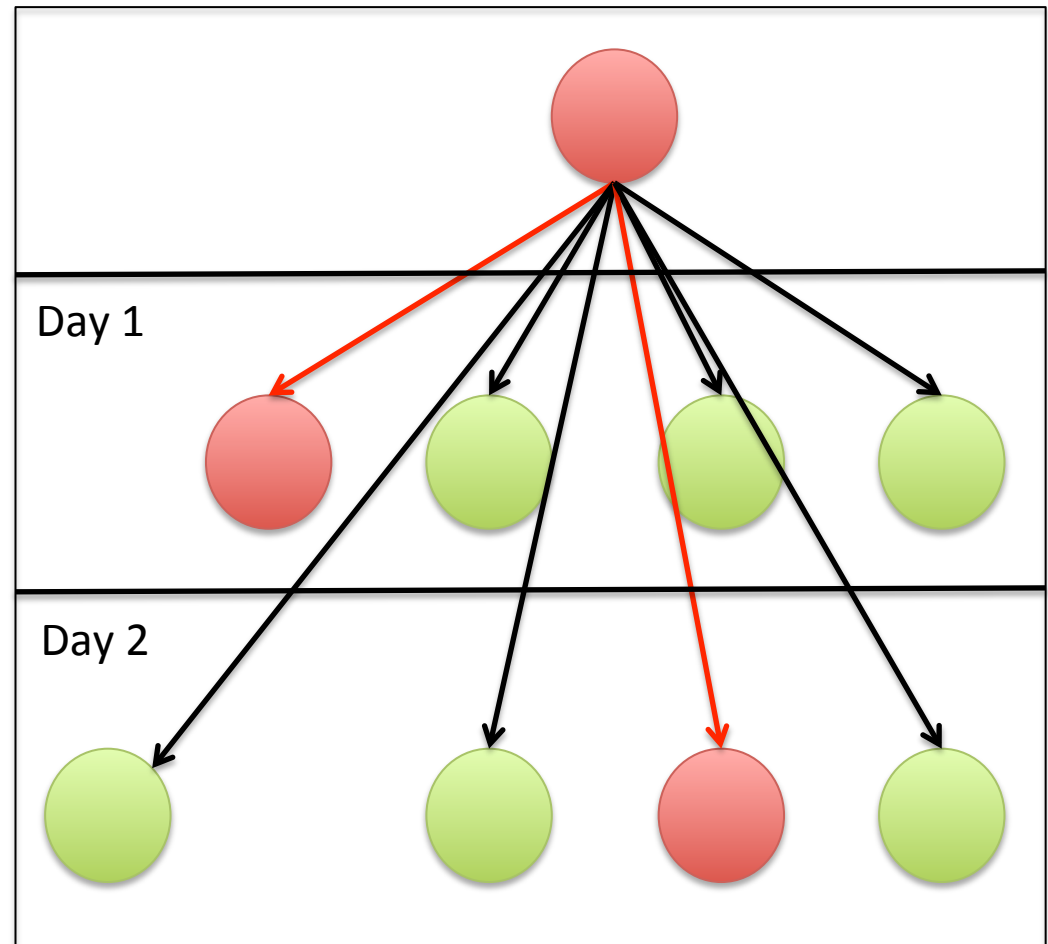
Three Additional Concepts

- Reproduction Number
- Herd Immunity
- Epidemic Thresholds

(1) Reproduction Number – R_0

- Formally defined as the average number of secondary infectious resulting from a typical infectious person being introduced to a totally susceptible population

$$R_0 = c_e D$$



$$D = 2 \quad c_e = 1 \quad R_0 = 2$$

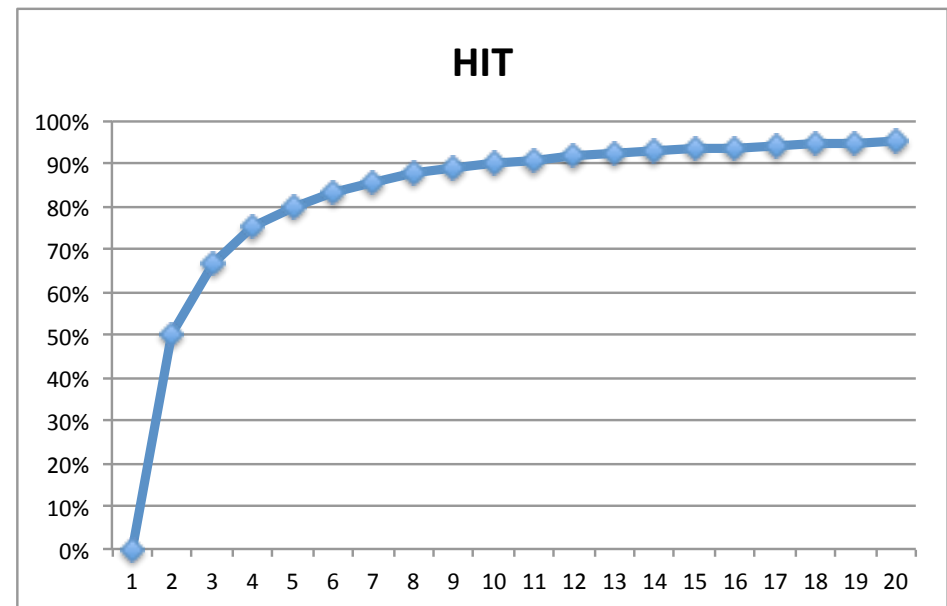
Challenge 10.2

- Suppose we have a town with 10,000 ($=N$) individuals, of which 1% were infectious with measles, with $R_0 = 13$ and $D=7$ Days
- Calculate the force of infection λ

(2) Herd Immunity Threshold

- Depends on R_0
- The proportion of the population which needs to be immune for the infection incidence to be stable
- To eradicate an infection, the proportion of the population that is immune must exceed this threshold value

$$HIT = 1 - \frac{1}{R_0}$$

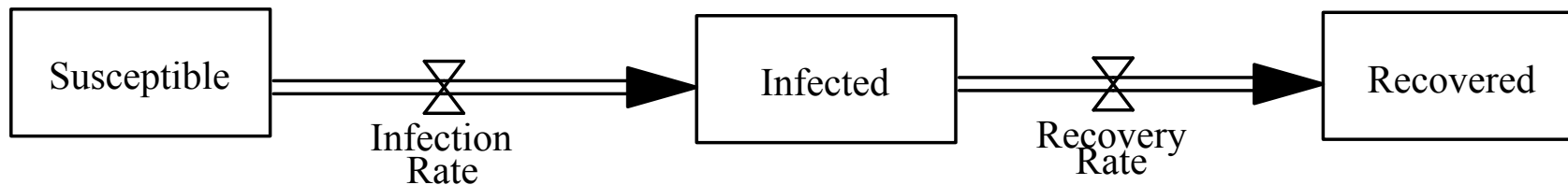


Approximate data for common potentially vaccine-preventable diseases

Infection	R_0	Herd Immunity
Diphtheria	6-7	85
Influenza	2-4	50-75
Malaria	5-100	80-99
Measles	12-18	83-94
Pertussis	12-17	92-94

(3) Threshold Dynamics for SIR Model

For the number of infectious people to increase, the inflow must be greater than the outflow.



$$IR > RR \longrightarrow \lambda S > \frac{I}{D} \longrightarrow \beta I S > \frac{I}{D}$$

$S = N$ for totally susceptible population

$$\beta N D > 1 \longrightarrow c_e D > 1 \quad \text{Given that } \beta = \frac{c_e}{N}$$

$$R_0 > 1$$

Challenge 10.3

- Introduce vaccinations and quarantine into the SIR model. Provide the appropriate rate equations.

