CT561: Systems Modelling and Simulation

Week 2: Integration

https://github.com/JimDuggan/CT561

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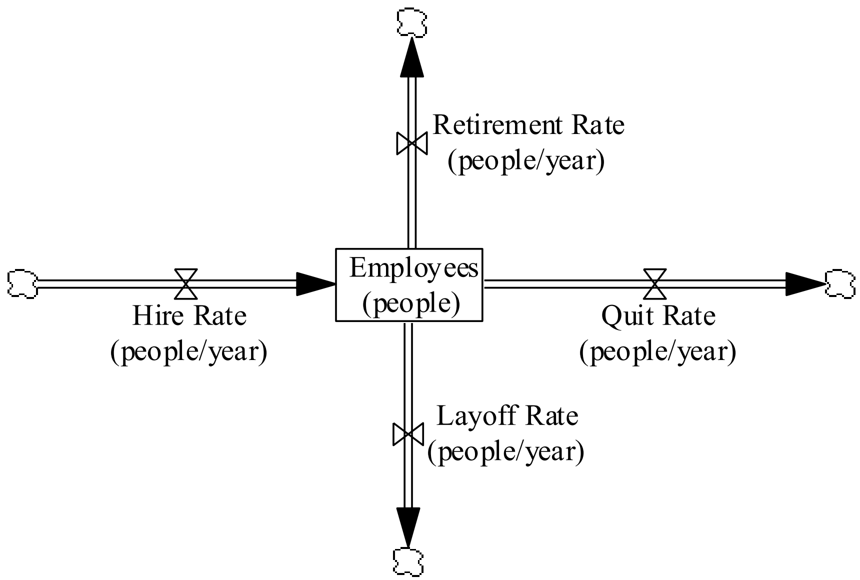
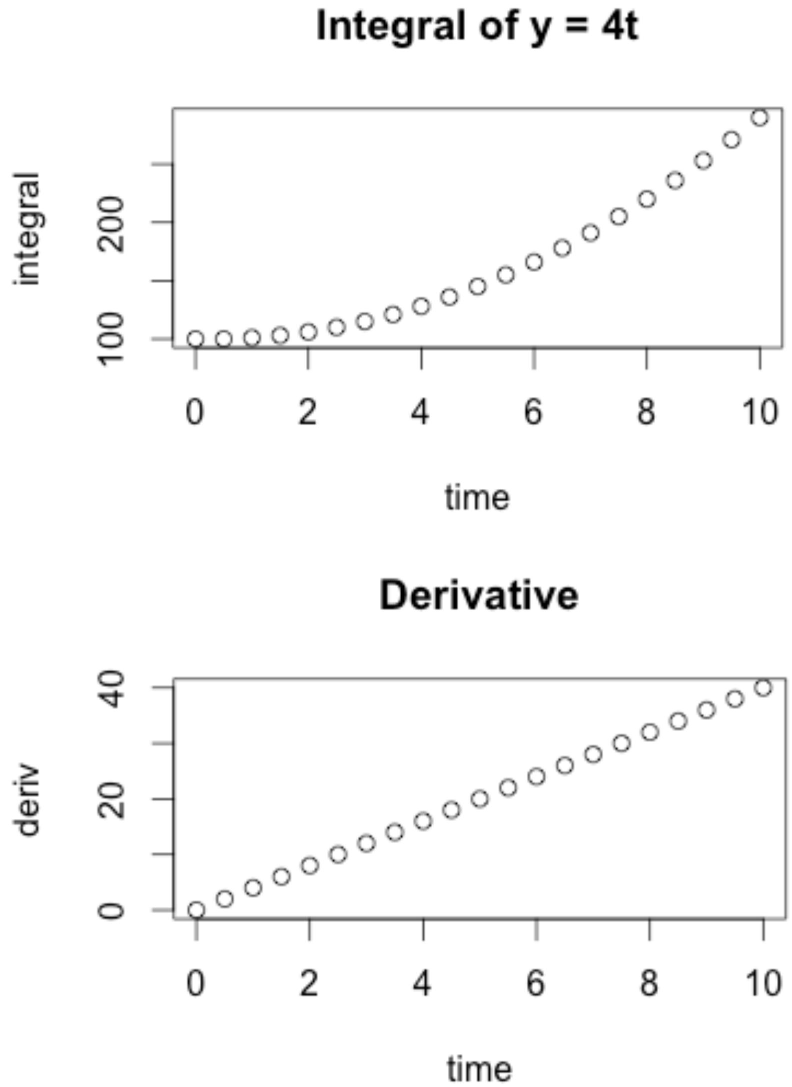
Stock and Flow Systems

- All stock-flow systems share the same underlying structure.
- The stock accumulates its inflows to it, less the outflows from it.
- This is a fundamental concept of calculus (integrals and derivatives)
- Knowledge of calculus is not necessary to understand the idea of stocks and flows



Calculus

- Given the dynamic of the flows, what is the behaviour of the stock?
  - Integration
- From the dynamics of the stock, can you infer the behaviour of the flows
  - Differentiation
- Calculus
  - “quite intuitive… it is the use of unfamiliar notation and a focus on analytic solutions that deters many people from the study of calculus” (Sterman 2000)



General Principle of Stock/Flow Systems

- From this simple bathtub model you can deduce **several important principles** that extend to more complicated systems:
  - As long as the sum of all inflows exceeds the sum of all outflows, the level of the stock will **rise**.
  - As long as the sum of all outflows exceeds the sum of all inflows, the level of the stock will **fall**.
  - If the sum of all outflows equals the sum of all inflows, the stock level **will not change**; it will be held in dynamic equilibrium at whatever level it happened to be when the two sets of flows became equal.

Meadows, Donella H. *Thinking in systems: A primer*. Chelsea Green Publishing, 2008.

Recap - Challenge 1.1

- Solve, analytically and numerically, the following derivative. Assume the integral has an initial value of 100, and the time interval is [0-10]. Set DT = 0.5 for the numerical solution.

$$\frac{dy}{dt} = 4t$$

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c$$

Solution to 1.1

$$\int t^n \; dt = \frac{1}{n+1} \, t^{n+1} + c$$

$$\frac{dy}{dt} = \; 4t$$

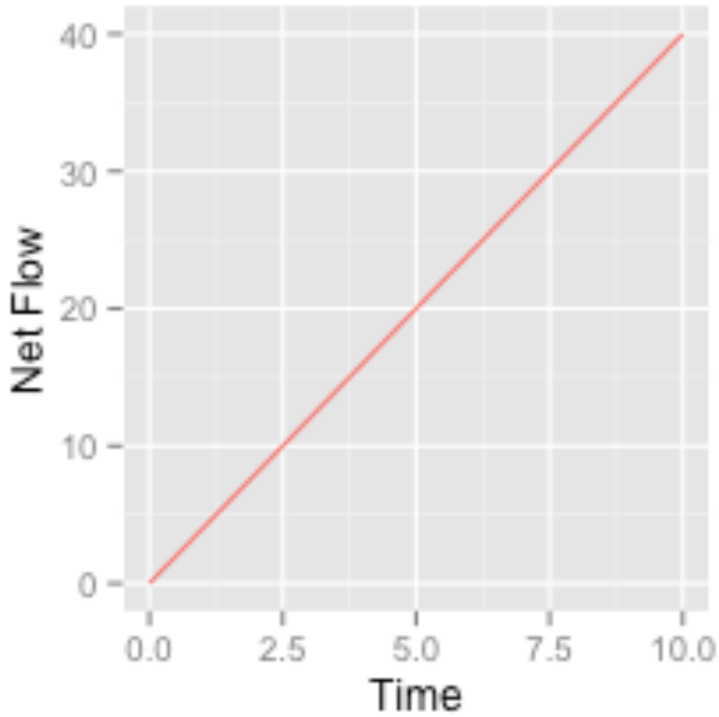
$$y = \; \int 4t \, dt = 4 \int t \, dt = 4 \left(\frac{1}{2}\right) t^2 + c$$

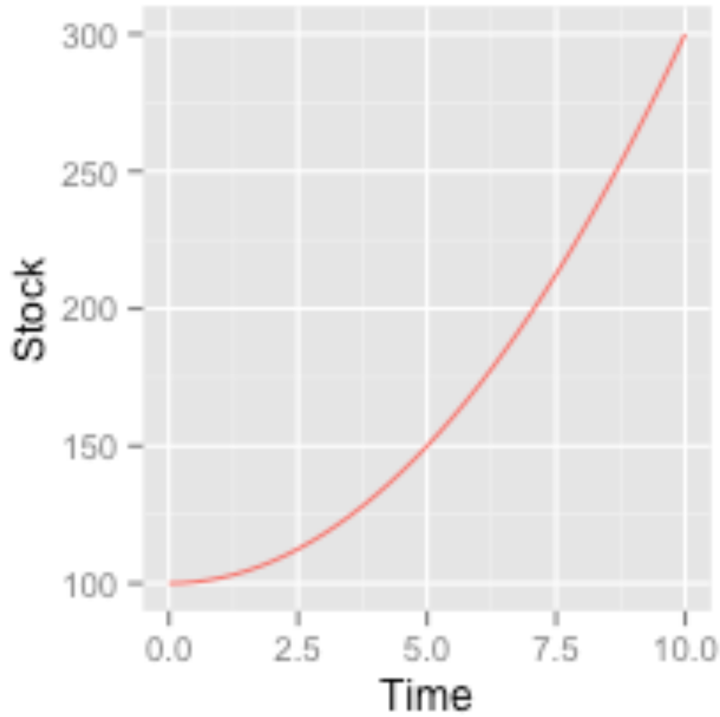
$$\boxed{\quad y_t \; = 2t^2 \begin{cases} t \\ 0 \end{cases}}$$

$$y_{10} = (100) + (2 \times 10^2) - (2 \times 0^2) = 300$$

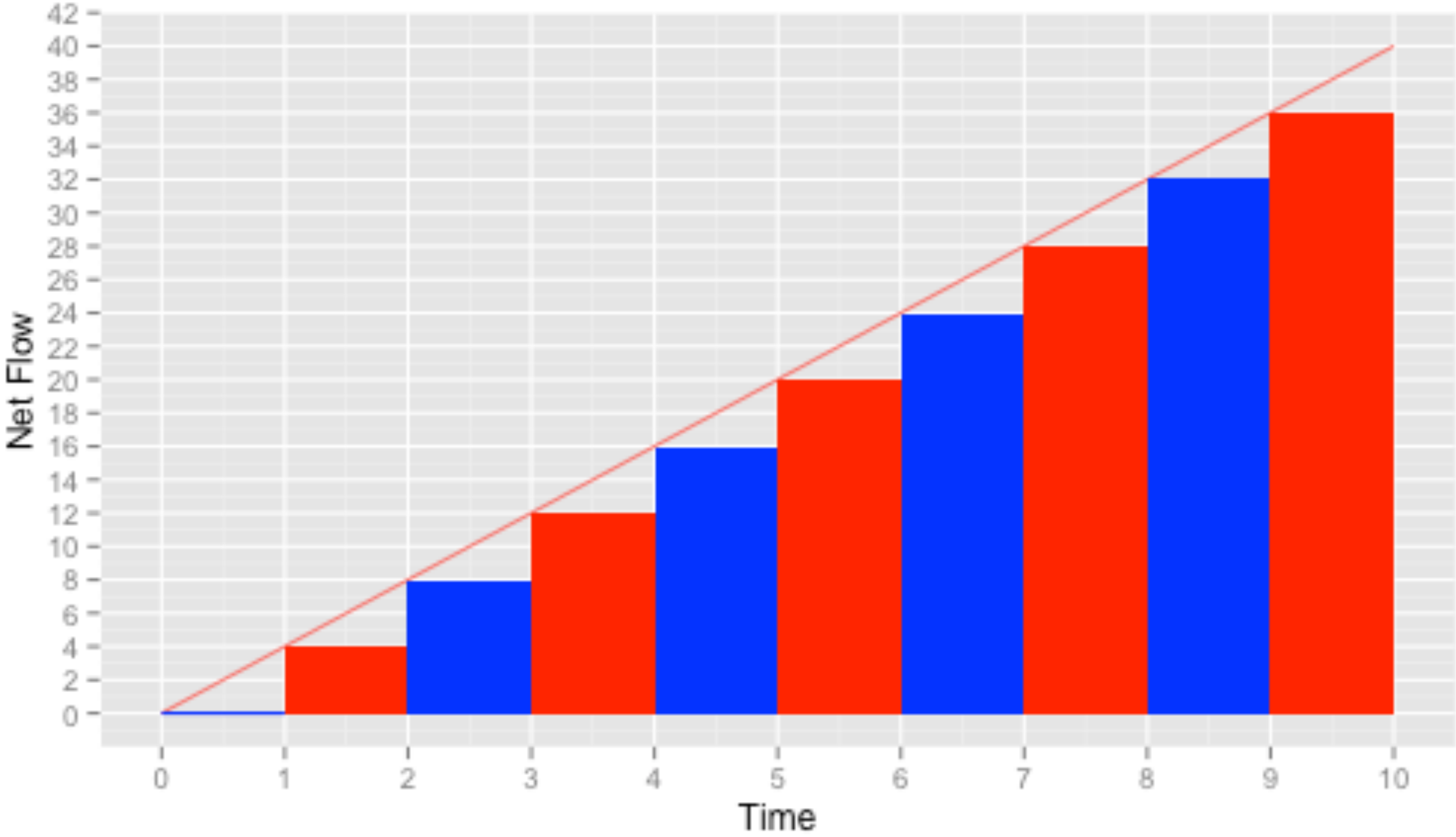
Analytical Model in R

```r
INIT<-100
DT<-0.25
time<-seq(0,10,by=DT)
flow<-4*time # dy/dt = 4t
stock<-INIT + 2*time^2
```



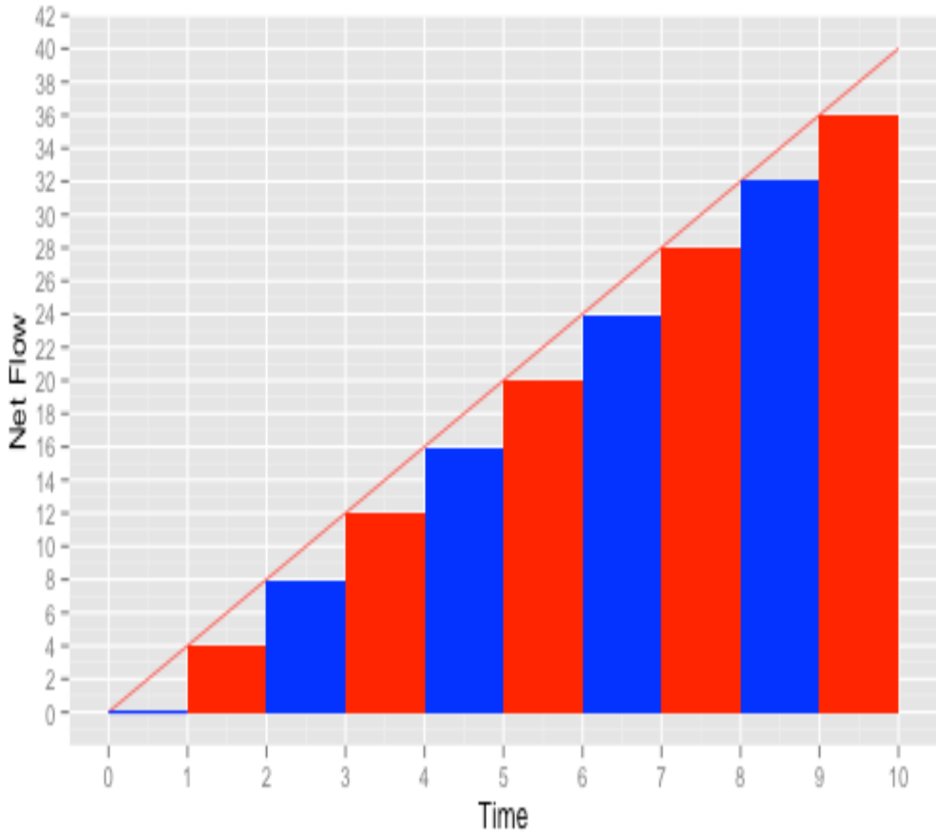


Numerical Integration: DT = 1

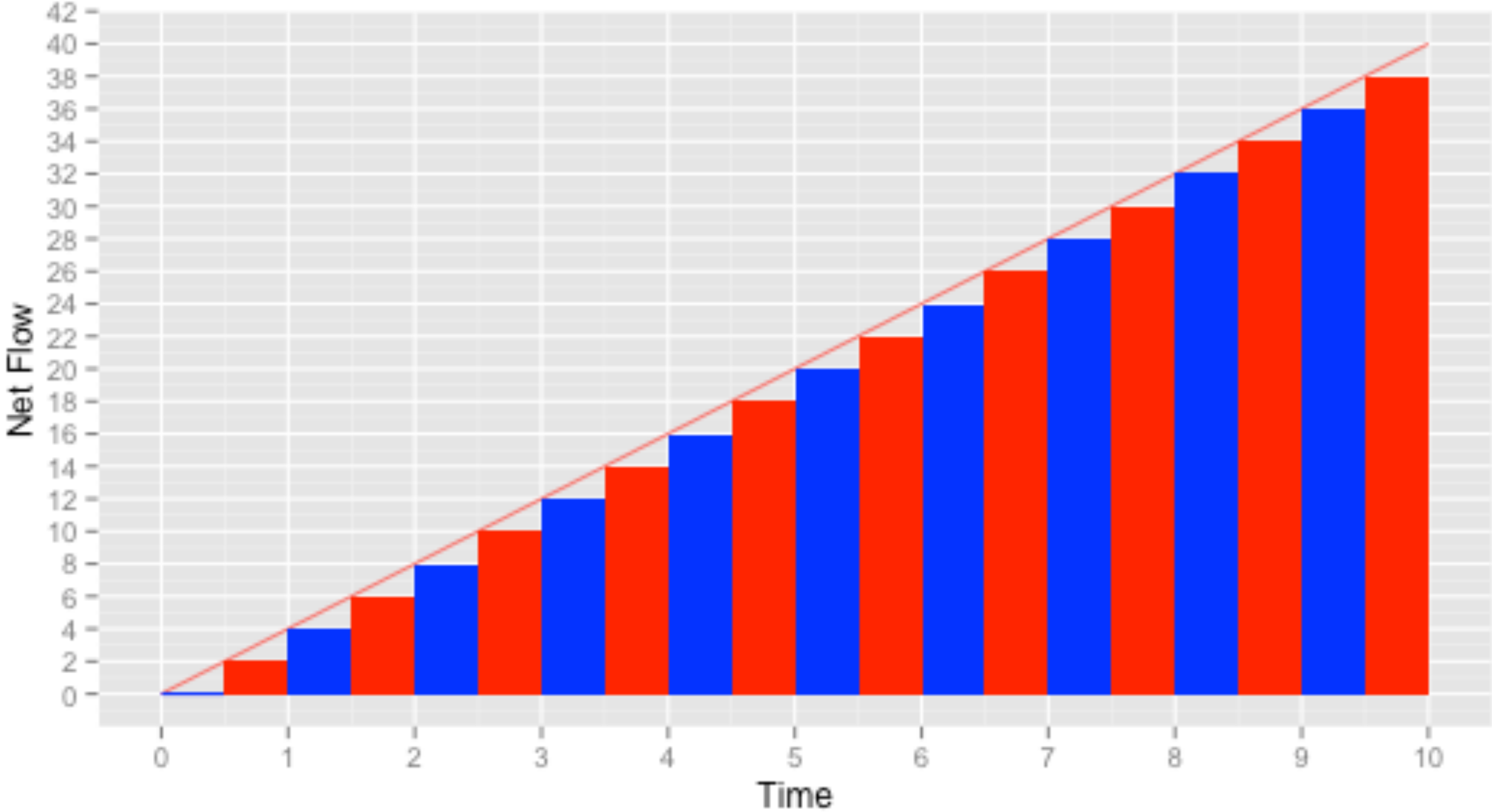


$$S_t = S_{t-dt} + NF_{t-dt} \times DT$$

Time	Stock	Flow	$Stock_A$
0	100	0	100
1	100	4	102
2	104	8	108
3	112	12	118
4	124	16	132
5	140	20	150
6	160	24	172
7	184	28	198
8	212	32	228
9	244	36	262
10	280	40	300

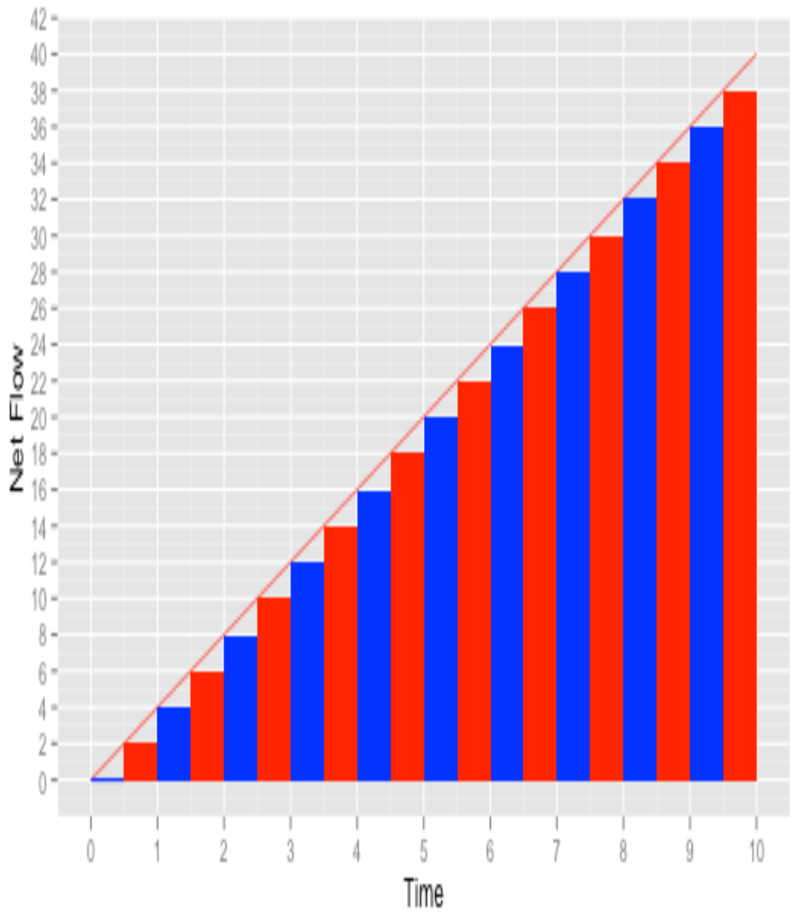


Numerical Integration: DT = 0.5

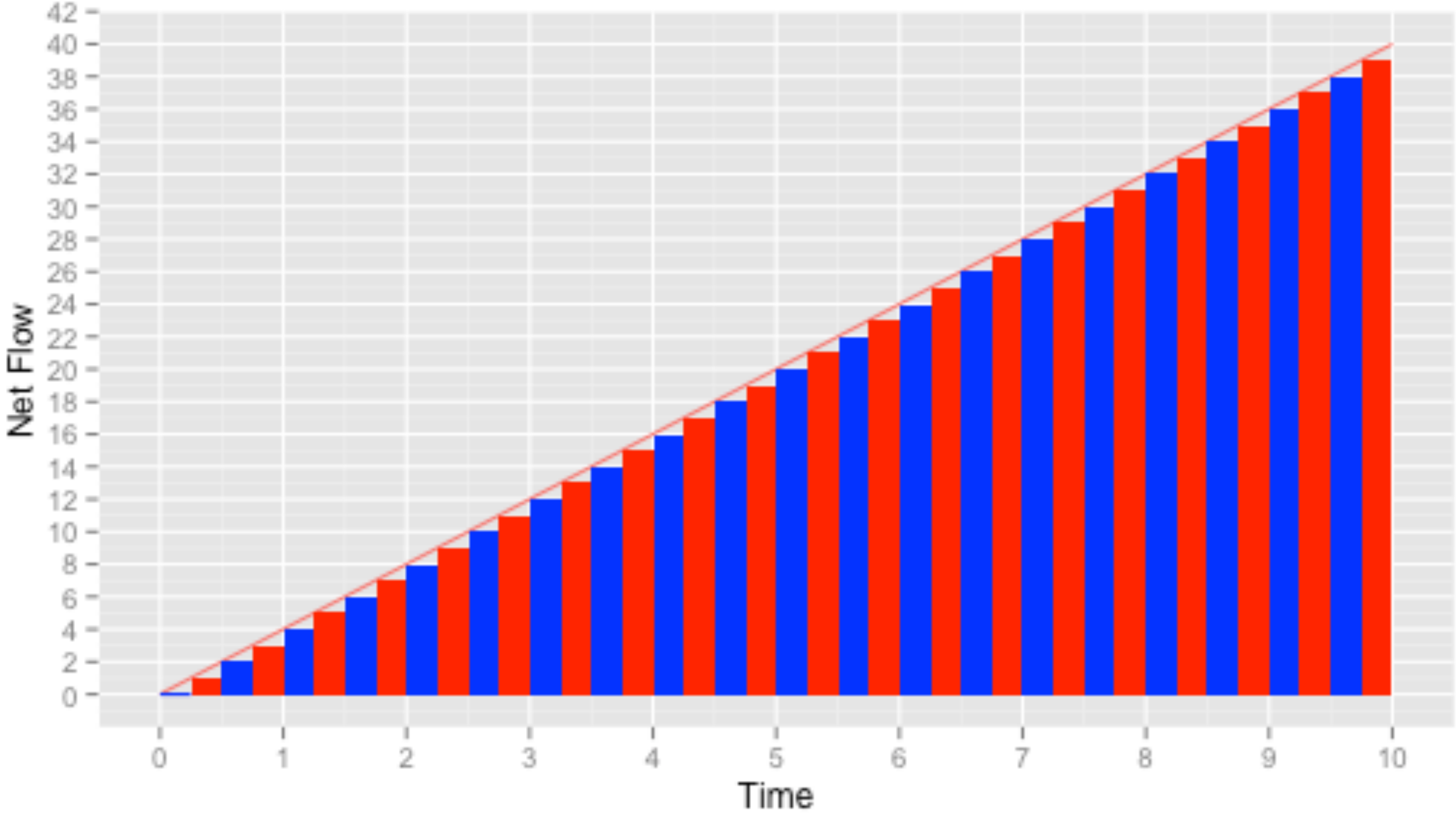


$$S_t = S_{t-dt} + NF_{t-dt} \times DT$$

Time	Stock	Flow	$Stock_A$
0.0	100	0	100.0
0.5	100	2	100.5
1.0	101	4	102.0
1.5	103	6	104.5
2.0	106	8	108.0
2.5	110	10	112.5
3.0	115	12	118.0
3.5	121	14	124.5
4.0	128	16	132.0
4.5	136	18	140.5
5.0	145	20	150.0

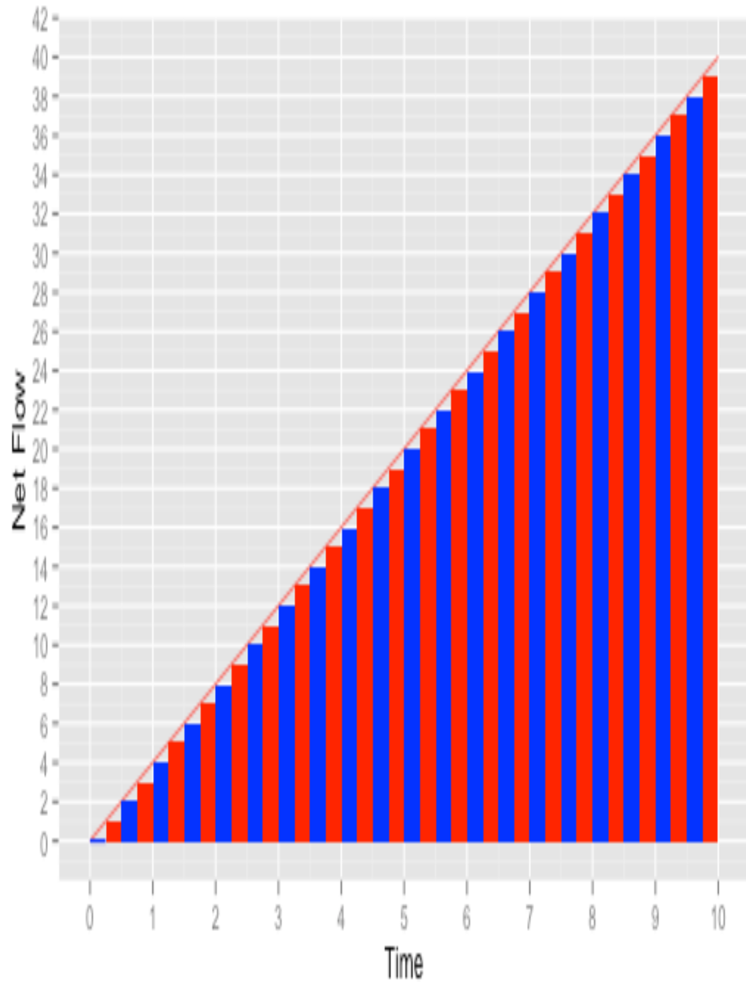


Numerical Integration: DT = 0.25



$$S_t = S_{t-dt} + NF_{t-dt} \times DT$$

Time (Month)	Y	"dY/dt"
0	100	0
0.25	100	1
0.5	100.2	2
0.75	100.8	3
1	101.5	4
1.25	102.5	5
1.5	103.8	6
1.75	105.2	7
2	107	8
2.25	109	9
2.5	111.2	10
2.75	113.8	11
3	116.5	12
3.25	119.5	13
3.5	122.8	14
3.75	126.2	15
4	130	16
4.25	134	17
4.5	138.2	18
4.75	142.8	19
5	147.5	20

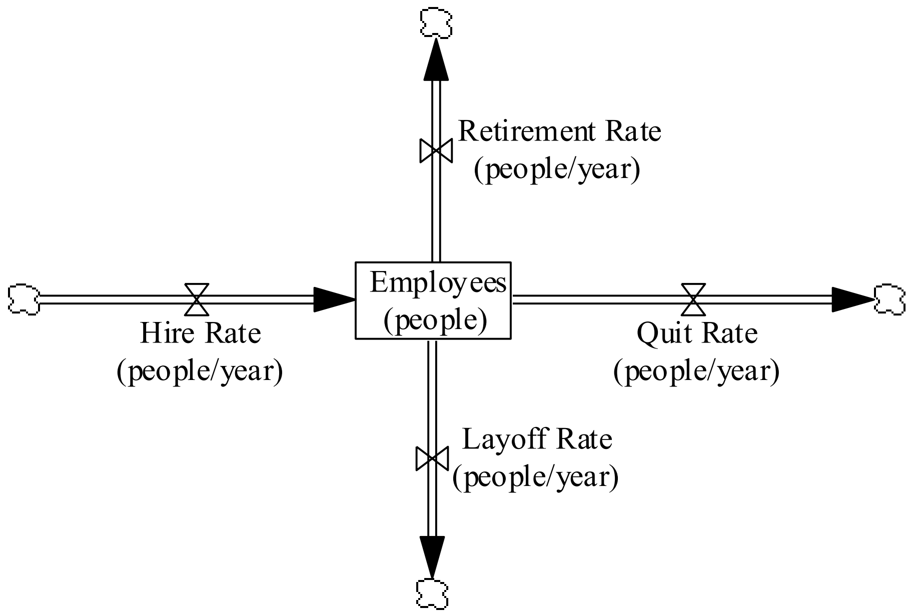


DT=0.001

Time (Month)	Y	"dY/dt"
4.986	149.7	19.94
4.987	149.7	19.95
4.988	149.8	19.95
4.989	149.8	19.96
4.99	149.8	19.96
4.991	149.8	19.96
4.992	149.8	19.97
4.993	149.9	19.97
4.994	149.9	19.98
4.995	149.9	19.98
4.996	149.9	19.98
4.997	149.9	19.99
4.998	150	19.99
4.999	150	20
5	150	20
5.001	150	20
5.002	150	20.01
5.003	150.1	20.01
5.004	150.1	20.02
5.005	150.1	20.02
5.006	150.1	20.02
5.007	150.1	20.03
5.008	150.2	20.03
5.009	150.2	20.04
5.01	150.2	20.04
5.011	150.2	20.04
5.012	150.2	20.05
5.013	150.3	20.05
5.014	150.3	20.06
5.015	150.3	20.06
5.016	150.3	20.06
5.017	150.3	20.07
5.018	150.4	20.07
5.019	150.4	20.08
5.02	150.4	20.08
5.021	150.4	20.08

Challenge 2.1

- A company has 1000 employees. Based on the following annual fractions, calculate the number of employees it will have after 5 years. Use DT=1.
  - 10% retire per year
  - 15% quit per year
  - 5% are laid off per year
  - 25% are hired each year



$$S_t = S_{t-dt} + NF_{t-dt} \times DT$$

Time	Employees	Hire Rate	Retire Rate	Layoff Rate	Quit Rate	Net flow
0	1,000					
1						
2						
3						
4						
5						

Graphical Integration

Key Idea:

S(20) = s(10) + Area under net flow
S(20) = 100 + 20*10 = 300

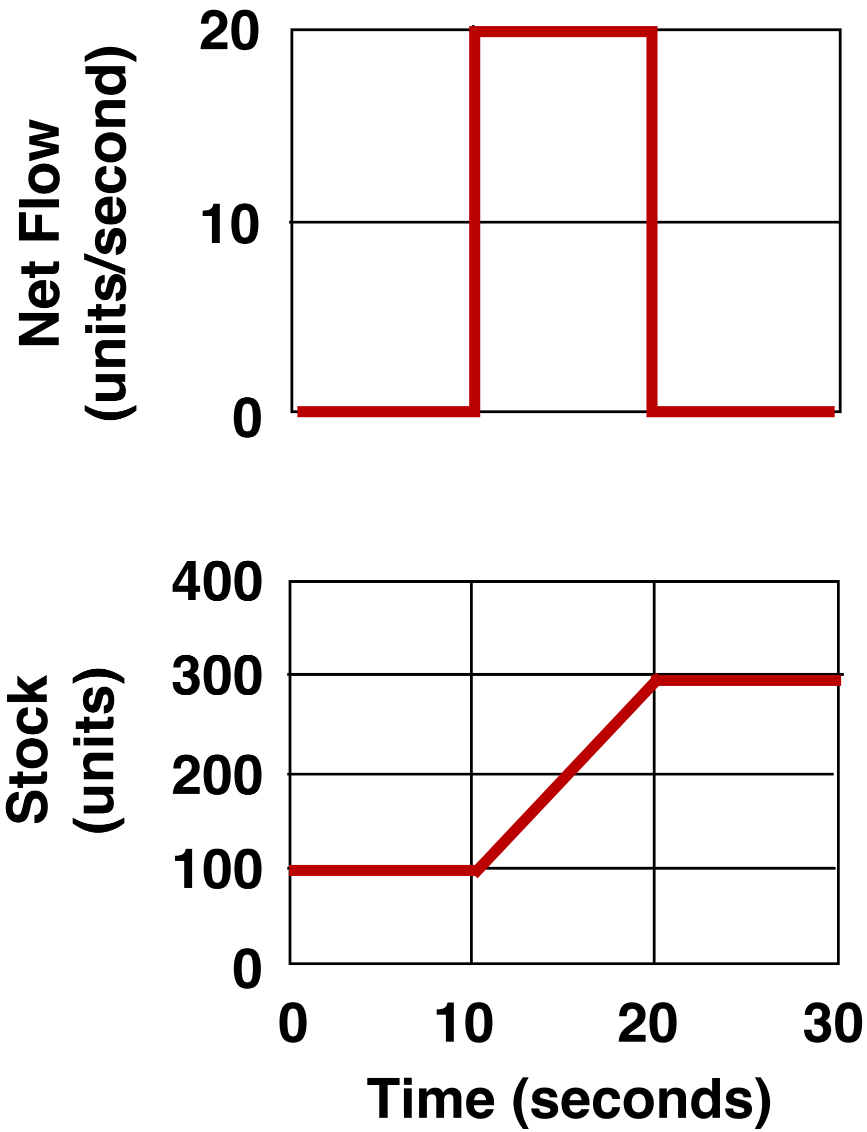
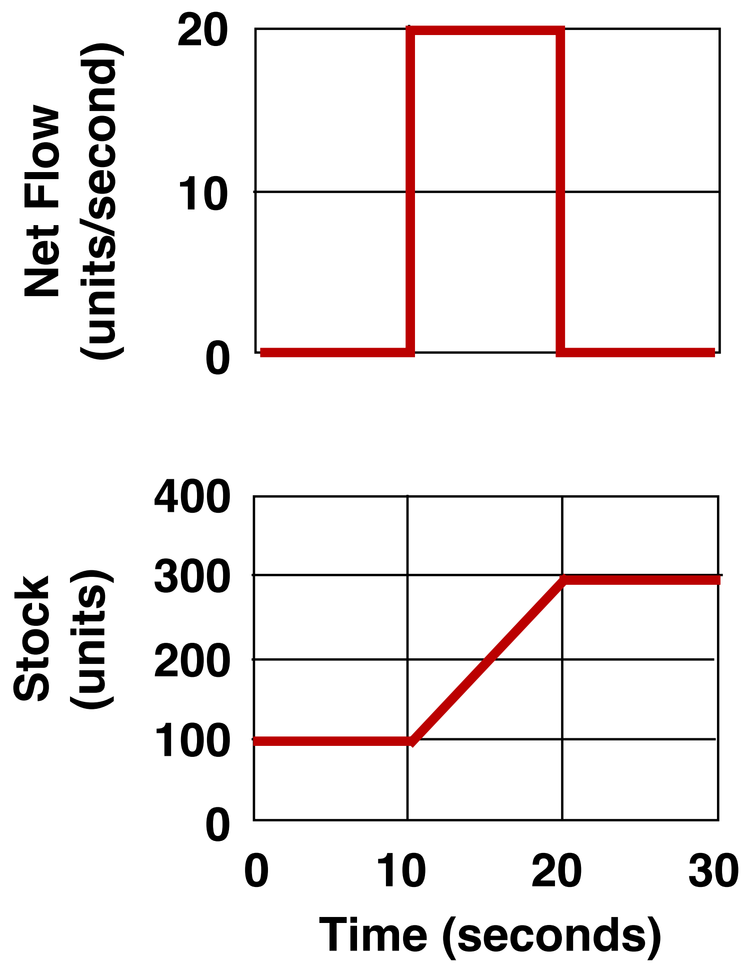


Figure 7-2 While the rate steps up and steps down, the stock rises and remains at a higher level. Note the different units of measure for the rate and stock.

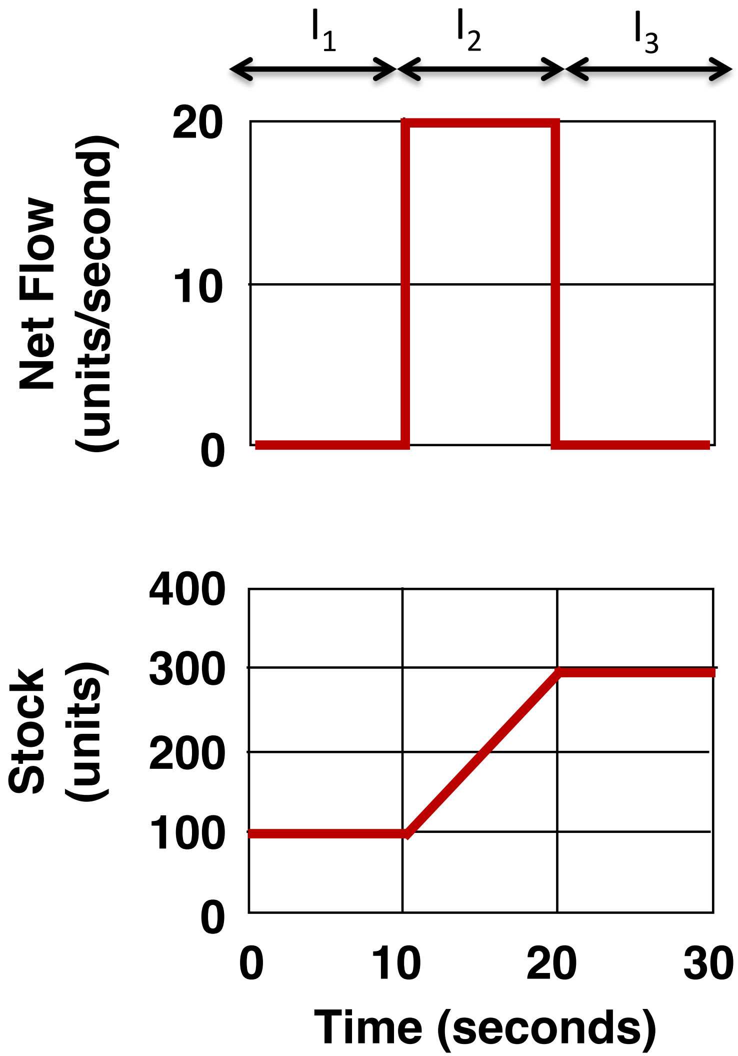
Graphical Integration (Sterman p. 236) (1/4)

1. Calculate and graph the total rate of inflow to the stock (sum of all inflows), and the total rate of outflow from the stock (sum of all outflows).
2. Calculate and graph the net rate of change of the stock (total inflow – total outflow).
3. Make a set of axes to graph the stock. Stocks (units) and flows (units per time period) have different units of measure, and must be graphed on different scales. Make a separate graph for the stock under the graph for the flows, with the time axes lined up.
4. Plot the initial value of the stock on the stock graph. The initial value MUST be specified.



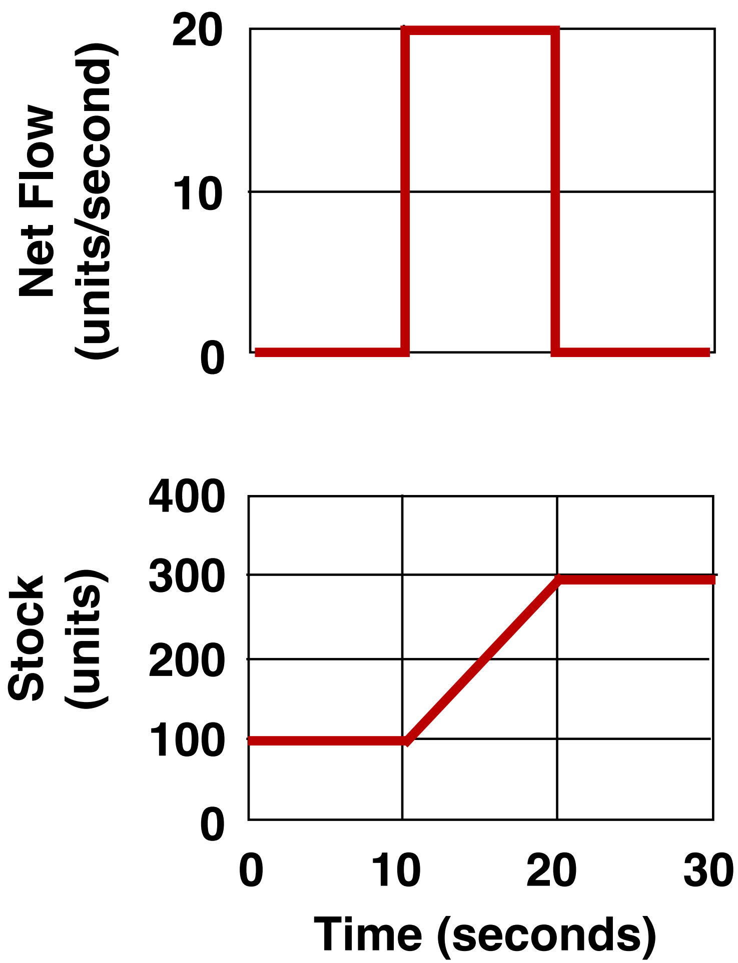
Graphical Integration (2/4)

5. Break the net flow into intervals with the same behaviour and calculate the amount added to the stock during the interval. The amount added or subtracted to the stock during an interval is the area under the net rate curve for that same interval. The total area is then added to the original value of the stock, and this point is then plotted on the stock graph.
6. Sketch the trajectory of the stock between the start and end of each interval. Find the value of the net rate at the start of the interval. If the net rate is positive, the stock will be increasing at that time, if it is negative, the stock will be decreasing.



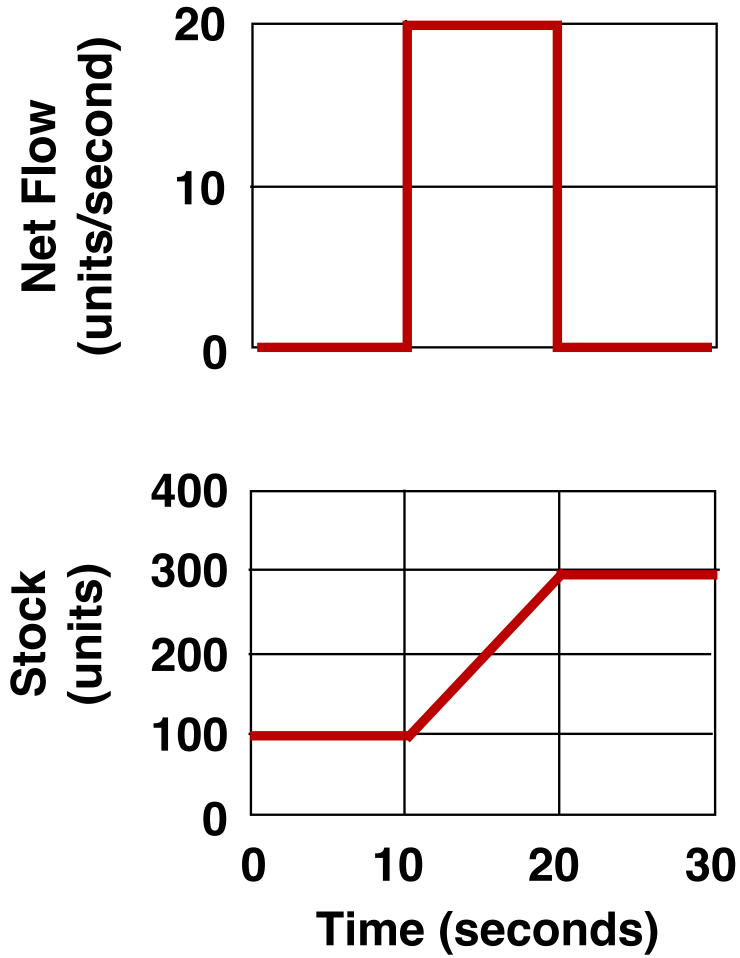
Graphical Integration (3/4)

7. The behaviour of the stock can be inferred from the net flow according to the following rules:
   - If the net rate is positive and increasing, the stock increases at an increasing rate (the stock accelerates upwards)
   - If the net rate is positive and decreasing, the stock increases at a decreasing rate (the stock is decelerating but still moving upwards)
   - If the net rate is negative and its magnitude is increasing (the net rate is becoming more negative), the stock decreases at an increasing rate.
   - If the net rate is negative and its magnitude is decreasing (becoming less negative), then the stock decreases at a decreasing rate.



Graphical Integration (4/4)

8. Whenever the net rate is zero, the stock is unchanging. Make sure that your graph of the stock shows no change in the stock everywhere the net rate is zero. At points where the net rate changes from positive to negative, the stock reaches a maximum as it ceases to rise and starts to fall. At points where the net rate changes from negative to positive, the stock reaches a minimum as it ceases to fall and starts to rise.
9. Repeat steps 5 through 8 until completion.



Challenge 2.2: Graph the Stock
Assume $S_0 = 100$