CT561: Systems Modelling & Simulation

Lecture 8: R Solution, Graphical Integration and the SIR Model

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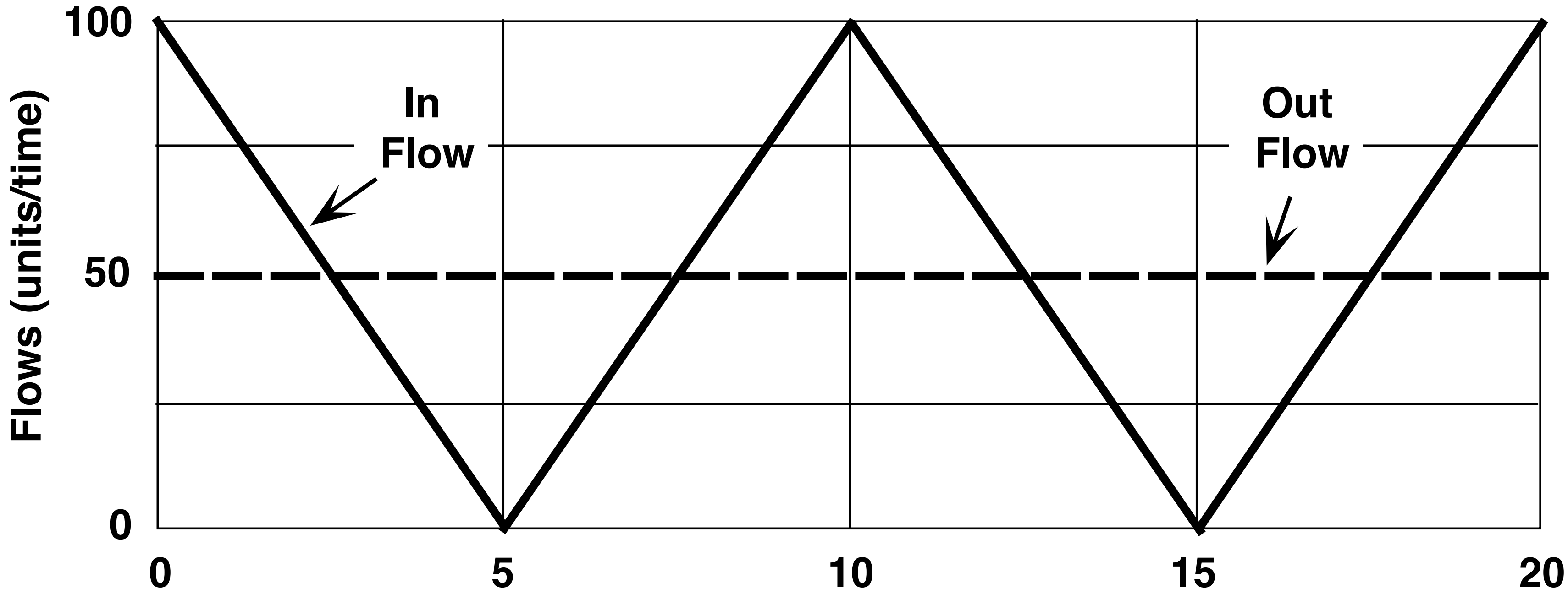
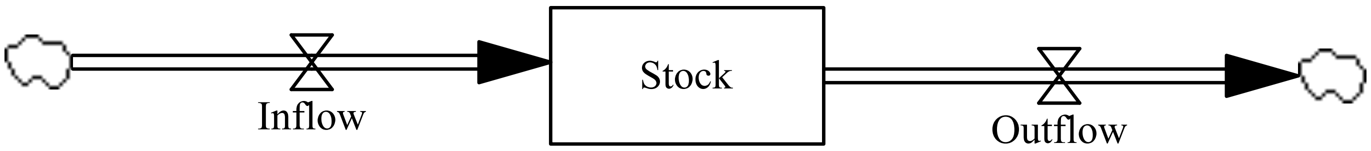
https://github.com/JimDuggan/SDMR

https://twitter.com/_jimduggan

Sample Solution using R
Init = 100, DT=0.25, Interval [0,20]

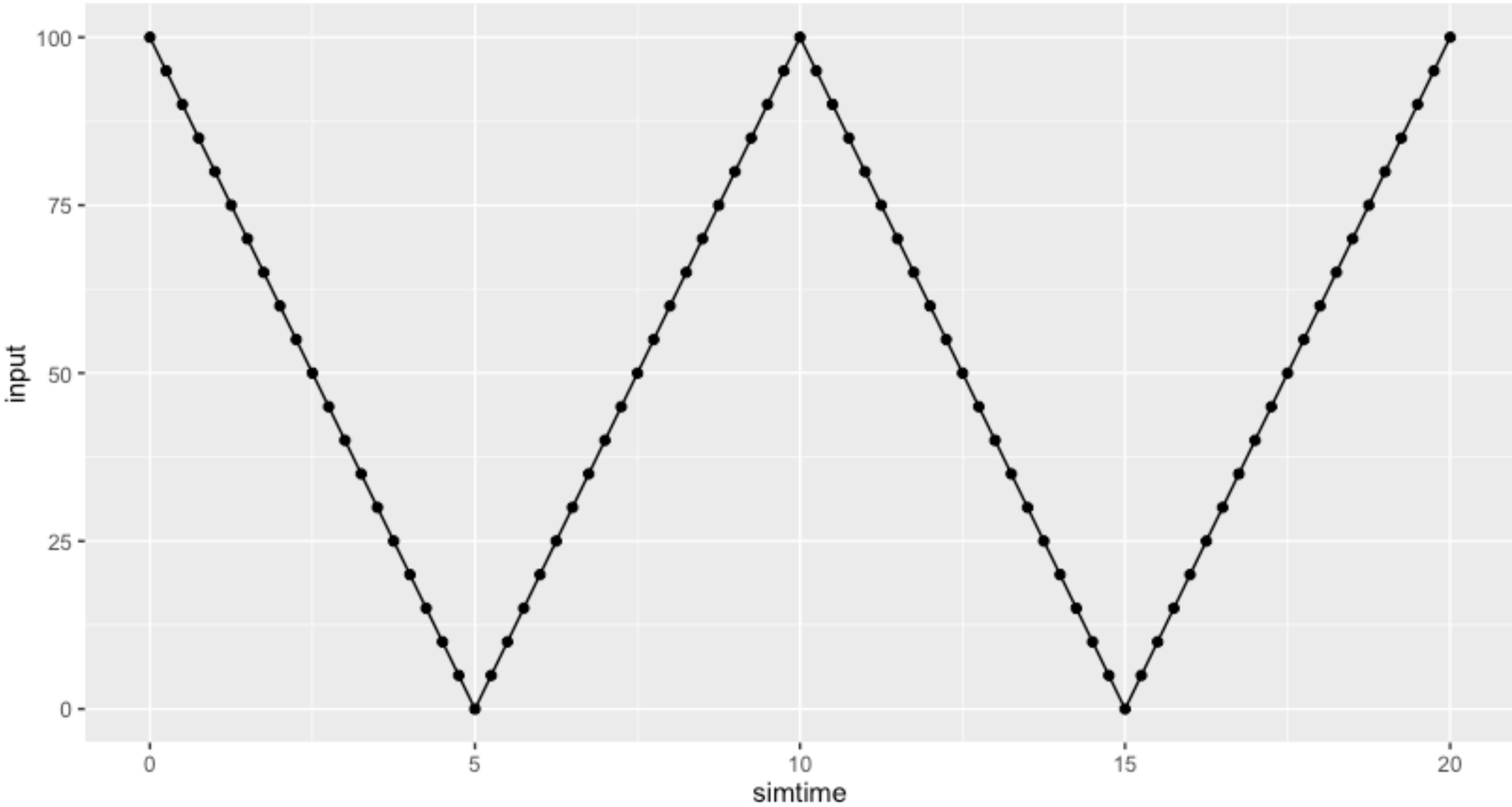




```r
library(deSolve)
library(ggplot2)

l1 <- seq(100,0,by=-5)
l2 <- seq(5,100,by=5)
l3 <- seq(95,0,by=-5)
l4 <- seq(5,100,by=5)

input <- c(l1, l2, l3, l4)

START<-0; FINISH<-20; STEP<-0.25
simtime <- seq(START, FINISH, by=STEP)

stocks  <- c(sStock=100)
auxs    <- c(aOutflow=50)
```

```
> input
 [1] 100  95  90  85  80  75  70  65  60  55  50  45  40  35  30  25  20  15
[19]  10   5   0   5  10  15  20  25  30  35  40  45  50  55  60  65  70  75
[37]  80  85  90  95 100  95  90  85  80  75  70  65  60  55  50  45  40  35
[55]  30  25  20  15  10   5   0   5  10  15  20  25  30  35  40  45  50  55
[73]  60  65  70  75  80  85  90  95 100
>
> simtime
 [1]  0.00  0.25  0.50  0.75  1.00  1.25  1.50  1.75  2.00  2.25  2.50  2.75
[13]  3.00  3.25  3.50  3.75  4.00  4.25  4.50  4.75  5.00  5.25  5.50  5.75
[25]  6.00  6.25  6.50  6.75  7.00  7.25  7.50  7.75  8.00  8.25  8.50  8.75
[37]  9.00  9.25  9.50  9.75 10.00 10.25 10.50 10.75 11.00 11.25 11.50 11.75
[49] 12.00 12.25 12.50 12.75 13.00 13.25 13.50 13.75 14.00 14.25 14.50 14.75
[61] 15.00 15.25 15.50 15.75 16.00 16.25 16.50 16.75 17.00 17.25 17.50 17.75
[73] 18.00 18.25 18.50 18.75 19.00 19.25 19.50 19.75 20.00
>
> stocks
sStock
   100
> auxs
aOutflow
      50
```

```
qplot(x=simtime,y=input,data=o) + geom_line()
```



The model function

```
model <- function(time, stocks, auxs){
  with(as.list(c(stocks, auxs)),{

    fInflow <- input[which(simtime==time)]

    fOutflow <- aOutflow

    dS_dt <- fInflow - fOutflow

    ans <- list(c(dS_dt),Inflow=fInflow,
                Outflow=fOutflow,
                NetFlow=dS_dt)
  })
}
```

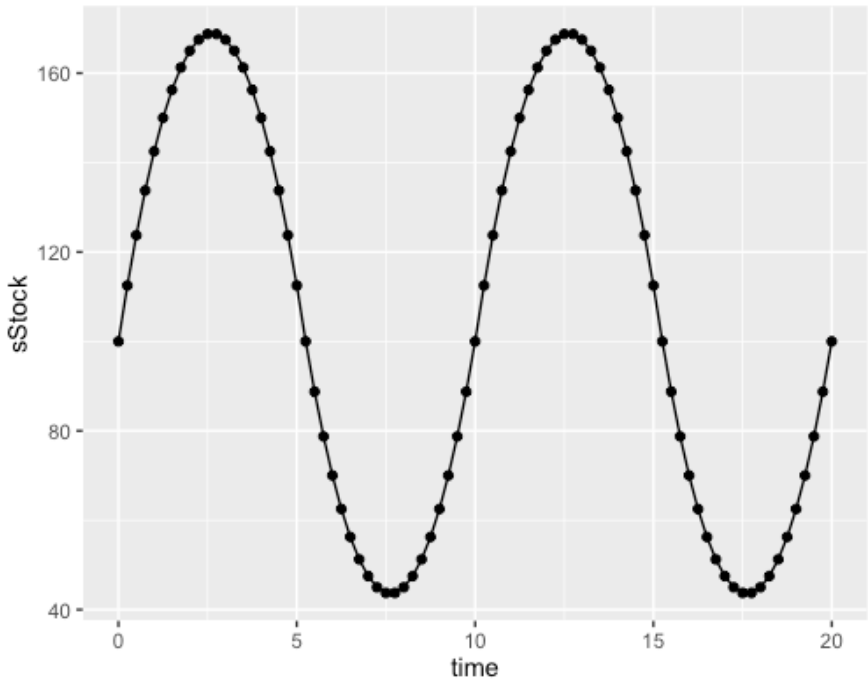
Running the simulation

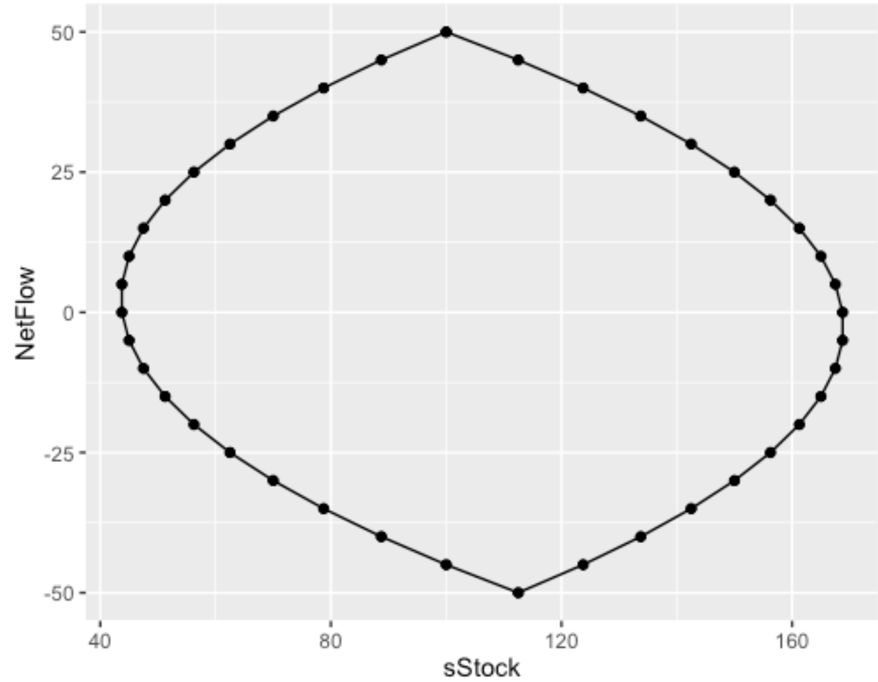
```
# Run simulation
o<-data.frame(ode(y=stocks, times=simtime, func = model,
                  parms=auxs, method='euler'))

qplot(x=time,y=sStock,data=o) + geom_line()
```

```
> o[1:9,]
  time sStock Inflow Outflow NetFlow
1 0.00 100.00    100      50      50
2 0.25 112.50     95      50      45
3 0.50 123.75     90      50      40
4 0.75 133.75     85      50      35
5 1.00 142.50     80      50      30
6 1.25 150.00     75      50      25
7 1.50 156.25     70      50      20
8 1.75 161.25     65      50      15
9 2.00 165.00     60      50      10
```

Output Graphs

```
qplot(x=time,y=sStock,data=o) +
  geom_line()
```

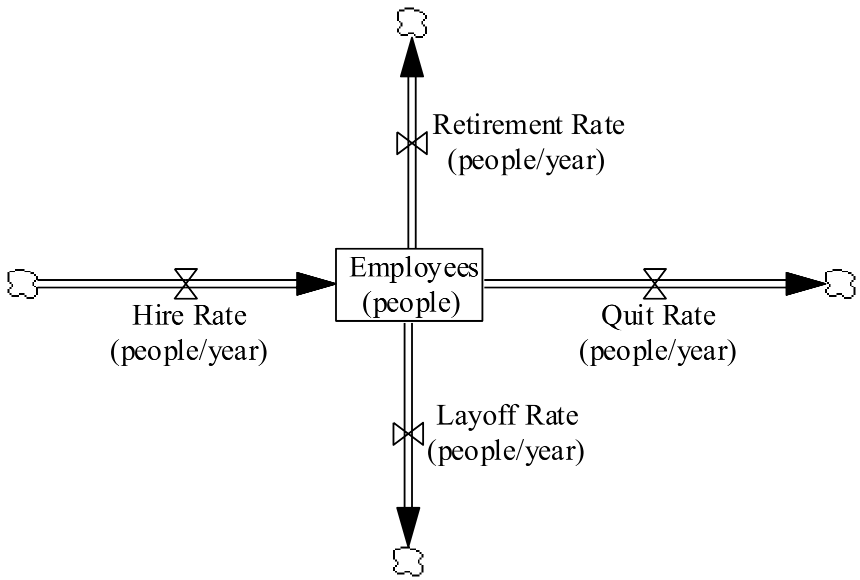





```
qplot(x=sStock,y=NetFlow,data=o) +
  geom_path()
```

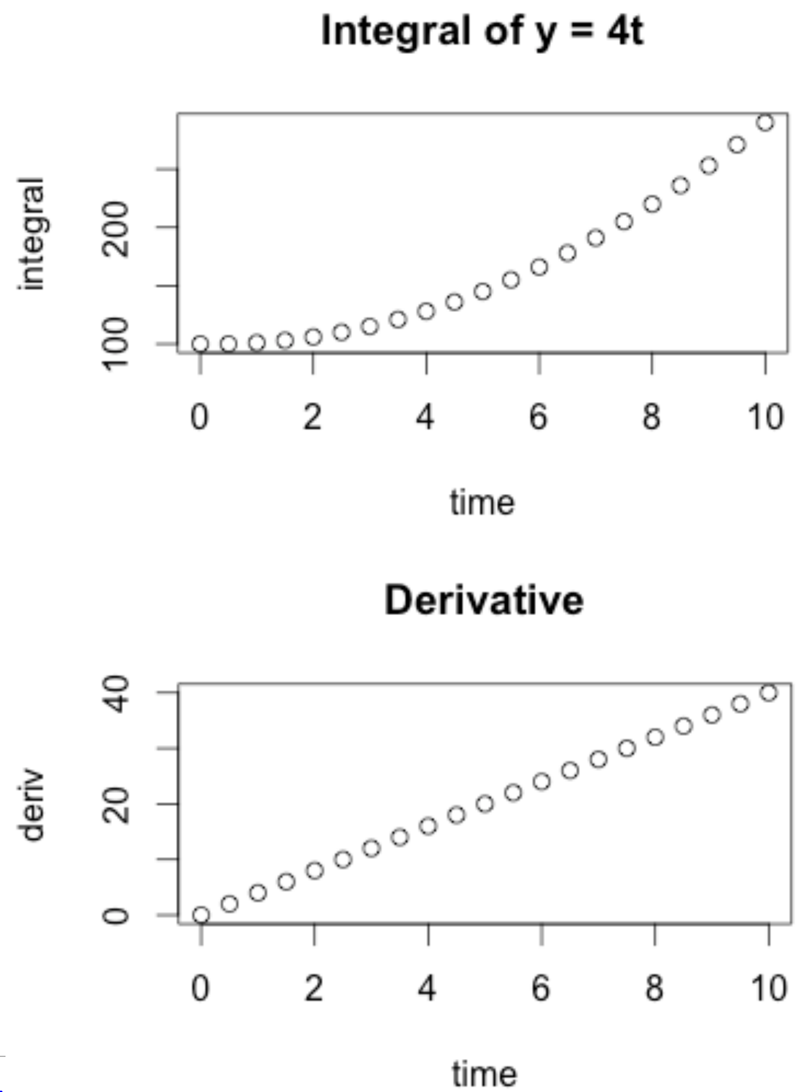
Stock and Flow Systems

- All stock-flow systems share the same underlying structure.
- The stock accumulates its inflows to it, less the outflows from it.
- This is a fundamental concept of calculus (integrals and derivatives)
- Knowledge of calculus is not necessary to understand the idea of stocks and flows



Calculus

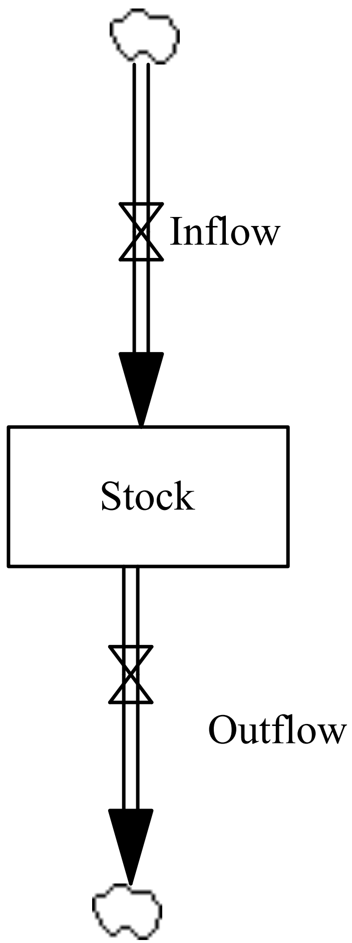
- Given the dynamic of the flows, what is the behaviour of the stock?
  - Integration
- From the dynamics of the stock, can you infer the behaviour of the flows
  - Differentiation
- Calculus
  - “quite intuitive… it is the use of unfamiliar notation and a focus on analytic solutions that deters many people from the study of calculus” (Sterman 2000)



General Principle of Stock/Flow Systems

Important principles that extend to more complicated systems:

- As long as the sum of all inflows exceeds the sum of all outflows, the level of the stock will **rise**.
- As long as the sum of all outflows exceeds the sum of all inflows, the level of the stock will **fall**.
- If the sum of all outflows equals the sum of all inflows, the stock level **will not change**; it will be held in dynamic equilibrium at whatever level it happened to be when the two sets of flows became equal.



Meadows, Donella H. *Thinking in systems: A primer*. Chelsea Green Publishing, 2008.

Graphical Integration

Key Idea:

S(20) = s(10) + Area under net flow
S(20) = 100 + 20*10 = 300

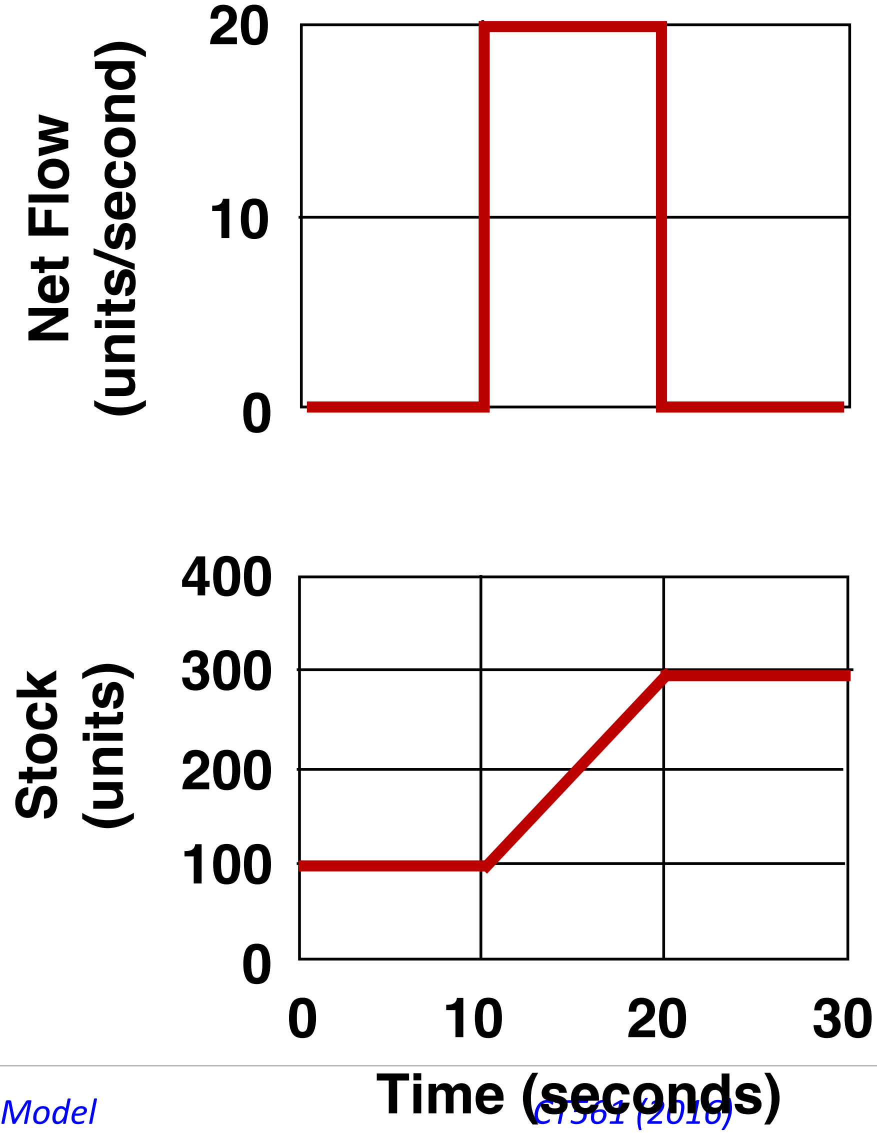
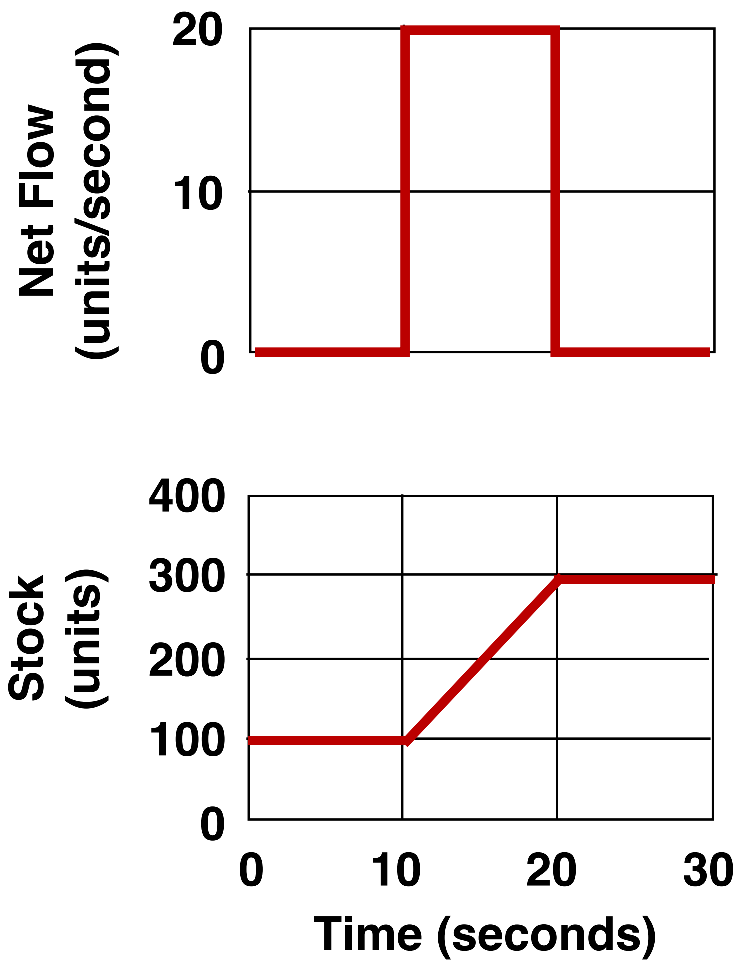


Figure 7-2 While the rate steps up and steps down, the stock rises and remains at a higher level. Note the different units of measure for the rate and stock.

Business Dynamics

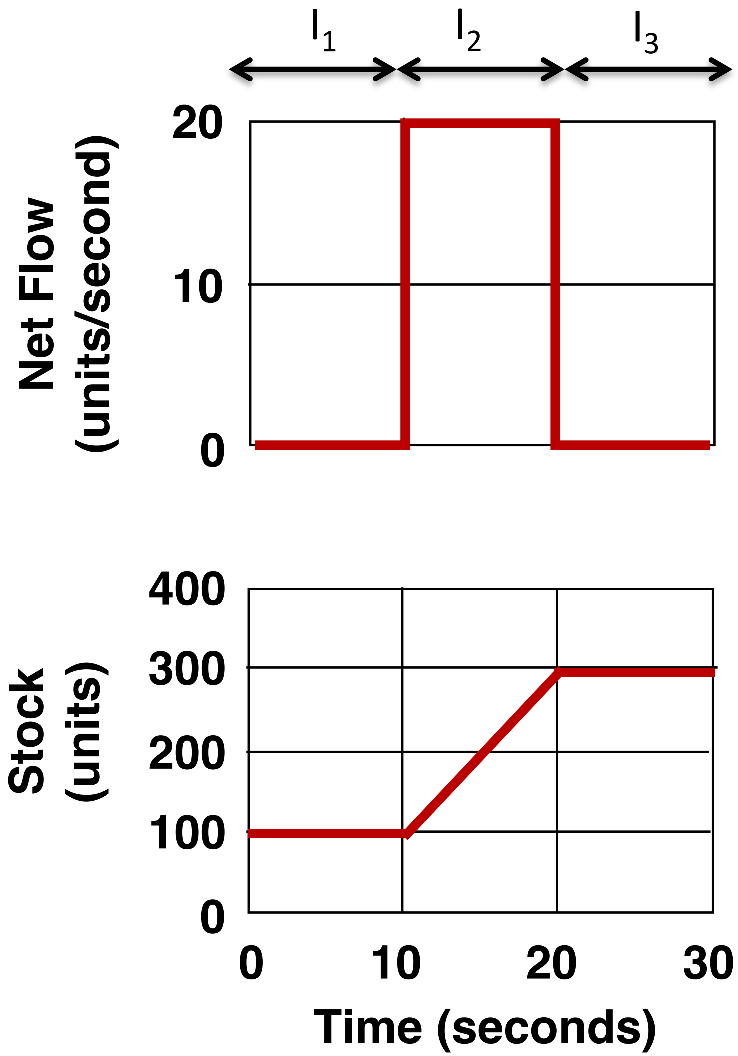
Graphical Integration (Sterman p. 236) (1/4)

1. Calculate and graph the total rate of inflow to the stock (sum of all inflows), and the total rate of outflow from the stock (sum of all outflows).
2. Calculate and graph the net rate of change of the stock (total inflow – total outflow).
3. Make a set of axes to graph the stock. Stocks (units) and flows (units per time period) have different units of measure, and must be graphed on different scales. **Make a separate graph for the stock under the graph for the flows, with the time axes lined up**.
4. Plot the initial value of the stock on the stock graph. The initial value MUST be specified.



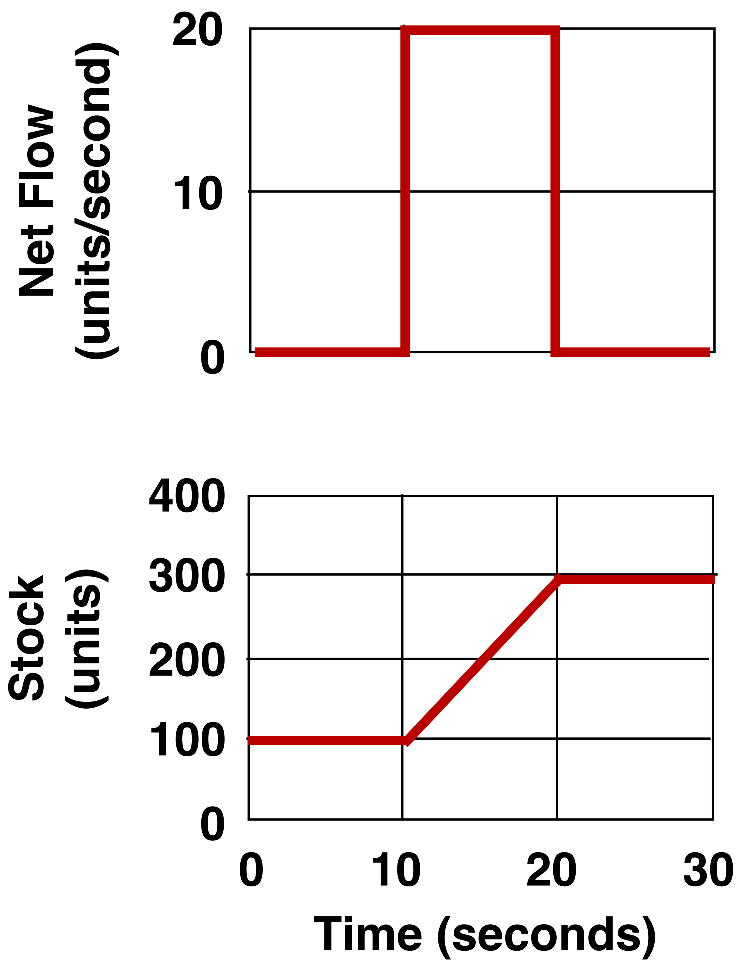
Graphical Integration (2/4)

5. Break the net flow into intervals with the same behaviour and calculate the amount added to the stock during the interval. The amount added or subtracted to the stock during an interval is the area under the net rate curve for that same interval. The total area is then added to the original value of the stock, and this point is then plotted on the stock graph.
6. Sketch the trajectory of the stock between the start and end of each interval. Find the value of the net rate at the start of the interval. If the net rate is positive, the stock will be increasing at that time, if it is negative, the stock will be decreasing.



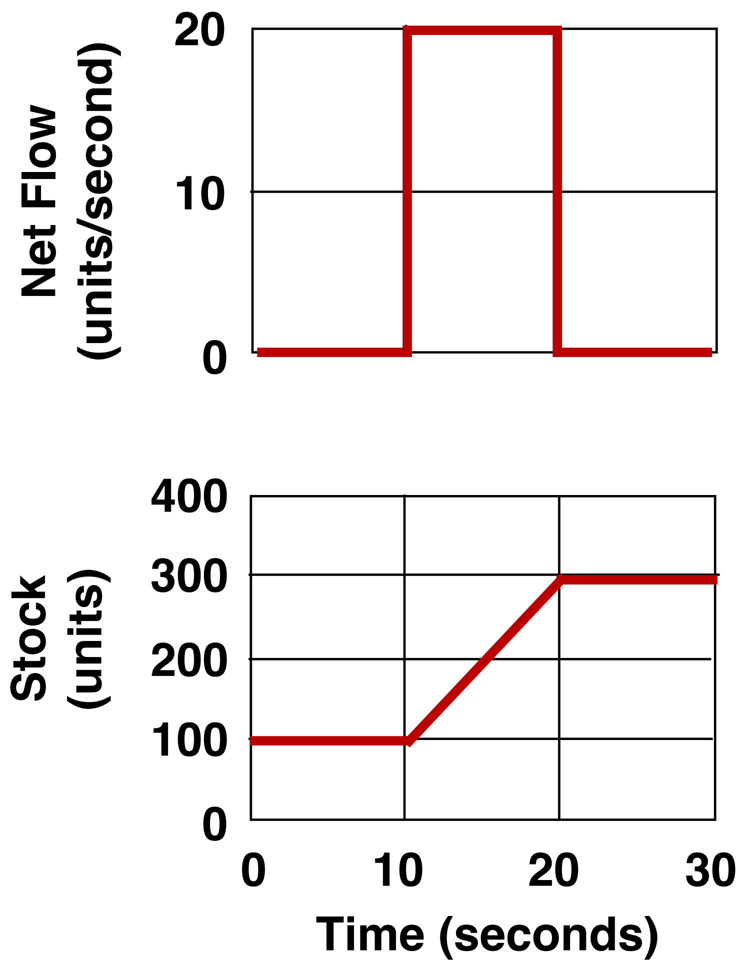
Graphical Integration (3/4)

7. The behaviour of the stock can be inferred from the net flow according to the following rules:
   - If the net rate is positive and increasing, the stock increases at an increasing rate (the stock accelerates upwards)
   - If the net rate is positive and decreasing, the stock increases at a decreasing rate (the stock is decelerating but still moving upwards)
   - If the net rate is negative and its magnitude is increasing (the net rate is becoming more negative), the stock decreases at an increasing rate.
   - If the net rate is negative and its magnitude is decreasing (becoming less negative), then the stock decreases at a decreasing rate.

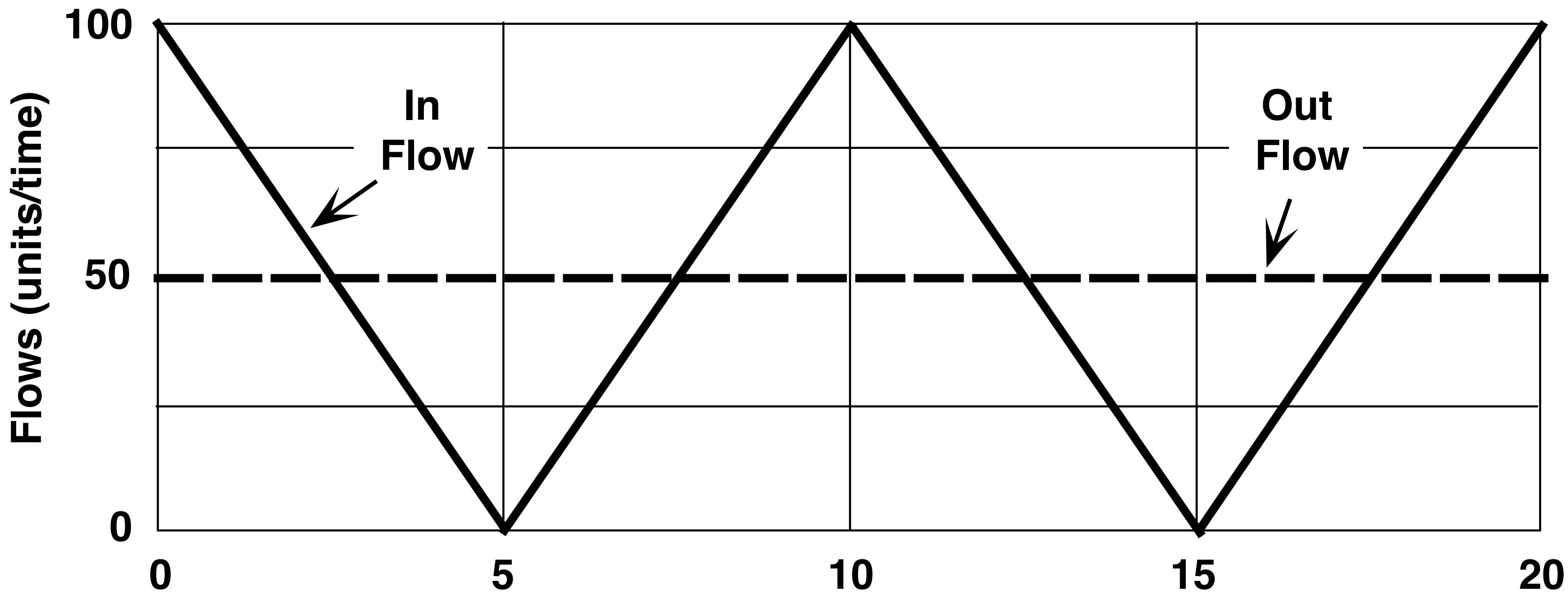


Graphical Integration (4/4)

8. Whenever the net rate is zero, the stock is unchanging. Make sure that your graph of the stock shows no change in the stock everywhere the net rate is zero. At points where the net rate changes from positive to negative, the stock reaches a maximum as it ceases to rise and starts to fall. At points where the net rate changes from negative to positive, the stock reaches a minimum as it ceases to fall and starts to rise.
9. Repeat steps 5 through 8 until completion.



Challenge 8.1: Graph the Stock
Assume $S_0 = 100$



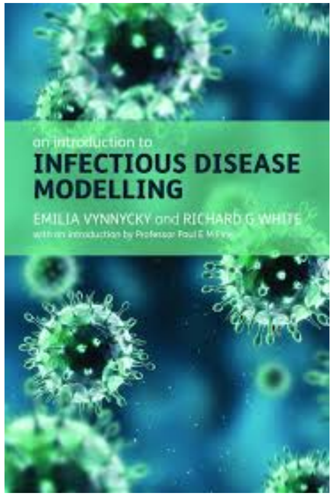
Business Dynamics

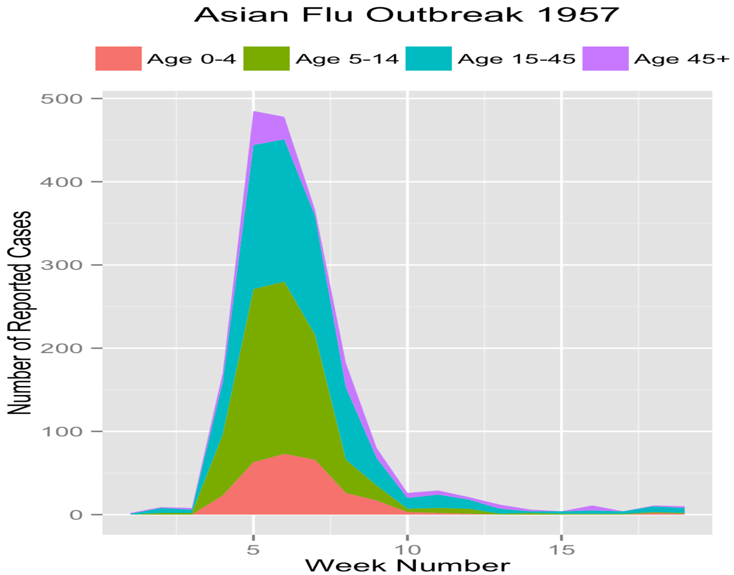
Public Health: Modelling Infectious Disease Outbreaks

- Public Health

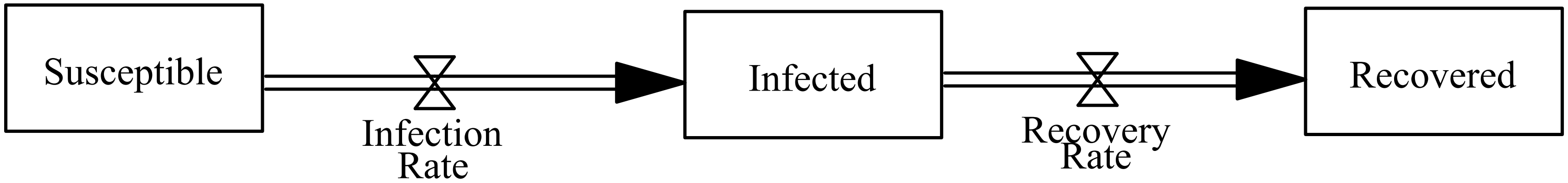
- Modelling transmission of infectious diseases

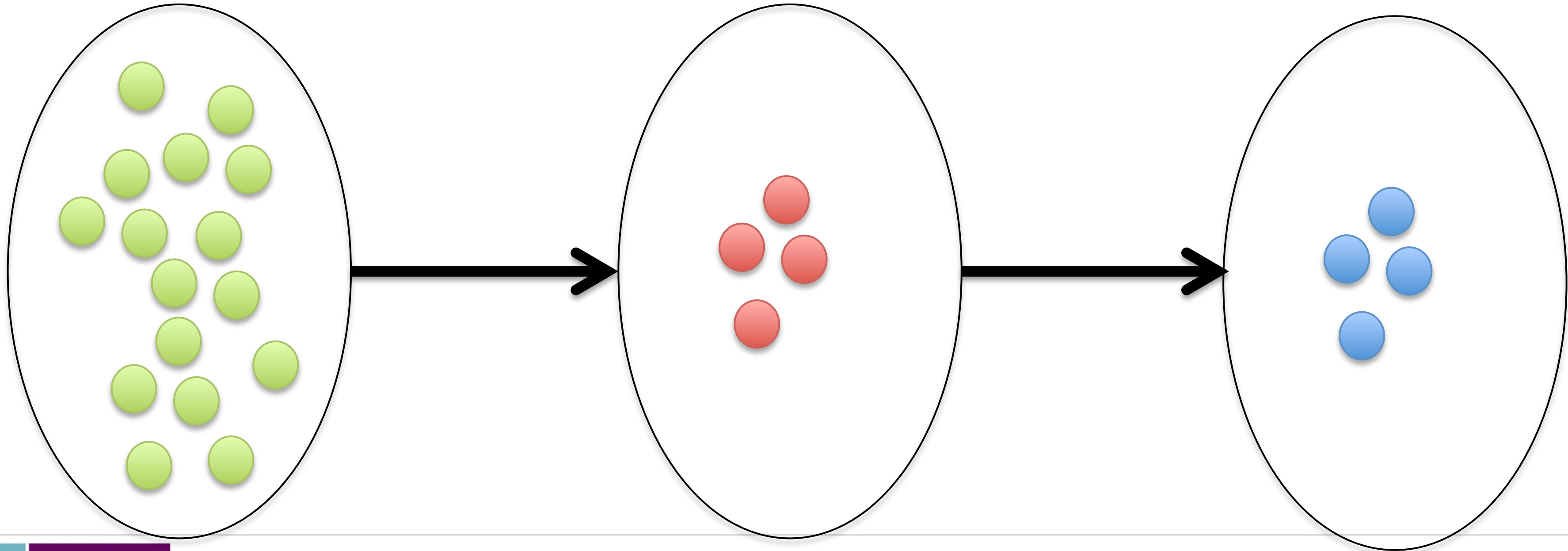
- SIR Model (3 stocks, 2 flows)





Need a model structure (Stocks and Flows)

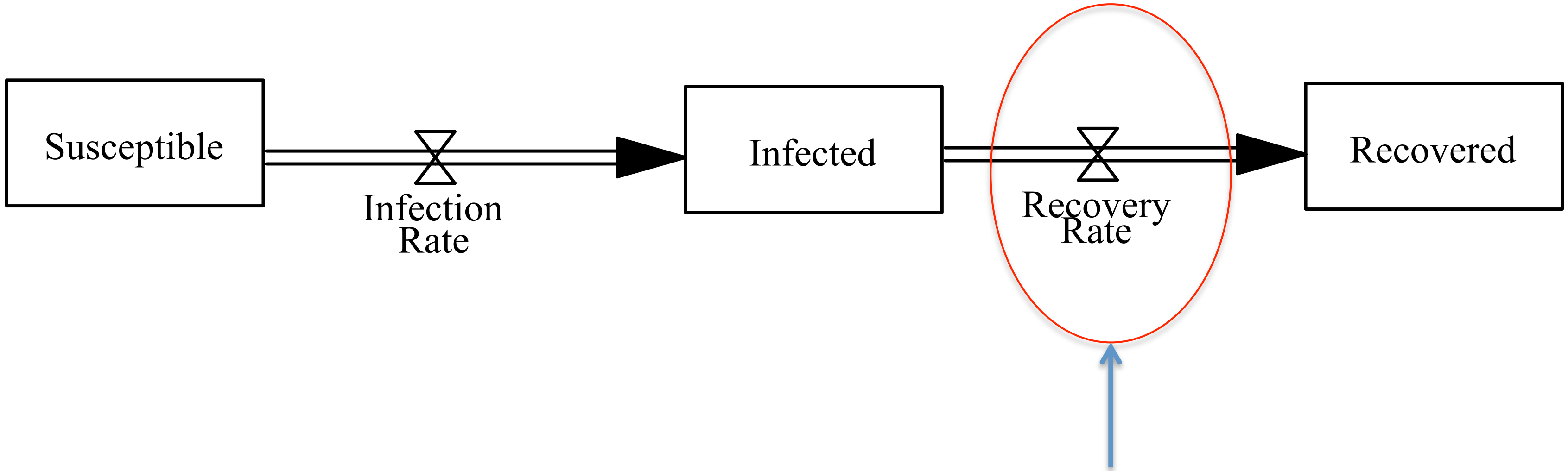




Stock Equations

$$Susceptible\ (S) = \ INTEGRAL(-IR, 99999)$$
$$Infected\ (I) = \ INTEGRAL(IR - RR, 1)$$
$$Recovered\,(R) = \ INTEGRAL(RR, 0)$$

What about the flows?

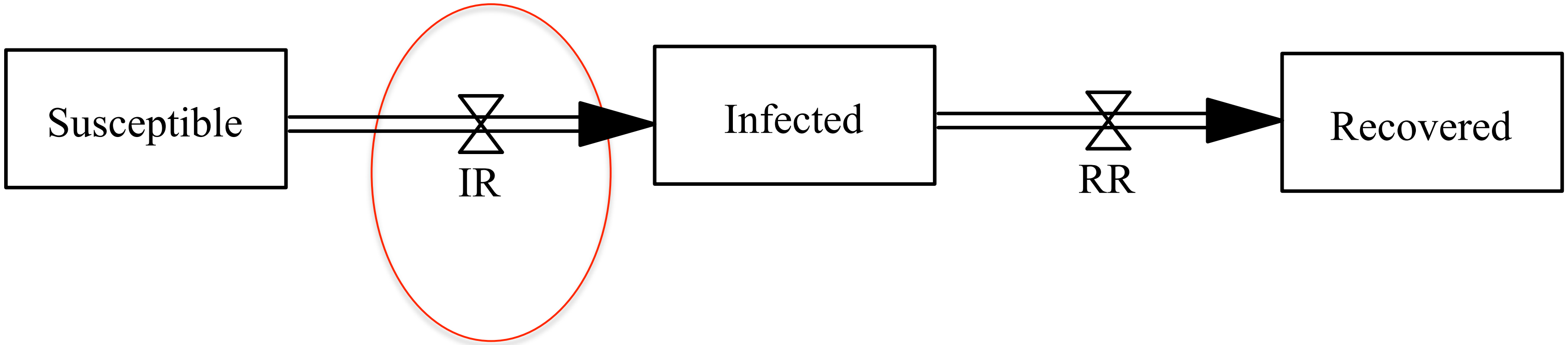


First Order Delay Structure

$$RR = \frac{I}{D}$$

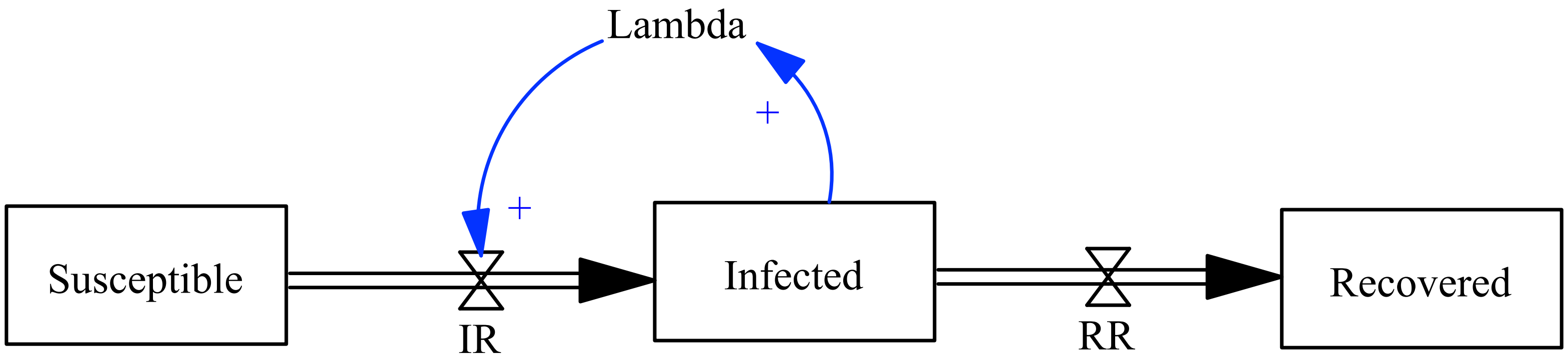
$$Delay\ (D) = 2$$

Infection Rate?



- Infection spreads through contact
- As the number of infected increase, so to does the infection rate
- A positive feedback process

Lambda – Force of Infection (Attack Rate)

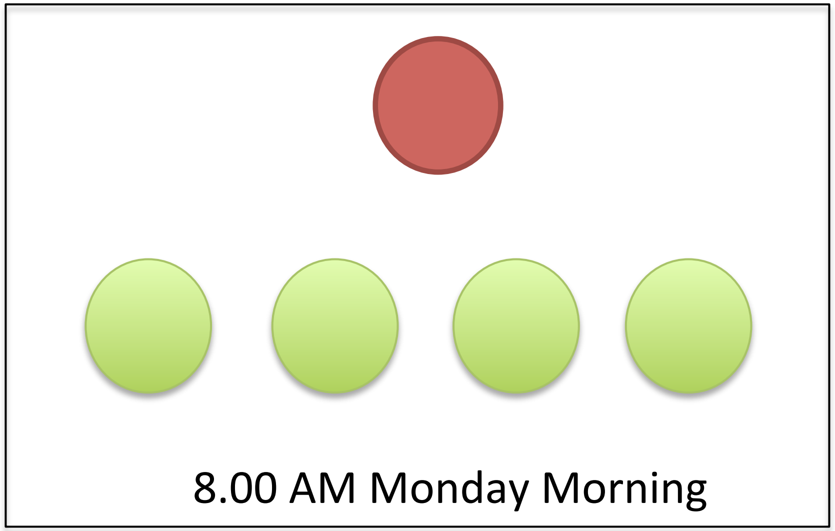


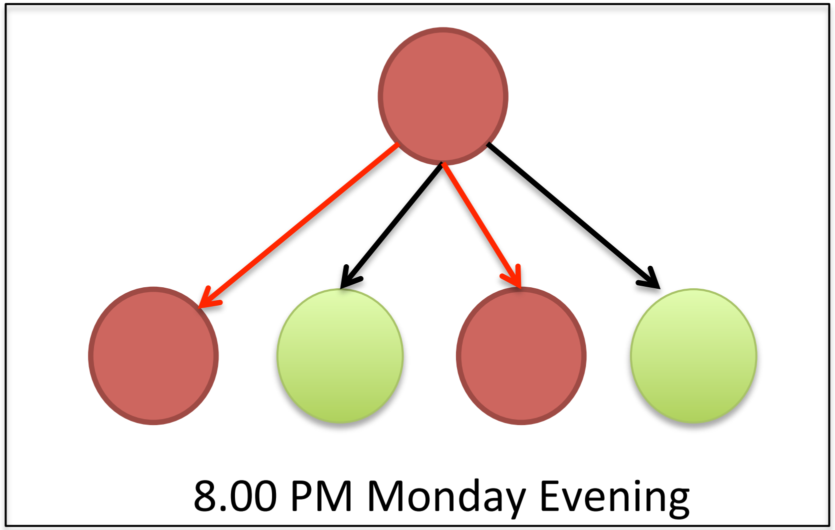
↑	Infected	→	Lambda	↑
↑	Lambda	→	IR	↑
↑	IR	→	Infected	↑

The rate at which susceptible individuals become infected per unit time
Proportional to the number infected

Effective Contact (C_e)

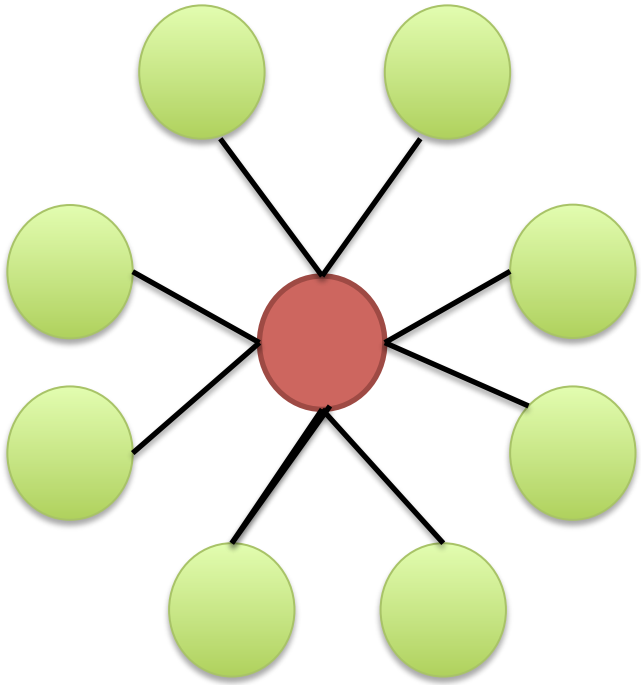
- Defined a one which is sufficient to lead to infection, were it to occur between a **susceptible** and **infectious** individuals
- For example, if $C_e = 2$
  - An infectious person will infect two susceptible people in one day
  - They could meet 4 people, and pass on the virus with probability (0.50)





Beta (β)

- **Per capita rate** at which two specific individuals come into effective contact per unit time
- An important parameter used to model disease transmission



$$\beta = \frac{c_e}{N}$$

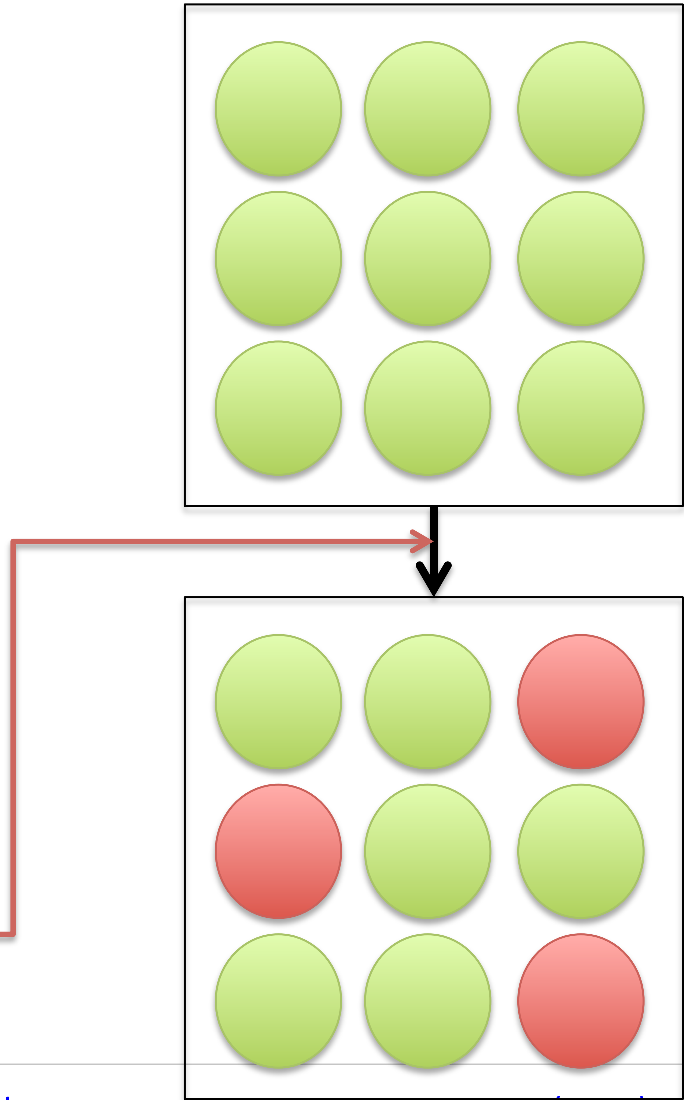
$$\beta = \frac{2}{10000} = 0.0002$$

Lambda – Force of Infection

- The rate at which susceptible individuals become infected per unit time
- Also know as the hazard rate or incidence rate

$$\lambda = \beta I$$

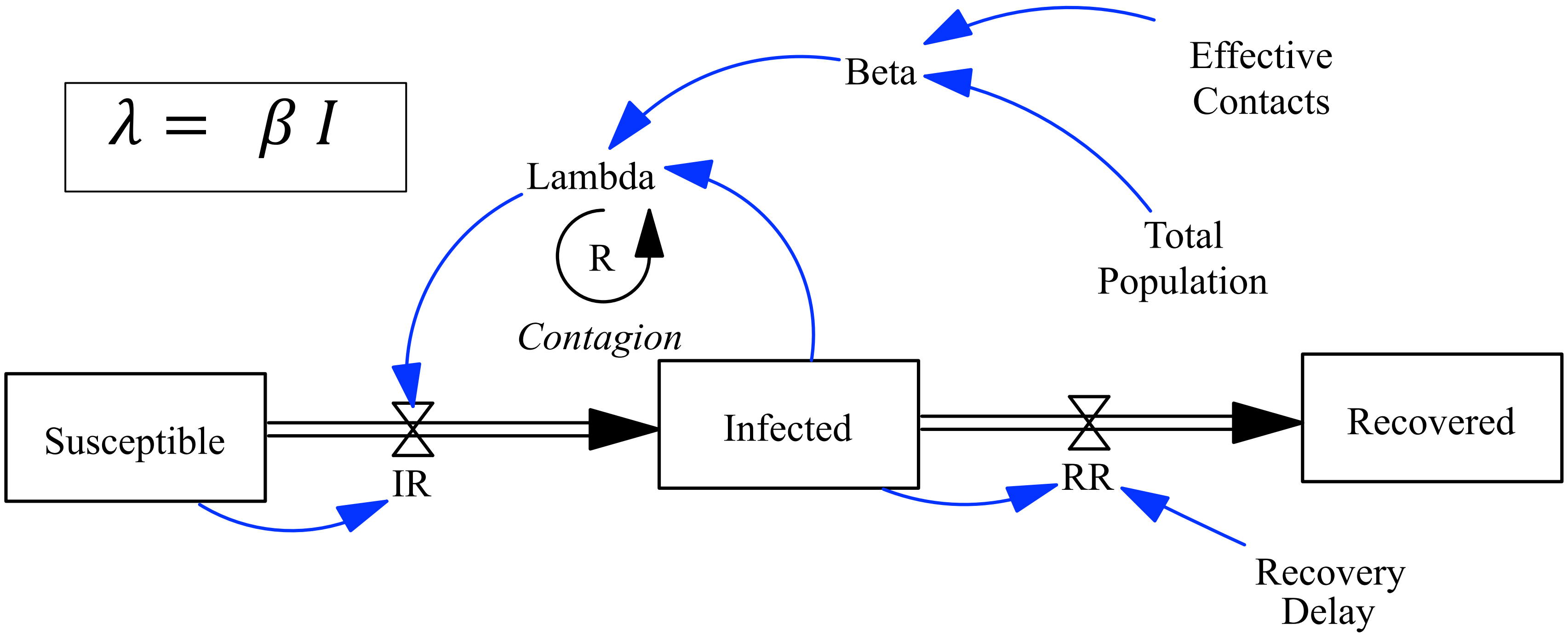
$$\lambda = \frac{1}{3}$$



A Stock and Flow Model

$$\beta = \frac{c_e}{N}$$

$$\lambda = \beta I$$



Model Equations

Total Population = 10000

Effective Contacts=2

Beta= Effective Contacts/Total Population

Lambda = Beta*Infected

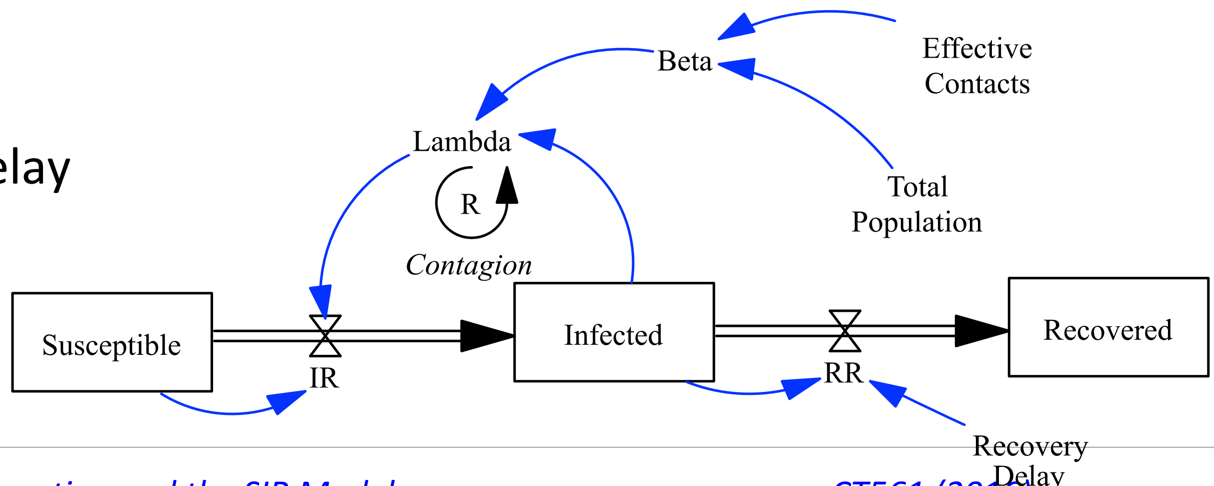
IR=Lambda*Susceptible

RR=Infected/Recovery Delay

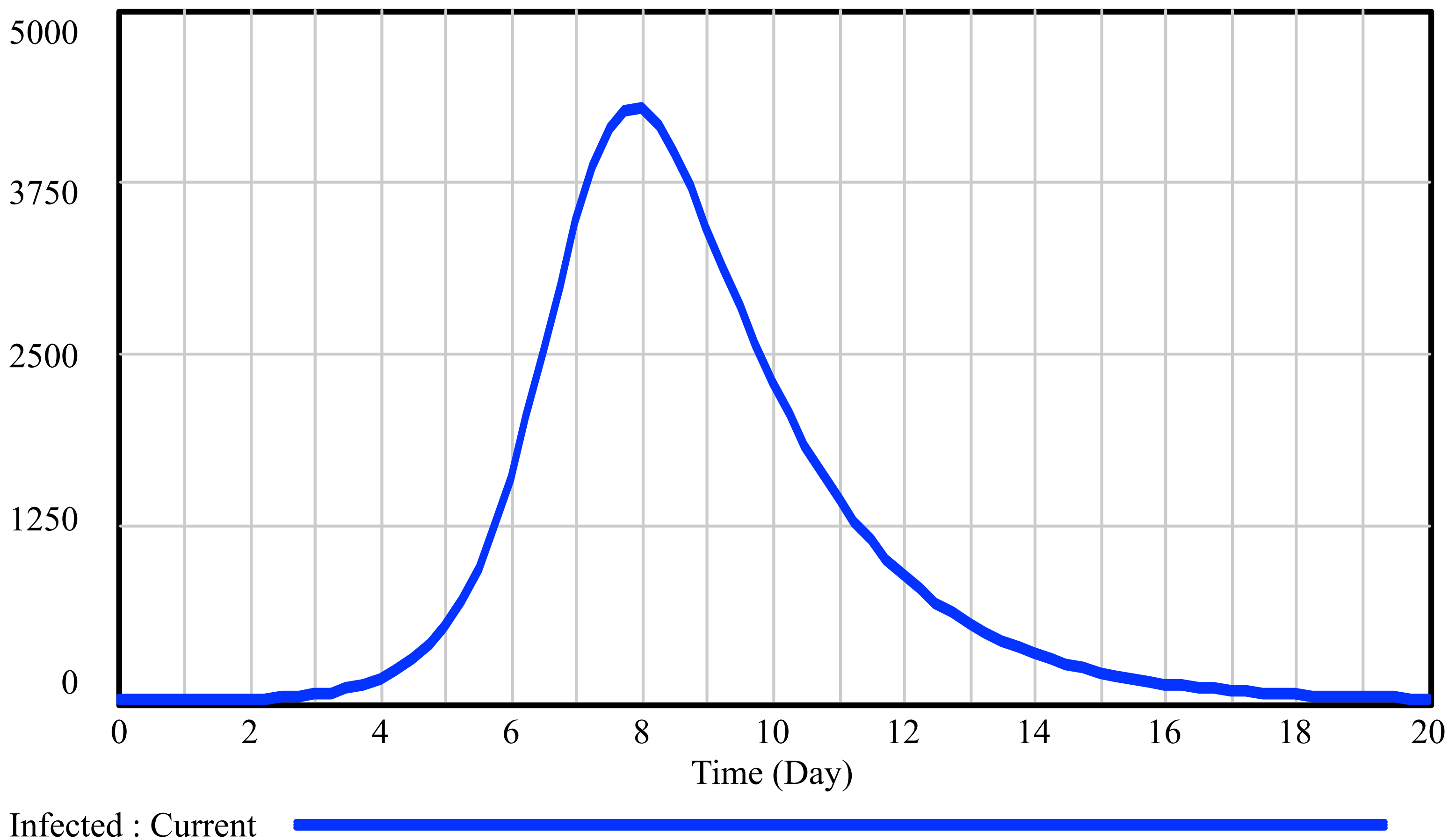
Susceptible= INTEG (-IR, 9999)

Infected= INTEG (IR-RR, 1)

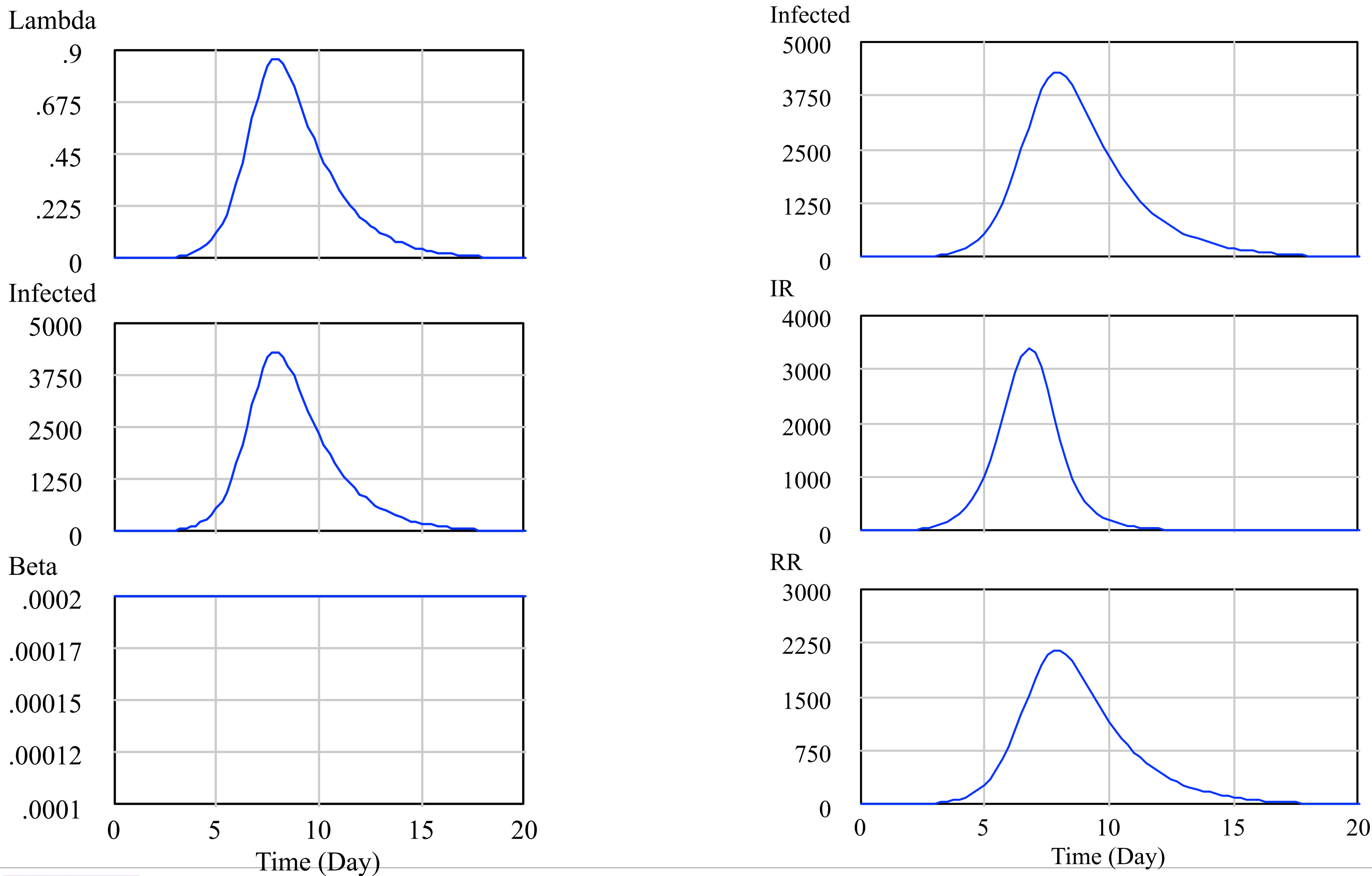
Recovered= INTEG (RR, 0)



Simulation Output



Exploring Variables…



Challenge 8.2

- Draw the stock and flow model the corresponds to the following equations

$$\frac{dS}{dt} = -\lambda S$$

$$\frac{dI}{dt} = fE - rI$$

$$\frac{dE}{dt} = \lambda S - fE$$

$$\frac{dR}{dt} = rI$$

The rate at which something occurs is 1 / Average time to the event
f and *r* are rates.

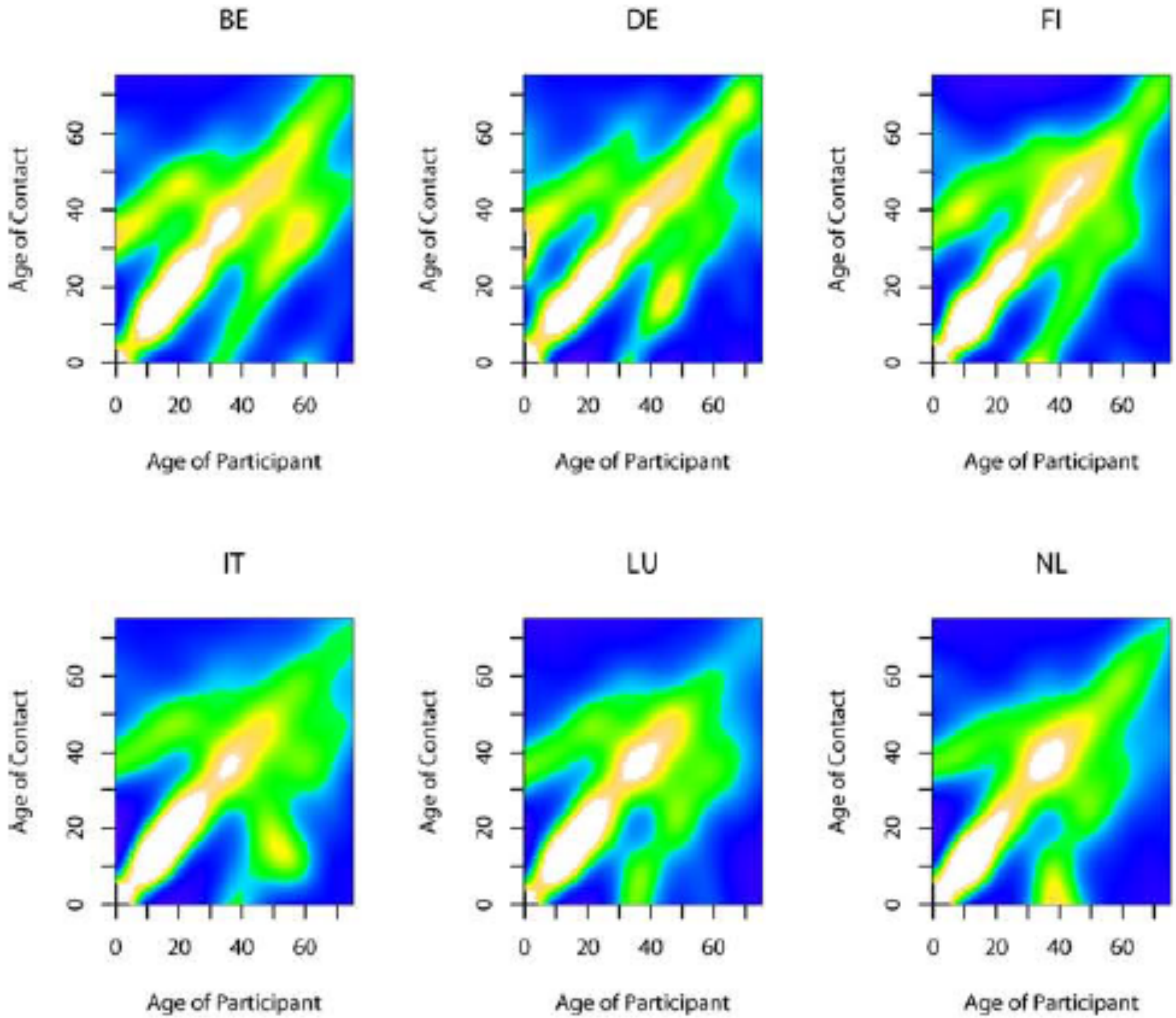
OPEN ACCESS Freely available online PLOS MEDICINE

Social Contacts and Mixing Patterns Relevant to the Spread of Infectious Diseases

Joël Mossong[1,2*], Niel Hens[3], Mark Jit[4], Philippe Beutels[5], Kari Auranen[6], Rafael Mikolajczyk[7], Marco Massari[8], Stefania Salmaso[8], Gianpaolo Scalia Tomba[9], Jacco Wallinga[10], Janneke Heijne[10], Malgorzata Sadkowska-Todys[11], Magdalena Rosinska[11], W. John Edmunds[4]

Methods and Findings

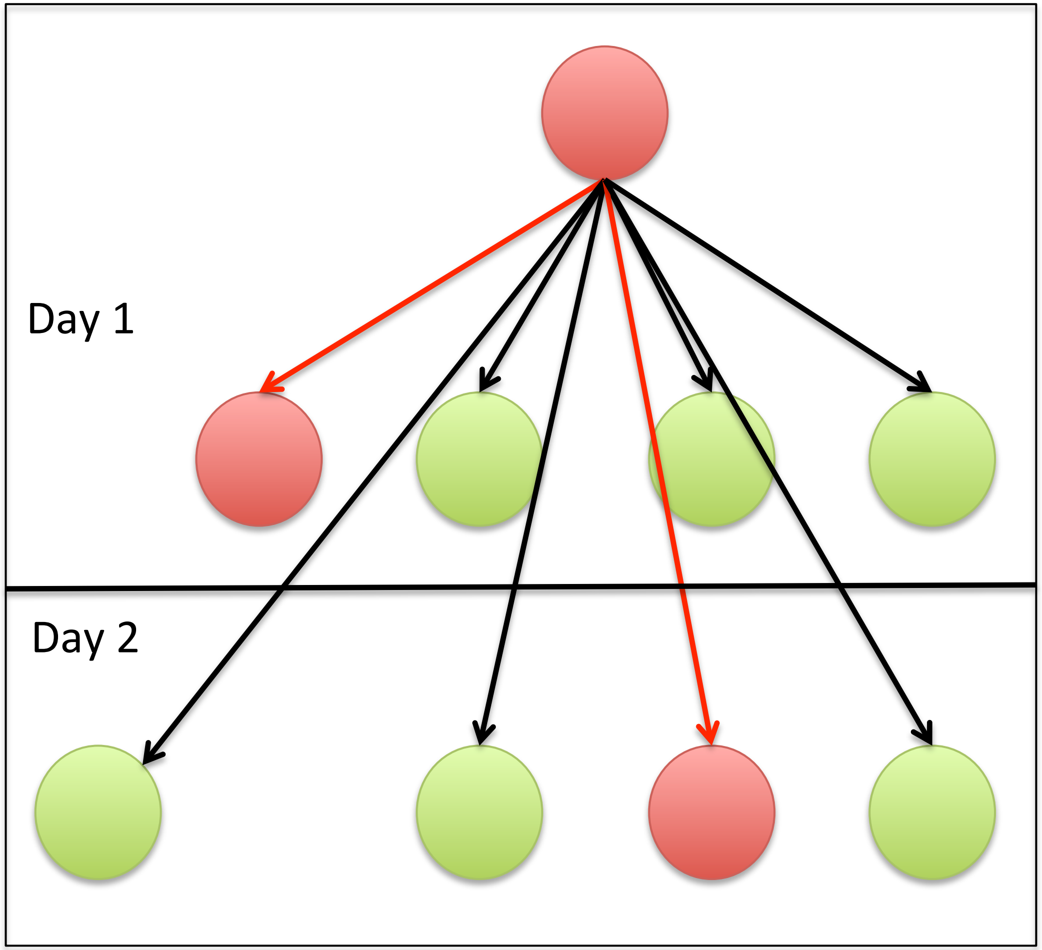
7,290 participants recorded characteristics of 97,904 contacts with different individuals during one day, including age, sex, location, duration, frequency, and occurrence of physical contact. We found that mixing patterns and contact characteristics were remarkably similar across different European countries. Contact patterns were highly assortative with age: schoolchildren and young adults in particular tended to mix with people of the same age. Contacts lasting at least one hour or occurring on a daily basis mostly involved physical contact, while short duration and infrequent contacts tended to be nonphysical. Contacts at home, school, or leisure were more likely to be physical than contacts at the workplace or while travelling. Preliminary modelling indicates that 5- to 19-year-olds are expected to suffer the highest incidence during the initial epidemic phase of an emerging infection transmitted through social contacts measured here when the population is completely susceptible.


BE
DE
FI
IT
LU
NL
Age of Contact
Age of Participant
0
20
40
60

Reproduction Number – R_0

- Formally defined as the average number of secondary infectious resulting from a typical infectious person being introduced to a totally susceptible population

$$\boxed{R_0 = c_e D}$$



$D = 2 \qquad c_e = 1 \qquad R_0 = 2$

Challenge 8.3

- Suppose we have a town with 10,000 (=N) individuals, of which 1% were infectious with measles, with R_0 = 13 and D=7 Days
- Calculate the force of infection λ

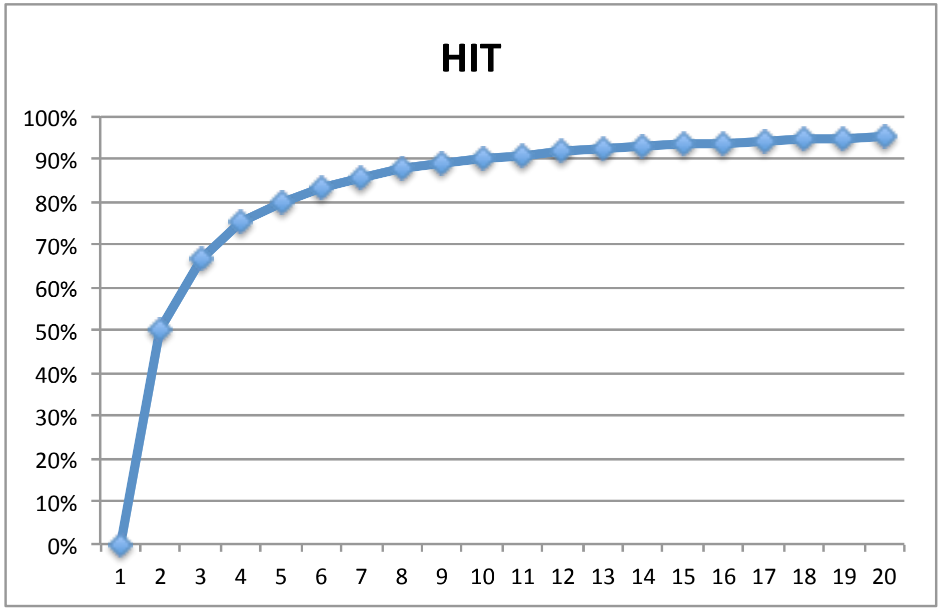
$$\boxed{R_0 = \; c_e \; D}$$

$$\boxed{\beta = \frac{c_e}{N}}$$

Herd Immunity Threshold

- Depends on R_0
- The proportion of the population which needs to be immune for the infection incidence to be stable
- To eradicate an infection, the proportion of the population that is immune must exceed this threshold value

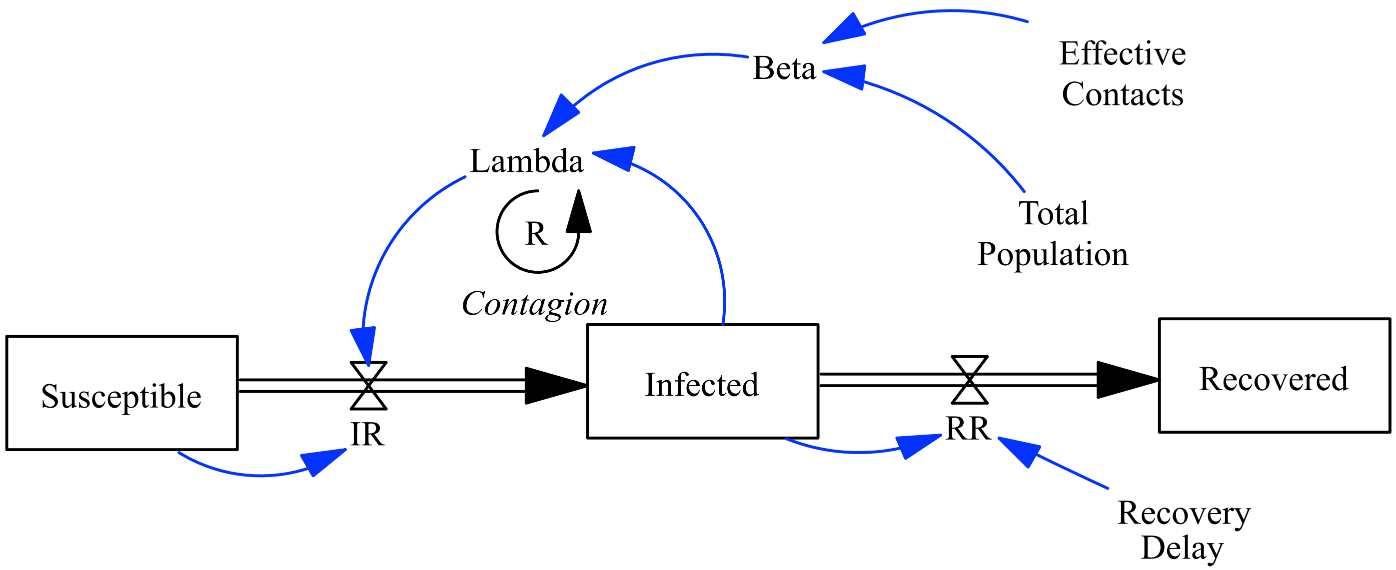
$$HIT = 1 - \frac{1}{R_0}$$



Approximate data for common potentially vaccine-preventable diseases

Infection	R_o	Herd Immunity
Diphtheria	6-7	85
Influenza	2-4	50-75
Malaria	5-100	80-99
Measles	12-18	83-94
Pertussis	12-17	92-94

Challenge 8.4



- Explore the SIR Model
- What model conditions will stop disease spread?