# Introducción a elementos finitos
# Tarea 6 I-2016

Aplicando el método de Newton-Cotes construir la tabla de pesos y puntos de muestreo hasta $n = 4$

## n = 1

$$k = 1 - 1 = 0$$

Calculando $r_i$

$$\int_{-1}^{+1} P(r) \, r^0 \, dr = 0$$

El polinomio es

$$P(r) = r - r_1$$

Reemplazando

$$\int_{-1}^{+1} r - r_1 \, dr = 0$$

Integrando

$$\int_{-1}^{+1} r - r_1 \, dr = \left. \left( \frac{1}{2}r^2 - r_1 r \right) \right|_{-1}^{+1} = -2r_1$$

Despejando

$$r_1 = 0$$

Calculando $w_i$

$$w_1 = \int_{-1}^{+1} dr = 2$$

## n = 2

$$k = 2 - 1 = 1$$

Calculando $r_i$

$$\int_{-1}^{+1} P(r) \, r^0 \, dr = 0$$
$$\int_{-1}^{+1} P(r) \, r^1 \, dr = 0$$

El polinomio es

$$P(r) = (r - r_1)(r - r_2)$$

Reemplazando

$$\int_{-1}^{+1} (r - r_1)(r - r_2) \, dr = 0$$

$$\int_{-1}^{+1} (r - r_1)(r - r_2)r \, dr = 0$$

Integrando

$$\left( \frac{1}{3}r^3 - \frac{r_1 + r_2}{2}r^2 + r_1 r_2 r \right)\Big|_{-1}^{+1} = 2\left( r_1 r_2 + \frac{1}{3} \right)$$

$$\left( \frac{1}{4}r^4 - \frac{r_1 + r_2}{3}r^3 + \frac{r_1 r_2}{2}r^2 \right)\Big|_{-1}^{+1} = -\frac{2}{3}(r_1 + r_2)$$

Formando el sistema de ecuaciones

$$r_1 r_2 = -\frac{1}{3}$$

$$r_1 + r_2 = 0$$

Resolviendo

$$r_1 = -\sqrt{\frac{1}{3}}$$

$$r_2 = \sqrt{\frac{1}{3}}$$

Calculando $w_i$

$$w_1 = \int_{-1}^{+1} \frac{r - r_2}{r_1 - r_2} \, dr$$

$$w_2 = \int_{-1}^{+1} \frac{r - r_1}{r_2 - r_1} \, dr$$

Reemplazando

$$w_1 = \int_{-1}^{+1} \frac{r - \sqrt{\frac{1}{3}}}{-\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{3}}} \, dr$$

$$w_2 = \int_{-1}^{+1} \frac{r + \sqrt{\frac{1}{3}}}{\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{3}}} \, dr$$

Integrando

$$w_1 = \left( -\frac{\sqrt{3}}{4}r^2 + \frac{1}{2}r \right)\Big|_{-1}^{+1} = 1$$

$$w_2 = \left( \frac{\sqrt{3}}{4}r^2 + \frac{1}{2}r \right)\Big|_{-1}^{+1} = 1$$

# n = 3

$$k = 3 - 1 = 2$$

Calculando $r_i$

$$\int_{-1}^{+1} P(r) \ r^0 \ dr = 0$$

$$\int_{-1}^{+1} P(r) \ r^1 \ dr = 0$$

$$\int_{-1}^{+1} P(r) \ r^2 \ dr = 0$$

El polinomio es

$$P(r) = (r - r_1)(r - r_2)(r - r_3)$$

Reemplazando

$$\int_{-1}^{+1} (r - r_1)(r - r_2)(r - r_3) \ dr = 0$$

$$\int_{-1}^{+1} (r - r_1)(r - r_2)(r - r_3)r \ dr = 0$$

$$\int_{-1}^{+1} (r - r_1)(r - r_2)(r - r_3)r^2 \ dr = 0$$

Integrando

$$\left[\frac{1}{4}r^4 - \frac{r_1 + r_2 + r_3}{3}r^3 + \frac{r_1r_2 + r_1r_3 + r_2r_3}{2}r^2 - r_1r_2r_3r\right]\Bigg|_{-1}^{+1} = -\frac{2}{3}(3r_1r_2r_3 + r_1 + r_2 + r_3)$$

$$\left[\frac{1}{5}r^5 - \frac{r_1 + r_2 + r_3}{4}r^4 + \frac{r_1r_2 + r_1r_3 + r_2r_3}{3}r^3 - \frac{r_1r_2r_3}{2}r^2\right]\Bigg|_{-1}^{+1} = \frac{2}{15}(5r_1r_2 + 5r_1r_3 + 5r_2r_3 + 3)$$

$$\left[\frac{1}{6}r^6 - \frac{r_1 + r_2 + r_3}{5}r^5 + \frac{r_1r_2 + r_1r_3 + r_2r_3}{4}r^4 - \frac{r_1r_2r_3}{3}r^3\right]\Bigg|_{-1}^{+1} = -\frac{2}{15}(5r_1r_2r_3 + 3r_1 + 3r_2 + 3r_3)$$

Formando el sistema de ecuaciones

$$3r_1r_2r_3 + r_1 + r_2 + r_3 = 0$$
$$5r_1r_2 + 5r_1r_3 + 5r_2r_3 = -3$$
$$5r_1r_2r_3 + 3r_1 + 3r_2 + 3r_3 = 0$$

Resolviendo

$$r_1 = -\sqrt{\frac{3}{5}}$$

$$r_2 = 0$$

$$r_3 = \sqrt{\frac{3}{5}}$$

Calculando $w_i$

$$w_1 = \int_{-1}^{+1} \frac{r - r_2}{r_1 - r_2} \cdot \frac{r - r_3}{r_1 - r_3} \ dr$$

$$w_2 = \int_{-1}^{+1} \frac{r - r_1}{r_2 - r_1} \cdot \frac{r - r_3}{r_2 - r_3} \ dr$$

$$w_3 = \int_{-1}^{+1} \frac{r - r_2}{r_3 - r_2} \cdot \frac{r - r_1}{r_3 - r_1} \ dr$$

Reemplazando

$$w_1 = \int_{-1}^{+1} \frac{r - 0}{-\sqrt{\frac{3}{5}} - 0} \cdot \frac{r - \sqrt{\frac{3}{5}}}{-\sqrt{\frac{3}{5}} - \sqrt{\frac{3}{5}}} \ dr$$

$$w_2 = \int_{-1}^{+1} \frac{r + \sqrt{\frac{3}{5}}}{0 + \sqrt{\frac{3}{5}}} \cdot \frac{r - \sqrt{\frac{3}{5}}}{0 - \sqrt{\frac{3}{5}}} \ dr$$

$$w_3 = \int_{-1}^{+1} \frac{r - 0}{\sqrt{\frac{3}{5}} - 0} \cdot \frac{r + \sqrt{\frac{3}{5}}}{\sqrt{\frac{3}{5}} + \sqrt{\frac{3}{5}}} \ dr$$

Integrando

$$w_1 = \left( \frac{5}{18}r^3 - \frac{\sqrt{15}}{12}r^2 \right)\Big|_{-1}^{+1} = \frac{5}{9}$$

$$w_2 = \left( -\frac{5}{9}r^3 + r \right)\Big|_{-1}^{+1} = \frac{8}{9}$$

$$w_3 = \left( \frac{5}{18}r^3 + \frac{\sqrt{15}}{12}r^2 \right)\Big|_{-1}^{+1} = \frac{5}{9}$$

# n = 4

$$k = 4 - 1 = 3$$

Calculando $r_i$

$$\int_{-1}^{+1} P(r) \ r^0 \ dr = 0$$

$$\int_{-1}^{+1} P(r) \ r^1 \ dr = 0$$

$$\int_{-1}^{+1} P(r) \ r^2 \ dr = 0$$

$$\int_{-1}^{+1} P(r) \ r^3 \ dr = 0$$

El polinomio es

$$P(r) = (r - r_1)(r - r_2)(r - r_3)(r - r_4)$$

Reemplazando

$$\int_{-1}^{+1} (r - r_1)(r - r_2)(r - r_3)(r - r_4) \, dr = 0$$

$$\int_{-1}^{+1} (r - r_1)(r - r_2)(r - r_3)(r - r_4)r \, dr = 0$$

$$\int_{-1}^{+1} (r - r_1)(r - r_2)(r - r_3)(r - r_4)r^2 \, dr = 0$$

$$\int_{-1}^{+1} (r - r_1)(r - r_2)(r - r_3)(r - r_4)r^3 \, dr = 0$$

Integrando

$$\left[ \frac{1}{5}r^5 - \frac{r_1 + r_2 + r_3 + r_4}{4}r^4 + \frac{r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4}{3}r^3 - \frac{r_1 r_2 r_3 + r_1 r_2 r_4 + r_1 r_3 r_4 + r_2 r_3 r_4}{2}r^2 \right.$$
$$\left. + r_1 r_2 r_3 r_4 r \right]\Bigg|_{-1}^{+1} = \frac{2}{15}(15 r_1 r_2 r_3 r_4 + 5 r_1 r_2 + 5 r_1 r_3 + 5 r_1 r_4 + 5 r_2 r_3 + 5 r_2 r_4 + 5 r_3 r_4 + 3)$$

$$\left[ \frac{1}{6}r^6 - \frac{r_1 + r_2 + r_3 + r_4}{5}r^5 + \frac{r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4}{4}r^4 - \frac{r_1 r_2 r_3 + r_1 r_2 r_4 + r_1 r_3 r_4 + r_2 r_3 r_4}{3}r^3 \right.$$
$$\left. + \frac{r_1 r_2 r_3 r_4}{2}r^2 \right]\Bigg|_{-1}^{+1} = -\frac{2}{15}(5 r_1 r_2 r_3 + 5 r_1 r_2 r_4 + 5 r_1 r_3 r_4 + 5 r_2 r_3 r_4 + 3 r_1 + 3 r_2 + 3 r_3 + 3 r_4)$$

$$\left[ \frac{1}{7}r^7 - \frac{r_1 + r_2 + r_3 + r_4}{6}r^6 + \frac{r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4}{5}r^5 - \frac{r_1 r_2 r_3 + r_1 r_2 r_4 + r_1 r_3 r_4 + r_2 r_3 r_4}{4}r^4 \right.$$
$$\left. + \frac{r_1 r_2 r_3 r_4}{3}r^3 \right]\Bigg|_{-1}^{+1} = \frac{2}{105}(35 r_1 r_2 r_3 r_4 + 21 r_1 r_2 + 21 r_1 r_3 + 21 r_1 r_4 + 21 r_2 r_3 + 21 r_2 r_4 + 21 r_3 r_4 + 15)$$

$$\left[ \frac{1}{8}r^8 - \frac{r_1 + r_2 + r_3 + r_4}{7}r^7 + \frac{r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4}{6}r^6 - \frac{r_1 r_2 r_3 + r_1 r_2 r_4 + r_1 r_3 r_4 + r_2 r_3 r_4}{5}r^5 \right.$$
$$\left. + \frac{r_1 r_2 r_3 r_4}{4}r^4 \right]\Bigg|_{-1}^{+1} = -\frac{2}{35}(7 r_1 r_2 r_3 + 7 r_1 r_2 r_4 + 7 r_1 r_3 r_4 + 7 r_2 r_3 r_4 + 5 r_1 + 5 r_2 + 5 r_3 + 5 r_4)$$

Formando el sistema de ecuaciones

$$15 r_1 r_2 r_3 r_4 + 5 r_1 r_2 + 5 r_1 r_3 + 5 r_1 r_4 + 5 r_2 r_3 + 5 r_2 r_4 + 5 r_3 r_4 = -3$$
$$5 r_1 r_2 r_3 + 5 r_1 r_2 r_4 + 5 r_1 r_3 r_4 + 5 r_2 r_3 r_4 + 3 r_1 + 3 r_2 + 3 r_3 + 3 r_4 = 0$$
$$35 r_1 r_2 r_3 r_4 + 21 r_1 r_2 + 21 r_1 r_3 + 21 r_1 r_4 + 21 r_2 r_3 + 21 r_2 r_4 + 21 r_3 r_4 = -15$$
$$7 r_1 r_2 r_3 + 7 r_1 r_2 r_4 + 7 r_1 r_3 r_4 + 7 r_2 r_3 r_4 + 5 r_1 + 5 r_2 + 5 r_3 + 5 r_4 = 0$$

Resolviendo

$$r_1 = -\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}$$

$$r_2 = -\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}$$

$$r_3 = \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}$$

$$r_4 = \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}$$

Calculando $w_i$

$$w_1 = \int_{-1}^{+1} \frac{r - r_2}{r_1 - r_2} \cdot \frac{r - r_3}{r_1 - r_3} \cdot \frac{r - r_4}{r_1 - r_4}\ dr = -\frac{6r_2 r_3 r_4 + 2r_2 + 2r_3 + 2r_4}{3(r_1 - r_2)(r_1 - r_3)(r_1 - r_4)}$$

$$w_2 = \int_{-1}^{+1} \frac{r - r_1}{r_2 - r_1} \cdot \frac{r - r_3}{r_2 - r_3} \cdot \frac{r - r_4}{r_2 - r_4}\ dr = \frac{6r_1 r_3 r_4 + 2r_1 + 2r_3 + 2r_4}{3(r_1 - r_2)(r_2 - r_3)(r_2 - r_4)}$$

$$w_3 = \int_{-1}^{+1} \frac{r - r_2}{r_3 - r_2} \cdot \frac{r - r_1}{r_3 - r_1} \cdot \frac{r - r_4}{r_3 - r_4}\ dr = -\frac{6r_1 r_2 r_4 + 2r_1 + 2r_2 + 2r_4}{3(r_2 - r_3)(r_1 - r_3)(r_3 - r_4)}$$

$$w_4 = \int_{-1}^{+1} \frac{r - r_3}{r_4 - r_3} \cdot \frac{r - r_2}{r_4 - r_2} \cdot \frac{r - r_1}{r_4 - r_1}\ dr = \frac{6r_1 r_2 r_3 + 2r_1 + 2r_2 + 2r_3}{3(r_3 - r_4)(r_2 - r_4)(r_1 - r_4)}$$

Reemplazando

$$w_1 = -\frac{6\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} + 2\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}} + 2\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}} + 2\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}}{3\left(-\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} + \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}\right)\left(-\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} - \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}\right)\left(-\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} - \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}\right)}$$

$$w_2 = \frac{-6\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} - 2\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} + 2\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}} + 2\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}}{3\left(-\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} + \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}\right)\left(-\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}} - \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}\right)\left(-\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}} - \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}\right)}$$

$$w_3 = -\frac{6\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} - 2\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} - 2\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}} + 2\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}}{3\left(-\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}} - \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}\right)\left(-\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} - \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}\right)\left(\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}} - \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}\right)}$$

$$w_4 = \frac{6\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}} - 2\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} - 2\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}} + 2\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}}{3\left(\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}} - \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}\right)\left(-\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}} - \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}\right)\left(-\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} - \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}\right)}$$

Simplificando

$$w_1 = \frac{18 - \sqrt{30}}{36}$$

$$w_2 = \frac{18 + \sqrt{30}}{36}$$

$$w_3 = \frac{18 + \sqrt{30}}{36}$$

$$w_4 = \frac{18 - \sqrt{30}}{36}$$

Tabla resumen

| $n$ | $r$ | $w$ |
|---|---|---|
| 1 | $0$ | $2$ |
| 2 | $-\sqrt{\frac{1}{3}}$ | $1$ |
|   | $\sqrt{\frac{1}{3}}$ | $1$ |
| 3 | $-\sqrt{\frac{3}{5}}$ | $\frac{5}{9}$ |
|   | $0$ | $\frac{8}{9}$ |
|   | $\sqrt{\frac{3}{5}}$ | $\frac{5}{9}$ |
| 4 | $-\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}$ | $\frac{18-\sqrt{30}}{36}$ |
|   | $-\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}$ | $\frac{18+\sqrt{30}}{36}$ |
|   | $\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}$ | $\frac{18+\sqrt{30}}{36}$ |
|   | $\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}$ | $\frac{18-\sqrt{30}}{36}$ |