Esquema implícito de Euler

Hallar el perfil de flujo usando $\Delta x = 40$ m, $\Delta t = 10$ h y $D = 1 \times 10^{-3}$ m^2/s, para un tiempo final de 20 h

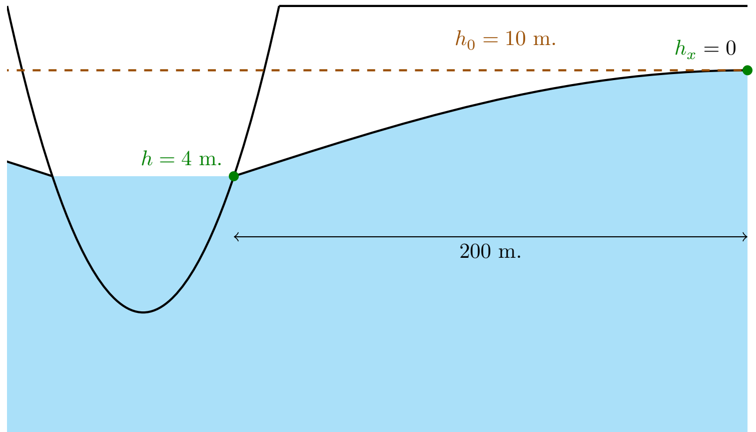

Figura 1: Representación gráfica del ejemplo

La ecuación y las condiciones de contorno serán

$$\frac{\partial h}{\partial t} - D\,\frac{\partial^2 h}{\partial x^2} = 0 \tag{1}$$

$$h(x,0) = 10 \tag{2}$$

$$h(0,t) = 4 \tag{3}$$

$$h_x(200,t) = 0 \tag{4}$$

Discretización espacial

$$N_{\text{elementos}} = \frac{L}{\Delta x} = \frac{200}{40} = 5$$

$$N_{\text{puntos}} = N_{\text{elementos}} + 1 = 5 + 1 = 6$$

Discretización temporal

$$N_{\text{elementos}} = \frac{t}{\Delta t} = \frac{20}{10} = 2$$

$$N_{\text{puntos}} = N_{\text{elementos}} + 1 = 2 + 1 = 3$$

Discretización numérica

$$\frac{\partial h}{\partial t} = \frac{h_i^{n+1} - h_i^n}{\Delta t}$$

$$\frac{\partial^2 h}{\partial x^2} = \frac{h_{i-1}^{n+1} - 2h_i^{n+1} + h_{i+1}^{n+1}}{\Delta x^2}$$

Reemplazando en (1)

$$\frac{h_i^{n+1} - h_i^n}{\Delta t} - D\left(\frac{h_{i-1}^{n+1} - 2h_i^{n+1} + h_{i+1}^{n+1}}{\Delta x^2}\right) = 0$$

Reordenando

$$-D\frac{\Delta t}{\Delta x^2}h_{i-1}^{n+1} + \left(1 + 2D\frac{\Delta t}{\Delta x^2}\right)h_i^{n+1} - D\frac{\Delta t}{\Delta x^2}h_{i+1}^{n+1} = h_i^n$$

Realizando un cambio de variable

$$a = -D\frac{\Delta t}{\Delta x^2}$$
$$b = 1 + 2D\frac{\Delta t}{\Delta x^2}$$
$$c = -D\frac{\Delta t}{\Delta x^2}$$

El esquema será

$$a\,h_{i-1}^{n+1} + b\,h_i^{n+1} + c\,h_{i+1}^{n+1} = h_i^n$$

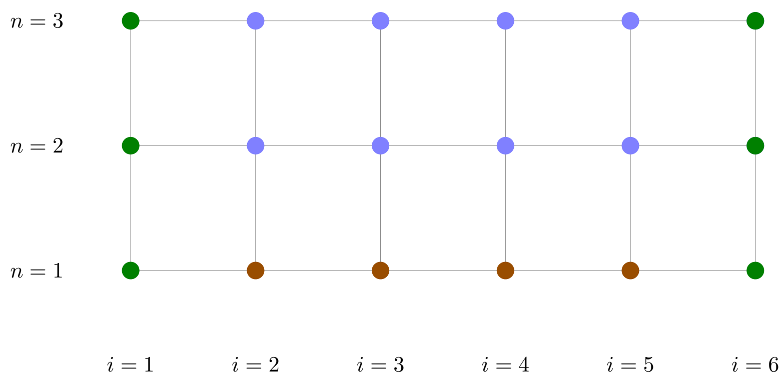

Figura 2: Mallado

3	h_1^3	h_2^3	h_3^3	h_4^3	h_5^3	h_6^3
2	h_1^2	h_2^2	h_3^2	h_4^2	h_5^2	h_6^2
1	h_1^1	h_2^1	h_3^1	h_4^1	h_5^1	h_6^1
	1	2	3	4	5	6

Figura 3: Matriz solución

El esquema es incondicionalmente estable para cualquier λ

$$D\frac{\Delta t}{\Delta x^2} = 0.001\left(\frac{36000}{40^2}\right) = 0.0225$$

Reemplazando las condiciones de contorno, para $i = 1$ y $n = 1, 2, 3$

$$h_1^1 = 4$$
$$h_1^2 = 4$$
$$h_1^3 = 4$$

Para $i = 2, 3, 4, 5$ y $n = 1$

$$\begin{aligned} h_2^1 &= 10 \\ h_3^1 &= 10 \\ h_4^1 &= 10 \\ h_5^1 &= 10 \end{aligned}$$

Para $i = 6$ y $n = 1$, usando un esquema hacia atrás

$$h_x = \frac{\partial h}{\partial x} = \frac{h_6^1 - h_5^1}{\Delta x} = 0$$

reordenando

$$\begin{aligned} h_6^1 &= h_5^1 \\ &= 10 \end{aligned}$$

3	4	h_2^3	h_3^3	h_4^3	h_5^3	h_6^3
2	4	h_2^2	h_3^2	h_4^2	h_5^2	h_6^2
1	4	10	10	10	10	10
	1	2	3	4	5	6

Figura 4: Matriz solución para $t = 0$ h

Las constantes a, b, c serán

$$\begin{aligned} a &= -0.0225 \\ b &= 1 + 2(0.0225) = 1.045 \\ c &= -0.0225 \end{aligned}$$

Usando el esquema elegido, para $i = 2$ y $n = 1$

$$-0.0225h_1^2 + 1.045h_2^2 - 0.0225h_3^2 = 10$$

Para $i = 3$ y $n = 1$

$$-0.0225h_2^2 + 1.045h_3^2 - 0.0225h_4^2 = 10$$

Para $i = 4$ y $n = 1$

$$-0.0225h_3^2 + 1.045h_4^2 - 0.0225h_5^2 = 10$$

Para $i = 5$ y $n = 1$

$$-0.0225h_4^2 + 1.045h_5^2 - 0.0225h_6^2 = 10$$

Formando un sistema de ecuaciones

$$
\begin{aligned}
-0.0225\,h_1^2 + \ 1.045\,h_2^2 - 0.0225\,h_3^2 &= 10\\
-0.0225\,h_2^2 + \ 1.045\,h_3^2 - 0.0225\,h_4^2 &= 10\\
-0.0225\,h_3^2 + \ 1.045\,h_4^2 - 0.0225\,h_5^2 &= 10\\
-0.0225\,h_4^2 + \ 1.045\,h_5^2 - 0.0225\,h_6^2 &= 10
\end{aligned}
$$

Reemplazando $h_1^2 = 4$ y $h_6^2 = h_5^2$

$$
\begin{aligned}
-0.0225(4) + \ 1.045\,h_2^2 - 0.0225\,h_3^2 &= 10\\
-0.0225\,h_2^2 + \ 1.045\,h_3^2 - 0.0225\,h_4^2 &= 10\\
-0.0225\,h_3^2 + \ 1.045\,h_4^2 - 0.0225\,h_5^2 &= 10\\
-0.0225\,h_4^2 + \ 1.045\,h_5^2 - 0.0225\,h_5^2 &= 10
\end{aligned}
$$

Simplificando y reordenando

$$
\begin{aligned}
1.045\,h_2^2 - 0.0225\,h_3^2 &= 10.09\\
-0.0225\,h_2^2 + \ 1.045\,h_3^2 - 0.0225\,h_4^2 &= 10\\
-0.0225\,h_3^2 + \ 1.045\,h_4^2 - 0.0225\,h_5^2 &= 10\\
-0.0225\,h_4^2 + 1.0225\,h_5^2 &= 10
\end{aligned}
$$

En forma matricial

$$
\begin{bmatrix}
1.045 & -0.0225 & 0 & 0\\
-0.0225 & 1.045 & -0.0225 & 0\\
0 & -0.0225 & 1.045 & -0.0225\\
0 & 0 & -0.0225 & 1.0225
\end{bmatrix}
\begin{bmatrix} h_2^2\\ h_3^2\\ h_4^2\\ h_5^2 \end{bmatrix}
=
\begin{bmatrix} 10.09\\ 10\\ 10\\ 10 \end{bmatrix}
$$

Resolviendo

$$
\begin{bmatrix} h_2^2\\ h_3^2\\ h_4^2\\ h_5^2 \end{bmatrix}
=
\begin{bmatrix} 9.87075\\ 9.99721\\ 9.99994\\ 9.99999 \end{bmatrix}
$$

3	4	h_2^3	h_3^3	h_4^3	h_5^3	h_6^3
2	4	9.87075	9.99721	9.99994	9.99999	9.99999
1	4	10	10	10	10	10
	1	2	3	4	5	6

Figura 5: Matriz solución para $t = 10$ h

Para $i = 2$ y $n = 2$

$$-0.0225h_1^3 + 1.045h_2^3 - 0.0225h_3^3 = 9.87075$$

Para $i = 3$ y $n = 2$

$$-0.0225h_2^3 + 1.045h_3^3 - 0.0225h_4^3 = 9.99721$$

Para $i = 4$ y $n = 2$

$$-0.0225h_3^3 + 1.045h_4^3 - 0.0225h_5^3 = 9.99994$$

Para $i = 5$ y $n = 2$

$$-0.0225h_4^3 + 1.045h_5^3 - 0.0225h_6^3 = 9.99999$$

Formando un sistema de ecuaciones

$$\begin{aligned}
-0.0225\,h_1^3 + 1.045\,h_2^3 - 0.0225\,h_3^3 &= 9.87075\\
-0.0225\,h_2^3 + 1.045\,h_3^3 - 0.0225\,h_4^3 &= 9.99721\\
-0.0225\,h_3^3 + 1.045\,h_4^3 - 0.0225\,h_5^3 &= 9.99994\\
-0.0225\,h_4^3 + 1.045\,h_5^3 - 0.0225\,h_6^3 &= 9.99999
\end{aligned}$$

Reemplazando $h_1^3 = 4$ y $h_6^3 = h_5^3$

$$\begin{aligned}
-0.0225(4) + 1.045\,h_2^3 - 0.0225\,h_3^3 &= 9.87075\\
-0.0225\,h_2^3 + 1.045\,h_3^3 - 0.0225\,h_4^3 &= 9.99721\\
-0.0225\,h_3^3 + 1.045\,h_4^3 - 0.0225\,h_5^3 &= 9.99994\\
-0.0225\,h_4^3 + 1.045\,h_5^3 - 0.0225\,h_5^3 &= 9.99999
\end{aligned}$$

Simplificando y reordenando

$$\begin{aligned}
1.045\,h_2^3 - 0.0225\,h_3^3 &= 9.96075\\
-0.0225\,h_2^3 + 1.045\,h_3^3 - 0.0225\,h_4^3 &= 9.99721\\
-0.0225\,h_3^3 + 1.045\,h_4^3 - 0.0225\,h_5^3 &= 9.99994\\
-0.0225\,h_4^3 + 1.0225\,h_5^3 &= 9.99999
\end{aligned}$$

En forma matricial

$$\begin{bmatrix} 1.045 & -0.0225 & 0 & 0 \\ -0.0225 & 1.045 & -0.0225 & 0 \\ 0 & -0.0225 & 1.045 & -0.0225 \\ 0 & 0 & -0.0225 & 1.0225 \end{bmatrix} \begin{bmatrix} h_2^3 \\ h_3^3 \\ h_4^3 \\ h_5^3 \end{bmatrix} = \begin{bmatrix} 9.96075 \\ 9.99721 \\ 9.99994 \\ 9.99999 \end{bmatrix}$$

Resolviendo

$$\begin{bmatrix} h_2^3 \\ h_3^3 \\ h_4^3 \\ h_5^3 \end{bmatrix} = \begin{bmatrix} 9.74695 \\ 9.99187 \\ 9.99976 \\ 9.99998 \end{bmatrix}$$

3	4	9.74695	9.99187	9.99976	9.99998	9.99998
2	4	9.87075	9.99721	9.99994	9.99999	9.99999
1	4	10	10	10	10	10
	1	2	3	4	5	6

Figura 6: Matriz solución para $t = 20$ h