Esquema de Crank-Nicolson

Hallar el perfil de flujo usando $\Delta x = 40$ m, $\Delta t = 10$ h y $D = 1 \times 10^{-3}$ m^2/s, para un tiempo final de 20 h

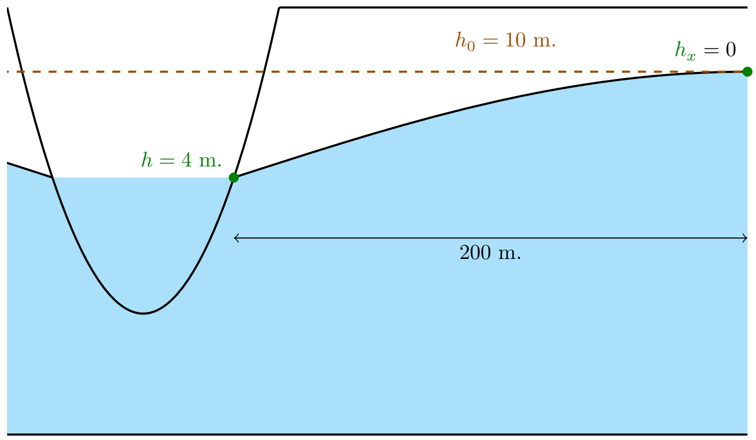

Figura 1: Representación gráfica del ejemplo

La ecuación y las condiciones de contorno serán

$$\frac{\partial h}{\partial t} - D\,\frac{\partial^2 h}{\partial x^2} = 0 \tag{1}$$
$$h(x,0) = 10 \tag{2}$$
$$h(0,t) = 4 \tag{3}$$
$$h_x(200,t) = 0 \tag{4}$$

Discretización espacial

$$N_{\text{elementos}} = \frac{L}{\Delta x} = \frac{200}{40} = 5$$
$$N_{\text{puntos}} = N_{\text{elementos}} + 1 = 5 + 1 = 6$$

Discretización temporal

$$N_{\text{elementos}} = \frac{t}{\Delta t} = \frac{20}{10} = 2$$
$$N_{\text{puntos}} = N_{\text{elementos}} + 1 = 2 + 1 = 3$$

Discretización numérica

$$\frac{\partial h}{\partial t} = \frac{h_i^{n+1} - h_i^n}{\Delta t}$$

a partir del esquema θ o esquema generalizado de Crank-Nicolson

$$\frac{\partial^2 h}{\partial x^2} = \theta\left(\frac{h_{i-1}^{n+1} - 2h_i^{n+1} + h_{i+1}^{n+1}}{\Delta x^2}\right) + (1-\theta)\left(\frac{h_{i-1}^{n} - 2h_i^{n} + h_{i+1}^{n}}{\Delta x^2}\right)$$

reemplazando $\theta = \frac{1}{2}$

$$\frac{\partial^2 h}{\partial x^2} = \frac{h_{i-1}^{n+1} - 2h_i^{n+1} + h_{i+1}^{n+1}}{2\Delta x^2} + \frac{h_{i-1}^{n} - 2h_i^{n} + h_{i+1}^{n}}{2\Delta x^2}$$

Reemplazando en (1)

$$\frac{h_i^{n+1} - h_i^n}{\Delta t} - D\left(\frac{h_{i-1}^{n+1} - 2h_i^{n+1} + h_{i+1}^{n+1}}{2\Delta x^2} + \frac{h_{i-1}^{n} - 2h_i^{n} + h_{i+1}^{n}}{2\Delta x^2}\right) = 0$$

Reordenando

$$-D\frac{\Delta t}{2\Delta x^2}h_{i-1}^{n+1} + \left(1 + D\frac{\Delta t}{\Delta x^2}\right)h_i^{n+1} - D\frac{\Delta t}{2\Delta x^2}h_{i+1}^{n+1} = h_i^n + D\frac{\Delta t}{2\Delta x^2}\left(h_{i-1}^n - 2h_i^n + h_{i+1}^n\right)$$

Realizando un cambio de variable

$$\begin{aligned} a &= -D\frac{\Delta t}{2\Delta x^2} \\ b &= 1 + D\frac{\Delta t}{\Delta x^2} \\ c &= -D\frac{\Delta t}{2\Delta x^2} \\ d_i &= h_i^n + D\frac{\Delta t}{2\Delta x^2}\left(h_{i-1}^n - 2h_i^n + h_{i+1}^n\right) \end{aligned}$$

El esquema será

$$a\,h_{i-1}^{n+1} + b\,h_i^{n+1} + c\,h_{i+1}^{n+1} = d_i$$

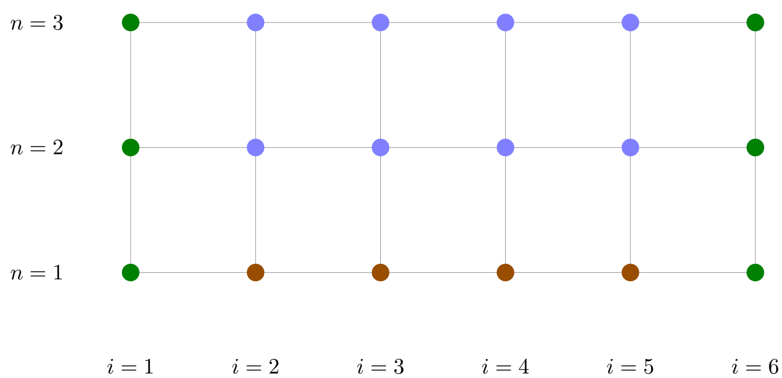

Figura 2: Mallado

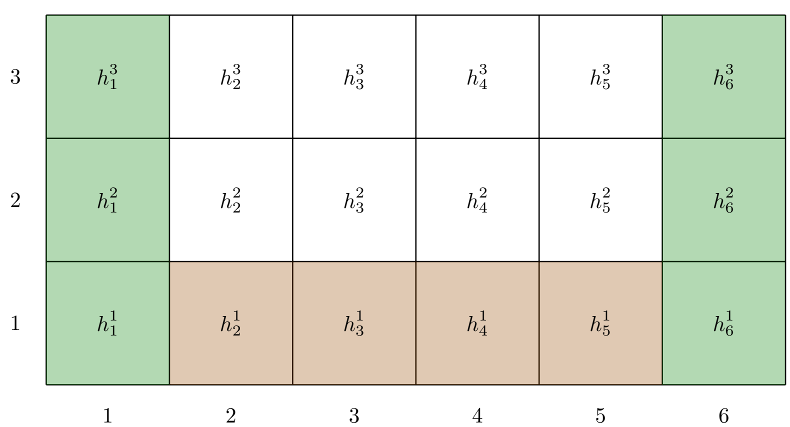

Figura 3: Matriz solución

El esquema es incondicionalmente estable para cualquier λ

$$D\frac{\Delta t}{\Delta x^2} = 0.001\left(\frac{36000}{40^2}\right) = 0.0225$$

Reemplazando las condiciones de contorno, para $i = 1$ y $n = 1, 2, 3$

$$\begin{aligned} h_1^1 &= 4 \\ h_1^2 &= 4 \\ h_1^3 &= 4 \end{aligned}$$

Para $i = 2, 3, 4, 5$ y $n = 1$

$$\begin{aligned} h_2^1 &= 10 \\ h_3^1 &= 10 \\ h_4^1 &= 10 \\ h_5^1 &= 10 \end{aligned}$$

Para $i = 6$ y $n = 1$, usando un esquema hacia atrás

$$h_x = \frac{\partial h}{\partial x} = \frac{h_6^1 - h_5^1}{\Delta x} = 0$$

reordenando

$$\begin{aligned} h_6^1 &= h_5^1 \\ &= 10 \end{aligned}$$

3	4	h_2^3	h_3^3	h_4^3	h_5^3	h_6^3
2	4	h_2^2	h_3^2	h_4^2	h_5^2	h_6^2
1	4	10	10	10	10	10
	1	2	3	4	5	6

Figura 4: Matriz solución para $t = 0$ h

Las constantes a, b, c serán

$$\begin{aligned} a &= -\frac{0.0225}{2} = -0.01125 \\ b &= 1 + 0.0225 = 1.0225 \\ c &= -\frac{0.0225}{2} = -0.01125 \end{aligned}$$

Usando el esquema elegido, para $i = 2$ y $n = 1$

$$-0.01125h_1^2 + 1.0225h_2^2 - 0.01125h_3^2 = d_2$$

$$d_2 = h_2^1 + D\frac{\Delta t}{2\Delta x^2}\left(h_1^1 - 2h_2^1 + h_3^1\right) = 10 + \frac{0.0225}{2}[4 - 2(10) + 10] = 9.9325$$

Para $i = 3$ y $n = 1$

$$-0.01125h_2^2 + 1.0225h_3^2 - 0.01125h_4^2 = d_3$$

$$d_3 = h_3^1 + D\frac{\Delta t}{2\Delta x^2}\left(h_2^1 - 2h_3^1 + h_4^1\right) = 10 + \frac{0.0225}{2}[10 - 2(10) + 10] = 10$$

Para $i = 4$ y $n = 1$

$$-0.01125h_3^2 + 1.0225h_4^2 - 0.01125h_5^2 = d_4$$

$$d_4 = h_4^1 + D\frac{\Delta t}{2\Delta x^2}\left(h_3^1 - 2h_4^1 + h_5^1\right) = 10 + \frac{0.0225}{2}[10 - 2(10) + 10] = 10$$

Para $i = 5$ y $n = 1$

$$-0.01125h_4^2 + 1.0225h_5^2 - 0.01125h_6^2 = d_5$$

$$d_5 = h_5^1 + D\frac{\Delta t}{2\Delta x^2}\left(h_4^1 - 2h_5^1 + h_6^1\right) = 10 + \frac{0.0225}{2}[10 - 2(10) + 10] = 10$$

Formando un sistema de ecuaciones

$$\begin{aligned}
-0.01125\,h_1^2 + 1.0225\,h_2^2 - 0.01125\,h_3^2 &= 9.9325\\
-0.01125\,h_2^2 + 1.0225\,h_3^2 - 0.01125\,h_4^2 &= 10\\
-0.01125\,h_3^2 + 1.0225\,h_4^2 - 0.01125\,h_5^2 &= 10\\
-0.01125\,h_4^2 + 1.0225\,h_5^2 - 0.01125\,h_6^2 &= 10
\end{aligned}$$

Reemplazando $h_1^2 = 4$ y $h_6^2 = h_5^2$

$$\begin{aligned}
-0.01125(4) + 1.0225\,h_2^2 - 0.01125\,h_3^2 &= 9.9325\\
-0.01125\,h_2^2 + 1.0225\,h_3^2 - 0.01125\,h_4^2 &= 10\\
-0.01125\,h_3^2 + 1.0225\,h_4^2 - 0.01125\,h_5^2 &= 10\\
-0.01125\,h_4^2 + 1.0225\,h_5^2 - 0.01125\,h_5^2 &= 10
\end{aligned}$$

Simplificando y reordenando

$$\begin{aligned}
1.0225\,h_2^2 - 0.01125\,h_3^2 &= 9.9775\\
-0.01125\,h_2^2 + 1.0225\,h_3^2 - 0.01125\,h_4^2 &= 10\\
-0.01125\,h_3^2 + 1.0225\,h_4^2 - 0.01125\,h_5^2 &= 10\\
-0.01125\,h_4^2 + 1.01125\,h_5^2 &= 10
\end{aligned}$$

En forma matricial

$$\begin{bmatrix} 1.0225 & -0.01125 & 0 & 0\\ -0.01125 & 1.0225 & -0.01125 & 0\\ 0 & -0.01125 & 1.0225 & -0.01125\\ 0 & 0 & -0.01125 & 1.01125 \end{bmatrix}\begin{bmatrix} h_2^2\\ h_3^2\\ h_4^2\\ h_5^2 \end{bmatrix} = \begin{bmatrix} 9.9775\\ 10\\ 10\\ 10 \end{bmatrix}$$

Resolviendo

$$\begin{bmatrix} h_2^2 \\ h_3^2 \\ h_4^2 \\ h_5^2 \end{bmatrix} = \begin{bmatrix} 9.86795 \\ 9.99854 \\ 9.99998 \\ 9.99999 \end{bmatrix}$$

3	4	h_2^3	h_3^3	h_4^3	h_5^3	h_6^3
2	4	9.86795	9.99854	9.99998	9.99999	9.99999
1	4	10	10	10	10	10
	1	2	3	4	5	6

Figura 5: Matriz solución para $t = 10$ h

Para $i = 2$ y $n = 2$

$$d_2 = h_2^2 + D\frac{\Delta t}{2\Delta x^2}\left(h_1^2 - 2h_2^2 + h_3^2\right) = 9.86795 + \frac{0.0225}{2}[4 - 2(9.86795) + 9.99854] = 9.80340$$

Para $i = 3$ y $n = 2$

$$d_3 = h_3^2 + D\frac{\Delta t}{2\Delta x^2}\left(h_2^2 - 2h_3^2 + h_4^2\right) = 9.99854 + \frac{0.0225}{2}[9.86795 - 2(9.99854) + 9.99998] = 9.99708$$

Para $i = 4$ y $n = 2$

$$d_4 = h_4^2 + D\frac{\Delta t}{2\Delta x^2}\left(h_3^2 - 2h_4^2 + h_5^2\right) = 9.99998 + \frac{0.0225}{2}[9.99854 - 2(9.99998) + 9.99999] = 9.99996$$

Para $i = 5$ y $n = 2$

$$d_5 = h_5^2 + D\frac{\Delta t}{2\Delta x^2}\left(h_4^2 - 2h_5^2 + h_6^2\right) = 9.99999 + \frac{0.0225}{2}[9.99998 - 2(9.99999) + 9.99999] = 9.99998$$

Formando un sistema de ecuaciones

$$\begin{aligned}
-0.01125\,h_1^2 + 1.0225\,h_2^2 - 0.01125\,h_3^2 &= 9.80340 \\
-0.01125\,h_2^2 + 1.0225\,h_3^2 - 0.01125\,h_4^2 &= 9.99708 \\
-0.01125\,h_3^2 + 1.0225\,h_4^2 - 0.01125\,h_5^2 &= 9.99996 \\
-0.01125\,h_4^2 + 1.0225\,h_5^2 - 0.01125\,h_6^2 &= 9.99998
\end{aligned}$$

Reemplazando $h_1^2 = 4$ y $h_6^2 = h_5^2$

$$\begin{aligned}
-0.01125(4) + 1.0225\,h_2^2 - 0.01125\,h_3^2 &= 9.80340 \\
-0.01125\,h_2^2 + 1.0225\,h_3^2 - 0.01125\,h_4^2 &= 9.99708 \\
-0.01125\,h_3^2 + 1.0225\,h_4^2 - 0.01125\,h_5^2 &= 9.99996 \\
-0.01125\,h_4^2 + 1.0225\,h_5^2 - 0.01125\,h_5^2 &= 9.99998
\end{aligned}$$

Simplificando y reordenando

$$\begin{aligned}
1.0225\,h_2^2 - 0.01125\,h_3^2 \qquad\qquad\qquad\qquad\qquad &= 9.84840\\
-\,0.01125\,h_2^2 + \;\; 1.0225\,h_3^2 - 0.01125\,h_4^2 \qquad\qquad &= 9.99708\\
-\,0.01125\,h_3^2 + \;\; 1.0225\,h_4^2 - 0.01125\,h_5^2 &= 9.99996\\
-\,0.01125\,h_4^2 + 1.01125\,h_5^2 &= 9.99998
\end{aligned}$$

En forma matricial

$$\begin{bmatrix} 1.0225 & -0.01125 & 0 & 0 \\ -0.01125 & 1.0225 & -0.01125 & 0 \\ 0 & -0.01125 & 1.0225 & -0.01125 \\ 0 & 0 & -0.01125 & 1.01125 \end{bmatrix} \begin{bmatrix} h_2^2 \\ h_3^2 \\ h_4^2 \\ h_5^2 \end{bmatrix} = \begin{bmatrix} 9.84840 \\ 9.99708 \\ 9.99996 \\ 9.99998 \end{bmatrix}$$

Resolviendo

$$\begin{bmatrix} h_2^2 \\ h_3^2 \\ h_4^2 \\ h_5^2 \end{bmatrix} = \begin{bmatrix} 9.74164 \\ 9.99430 \\ 9.99989 \\ 9.99997 \end{bmatrix}$$

3	4	9.74164	9.99430	9.99989	9.99997	9.99997
2	4	9.86792	9.99894	9.99993	9.99995	9.99995
1	4	10	10	10	10	10
	1	2	3	4	5	6

Figura 6: Matriz solución para $t = 20$ h