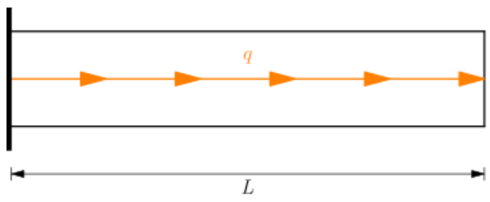
Introducción a elementos finitos
Examen final I-2016 Segunda opción

1. Resolver la estructura con E, I, A constantes por el método de Ritz



Solución

La solución exacta es un polinomio de segundo grado, la aproximación del campo de desplazamientos será

$$u(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$$

Reemplazando $u(0) = 0$

$$\alpha_0 + \alpha_1 (0) + \alpha_2 (0)^2 = 0$$

Resolviendo

$$\alpha_0 = 0$$

Reemplazando en el campo de desplazamientos

$$u = \alpha_1 x + \alpha_2 x^2$$

La normal es

$$N = q$$

La deformación unitaria es

$$\varepsilon = \frac{du}{dx} = \alpha_1 + 2\alpha_2 x$$

El funcional de energía es

$$\pi = \int_0^L \frac{1}{2} \varepsilon \, \sigma \, dV - \int_0^L N \, u \, dx = \int_0^L \frac{1}{2} EA \, \varepsilon^2 \, dx - \int_0^L N \, u \, dx$$

Reemplazando

$$\pi = \int_0^L \frac{EA}{2} (\alpha_1 + 2\alpha_2 x)^2 \, dx - \int_0^L q \left(\alpha_1 x + \alpha_2 x^2\right) dx$$

Integrando

$$\pi = \frac{EAL}{2} \alpha_1^2 - \frac{qL^2}{2} \alpha_1 + EAL^2 \, \alpha_1 \alpha_2 - \frac{qL^3}{3} \alpha_2 + \frac{2EAL^3}{3} \alpha_2^2$$

Minimizando el funcional

$$\frac{\partial \pi}{\partial \alpha_1} = EAL \, \alpha_1 + EAL^2 \, \alpha_2 - \frac{qL^2}{2} = 0$$
$$\frac{\partial \pi}{\partial \alpha_2} = EAL^2 \, \alpha_1 + \frac{4EAL^3}{3} \alpha_2 - \frac{qL^3}{3} = 0$$

Formando el sistema de ecuaciones

$$EAL \, \alpha_1 + EAL^2 \, \alpha_2 = \frac{qL^2}{2}$$
$$EAL^2 \, \alpha_1 + \frac{4EAL^3}{3} \alpha_2 = \frac{qL^3}{3}$$

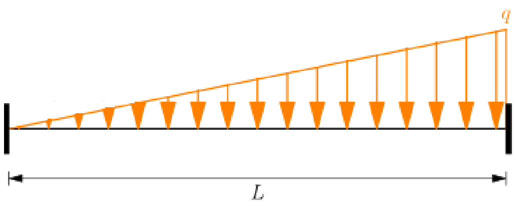
Resolviendo

$$\alpha_1 = \frac{qL}{EA}$$
$$\alpha_2 = -\frac{q}{2EA}$$

Reemplazando en u

$$u = \frac{qL}{EA} x - \frac{q}{2EA} x^2 = \frac{q}{EA} \left(Lx - \frac{1}{2} x^2\right)$$

2. Calcular las funciones de forma $\boldsymbol{N}$ y el vector de carga $\boldsymbol{F}$ mediante el método de balance de energía



Solución

Funciones de forma

Aproximación del campo de desplazamientos

$$v = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

Desplazamiento angular

$$\theta = \frac{dv}{dx} = \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2$$

Reemplazando $v(0) = v_1$, $\theta(0) = \theta_1$, $v(L) = v_2$ y $\theta(L) = \theta_2$

$$\begin{aligned} \alpha_0 + \alpha_1(0) + \alpha_2(0)^2 + \alpha_3(0)^3 &= v_1 \\ \alpha_1 + 2\alpha_2(0) + 3\alpha_3(0)^2 &= \theta_1 \\ \alpha_0 + \alpha_1(L) + \alpha_2(L)^2 + \alpha_3(L)^3 &= v_2 \\ \alpha_1 + 2\alpha_2(L) + 3\alpha_3(L)^2 &= \theta_2 \end{aligned}$$

Simplificando

$$\begin{aligned} \alpha_0 &= v_1 \\ \alpha_1 &= \theta_1 \\ \alpha_0 + L\alpha_1 + L^2\alpha_2 + L^3\alpha_3 &= v_2 \\ \alpha_1 + 2L\alpha_2 + 3L^2\alpha_3 &= \theta_2 \end{aligned}$$

En forma matricial

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$

Resolviendo

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{L^2} & -\frac{2}{L} & \frac{3}{L^2} & -\frac{1}{L} \\ \frac{2}{L^3} & \frac{1}{L^2} & -\frac{2}{L^3} & \frac{1}{L^2} \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$

Reemplazando en el campo de desplazamientos

$$v = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{L^2} & -\frac{2}{L} & \frac{3}{L^2} & -\frac{1}{L} \\ \frac{2}{L^3} & \frac{1}{L^2} & -\frac{2}{L^3} & \frac{1}{L^2} \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$

Multiplicando

$$v = \begin{bmatrix} 1 - \frac{3}{L^2}x^2 + \frac{2}{L^3}x^3 & x - \frac{2}{L}x^2 + \frac{1}{L^2}x^3 & \frac{3}{L^2}x^2 - \frac{2}{L^3}x^3 & -\frac{1}{L}x^2 + \frac{1}{L^2}x^3 \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$

Las funciones de forma son

$$\begin{aligned} N_1 &= 1 - \frac{3}{L^2}x^2 + \frac{2}{L^3}x^3 \\ N_2 &= x - \frac{2}{L}x^2 + \frac{1}{L^2}x^3 \\ N_3 &= \frac{3}{L^2}x^2 - \frac{2}{L^3}x^3 \\ N_4 &= -\frac{1}{L}x^2 + \frac{1}{L^2}x^3 \end{aligned}$$

Vector de carga

$$\boldsymbol{F} = \int_0^L f\, \boldsymbol{N}^{\mathrm{T}}\, dx$$

La carga es

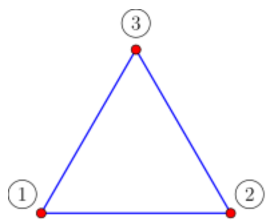
$$f = \frac{q}{L}x$$

Reemplazando e integrando

$$\boldsymbol{F} = \int_0^L \frac{q}{L} x \begin{bmatrix} 1 - \frac{3}{L^2}x^2 + \frac{2}{L^3}x^3 \\ x - \frac{2}{L}x^2 + \frac{1}{L^2}x^3 \\ \frac{3}{L^2}x^2 - \frac{2}{L^3}x^3 \\ -\frac{1}{L}x^2 + \frac{1}{L^2}x^3 \end{bmatrix} dx = \begin{bmatrix} \frac{3qL}{20} \\ \frac{qL^2}{30} \\ \frac{7qL}{20} \\ -\frac{qL^2}{20} \end{bmatrix}$$

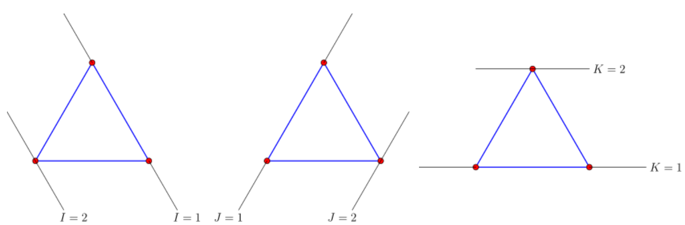
Reemplazando

3. Calcular las funciones de forma $\boldsymbol{N}$



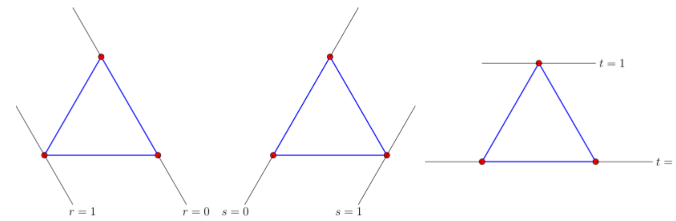
Solución

Numeración de nodos



$$① = [I_1, J_1, K_1] = [2, 1, 1] \quad ③ = [I_3, J_3, K_3] = [1, 1, 2]$$
$$② = [I_2, J_2, K_2] = [1, 2, 1]$$

Coordenadas de nodos



$$① = [r_2, s_1, t_1] = [1, 0, 0] \quad ③ = [r_1, s_1, s_2] = \left[0, 0, 1\right]$$
$$② = [r_1, s_2, t_1] = [0, 1, 0]$$

Nodo ①, $I = 2,\ J = 1,\ K = 1$

$$N_1(r, s, t) = T_2(r)T_1(s)T_1(t)$$

Reemplazando coordenadas

$$T_2(r) = \frac{r - r_1}{r_2 - r_1} = \frac{r - 0}{1 - 0} = r$$
$$T_1(s) = 1$$
$$T_1(t) = 1$$

Reemplazando polinomios

$$N_1 = r \cdot 1 \cdot 1 = r$$

Nodo ②, $I = 1,\ J = 2,\ K = 1$

$$N_2(r, s, t) = T_1(r)T_2(s)T_1(t)$$

Reemplazando coordenadas

$$T_1(r) = 1$$
$$T_2(s) = \frac{s - s_1}{s_2 - s_1} = \frac{s - 0}{1 - 0} = s$$
$$T_1(t) = 1$$

Reemplazando polinomios

$$N_2 = 1 \cdot s \cdot 1 = s$$

Nodo 3, $I = 1,\ J = 1,\ K = 2$

$$N_3(r, s, t) = T_1(r)T_1(s)T_2(t)$$

Reemplazando coordenadas

$$\begin{aligned}
T_1(r) &= 1 \\
T_1(s) &= 1 \\
T_2(t) &= \frac{t - t_1}{t_2 - t_1} = \frac{t - 0}{1 - 0} = t
\end{aligned}$$

Reemplazando polinomios

$$N_3 = 1 \cdot 1 \cdot t = t$$

4. Defina que es funcional

Solución

Un funcional es una función de funciones