Esquema de Crank-Nicolson

Hallar el perfil de flujo usando $\Delta x = 40$ m, $\Delta t = 10$ h y $D = 1 \times 10^{-3}$ m^2/s, para un tiempo final de 20 h

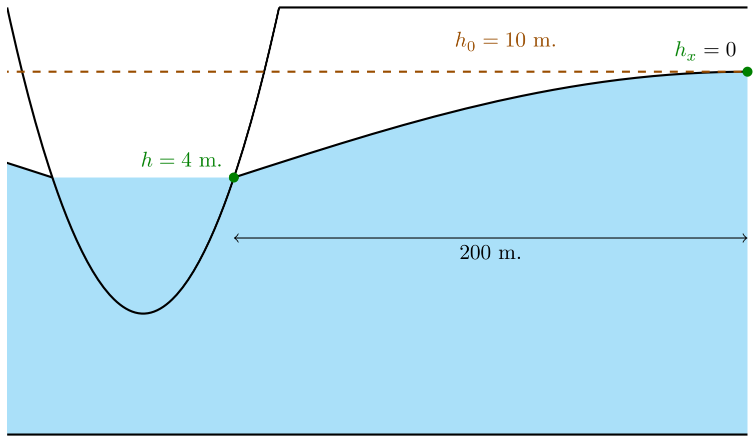

Figura 1: Representación gráfica del ejemplo

La ecuación y las condiciones de contorno serán

$$\frac{\partial h}{\partial t} - D\,\frac{\partial^2 h}{\partial x^2} = 0 \tag{1}$$
$$h(x, 0) = 10 \tag{2}$$
$$h(0, t) = 4 \tag{3}$$
$$h_x(200, t) = 0 \tag{4}$$

Discretización espacial

$$N_{\text{elementos}} = \frac{L}{\Delta x} = \frac{200}{40} = 5$$
$$N_{\text{puntos}} = N_{\text{elementos}} + 1 = 5 + 1 = 6$$

Discretización temporal

$$N_{\text{elementos}} = \frac{t}{\Delta t} = \frac{20}{10} = 2$$
$$N_{\text{puntos}} = N_{\text{elementos}} + 1 = 2 + 1 = 3$$

Discretización numérica

$$\frac{\partial h}{\partial t} = \frac{h_i^{n+1} - h_i^n}{\Delta t}$$

a partir del esquema θ o esquema generalizado de Crank-Nicolson

$$\frac{\partial^2 h}{\partial x^2} = \theta\left(\frac{h_{i-1}^{n+1} - 2h_i^{n+1} + h_{i+1}^{n+1}}{\Delta x^2}\right) + (1-\theta)\left(\frac{h_{i-1}^{n} - 2h_i^{n} + h_{i+1}^{n}}{\Delta x^2}\right)$$

reemplazando $\theta = \frac{1}{2}$

$$\frac{\partial^2 h}{\partial x^2} = \frac{h_{i-1}^{n+1} - 2h_i^{n+1} + h_{i+1}^{n+1}}{2\Delta x^2} + \frac{h_{i-1}^{n} - 2h_i^{n} + h_{i+1}^{n}}{2\Delta x^2}$$

Reemplazando en (1)

$$\frac{h_i^{n+1} - h_i^n}{\Delta t} - D\left(\frac{h_{i-1}^{n+1} - 2h_i^{n+1} + h_{i+1}^{n+1}}{2\Delta x^2} + \frac{h_{i-1}^{n} - 2h_i^{n} + h_{i+1}^{n}}{2\Delta x^2}\right) = 0$$

Reordenando

$$-D\frac{\Delta t}{2\Delta x^2}h_{i-1}^{n+1} + \left(1 + D\frac{\Delta t}{\Delta x^2}\right)h_i^{n+1} - D\frac{\Delta t}{2\Delta x^2}h_{i+1}^{n+1} = h_i^n + D\frac{\Delta t}{2\Delta x^2}\left(h_{i-1}^n - 2h_i^n + h_{i+1}^n\right)$$

Realizando un cambio de variable

$$\begin{aligned} a &= -D\frac{\Delta t}{2\Delta x^2} \\ b &= 1 + D\frac{\Delta t}{\Delta x^2} \\ c &= -D\frac{\Delta t}{2\Delta x^2} \\ d_i &= h_i^n + D\frac{\Delta t}{2\Delta x^2}\left(h_{i-1}^n - 2h_i^n + h_{i+1}^n\right) \end{aligned}$$

El esquema será

$$a\,h_{i-1}^{n+1} + b\,h_i^{n+1} + c\,h_{i+1}^{n+1} = d_i$$

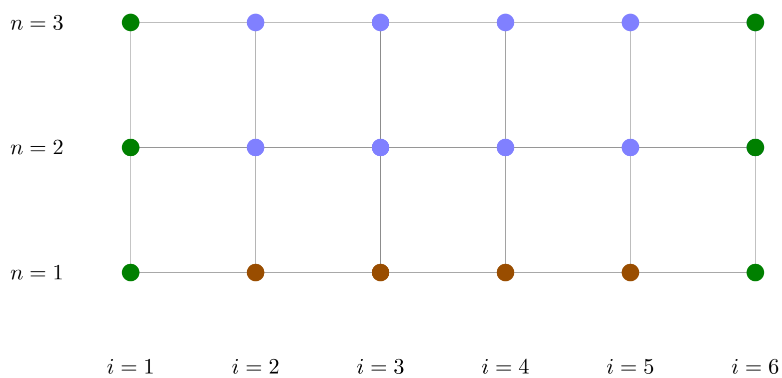

Figura 2: Mallado

3	h_1^3	h_2^3	h_3^3	h_4^3	h_5^3	h_6^3
2	h_1^2	h_2^2	h_3^2	h_4^2	h_5^2	h_6^2
1	h_1^1	h_2^1	h_3^1	h_4^1	h_5^1	h_6^1
	1	2	3	4	5	6

Figura 3: Matriz solución

El esquema es incondicionalmente estable para cualquier λ

$$D\frac{\Delta t}{\Delta x^2} = 0.001\left(\frac{36000}{40^2}\right) = 0.0225$$

Reemplazando las condiciones de contorno, para $i = 1$ y $n = 1, 2, 3$

$$h_1^1 = 4$$
$$h_1^2 = 4$$
$$h_1^3 = 4$$

Para $i = 2, 3, 4, 5$ y $n = 1$

$$h_2^1 = 10$$
$$h_3^1 = 10$$
$$h_4^1 = 10$$
$$h_5^1 = 10$$

Para $i = 6$ y $n = 1$, usando un esquema hacia atrás

$$h_x = \frac{\partial h}{\partial x} = \frac{h_6^1 - h_5^1}{\Delta x} = 0$$

reordenando

$$\begin{aligned} h_6^1 &= h_5^1 \\ &= 10 \end{aligned}$$

3	4	h_2^3	h_3^3	h_4^3	h_5^3	h_6^3
2	4	h_2^2	h_3^2	h_4^2	h_5^2	h_6^2
1	4	10	10	10	10	10
	1	2	3	4	5	6

Figura 4: Matriz solución para $t = 0$ h

Las constantes a, b, c serán

$$\begin{aligned} a &= -\frac{0.0225}{2} = -0.01125 \\ b &= 1 + 0.0225 = 1.0225 \\ c &= -\frac{0.0225}{2} = -0.01125 \end{aligned}$$

Usando el esquema elegido, para $i = 2$ y $n = 1$

$$
\begin{aligned}
-0.01125h_1^2 + 1.0225h_2^2 - 0.01125h_3^2 &= d_2 \\
d_2 = h_2^1 + D\frac{\Delta t}{2\Delta x^2}\left(h_1^1 - 2h_2^1 + h_3^1\right) &= 10 + \frac{0.0225}{2}[4 - 2(10) + 10] = 9.9325
\end{aligned}
$$

Para $i = 3$ y $n = 1$

$$
\begin{aligned}
-0.01125h_2^2 + 1.0225h_3^2 - 0.01125h_4^2 &= d_3 \\
d_3 = h_3^1 + D\frac{\Delta t}{2\Delta x^2}\left(h_2^1 - 2h_3^1 + h_4^1\right) &= 10 + \frac{0.0225}{2}[10 - 2(10) + 10] = 10
\end{aligned}
$$

Para $i = 4$ y $n = 1$

$$
\begin{aligned}
-0.01125h_3^2 + 1.0225h_4^2 - 0.01125h_5^2 &= d_4 \\
d_4 = h_4^1 + D\frac{\Delta t}{2\Delta x^2}\left(h_3^1 - 2h_4^1 + h_5^1\right) &= 10 + \frac{0.0225}{2}[10 - 2(10) + 10] = 10
\end{aligned}
$$

Para $i = 5$ y $n = 1$

$$
\begin{aligned}
-0.01125h_4^2 + 1.0225h_5^2 - 0.01125h_6^2 &= d_5 \\
d_5 = h_5^1 + D\frac{\Delta t}{2\Delta x^2}\left(h_4^1 - 2h_5^1 + h_6^1\right) &= 10 + \frac{0.0225}{2}[10 - 2(10) + 10] = 10
\end{aligned}
$$

Formando un sistema de ecuaciones

$$
\begin{array}{rcl}
-0.01125\,h_1^2 + 1.0225\,h_2^2 - 0.01125\,h_3^2 & & = 9.9325 \\
-0.01125\,h_2^2 + 1.0225\,h_3^2 - 0.01125\,h_4^2 & & = 10 \\
-0.01125\,h_3^2 + 1.0225\,h_4^2 - 0.01125\,h_5^2 & & = 10 \\
-0.01125\,h_4^2 + 1.0225\,h_5^2 - 0.01125\,h_6^2 & & = 10
\end{array}
$$

En forma matricial

$$
\begin{bmatrix}
-0.01125 & 1.0225 & -0.01125 & 0 & 0 & 0 \\
0 & -0.01125 & 1.0225 & -0.01125 & 0 & 0 \\
0 & 0 & -0.01125 & 1.0225 & -0.01125 & 0 \\
0 & 0 & 0 & -0.01125 & 1.0225 & -0.01125
\end{bmatrix}
\begin{bmatrix} h_1^2 \\ h_2^2 \\ h_3^2 \\ h_4^2 \\ h_5^2 \\ h_6^2 \end{bmatrix}
=
\begin{bmatrix} 9.9325 \\ 10 \\ 10 \\ 10 \end{bmatrix}
$$

Agregando las dos ecuaciones faltantes

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-0.01125 & 1.0225 & -0.01125 & 0 & 0 & 0 \\
0 & -0.01125 & 1.0225 & -0.01125 & 0 & 0 \\
0 & 0 & -0.01125 & 1.0225 & -0.01125 & 0 \\
0 & 0 & 0 & -0.01125 & 1.0225 & -0.01125 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} h_1^2 \\ h_2^2 \\ h_3^2 \\ h_4^2 \\ h_5^2 \\ h_6^2 \end{bmatrix}
=
\begin{bmatrix} 4 \\ 9.9325 \\ 10 \\ 10 \\ 10 \\ h_6^2 \end{bmatrix}
$$

El sistema anterior puede transformarse en una tabla para aplicar el algoritmo de Thomas

	a	b	c	d	e	f
1		1	0	4		
2	-0.01125	1.0225	-0.01125	9.9325		
3	-0.01125	1.0225	-0.01125	10		
4	-0.01125	1.0225	-0.01125	10		
5	-0.01125	1.0225	-0.01125	10		
6	0	1		h_6^2		

Constantes e y f, hacia adelante

$$e_1 = \frac{d_1}{b_1} = \frac{4}{1} = 4 \qquad f_1 = -\frac{c_1}{b_1} = -\frac{0}{1} = 0$$

$$e_2 = \frac{d_2 - a_2 e_1}{b_2 + a_2 f_1} = \frac{9.9325 - (-0.01125)(4)}{1.0225 + (-0.01125)(0)} = 9.75794 \qquad f_2 = -\frac{c_2}{b_2 + a_2 f_1} = -\frac{-0.01125}{1.0225 + (-0.01125)(0)} = 0.01100$$

$$e_3 = \frac{d_3 - a_3 e_2}{b_3 + a_3 f_2} = \frac{10 - (-0.01125)(9.75794)}{1.0225 + (-0.01125)(0.01100)} = 9.88850 \qquad f_3 = -\frac{c_3}{b_3 + a_3 f_2} = -\frac{-0.01125}{1.0225 + (-0.01125)(0.01100)} = 0.01100$$

$$e_4 = \frac{d_4 - a_4 e_3}{b_4 + a_4 f_3} = \frac{10 - (-0.01125)(9.88850)}{1.0225 + (-0.01125)(0.01100)} = 9.88994 \qquad f_4 = -\frac{c_4}{b_4 + a_4 f_3} = -\frac{-0.01125}{1.0225 + (-0.01125)(0.01100)} = 0.01100$$

$$e_5 = \frac{d_5 - a_5 e_4}{b_5 + a_5 f_4} = \frac{10 - (-0.01125)(9.88994)}{1.0225 + (-0.01125)(0.01100)} = 9.88996 \qquad f_5 = -\frac{c_5}{b_5 + a_5 f_4} = -\frac{-0.01125}{1.0225 + (-0.01125)(0.01100)} = 0.01100$$

Incógnitas, hacia atrás

$$\begin{aligned}
h_6^2 &= h_6^2 \\
h_5^2 &= e_5 + f_5 h_6^2 \\
h_4^2 &= e_4 + f_4 h_5^2 \\
h_3^2 &= e_3 + f_3 h_4^2 \\
h_2^2 &= e_2 + f_2 h_3^2 \\
h_1^2 &= e_1 + f_1 h_2^2
\end{aligned}$$

Debido a la condición de contorno del lado derecho, la primera ecuación cambia

$$\begin{aligned}
&h_6^2 = h_5^2 \\
&h_5^2 = e_5 + f_5 h_5^2 = 9.88996 + 0.0110 h_5^2 = 9.99995 \\
&h_4^2 = e_4 + f_4 h_5^2 = 9.88994 + 0.0110(9.99995) = 9.99993 \\
&h_3^2 = e_3 + f_3 h_4^2 = 9.88850 + 0.0110(9.99993) = 9.99849 \\
&h_2^2 = e_2 + f_2 h_3^2 = 9.75794 + 0.0110(9.99849) = 9.86792 \\
&h_1^2 = e_1 + f_1 h_2^2 = 4 + 0(9.86792) = 4
\end{aligned}$$

3	4	h_2^3	h_3^3	h_4^3	h_5^3	h_6^3
2	4	9.86792	9.99894	9.99993	9.99995	9.99995
1	4	10	10	10	10	10
	1	2	3	4	5	6

Figura 5: Matriz solución para $t = 10$ h

Para $i = 2$ y $n = 2$

$$d_2 = h_2^2 + D\frac{\Delta t}{2\Delta x^2}\left(h_1^2 - 2h_2^2 + h_3^2\right) = 9.86792 + \frac{0.0225}{2}[4 - 2(9.86792) + 9.99894] = 9.80337$$

Para $i = 3$ y $n = 2$

$$d_3 = h_3^2 + D\frac{\Delta t}{2\Delta x^2}\left(h_2^2 - 2h_3^2 + h_4^2\right) = 9.99894 + \frac{0.0225}{2}[9.86792 - 2(9.99894) + 9.99993] = 9.99747$$

Para $i = 4$ y $n = 2$

$$d_4 = h_4^2 + D\frac{\Delta t}{2\Delta x^2}\left(h_3^2 - 2h_4^2 + h_5^2\right) = 9.99993 + \frac{0.0225}{2}[9.99894 - 2(9.99993) + 9.99995] = 9.99991$$

Para $i = 5$ y $n = 2$

$$d_5 = h_5^2 + D\frac{\Delta t}{2\Delta x^2}\left(h_4^2 - 2h_5^2 + h_6^2\right) = 9.99995 + \frac{0.0225}{2}[9.99993 - 2(9.99995) + 9.99995] = 9.99994$$

En forma matricial

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -0.01125 & 1.0225 & -0.01125 & 0 & 0 & 0 \\ 0 & -0.01125 & 1.0225 & -0.01125 & 0 & 0 \\ 0 & 0 & -0.01125 & 1.0225 & -0.01125 & 0 \\ 0 & 0 & 0 & -0.01125 & 1.0225 & -0.01125 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} h_1^3 \\ h_2^3 \\ h_3^3 \\ h_4^3 \\ h_5^3 \\ h_6^3 \end{bmatrix} = \begin{bmatrix} 4 \\ 9.80337 \\ 9.99747 \\ 9.99991 \\ 9.99994 \\ h_6^3 \end{bmatrix}$$

En tabla para aplicar el algoritmo de Thomas

	a	b	c	d	e	f
1		1	0	4		
2	-0.01125	1.0225	-0.01125	9.80337		
3	-0.01125	1.0225	-0.01125	9.99747		
4	-0.01125	1.0225	-0.01125	9.99991		
5	-0.01125	1.0225	-0.01125	9.99994		
6	0	1		h_6^3		

Constantes e y f, hacia adelante

$$e_1 = \frac{d_1}{b_1} = \frac{4}{1} = 4$$

$$e_2 = \frac{d_2 - a_2 e_1}{b_2 + a_2 f_1} = \frac{9.80337 - (-0.01125)(4)}{1.0225 + (-0.01125)(0)} = 9.63165$$

$$e_3 = \frac{d_3 - a_3 e_2}{b_3 + a_3 f_2} = \frac{9.99747 - (-0.01125)(9.63165)}{1.0225 + (-0.01125)(0.01100)} = 9.88464$$

$$e_4 = \frac{d_4 - a_4 e_3}{b_4 + a_4 f_3} = \frac{9.99991 - (-0.01125)(9.88464)}{1.0225 + (-0.01125)(0.01100)} = 9.88981$$

$$e_5 = \frac{d_5 - a_5 e_4}{b_5 + a_5 f_4} = \frac{9.99994 - (-0.01125)(9.88981)}{1.0225 + (-0.01125)(0.01100)} = 9.88990$$

$$f_1 = -\frac{c_1}{b_1} = -\frac{0}{1} = 0$$

$$f_2 = -\frac{c_2}{b_2 + a_2 f_1} = -\frac{-0.01125}{1.0225 + (-0.01125)(0)} = 0.01100$$

$$f_3 = -\frac{c_3}{b_3 + a_3 f_2} = -\frac{-0.01125}{1.0225 + (-0.01125)(0.01100)} = 0.01100$$

$$f_4 = -\frac{c_4}{b_4 + a_4 f_3} = -\frac{-0.01125}{1.0225 + (-0.01125)(0.01100)} = 0.01100$$

$$f_5 = -\frac{c_5}{b_5 + a_5 f_4} = -\frac{-0.01125}{1.0225 + (-0.01125)(0.01100)} = 0.01100$$

Incógnitas, hacia atrás

$$h_6^3 = h_5^3$$
$$h_5^3 = e_5 + f_5 h_5^3 = 9.88990 + 0.0110 h_5^3 = 9.99989$$
$$h_4^3 = e_4 + f_4 h_5^3 = 9.88981 + 0.0110(9.99989) = 9.99980$$
$$h_3^3 = e_3 + f_3 h_4^3 = 9.88464 + 0.0110(9.99980) = 9.99463$$
$$h_2^3 = e_2 + f_2 h_3^3 = 9.63165 + 0.0110(9.99463) = 9.74159$$
$$h_1^3 = e_1 + f_1 h_2^3 = 4 + 0(9.74159) = 4$$

3	4	9.74159	9.99463	9.99980	9.99989	9.99989
2	4	9.86792	9.99894	9.99993	9.99995	9.99995
1	4	10	10	10	10	10
	1	2	3	4	5	6

Figura 6: Matriz solución para $t = 20$ h