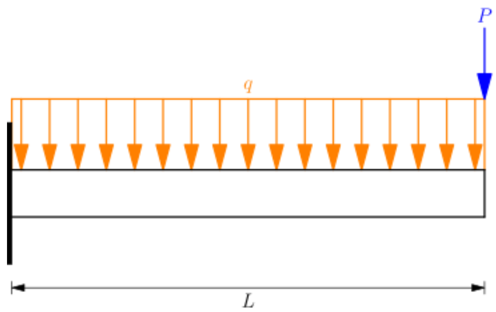
Ejemplo 5



Resolver

$$
\begin{aligned}
EI\,\frac{d^4 v}{dx^4} + q &= 0 \\
v(0) &= 0 \qquad EIv''(L) = 0 \\
v'(0) &= 0 \qquad EIv'''(L) = P
\end{aligned}
$$

Solución exacta

$$v(x) = -\frac{qL^2 + 2PL}{4EI}x^2 + \frac{qL + P}{6EI}x^3 - \frac{q}{24EI}x^4$$

Solución aproximada cúbica

La forma débil de la ecuación diferencial es

$$\int_0^L R(x)\,W(x)\,dx = \int_0^L \left(EI\,\frac{d^4\hat{v}}{dx^4} + q\right) W\,dx = 0$$

reduciendo el grado de las derivadas

$$\int_0^L \frac{d^2W}{dx^2}\,EI\,\frac{d^2\hat{v}}{dx^2}\,dx = -\int_0^L W\,q\,dx - W(L)\,EI\frac{d^3\hat{v}(L)}{dx^3} + W(0)\,EI\frac{d^3\hat{v}(0)}{dx^3} + \frac{dW(L)}{dx}\,EI\frac{d^2\hat{v}(L)}{dx^2} - \frac{dW(0)}{dx}\,EI\frac{d^2\hat{v}(0)}{dx^2}$$

usando bases cúbicas en coordenadas locales

$$v(x) \approx \hat{v}(x) = v_1\left(1 - \frac{3}{L^2}x^2 + \frac{2}{L^3}x^3\right) + \theta_1\left(x - \frac{2}{L}x^2 + \frac{1}{L^2}x^3\right) + v_2\left(\frac{3}{L^2}x^2 - \frac{2}{L^3}x^3\right) + \theta_2\left(-\frac{1}{L}x^2 + \frac{1}{L^2}x^3\right)$$

$\hat{v}_{xx}$ es

$$\frac{d^2\hat{v}}{dx^2} = \left(-\frac{6}{L^2} + \frac{12}{L^3}x\right)v_1 + \left(-\frac{4}{L} + \frac{6}{L^2}x\right)\theta_1 + \left(\frac{6}{L^2} - \frac{12}{L^3}x\right)v_2 + \left(-\frac{2}{L} + \frac{6}{L^2}x\right)\theta_2$$

las funciones ponderadas son

$$
\begin{aligned}
W_1 &= \frac{d\hat{v}}{dv_1} = 1 - \frac{3}{L^2}x^2 + \frac{2}{L^3}x^3 \\
W_2 &= \frac{d\hat{v}}{d\theta_1} = x - \frac{2}{L}x^2 + \frac{1}{L^2}x^3 \\
W_3 &= \frac{d\hat{v}}{dv_2} = \frac{3}{L^2}x^2 - \frac{2}{L^3}x^3 \\
W_4 &= \frac{d\hat{v}}{d\theta_2} = -\frac{1}{L}x^2 + \frac{1}{L^2}x^3
\end{aligned}
$$

formando el sistema de ecuaciones

$$\int_0^L \frac{d^2W_1}{dx^2}\,EI\,\frac{d^2\hat{v}}{dx^2}\,dx = -\int_0^L W_1\,q\,dx - W_1(L)\,EI\frac{d^3\hat{v}(L)}{dx^3} + W_1(0)\,EI\frac{d^3\hat{v}(0)}{dx^3} + \frac{dW_1(L)}{dx}\,EI\frac{d^2\hat{v}(L)}{dx^2} - \frac{dW_1(0)}{dx}\,EI\frac{d^2\hat{v}(0)}{dx^2}$$
$$\int_0^L \frac{d^2W_2}{dx^2}\,EI\,\frac{d^2\hat{v}}{dx^2}\,dx = -\int_0^L W_2\,q\,dx - W_2(L)\,EI\frac{d^3\hat{v}(L)}{dx^3} + W_2(0)\,EI\frac{d^3\hat{v}(0)}{dx^3} + \frac{dW_2(L)}{dx}\,EI\frac{d^2\hat{v}(L)}{dx^2} - \frac{dW_2(0)}{dx}\,EI\frac{d^2\hat{v}(0)}{dx^2}$$
$$\int_0^L \frac{d^2W_3}{dx^2}\,EI\,\frac{d^2\hat{v}}{dx^2}\,dx = -\int_0^L W_3\,q\,dx - W_3(L)\,EI\frac{d^3\hat{v}(L)}{dx^3} + W_3(0)\,EI\frac{d^3\hat{v}(0)}{dx^3} + \frac{dW_3(L)}{dx}\,EI\frac{d^2\hat{v}(L)}{dx^2} - \frac{dW_3(0)}{dx}\,EI\frac{d^2\hat{v}(0)}{dx^2}$$
$$\int_0^L \frac{d^2W_4}{dx^2}\,EI\,\frac{d^2\hat{v}}{dx^2}\,dx = -\int_0^L W_4\,q\,dx - W_4(L)\,EI\frac{d^3\hat{v}(L)}{dx^3} + W_4(0)\,EI\frac{d^3\hat{v}(0)}{dx^3} + \frac{dW_4(L)}{dx}\,EI\frac{d^2\hat{v}(L)}{dx^2} - \frac{dW_4(0)}{dx}\,EI\frac{d^2\hat{v}(0)}{dx^2}$$

funciones ponderadas y sus derivadas

$$W_1 = 1 - \frac{3}{L^2}x^2 + \frac{2}{L^3}x^3 \quad \frac{d^2W_1}{dx^2} = -\frac{6}{L^2} + \frac{12}{L^3}x \qquad W_2 = x - \frac{2}{L}x^2 + \frac{1}{L^2}x^3 \quad \frac{d^2W_2}{dx^2} = -\frac{4}{L} + \frac{6}{L^2}x$$
$$W_3 = \frac{3}{L^2}x^2 - \frac{2}{L^3}x^3 \quad \frac{d^2W_3}{dx^2} = \frac{6}{L^2} - \frac{12}{L^3}x \qquad W_4 = -\frac{1}{L}x^2 + \frac{1}{L^2}x^3 \quad \frac{d^2W_4}{dx^2} = -\frac{2}{L} + \frac{6}{L^2}x$$

valores de las funciones ponderadas en los nodos

$$W_1(L) = 0 \quad W_1(0) = 1 \qquad \frac{dW_1(L)}{dx} = 0 \quad \frac{dW_1(0)}{dx} = 0$$
$$W_2(L) = 0 \quad W_2(0) = 0 \qquad \frac{dW_2(L)}{dx} = 0 \quad \frac{dW_2(0)}{dx} = 1$$
$$W_3(L) = 1 \quad W_3(0) = 0 \qquad \frac{dW_3(L)}{dx} = 0 \quad \frac{dW_3(0)}{dx} = 0$$
$$W_4(L) = 0 \quad W_4(0) = 0 \qquad \frac{dW_4(L)}{dx} = 1 \quad \frac{dW_4(0)}{dx} = 0$$

cortante y momento en los nodos

$$EI\frac{d^3\hat{v}(L)}{dx^3} = V_2 \qquad EI\frac{d^3\hat{v}(0)}{dx^3} = V_1$$
$$EI\frac{d^2\hat{v}(L)}{dx^2} = M_2 \qquad EI\frac{d^2\hat{v}(0)}{dx^2} = M_1$$

reemplazando

$$\int_0^L \left(-\frac{6}{L^2} + \frac{12}{L^3}\right) EI\left[\left(-\frac{6}{L^2} + \frac{12}{L^3}x\right)v_1 + \left(-\frac{4}{L} + \frac{6}{L^2}x\right)\theta_1 + \left(\frac{6}{L^2} - \frac{12}{L^3}x\right)v_2 + \left(-\frac{2}{L} + \frac{6}{L^2}x\right)\theta_2\right] dx$$
$$= -\int_0^L \left(1 - \frac{3}{L^2}x^2 + \frac{2}{L^3}x^3\right) q\,dx - 0(V_2) + 1(V_1) + 0(M_2) - 0(M_1)$$
$$\int_0^L \left(-\frac{4}{L} + \frac{6}{L^2}x\right) EI\left[\left(-\frac{6}{L^2} + \frac{12}{L^3}x\right)v_1 + \left(-\frac{4}{L} + \frac{6}{L^2}x\right)\theta_1 + \left(\frac{6}{L^2} - \frac{12}{L^3}x\right)v_2 + \left(-\frac{2}{L} + \frac{6}{L^2}x\right)\theta_2\right] dx$$
$$= -\int_0^L \left(x - \frac{2}{L}x^2 + \frac{1}{L^2}x^3\right) q\,dx - 0(V_2) + 0(V_1) + 0(M_2) - 1(M_1)$$
$$\int_0^L \left(\frac{6}{L^2} - \frac{12}{L^3}x\right) EI\left[\left(-\frac{6}{L^2} + \frac{12}{L^3}x\right)v_1 + \left(-\frac{4}{L} + \frac{6}{L^2}x\right)\theta_1 + \left(\frac{6}{L^2} - \frac{12}{L^3}x\right)v_2 + \left(-\frac{2}{L} + \frac{6}{L^2}x\right)\theta_2\right] dx$$
$$= -\int_0^L \left(\frac{3}{L^2}x^2 - \frac{2}{L^3}x^3\right) q\,dx - 1(V_2) + 0(V_1) + 0(M_2) - 0(M_1)$$
$$\int_0^L \left(-\frac{2}{L} + \frac{6}{L^2}x\right) EI\left[\left(-\frac{6}{L^2} + \frac{12}{L^3}x\right)v_1 + \left(-\frac{4}{L} + \frac{6}{L^2}x\right)\theta_1 + \left(\frac{6}{L^2} - \frac{12}{L^3}x\right)v_2 + \left(-\frac{2}{L} + \frac{6}{L^2}x\right)\theta_2\right] dx$$
$$= -\int_0^L \left(-\frac{1}{L}x^2 + \frac{1}{L^2}x^3\right) q\,dx - 0(V_2) + 0(V_1) + 1(M_2) - 0(M_1)$$

integrando

$$
\begin{aligned}
\frac{EI}{L^3}(12v_1 + 6L\theta_1 - 12v_2 + 6L\theta_2) &= -\frac{qL}{2} + V_1 \\
\frac{EI}{L^3}(6Lv_1 + 4L^2\theta_1 - 6Lv_2 + 2L^2\theta_2) &= -\frac{qL^2}{12} - M_1 \\
\frac{EI}{L^3}(-12v_1 - 6L\theta_1 + 12v_2 - 6L\theta_2) &= -\frac{qL}{2} - V_2 \\
\frac{EI}{L^3}(6Lv_1 + 2L^2\theta_1 - 6Lv_2 + 4L^2\theta_2) &= \frac{qL^2}{12} + M_2
\end{aligned}
$$

en forma matricial

$$
\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} -\frac{qL}{2} \\ -\frac{qL^2}{12} \\ -\frac{qL}{2} \\ \frac{qL^2}{12} \end{bmatrix} + \begin{bmatrix} V_1 \\ -M_1 \\ -V_2 \\ M_2 \end{bmatrix}
$$

reemplazando fuerzas y desplazamientos

$$
\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ v_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} -\frac{qL}{2} \\ -\frac{qL^2}{12} \\ -\frac{qL}{2} \\ \frac{qL^2}{12} \end{bmatrix} + \begin{bmatrix} V_1 \\ -M_1 \\ -P \\ 0 \end{bmatrix}
$$

resolviendo

$$
\begin{aligned}
v_2 &= -\frac{3qL^4 + 8PL^3}{24EI} \\
\theta_2 &= -\frac{qL^3 + 3PL^2}{6EI} \\
V_1 &= \frac{qL}{2} \\
M_1 &= -\frac{qL^2}{12}
\end{aligned}
$$

reemplazando en la solución aproximada

$$
\hat{v}(x) = \left(-\frac{3qL^4 + 8PL^3}{24EI}\right)\left(\frac{3}{L^2}x^2 - \frac{2}{L^3}x^3\right) + \left(-\frac{qL^3 + 3PL^2}{6EI}\right)\left(-\frac{1}{L}x^2 + \frac{1}{L^2}x^3\right) = -\frac{5qL^2 + 12PL}{24EI}x^2 + \frac{qL + 2P}{12EI}x^3
$$