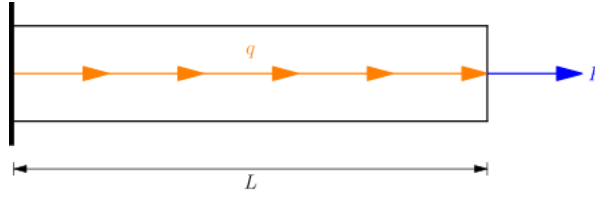


Ejemplo 3



Resolver

$$\begin{aligned} EA \frac{d^2 u}{dx^2} + q &= 0 \\ u(0) &= 0 \\ u'(L) &= \frac{P}{EA} \end{aligned}$$

Solución exacta

$$u(x) = \frac{P + qL}{EA} x - \frac{q}{2EA} x^2$$

Solución aproximada cuadrática

La forma débil de la ecuación diferencial es

$$\int_0^L R(x) W(x) dx = \int_0^L \left(EA \frac{d^2 \hat{u}}{dx^2} + q \right) W dx = 0$$

reduciendo el grado de las derivadas

$$\int_0^L \frac{dW}{dx} EA \frac{d\hat{u}}{dx} dx = \int_0^L W q dx + W(L) EA \frac{d\hat{u}(L)}{dx} - W(0) EA \frac{d\hat{u}(0)}{dx}$$

usando bases cuadráticas en coordenadas locales

$$u(x) \approx \hat{u}(x) = u_1 \left(1 - \frac{3}{L}x + \frac{2}{L^2}x^2 \right) + u_2 \left(\frac{4}{L}x - \frac{4}{L^2}x^2 \right) + u_3 \left(-\frac{1}{L}x + \frac{2}{L^2}x^2 \right)$$

\hat{u}_x es

$$\frac{d\hat{u}}{dx} = -\frac{3u_1 - 4u_2 + u_3}{L} + \frac{4u_1 - 8u_2 + 4u_3}{L^2}x$$

las funciones ponderadas son

$$W_1 = \frac{d\hat{u}}{du_1} = 1 - \frac{3}{L}x + \frac{2}{L^2}x^2$$

$$W_2 = \frac{d\hat{u}}{du_2} = \frac{4}{L}x - \frac{4}{L^2}x^2$$

$$W_3 = \frac{d\hat{u}}{du_3} = -\frac{1}{L}x + \frac{2}{L^2}x^2$$

formando el sistema de ecuaciones

$$\begin{aligned}
\int_0^L \frac{dW_1}{dx} EA \frac{d\hat{u}}{dx} dx &= \int_0^L W_1 q dx + W_1(L) EA \frac{d\hat{u}(L)}{dx} - W_1(0) EA \frac{d\hat{u}(0)}{dx} \\
\int_0^L \frac{dW_2}{dx} EA \frac{d\hat{u}}{dx} dx &= \int_0^L W_2 q dx + W_2(L) EA \frac{d\hat{u}(L)}{dx} - W_2(0) EA \frac{d\hat{u}(0)}{dx} \\
\int_0^L \frac{dW_3}{dx} EA \frac{d\hat{u}}{dx} dx &= \int_0^L W_3 q dx + W_3(L) EA \frac{d\hat{u}(L)}{dx} - W_3(0) EA \frac{d\hat{u}(0)}{dx}
\end{aligned}$$

funciones ponderadas y sus derivadas

$$\begin{aligned}
W_1 &= 1 - \frac{3}{L}x + \frac{2}{L^2}x^2 & \frac{dW_1}{dx} &= -\frac{3}{L} + \frac{4}{L^2}x & W_2 &= \frac{4}{L}x - \frac{4}{L^2}x^2 & \frac{dW_2}{dx} &= \frac{4}{L} - \frac{8}{L^2}x \\
W_3 &= -\frac{1}{L}x + \frac{2}{L^2}x^2 & \frac{dW_3}{dx} &= -\frac{1}{L} + \frac{4}{L^2}x
\end{aligned}$$

valores de las funciones ponderadas en los nodos

$$\begin{aligned}
W_1(L) &= 0 & W_1(0) &= 1 & W_2(L) &= 0 & W_2(0) &= 0 \\
W_3(L) &= 1 & W_3(0) &= 0
\end{aligned}$$

fuerzas en los nodos

$$EA \frac{d\hat{u}(L)}{dx} = F_3 \quad EA \frac{d\hat{u}(0)}{dx} = F_1$$

reemplazando

$$\begin{aligned}
\int_0^L \left(\frac{3}{L} + \frac{4}{L^2}x \right) EA \left(-\frac{3u_1 - 4u_2 + u_3}{L} + \frac{4u_1 - 8u_2 + 4u_3}{L^2}x \right) dx &= \int_0^L \left(1 - \frac{3}{L}x + \frac{2}{L^2}x^2 \right) q dx + 0(F_3) - 1(F_1) \\
\int_0^L \left(\frac{4}{L} - \frac{8}{L^2}x \right) EA \left(-\frac{3u_1 - 4u_2 + u_3}{L} + \frac{4u_1 - 8u_2 + 4u_3}{L^2}x \right) dx &= \int_0^L \left(\frac{4}{L}x - \frac{4}{L^2}x^2 \right) q dx + 0(F_3) - 0(F_1) \\
\int_0^L \left(-\frac{1}{L} + \frac{4}{L^2}x \right) EA \left(-\frac{3u_1 - 4u_2 + u_3}{L} + \frac{4u_1 - 8u_2 + 4u_3}{L^2}x \right) dx &= \int_0^L \left(-\frac{1}{L}x + \frac{2}{L^2}x^2 \right) q dx + 1(F_3) - 0(F_1)
\end{aligned}$$

reordenando

$$\begin{aligned}
\frac{EA}{L} \int_0^L \left(3 + \frac{4}{L}x \right) \left(-\frac{3u_1 - 4u_2 + u_3}{L} + \frac{4u_1 - 8u_2 + 4u_3}{L^2}x \right) dx &= \int_0^L \left(1 - \frac{3}{L}x + \frac{2}{L^2}x^2 \right) q dx - F_1 \\
\frac{EA}{L} \int_0^L \left(4 - \frac{8}{L}x \right) \left(-\frac{3u_1 - 4u_2 + u_3}{L} + \frac{4u_1 - 8u_2 + 4u_3}{L^2}x \right) dx &= \int_0^L \left(\frac{4}{L}x - \frac{4}{L^2}x^2 \right) q dx \\
\frac{EA}{L} \int_0^L \left(-1 + \frac{4}{L}x \right) \left(-\frac{3u_1 - 4u_2 + u_3}{L} + \frac{4u_1 - 8u_2 + 4u_3}{L^2}x \right) dx &= \int_0^L \left(-\frac{1}{L}x + \frac{2}{L^2}x^2 \right) q dx + F_3
\end{aligned}$$

integrando

$$\begin{aligned}
\frac{EA}{L} \left(\frac{7}{3}u_1 - \frac{8}{3}u_2 + \frac{1}{3}u_3 \right) &= \frac{qL}{6} - F_1 \\
\frac{EA}{L} \left(-\frac{8}{3}u_1 + \frac{16}{3}u_2 - \frac{8}{3}u_3 \right) &= \frac{2qL}{3} \\
\frac{EA}{L} \left(\frac{1}{3}u_1 - \frac{8}{3}u_2 + \frac{7}{3}u_3 \right) &= \frac{qL}{6} + F_3
\end{aligned}$$

en forma matricial

$$\frac{EA}{L} \begin{bmatrix} \frac{7}{3} & -\frac{8}{3} & \frac{1}{3} \\ -\frac{8}{3} & \frac{16}{3} & -\frac{8}{3} \\ \frac{1}{3} & -\frac{8}{3} & \frac{7}{3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{qL}{6} \\ \frac{2qL}{3} \\ \frac{qL}{6} \end{bmatrix} + \begin{bmatrix} -F_1 \\ 0 \\ F_3 \end{bmatrix}$$

reemplazando fuerzas y desplazamientos

$$\frac{EA}{L} \begin{bmatrix} \frac{7}{3} & -\frac{8}{3} & \frac{1}{3} \\ -\frac{8}{3} & \frac{16}{3} & -\frac{8}{3} \\ \frac{1}{3} & -\frac{8}{3} & \frac{7}{3} \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{qL}{6} \\ \frac{2qL}{3} \\ \frac{qL}{6} \end{bmatrix} + \begin{bmatrix} -F_1 \\ 0 \\ P \end{bmatrix}$$

resolviendo

$$\begin{aligned} u_2 &= \frac{3qL^2 + 4PL}{8EA} \\ u_3 &= \frac{qL^2 + 2PL}{2EA} \\ F_1 &= P + qL \end{aligned}$$

reemplazando en la solución aproximada

$$\hat{u}(x) = \left(\frac{3qL^2 + 4PL}{8EA} \right) \left(\frac{4}{L}x - \frac{4}{L^2}x^2 \right) + \left(\frac{qL^2 + 2PL}{2EA} \right) \left(-\frac{1}{L}x + \frac{2}{L^2}x^2 \right) = \frac{P + qL}{EA}x - \frac{q}{2EA}x^2$$