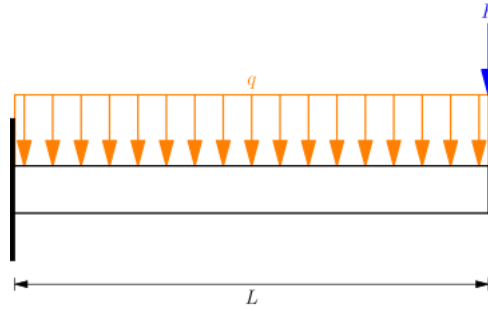


Ejemplo 5



Resolver

$$EI \frac{d^4 v}{dx^4} + q = 0$$

$$\begin{aligned} v(0) &= 0 & EI v''(L) &= 0 \\ v'(0) &= 0 & EI v'''(L) &= P \end{aligned}$$

Solución exacta

$$v(x) = -\frac{qL^2 + 2PL}{4EI}x^2 + \frac{qL + P}{6EI}x^3 - \frac{q}{24EI}x^4$$

Solución aproximada cúbica

La forma débil de la ecuación diferencial es

$$\int_0^L R(x) W(x) dx = \int_0^L \left(EI \frac{d^4 \hat{v}}{dx^4} + q \right) W dx = 0$$

reduciendo el grado de las derivadas

$$\int_0^L \frac{d^2 W}{dx^2} EI \frac{d^2 \hat{v}}{dx^2} dx = - \int_0^L W q dx - W(L) EI \frac{d^3 \hat{v}(L)}{dx^3} + W(0) EI \frac{d^3 \hat{v}(0)}{dx^3} + \frac{dW(L)}{dx} EI \frac{d^2 \hat{v}(L)}{dx^2} - \frac{dW(0)}{dx} EI \frac{d^2 \hat{v}(0)}{dx^2}$$

usando bases cúbicas en coordenadas locales

$$v(x) \approx \hat{v}(x) = v_1 \left(1 - \frac{3}{L^2}x^2 + \frac{2}{L^3}x^3 \right) + \theta_1 \left(x - \frac{2}{L}x^2 + \frac{1}{L^2}x^3 \right) + v_2 \left(\frac{3}{L^2}x^2 - \frac{2}{L^3}x^3 \right) + \theta_2 \left(-\frac{1}{L}x^2 + \frac{1}{L^2}x^3 \right)$$

\hat{v}_{xx} es

$$\frac{d^2 \hat{v}}{dx^2} = \left(-\frac{6}{L^2} + \frac{12}{L^3}x \right) v_1 + \left(-\frac{4}{L} + \frac{6}{L^2}x \right) \theta_1 + \left(\frac{6}{L^2} - \frac{12}{L^3}x \right) v_2 + \left(-\frac{2}{L} + \frac{6}{L^2}x \right) \theta_2$$

las funciones ponderadas son

$$W_1 = \frac{d\hat{v}}{dv_1} = 1 - \frac{3}{L^2}x^2 + \frac{2}{L^3}x^3$$

$$W_2 = \frac{d\hat{v}}{d\theta_1} = x - \frac{2}{L}x^2 + \frac{1}{L^2}x^3$$

$$W_3 = \frac{d\hat{v}}{dv_2} = \frac{3}{L^2}x^2 - \frac{2}{L^3}x^3$$

$$W_4 = \frac{d\hat{v}}{d\theta_2} = -\frac{1}{L}x^2 + \frac{1}{L^2}x^3$$

formando el sistema de ecuaciones

$$\begin{aligned}\int_0^L \frac{d^2 W_1}{dx^2} EI \frac{d^2 \hat{v}}{dx^2} dx &= - \int_0^L W_1 q dx - W_1(L) EI \frac{d^3 \hat{v}(L)}{dx^3} + W_1(0) EI \frac{d^3 \hat{v}(0)}{dx^3} + \frac{dW_1(L)}{dx} EI \frac{d^2 \hat{v}(L)}{dx^2} - \frac{dW_1(0)}{dx} EI \frac{d^2 \hat{v}(0)}{dx^2} \\ \int_0^L \frac{d^2 W_2}{dx^2} EI \frac{d^2 \hat{v}}{dx^2} dx &= - \int_0^L W_2 q dx - W_2(L) EI \frac{d^3 \hat{v}(L)}{dx^3} + W_2(0) EI \frac{d^3 \hat{v}(0)}{dx^3} + \frac{dW_2(L)}{dx} EI \frac{d^2 \hat{v}(L)}{dx^2} - \frac{dW_2(0)}{dx} EI \frac{d^2 \hat{v}(0)}{dx^2} \\ \int_0^L \frac{d^2 W_3}{dx^2} EI \frac{d^2 \hat{v}}{dx^2} dx &= - \int_0^L W_3 q dx - W_3(L) EI \frac{d^3 \hat{v}(L)}{dx^3} + W_3(0) EI \frac{d^3 \hat{v}(0)}{dx^3} + \frac{dW_3(L)}{dx} EI \frac{d^2 \hat{v}(L)}{dx^2} - \frac{dW_3(0)}{dx} EI \frac{d^2 \hat{v}(0)}{dx^2} \\ \int_0^L \frac{d^2 W_4}{dx^2} EI \frac{d^2 \hat{v}}{dx^2} dx &= - \int_0^L W_4 q dx - W_4(L) EI \frac{d^3 \hat{v}(L)}{dx^3} + W_4(0) EI \frac{d^3 \hat{v}(0)}{dx^3} + \frac{dW_4(L)}{dx} EI \frac{d^2 \hat{v}(L)}{dx^2} - \frac{dW_4(0)}{dx} EI \frac{d^2 \hat{v}(0)}{dx^2}\end{aligned}$$

funciones ponderadas y sus derivadas

$$\begin{aligned}W_1 &= 1 - \frac{3}{L^2}x^2 + \frac{2}{L^3}x^3 & \frac{d^2 W_1}{dx^2} &= -\frac{6}{L^2} + \frac{12}{L^3}x & W_2 &= x - \frac{2}{L}x^2 + \frac{1}{L^2}x^3 & \frac{d^2 W_2}{dx^2} &= -\frac{4}{L} + \frac{6}{L^2}x \\ W_3 &= \frac{3}{L^2}x^2 - \frac{2}{L^3}x^3 & \frac{d^2 W_3}{dx^2} &= \frac{6}{L^2} - \frac{12}{L^3}x & W_4 &= -\frac{1}{L}x^2 + \frac{1}{L^2}x^3 & \frac{d^2 W_4}{dx^2} &= -\frac{2}{L} + \frac{6}{L^2}x\end{aligned}$$

valores de las funciones ponderadas en los nodos

$$\begin{aligned}W_1(L) &= 0 & W_1(0) &= 1 & \frac{dW_1(L)}{dx} &= 0 & \frac{dW_1(0)}{dx} &= 0 \\ W_2(L) &= 0 & W_2(0) &= 0 & \frac{dW_2(L)}{dx} &= 0 & \frac{dW_2(0)}{dx} &= 1 \\ W_3(L) &= 1 & W_3(0) &= 0 & \frac{dW_3(L)}{dx} &= 0 & \frac{dW_3(0)}{dx} &= 0 \\ W_4(L) &= 0 & W_4(0) &= 0 & \frac{dW_4(L)}{dx} &= 1 & \frac{dW_4(0)}{dx} &= 0\end{aligned}$$

cortante y momento en los nodos

$$\begin{aligned}EI \frac{d^3 \hat{v}(L)}{dx^3} &= V_2 & EI \frac{d^3 \hat{v}(0)}{dx^3} &= V_1 \\ EI \frac{d^2 \hat{v}(L)}{dx^2} &= M_2 & EI \frac{d^2 \hat{v}(0)}{dx^2} &= M_1\end{aligned}$$

reemplazando

$$\begin{aligned}&\int_0^L \left(1 - \frac{3}{L^2}x^2 + \frac{2}{L^3}x^3\right) EI \left[\left(-\frac{6}{L^2} + \frac{12}{L^3}x\right) v_1 + \left(-\frac{4}{L} + \frac{6}{L^2}x\right) \theta_1 + \left(\frac{6}{L^2} - \frac{12}{L^3}x\right) v_2 + \left(-\frac{2}{L} + \frac{6}{L^2}x\right) \theta_2 \right] dx \\ &= - \int_0^L \left(1 - \frac{3}{L^2}x^2 + \frac{2}{L^3}x^3\right) q dx - 0(V_2) + 1(V_1) + 0(M_2) - 0(M_1) \\ &\int_0^L \left(x - \frac{2}{L}x^2 + \frac{1}{L^2}x^3\right) EI \left[\left(-\frac{6}{L^2} + \frac{12}{L^3}x\right) v_1 + \left(-\frac{4}{L} + \frac{6}{L^2}x\right) \theta_1 + \left(\frac{6}{L^2} - \frac{12}{L^3}x\right) v_2 + \left(-\frac{2}{L} + \frac{6}{L^2}x\right) \theta_2 \right] dx \\ &= - \int_0^L \left(x - \frac{2}{L}x^2 + \frac{1}{L^2}x^3\right) q dx - 0(V_2) + 0(V_1) + 0(M_2) - 1(M_1) \\ &\int_0^L \left(\frac{3}{L^2}x^2 - \frac{2}{L^3}x^3\right) EI \left[\left(-\frac{6}{L^2} + \frac{12}{L^3}x\right) v_1 + \left(-\frac{4}{L} + \frac{6}{L^2}x\right) \theta_1 + \left(\frac{6}{L^2} - \frac{12}{L^3}x\right) v_2 + \left(-\frac{2}{L} + \frac{6}{L^2}x\right) \theta_2 \right] dx \\ &= - \int_0^L \left(\frac{3}{L^2}x^2 - \frac{2}{L^3}x^3\right) q dx - 1(V_2) + 0(V_1) + 0(M_2) - 0(M_1) \\ &\int_0^L \left(-\frac{1}{L}x^2 + \frac{1}{L^2}x^3\right) EI \left[\left(-\frac{6}{L^2} + \frac{12}{L^3}x\right) v_1 + \left(-\frac{4}{L} + \frac{6}{L^2}x\right) \theta_1 + \left(\frac{6}{L^2} - \frac{12}{L^3}x\right) v_2 + \left(-\frac{2}{L} + \frac{6}{L^2}x\right) \theta_2 \right] dx \\ &= - \int_0^L \left(-\frac{1}{L}x^2 + \frac{1}{L^2}x^3\right) q dx - 0(V_2) + 0(V_1) + 1(M_2) - 0(M_1)\end{aligned}$$

integrando

$$\begin{aligned}\frac{EI}{L^3}(12v_1 + 6L\theta_1 - 12v_2 + 6L\theta_2) &= -\frac{qL}{2} + V_1 \\ \frac{EI}{L^3}(6Lv_1 + 4L^2\theta_1 - 6Lv_2 + 2L^2\theta_2) &= -\frac{qL^2}{12} - M_1 \\ \frac{EI}{L^3}(-12v_1 - 6L\theta_1 + 12v_2 - 6L\theta_2) &= -\frac{qL}{2} - V_2 \\ \frac{EI}{L^3}(6Lv_1 + 2L^2\theta_1 - 6Lv_2 + 4L^2\theta_2) &= \frac{qL^2}{12} + M_2\end{aligned}$$

en forma matricial

$$\frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} -\frac{qL}{2} \\ -\frac{qL^2}{12} \\ -\frac{qL}{2} \\ \frac{qL^2}{12} \end{bmatrix} + \begin{bmatrix} V_1 \\ -M_1 \\ -V_2 \\ M_2 \end{bmatrix}$$

reemplazando fuerzas y desplazamientos

$$\frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ v_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} -\frac{qL}{2} \\ -\frac{qL^2}{12} \\ -\frac{qL}{2} \\ \frac{qL^2}{12} \end{bmatrix} + \begin{bmatrix} V_1 \\ -M_1 \\ -P \\ 0 \end{bmatrix}$$

resolviendo

$$\begin{aligned}v_2 &= -\frac{3qL^4 + 8PL^3}{24EI} \\ \theta_2 &= -\frac{qL^3 + 3PL^2}{6EI} \\ V_1 &= \frac{qL}{2} \\ M_1 &= -\frac{qL^2}{12}\end{aligned}$$

reemplazando en la solución aproximada

$$\hat{v}(x) = \left(-\frac{3qL^4 + 8PL^3}{24EI}\right)\left(\frac{3}{L^2}x^2 - \frac{2}{L^3}x^3\right) + \left(-\frac{qL^3 + 3PL^2}{6EI}\right)\left(-\frac{1}{L}x^2 + \frac{1}{L^2}x^3\right) = -\frac{5qL^2 + 12PL}{24EI}x^2 + \frac{qL + 2P}{12EI}x^3$$