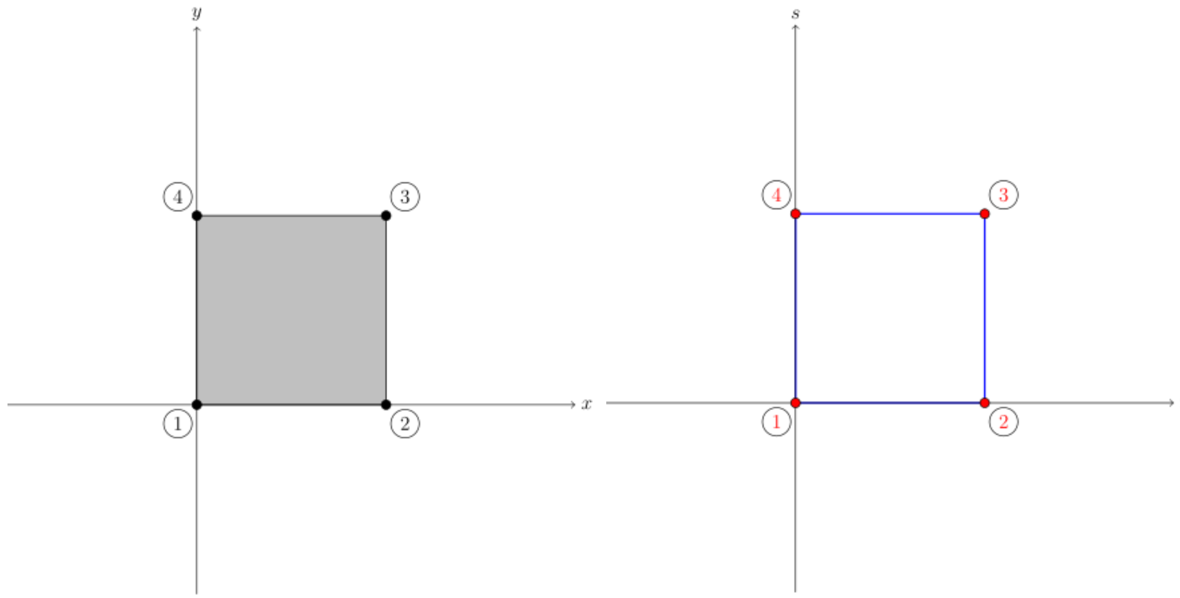
Introducción a elementos finitos
Tarea 8 I-2017

Calcular la matriz de rigidez de la placa triangular de acero A36 con base 250 mm, altura 250 mm, espesor 20 mm, coeficiente de Poisson 0.26 y módulo de elasticidad 200 GPa sujeta a esfuerzo plano, usando un elemento en coordenadas naturales



Solución 1

La matriz de rigidez es

$$K = \int_A B^{\mathrm{T}}\, C\, B\, \det J\, t\, dA$$

La matriz constitutiva es

$$C = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \frac{200000}{1-0.26^2}\begin{bmatrix} 1 & 0.26 & 0 \\ 0.26 & 1 & 0 \\ 0 & 0 & \frac{1-0.26}{2} \end{bmatrix} = \begin{bmatrix} 214500.214 & 55770.056 & 0 \\ 55770.056 & 214500.214 & 0 \\ 0 & 0 & 79365.079 \end{bmatrix}$$

Coordenadas de los nodos del elemento triangular

$$
\begin{aligned}
&① = [r_1, s_1] = [0,0] \quad ③ = [r_3, s_3] = [0,1]\\
&② = [r_2, s_2] = [1,0]
\end{aligned}
$$

Funciones que interpolan los desplazamientos

$$N = \begin{bmatrix} r & s & 1-r-s \end{bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}$$

Coordenadas de los nodos de la placa

$$
\begin{aligned}
&① = [x_1, y_1] = [0,0] \qquad ③ = [x_3, y_3] = [0,250]\\
&② = [x_2, y_2] = [250,0]
\end{aligned}
$$

Funciones que interpolan la geometría

$$
\begin{aligned}
x &= N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4\\
y &= N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4
\end{aligned}
$$

Reemplazando las coordenadas de los nodos

$$
\begin{aligned}
x &= 250s\\
y &= -250r - 250s + 250
\end{aligned}
$$

El jacobiano y el jacobiano inverso son

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} \quad J^{-1} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{bmatrix}$$

Reemplazando derivadas

$$J = \begin{bmatrix} 0 & -250 \\ 250 & -250 \end{bmatrix} \quad J^{-1} = \begin{bmatrix} -\frac{1}{250} & \frac{1}{250} \\ -\frac{1}{250} & 0 \end{bmatrix}$$

Determinante del jacobiano

$$\det J = 62500$$

La matriz de deformaciones es

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix}$$

Debido a que las funciones de forma están en función de r y s, se usará la regla de la cadena

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_i}{\partial s}\frac{\partial s}{\partial x}$$
$$\frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_i}{\partial s}\frac{\partial s}{\partial y}$$

Reemplazando en B_i

$$B_1 = \begin{bmatrix} \frac{\partial N_1}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_1}{\partial s}\frac{\partial s}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_1}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial N_1}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_1}{\partial s}\frac{\partial s}{\partial y} & \frac{\partial N_1}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_1}{\partial s}\frac{\partial s}{\partial x} \end{bmatrix} = \begin{bmatrix} -\frac{1}{250} & 0 \\ 0 & -\frac{1}{250} \\ -\frac{1}{250} & -\frac{1}{250} \end{bmatrix}$$
$$B_2 = \begin{bmatrix} \frac{\partial N_2}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_2}{\partial s}\frac{\partial s}{\partial x} & 0 \\ 0 & \frac{\partial N_2}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_2}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial N_2}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_2}{\partial s}\frac{\partial s}{\partial y} & \frac{\partial N_2}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_2}{\partial s}\frac{\partial s}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{1}{250} & 0 \\ 0 & 0 \\ 0 & \frac{1}{250} \end{bmatrix}$$
$$B_3 = \begin{bmatrix} \frac{\partial N_3}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_3}{\partial s}\frac{\partial s}{\partial x} & 0 \\ 0 & \frac{\partial N_3}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_3}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial N_3}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_3}{\partial s}\frac{\partial s}{\partial y} & \frac{\partial N_3}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_3}{\partial s}\frac{\partial s}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{250} \\ \frac{1}{250} & 0 \end{bmatrix}$$

Reemplazando en B

$$B = \begin{bmatrix} -\frac{1}{250} & 0 & \frac{1}{250} & 0 & 0 & 0 \\ 0 & -\frac{1}{250} & 0 & 0 & 0 & \frac{1}{250} \\ -\frac{1}{250} & -\frac{1}{250} & 0 & \frac{1}{250} & \frac{1}{250} & 0 \end{bmatrix}$$

Reemplazando en la fórmula de la matriz de rigidez

$$K = \int_A \begin{bmatrix} -\frac{1}{250} & 0 & -\frac{1}{250} \\ 0 & -\frac{1}{250} & -\frac{1}{250} \\ \frac{1}{250} & 0 & 0 \\ 0 & 0 & \frac{1}{250} \\ 0 & 0 & \frac{1}{250} \\ 0 & \frac{1}{250} & 0 \end{bmatrix} \begin{bmatrix} 214500.214 & 55770.056 & 0 \\ 55770.056 & 214500.214 & 0 \\ 0 & 0 & 79365.079 \end{bmatrix} \begin{bmatrix} -\frac{1}{250} & 0 & \frac{1}{250} & 0 & 0 & 0 \\ 0 & -\frac{1}{250} & 0 & 0 & 0 & \frac{1}{250} \\ -\frac{1}{250} & -\frac{1}{250} & 0 & \frac{1}{250} & \frac{1}{250} & 0 \end{bmatrix} 62500 \cdot 20 \, dA$$

Reordenando

$$K = \begin{bmatrix} -\frac{1}{250} & 0 & -\frac{1}{250} \\ 0 & -\frac{1}{250} & -\frac{1}{250} \\ \frac{1}{250} & 0 & 0 \\ 0 & 0 & \frac{1}{250} \\ 0 & 0 & \frac{1}{250} \\ 0 & \frac{1}{250} & 0 \end{bmatrix} \begin{bmatrix} 214500.214 & 55770.056 & 0 \\ 55770.056 & 214500.214 & 0 \\ 0 & 0 & 79365.079 \end{bmatrix} \begin{bmatrix} -\frac{1}{250} & 0 & \frac{1}{250} & 0 & 0 & 0 \\ 0 & -\frac{1}{250} & 0 & 0 & 0 & \frac{1}{250} \\ -\frac{1}{250} & -\frac{1}{250} & 0 & \frac{1}{250} & \frac{1}{250} & 0 \end{bmatrix} 62500 \cdot 20 \int\limits_A dA$$

El área del elemento triangular es

$$A = \frac{b\,h}{2} = \frac{1 \cdot 1}{2} = \frac{1}{2}$$

Reemplazando

$$K = \begin{bmatrix} -\frac{1}{250} & 0 & -\frac{1}{250} \\ 0 & -\frac{1}{250} & -\frac{1}{250} \\ \frac{1}{250} & 0 & 0 \\ 0 & 0 & \frac{1}{250} \\ 0 & 0 & \frac{1}{250} \\ 0 & \frac{1}{250} & 0 \end{bmatrix} \begin{bmatrix} 214500.214 & 55770.056 & 0 \\ 55770.056 & 214500.214 & 0 \\ 0 & 0 & 79365.079 \end{bmatrix} \begin{bmatrix} -\frac{1}{250} & 0 & \frac{1}{250} & 0 & 0 & 0 \\ 0 & -\frac{1}{250} & 0 & 0 & 0 & \frac{1}{250} \\ -\frac{1}{250} & -\frac{1}{250} & 0 & \frac{1}{250} & \frac{1}{250} & 0 \end{bmatrix} 62500 \cdot 20 \cdot \frac{1}{2}$$

Multiplicando

$$K = \begin{bmatrix} 2938652.94 & 1351351.35 & -2145002.15 & -793650.794 & -793650.794 & -557700.558 \\ 1351351.35 & 2938652.94 & -557700.558 & -793650.794 & -793650.794 & -2145002.15 \\ -2145002.15 & -557700.558 & 2145002.15 & 0 & 0 & 557700.558 \\ -793650.794 & -793650.794 & 0 & 793650.794 & 793650.794 & 0 \\ -793650.794 & -793650.794 & 0 & 793650.794 & 793650.794 & 0 \\ -557700.558 & -2145002.15 & 557700.558 & 0 & 0 & 2145002.15 \end{bmatrix}$$

Solución 2

La matriz de rigidez es

$$K = \int\limits_A B^{\mathrm{T}}\, C\, B\, \det J\, t\, dA$$

La matriz constitutiva es

$$C = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \frac{200000}{1-0.26^2}\begin{bmatrix} 1 & 0.26 & 0 \\ 0.26 & 1 & 0 \\ 0 & 0 & \frac{1-0.26}{2} \end{bmatrix} = \begin{bmatrix} 214500.214 & 55770.056 & 0 \\ 55770.056 & 214500.214 & 0 \\ 0 & 0 & 79365.079 \end{bmatrix}$$

Coordenadas de los nodos del elemento triangular

$$\textcircled{1} = [r_1, s_1] = [0, 0] \quad \textcircled{3} = [r_3, s_3] = [0, 1]$$
$$\textcircled{2} = [r_2, s_2] = [1, 0]$$

Funciones que interpolan los desplazamientos

$$N = \begin{bmatrix} r & s & 1-r-s \end{bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}$$

Coordenadas de los nodos de la placa

$$\textcircled{1} = [x_1, y_1] = [0, 0] \qquad \textcircled{3} = [x_3, y_3] = [0, 250]$$
$$\textcircled{2} = [x_2, y_2] = [250, 0]$$

Funciones que interpolan la geometría

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

Reemplazando las coordenadas de los nodos

$$x = 250s$$
$$y = -250r - 250s + 250$$

El jacobiano y el jacobiano inverso son

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} \quad J^{-1} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{bmatrix}$$

Reemplazando derivadas

$$J = \begin{bmatrix} 0 & -250 \\ 250 & -250 \end{bmatrix} \quad J^{-1} = \begin{bmatrix} -\frac{1}{250} & \frac{1}{250} \\ -\frac{1}{250} & 0 \end{bmatrix}$$

Determinante del jacobiano

$$\det J = 62500$$

La matriz de deformaciones es

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix}$$

Las derivadas de las funciones de forma se calcularán en forma matricial

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial N_i}{\partial r} \\ \frac{\partial N_i}{\partial s} \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial r} \\ \frac{\partial N_i}{\partial s} \end{bmatrix}$$

Reemplazando en N_i

$$\begin{bmatrix} \frac{\partial N_1}{\partial x} \\ \frac{\partial N_1}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial r} \\ \frac{\partial N_1}{\partial s} \end{bmatrix} = \begin{bmatrix} -\frac{1}{250} & \frac{1}{250} \\ -\frac{1}{250} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{250} \\ -\frac{1}{250} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_2}{\partial x} \\ \frac{\partial N_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial N_2}{\partial r} \\ \frac{\partial N_2}{\partial s} \end{bmatrix} = \begin{bmatrix} -\frac{1}{250} & \frac{1}{250} \\ -\frac{1}{250} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{250} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_3}{\partial x} \\ \frac{\partial N_3}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial N_3}{\partial r} \\ \frac{\partial N_3}{\partial s} \end{bmatrix} = \begin{bmatrix} -\frac{1}{250} & \frac{1}{250} \\ -\frac{1}{250} & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{250} \end{bmatrix}$$

Reemplazando en B

$$B = \begin{bmatrix} -\frac{1}{250} & 0 & \frac{1}{250} & 0 & 0 & 0 \\ 0 & -\frac{1}{250} & 0 & 0 & 0 & \frac{1}{250} \\ -\frac{1}{250} & -\frac{1}{250} & 0 & \frac{1}{250} & \frac{1}{250} & 0 \end{bmatrix}$$

Reemplazando en la fórmula de la matriz de rigidez

$$K = \int_A \begin{bmatrix} -\frac{1}{250} & 0 & -\frac{1}{250} \\ 0 & -\frac{1}{250} & -\frac{1}{250} \\ \frac{1}{250} & 0 & 0 \\ 0 & 0 & \frac{1}{250} \\ 0 & 0 & \frac{1}{250} \\ 0 & \frac{1}{250} & 0 \end{bmatrix} \begin{bmatrix} 214500.214 & 55770.056 & 0 \\ 55770.056 & 214500.214 & 0 \\ 0 & 0 & 79365.079 \end{bmatrix} \begin{bmatrix} -\frac{1}{250} & 0 & \frac{1}{250} & 0 & 0 & 0 \\ 0 & -\frac{1}{250} & 0 & 0 & 0 & \frac{1}{250} \\ -\frac{1}{250} & -\frac{1}{250} & 0 & \frac{1}{250} & \frac{1}{250} & 0 \end{bmatrix} 62500 \cdot 20 \, dA$$

Reordenando

$$K = \begin{bmatrix} -\frac{1}{250} & 0 & -\frac{1}{250} \\ 0 & -\frac{1}{250} & -\frac{1}{250} \\ \frac{1}{250} & 0 & 0 \\ 0 & 0 & \frac{1}{250} \\ 0 & 0 & \frac{1}{250} \\ 0 & \frac{1}{250} & 0 \end{bmatrix} \begin{bmatrix} 214500.214 & 55770.056 & 0 \\ 55770.056 & 214500.214 & 0 \\ 0 & 0 & 79365.079 \end{bmatrix} \begin{bmatrix} -\frac{1}{250} & 0 & \frac{1}{250} & 0 & 0 & 0 \\ 0 & -\frac{1}{250} & 0 & 0 & 0 & \frac{1}{250} \\ -\frac{1}{250} & -\frac{1}{250} & 0 & \frac{1}{250} & \frac{1}{250} & 0 \end{bmatrix} 62500 \cdot 20 \int\limits_A dA$$

El área del elemento triangular es

$$A = \frac{b\,h}{2} = \frac{1 \cdot 1}{2} = \frac{1}{2}$$

Reemplazando

$$K = \begin{bmatrix} -\frac{1}{250} & 0 & -\frac{1}{250} \\ 0 & -\frac{1}{250} & -\frac{1}{250} \\ \frac{1}{250} & 0 & 0 \\ 0 & 0 & \frac{1}{250} \\ 0 & 0 & \frac{1}{250} \\ 0 & \frac{1}{250} & 0 \end{bmatrix} \begin{bmatrix} 214500.214 & 55770.056 & 0 \\ 55770.056 & 214500.214 & 0 \\ 0 & 0 & 79365.079 \end{bmatrix} \begin{bmatrix} -\frac{1}{250} & 0 & \frac{1}{250} & 0 & 0 & 0 \\ 0 & -\frac{1}{250} & 0 & 0 & 0 & \frac{1}{250} \\ -\frac{1}{250} & -\frac{1}{250} & 0 & \frac{1}{250} & \frac{1}{250} & 0 \end{bmatrix} 62500 \cdot 20 \cdot \frac{1}{2}$$

Multiplicando

$$K = \begin{bmatrix} 2938652.94 & 1351351.35 & -2145002.15 & -793650.794 & -793650.794 & -557700.558 \\ 1351351.35 & 2938652.94 & -557700.558 & -793650.794 & -793650.794 & -2145002.15 \\ -2145002.15 & -557700.558 & 2145002.15 & 0 & 0 & 557700.558 \\ -793650.794 & -793650.794 & 0 & 793650.794 & 793650.794 & 0 \\ -793650.794 & -793650.794 & 0 & 793650.794 & 793650.794 & 0 \\ -557700.558 & -2145002.15 & 557700.558 & 0 & 0 & 2145002.15 \end{bmatrix}$$

Solución 3

La matriz de rigidez es

$$K = \int\limits_A B^{\mathrm{T}}\, C\, B \;\det J\, t\, dA$$

La matriz constitutiva es

$$C = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix} = \frac{200000}{1-0.26^2}\begin{bmatrix}1 & 0.26 & 0\\ 0.26 & 1 & 0\\ 0 & 0 & \frac{1-0.26}{2}\end{bmatrix} = \begin{bmatrix}214500.214 & 55770.056 & 0\\ 55770.056 & 214500.214 & 0\\ 0 & 0 & 79365.079\end{bmatrix}$$

Coordenadas de los nodos del elemento triangular

$$1 = [r_1, s_1] = [0,0] \quad 3 = [r_3, s_3] = [0,1]$$
$$2 = [r_2, s_2] = [1,0]$$

Funciones que interpolan los desplazamientos

$$N = \begin{bmatrix}r & s & 1-r-s\end{bmatrix} = \begin{bmatrix}N_1 & N_2 & N_3\end{bmatrix}$$

Coordenadas de los nodos de la placa

$$1 = [x_1, y_1] = [0,0] \qquad 3 = [x_3, y_3] = [0,250]$$
$$2 = [x_2, y_2] = [250,0]$$

Funciones que interpolan la geometría

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

Reemplazando las coordenadas de los nodos

$$x = 250s$$
$$y = -250r - 250s + 250$$

El jacobiano y el jacobiano inverso son

$$J = \begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial y}{\partial r}\\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s}\end{bmatrix} \quad J^{-1} = \begin{bmatrix}\frac{\partial r}{\partial x} & \frac{\partial s}{\partial x}\\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y}\end{bmatrix}$$

Reemplazando derivadas

$$J = \begin{bmatrix}0 & -250\\ 250 & -250\end{bmatrix} \quad J^{-1} = \begin{bmatrix}-\frac{1}{250} & \frac{1}{250}\\ -\frac{1}{250} & 0\end{bmatrix}$$

Determinante del jacobiano

$$\det J = 62500$$

La matriz de deformaciones es

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix}$$

Las derivadas de las funciones de forma se calcularán usando una forma alternativa del jacobiano inverso

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = \frac{1}{\det J} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial r} \\ \frac{\partial N_i}{\partial s} \end{bmatrix} = \frac{1}{\det J} \begin{bmatrix} \frac{\partial y}{\partial s} & -\frac{\partial y}{\partial r} \\ -\frac{\partial x}{\partial s} & \frac{\partial x}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial r} \\ \frac{\partial N_i}{\partial s} \end{bmatrix}$$

Reemplazando en N_i

$$\begin{bmatrix} \frac{\partial N_1}{\partial x} \\ \frac{\partial N_1}{\partial y} \end{bmatrix} = \frac{1}{\det J} \begin{bmatrix} \frac{\partial y}{\partial s} & -\frac{\partial y}{\partial r} \\ -\frac{\partial x}{\partial s} & \frac{\partial x}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial r} \\ \frac{\partial N_1}{\partial s} \end{bmatrix} = \frac{1}{62500} \begin{bmatrix} -250 & 250 \\ -250 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{250} \\ -\frac{1}{250} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_2}{\partial x} \\ \frac{\partial N_2}{\partial y} \end{bmatrix} = \frac{1}{\det J} \begin{bmatrix} \frac{\partial y}{\partial s} & -\frac{\partial y}{\partial r} \\ -\frac{\partial x}{\partial s} & \frac{\partial x}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{\partial N_2}{\partial r} \\ \frac{\partial N_2}{\partial s} \end{bmatrix} = \frac{1}{62500} \begin{bmatrix} -250 & 250 \\ -250 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{250} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_3}{\partial x} \\ \frac{\partial N_3}{\partial y} \end{bmatrix} = \frac{1}{\det J} \begin{bmatrix} \frac{\partial y}{\partial s} & -\frac{\partial y}{\partial r} \\ -\frac{\partial x}{\partial s} & \frac{\partial x}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{\partial N_3}{\partial r} \\ \frac{\partial N_3}{\partial s} \end{bmatrix} = \frac{1}{62500} \begin{bmatrix} -250 & 250 \\ -250 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{250} \end{bmatrix}$$

Reemplazando en B

$$B = \begin{bmatrix} -\frac{1}{250} & 0 & \frac{1}{250} & 0 & 0 & 0 \\ 0 & -\frac{1}{250} & 0 & 0 & 0 & \frac{1}{250} \\ -\frac{1}{250} & -\frac{1}{250} & 0 & \frac{1}{250} & \frac{1}{250} & 0 \end{bmatrix}$$

Reemplazando en la fórmula de la matriz de rigidez

$$K = \int_A \begin{bmatrix} -\frac{1}{250} & 0 & -\frac{1}{250} \\ 0 & -\frac{1}{250} & -\frac{1}{250} \\ \frac{1}{250} & 0 & 0 \\ 0 & 0 & \frac{1}{250} \\ 0 & 0 & \frac{1}{250} \\ 0 & \frac{1}{250} & 0 \end{bmatrix} \begin{bmatrix} 214500.214 & 55770.056 & 0 \\ 55770.056 & 214500.214 & 0 \\ 0 & 0 & 79365.079 \end{bmatrix} \begin{bmatrix} -\frac{1}{250} & 0 & \frac{1}{250} & 0 & 0 & 0 \\ 0 & -\frac{1}{250} & 0 & 0 & 0 & \frac{1}{250} \\ -\frac{1}{250} & -\frac{1}{250} & 0 & \frac{1}{250} & \frac{1}{250} & 0 \end{bmatrix} 62500 \cdot 20 \, dA$$

Reordenando

$$K = \begin{bmatrix} -\frac{1}{250} & 0 & -\frac{1}{250} \\ 0 & -\frac{1}{250} & -\frac{1}{250} \\ \frac{1}{250} & 0 & 0 \\ 0 & 0 & \frac{1}{250} \\ 0 & 0 & \frac{1}{250} \\ 0 & \frac{1}{250} & 0 \end{bmatrix} \begin{bmatrix} 214500.214 & 55770.056 & 0 \\ 55770.056 & 214500.214 & 0 \\ 0 & 0 & 79365.079 \end{bmatrix} \begin{bmatrix} -\frac{1}{250} & 0 & \frac{1}{250} & 0 & 0 & 0 \\ 0 & -\frac{1}{250} & 0 & 0 & 0 & \frac{1}{250} \\ -\frac{1}{250} & -\frac{1}{250} & 0 & \frac{1}{250} & \frac{1}{250} & 0 \end{bmatrix} 62500 \cdot 20 \int\limits_A dA$$

El área del elemento triangular es

$$A = \frac{b\,h}{2} = \frac{1 \cdot 1}{2} = \frac{1}{2}$$

Reemplazando

$$K = \begin{bmatrix} -\frac{1}{250} & 0 & -\frac{1}{250} \\ 0 & -\frac{1}{250} & -\frac{1}{250} \\ \frac{1}{250} & 0 & 0 \\ 0 & 0 & \frac{1}{250} \\ 0 & 0 & \frac{1}{250} \\ 0 & \frac{1}{250} & 0 \end{bmatrix} \begin{bmatrix} 214500.214 & 55770.056 & 0 \\ 55770.056 & 214500.214 & 0 \\ 0 & 0 & 79365.079 \end{bmatrix} \begin{bmatrix} -\frac{1}{250} & 0 & \frac{1}{250} & 0 & 0 & 0 \\ 0 & -\frac{1}{250} & 0 & 0 & 0 & \frac{1}{250} \\ -\frac{1}{250} & -\frac{1}{250} & 0 & \frac{1}{250} & \frac{1}{250} & 0 \end{bmatrix} 62500 \cdot 20 \cdot \frac{1}{2}$$

Multiplicando

$$K = \begin{bmatrix} 2938652.94 & 1351351.35 & -2145002.15 & -793650.794 & -793650.794 & -557700.558 \\ 1351351.35 & 2938652.94 & -557700.558 & -793650.794 & -793650.794 & -2145002.15 \\ -2145002.15 & -557700.558 & 2145002.15 & 0 & 0 & 557700.558 \\ -793650.794 & -793650.794 & 0 & 793650.794 & 793650.794 & 0 \\ -793650.794 & -793650.794 & 0 & 793650.794 & 793650.794 & 0 \\ -557700.558 & -2145002.15 & 557700.558 & 0 & 0 & 2145002.15 \end{bmatrix}$$

Solución 4

La matriz de rigidez es

$$K = \int\limits_A B^{\mathrm{T}}\, C\, B\, \det J\, t\, dA$$

La matriz constitutiva es

$$C = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix} = \frac{200000}{1-0.26^2}\begin{bmatrix}1 & 0.26 & 0\\ 0.26 & 1 & 0\\ 0 & 0 & \frac{1-0.26}{2}\end{bmatrix} = \begin{bmatrix}214500.214 & 55770.056 & 0\\ 55770.056 & 214500.214 & 0\\ 0 & 0 & 79365.079\end{bmatrix}$$

Coordenadas de los nodos del elemento triangular

$$1 = [r_1, s_1] = [0,0] \quad 3 = [r_3, s_3] = [0,1]$$
$$2 = [r_2, s_2] = [1,0]$$

Funciones que interpolan los desplazamientos

$$N = \begin{bmatrix}r & s & 1-r-s\end{bmatrix} = \begin{bmatrix}N_1 & N_2 & N_3\end{bmatrix}$$

Coordenadas de los nodos de la placa

$$1 = [x_1, y_1] = [0,0] \qquad 3 = [x_3, y_3] = [0,250]$$
$$2 = [x_2, y_2] = [250,0]$$

Funciones que interpolan la geometría

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$$
$$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4$$

Reemplazando las coordenadas de los nodos

$$x = 250s$$
$$y = -250r - 250s + 250$$

El jacobiano y el jacobiano inverso son

$$J = \begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial y}{\partial r}\\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s}\end{bmatrix} \quad J^{-1} = \begin{bmatrix}\frac{\partial r}{\partial x} & \frac{\partial s}{\partial x}\\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y}\end{bmatrix}$$

Reemplazando derivadas

$$J = \begin{bmatrix}0 & -250\\ 250 & -250\end{bmatrix} \quad J^{-1} = \begin{bmatrix}-\frac{1}{250} & \frac{1}{250}\\ -\frac{1}{250} & 0\end{bmatrix}$$

Determinante del jacobiano

$$\det J = 62500$$

La matriz de deformaciones es

$$B = M_1\, M_2\, M_3$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} & 0 & 0 \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} & 0 & 0 \\ 0 & 0 & \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ 0 & 0 & \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{bmatrix}$$

$$M_3 = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 \\ \frac{\partial N_1}{\partial s} & 0 & \frac{\partial N_2}{\partial s} & 0 & \frac{\partial N_3}{\partial s} & 0 \\ 0 & \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} \\ 0 & \frac{\partial N_1}{\partial s} & 0 & \frac{\partial N_2}{\partial s} & 0 & \frac{\partial N_3}{\partial s} \end{bmatrix}$$

Reemplazando en M_2 y M_3

$$M_2 = \begin{bmatrix} -\frac{1}{250} & \frac{1}{250} & 0 & 0 \\ -\frac{1}{250} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{250} & \frac{1}{250} \\ 0 & 0 & -\frac{1}{250} & 0 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

Reemplazando en B

$$B = \begin{bmatrix} -\frac{1}{250} & 0 & \frac{1}{250} & 0 & 0 & 0 \\ 0 & -\frac{1}{250} & 0 & 0 & 0 & \frac{1}{250} \\ -\frac{1}{250} & -\frac{1}{250} & 0 & \frac{1}{250} & \frac{1}{250} & 0 \end{bmatrix}$$

Reemplazando en la fórmula de la matriz de rigidez

$$K = \int_A \begin{bmatrix} -\frac{1}{250} & 0 & -\frac{1}{250} \\ 0 & -\frac{1}{250} & -\frac{1}{250} \\ \frac{1}{250} & 0 & 0 \\ 0 & 0 & \frac{1}{250} \\ 0 & 0 & \frac{1}{250} \\ 0 & \frac{1}{250} & 0 \end{bmatrix} \begin{bmatrix} 214500.214 & 55770.056 & 0 \\ 55770.056 & 214500.214 & 0 \\ 0 & 0 & 79365.079 \end{bmatrix} \begin{bmatrix} -\frac{1}{250} & 0 & \frac{1}{250} & 0 & 0 & 0 \\ 0 & -\frac{1}{250} & 0 & 0 & 0 & \frac{1}{250} \\ -\frac{1}{250} & -\frac{1}{250} & 0 & \frac{1}{250} & \frac{1}{250} & 0 \end{bmatrix} 62500 \cdot 20 \, dA$$

Reordenando

$$K = \begin{bmatrix} -\frac{1}{250} & 0 & -\frac{1}{250} \\ 0 & -\frac{1}{250} & -\frac{1}{250} \\ \frac{1}{250} & 0 & 0 \\ 0 & 0 & \frac{1}{250} \\ 0 & 0 & \frac{1}{250} \\ 0 & \frac{1}{250} & 0 \end{bmatrix} \begin{bmatrix} 214500.214 & 55770.056 & 0 \\ 55770.056 & 214500.214 & 0 \\ 0 & 0 & 79365.079 \end{bmatrix} \begin{bmatrix} -\frac{1}{250} & 0 & \frac{1}{250} & 0 & 0 & 0 \\ 0 & -\frac{1}{250} & 0 & 0 & 0 & \frac{1}{250} \\ -\frac{1}{250} & -\frac{1}{250} & 0 & \frac{1}{250} & \frac{1}{250} & 0 \end{bmatrix} 62500 \cdot 20 \int_A dA$$

El área del elemento triangular es

$$A = \frac{b\,h}{2} = \frac{1 \cdot 1}{2} = \frac{1}{2}$$

Reemplazando

$$K = \begin{bmatrix} -\frac{1}{250} & 0 & -\frac{1}{250} \\ 0 & -\frac{1}{250} & -\frac{1}{250} \\ \frac{1}{250} & 0 & 0 \\ 0 & 0 & \frac{1}{250} \\ 0 & 0 & \frac{1}{250} \\ 0 & \frac{1}{250} & 0 \end{bmatrix} \begin{bmatrix} 214500.214 & 55770.056 & 0 \\ 55770.056 & 214500.214 & 0 \\ 0 & 0 & 79365.079 \end{bmatrix} \begin{bmatrix} -\frac{1}{250} & 0 & \frac{1}{250} & 0 & 0 & 0 \\ 0 & -\frac{1}{250} & 0 & 0 & 0 & \frac{1}{250} \\ -\frac{1}{250} & -\frac{1}{250} & 0 & \frac{1}{250} & \frac{1}{250} & 0 \end{bmatrix} 62500 \cdot 20 \cdot \frac{1}{2}$$

Multiplicando

$$K = \begin{bmatrix} 2938652.94 & 1351351.35 & -2145002.15 & -793650.794 & -793650.794 & -557700.558 \\ 1351351.35 & 2938652.94 & -557700.558 & -793650.794 & -793650.794 & -2145002.15 \\ -2145002.15 & -557700.558 & 2145002.15 & 0 & 0 & 557700.558 \\ -793650.794 & -793650.794 & 0 & 793650.794 & 793650.794 & 0 \\ -793650.794 & -793650.794 & 0 & 793650.794 & 793650.794 & 0 \\ -557700.558 & -2145002.15 & 557700.558 & 0 & 0 & 2145002.15 \end{bmatrix}$$