Esquema FTCS

Hallar el perfil de flujo usando $\Delta x = 40$ m, $\Delta t = 10$ h y $D = 1 \times 10^{-3}$ m^2/s, para un tiempo final de 20 h

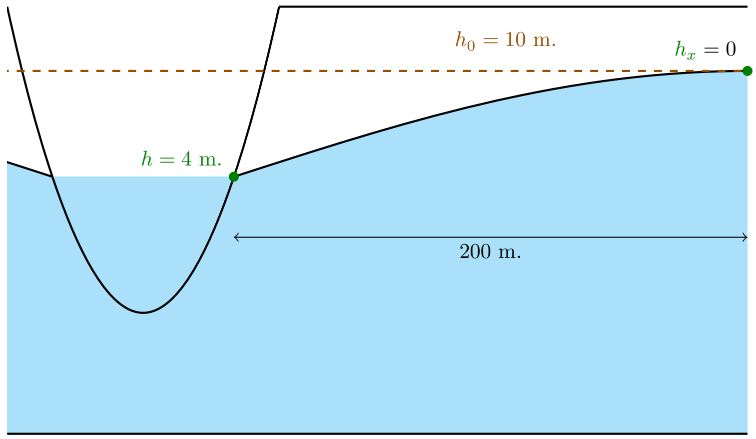

Figura 1: Representación gráfica del ejemplo

La ecuación y las condiciones de contorno serán

$$\frac{\partial h}{\partial t} - D\,\frac{\partial^2 h}{\partial x^2} = 0 \tag{1}$$
$$h(x, 0) = 10 \tag{2}$$
$$h(0, t) = 4 \tag{3}$$
$$h_x(200, t) = 0 \tag{4}$$

Discretización espacial

$$N_{\text{elementos}} = \frac{L}{\Delta x} = \frac{200}{40} = 5$$
$$N_{\text{puntos}} = N_{\text{elementos}} + 1 = 5 + 1 = 6$$

Discretización temporal

$$N_{\text{elementos}} = \frac{t}{\Delta t} = \frac{20}{10} = 2$$
$$N_{\text{puntos}} = N_{\text{elementos}} + 1 = 2 + 1 = 3$$

Discretización numérica

$$\frac{\partial h}{\partial t} = \frac{h_i^{n+1} - h_i^n}{\Delta t}$$
$$\frac{\partial^2 h}{\partial x^2} = \frac{\dfrac{h_{i+1}^n - h_i^n}{\Delta x} - \dfrac{h_i^n - h_{i-1}^n}{\Delta x}}{\Delta x} = \frac{h_{i-1}^n - 2h_i^n + h_{i+1}^n}{\Delta x^2}$$

Reemplazando en (1)

$$\frac{h_i^{n+1} - h_i^n}{\Delta t} - D\left(\frac{h_{i-1}^n - 2h_i^n + h_{i+1}^n}{\Delta x^2}\right) = 0$$

Reordenando

$$h_i^{n+1} = h_i^n + D\frac{\Delta t}{\Delta x^2}\left(h_{i-1}^n - 2h_i^n + h_{i+1}^n\right)$$

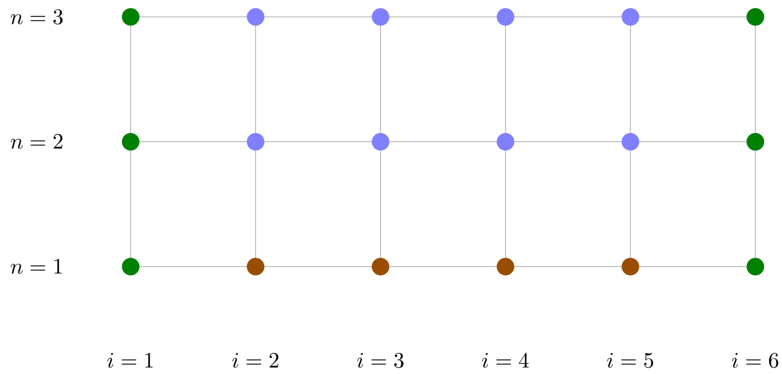

Figura 2: Mallado

3	h_1^3	h_2^3	h_3^3	h_4^3	h_5^3	h_6^3
2	h_1^2	h_2^2	h_3^2	h_4^2	h_5^2	h_6^2
1	h_1^1	h_2^1	h_3^1	h_4^1	h_5^1	h_6^1
	1	2	3	4	5	6

Figura 3: Matriz solución

Verificando si es estable $\lambda \leqslant 0.5$

$$D\frac{\Delta t}{\Delta x^2} = 0.001\left(\frac{36000}{40^2}\right) = 0.0225$$

Reemplazando las condiciones de contorno, para $i = 1$ y $n = 1, 2, 3$

$$h_1^1 = 4$$
$$h_1^2 = 4$$
$$h_1^3 = 4$$

Para $i = 2, 3, 4, 5$ y $n = 1$

$$h_2^1 = 10$$
$$h_3^1 = 10$$
$$h_4^1 = 10$$
$$h_5^1 = 10$$

Para $i = 6$ y $n = 1$, usando un esquema hacia atrás

$$h_x = \frac{\partial h}{\partial x} = \frac{h_6^1 - h_5^1}{\Delta x} = 0$$

reordenando

$$\begin{aligned} h_6^1 &= h_5^1 \\ &= 10 \end{aligned}$$

3	4	h_2^3	h_3^3	h_4^3	h_5^3	h_6^3
2	4	h_2^2	h_3^2	h_4^2	h_5^2	h_6^2
1	4	10	10	10	10	10
	1	2	3	4	5	6

Figura 4: Matriz solución para $t = 0$ h

Usando el esquema elegido, para $i = 2$ y $n = 1$

$$h_2^2 = h_2^1 + D\frac{\Delta t}{\Delta x^2}\left(h_1^1 - 2h_2^1 + h_3^1\right) = 10 + 0.0225[4 - 2(10) + 10] = 9.865$$

Para $i = 3$ y $n = 1$

$$h_3^2 = h_3^1 + D\frac{\Delta t}{\Delta x^2}\left(h_2^1 - 2h_3^1 + h_4^1\right) = 10 + 0.0225[10 - 2(10) + 10] = 10$$

Para $i = 4$ y $n = 1$

$$h_4^2 = h_4^1 + D\frac{\Delta t}{\Delta x^2}\left(h_3^1 - 2h_4^1 + h_5^1\right) = 10 + 0.0225[10 - 2(10) + 10] = 10$$

Para $i = 5$ y $n = 1$

$$h_5^2 = h_5^1 + D\frac{\Delta t}{\Delta x^2}\left(h_4^1 - 2h_5^1 + h_6^1\right) = 10 + 0.0225[10 - 2(10) + 10] = 10$$

Para $i = 6$ y $n = 1$

$$\begin{aligned} h_6^2 &= h_5^2 \\ &= 10 \end{aligned}$$

3	4	h_2^3	h_3^3	h_4^3	h_5^3	h_6^3
2	4	9.865	10	10	10	10
1	4	10	10	10	10	10
	1	2	3	4	5	6

Figura 5: Matriz solución para $t = 10$ h

Para $i = 2$ y $n = 2$

$$h_2^3 = h_2^2 + D\frac{\Delta t}{\Delta x^2}\left(h_1^2 - 2h_2^2 + h_3^2\right) = 9.865 + 0.0225[4 - 2(9.865) + 10] = 9.736$$

Para $i = 3$ y $n = 2$

$$h_3^3 = h_3^2 + D\frac{\Delta t}{\Delta x^2}\left(h_2^2 - 2h_3^2 + h_4^2\right) = 10 + 0.0225[9.865 - 2(10) + 10] = 9.997$$

Para $i = 4$ y $n = 2$

$$h_4^3 = h_4^2 + D\frac{\Delta t}{\Delta x^2}\left(h_3^2 - 2h_4^2 + h_5^2\right) = 10 + 0.0225[10 - 2(10) + 10] = 10$$

Para $i = 5$ y $n = 2$

$$h_5^3 = h_5^2 + D\frac{\Delta t}{\Delta x^2}\left(h_4^2 - 2h_5^2 + h_6^2\right) = 10 + 0.0225[10 - 2(10) + 10] = 10$$

Para $i = 6$ y $n = 2$

$$\begin{aligned} h_6^3 &= h_5^3 \\ &= 10 \end{aligned}$$

3	4	9.736	9.997	10	10	10
2	4	9.865	10	10	10	10
1	4	10	10	10	10	10
	1	2	3	4	5	6

Figura 6: Matriz solución para $t = 20$ h