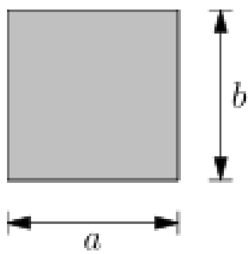
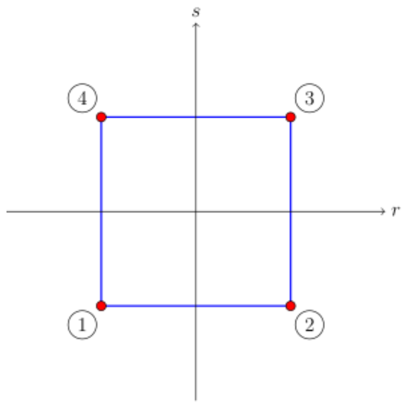
Introducción a elementos finitos
Segundo Parcial II-2016

1. Calcular la matriz de rigidez de la placa de espesor t sujeta a esfuerzo plano, usando un elemento en coordenadas naturales





Solución 1

La matriz de rigidez es

$$K = \int_{-1}^{1}\int_{-1}^{1} B^{\mathrm{T}}\, C\, B \,\det J\, t\, ds\, dr$$

La matriz constitutiva es

$$C = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Funciones de forma unidimensional de dos nodos

$$N(r) = \begin{bmatrix} \frac{1}{2} - \frac{1}{2}r & \frac{1}{2} + \frac{1}{2}r \end{bmatrix}$$
$$N(s) = \begin{bmatrix} \frac{1}{2} - \frac{1}{2}s & \frac{1}{2} + \frac{1}{2}s \end{bmatrix}$$

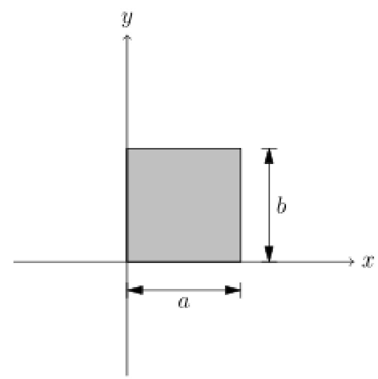
Multiplicando

$$\begin{bmatrix} \frac{1}{2} - \frac{1}{2}r \\ \frac{1}{2} + \frac{1}{2}r \end{bmatrix} \begin{bmatrix} \frac{1}{2} - \frac{1}{2}s & \frac{1}{2} + \frac{1}{2}s \end{bmatrix} = \begin{bmatrix} \frac{1}{4}(r-1)(s-1) & -\frac{1}{4}(r-1)(s+1) \\ -\frac{1}{4}(r+1)(s-1) & \frac{1}{4}(r+1)(s+1) \end{bmatrix}$$
$$= \begin{bmatrix} N_1 & N_4 \\ N_2 & N_3 \end{bmatrix}$$

Escribiendo en la forma estándar

$$N^{\mathrm{T}} = \begin{bmatrix} \frac{1}{4}(r-1)(s-1) \\ -\frac{1}{4}(r+1)(s-1) \\ \frac{1}{4}(r+1)(s+1) \\ -\frac{1}{4}(r-1)(s+1) \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}$$

Colocando un sistema de referencia



Coordenadas de los nodos

$$[r_1, s_1] = [0, 0] \quad [r_3, s_3] = [a, b]$$
$$[r_2, s_2] = [a, 0] \quad [r_4, s_4] = [0, b]$$

Funciones que interpolan la geometría

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

Reemplazando las coordenadas de los nodos

$$x = \frac{a}{2}(r+1)$$
$$y = \frac{b}{2}(s+1)$$

El jacobiano y el jacobiano inverso son

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{bmatrix} \quad J^{-1} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{bmatrix}$$

Reemplazando derivadas

$$J = \begin{bmatrix} \frac{a}{2} & 0 \\ 0 & \frac{b}{2} \end{bmatrix} \quad J^{-1} = \begin{bmatrix} \frac{2}{a} & 0 \\ 0 & \frac{2}{b} \end{bmatrix}$$

Determinante del jacobiano

$$\det J = \frac{ab}{4}$$

La matriz de deformaciones es

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix}$$

Debido a que las funciones de forma están en función de r y s, se usará la regla de la cadena

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_i}{\partial s}\frac{\partial s}{\partial x}$$
$$\frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_i}{\partial s}\frac{\partial s}{\partial y}$$

Reemplazando en B_i

$$B_1 = \begin{bmatrix} \frac{\partial N_1}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_1}{\partial s}\frac{\partial s}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_1}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial N_1}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_1}{\partial s}\frac{\partial s}{\partial y} & \frac{\partial N_1}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_1}{\partial s}\frac{\partial s}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{s-1}{2a} & 0 \\ 0 & \frac{r-1}{2b} \\ \frac{r-1}{2b} & \frac{s-1}{2a} \end{bmatrix}$$

$$B_2 = \begin{bmatrix} \frac{\partial N_2}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_2}{\partial s}\frac{\partial s}{\partial x} & 0 \\ 0 & \frac{\partial N_2}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_2}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial N_2}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_2}{\partial s}\frac{\partial s}{\partial y} & \frac{\partial N_2}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_2}{\partial s}\frac{\partial s}{\partial x} \end{bmatrix} = \begin{bmatrix} -\frac{s-1}{2a} & 0 \\ 0 & -\frac{r+1}{2b} \\ -\frac{r+1}{2b} & -\frac{s-1}{2a} \end{bmatrix}$$

$$B_3 = \begin{bmatrix} \frac{\partial N_3}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_3}{\partial s}\frac{\partial s}{\partial x} & 0 \\ 0 & \frac{\partial N_3}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_3}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial N_3}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_3}{\partial s}\frac{\partial s}{\partial y} & \frac{\partial N_3}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_3}{\partial s}\frac{\partial s}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{s+1}{2a} & 0 \\ 0 & \frac{r+1}{2b} \\ \frac{r+1}{2b} & \frac{s+1}{2a} \end{bmatrix}$$

$$B_4 = \begin{bmatrix} \frac{\partial N_4}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_4}{\partial s}\frac{\partial s}{\partial x} & 0 \\ 0 & \frac{\partial N_4}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_4}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial N_4}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_4}{\partial s}\frac{\partial s}{\partial y} & \frac{\partial N_4}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_4}{\partial s}\frac{\partial s}{\partial x} \end{bmatrix} = \begin{bmatrix} -\frac{s+1}{2a} & 0 \\ 0 & -\frac{r-1}{2b} \\ -\frac{r-1}{2b} & -\frac{s+1}{2a} \end{bmatrix}$$

Reemplazando en B

$$B = \begin{bmatrix} \frac{s-1}{2a} & 0 & -\frac{s-1}{2a} & 0 & \frac{s+1}{2a} & 0 & -\frac{s+1}{2a} & 0 \\ 0 & \frac{r-1}{2b} & 0 & -\frac{r+1}{2b} & 0 & \frac{r+1}{2b} & 0 & -\frac{r-1}{2b} \\ \frac{r-1}{2b} & \frac{s-1}{2a} & -\frac{r+1}{2b} & -\frac{s-1}{2a} & \frac{r+1}{2b} & \frac{s+1}{2a} & -\frac{r-1}{2b} & -\frac{s+1}{2a} \end{bmatrix}$$

Después de multiplicar e integrar la matriz de rigidez es de 8×8

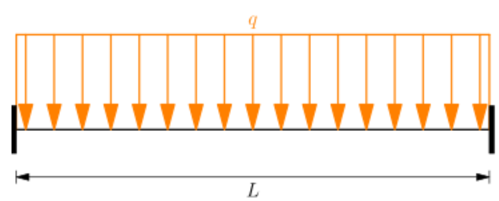
Solución 2

Pendiente

Solución 3

Pendiente

2. Obtener la expresión de la matriz de rigidez con E, I, A constantes por el método de Galerkin



Solución

Ecuación diferencial de la viga

$$EI\frac{d^4v}{dx^4} - q(x) = 0$$

La función ponderada será

$$W = \phi(x)$$

Aplicando el método de Galerkin

$$\int_0^L \left(EI\frac{d^4v}{dx^4} - q\right)\phi\,dx = \int_0^L \phi\,EI\frac{d^4v}{dx^4}\,dx - \int_0^L \phi\,q\,dx = 0$$

Usando el teorema de Gauss o integrando por partes

$$\left(\phi\,EI\frac{d^3v}{dx^3} - \frac{d\phi}{dx}EI\frac{d^2v}{dx^2}\right)\Bigg|_0^L + \int_0^L \frac{d^2\phi}{dx^2}EI\frac{d^2v}{dx^2}\,dx - \int_0^L \phi\,q\,dx = 0$$

Cortante y momento

$$V = EI\frac{d^3v}{dx^3}$$
$$M = EI\frac{d^2v}{dx^2}$$

Reemplazando

$$\left(\phi\,V - \frac{d\phi}{dx}\,M\right)\Bigg|_0^L + EI\int_0^L \frac{d^2\phi}{dx^2}\frac{d^2v}{dx^2}\,dx - \int_0^L \phi\,q\,dx = 0$$

Reordenando

$$EI\int_0^L \frac{d^2\phi}{dx^2}\frac{d^2v}{dx^2}\,dx = \int_0^L \phi\,q\,dx + \left(-\phi\,V + \frac{d\phi}{dx}\,M\right)\Bigg|_0^L$$

La aproximación de los desplazamientos será

$$v(x) = \phi(x)$$

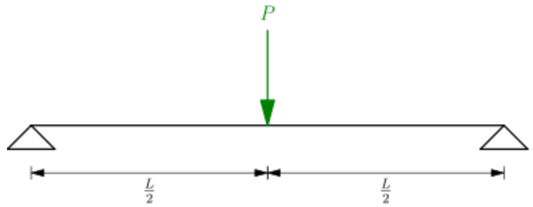
Reemplazando

$$EI\int_0^L \frac{d^2\phi}{dx^2}\frac{d^2\phi}{dx^2}\,dx = \int_0^L \phi\,q\,dx - \left(\phi\,V - \frac{d\phi}{dx}\,M\right)\Bigg|_0^L$$

La matriz de rigidez es

$$K = EI\int_0^L \frac{d^2\phi}{dx^2}\frac{d^2\phi}{dx^2}\,dx = EI\int_0^L \left(\frac{d^2\phi}{dx^2}\right)^2 dx$$

3. Resolver la estructura con E, I, A constantes por el método de Rayleigh-Ritz



Solución

La solución exacta es un polinomio de tercer grado, la aproximación del campo de desplazamientos será

$$v(x) = \begin{cases} \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 & \text{para} \quad 0 \leqslant x \leqslant \frac{L}{2} \\ \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 & \text{para} \quad \frac{L}{2} \leqslant x \leqslant L \end{cases}$$

Tramo $0 \leqslant x \leqslant \frac{L}{2}$

Reemplazando $v(0) = 0$ y $v'(\frac{L}{2}) = 0$

$$\alpha_0 + \alpha_1(0) + \alpha_2(0)^2 + \alpha_3(0)^3 = 0$$
$$\alpha_1 + 2\alpha_2\left(\frac{L}{2}\right) + 3\alpha_3\left(\frac{L}{2}\right)^2 = 0$$

Resolviendo

$$\alpha_0 = 0$$
$$\alpha_1 = -\left(L\alpha_2 + \frac{3L^2}{4}\alpha_3\right)$$

Reemplazando en el campo de desplazamientos

$$v = -\left(L\alpha_2 + \frac{3L^2}{4}\alpha_3\right)x + \alpha_2 x^2 + \alpha_3 x^3$$

La curvatura es

$$\kappa = \frac{d^2 v}{dx^2} = 2\alpha_2 + 6\alpha_3 x$$

El funcional de energía es

$$\pi = \int_0^{\frac{L}{2}} \frac{EI}{2}\,\kappa^2\,dx - \frac{P}{2}\,v(\tfrac{L}{2})$$

Reemplazando

$$\begin{aligned}\pi = &\int_0^{\frac{L}{2}} \frac{EI}{2}\,(2\alpha_2 + 6\alpha_3 x)^2\,dx \\ &- \frac{P}{2}\left[-\left(L\alpha_2 + \frac{3L^2}{4}\alpha_3\right)\left(\frac{L}{2}\right) + \alpha_2\left(\frac{L}{2}\right)^2 + \alpha_3\left(\frac{L}{2}\right)^3\right]\end{aligned}$$

Integrando

$$\pi = \frac{PL^2}{8}\alpha_2 + \frac{PL^3}{8}\alpha_3 + EIL\alpha_2^2 + \frac{3EIL^2}{2}\alpha_2\alpha_3 + \frac{3EIL^3}{4}\alpha_3^2$$

Minimizando el funcional

$$\begin{aligned}\frac{\partial \pi}{\partial \alpha_2} &= 2EIL\,\alpha_2 + \frac{3}{2}EIL^2\,\alpha_3 + \frac{PL^2}{8} = 0 \\ \frac{\partial \pi}{\partial \alpha_3} &= \frac{3}{2}EIL^2\,\alpha_2 + \frac{3}{2}EIL^3\,\alpha_3 + \frac{PL^3}{8} = 0\end{aligned}$$

Formando el sistema de ecuaciones

$$\begin{aligned}4EIL\,\alpha_2 + 3EIL^2\,\alpha_3 &= -\frac{PL^2}{4} \\ EIL^2\,\alpha_2 + EIL^3\,\alpha_3 &= -\frac{PL^3}{12}\end{aligned}$$

Resolviendo

$$\begin{aligned}\alpha_2 &= 0 \\ \alpha_3 &= -\frac{P}{12EI}\end{aligned}$$

Reemplazando en v

$$v(x) = \frac{PL^2}{16EI}x - \frac{P}{12EI}x^3$$

Factorizando

$$v(x) = \frac{P}{48EI}\,x(3L^2 - 4x^2)$$

Tramo $\frac{L}{2} \leqslant x \leqslant L$

Reemplazando $v(L) = 0$ y $v'(\frac{L}{2}) = 0$

$$\begin{aligned}\beta_0 + \beta_1(L) + \beta_2(L)^2 + \beta_3(L)^3 &= 0 \\ \beta_1 + 2\beta_2\left(\frac{L}{2}\right) + 3\beta_3\left(\frac{L}{2}\right)^2 &= 0\end{aligned}$$

Resolviendo

$$\begin{aligned}\beta_0 &= -\frac{L^3}{4}\beta_3 \\ \beta_1 &= -\left(L\beta_2 + \frac{3L^2}{4}\beta_3\right)\end{aligned}$$

Reemplazando en el campo de desplazamientos

$$v = -\frac{L^3}{4}\beta_3 - \left(L\beta_2 + \frac{3L^2}{4}\beta_3\right)x + \beta_2 x^2 + \beta_3 x^3$$

La curvatura es

$$\kappa = \frac{d^2 v}{dx^2} = 2\beta_2 + 6\beta_3 x$$

El funcional de energía es

$$\pi = \int_{\frac{L}{2}}^{L} \frac{EI}{2}\,\kappa^2\,dx - \frac{P}{2}\,v(\tfrac{L}{2})$$

Reemplazando

$$\begin{aligned}\pi = &\int_{\frac{L}{2}}^{L} \frac{EI}{2}\,(2\beta_2 + 6\beta_3 x)^2\,dx \\ &- \frac{P}{2}\left[-\frac{L^3}{4}\beta_3 - \left(L\beta_2 + \frac{3L^2}{4}\beta_3\right)\left(\frac{L}{2}\right) + \beta_2\left(\frac{L}{2}\right)^2 + \beta_3\left(\frac{L}{2}\right)^3\right]\end{aligned}$$

Integrando

$$\pi = \frac{PL^2}{8}\beta_2 + \frac{PL^3}{4}\beta_3 + EIL\beta_2^2 + \frac{9EIL^2}{2}\beta_2\beta_3 + \frac{21EIL^3}{4}\beta_3^2$$

Minimizando el funcional

$$\begin{aligned}\frac{\partial\pi}{\partial\beta_2} &= 2EIL\,\beta_2 + \frac{9}{2}EIL^2\,\beta_3 + \frac{PL^2}{8} = 0 \\ \frac{\partial\pi}{\partial\beta_3} &= \frac{9}{2}EIL^2\,\beta_2 + \frac{21}{2}EIL^3\,\beta_3 + \frac{PL^3}{4} = 0\end{aligned}$$

Formando el sistema de ecuaciones

$$\begin{aligned}4EIL\,\beta_2 + 9EIL^2\,\beta_3 &= -\frac{PL^2}{4} \\ 9EIL^2\,\beta_2 + 21EIL^3\,\beta_3 &= -\frac{PL^3}{2}\end{aligned}$$

Resolviendo

$$\begin{aligned}\beta_2 &= -\frac{PL}{4EI} \\ \beta_3 &= \frac{P}{12EI}\end{aligned}$$

Reemplazando en v

$$v(x) = -\frac{PL^3}{48EI} + \frac{3PL^2}{16EI}x - \frac{PL}{4EI}x^2 + \frac{P}{12EI}x^3$$

Factorizando

$$v(x) = \frac{P}{48EI}\,(L-x)(L^2 - 8Lx + 4x^2)$$

La solución completa será

$$v(x) = \begin{cases} \frac{P}{48EI}\,x(3L^2 - 4x^2) & \text{para} \quad 0 \leqslant x \leqslant \frac{L}{2} \\ \frac{P}{48EI}\,(L-x)(L^2 - 8Lx + 4x^2) & \text{para} \quad \frac{L}{2} \leqslant x \leqslant L \end{cases}$$

4. Obtener la expresión de la matriz de rigidez de un elemento cualquiera mediante la energía potencial total

Solución

Funcional de energía

$$\pi = \iiint\limits_V \frac{1}{2}\,\varepsilon^{\mathrm{T}}\,\sigma\,dV - \iiint\limits_V u^{\mathrm{T}} f_V\,dV - \iint\limits_\Omega u^{\mathrm{T}} f_\Omega\,d\Omega - \sum_{i=1}^{n} u_i^{\mathrm{T}} P_i$$

Reemplazando la ley de Hooke generalizada y los campos de aproximación para desplazamientos y deformaciones unitarias

$$\begin{aligned}\pi = &\iiint\limits_V \frac{1}{2}\,(B\,u_i)^{\mathrm{T}}\,C\,(B\,u_i)\,dV - \iiint\limits_V (N\,u_i)^{\mathrm{T}} f_V\,dV \\ &- \iint\limits_\Omega (N\,u_i)^{\mathrm{T}} f_\Omega\,d\Omega - \sum_{i=1}^{n} u_i^{\mathrm{T}} P_i\end{aligned}$$

Reordenando términos

$$\begin{aligned}\pi = &\iiint\limits_V \frac{1}{2}\,u_i^{\mathrm{T}} B^{\mathrm{T}}\,C\,B\,u_i\,dV - \iiint\limits_V u_i^{\mathrm{T}} N^{\mathrm{T}} f_V\,dV \\ &- \iint\limits_\Omega u_i^{\mathrm{T}} N^{\mathrm{T}} f_\Omega\,d\Omega - \sum_{i=1}^{n} u_i^{\mathrm{T}} P_i\end{aligned}$$

Las constantes salen del integrando

$$\pi = \frac{1}{2}\, u_i^{\mathrm{T}} \iiint\limits_V B^{\mathrm{T}}\, C\, B\, dV\, u_i - \, u_i^{\mathrm{T}} \iiint\limits_V N^{\mathrm{T}} f_V\, dV$$
$$- \, u_i^{\mathrm{T}} \iint\limits_\Omega N^{\mathrm{T}} f_\Omega\, d\Omega - \, u_i^{\mathrm{T}} \sum_{i=1}^{n} P_i$$

Minimizando el funcional

$$\frac{\partial \pi}{\partial u_i} = \iiint\limits_V B^{\mathrm{T}}\, C\, B\, dV\, u_i - \iiint\limits_V N^{\mathrm{T}} f_V\, dV - \iint\limits_\Omega N^{\mathrm{T}} f_\Omega\, d\Omega - \sum_{i=1}^{n} P_i = 0$$

Reordenando

$$\iiint\limits_V B^{\mathrm{T}}\, C\, B\, dV\, u_i = \iiint\limits_V N^{\mathrm{T}} f_V\, dV + \iint\limits_\Omega N^{\mathrm{T}} f_\Omega\, d\Omega + \sum_{i=1}^{n} P_i$$

La matriz de rigidez es

$$K = \iiint\limits_V B^{\mathrm{T}}\, C\, B\, dV$$