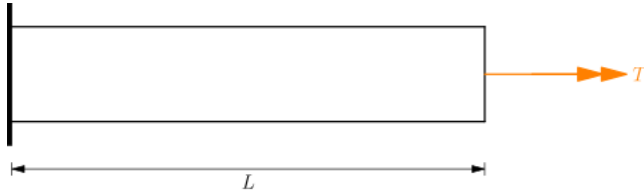


Introducción a elementos finitos

Examen final I-2016

1. Calcular la matriz de rigidez \mathbf{K} mediante el método de balance de energía



Solución

Deformación de la barra

$$\phi = \frac{TL}{GI_p}$$

Despejando T

$$T = \frac{GI_p}{L} \phi$$

Energía de deformación por torsión

$$U_i = \frac{1}{2} T \phi dx$$

Reemplazando T en U_i

$$U_i = \frac{1}{2} \frac{GI_p}{L} \phi^2$$

La deformación unitaria es

$$\theta = \frac{\phi}{L}$$

Despejando ϕ

$$\phi = L \theta$$

Reemplazando ϕ en U_i

$$U_i = \frac{1}{2} GI_p L \theta^2$$

Para condiciones variables

$$U_i = \int \frac{1}{2} GI_p \theta^2 dx$$

Funcional de energía

$$\pi = \int_0^L \frac{1}{2} GI_p \theta^2 dx - T(\theta_2 - \theta_1) = \int_0^L \frac{1}{2} \theta^T GI_p \theta dx - \sum_{i=1}^2 \theta_i T_i$$

Aproximación del campo de desplazamientos

$$\phi = \alpha_0 + \alpha_1 x = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$$

Reemplazando $\phi(0) = \phi_1$ y $\phi(L) = \phi_2$

$$\alpha_0 + \alpha_1(0) = \phi_1$$

$$\alpha_0 + \alpha_1(L) = \phi_2$$

Simplificando

$$\alpha_0 = \phi_1$$

$$\alpha_0 + L\alpha_1 = \phi_2$$

En forma matricial

$$\begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

Resolviendo

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

Reemplazando

$$\phi = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{L}x & \frac{1}{L}x \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \mathbf{N} \phi_i$$

Deformación angular

$$\theta = \frac{d\phi}{dx} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \mathbf{B} \phi_i$$

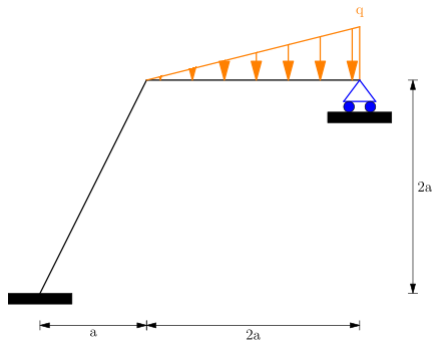
Matriz de rigidez

$$\mathbf{K} = \int_0^L \mathbf{B}^T G I_p \mathbf{B} dx$$

Reemplazando e integrando

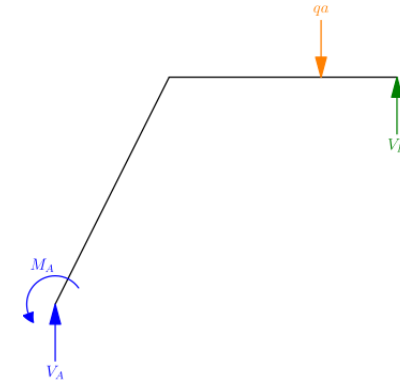
$$\mathbf{K} = \int_0^L \begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} G I_p \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} dx = \frac{G I_p}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

2. Resolver la estructura por cualquier método



Solución

Estructura equivalente



Sumatoria de fuerzas y momentos

$$\begin{aligned} V_A + V_B - qa &= 0 \\ M_A - qa\left(\frac{7}{3}a\right) + V_B(3a) &= 0 \\ qa\left(\frac{2}{3}a\right) - V_A(3a) &= 0 \end{aligned}$$

Resolviendo

$$\begin{aligned} V_A &= \frac{2}{9} qa \\ M_A &= \frac{7}{6} qa^2 \\ V_B &= \frac{7}{9} qa \end{aligned}$$

3. Calcular la integral mediante la cuadratura de Newton-Cotes para $n = 2$, los pesos w_i y los puntos de muestreo r_i

$$I = \int_0^2 x^2 - 5x dx$$

Solución

Número de términos

$$k = n - 1 = 2 - 1 = 1$$

Calculando r_i

$$\int_{-1}^{+1} P(r) r^0 dr = 0$$

$$\int_{-1}^{+1} P(r) r^1 dr = 0$$

$$w_1 = \int_{-1}^{+1} \frac{r - r_2}{r_1 - r_2} dr$$

$$w_2 = \int_{-1}^{+1} \frac{r - r_1}{r_2 - r_1} dr$$

El polinomio es

$$P(r) = (r - r_1)(r - r_2)$$

Reemplazando

$$\int_{-1}^{+1} (r - r_1)(r - r_2) dr = 0$$

$$\int_{-1}^{+1} (r - r_1)(r - r_2)r dr = 0$$

Integrando

$$\left(\frac{1}{3}r^3 - \frac{r_1 + r_2}{2}r^2 + r_1r_2r \right) \Big|_{-1}^{+1} = 2\left(r_1r_2 + \frac{1}{3} \right)$$

$$\left(\frac{1}{4}r^4 - \frac{r_1 + r_2}{3}r^3 + \frac{r_1r_2}{2}r^2 \right) \Big|_{-1}^{+1} = -\frac{2}{3}(r_1 + r_2)$$

Formando el sistema de ecuaciones

$$r_1r_2 = -\frac{1}{3}$$

$$r_1 + r_2 = 0$$

Resolviendo

$$r_1 = -\sqrt{\frac{1}{3}}$$

$$r_2 = \sqrt{\frac{1}{3}}$$

Calculando w_i

Reemplazando e integrando

$$w_1 = \int_{-1}^{+1} \frac{r - \sqrt{\frac{1}{3}}}{-\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{3}}} dr = \left(-\frac{\sqrt{3}}{4}r^2 + \frac{1}{2}r \right) \Big|_{-1}^{+1} = 1$$

$$w_2 = \int_{-1}^{+1} \frac{r + \sqrt{\frac{1}{3}}}{\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{3}}} dr = \left(\frac{\sqrt{3}}{4}r^2 + \frac{1}{2}r \right) \Big|_{-1}^{+1} = 1$$

Usando la fórmula

$$I = w'_1 f(r'_1) + w'_2 f(r'_2)$$

Puntos de muestreo

$$r'_1 = \frac{b+a}{2} + \frac{b-a}{2}r_1 = \frac{2+0}{2} + \frac{2-0}{2} \left(-\sqrt{\frac{1}{3}} \right) = 0.42265$$

$$r'_2 = \frac{b+a}{2} + \frac{b-a}{2}r_2 = \frac{2+0}{2} + \frac{2-0}{2} \left(\sqrt{\frac{1}{3}} \right) = 1.57735$$

Pesos

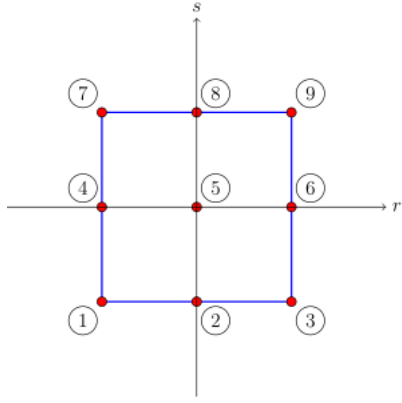
$$w'_1 = \frac{b-a}{2}w_1 = \frac{2-0}{2}(1) = 1$$

$$w'_2 = \frac{b-a}{2}w_2 = \frac{2-0}{2}(1) = 1$$

Reemplazando

$$I = 1 \left[0.42265^2 - 5(0.42265) \right] + 1 \left[1.57735^2 - 5(1.57735) \right] = -7.33333$$

4. Calcular las funciones de forma N



Solución

Coordenadas de los nodos

$$\begin{aligned} \textcircled{1} &= [r_1, s_1] = [-1, -1] & \textcircled{6} &= [r_6, s_6] = [1, 0] \\ \textcircled{2} &= [r_2, s_2] = [0, -1] & \textcircled{7} &= [r_7, s_7] = [-1, 1] \\ \textcircled{3} &= [r_3, s_3] = [1, -1] & \textcircled{8} &= [r_8, s_8] = [0, 1] \\ \textcircled{4} &= [r_4, s_4] = [-1, 0] & \textcircled{9} &= [r_9, s_9] = [1, 1] \\ \textcircled{5} &= [r_5, s_5] = [0, 0] \end{aligned}$$

Reemplazando valores

$$\begin{aligned} N_1 &= \frac{r-r_2}{r_1-r_2} \cdot \frac{r-r_3}{r_1-r_3} \cdot \frac{s-s_4}{s_1-s_4} \cdot \frac{s-s_7}{s_1-s_7} = \frac{r-0}{-1-0} \cdot \frac{r-1}{-1-1} \cdot \frac{s-0}{-1-0} \cdot \frac{s-1}{-1-1} \\ &= \frac{1}{4} r(r-1)s(s-1) \\ N_2 &= \frac{r-r_1}{r_2-r_1} \cdot \frac{r-r_3}{r_2-r_3} \cdot \frac{s-s_5}{s_2-s_5} \cdot \frac{s-s_8}{s_2-s_8} = \frac{r-(-1)}{0-(-1)} \cdot \frac{r-1}{0-1} \cdot \frac{s-0}{-1-0} \cdot \frac{s-1}{-1-1} \\ &= -\frac{1}{2} r(r+1)s(s-1) \end{aligned}$$

$$\begin{aligned} N_3 &= \frac{r-r_1}{r_3-r_1} \cdot \frac{r-r_2}{r_3-r_2} \cdot \frac{s-s_6}{s_3-s_6} \cdot \frac{s-s_9}{s_3-s_9} = \frac{r-(-1)}{1-(-1)} \cdot \frac{r-0}{1-0} \cdot \frac{s-0}{-1-0} \cdot \frac{s-1}{-1-1} \\ &= \frac{1}{4} r(r+1)s(s-1) \\ N_4 &= \frac{r-r_5}{r_4-r_5} \cdot \frac{r-r_6}{r_4-r_6} \cdot \frac{s-s_1}{s_4-s_1} \cdot \frac{s-s_7}{s_4-s_7} = \frac{r-0}{-1-0} \cdot \frac{r-1}{-1-1} \cdot \frac{s-(-1)}{0-(-1)} \cdot \frac{s-1}{0-1} \\ &= -\frac{1}{2} r(r-1)(s+1)(s-1) \\ N_5 &= \frac{r-r_4}{r_5-r_4} \cdot \frac{r-r_6}{r_5-r_6} \cdot \frac{s-s_2}{s_5-s_2} \cdot \frac{s-s_8}{s_5-s_8} = \frac{r-(-1)}{0-(-1)} \cdot \frac{r-1}{0-1} \cdot \frac{s-(-1)}{0-(-1)} \cdot \frac{s-1}{0-1} \\ &= (r+1)(r-1)(s+1)s(s-1) \\ N_6 &= \frac{r-r_4}{r_6-r_4} \cdot \frac{r-r_5}{r_6-r_5} \cdot \frac{s-s_3}{s_6-s_3} \cdot \frac{s-s_9}{s_6-s_9} = \frac{r-(-1)}{1-(-1)} \cdot \frac{r-0}{1-0} \cdot \frac{s-(-1)}{0-(-1)} \cdot \frac{s-1}{0-1} \\ &= -\frac{1}{2} r(r-1)(s+1)(s-1) \\ N_7 &= \frac{r-r_8}{r_7-r_8} \cdot \frac{r-r_9}{r_7-r_9} \cdot \frac{s-s_1}{s_7-s_1} \cdot \frac{s-s_4}{s_7-s_4} = \frac{r-0}{-1-0} \cdot \frac{r-1}{-1-1} \cdot \frac{s-(-1)}{1-(-1)} \cdot \frac{s-0}{1-0} \\ &= \frac{1}{4} r(r-1)s(s+1) \\ N_8 &= \frac{r-r_7}{r_8-r_7} \cdot \frac{r-r_9}{r_8-r_9} \cdot \frac{s-s_2}{s_8-s_2} \cdot \frac{s-s_5}{s_8-s_5} = \frac{r-(-1)}{0-(-1)} \cdot \frac{r-1}{0-1} \cdot \frac{s-(-1)}{1-(-1)} \cdot \frac{s-0}{1-0} \\ &= -\frac{1}{2} (r+1)(r-1)s(s+1) \\ N_9 &= \frac{r-r_7}{r_9-r_7} \cdot \frac{r-r_8}{r_9-r_8} \cdot \frac{s-s_3}{s_9-s_3} \cdot \frac{s-s_6}{s_9-s_6} = \frac{r-(-1)}{1-(-1)} \cdot \frac{r-0}{1-0} \cdot \frac{s-(-1)}{1-(-1)} \cdot \frac{s-0}{1-0} \\ &= \frac{1}{4} r(r+1)s(s+1) \end{aligned}$$