Introducción al Método de Diferencias Finitas

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Capítulo 1

Diferenciación numérica

Capítulo 2

Ecuaciones diferenciales parciales

Capítulo 3

Ecuación lineal elíptica

Capítulo 4

Ecuación lineal hiperbólica

$$\frac{\partial T}{\partial t} + u\,\frac{\partial T}{\partial x} = 0$$

4.1. Métodos explícitos

4.1.1. FTCS

Aproximación hacia adelante de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t+\Delta t) - T(t)}{\Delta t}$$

Aproximación centrada de T_x

$$\frac{\partial T}{\partial x} = \frac{T(x+\Delta x) - T(x-\Delta x)}{2\Delta x}$$

Reemplazando

$$\frac{T(t+\Delta t) - T(t)}{\Delta t} + u\,\frac{T(x+\Delta x) - T(x-\Delta x)}{2\Delta x} = 0$$

Reordenando

$$T(t+\Delta t) - T(t) + \frac{u\Delta t}{2\Delta x}[T(x+\Delta x) - T(x-\Delta x)] = 0$$

Se reemplaza por C, el número de Courant-Friedrich-Levy

$$T(t+\Delta t) - T(t) + \frac{C}{2}[T(x+\Delta x) - T(x-\Delta x)] = 0$$

Reordenando

$$T(t+\Delta t) = T(t) - \frac{C}{2}[T(x+\Delta x) - T(x-\Delta x)]$$

Reescribiendo para su formulación matricial

$$T(t+\Delta t, x) = T(t,x) - \frac{C}{2}[T(t, x+\Delta x) - T(t, x-\Delta x)]$$

Intercambiando por los índices del mallado

$$T(n+1, j) = T(n,j) - \frac{C}{2}[T(n, j+1) - T(n, j-1)]$$

Reordenando

$$T(n+1, j) = \frac{C}{2}T(n, j-1) + T(n,j) - \frac{C}{2}T(n, j+1) \tag{4.1}$$

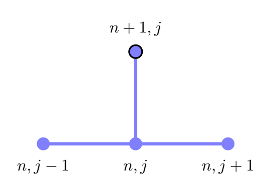

Figura 4.1: Stencil FTCS

Ejemplo 4.1.1.
Resolver la siguiente ecuación diferencial

$$\frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} = 0$$
$$T(x,0) = 0$$
$$T(0,t) = 10$$
$$T(6,t) = 3$$

Usando el esquema FTCS con $\Delta x = 1$ y $\Delta t = 1$

Solución
Verificando el número Courant

$$C = \frac{u\Delta t}{\Delta x} = \frac{1 \cdot 1}{1} = 1 \leqslant 1$$

Dibujando la malla

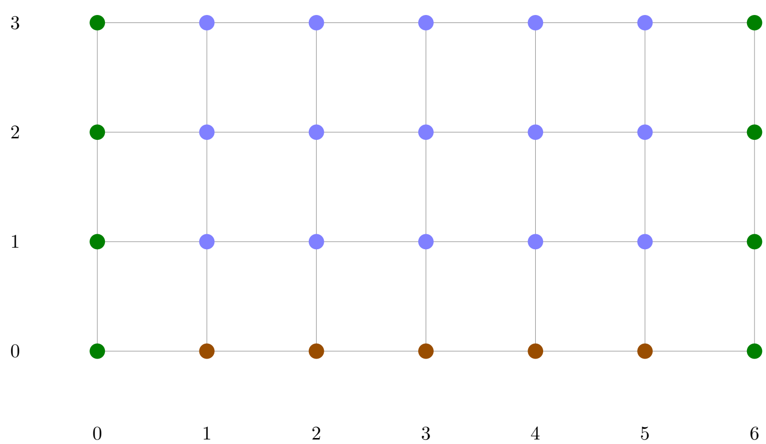

Figura 4.2: Mallado del problema

Creando una matriz de valores nulos

3	$T(0,3)$	$T(1,3)$	$T(2,3)$	$T(3,3)$	$T(4,3)$	$T(5,3)$	$T(6,3)$
2	$T(0,2)$	$T(1,2)$	$T(2,2)$	$T(3,2)$	$T(4,2)$	$T(5,2)$	$T(6,2)$
1	$T(0,1)$	$T(1,1)$	$T(2,1)$	$T(3,1)$	$T(4,1)$	$T(5,1)$	$T(6,1)$
0	$T(0,0)$	$T(1,0)$	$T(2,0)$	$T(3,0)$	$T(4,0)$	$T(5,0)$	$T(6,0)$
	0	1	2	3	4	5	6

Figura 4.3: Matriz solución sin valores

Rellenando los valores conocidos

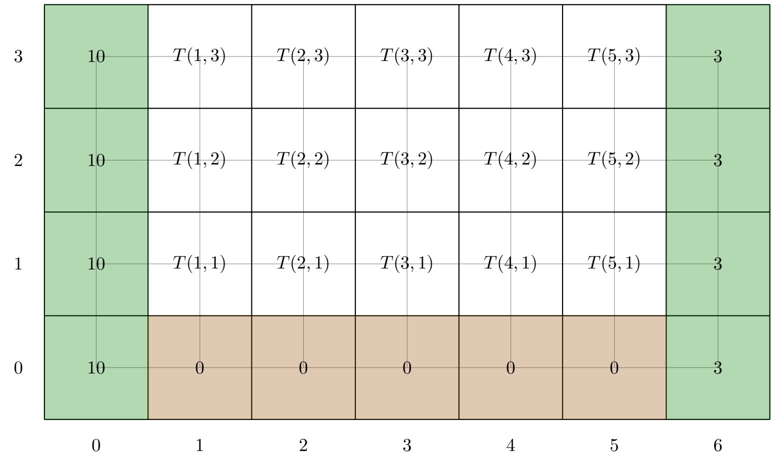

Figura 4.4: Matriz solución para $t = 0$

Para cinco pasos se requieren cinco ecuaciones, para $t = 1$

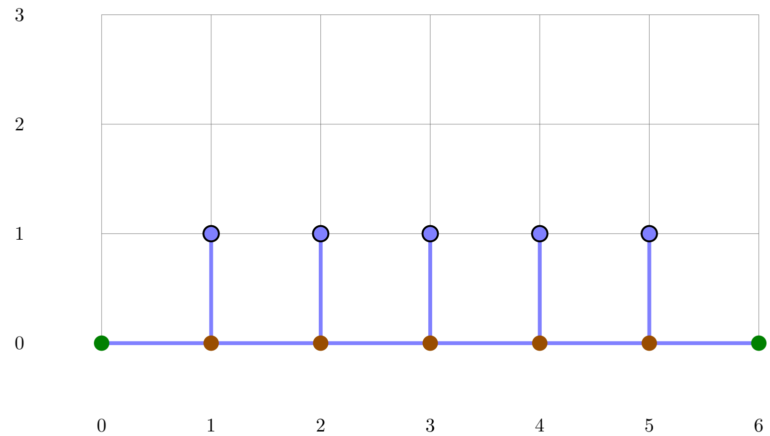

Figura 4.5: Movimiento del stencil para $t = 1$

$$T(1,1) = 0.5T(0,0) + T(0,1) - 0.5T(0,2) = 0.5(10) + 0 - 0.5(0) = 5$$
$$T(1,2) = 0.5T(0,1) + T(0,2) - 0.5T(0,3) = 0.5(0) + 0 - 0.5(0) = 0$$
$$T(1,3) = 0.5T(0,2) + T(0,3) - 0.5T(0,4) = 0.5(0) + 0 - 0.5(0) = 0$$
$$T(1,4) = 0.5T(0,3) + T(0,4) - 0.5T(0,5) = 0.5(0) + 0 - 0.5(0) = 0$$
$$T(1,5) = 0.5T(0,4) + T(0,5) - 0.5T(0,6) = 0.5(0) + 0 - 0.5(3) = -1.5$$

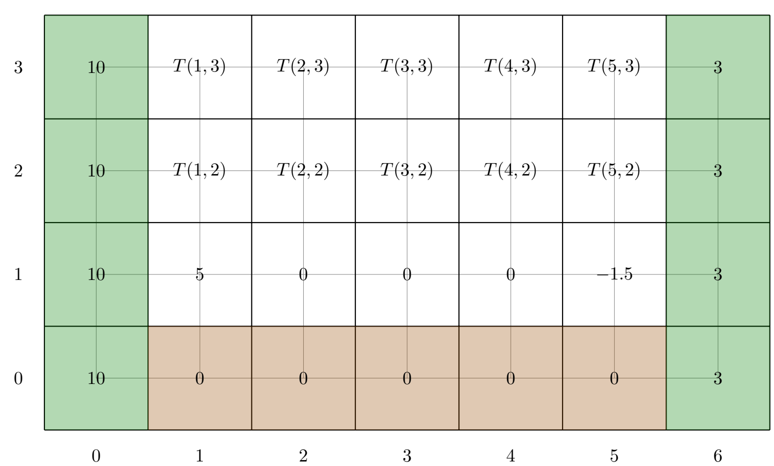

Figura 4.6: Matriz solución para $t = 1$

Para $t = 2$

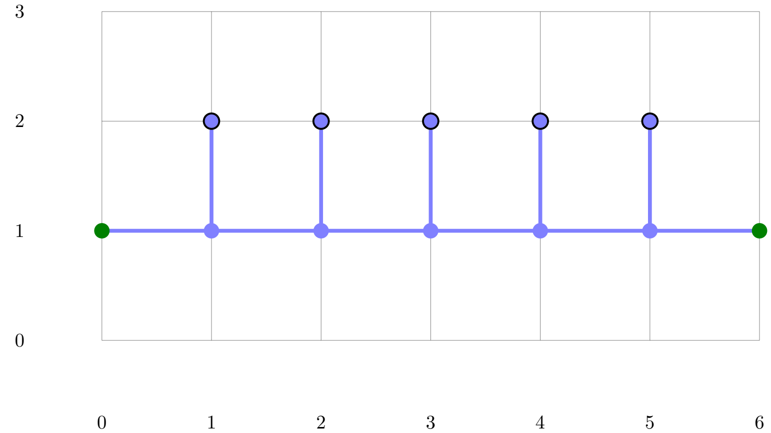

Figura 4.7: Movimiento del stencil para $t = 2$

$$
\begin{aligned}
T(1,1) &= 0.5T(0,0) + T(0,1) - 0.5T(0,2) = 0.5(10) + 5 - 0.5(0) = 10\\
T(1,2) &= 0.5T(0,1) + T(0,2) - 0.5T(0,3) = 0.5(5) + 0 - 0.5(0) = 2.5\\
T(1,3) &= 0.5T(0,2) + T(0,3) - 0.5T(0,4) = 0.5(0) + 0 - 0.5(0) = 0\\
T(1,4) &= 0.5T(0,3) + T(0,4) - 0.5T(0,5) = 0.5(0) + 0 - 0.5(-1.5) = 0.75\\
T(1,5) &= 0.5T(0,4) + T(0,5) - 0.5T(0,6) = 0.5(0) - 1.5 - 0.5(3) = -3
\end{aligned}
$$

3	10	$T(1,3)$	$T(2,3)$	$T(3,3)$	$T(4,3)$	$T(5,3)$	3
2	10	10	2.5	0	0.75	-3	3
1	10	5	0	0	0	-1.5	3
0	10	0	0	0	0	0	3
	0	1	2	3	4	5	6

Figura 4.8: Matriz solución para $t = 2$

Para $t = 3$

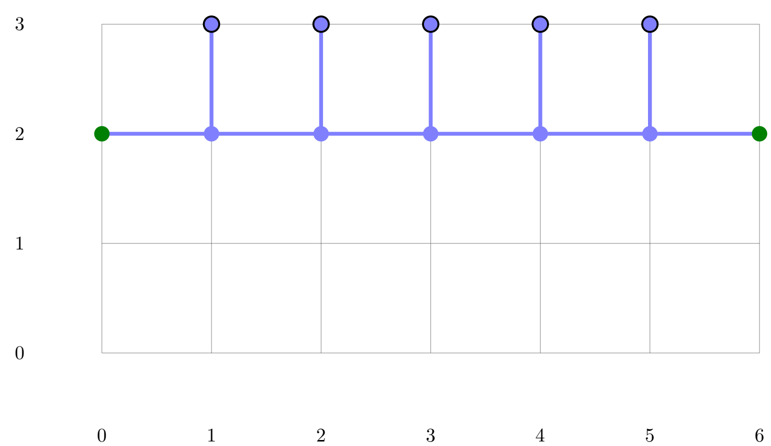

Figura 4.9: Movimiento del stencil para $t = 3$

$$
\begin{aligned}
T(1,1) &= 0.5T(0,0) + T(0,1) - 0.5T(0,2) = 0.5(10) + 10 - 0.5(2.5) = 13.75 \\
T(1,2) &= 0.5T(0,1) + T(0,2) - 0.5T(0,3) = 0.5(10) + 2.5 - 0.5(0) = 7.5 \\
T(1,3) &= 0.5T(0,2) + T(0,3) - 0.5T(0,4) = 0.5(2.5) + 0 - 0.5(0.75) = 0.875 \\
T(1,4) &= 0.5T(0,3) + T(0,4) - 0.5T(0,5) = 0.5(0) + 0.75 - 0.5(-3) = 2.25 \\
T(1,5) &= 0.5T(0,4) + T(0,5) - 0.5T(0,6) = 0.5(0.75) - 3 - 0.5(3) = -4.125
\end{aligned}
$$

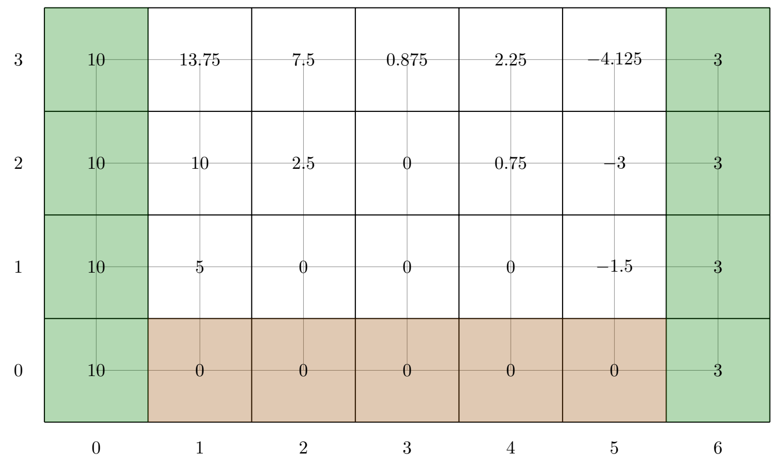

Figura 4.10: Matriz solución para $t = 3$

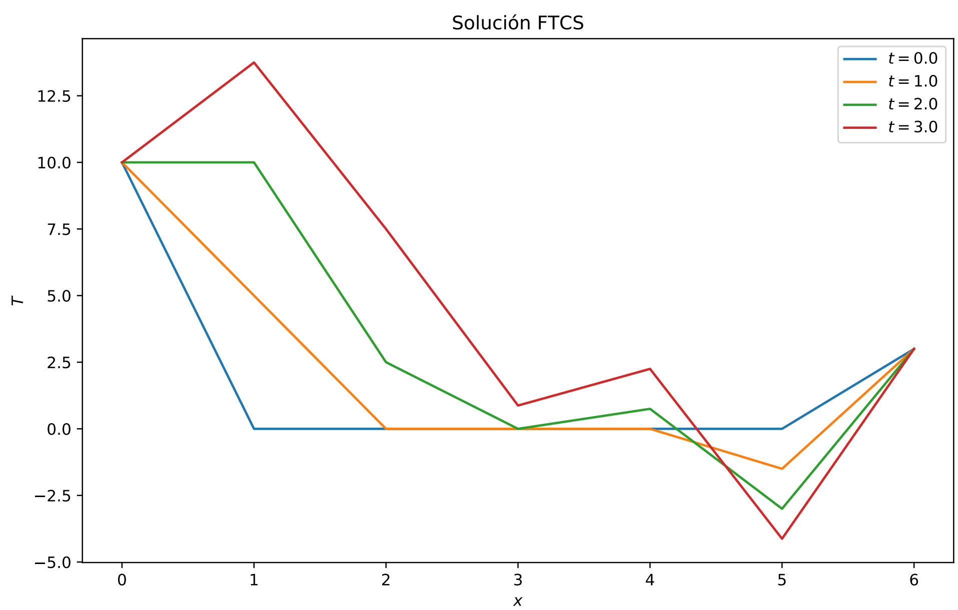

Figura 4.11: Solución numérica $\Delta x = 1$ y $\Delta t = 1$

4.1.2. FTFS o Downwind

Aproximación hacia adelante de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t+\Delta t) - T(t)}{\Delta t}$$

Aproximación hacia adelante de T_x

$$\frac{\partial T}{\partial x} = \frac{T(x+\Delta x) - T(x)}{\Delta x}$$

Reemplazando

$$\frac{T(t+\Delta t) - T(t)}{\Delta t} + u\,\frac{T(x+\Delta x) - T(x)}{\Delta x} = 0$$

Reordenando

$$T(t+\Delta t) - T(t) + \frac{u\Delta t}{\Delta x}[T(x+\Delta x) - T(x)] = 0$$

Se reemplaza por el término C

$$T(t+\Delta t) - T(t) + C[T(x+\Delta x) - T(x)] = 0$$

Reordenando

$$T(t+\Delta t) = T(t) - C[T(x+\Delta x) - T(x)]$$

Reescribiendo para su formulación matricial

$$T(t+\Delta t, x) = T(t,x) - C[T(t, x+\Delta x) - T(t,x)]$$

Intercambiando por los índices del mallado

$$T(n+1,j) = T(n,j) - C[T(n,j+1) - T(n,j)]$$

Reordenando

$$T(n+1,j) = (1+C)T(n,j) - CT(n,j+1) \tag{4.2}$$

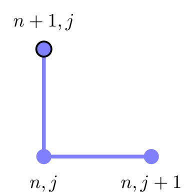

Figura 4.12: Stencil FTFS

4.1.3. FTBS o Upwind

Aproximación hacia adelante de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t+\Delta t) - T(t)}{\Delta t}$$

Aproximación hacia atrás de T_x

$$\frac{\partial T}{\partial x} = \frac{T(x) - T(x-\Delta x)}{\Delta x}$$

Reemplazando

$$\frac{T(t+\Delta t) - T(t)}{\Delta t} + u\,\frac{T(x) - T(x-\Delta x)}{\Delta x} = 0$$

Reordenando

$$T(t+\Delta t) - T(t) + \frac{u\Delta t}{\Delta x}[T(x) - T(x-\Delta x)] = 0$$

Se reemplaza por el término C

$$T(t+\Delta t) - T(t) + C[T(x) - T(x-\Delta x)] = 0$$

Reordenando

$$T(t+\Delta t) = T(t) - C[T(x) - T(x-\Delta x)]$$

Reescribiendo para su formulación matricial

$$T(t+\Delta t, x) = T(t,x) - C[T(t,x) - T(t, x-\Delta x)]$$

Intercambiando por los índices del mallado

$$T(n+1,j) = T(n,j) - C[T(n,j) - T(n,j-1)]$$

Reordenando

$$T(n+1,j) = CT(n,j-1) + (1-C)T(n,j) \tag{4.3}$$

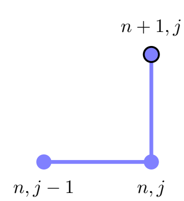

Figura 4.13: Stencil FTBS

4.1.4. CTCS o Leap-Frog

Aproximación centrada de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t+\Delta t) - T(t-\Delta t)}{2\Delta t}$$

Aproximación centrada de T_x

$$\frac{\partial T}{\partial x} = \frac{T(x+\Delta x) - T(x-\Delta x)}{2\Delta x}$$

Reemplazando

$$\frac{T(t+\Delta t) - T(t-\Delta t)}{2\Delta t} + u\,\frac{T(x+\Delta x) - T(x-\Delta x)}{2\Delta x} = 0$$

Reordenando

$$T(t+\Delta t) - T(t-\Delta t) + \frac{u\Delta t}{\Delta x}[T(x+\Delta x) - T(x-\Delta x)] = 0$$

Se reemplaza por el término C

$$T(t+\Delta t) - T(t-\Delta t) + C[T(x+\Delta x) - T(x-\Delta x)] = 0$$

Reordenando

$$T(t+\Delta t) = T(t-\Delta t) - C[T(x+\Delta x) - T(x-\Delta x)]$$

Reescribiendo para su formulación matricial

$$T(t+\Delta t, x) = T(t-\Delta t, x) - C[T(t, x+\Delta x) - T(t, x-\Delta x)]$$

Intercambiando por los índices del mallado

$$T(n+1, j) = T(n-1, j) - C[T(n, j+1) - T(n, j-1)]$$

Reordenando

$$T(n+1, j) = T(n-1, j) - CT(n, j+1) + CT(n, j-1) \tag{4.4}$$

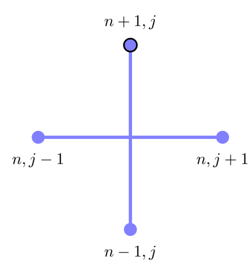

Figura 4.14: Stencil CTCS

4.1.5. Lax

Se deriva de FTCS

$$T(n+1, j) = T(n, j) - \frac{C}{2}[T(n, j+1) - T(n, j-1)]$$

El nodo actual se reemplaza por el promedio de los nodos adyacentes

$$T(n, j) = \frac{T(n, j-1) + T(n, j+1)}{2}$$

Reemplazando

$$T(n+1, j) = \frac{1}{2}[T(n, j-1) + T(n, j+1)] - \frac{C}{2}[T(n, j+1) - T(n, j-1)]$$

Reordenando

$$T(n+1, j) = \left(\frac{1+C}{2}\right)T(n, j-1) + \left(\frac{1-C}{2}\right)T(n, j+1) \tag{4.5}$$

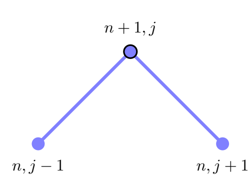

Figura 4.15: Stencil Lax

4.1.6. Lax-Friedrichs

Se deriva de FTCS

$$T(n+1,j) = T(n,j) - \frac{C}{2}[T(n,j+1) - T(n,j-1)]$$

El nodo actual se reemplaza por

$$T(n,j) = T(n,j) + \alpha T(n,j) - \alpha T(n,j)$$

Luego se reemplaza por el promedio ponderado de los nodos adyacentes

$$\alpha T(n,j) = \frac{\alpha T(n,j+1) + \alpha T(n,j-1)}{2}$$

Reemplazando y reordenando

$$T(n,j) = \frac{\alpha}{2}[T(n,j+1) + T(n,j-1)] + (1-\alpha)T(n,j)$$

Reemplazando en el método FTCS

$$T(n+1,j) = \frac{\alpha}{2}[T(n,j+1) + T(n,j-1)] + (1-\alpha)T(n,j) - \frac{C}{2}[T(n,j+1) - T(n,j-1)]$$

Reordenando

$$T(n+1,j) = \left(\frac{\alpha + C}{2}\right)T(n,j-1) + (1-\alpha)T(n,j) + \left(\frac{\alpha - C}{2}\right)T(n,j+1) \tag{4.6}$$

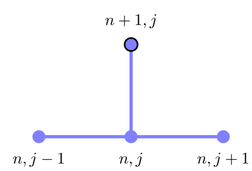

Figura 4.16: Stencil Lax-Friedrichs

4.1.7. Lax-Wendroff

Este método usa la serie de Taylor para mejorar el orden de aproximación

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2}(z-a)^2$$

Reemplazando $a = t$ y $z - a = \Delta t$

$$T(t+\Delta t, x) = T(t,x) + \Delta t\,\frac{\partial}{\partial t}T(t,x) + \frac{(\Delta t)^2}{2}\,\frac{\partial^2}{\partial t^2}T(t,x)$$

Las derivadas son

$$\frac{\partial T}{\partial t} = -u\,\frac{\partial T}{\partial x}$$
$$\frac{\partial^2 T}{\partial t^2} = \frac{\partial}{\partial t}\left(-u\,\frac{\partial T}{\partial x}\right) = -u\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial x}\right) = -u\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial t}\right) = u^2\,\frac{\partial^2 T}{\partial x^2}$$

Reemplazando

$$T(t+\Delta t, x) = T(t,x) - u\Delta t\,\frac{\partial}{\partial x}T(t,x) + \frac{(u\Delta t)^2}{2}\,\frac{\partial^2}{\partial x^2}T(t,x)$$

Aproximaciones centradas de T_x y T_{xx}

$$\frac{\partial T}{\partial x} = \frac{T(t,x+\Delta x) - T(t,x-\Delta x)}{2\Delta x}$$
$$\frac{\partial T^2}{\partial x^2} = \frac{T(t,x+\Delta x) - 2T(t,x) + T(t,x-\Delta x)}{(\Delta x)^2}$$

Reemplazando

$$T(t+\Delta t,x) = T(t,x) - \frac{u\Delta t}{2\Delta x}[T(t,x+\Delta x) - T(t,x-\Delta x)] + \frac{1}{2}\left(\frac{u\Delta t}{\Delta x}\right)^2[T(t,x+\Delta x) - 2T(t,x) + T(t,x-\Delta x)]$$

Reemplazando por el término C

$$T(t+\Delta t,x) = T(t,x) - \frac{C}{2}[T(t,x+\Delta x) - T(t,x-\Delta x)] + \frac{C^2}{2}[T(t,x+\Delta x) - 2T(t,x) + T(t,x-\Delta x)]$$

Intercambiando por los índices del mallado

$$T(n+1,j) = T(t,j) - \frac{C}{2}[T(n,j+1) - T(n,j-1)] + \frac{C^2}{2}[T(n,j+1) - 2T(n,j) + T(n,j-1)]$$

Reordenando

$$T(n+1,j) = \left(\frac{C^2+C}{2}\right)T(n,j-1) - (C^2-1)T(n,j) + \left(\frac{C^2-C}{2}\right)T(n,j+1)$$

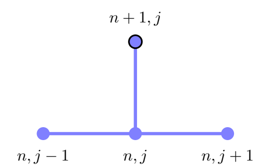

Figura 4.17: Stencil Lax-Wendroff

4.2. Métodos implícitos

4.2.1. FTCS implícito o Laasonen

Se deriva de FTCS

$$T(t+\Delta t) = T(t) - \frac{C}{2}[T(x+\Delta x) - T(x-\Delta x)]$$

Reordenando

$$T(t) = T(t+\Delta t) + \frac{C}{2}[T(x+\Delta x) - T(x-\Delta x)]$$

Reescribiendo

$$T(t,x) = T(t+\Delta t,x) + \frac{C}{2}[T(t,x+\Delta x) - T(t,x-\Delta x)]$$

Se reemplazará $t = t+\Delta t$ en los elementos multiplicados por $\frac{C}{2}$

$$T(t,x) = T(t+\Delta t,x) + \frac{C}{2}[T(t+\Delta t,x+\Delta x) - T(t+\Delta t,x-\Delta x)]$$

Simplificando

$$T(t,x) = -\frac{C}{2}T(t+\Delta t,x-\Delta x) + T(t+\Delta t,x) + \frac{C}{2}T(t+\Delta t,x+\Delta x)$$

Intercambiando por los índices del mallado

$$T(n,j) = -\frac{C}{2}T(n+1,j-1) + T(n+1,j) + \frac{C}{2}T(n+1,j+1)$$

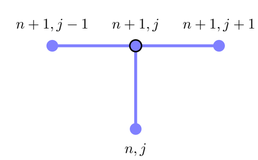

Figura 4.18: Stencil FTCS implícito

Para generalizar el método se usará tres pasos en el espacio

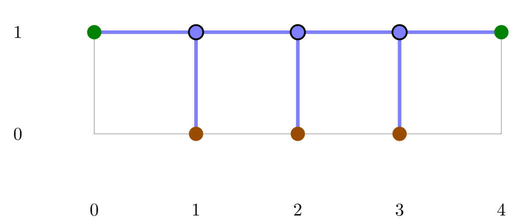

Figura 4.19: Movimiento del stencil para $t = 1$

Escribiendo una ecuación por cada paso

$$-\frac{C}{2}T[1,0]+T[1,1]+\frac{C}{2}T[1,2]=T[0,1]$$
$$-\frac{C}{2}T[1,1]+T[1,2]+\frac{C}{2}T[1,3]=T[0,2]$$
$$-\frac{C}{2}T[1,2]+T[1,3]+\frac{C}{2}T[1,4]=T[0,3]$$

Reescribiendo

$$\begin{aligned}
-\frac{C}{2}T[1,0]+\quad T[1,1]+\frac{C}{2}T[1,2]+\quad 0T[1,3]+\quad 0T[1,4]&=T[0,1]\\
0T[1,0]-\frac{C}{2}T[1,1]+\quad T[1,2]+\frac{C}{2}T[1,3]+\quad 0T[1,4]&=T[0,2]\\
0T[1,0]+\quad 0T[1,1]-\frac{C}{2}T[1,2]+\quad T[1,3]+\frac{C}{2}T[1,4]&=T[0,3]
\end{aligned}$$

Reordenando

$$\begin{aligned}
T[1,1]+\frac{C}{2}T[1,2]+\quad 0T[1,3]&=T[0,1]+\frac{C}{2}T[1,0]+\quad 0T[1,4]\\
-\frac{C}{2}T[1,1]+\quad T[1,2]+\frac{C}{2}T[1,3]&=T[0,2]+\quad 0T[1,0]+\quad 0T[1,4]\\
0T[1,1]-\frac{C}{2}T[1,2]+\quad T[1,3]&=T[0,3]+\quad 0T[1,0]-\frac{C}{2}T[1,4]
\end{aligned}$$

Simplificando

$$\begin{aligned}
T[1,1]+\frac{C}{2}T[1,2]+\quad 0T[1,3]&=T[0,1]+\frac{C}{2}T[1,0]\\
-\frac{C}{2}T[1,1]+\quad T[1,2]+\frac{C}{2}T[1,3]&=T[0,2]\\
0T[1,1]-\frac{C}{2}T[1,2]+\quad T[1,3]&=T[0,3]-\frac{C}{2}T[1,4]
\end{aligned}$$

En forma matricial

$$\begin{bmatrix}1 & \frac{C}{2} & 0\\ -\frac{C}{2} & 1 & \frac{C}{2}\\ 0 & -\frac{C}{2} & 1\end{bmatrix}\begin{bmatrix}T[1,1]\\T[1,2]\\T[1,3]\end{bmatrix}=\begin{bmatrix}T[0,1]\\T[0,2]\\T[0,3]\end{bmatrix}+\frac{C}{2}\begin{bmatrix}T[1,0]\\0\\0\end{bmatrix}-\frac{C}{2}\begin{bmatrix}0\\0\\T[1,4]\end{bmatrix}$$

Al resolver el sistema se obtienen los valores $T[1,x]$, también puede escribirse como

$$\begin{bmatrix}1 & \frac{C}{2} & 0\\ -\frac{C}{2} & 1 & \frac{C}{2}\\ 0 & -\frac{C}{2} & 1\end{bmatrix}\begin{bmatrix}T[1,1]\\T[1,2]\\T[1,3]\end{bmatrix}=\begin{bmatrix}T[0,1]+\frac{C}{2}T[1,0]\\T[0,2]\\T[0,3]-\frac{C}{2}T[1,4]\end{bmatrix}$$

Generalizando el método para s pasos

$$\begin{bmatrix}
1 & \frac{C}{2} & 0 & \cdots & \cdots & \cdots & 0\\
-\frac{C}{2} & 1 & \frac{C}{2} & \ddots & & & \vdots\\
0 & -\frac{C}{2} & 1 & \ddots & \ddots & & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & & \ddots & \ddots & 1 & \frac{C}{2} & 0\\
\vdots & & & \ddots & -\frac{C}{2} & 1 & \frac{C}{2}\\
0 & \cdots & \cdots & \cdots & 0 & -\frac{C}{2} & 1
\end{bmatrix}
\begin{bmatrix}T[n+1,1]\\T[n+1,2]\\\vdots\\\vdots\\\vdots\\T[n+1,s-1]\\T[n+1,s]\end{bmatrix}
=\begin{bmatrix}T[n,1]+\frac{C}{2}T[n+1,0]\\T[n,2]\\\vdots\\\vdots\\\vdots\\T[n,s-1]\\T[n,s]-\frac{C}{2}T[n+1,s+1]\end{bmatrix}\tag{4.7}$$

Ejemplo 4.2.1.
Resolver la siguiente ecuación diferencial

$$\frac{\partial T}{\partial t}+\frac{\partial T}{\partial x}=0$$
$$T(x,0)=0$$
$$T(0,t)=10$$
$$T(6,t)=3$$

Usando el esquema FTCS con $\Delta x=1$ y $\Delta t=1$

Solución
Verificando el número de Courant

$$C = \frac{u\Delta t}{\Delta x} = \frac{1 \cdot 1}{1} = 1 \leqslant 1$$

Dibujando la malla

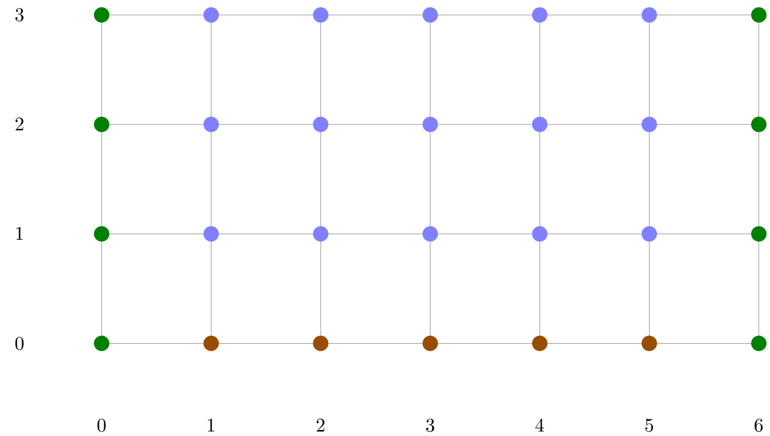

Figura 4.20: Mallado del problema

Creando una matriz de valores nulos

3	$T(0,3)$	$T(1,3)$	$T(2,3)$	$T(3,3)$	$T(4,3)$	$T(5,3)$	$T(6,3)$
2	$T(0,2)$	$T(1,2)$	$T(2,2)$	$T(3,2)$	$T(4,2)$	$T(5,2)$	$T(6,2)$
1	$T(0,1)$	$T(1,1)$	$T(2,1)$	$T(3,1)$	$T(4,1)$	$T(5,1)$	$T(6,1)$
0	$T(0,0)$	$T(1,0)$	$T(2,0)$	$T(3,0)$	$T(4,0)$	$T(5,0)$	$T(6,0)$
	0	1	2	3	4	5	6

Figura 4.21: Matriz solución sin valores

Rellenando los valores conocidos

3	10	$T(1,3)$	$T(2,3)$	$T(3,3)$	$T(4,3)$	$T(5,3)$	3
2	10	$T(1,2)$	$T(2,2)$	$T(3,2)$	$T(4,2)$	$T(5,2)$	3
1	10	$T(1,1)$	$T(2,1)$	$T(3,1)$	$T(4,1)$	$T(5,1)$	3
0	10	0	0	0	0	0	3
	0	1	2	3	4	5	6

Figura 4.22: Matriz solución para $t = 0$

Para cinco pasos se requieren cinco ecuaciones, para $t = 1$

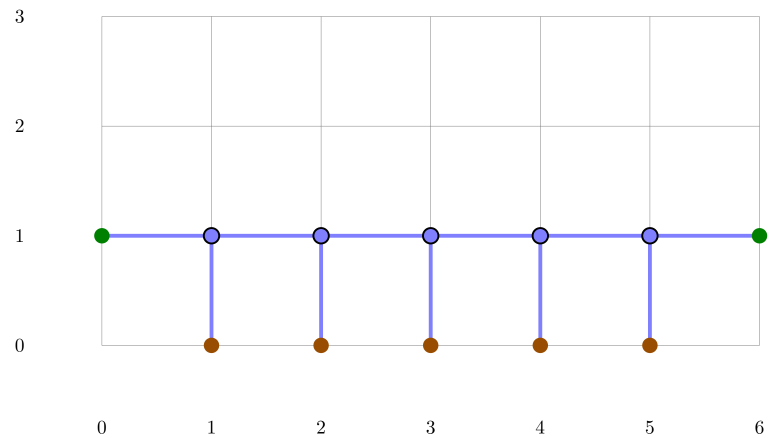

Figura 4.23: Movimiento del stencil para $t = 1$

Usando la fórmula

$$\begin{bmatrix} 1 & \frac{C}{2} & 0 & 0 & 0 \\ -\frac{C}{2} & 1 & \frac{C}{2} & 0 & 0 \\ 0 & -\frac{C}{2} & 1 & \frac{C}{2} & 0 \\ 0 & 0 & -\frac{C}{2} & 1 & \frac{C}{2} \\ 0 & 0 & 0 & -\frac{C}{2} & 1 \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \\ T[1,4] \\ T[1,5] \end{bmatrix} = \begin{bmatrix} T[0,1] + \frac{C}{2}T[1,0] \\ T[0,2] \\ T[0,3] \\ T[0,4] \\ T[0,5] - \frac{C}{2}T[1,6] \end{bmatrix}$$

Reemplazando valores

$$\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 \\ -0.5 & 1 & 0.5 & 0 & 0 \\ 0 & -0.5 & 1 & 0.5 & 0 \\ 0 & 0 & -0.5 & 1 & 0.5 \\ 0 & 0 & 0 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \\ T[1,4] \\ T[1,5] \end{bmatrix} = \begin{bmatrix} 0 + 0.5(10) \\ 0 \\ 0 \\ 0 \\ 0 - 0.5(3) \end{bmatrix}$$

Simplificando y resolviendo

$$\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 \\ -0.5 & 1 & 0.5 & 0 & 0 \\ 0 & -0.5 & 1 & 0.5 & 0 \\ 0 & 0 & -0.5 & 1 & 0.5 \\ 0 & 0 & 0 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \\ T[1,4] \\ T[1,5] \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ -1.5 \end{bmatrix} \qquad T_1 = \begin{bmatrix} 4.1 \\ 1.8 \\ 0.5 \\ 0.8 \\ -1.1 \end{bmatrix}$$

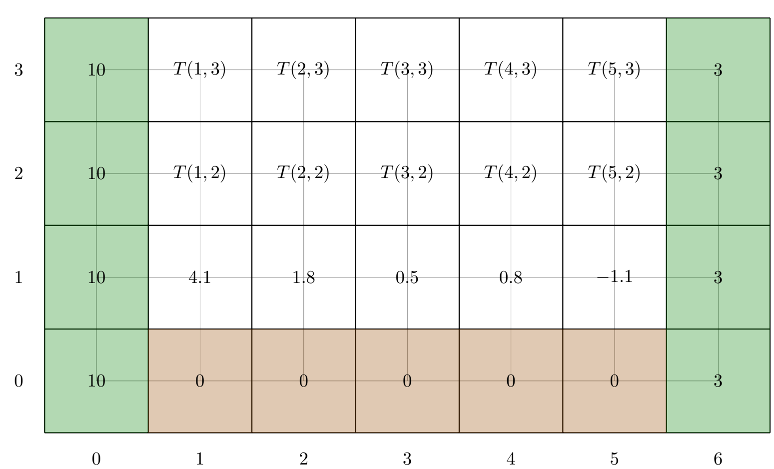

Figura 4.24: Matriz solución para $t = 1$

Para $t = 2$

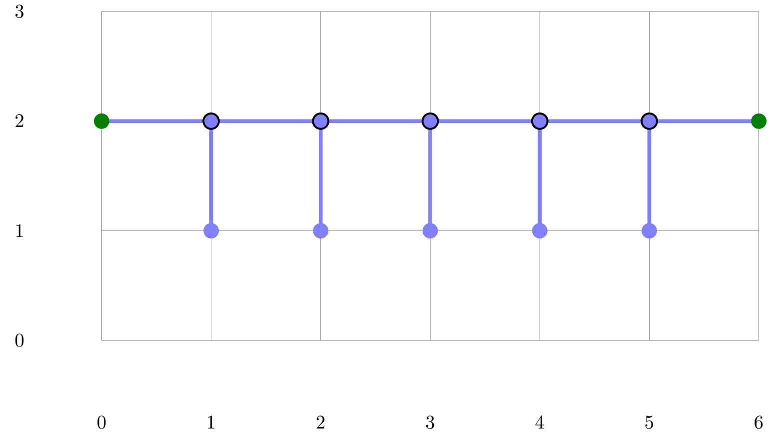

Figura 4.25: Movimiento del stencil para $t = 2$

Reemplazando valores

$$\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 \\ -0.5 & 1 & 0.5 & 0 & 0 \\ 0 & -0.5 & 1 & 0.5 & 0 \\ 0 & 0 & -0.5 & 1 & 0.5 \\ 0 & 0 & 0 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} T[2,1] \\ T[2,2] \\ T[2,3] \\ T[2,4] \\ T[2,5] \end{bmatrix} = \begin{bmatrix} 4.1 + 0.5(10) \\ 1.8 \\ 0.5 \\ 0.8 \\ -1.1 - 0.5(3) \end{bmatrix}$$

Simplificando y resolviendo

$$\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 \\ -0.5 & 1 & 0.5 & 0 & 0 \\ 0 & -0.5 & 1 & 0.5 & 0 \\ 0 & 0 & -0.5 & 1 & 0.5 \\ 0 & 0 & 0 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} T[2,1] \\ T[2,2] \\ T[2,3] \\ T[2,4] \\ T[2,5] \end{bmatrix} = \begin{bmatrix} 9.1 \\ 1.8 \\ 0.5 \\ 0.8 \\ -2.6 \end{bmatrix} \qquad T_2 = \begin{bmatrix} 6.874 \\ 4.451 \\ 1.571 \\ 2.308 \\ -1.446 \end{bmatrix}$$

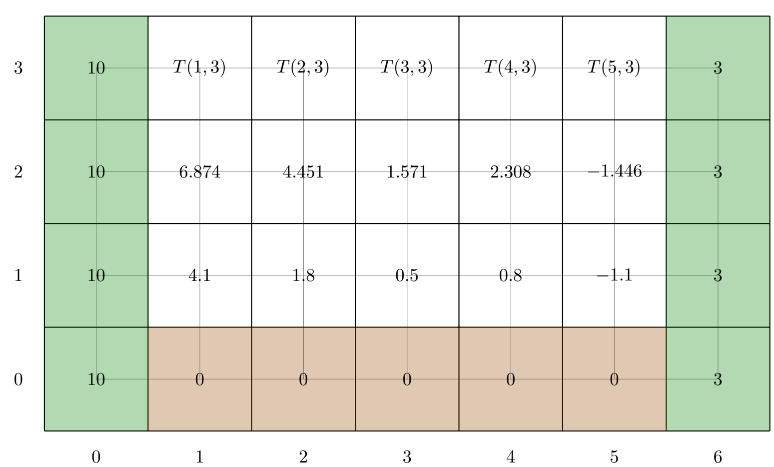

Figura 4.26: Matriz solución para $t = 2$

Para $t = 3$

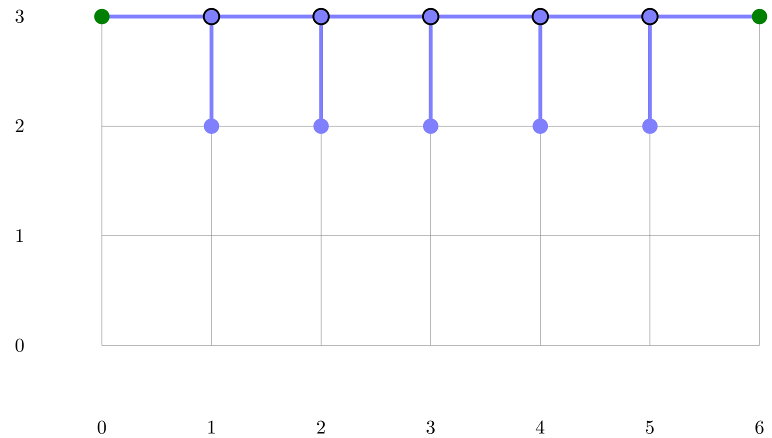

Figura 4.27: Movimiento del stencil para $t = 3$

Reemplazando valores

$$\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 \\ -0.5 & 1 & 0.5 & 0 & 0 \\ 0 & -0.5 & 1 & 0.5 & 0 \\ 0 & 0 & -0.5 & 1 & 0.5 \\ 0 & 0 & 0 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} T[3,1] \\ T[3,2] \\ T[3,3] \\ T[3,4] \\ T[3,5] \end{bmatrix} = \begin{bmatrix} 6.874 + 0.5(10) \\ 4.451 \\ 1.571 \\ 2.308 \\ -1.446 - 0.5(3) \end{bmatrix}$$

Simplificando y resolviendo

$$\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 \\ -0.5 & 1 & 0.5 & 0 & 0 \\ 0 & -0.5 & 1 & 0.5 & 0 \\ 0 & 0 & -0.5 & 1 & 0.5 \\ 0 & 0 & 0 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} T[3,1] \\ T[3,2] \\ T[3,3] \\ T[3,4] \\ T[3,5] \end{bmatrix} = \begin{bmatrix} 11.874 \\ 4.451 \\ 1.571 \\ 2.308 \\ -2.946 \end{bmatrix} \qquad T_3 = \begin{bmatrix} 8.321 \\ 7.107 \\ 3.010 \\ 4.229 \\ -0.831 \end{bmatrix}$$

3	10	8.321	7.107	3.010	4.229	-0.831	3
2	10	6.874	4.451	1.571	2.308	-1.446	3
1	10	4.1	1.8	0.5	0.8	-1.1	3
0	10	0	0	0	0	0	3
	0	1	2	3	4	5	6

Figura 4.28: Matriz solución para $t = 3$

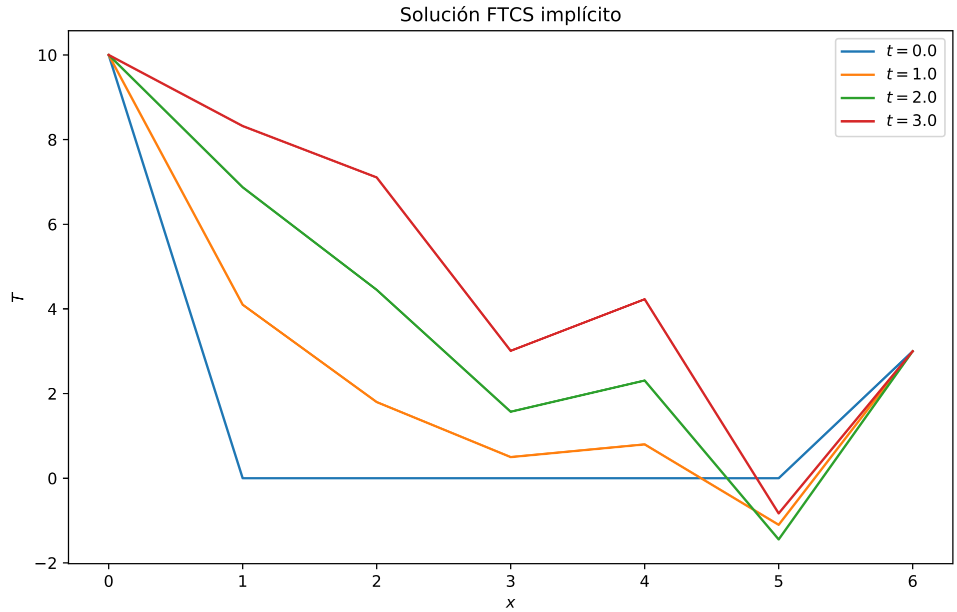

Figura 4.29: Solución numérica $\Delta x = 1$ y $\Delta t = 1$

4.2.2. BTCS

Aproximación hacia atrás de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t) - T(t-\Delta t)}{\Delta t}$$

Aproximación centrada de T_x

$$\frac{\partial T}{\partial x} = \frac{T(x+\Delta x) - T(x-\Delta x)}{2\Delta x}$$

Reemplazando

$$\frac{T(t) - T(t-\Delta t)}{\Delta t} + u\,\frac{T(x+\Delta x) - T(x-\Delta x)}{2\Delta x} = 0$$

Reordenando

$$T(t) - T(t-\Delta t) + \frac{u\Delta t}{2\Delta x}[T(x+\Delta x) - T(x-\Delta x)] = 0$$

Se reemplaza por el término C

$$T(t) - T(t-\Delta t) + \frac{C}{2}[T(x+\Delta x) - T(x-\Delta x)] = 0$$

Reordenando

$$T(t) = T(t-\Delta t) - \frac{C}{2}[T(x+\Delta x) - T(x-\Delta x)]$$

Reescribiendo para su formulación matricial

$$T(t,x) = T(t-\Delta t, x) - \frac{C}{2}[T(t, x+\Delta x) - T(t, x-\Delta x)]$$

Intercambiando por los índices del mallado

$$T(n,j) = T(n-1,j) - \frac{C}{2}[T(n,j+1) - T(n,j-1)]$$

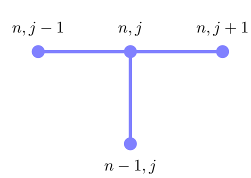

Figura 4.30: Stencil BTCS

4.2.3. Crank-Nicolson

Puede derivarse de métodos FTCS

$$T(t+\Delta t, x) = T(t,x) - \frac{C}{2}[T(t, x+\Delta x) - T(t, x-\Delta x)] \qquad \text{explícito}$$
$$T(t,x) = T(t+\Delta t, x) + \frac{C}{2}[T(t+\Delta t, x+\Delta x) - T(t+\Delta t, x-\Delta x)] \qquad \text{implícito}$$

Reordenando

$$T(t+\Delta t, x) - T(t,x) = -\frac{C}{2}[T(t, x+\Delta x) - T(t, x-\Delta x)]$$
$$T(t+\Delta t, x) - T(t,x) = -\frac{C}{2}[T(t+\Delta t, x+\Delta x) - T(t+\Delta t, x-\Delta x)]$$

Multiplicando por $\frac{1}{2}$ y sumando

$$T(t+\Delta t, x) - T(t,x) = -\frac{C}{4}[T(t, x+\Delta x) - T(t, x-\Delta x)] - \frac{C}{4}[T(t+\Delta t, x+\Delta x) - T(t+\Delta t, x-\Delta x)]$$

Simplificando

$$T(t+\Delta t, x) - T(t,x) = -\frac{C}{4}[T(t, x+\Delta x) - T(t, x-\Delta x) + T(t+\Delta t, x+\Delta x) - T(t+\Delta t, x-\Delta x)]$$

Reordenando

$$-\frac{C}{4}T(t, x-\Delta x) - T(t,x) + \frac{C}{4}T(t, x+\Delta x) = \frac{C}{4}T(t+\Delta t, x-\Delta x) - T(t+\Delta t, x) - \frac{C}{4}T(t+\Delta t, x+\Delta x)$$

Intercambiando por los índices del mallado

$$-\frac{C}{4}T(n, j-1) - T(n,j) + \frac{C}{4}T(n, j+1) = \frac{C}{4}T(n+1, j-1) - T(n+1, j) - \frac{C}{4}T(n+1, j+1)$$

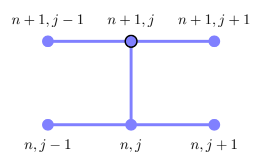

Figura 4.31: Stencil Crank-Nicolson

Para generalizar el método se usará tres pasos en el espacio

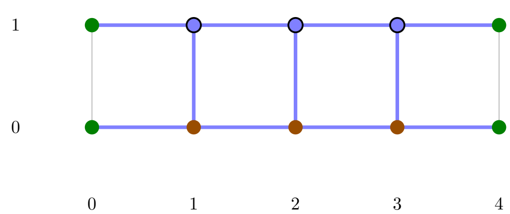

Figura 4.32: Movimiento del stencil para $t = 1$

Escribiendo una ecuación por cada paso

$$\begin{aligned}
\frac{C}{4}T[1,0] - T[1,1] - \frac{C}{4}T[1,2] &= -\frac{C}{4}T[0,0] - T[0,1] + \frac{C}{4}T[0,2] \\
\frac{C}{4}T[1,1] - T[1,2] - \frac{C}{4}T[1,3] &= -\frac{C}{4}T[0,1] - T[0,2] + \frac{C}{4}T[0,3] \\
\frac{C}{4}T[1,2] - T[1,3] - \frac{C}{4}T[1,4] &= -\frac{C}{4}T[0,2] - T[0,3] + \frac{C}{4}T[0,4]
\end{aligned}$$

Reordenando

$$\begin{aligned}
- \quad T[1,1] - \frac{C}{4}T[1,2] + \; 0T[1,3] &= - \quad T[0,1] + \frac{C}{4}T[0,2] + \; 0T[0,3] - \frac{C}{4}T[0,0] - \frac{C}{4}T[1,0] \\
\frac{C}{4}T[1,1] - \quad T[1,2] - \frac{C}{4}T[1,3] &= -\frac{C}{4}T[0,1] - \quad T[0,2] + \frac{C}{4}T[0,3] \\
0T[1,1] + \frac{C}{4}T[1,2] - \quad T[1,3] &= \quad T[0,1] - \frac{C}{4}T[0,2] - \quad T[0,3] + \frac{C}{4}T[0,4] + \frac{C}{4}T[1,4]
\end{aligned}$$

En forma matricial

$$\begin{bmatrix} -1 & -\frac{C}{4} & 0 \\ \frac{C}{4} & -1 & -\frac{C}{4} \\ 0 & \frac{C}{4} & -1 \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \end{bmatrix} = \begin{bmatrix} -1 & \frac{C}{4} & 0 \\ -\frac{C}{4} & -1 & \frac{C}{4} \\ 0 & -\frac{C}{4} & -1 \end{bmatrix} \begin{bmatrix} T[0,1] \\ T[0,2] \\ T[0,3] \end{bmatrix} - \frac{C}{4} \begin{bmatrix} T[0,0] \\ 0 \\ 0 \end{bmatrix} + \frac{C}{4} \begin{bmatrix} 0 \\ 0 \\ T[0,4] \end{bmatrix} - \frac{C}{4} \begin{bmatrix} T[1,0] \\ 0 \\ 0 \end{bmatrix} + \frac{C}{4} \begin{bmatrix} 0 \\ 0 \\ T[1,4] \end{bmatrix}$$

Al resolver el sistema se obtienen los valores $T[1,x]$, también puede escribirse como

$$\begin{bmatrix} -1 & -\frac{C}{4} & 0 \\ \frac{C}{4} & -1 & -\frac{C}{4} \\ 0 & \frac{C}{4} & -1 \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \end{bmatrix} = \begin{bmatrix} -1 & \frac{C}{4} & 0 \\ -\frac{C}{4} & -1 & \frac{C}{4} \\ 0 & -\frac{C}{4} & -1 \end{bmatrix} \begin{bmatrix} T[0,1] \\ T[0,2] \\ T[0,3] \end{bmatrix} + \frac{C}{4} \begin{bmatrix} -T[0,0] - T[1,0] \\ 0 \\ T[0,4] + T[1,4] \end{bmatrix}$$

Generalizando el método para s pasos

$$\begin{aligned} &\begin{bmatrix} -1 & -\frac{C}{4} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \frac{C}{4} & -1 & -\frac{C}{4} & \ddots & & & & \vdots \\ 0 & \frac{C}{4} & -1 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & -1 & -\frac{C}{4} & 0 \\ \vdots & & & & \ddots & \frac{C}{4} & -1 & -\frac{C}{4} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \frac{C}{4} & -1 \end{bmatrix} \begin{bmatrix} T[n+1,1] \\ T[n+1,2] \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ T[n+1,s-1] \\ T[n+1,s] \end{bmatrix} \\ &= \begin{bmatrix} -1 & \frac{C}{4} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -\frac{C}{4} & -1 & \frac{C}{4} & \ddots & & & & \vdots \\ 0 & -\frac{C}{4} & -1 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & -1 & \frac{C}{4} & 0 \\ \vdots & & & & \ddots & -\frac{C}{4} & -1 & \frac{C}{4} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & -\frac{C}{4} & -1 \end{bmatrix} \begin{bmatrix} T[n,1] \\ T[n,2] \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ T[n,s-1] \\ T[n,s] \end{bmatrix} + \frac{C}{4} \begin{bmatrix} -T[n,0] - T[n+1,0] \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ T[n,s+1] + T[n+1,s+1] \end{bmatrix} \end{aligned} \tag{4.8}$$

4.2.4. θ

Reemplazando en el método de Crank-Nicolson, $\theta = \frac{1}{4}$

$$-\theta C T(n,j-1) - T(n,j) + \theta C T(n,j+1) = \theta C T(n+1,j-1) - T(n+1,j) - \theta C T(n+1,j+1)$$

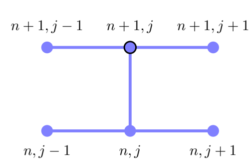

Figura 4.33: Stencil θ

Realizando los mismos pasos para los métodos anteriores, se obtiene

$$\begin{bmatrix} -1 & -\theta C & 0 \\ \theta C & -1 & -\theta C \\ 0 & \theta C & -1 \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \end{bmatrix} = \begin{bmatrix} -1 & \theta C & 0 \\ -\theta C & -1 & \theta C \\ 0 & -\theta C & -1 \end{bmatrix} \begin{bmatrix} T[0,1] \\ T[0,2] \\ T[0,3] \end{bmatrix} + \theta C \begin{bmatrix} -T[0,0] - T[1,0] \\ 0 \\ T[0,4] + T[1,4] \end{bmatrix}$$

Generalizando el método para s pasos

$$
\begin{bmatrix}
-1 & -\theta C & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\theta C & -1 & -\theta C & \ddots & & & & \vdots \\
0 & \theta C & -1 & \ddots & \ddots & & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & \ddots & \ddots & -1 & -\theta C & 0 \\
\vdots & & & & \ddots & \theta C & -1 & -\theta C \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & \theta C & -1
\end{bmatrix}
\begin{bmatrix}
T[n+1,1] \\ T[n+1,2] \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ T[n+1,s-1] \\ T[n+1,s]
\end{bmatrix}
$$

$$
= \begin{bmatrix}
-1 & \theta C & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
-\theta C & -1 & \theta C & \ddots & & & & \vdots \\
0 & -\theta C & -1 & \ddots & \ddots & & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & \ddots & \ddots & -1 & \theta C & 0 \\
\vdots & & & & \ddots & -\theta C & -1 & \theta C \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & -\theta C & -1
\end{bmatrix}
\begin{bmatrix}
T[n,1] \\ T[n,2] \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ T[n,s-1] \\ T[n,s]
\end{bmatrix}
+ \theta C
\begin{bmatrix}
-T[n,0] - T[n+1,0] \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ T[n,s+1] + T[n+1,s+1]
\end{bmatrix}
\tag{4.9}
$$

Capítulo 5

Ecuación lineal parabólica

$$\frac{\partial T}{\partial t} = D\,\frac{\partial^2 T}{\partial x^2}$$

5.1. Métodos explícitos

5.1.1. FTCS

Aproximación hacia adelante de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t+\Delta t) - T(t)}{\Delta t}$$

Aproximación centrada de T_{xx}

$$\frac{\partial^2 T}{\partial x^2} = \frac{T(x+\Delta x) - 2T(x) + T(x-\Delta x)}{\Delta x^2}$$

Reemplazando

$$\frac{T(t+\Delta t) - T(t)}{\Delta t} - D\,\frac{T(x+\Delta x) - 2T(x) + T(x-\Delta x)}{\Delta x^2} = 0$$

Reordenando

$$T(t+\Delta t) - T(t) - \frac{D\Delta t}{\Delta x^2}[T(x+\Delta x) - 2T(x) + T(x-\Delta x)] = 0$$

Se reemplaza por S, el número de difusión

$$T(t+\Delta t) - T(t) - S[T(x+\Delta x) - 2T(x) + T(x-\Delta x)] = 0$$

Reordenando

$$T(t+\Delta t) = T(t) + S[T(x+\Delta x) - 2T(x) + T(x-\Delta x)]$$

Reescribiendo para su formulación matricial

$$T(t+\Delta t, x) = T(t,x) + S[T(t, x+\Delta x) - 2T(t,x) + T(t, x-\Delta x)]$$

Intercambiando por los índices del mallado

$$T(n+1, j) = T(n,j) + S[T(n, j+1) - 2T(n,j) + T(n, j-1)]$$

Reordenando

$$T(n+1, j) = ST(n, j-1) + (1-2S)T(n,j) + ST(n, j+1) \tag{5.1}$$

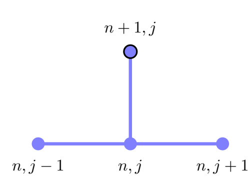

Figura 5.1: Stencil FTCS

Ejemplo 5.1.1.
Resolver la siguiente ecuación diferencial

$$\begin{aligned} \frac{\partial T}{\partial t} &= 0.4\frac{\partial^2 T}{\partial x^2} \\ T(x,0) &= 0 \\ T(0,t) &= 10 \\ T(6,t) &= 3 \end{aligned}$$

Usando el esquema FTCS con $\Delta x = 1$ y $\Delta t = 1$

Solución
Verificando el número de difusión

$$S = \frac{D\Delta t}{\Delta x^2} = \frac{0.4 \cdot 1}{1^2} = 0.4 \leqslant 0.5$$

Dibujando la malla

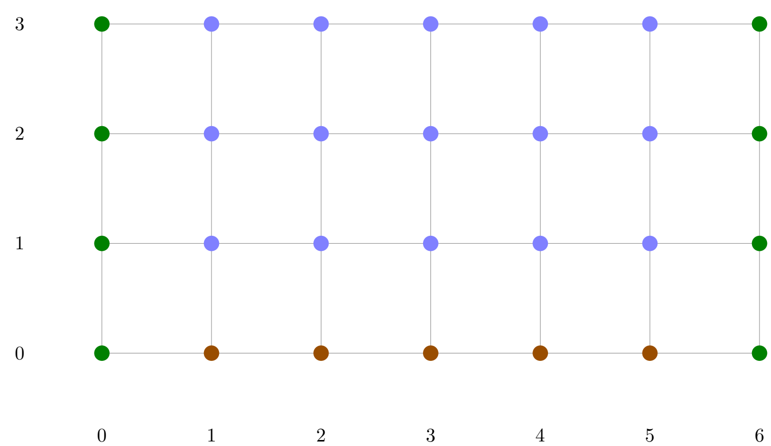

Figura 5.2: Mallado del problema

Creando una matriz de valores nulos

3	$T(0,3)$	$T(1,3)$	$T(2,3)$	$T(3,3)$	$T(4,3)$	$T(5,3)$	$T(6,3)$
2	$T(0,2)$	$T(1,2)$	$T(2,2)$	$T(3,2)$	$T(4,2)$	$T(5,2)$	$T(6,2)$
1	$T(0,1)$	$T(1,1)$	$T(2,1)$	$T(3,1)$	$T(4,1)$	$T(5,1)$	$T(6,1)$
0	$T(0,0)$	$T(1,0)$	$T(2,0)$	$T(3,0)$	$T(4,0)$	$T(5,0)$	$T(6,0)$
	0	1	2	3	4	5	6

Figura 5.3: Matriz solución sin valores

Rellenando los valores conocidos

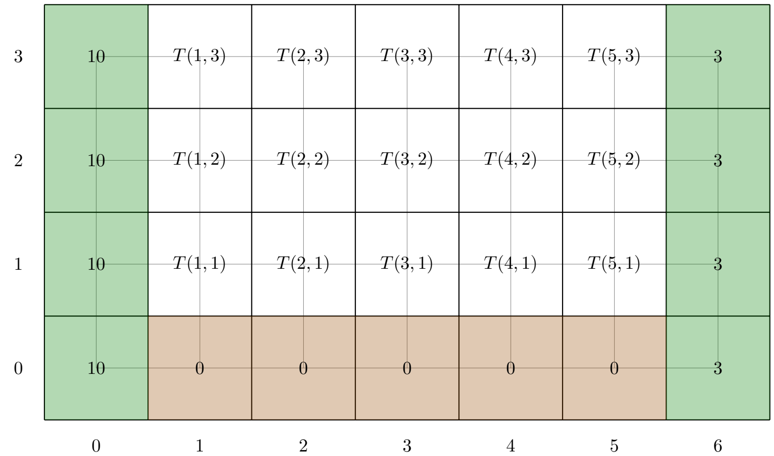

Figura 5.4: Matriz solución para $t = 0$

Para cinco pasos se requieren cinco ecuaciones, para $t = 1$

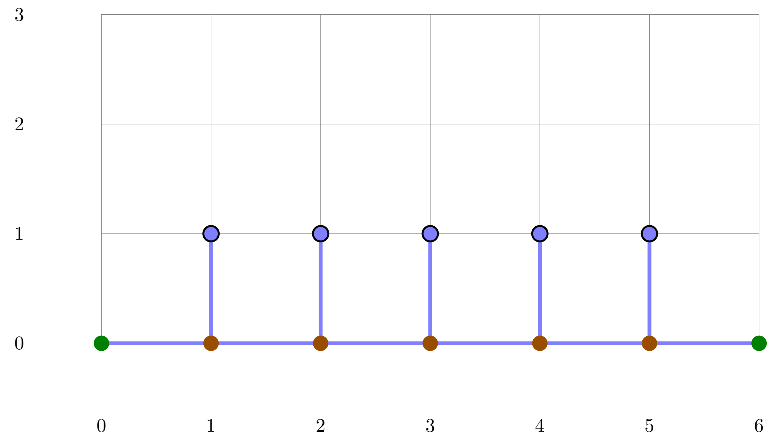

Figura 5.5: Movimiento del stencil para $t = 1$

$$
\begin{aligned}
T(1,1) &= 0.4T(0,0) + 0.2T(0,1) + 0.4T(0,2) = 0.4(10) + 0.2(0) + 0.4(0) = 4 \\
T(1,2) &= 0.4T(0,1) + 0.2T(0,2) + 0.4T(0,3) = 0.4(0) + 0.2(0) + 0.4(0) = 0 \\
T(1,3) &= 0.4T(0,2) + 0.2T(0,3) + 0.4T(0,4) = 0.4(0) + 0.2(0) + 0.4(0) = 0 \\
T(1,4) &= 0.4T(0,3) + 0.2T(0,4) + 0.4T(0,5) = 0.4(0) + 0.2(0) + 0.4(0) = 0 \\
T(1,5) &= 0.4T(0,4) + 0.2T(0,5) + 0.4T(0,6) = 0.4(0) + 0.2(0) + 0.4(3) = 1.2
\end{aligned}
$$

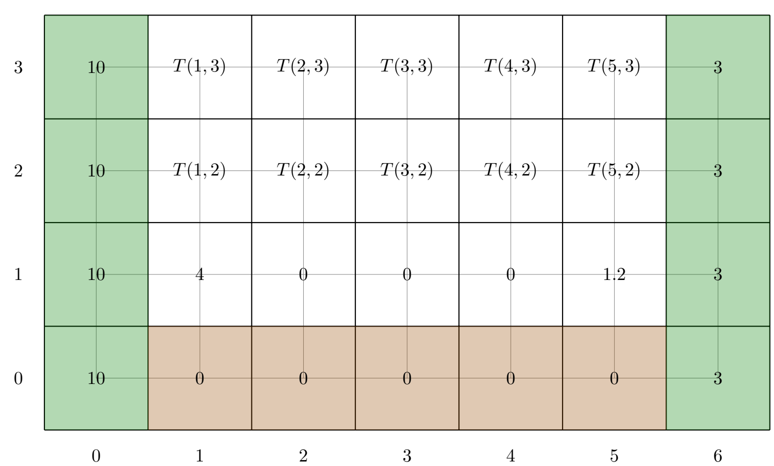

Figura 5.6: Matriz solución para $t = 1$

Para $t = 2$

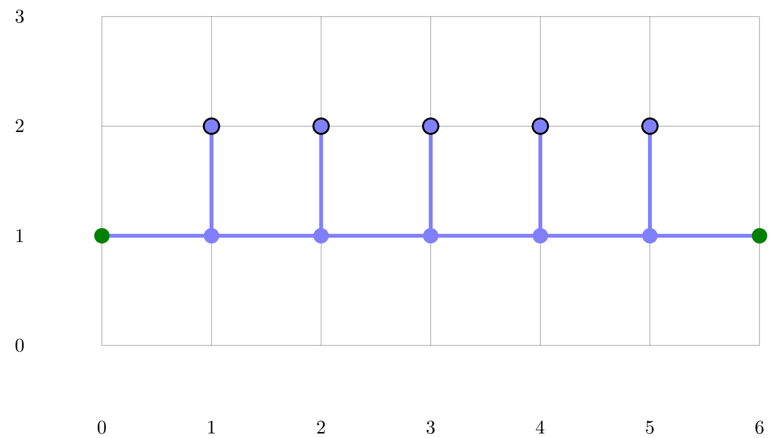

Figura 5.7: Movimiento del stencil para $t = 2$

$$
\begin{aligned}
T(2,1) &= 0.4T(1,0) + 0.2T(1,1) + 0.4T(1,2) = 0.4(10) + 0.2(4) + 0.4(0) = 4.8\\
T(2,2) &= 0.4T(1,1) + 0.2T(1,2) + 0.4T(1,3) = 0.4(4) + 0.2(0) + 0.4(0) = 1.6\\
T(2,3) &= 0.4T(1,2) + 0.2T(1,3) + 0.4T(1,4) = 0.4(0) + 0.2(0) + 0.4(0) = 0\\
T(2,4) &= 0.4T(1,3) + 0.2T(1,4) + 0.4T(1,5) = 0.4(0) + 0.2(0) + 0.4(1.2) = 0.48\\
T(2,5) &= 0.4T(1,4) + 0.2T(1,5) + 0.4T(1,6) = 0.4(0) + 0.2(1.2) + 0.4(3) = 1.44
\end{aligned}
$$

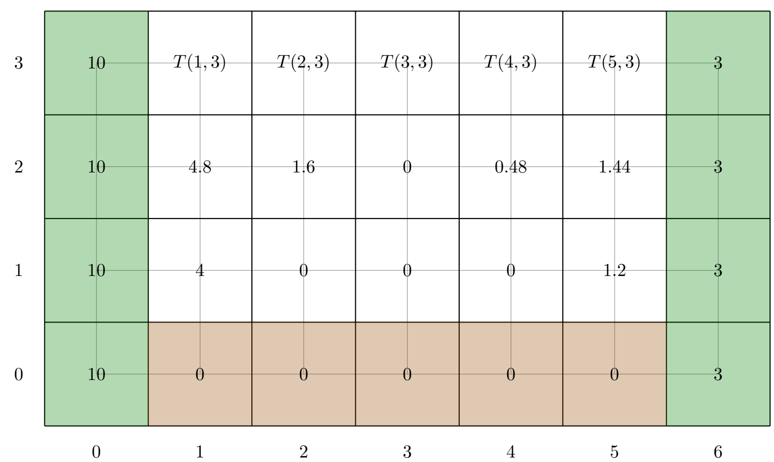

Figura 5.8: Matriz solución para $t = 2$

Para $t = 3$

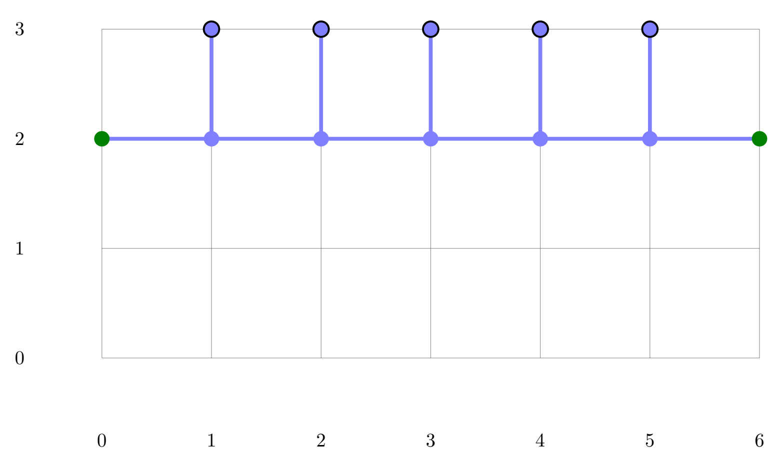

Figura 5.9: Movimiento del stencil para $t = 3$

$$T(3,1) = 0.4T(2,0) + 0.2T(2,1) + 0.4T(2,2) = 0.4(10) + 0.2(4.8) + 0.4(1.6) = 5.6$$
$$T(3,2) = 0.4T(2,1) + 0.2T(2,2) + 0.4T(2,3) = 0.4(4.8) + 0.2(1.6) + 0.4(0) = 2.24$$
$$T(3,3) = 0.4T(2,2) + 0.2T(2,3) + 0.4T(2,4) = 0.4(1.6) + 0.2(0) + 0.4(0.48) = 0.832$$
$$T(3,4) = 0.4T(2,3) + 0.2T(2,4) + 0.4T(2,5) = 0.4(0) + 0.2(0.48) + 0.4(1.44) = 0.672$$
$$T(3,5) = 0.4T(2,4) + 0.2T(2,5) + 0.4T(2,6) = 0.4(0.48) + 0.2(1.44) + 0.4(3) = 1.68$$

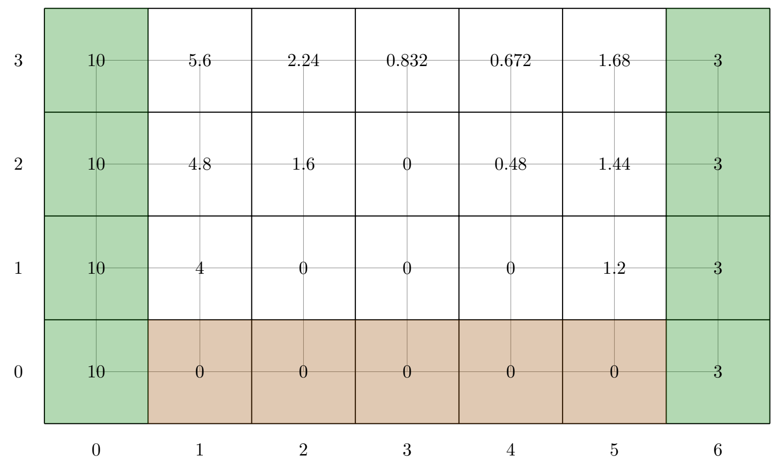

Figura 5.10: Matriz solución para $t = 3$

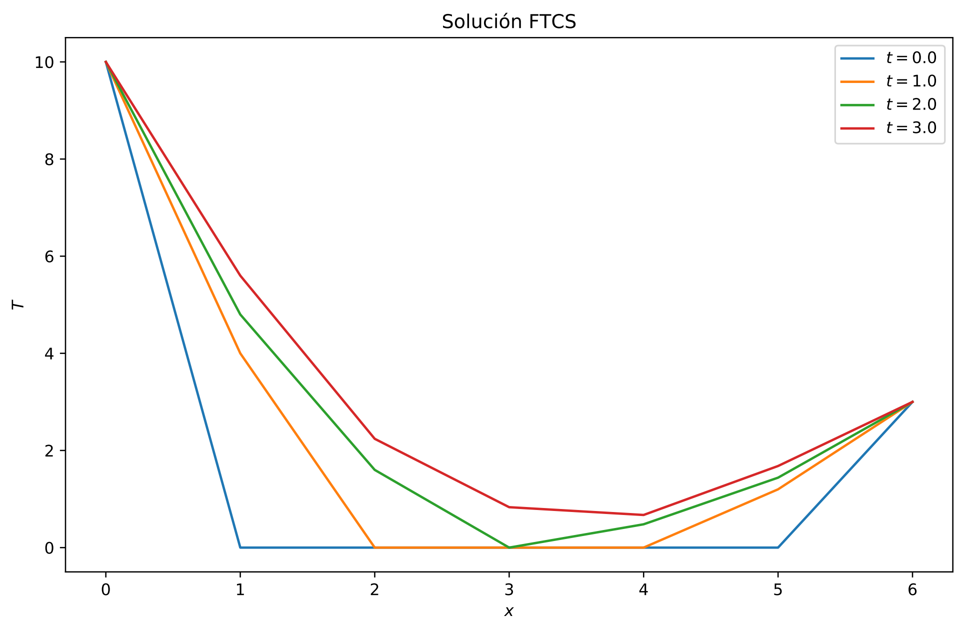

Figura 5.11: Solución numérica $\Delta x = 1$ y $\Delta t = 1$

5.1.2. FTFS o Downwind

Aproximación hacia adelante de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t + \Delta t) - T(t)}{\Delta t}$$

Aproximación hacia adelante de T_{xx}

$$\frac{\partial^2 T}{\partial x^2} = \frac{T(x + 2\Delta x) - 2T(x + \Delta x) + T(x)}{\Delta x^2}$$

Reemplazando

$$\frac{T(t + \Delta t) - T(t)}{\Delta t} - D\,\frac{T(x + 2\Delta x) - 2T(x + \Delta x) + T(x)}{\Delta x^2} = 0$$

Reordenando

$$T(t+\Delta t) - T(t) - \frac{D\Delta t}{\Delta x^2}[T(x+2\Delta x) - 2T(x+\Delta x) + T(x)] = 0$$

Se reemplaza por el término S

$$T(t+\Delta t) - T(t) - S[T(x+2\Delta x) - 2T(x+\Delta x) + T(x)] = 0$$

Reordenando

$$T(t+\Delta t) = T(t) + S[T(x+2\Delta x) - 2T(x+\Delta x) + T(x)]$$

Reescribiendo para su formulación matricial

$$T(t+\Delta t, x) = T(t,x) + S[T(t, x+2\Delta x) - 2T(t, x+\Delta x) + T(t,x)]$$

Intercambiando por los índices del mallado

$$T(n+1,j) = T(n,j) + S[T(n,j+2) - 2T(n,j+1) + T(n,j)]$$

Reordenando

$$T(n+1,j) = (1+S)T(n,j) - 2ST(n,j+1) + ST(n,j+2) \tag{5.2}$$

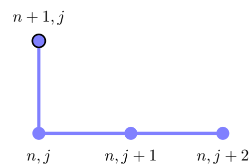

Figura 5.12: Stencil FTFS

5.1.3. FTBS o Upwind

Aproximación hacia adelante de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t+\Delta t) - T(t)}{\Delta t}$$

Aproximación hacia atrás de T_{xx}

$$\frac{\partial^2 T}{\partial x^2} = \frac{T(x) - 2T(x-\Delta x) + T(x-2\Delta x)}{\Delta x^2}$$

Reemplazando

$$\frac{T(t+\Delta t) - T(t)}{\Delta t} - D\,\frac{T(x) - 2T(x-\Delta x) + T(x-2\Delta x)}{\Delta x^2} = 0$$

Reordenando

$$T(t+\Delta t) - T(t) - \frac{D\Delta t}{\Delta x^2}[T(x) - 2T(x-\Delta x) + T(x-2\Delta x)] = 0$$

Se reemplaza por el término S

$$T(t+\Delta t) - T(t) - S[T(x) - 2T(x-\Delta x) + T(x-2\Delta x)] = 0$$

Reordenando

$$T(t+\Delta t) = T(t) + S[T(x) - 2T(x-\Delta x) + T(x-2\Delta x)]$$

Reescribiendo para su formulación matricial

$$T(t+\Delta t, x) = T(t,x) + S[T(t,x) - 2T(t, x-\Delta x) + T(t, x-2\Delta x)]$$

Intercambiando por los índices del mallado

$$T(n+1,j) = T(n,j) + S[T(n,j) - 2T(n,j-1) + T(n,j-2)]$$

Reordenando

$$T(n+1,j) = ST(n,j-2) - 2ST(n,j-1) + (1+S)T(n,j) \tag{5.3}$$

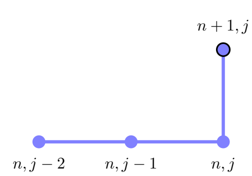

Figura 5.13: Stencil FTFS

5.1.4. CTCS o Leap-Frog

Aproximación centrada de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t+\Delta t) - T(t-\Delta t)}{2\Delta t}$$

Aproximación centrada de T_{xx}

$$\frac{\partial^2 T}{\partial x^2} = \frac{T(x+\Delta x) - 2T(x) + T(x-\Delta x)}{\Delta x^2}$$

Reemplazando

$$\frac{T(t+\Delta t) - T(t-\Delta t)}{2\Delta t} - D\,\frac{T(x+\Delta x) - 2T(x) + T(x-\Delta x)}{\Delta x^2} = 0$$

Reordenando

$$T(t+\Delta t) - T(t-\Delta t) - \frac{2D\Delta t}{\Delta x^2}[T(x+\Delta x) - 2T(x) + T(x-\Delta x)] = 0$$

Se reemplaza por el término S

$$T(t+\Delta t) - T(t-\Delta t) - 2S[T(x+\Delta x) - 2T(x) + T(x-\Delta x)] = 0$$

Reordenando

$$T(t+\Delta t) = T(t-\Delta t) + 2S[T(x+\Delta x) - 2T(x) + T(x-\Delta x)]$$

Reescribiendo para su formulación matricial

$$T(t+\Delta t, x) = T(t-\Delta t, x) + 2S[T(t, x+\Delta x) - 2T(t,x) + T(t, x-\Delta x)]$$

Intercambiando por los índices del mallado

$$T(n+1,j) = T(n-1,j) + 2S[T(n,j+1) - 2T(n,j) + T(n,j-1)]$$

Reordenando

$$T(n+1,j) = T(n-1,j) + 2ST(n,j-1) - 4ST(n,j) + 2ST(n,j+1) \tag{5.4}$$

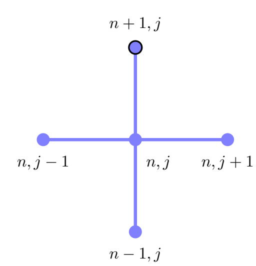

Figura 5.14: Stencil CTCS

5.1.5. Du Fort-Frankel

Se deriva de Leap-Frog

$$T(n+1,j) = T(n-1,j) + 2S[T(n,j-1) - 2T(n,j) + T(n,j+1)]$$

El nodo actual se reemplaza por el promedio de los nodos adyacentes

$$T(n,j) = \frac{T(n-1,j) + T(n+1,j)}{2}$$

Reemplazando

$$T(n+1,j) = T(n-1,j) + 2S[T(n,j-1) - T(n-1,j) - T(n+1,j) + T(n,j+1)]$$

Reordenando

$$T(n+1,j) = \left(\frac{1-2S}{1+2S}\right)T(n-1,j) + \left(\frac{2S}{1+2S}\right)T(n,j-1) + \left(\frac{2S}{1+2S}\right)T(n,j+1) \tag{5.5}$$

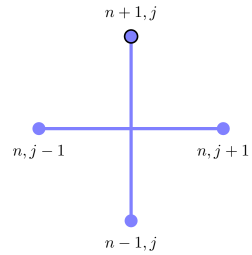

Figura 5.15: Stencil Du Fort-Frankel

5.1.6. Lax

Se deriva de FTCS

$$T(n+1,j) = T(n,j) + S[T(n,j+1) - 2T(n,j) + T(n,j-1)]$$

El nodo actual se reemplaza por el promedio de los nodos adyacentes

$$T(n,j) = \frac{T(n,j-1) + T(n,j) + T(n,j+1)}{3}$$

Reemplazando

$$T(n+1,j) = \frac{1}{3}[T(n,j-1) + T(n,j) + T(n,j+1)] + S[T(n,j+1) - 2T(n,j) + T(n,j-1)]$$

Reordenando

$$T(n+1,j) = \left(\frac{1}{3} + S\right)T(n,j-1) + \left(\frac{1}{3} - 2S\right)T(n,j) + \left(\frac{1}{3} + S\right)T(n,j+1) \tag{5.6}$$

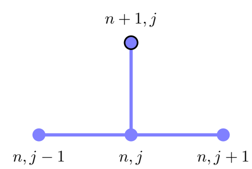

Figura 5.16: Stencil Lax

5.1.7. Lax-Friedrichs

Se deriva de FTCS

$$T(n+1,j) = T(n,j) + S[T(n,j+1) - 2T(n,j) + T(n,j-1)]$$

El nodo actual se reemplaza por

$$T(n,j) = T(n,j) + \alpha T(n,j) - \alpha T(n,j)$$

Luego se reemplaza por el promedio ponderado de los nodos adyacentes

$$\alpha T(n,j) = \frac{\alpha T(n,j+1) + \alpha T(n,j) + \alpha T(n,j-1)}{3}$$

Reemplazando y reordenando

$$T(n,j) = \frac{\alpha}{3}[T(n,j+1) + T(n,j) + T(n,j-1)] + (1-\alpha)T(n,j)$$

Reemplazando

$$T(n+1,j) = \frac{\alpha}{3}[T(n,j+1) + T(n,j) + T(n,j-1)] + (1-\alpha)T(n,j) + S[T(n,j+1) - 2T(n,j) + T(n,j-1)]$$

Reordenando

$$T(n+1,j) = \left(\frac{\alpha}{3} + S\right)T(n,j-1) + \left(1 - \frac{2}{3}\alpha - 2S\right)T(n,j) + \left(\frac{\alpha}{3} + S\right)T(n,j+1) \tag{5.7}$$

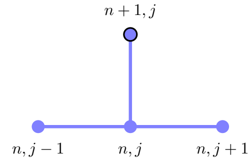

Figura 5.17: Stencil Lax-Friedrichs

5.2. Métodos implícitos

5.2.1. FTCS implícito o Laasonen

Se deriva de FTCS

$$T(t+\Delta t) = T(t) + S[T(x+\Delta x) - 2T(x) + T(x-\Delta x)]$$

Reordenando

$$T(t) = T(t+\Delta t) - S[T(x+\Delta x) - 2T(x) + T(x-\Delta x)]$$

Reescribiendo

$$T(t,x) = T(t+\Delta t, x) - S[T(t, x+\Delta x) - 2T(t,x) + T(t, x-\Delta x)]$$

Se reemplazará $t = t + \Delta t$ en los elementos multiplicados por s

$$T(t,x) = T(t+\Delta t, x) - S[T(t+\Delta t, x+\Delta x) - 2T(t+\Delta t, x) + T(t+\Delta t, x-\Delta x)]$$

Simplificando

$$T(t,x) = -ST(t+\Delta t, x+\Delta x) + (1+2S)T(t+\Delta t, x) - ST(t+\Delta t, x-\Delta x)$$

Intercambiando por los índices del mallado

$$T(n,j) = -ST(n+1, j+1) + (1+2S)T(n+1, j) - ST(n+1, j-1)$$

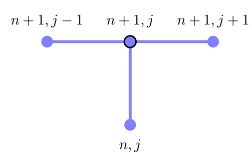

Figura 5.18: Stencil FTCS implícito

Para generalizar el método se usará tres pasos en el espacio

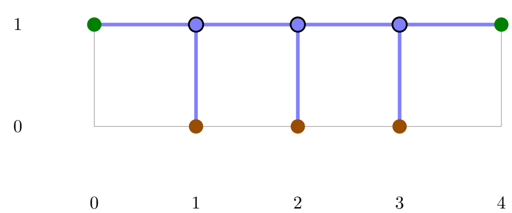

Figura 5.19: Movimiento del stencil para $t = 1$

Escribiendo una ecuación por cada paso

$$\begin{aligned}
-ST[1,0] + (1+2S)T[1,1] - ST[1,2] &= T[0,1] \\
-ST[1,1] + (1+2S)T[1,2] - ST[1,3] &= T[0,2] \\
-ST[1,2] + (1+2S)T[1,3] - ST[1,4] &= T[0,3]
\end{aligned}$$

Reescribiendo

$$\begin{aligned}
-ST[1,0] + (1+2S)T[1,1] - ST[1,2] + 0T[1,3] + 0T[1,4] &= T[0,1] \\
0T[1,0] - ST[1,1] + (1+2S)T[1,2] - ST[1,3] + 0T[1,4] &= T[0,2] \\
0T[1,0] + 0T[1,1] - ST[1,2] + (1+2S)T[1,3] - ST[1,4] &= T[0,3]
\end{aligned}$$

Reordenando

$$\begin{aligned} (1+2S)T[1,1] - \quad & ST[1,2] + \quad & 0T[1,3] &= T[0,1] + ST[1,0] + 0T[1,0] \\ - \quad ST[1,1] + & (1+2S)T[1,2] - \quad & ST[1,3] &= T[0,2] + 0T[1,0] + 0T[1,4] \\ 0T[1,1] - \quad & ST[1,2] + & (1+2S)T[1,3] &= T[0,3] + 0T[1,0] + ST[1,4] \end{aligned}$$

Simplificando

$$\begin{aligned} (1+2S)T[1,1] - \quad & ST[1,2] + \quad & 0T[1,3] &= T[0,1] + ST[1,0] \\ - \quad ST[1,1] + & (1+2S)T[1,2] - \quad & ST[1,3] &= T[0,2] \\ 0T[1,1] - \quad & ST[1,2] + & (1+2S)T[1,3] &= T[0,3] + ST[1,4] \end{aligned}$$

En forma matricial

$$\begin{bmatrix} 1+2S & -S & 0 \\ -S & 1+2S & -S \\ 0 & -S & 1+2S \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \end{bmatrix} = \begin{bmatrix} T[0,1] \\ T[0,2] \\ T[0,3] \end{bmatrix} + S \begin{bmatrix} T[1,0] \\ 0 \\ T[1,4] \end{bmatrix}$$

Al resolver el sistema se obtienen los valores $T[1,x]$, también puede escribirse como

$$\begin{bmatrix} 1+2S & -S & 0 \\ -S & 1+2S & -S \\ 0 & -S & 1+2S \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \end{bmatrix} = \begin{bmatrix} T[0,1] + ST[1,0] \\ T[0,2] \\ T[0,3] + ST[1,4] \end{bmatrix}$$

Generalizando el método para s pasos

$$\begin{bmatrix} 1+2S & -S & 0 & \cdots & \cdots & \cdots & 0 \\ -S & 1+2S & -S & \ddots & & & \vdots \\ 0 & -S & 1+2S & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1+2S & -S & 0 \\ \vdots & & & \ddots & -S & 1+2S & -S \\ 0 & \cdots & \cdots & \cdots & 0 & -S & 1+2S \end{bmatrix} \begin{bmatrix} T[n+1,1] \\ T[n+1,2] \\ \vdots \\ \vdots \\ \vdots \\ T[n+1,s-1] \\ T[n+1,s] \end{bmatrix} = \begin{bmatrix} T[n,1] + ST[n+1,0] \\ T[n,2] \\ \vdots \\ \vdots \\ \vdots \\ T[n,s-1] \\ T[n,s] + ST[n+1,s+1] \end{bmatrix} \tag{5.8}$$

Ejemplo 5.2.1.
Resolver la siguiente ecuación diferencial

$$\begin{aligned} \frac{\partial T}{\partial t} &= 0.4\frac{\partial^2 T}{\partial x^2} \\ T(x,0) &= 0 \\ T(0,t) &= 10 \\ T(6,t) &= 3 \end{aligned}$$

Usando el esquema FTCS implícito con $\Delta x = 1$ y $\Delta t = 1$

Solución
Verificando el número de difusión

$$S = \frac{D\Delta t}{\Delta x^2} = \frac{0.4 \cdot 1}{1^2} = 0.4 \leqslant 0.5$$

Dibujando la malla

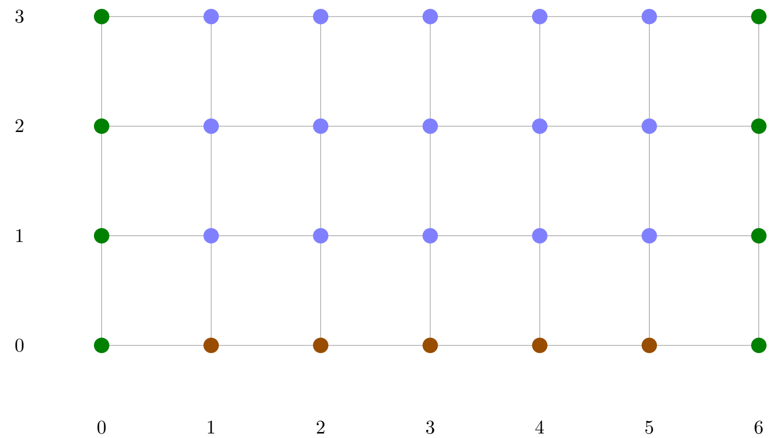

Figura 5.20: Mallado del problema

Creando una matriz de valores nulos

3	$T(0,3)$	$T(1,3)$	$T(2,3)$	$T(3,3)$	$T(4,3)$	$T(5,3)$	$T(6,3)$
2	$T(0,2)$	$T(1,2)$	$T(2,2)$	$T(3,2)$	$T(4,2)$	$T(5,2)$	$T(6,2)$
1	$T(0,1)$	$T(1,1)$	$T(2,1)$	$T(3,1)$	$T(4,1)$	$T(5,1)$	$T(6,1)$
0	$T(0,0)$	$T(1,0)$	$T(2,0)$	$T(3,0)$	$T(4,0)$	$T(5,0)$	$T(6,0)$
	0	1	2	3	4	5	6

Figura 5.21: Matriz solución sin valores

Rellenando los valores conocidos

3	10	$T(1,3)$	$T(2,3)$	$T(3,3)$	$T(4,3)$	$T(5,3)$	3
2	10	$T(1,2)$	$T(2,2)$	$T(3,2)$	$T(4,2)$	$T(5,2)$	3
1	10	$T(1,1)$	$T(2,1)$	$T(3,1)$	$T(4,1)$	$T(5,1)$	3
0	10	0	0	0	0	0	3
	0	1	2	3	4	5	6

Figura 5.22: Matriz solución para $t = 0$

Para cinco pasos se requieren cinco ecuaciones, para $t = 1$

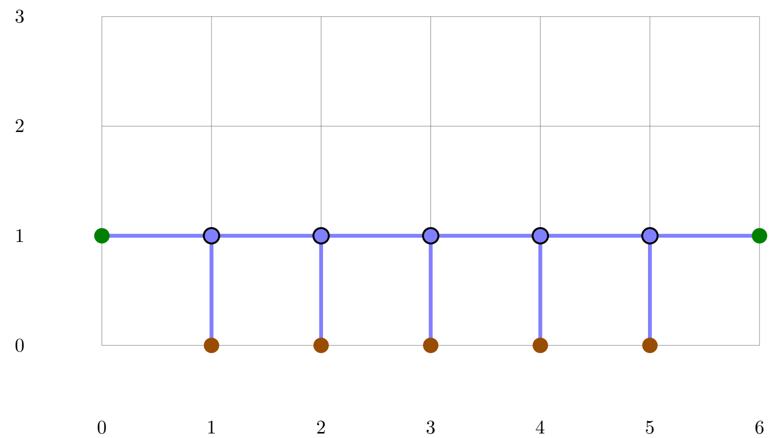

Figura 5.23: Movimiento del stencil para $t = 1$

Usando la fórmula

$$\begin{bmatrix} 1+2S & -S & 0 & 0 & 0 \\ -S & 1+2S & -S & 0 & 0 \\ 0 & -S & 1+2S & -S & 0 \\ 0 & 0 & -S & 1+2S & -S \\ 0 & 0 & 0 & -S & 1+2S \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \\ T[1,4] \\ T[1,5] \end{bmatrix} = \begin{bmatrix} T[0,1] + ST[1,0] \\ T[0,2] \\ T[0,3] \\ T[0,4] \\ T[0,5] + ST[1,6] \end{bmatrix}$$

Reemplazando valores

$$\begin{bmatrix} 1.8 & -0.4 & 0 & 0 & 0 \\ -0.4 & 1.8 & -0.4 & 0 & 0 \\ 0 & -0.4 & 1.8 & -0.4 & 0 \\ 0 & 0 & -0.4 & 1.8 & -0.4 \\ 0 & 0 & 0 & -0.4 & 1.8 \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \\ T[1,4] \\ T[1,5] \end{bmatrix} = \begin{bmatrix} 0 + 0.4(10) \\ 0 \\ 0 \\ 0 \\ 0 + 0.4(3) \end{bmatrix}$$

Simplificando y resolviendo

$$\begin{bmatrix} 1.8 & -0.4 & 0 & 0 & 0 \\ -0.4 & 1.8 & -0.4 & 0 & 0 \\ 0 & -0.4 & 1.8 & -0.4 & 0 \\ 0 & 0 & -0.4 & 1.8 & -0.4 \\ 0 & 0 & 0 & -0.4 & 1.8 \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \\ T[1,4] \\ T[1,5] \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 1.2 \end{bmatrix} \qquad T_1 = \begin{bmatrix} 2.346 \\ 0.559 \\ 0.167 \\ 0.195 \\ 0.710 \end{bmatrix}$$

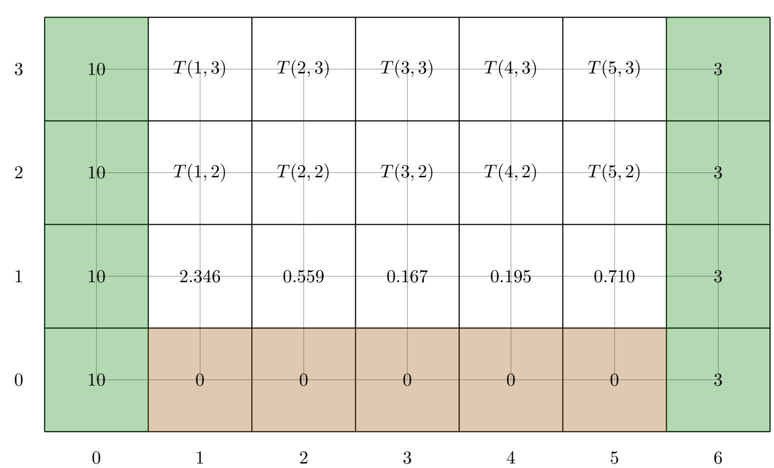

Figura 5.24: Matriz solución para $t = 1$

Para $t = 2$

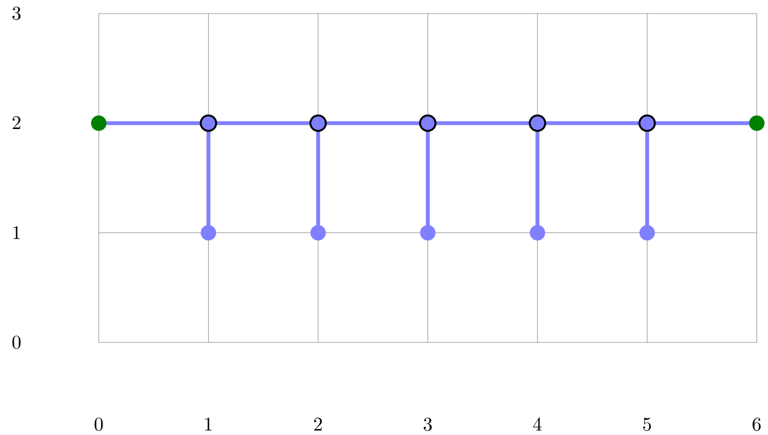

Figura 5.25: Movimiento del stencil para $t = 2$

Reemplazando valores

$$\begin{bmatrix} 1.8 & -0.4 & 0 & 0 & 0 \\ -0.4 & 1.8 & -0.4 & 0 & 0 \\ 0 & -0.4 & 1.8 & -0.4 & 0 \\ 0 & 0 & -0.4 & 1.8 & -0.4 \\ 0 & 0 & 0 & -0.4 & 1.8 \end{bmatrix} \begin{bmatrix} T[2,1] \\ T[2,2] \\ T[2,3] \\ T[2,4] \\ T[2,5] \end{bmatrix} = \begin{bmatrix} 2.346 + 0.4(10) \\ 0.559 \\ 0.167 \\ 0.195 \\ 0.710 + 0.4(3) \end{bmatrix}$$

Simplificando y resolviendo

$$\begin{bmatrix} 1.8 & -0.4 & 0 & 0 & 0 \\ -0.4 & 1.8 & -0.4 & 0 & 0 \\ 0 & -0.4 & 1.8 & -0.4 & 0 \\ 0 & 0 & -0.4 & 1.8 & -0.4 \\ 0 & 0 & 0 & -0.4 & 1.8 \end{bmatrix} \begin{bmatrix} T[2,1] \\ T[2,2] \\ T[2,3] \\ T[2,4] \\ T[2,5] \end{bmatrix} = \begin{bmatrix} 6.346 \\ 0.559 \\ 0.167 \\ 0.195 \\ 1.910 \end{bmatrix} \qquad T_2 = \begin{bmatrix} 3.806 \\ 1.263 \\ 0.479 \\ 0.474 \\ 1.166 \end{bmatrix}$$

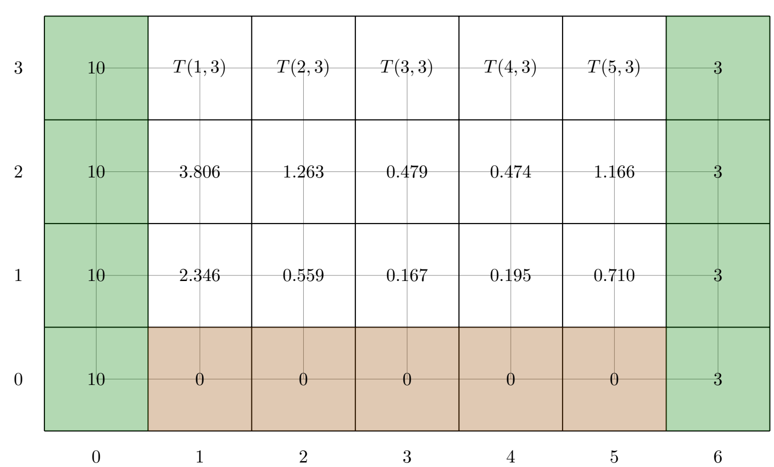

Figura 5.26: Matriz solución para $t = 2$

Para $t = 3$

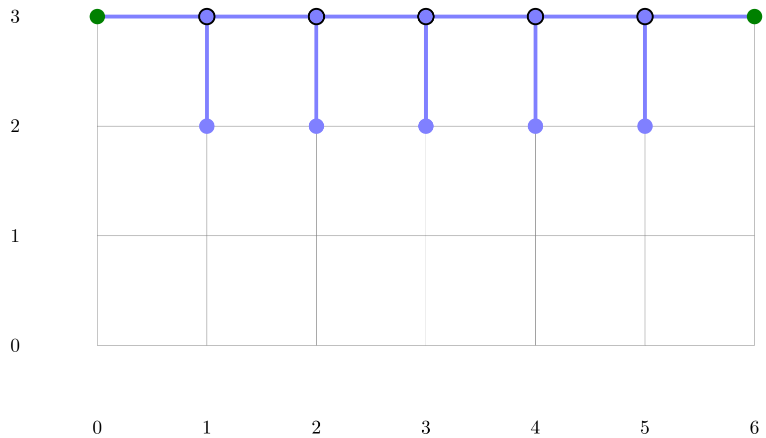

Figura 5.27: Movimiento del stencil para $t = 3$

Reemplazando valores

$$\begin{bmatrix} 1.8 & -0.4 & 0 & 0 & 0 \\ -0.4 & 1.8 & -0.4 & 0 & 0 \\ 0 & -0.4 & 1.8 & -0.4 & 0 \\ 0 & 0 & -0.4 & 1.8 & -0.4 \\ 0 & 0 & 0 & -0.4 & 1.8 \end{bmatrix} \begin{bmatrix} T[3,1] \\ T[3,2] \\ T[3,3] \\ T[3,4] \\ T[3,5] \end{bmatrix} = \begin{bmatrix} 3.806 + 0.4(10) \\ 1.263 \\ 0.479 \\ 0.474 \\ 1.166 + 0.4(3) \end{bmatrix}$$

Simplificando y resolviendo

$$\begin{bmatrix} 1.8 & -0.4 & 0 & 0 & 0 \\ -0.4 & 1.8 & -0.4 & 0 & 0 \\ 0 & -0.4 & 1.8 & -0.4 & 0 \\ 0 & 0 & -0.4 & 1.8 & -0.4 \\ 0 & 0 & 0 & -0.4 & 1.8 \end{bmatrix} \begin{bmatrix} T[3,1] \\ T[3,2] \\ T[3,3] \\ T[3,4] \\ T[3,5] \end{bmatrix} = \begin{bmatrix} 7.806 \\ 1.263 \\ 0.479 \\ 0.474 \\ 2.366 \end{bmatrix} \qquad T_3 = \begin{bmatrix} 4.772 \\ 1.956 \\ 0.876 \\ 0.789 \\ 1.490 \end{bmatrix}$$

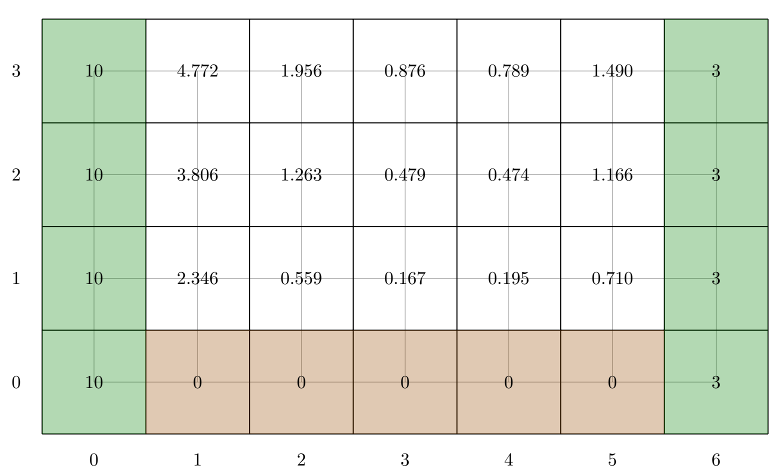

Figura 5.28: Matriz solución para $t = 3$

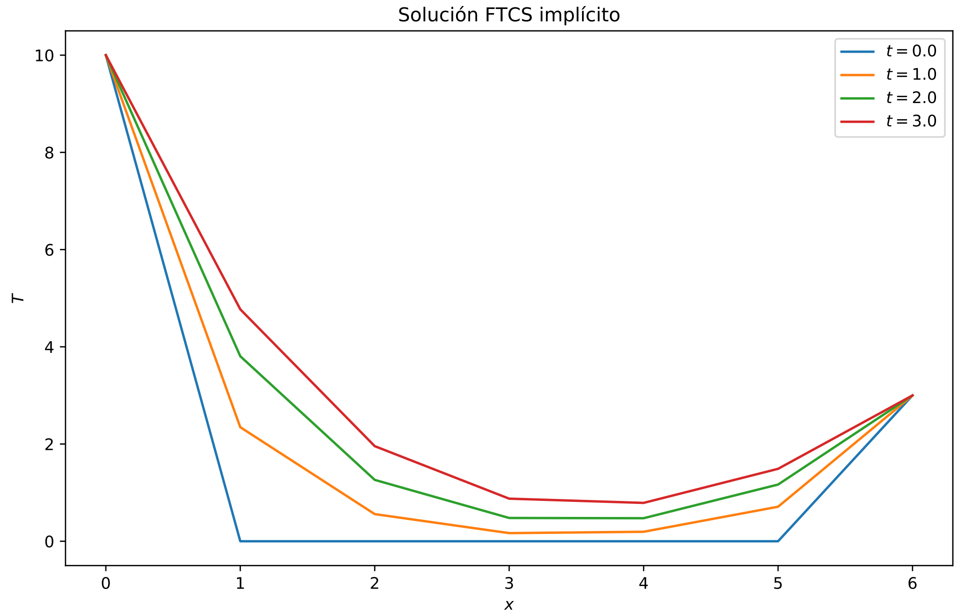

Figura 5.29: Solución numérica $\Delta x = 1$ y $\Delta t = 1$

5.2.2. BTCS

Aproximación hacia atrás de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t) - T(t - \Delta t)}{\Delta t}$$

Aproximación centrada de T_{xx}

$$\frac{\partial^2 T}{\partial x^2} = \frac{T(x + \Delta x) - 2T(x) + T(x - \Delta x)}{\Delta x^2}$$

Reemplazando

$$\frac{T(t) - T(t - \Delta t)}{\Delta t} - D\,\frac{T(x + \Delta x) - 2T(x) + T(x - \Delta x)}{\Delta x^2} = 0$$

Reordenando

$$T(t) - T(t - \Delta t) - \frac{D\Delta t}{\Delta x^2}[T(x + \Delta x) - 2T(x) + T(x - \Delta x)] = 0$$

Se reemplaza por el término S

$$T(t) - T(t - \Delta t) - S[T(x + \Delta x) - 2T(x) + T(x - \Delta x)] = 0$$

Reordenando

$$T(t) = T(t - \Delta t) + S[T(x + \Delta x) - 2T(x) + T(x - \Delta x)]$$

Reescribiendo para su formulación matricial

$$T(t, x) = T(t - \Delta t, x) + S[T(t, x + \Delta x) - 2T(t, x) + T(t, x - \Delta x)]$$

Intercambiando por los índices del mallado

$$T(n, j) = T(n - 1, j) + S[T(n, j + 1) - 2T(n, j) + T(n, j - 1)]$$

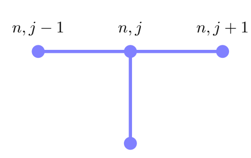

Figura 5.30: Stencil BTCS

5.2.3. Crank-Nicolson

Puede derivarse de métodos FTCS

$$\begin{aligned} T(t+\Delta t,x) &= T(t,x) + S[T(t,x+\Delta x) - 2T(t,x) + T(t,x-\Delta x)] && \text{explícito} \\ T(t,x) &= T(t+\Delta t,x) - S[T(t+\Delta t,x+\Delta x) - 2T(t+\Delta t,x) + T(t+\Delta t,x-\Delta x)] && \text{implícito} \end{aligned}$$

Reordenando

$$\begin{aligned} T(t+\Delta t,x) - T(t,x) &= S[T(t,x+\Delta x) - 2T(t,x) + T(t,x-\Delta x)] \\ T(t+\Delta t,x) - T(t,x) &= S[T(t+\Delta t,x+\Delta x) - 2T(t+\Delta t,x) + T(t+\Delta t,x-\Delta x)] \end{aligned}$$

Multiplicando por $\frac{1}{2}$ y sumando

$$T(t+\Delta t,x)-T(t,x) = \frac{S}{2}[T(t,x+\Delta x)-2T(t,x)+T(t,x-\Delta x)]+\frac{S}{2}[T(t+\Delta t,x+\Delta x)-2T(t+\Delta t,x)+T(t+\Delta t,x-\Delta x)]$$

Simplificando

$$T(t+\Delta t,x)-T(t,x) = \frac{S}{2}[T(t,x+\Delta x)-2T(t,x)+T(t,x-\Delta x)+T(t+\Delta t,x+\Delta x)-2T(t+\Delta t,x)+T(t+\Delta t,x-\Delta x)]$$

Reordenando

$$\frac{S}{2}T(t,x+\Delta x)+(1-S)T(t,x)+\frac{S}{2}T(t,x-\Delta x) = -\frac{S}{2}T(t+\Delta t,x+\Delta x)+(1+S)T(t+\Delta t,x)-\frac{S}{2}T(t+\Delta t,x-\Delta x)$$

Intercambiando por los índices del mallado

$$\frac{S}{2}T(n,j+1) + (1-S)T(n,j) + \frac{S}{2}T(n,j-1) = -\frac{S}{2}T(n+1,j+1) + (1+S)T(n+1,j) - \frac{S}{2}T(n+1,j-1)$$

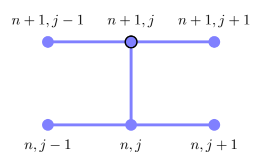

Figura 5.31: Stencil Crank-Nicolson

Para generalizar el método se usará tres pasos en el espacio

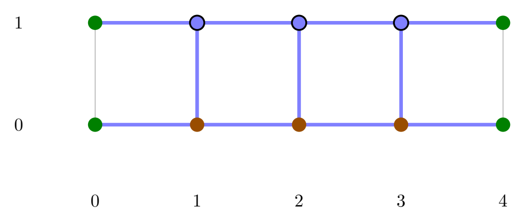

Figura 5.32: Movimiento del stencil para $t = 1$

Escribiendo una ecuación por cada paso

$$\begin{aligned} -\frac{S}{2}T[1,0] + (1+S)T[1,1] - \frac{S}{2}T[1,2] &= \frac{S}{2}T[0,0] + (1-S)T[0,1] + \frac{S}{2}T[0,2] \\ -\frac{S}{2}T[1,1] + (1+S)T[1,2] - \frac{S}{2}T[1,3] &= \frac{S}{2}T[0,1] + (1-S)T[0,2] + \frac{S}{2}T[0,3] \\ -\frac{S}{2}T[1,2] + (1+S)T[1,3] - \frac{S}{2}T[1,4] &= \frac{S}{2}T[0,2] + (1-S)T[0,3] + \frac{S}{2}T[0,4] \end{aligned}$$

Reordenando

$$\begin{array}{rrrcrrrrr} (1+S)T[1,1] - & \frac{S}{2}T[1,2] + & 0T[1,3] = & (1-S)T[0,1] + & \frac{S}{2}T[0,2] + & 0T[0,3] + & \frac{S}{2}T[0,0] + & \frac{S}{2}T[1,0] \\ -\frac{S}{2}T[1,1] + & (1+S)T[1,2] - & \frac{S}{2}T[1,3] = & \frac{S}{2}T[0,1] + & (1-S)T[0,2] + & \frac{S}{2}T[0,3] & & \\ 0T[1,1] - & \frac{S}{2}T[1,2] + & (1+S)T[1,3] = & 0T[0,1] + & \frac{S}{2}T[0,2] + & (1-S)T[0,3] + & \frac{S}{2}T[0,4] + & \frac{S}{2}T[1,4] \end{array}$$

En forma matricial

$$\begin{bmatrix} 1+S & -\frac{S}{2} & 0 \\ -\frac{S}{2} & 1+S & -\frac{S}{2} \\ 0 & -\frac{S}{2} & 1+S \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \end{bmatrix} = \begin{bmatrix} 1-S & \frac{S}{2} & 0 \\ \frac{S}{2} & 1-S & \frac{S}{2} \\ 0 & \frac{S}{2} & 1-S \end{bmatrix} \begin{bmatrix} T[0,1] \\ T[0,2] \\ T[0,3] \end{bmatrix} + \frac{S}{2} \begin{bmatrix} T[0,0] \\ 0 \\ T[0,4] \end{bmatrix} + \frac{S}{2} \begin{bmatrix} T[1,0] \\ 0 \\ T[1,4] \end{bmatrix}$$

Al resolver el sistema se obtienen los valores $T[1,x]$, también puede escribirse como

$$\begin{bmatrix} 1+S & -\frac{S}{2} & 0 \\ -\frac{S}{2} & 1+S & -\frac{S}{2} \\ 0 & -\frac{S}{2} & 1+S \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \end{bmatrix} = \begin{bmatrix} 1-S & \frac{S}{2} & 0 \\ \frac{S}{2} & 1-S & \frac{S}{2} \\ 0 & \frac{S}{2} & 1-S \end{bmatrix} \begin{bmatrix} T[0,1] \\ T[0,2] \\ T[0,3] \end{bmatrix} + \frac{S}{2} \begin{bmatrix} T[0,0]+T[1,0] \\ 0 \\ T[0,4]+T[1,4] \end{bmatrix}$$

Generalizando el método para s pasos

$$\begin{bmatrix} 1+S & -\frac{S}{2} & 0 & \cdots & \cdots & \cdots & 0 \\ -\frac{S}{2} & 1+S & -\frac{S}{2} & \ddots & & & \vdots \\ 0 & -\frac{S}{2} & 1+S & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1+S & -\frac{S}{2} & 0 \\ \vdots & & & \ddots & -\frac{S}{2} & 1+S & -\frac{S}{2} \\ 0 & \cdots & \cdots & \cdots & 0 & -\frac{S}{2} & 1+S \end{bmatrix} \begin{bmatrix} T[n+1,1] \\ T[n+1,2] \\ \vdots \\ \vdots \\ \vdots \\ T[n+1,s-1] \\ T[n+1,s] \end{bmatrix}$$

$$= \begin{bmatrix} 1-S & \frac{S}{2} & 0 & \cdots & \cdots & \cdots & 0 \\ \frac{S}{2} & 1-S & \frac{S}{2} & \ddots & & & \vdots \\ 0 & \frac{S}{2} & 1-S & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1-S & \frac{S}{2} & 0 \\ \vdots & & & \ddots & \frac{S}{2} & 1-S & \frac{S}{2} \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{S}{2} & 1-S \end{bmatrix} \begin{bmatrix} T[n,1] \\ T[n,2] \\ \vdots \\ \vdots \\ \vdots \\ T[n,s-1] \\ T[n,s] \end{bmatrix} + \frac{S}{2} \begin{bmatrix} T[n,0]+T[n+1,0] \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ T[n,s+1]+T[n+1,s+1] \end{bmatrix} \tag{5.9}$$

5.2.4. θ

Reemplazando en el método de Crank-Nicolson, $\theta = \frac{1}{2}$

$$\theta ST(n,j+1) + (1-2\theta S)T(n,j) + \theta ST(n,j-1) = -\theta ST(n+1,j+1) + (1+2\theta S)T(n+1,j) - \theta ST(n+1,j-1)$$

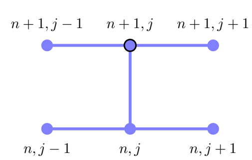

Figura 5.33: Stencil θ

Realizando los mismos pasos para los métodos anteriores, se obtiene

$$\begin{bmatrix} 1+2\theta S & -\theta S & 0 \\ -\theta S & 1+2\theta S & -\theta S \\ 0 & -\theta S & 1+2\theta S \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \end{bmatrix} = \begin{bmatrix} 1-2\theta S & \theta S & 0 \\ \theta S & 1-2\theta S & \theta S \\ 0 & \theta S & 1-2\theta S \end{bmatrix} \begin{bmatrix} T[0,1] \\ T[0,2] \\ T[0,3] \end{bmatrix} + \theta S \begin{bmatrix} T[0,0]+T[1,0] \\ 0 \\ T[0,4]+T[1,4] \end{bmatrix}$$

Generalizando el método para s pasos

$$\begin{bmatrix} 1+2\theta S & -\theta S & 0 & \cdots & \cdots & \cdots & 0 \\ -\theta S & 1+2\theta S & -\theta S & \ddots & & & \vdots \\ 0 & -\theta S & 1+2\theta S & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1+2\theta S & -\theta S & 0 \\ \vdots & & & \ddots & -\theta S & 1+2\theta S & -\theta S \\ 0 & \cdots & \cdots & \cdots & 0 & -\theta S & 1+2\theta S \end{bmatrix} \begin{bmatrix} T[n+1,1] \\ T[n+1,2] \\ \vdots \\ \vdots \\ \vdots \\ T[n+1,s-1] \\ T[n+1,s] \end{bmatrix}$$
$$= \begin{bmatrix} 1-2\theta S & \theta S & 0 & \cdots & \cdots & \cdots & 0 \\ \theta S & 1-2\theta S & \theta S & \ddots & & & \vdots \\ 0 & \theta S & 1-2\theta S & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1-2\theta S & \theta S & 0 \\ \vdots & & & \ddots & \theta S & 1-2\theta S & \theta S \\ 0 & \cdots & \cdots & \cdots & 0 & \theta S & 1-2\theta S \end{bmatrix} \begin{bmatrix} T[n,1] \\ T[n,2] \\ \vdots \\ \vdots \\ \vdots \\ T[n,s-1] \\ T[n,s] \end{bmatrix} \tag{5.10}$$
$$+\,\theta S \begin{bmatrix} T[n,0]+T[n+1,0] \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ T[n,s+1]+T[n+1,s+1] \end{bmatrix}$$

Capítulo 6

Aplicaciones en hidráulica

6.1. Advección-Difusión

$$\frac{\partial T}{\partial t} + u\,\frac{\partial T}{\partial x} = D\,\frac{\partial^2 T}{\partial x^2}$$

6.1.1. FTCS

Aproximación hacia adelante de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t+\Delta t) - T(t)}{\Delta t}$$

Aproximación centrada de T_x

$$\frac{\partial T}{\partial x} = \frac{T(x+\Delta x) - T(x-\Delta x)}{2\Delta x}$$

Aproximación centrada de T_{xx}

$$\frac{\partial^2 T}{\partial x^2} = \frac{T(x+\Delta x) - 2T(x) + T(x-\Delta x)}{\Delta x^2}$$

Reemplazando

$$\frac{T(t+\Delta t) - T(t)}{\Delta t} + u\frac{T(x+\Delta x) - T(x-\Delta x)}{2\Delta x} = D\,\frac{T(x+\Delta x) - 2T(x) + T(x-\Delta x)}{\Delta x^2}$$

Reordenando

$$T(t+\Delta t) - T(t) + \frac{u\Delta t}{2\Delta x}[T(x+\Delta x) - T(x-\Delta x)] = \frac{D\Delta t}{\Delta x^2}[T(x+\Delta x) - 2T(x) + T(x-\Delta x)]$$

Reemplazando por C y S

$$T(t+\Delta t) - T(t) + \frac{C}{2}[T(x+\Delta x) - T(x-\Delta x)] = S[T(x+\Delta x) - 2T(x) + T(x-\Delta x)]$$

Reordenando

$$T(t+\Delta t) = T(t) - \frac{C}{2}[T(x+\Delta x) - T(x-\Delta x)] + S[T(x+\Delta x) - 2T(x) + T(x-\Delta x)]$$

Reescribiendo para su formulación matricial

$$T(t+\Delta t, x) = T(t,x) - \frac{C}{2}[T(t,x+\Delta x) - T(t,x-\Delta x)] + S[T(t,x+\Delta x) - 2T(t,x) + T(t,x-\Delta x)]$$

Intercambiando por los índices del mallado

$$T(n+1,j) = T(n,j) - \frac{C}{2}[T(n,j+1) - T(n,j-1)] + S[T(n,j+1) - 2T(n,j) + T(n,j-1)]$$

Reordenando

$$T(n+1,j) = \left(S + \frac{C}{2}\right)T(n,j-1) + (1-2S)T(n,j) + \left(S - \frac{C}{2}\right)T(n,j+1) \tag{6.1}$$

6.1.2. FTCS implícito

Aproximación hacia atrás de T_t

$$\frac{\partial T}{\partial t} = \frac{T(t) - T(t - \Delta t)}{\Delta t}$$

Aproximación centrada de T_x

$$\frac{\partial T}{\partial x} = \frac{T(x + \Delta x) - T(x - \Delta x)}{2\Delta x}$$

Aproximación centrada de T_{xx}

$$\frac{\partial^2 T}{\partial x^2} = \frac{T(x + \Delta x) - 2T(x) + T(x - \Delta x)}{\Delta x^2}$$

Reemplazando

$$\frac{T(t) - T(t - \Delta t)}{\Delta t} + u\frac{T(x + \Delta x) - T(x - \Delta x)}{2\Delta x} = D\,\frac{T(x + \Delta x) - 2T(x) + T(x - \Delta x)}{\Delta x^2}$$

Reordenando

$$T(t) - T(t - \Delta t) + \frac{u\Delta t}{2\Delta x}[T(x + \Delta x) - T(x - \Delta x)] = \frac{D\Delta t}{\Delta x^2}[T(x + \Delta x) - 2T(x) + T(x - \Delta x)]$$

Reemplazando por C y S

$$T(t) - T(t - \Delta t) + \frac{C}{2}[T(x + \Delta x) - T(x - \Delta x)] = S[T(x + \Delta x) - 2T(x) + T(x - \Delta x)]$$

Reordenando

$$T(t) = T(t - \Delta t) - \frac{C}{2}[T(x + \Delta x) - T(x - \Delta x)] + S[T(x + \Delta x) - 2T(x) + T(x - \Delta x)]$$

Reescribiendo para su formulación matricial

$$T(t, x) = T(t - \Delta t, x) - \frac{C}{2}[T(t, x + \Delta x) - T(t, x - \Delta x)] + S[T(t, x + \Delta x) - 2T(t, x) + T(t, x - \Delta x)]$$

Reemplazando $t = t + \Delta t$

$$T(t{+}\Delta t, x) = T(t, x) - \frac{C}{2}[T(t{+}\Delta t, x{+}\Delta x) - T(t{+}\Delta t, x{-}\Delta x)] + S[T(t{+}\Delta t, x{+}\Delta x) - 2T(t{+}\Delta t, x) + T(t{+}\Delta t, x{-}\Delta x)]$$

Intercambiando por los índices del mallado

$$T(n + 1, j) = T(n, j) - \frac{C}{2}[T(n + 1, j + 1) - T(n + 1, j - 1)] + S[T(n + 1, j + 1) - 2T(n + 1, j) + T(n + 1, j - 1)]$$

Reordenando

$$T(n, j) = -\left(S + \frac{C}{2}\right)T(n + 1, j - 1) + (1 + 2S)T(n + 1, j) - \left(S - \frac{C}{2}\right)T(n + 1, j + 1) \tag{6.2}$$

Para generalizar el método se usará tres pasos en el espacio

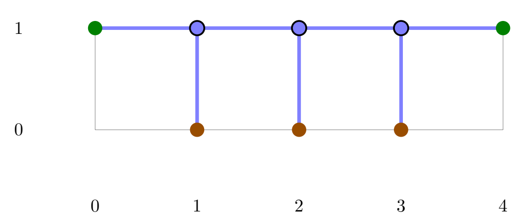

Figura 6.1: Movimiento del stencil para $t = 1$

Escribiendo una ecuación por cada paso

$$-\left(S + \frac{C}{2}\right)T[1, 0] + (1 + 2S)T[1, 1] - \left(S - \frac{C}{2}\right)T[1, 2] = T[0, 1]$$
$$-\left(S + \frac{C}{2}\right)T[1, 1] + (1 + 2S)T[1, 2] - \left(S - \frac{C}{2}\right)T[1, 3] = T[0, 2]$$
$$-\left(S + \frac{C}{2}\right)T[1, 2] + (1 + 2S)T[1, 3] - \left(S - \frac{C}{2}\right)T[1, 4] = T[0, 3]$$

Reescribiendo

$$\begin{aligned}
-\left(S+\frac{C}{2}\right)T[1,0]+\quad(1+2S)T[1,1]-\left(S-\frac{C}{2}\right)T[1,2]+\qquad 0T[1,3]+\qquad 0T[1,4]&=T[0,1]\\
0T[1,0]-\left(S+\frac{C}{2}\right)T[1,1]+\quad(1+2S)T[1,2]-\left(S-\frac{C}{2}\right)T[1,3]+\qquad 0T[1,4]&=T[0,2]\\
0T[1,0]+\qquad 0T[1,1]-\left(S+\frac{C}{2}\right)T[1,2]+\quad(1+2S)T[1,3]-\left(S-\frac{C}{2}\right)T[1,4]&=T[0,3]
\end{aligned}$$

Reordenando

$$\begin{aligned}
(1+2S)T[1,1]-\left(S-\frac{C}{2}\right)T[1,2]+\qquad 0T[1,3]&=T[0,1]+\left(S+\frac{C}{2}\right)T[1,0]+\qquad 0T[1,0]\\
-\left(S+\frac{C}{2}\right)T[1,1]+\quad(1+2S)T[1,2]-\left(S-\frac{C}{2}\right)T[1,3]&=T[0,2]+\qquad 0T[1,0]+\qquad 0T[1,4]\\
0T[1,1]-\left(S+\frac{C}{2}\right)T[1,2]+\quad(1+2S)T[1,3]&=T[0,3]+\qquad 0T[1,0]+\left(S-\frac{C}{2}\right)T[1,4]
\end{aligned}$$

Simplificando

$$\begin{aligned}
(1+2S)T[1,1]-\left(S-\frac{C}{2}\right)T[1,2]+\qquad 0T[1,3]&=T[0,1]+\left(S+\frac{C}{2}\right)T[1,0]\\
-\left(S+\frac{C}{2}\right)T[1,1]+\quad(1+2S)T[1,2]-\left(S-\frac{C}{2}\right)T[1,3]&=T[0,2]\\
0T[1,1]-\left(S+\frac{C}{2}\right)T[1,2]+\quad(1+2S)T[1,3]&=T[0,3]+\left(S-\frac{C}{2}\right)T[1,4]
\end{aligned}$$

En forma matricial

$$\begin{bmatrix} 1+2S & -\left(S-\frac{C}{2}\right) & 0\\ -\left(S+\frac{C}{2}\right) & 1+2S & -\left(S-\frac{C}{2}\right)\\ 0 & -\left(S+\frac{C}{2}\right) & 1+2S \end{bmatrix}\begin{bmatrix}T[1,1]\\T[1,2]\\T[1,3]\end{bmatrix}=\begin{bmatrix}T[0,1]\\T[0,2]\\T[0,3]\end{bmatrix}+S\begin{bmatrix}T[1,0]\\0\\T[1,4]\end{bmatrix}+\frac{C}{2}\begin{bmatrix}T[1,0]\\0\\-T[1,4]\end{bmatrix}$$

Al resolver el sistema se obtienen los valores $T[1,x]$, también puede escribirse como

$$\begin{bmatrix} 1+2S & -\left(S-\frac{C}{2}\right) & 0\\ -\left(S+\frac{C}{2}\right) & 1+2S & -\left(S-\frac{C}{2}\right)\\ 0 & -\left(S+\frac{C}{2}\right) & 1+2S \end{bmatrix}\begin{bmatrix}T[1,1]\\T[1,2]\\T[1,3]\end{bmatrix}=\begin{bmatrix}T[0,1]+\left(S+\frac{C}{2}\right)T[1,0]\\T[0,2]\\T[0,3]+\left(S-\frac{C}{2}\right)T[1,4]\end{bmatrix}$$

Generalizando el método para s pasos

$$\begin{bmatrix}
1+2S & -\left(S-\frac{C}{2}\right) & 0 & \cdots & \cdots & \cdots & 0\\
-\left(S+\frac{C}{2}\right) & 1+2S & -\left(S-\frac{C}{2}\right) & \ddots & & & \vdots\\
0 & -\left(S+\frac{C}{2}\right) & 1+2S & \ddots & \ddots & & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & & \ddots & \ddots & 1+2S & -\left(S-\frac{C}{2}\right) & 0\\
\vdots & & & \ddots & -\left(S+\frac{C}{2}\right) & 1+2S & -\left(S-\frac{C}{2}\right)\\
0 & \cdots & \cdots & \cdots & 0 & -\left(S+\frac{C}{2}\right) & 1+2S
\end{bmatrix}\begin{bmatrix}T[n+1,1]\\T[n+1,2]\\\vdots\\\vdots\\\vdots\\T[n+1,s-1]\\T[n+1,s]\end{bmatrix}$$

$$=\begin{bmatrix}T[n,1]+\left(S+\frac{C}{2}\right)T[n+1,0]\\T[n,2]\\\vdots\\\vdots\\\vdots\\T[n,s-1]\\T[n,s]+\left(S-\frac{C}{2}\right)T[n+1,s+1]\end{bmatrix} \tag{6.3}$$

6.1.3. Crank-Nicolson

Puede derivarse de métodos FTCS

$$T(n+1,j) = \left(S + \frac{C}{2}\right)T(n,j-1) + (1-2S)T(n,j) + \left(S - \frac{C}{2}\right)T(n,j+1) \qquad \text{explícito}$$

$$T(n,j) = -\left(S + \frac{C}{2}\right)T(n+1,j-1) + (1+2S)T(n+1,j) - \left(S - \frac{C}{2}\right)T(n+1,j+1) \qquad \text{implícito}$$

Reordenando

$$T(n+1,j) - T(n,j) = \left(S + \frac{C}{2}\right)T(n,j-1) - 2ST(n,j) + \left(S - \frac{C}{2}\right)T(n,j+1)$$

$$T(n+1,j) - T(n,j) = \left(S + \frac{C}{2}\right)T(n+1,j-1) - 2ST(n+1,j) + \left(S - \frac{C}{2}\right)T(n+1,j+1)$$

Multiplicando por $\frac{1}{2}$ y sumando

$$\begin{aligned} T(n+1,j) - T(n,j) = {} & \left(\frac{S}{2} + \frac{C}{4}\right)T(n,j-1) - ST(n,j) + \left(\frac{S}{2} - \frac{C}{4}\right)T(n,j+1) \\ & + \left(\frac{S}{2} + \frac{C}{4}\right)T(n+1,j-1) - ST(n+1,j) + \left(\frac{S}{2} - \frac{C}{4}\right)T(n+1,j+1) \end{aligned}$$

Reordenando

$$\begin{aligned} & \left(\frac{S}{2} + \frac{C}{4}\right)T(n,j-1) + (1-S)T(n,j) + \left(\frac{S}{2} - \frac{C}{4}\right)T(n,j+1) \\ & = -\left(\frac{S}{2} + \frac{C}{4}\right)T(n+1,j-1) + (1+S)T(n+1,j) - \left(\frac{S}{2} - \frac{C}{4}\right)T(n+1,j+1) \end{aligned}$$

Para generalizar el método se usará tres pasos en el espacio

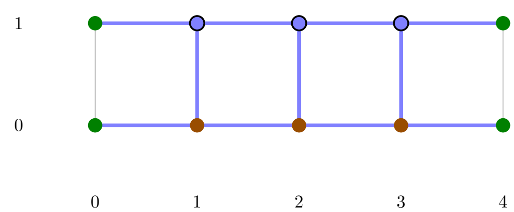

Figura 6.2: Movimiento del stencil para $t = 1$

Escribiendo una ecuación por cada paso

$$-\left(\frac{S}{2} + \frac{C}{4}\right)T[1,0] + (1+S)T[1,1] - \left(\frac{S}{2} - \frac{C}{4}\right)T[1,2] = \left(\frac{S}{2} + \frac{C}{4}\right)T[0,0] + (1-S)T[0,1] + \left(\frac{S}{2} - \frac{C}{4}\right)T[0,2]$$

$$-\left(\frac{S}{2} + \frac{C}{4}\right)T[1,1] + (1+S)T[1,2] - \left(\frac{S}{2} - \frac{C}{4}\right)T[1,3] = \left(\frac{S}{2} + \frac{C}{4}\right)T[0,1] + (1-S)T[0,2] + \left(\frac{S}{2} - \frac{C}{4}\right)T[0,3]$$

$$-\left(\frac{S}{2} + \frac{C}{4}\right)T[1,2] + (1+S)T[1,3] - \left(\frac{S}{2} - \frac{C}{4}\right)T[1,4] = \left(\frac{S}{2} + \frac{C}{4}\right)T[0,2] + (1-S)T[0,3] + \left(\frac{S}{2} - \frac{C}{4}\right)T[0,4]$$

Reordenando

$$(1+S)T[1,1] - \left(\frac{S}{2} - \frac{C}{4}\right)T[1,2] + 0T[1,3] = (1-S)T[0,1] + \left(\frac{S}{2} - \frac{C}{4}\right)T[0,2] + 0T[0,3] + \left(\frac{S}{2} + \right.$$

$$-\left(\frac{S}{2} + \frac{C}{4}\right)T[1,1] + (1+S)T[1,2] - \left(\frac{S}{2} - \frac{C}{4}\right)T[1,3] = \left(\frac{S}{2} + \frac{C}{4}\right)T[0,1] + (1-S)T[0,2] + \left(\frac{S}{2} - \frac{C}{4}\right)T[0,3]$$

$$0T[1,1] - \left(\frac{S}{2} + \frac{C}{4}\right)T[1,2] + (1+S)T[1,3] = 0T[0,1] + \left(\frac{S}{2} + \frac{C}{4}\right)T[0,2] + (1-S)T[0,3] + \left(\frac{S}{2} - \right.$$

En forma matricial

$$\begin{bmatrix} 1+S & -\left(\frac{S}{2} - \frac{C}{4}\right) & 0 \\ -\left(\frac{S}{2} + \frac{C}{4}\right) & 1+S & -\left(\frac{S}{2} - \frac{C}{4}\right) \\ 0 & -\left(\frac{S}{2} + \frac{C}{4}\right) & 1+S \end{bmatrix} \begin{bmatrix} T[1,1] \\ T[1,2] \\ T[1,3] \end{bmatrix} = \begin{bmatrix} 1-S & \frac{S}{2} - \frac{C}{4} & 0 \\ \frac{S}{2} + \frac{C}{4} & 1-S & \frac{S}{2} - \frac{C}{4} \\ 0 & \frac{S}{2} + \frac{C}{4} & 1-S \end{bmatrix} \begin{bmatrix} T[0,1] \\ T[0,2] \\ T[0,3] \end{bmatrix} + \begin{bmatrix} \left(\frac{S}{2} + \frac{C}{4}\right)(T[0,0] + T[1,0]) \\ 0 \\ \left(\frac{S}{2} - \frac{C}{4}\right)(T[0,4] + T[1,4]) \end{bmatrix}$$

Generalizando el método para s pasos

$$
\begin{aligned}
&\begin{bmatrix}
1+S & -\left(\frac{S}{2}-\frac{C}{4}\right) & 0 & \cdots & \cdots & \cdots & 0\\
-\left(\frac{S}{2}+\frac{C}{4}\right) & 1+S & -\left(\frac{S}{2}-\frac{C}{4}\right) & \ddots & & & \vdots\\
0 & -\left(\frac{S}{2}+\frac{C}{4}\right) & 1+S & \ddots & \ddots & & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & & \ddots & \ddots & 1+S & -\left(\frac{S}{2}-\frac{C}{4}\right) & 0\\
\vdots & & & \ddots & -\left(\frac{S}{2}+\frac{C}{4}\right) & 1+S & -\left(\frac{S}{2}-\frac{C}{4}\right)\\
0 & \cdots & \cdots & \cdots & 0 & -\left(\frac{S}{2}+\frac{C}{4}\right) & 1+S
\end{bmatrix}
\begin{bmatrix}
T[n+1,1]\\ T[n+1,2]\\ \vdots\\ \vdots\\ \vdots\\ T[n+1,s-1]\\ T[n+1,s]
\end{bmatrix}\\
&=\begin{bmatrix}
1-S & \frac{S}{2}-\frac{C}{4} & 0 & \cdots & \cdots & \cdots & 0\\
\frac{S}{2}+\frac{C}{4} & 1-S & \frac{S}{2}-\frac{C}{4} & \ddots & & & \vdots\\
0 & \frac{S}{2}+\frac{C}{4} & 1-S & \ddots & \ddots & & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & & \ddots & \ddots & 1-S & \frac{S}{2}-\frac{C}{4} & 0\\
\vdots & & & \ddots & \frac{S}{2}+\frac{C}{4} & 1-S & \frac{S}{2}-\frac{C}{4}\\
0 & \cdots & \cdots & \cdots & 0 & \frac{S}{2}+\frac{C}{4} & 1-S
\end{bmatrix}
\begin{bmatrix}
T[n,1]\\ T[n,2]\\ \vdots\\ \vdots\\ \vdots\\ T[n,s-1]\\ T[n,s]
\end{bmatrix}\\
&\quad+\begin{bmatrix}
\left(\frac{S}{2}+\frac{C}{4}\right)(T[n,0]+T[n+1,0])\\
0\\ \vdots\\ \vdots\\ \vdots\\ 0\\
\left(\frac{S}{2}-\frac{C}{4}\right)(T[n,s+1]+T[n+1,s+1])
\end{bmatrix}
\end{aligned}
\tag{6.4}
$$