Introducción a elementos finitos
Tarea 6 I-2016

Aplicando el método de Newton-Cotes construir la tabla de pesos y puntos de muestreo hasta $n = 4$

n = 1

$$k = 1 - 1 = 0$$

Calculando r_i

$$\int_{-1}^{+1} P(r)\ r^0\ dr = 0$$

El polinomio es

$$P(r) = r - r_1$$

Reemplazando

$$\int_{-1}^{+1} r - r_1\ dr = 0$$

Integrando

$$\int_{-1}^{+1} r - r_1\ dr = \left(\frac{1}{2}r^2 - r_1 r\right)\Big|_{-1}^{+1} = -2r_1$$

Despejando

$$r_1 = 0$$

Calculando w_i

$$w_1 = \int_{-1}^{+1} dr = 2$$

n = 2

$$k = 2 - 1 = 1$$

Calculando r_i

$$\int_{-1}^{+1} P(r)\ r^0\ dr = 0$$
$$\int_{-1}^{+1} P(r)\ r^1\ dr = 0$$

El polinomio es

$$P(r) = (r - r_1)(r - r_2)$$

Reemplazando

$$\int_{-1}^{+1} (r - r_1)(r - r_2)\ dr = 0$$
$$\int_{-1}^{+1} (r - r_1)(r - r_2) r\ dr = 0$$

Integrando

$$\left(\frac{1}{3}r^3 - \frac{r_1 + r_2}{2}r^2 + r_1 r_2 r\right)\Bigg|_{-1}^{+1} = 2\left(r_1 r_2 + \frac{1}{3}\right)$$
$$\left(\frac{1}{4}r^4 - \frac{r_1 + r_2}{3}r^3 + \frac{r_1 r_2}{2}r^2\right)\Bigg|_{-1}^{+1} = -\frac{2}{3}(r_1 + r_2)$$

Formando el sistema de ecuaciones

$$r_1 r_2 = -\frac{1}{3}$$
$$r_1 + r_2 = 0$$

Resolviendo

$$r_1 = -\sqrt{\frac{1}{3}}$$
$$r_2 = \sqrt{\frac{1}{3}}$$

Calculando w_i

$$w_1 = \int_{-1}^{+1} \frac{r - r_2}{r_1 - r_2}\ dr$$
$$w_2 = \int_{-1}^{+1} \frac{r - r_1}{r_2 - r_1}\ dr$$

Reemplazando

$$w_1 = \int_{-1}^{+1} \frac{r - \sqrt{\frac{1}{3}}}{-\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{3}}}\ dr$$
$$w_2 = \int_{-1}^{+1} \frac{r + \sqrt{\frac{1}{3}}}{\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{3}}}\ dr$$

Integrando

$$w_1 = \left(-\frac{\sqrt{3}}{4}r^2 + \frac{1}{2}r\right)\Bigg|_{-1}^{+1} = 1$$
$$w_2 = \left(\frac{\sqrt{3}}{4}r^2 + \frac{1}{2}r\right)\Bigg|_{-1}^{+1} = 1$$

n = 3

$$k = 3 - 1 = 2$$

Calculando r_i

$$\int_{-1}^{+1} P(r)\ r^0\ dr = 0$$
$$\int_{-1}^{+1} P(r)\ r^1\ dr = 0$$
$$\int_{-1}^{+1} P(r)\ r^2\ dr = 0$$

El polinomio es

$$P(r) = (r - r_1)(r - r_2)(r - r_3)$$

Reemplazando

$$\int_{-1}^{+1} (r - r_1)(r - r_2)(r - r_3)\ dr = 0$$
$$\int_{-1}^{+1} (r - r_1)(r - r_2)(r - r_3) r\ dr = 0$$
$$\int_{-1}^{+1} (r - r_1)(r - r_2)(r - r_3) r^2\ dr = 0$$

Integrando

$$\left[\frac{1}{4}r^4 - \frac{r_1 + r_2 + r_3}{3} r^3 + \frac{r_1r_2 + r_1r_3 + r_2r_3}{2} r^2 - r_1r_2r_3r\right]\Bigg|_{-1}^{+1} = -\frac{2}{3}(3r_1r_2r_3 + r_1 + r_2 + r_3)$$
$$\left[\frac{1}{5}r^5 - \frac{r_1 + r_2 + r_3}{4} r^4 + \frac{r_1r_2 + r_1r_3 + r_2r_3}{3} r^3 - \frac{r_1r_2r_3}{2} r^2\right]\Bigg|_{-1}^{+1} = \frac{2}{15}(5r_1r_2 + 5r_1r_3 + 5r_2r_3 + 3)$$
$$\left[\frac{1}{6}r^6 - \frac{r_1 + r_2 + r_3}{5} r^5 + \frac{r_1r_2 + r_1r_3 + r_2r_3}{4} r^4 - \frac{r_1r_2r_3}{3} r^3\right]\Bigg|_{-1}^{+1} = -\frac{2}{15}(5r_1r_2r_3 + 3r_1 + 3r_2 + 3r_3)$$

Formando el sistema de ecuaciones

$$3r_1r_2r_3 + r_1 + r_2 + r_3 = 0$$
$$5r_1r_2 + 5r_1r_3 + 5r_2r_3 = -3$$
$$5r_1r_2r_3 + 3r_1 + 3r_2 + 3r_3 = 0$$

Resolviendo

$$r_1 = -\sqrt{\frac{3}{5}}$$
$$r_2 = 0$$
$$r_3 = \sqrt{\frac{3}{5}}$$

Calculando w_i

$$w_1 = \int_{-1}^{+1} \frac{r - r_2}{r_1 - r_2} \cdot \frac{r - r_3}{r_1 - r_3} \ dr$$
$$w_2 = \int_{-1}^{+1} \frac{r - r_1}{r_2 - r_1} \cdot \frac{r - r_3}{r_2 - r_3} \ dr$$
$$w_3 = \int_{-1}^{+1} \frac{r - r_2}{r_3 - r_2} \cdot \frac{r - r_1}{r_3 - r_1} \ dr$$

Reemplazando

$$w_1 = \int_{-1}^{+1} \frac{r - 0}{-\sqrt{\frac{3}{5}} - 0} \cdot \frac{r - \sqrt{\frac{3}{5}}}{-\sqrt{\frac{3}{5}} - \sqrt{\frac{3}{5}}} \ dr$$
$$w_2 = \int_{-1}^{+1} \frac{r + \sqrt{\frac{3}{5}}}{0 + \sqrt{\frac{3}{5}}} \cdot \frac{r - \sqrt{\frac{3}{5}}}{0 - \sqrt{\frac{3}{5}}} \ dr$$
$$w_3 = \int_{-1}^{+1} \frac{r - 0}{\sqrt{\frac{3}{5}} - 0} \cdot \frac{r + \sqrt{\frac{3}{5}}}{\sqrt{\frac{3}{5}} + \sqrt{\frac{3}{5}}} \ dr$$

Integrando

$$w_1 = \left(\frac{5}{18} r^3 - \frac{\sqrt{15}}{12} r^2\right)\Bigg|_{-1}^{+1} = \frac{5}{9}$$
$$w_2 = \left(-\frac{5}{9} r^3 + r\right)\Bigg|_{-1}^{+1} = \frac{8}{9}$$
$$w_3 = \left(\frac{5}{18} r^3 + \frac{\sqrt{15}}{12} r^2\right)\Bigg|_{-1}^{+1} = \frac{5}{9}$$

$\mathbf{n = 4}$

$$k = 4 - 1 = 3$$

Calculando r_i

$$\int_{-1}^{+1} P(r) \ r^0 \ dr = 0$$
$$\int_{-1}^{+1} P(r) \ r^1 \ dr = 0$$
$$\int_{-1}^{+1} P(r) \ r^2 \ dr = 0$$
$$\int_{-1}^{+1} P(r) \ r^3 \ dr = 0$$

El polinomio es

$$P(r) = (r - r_1)(r - r_2)(r - r_3)(r - r_4)$$

Reemplazando

$$\begin{aligned}
\int_{-1}^{+1} (r - r_1)(r - r_2)(r - r_3)(r - r_4)\ dr = 0 \\
\int_{-1}^{+1} (r - r_1)(r - r_2)(r - r_3)(r - r_4) r\ dr = 0 \\
\int_{-1}^{+1} (r - r_1)(r - r_2)(r - r_3)(r - r_4) r^2\ dr = 0 \\
\int_{-1}^{+1} (r - r_1)(r - r_2)(r - r_3)(r - r_4) r^3\ dr = 0
\end{aligned}$$

Integrando

$$\begin{aligned}
&\Big[\frac{1}{5}r^5 - \frac{r_1 + r_2 + r_3 + r_4}{4}r^4 + \frac{r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4}{3}r^3 - \frac{r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4}{2}r^2 \\
&+ r_1r_2r_3r_4r\Big]\Big|_{-1}^{+1} = \frac{2}{15}(15r_1r_2r_3r_4 + 5r_1r_2 + 5r_1r_3 + 5r_1r_4 + 5r_2r_3 + 5r_2r_4 + 5r_3r_4 + 3) \\
&\Big[\frac{1}{6}r^6 - \frac{r_1 + r_2 + r_3 + r_4}{5}r^5 + \frac{r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4}{4}r^4 - \frac{r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4}{3}r^3 \\
&+ \frac{r_1r_2r_3r_4}{2}r^2\Big]\Big|_{-1}^{+1} = -\frac{2}{15}(5r_1r_2r_3 + 5r_1r_2r_4 + 5r_1r_3r_4 + 5r_2r_3r_4 + 3r_1 + 3r_2 + 3r_3 + 3r_4) \\
&\Big[\frac{1}{7}r^7 - \frac{r_1 + r_2 + r_3 + r_4}{6}r^6 + \frac{r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4}{5}r^5 - \frac{r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4}{4}r^4 \\
&+ \frac{r_1r_2r_3r_4}{3}r^3\Big]\Big|_{-1}^{+1} = \frac{2}{105}(35r_1r_2r_3r_4 + 21r_1r_2 + 21r_1r_3 + 21r_1r_4 + 21r_2r_3 + 21r_2r_4 + 21r_3r_4 + 15) \\
&\Big[\frac{1}{8}r^8 - \frac{r_1 + r_2 + r_3 + r_4}{7}r^7 + \frac{r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4}{6}r^6 - \frac{r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4}{5}r^5 \\
&+ \frac{r_1r_2r_3r_4}{4}r^4\Big]\Big|_{-1}^{+1} = -\frac{2}{35}(7r_1r_2r_3 + 7r_1r_2r_4 + 7r_1r_3r_4 + 7r_2r_3r_4 + 5r_1 + 5r_2 + 5r_3 + 5r_4)
\end{aligned}$$

Formando el sistema de ecuaciones

$$\begin{aligned}
15r_1r_2r_3r_4 + 5r_1r_2 + 5r_1r_3 + 5r_1r_4 + 5r_2r_3 + 5r_2r_4 + 5r_3r_4 = -3 \\
5r_1r_2r_3 + 5r_1r_2r_4 + 5r_1r_3r_4 + 5r_2r_3r_4 + 3r_1 + 3r_2 + 3r_3 + 3r_4 = 0 \\
35r_1r_2r_3r_4 + 21r_1r_2 + 21r_1r_3 + 21r_1r_4 + 21r_2r_3 + 21r_2r_4 + 21r_3r_4 = -15 \\
7r_1r_2r_3 + 7r_1r_2r_4 + 7r_1r_3r_4 + 7r_2r_3r_4 + 5r_1 + 5r_2 + 5r_3 + 5r_4 = 0
\end{aligned}$$

Resolviendo

$$r_1 = -\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}$$
$$r_2 = -\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}$$
$$r_3 = \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}$$
$$r_4 = \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}$$

Calculando w_i

$$w_1 = \int_{-1}^{+1} \frac{r - r_2}{r_1 - r_2} \cdot \frac{r - r_3}{r_1 - r_3} \cdot \frac{r - r_4}{r_1 - r_4}\ dr$$
$$w_2 = \int_{-1}^{+1} \frac{r - r_1}{r_2 - r_1} \cdot \frac{r - r_3}{r_2 - r_3} \cdot \frac{r - r_4}{r_2 - r_4}\ dr$$
$$w_3 = \int_{-1}^{+1} \frac{r - r_2}{r_3 - r_2} \cdot \frac{r - r_1}{r_3 - r_1} \cdot \frac{r - r_4}{r_3 - r_4}\ dr$$
$$w_4 = \int_{-1}^{+1} \frac{r - r_3}{r_4 - r_3} \cdot \frac{r - r_2}{r_4 - r_2} \cdot \frac{r - r_1}{r_4 - r_1}\ dr$$

Reemplazando

$$w_1 = \int_{-1}^{+1} \frac{r + \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}}{-\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} + \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}} \cdot \frac{r - \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}}{-\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} - \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}} \cdot \frac{r - \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}}{-\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} - \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}}\ dr$$
$$w_2 = \int_{-1}^{+1} \frac{r + \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}}{-\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}} + \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}} \cdot \frac{r - \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}}{-\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}} - \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}} \cdot \frac{r - \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}}{-\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}} - \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}}\ dr$$
$$w_3 = \int_{-1}^{+1} \frac{r + \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}}{\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}} + \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}} \cdot \frac{r + \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}}{\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}} + \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}} \cdot \frac{r - \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}}{\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}} - \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}}\ dr$$
$$w_4 = \int_{-1}^{+1} \frac{r - \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}}{\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} - \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}} \cdot \frac{r + \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}}{\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} + \sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}} \cdot \frac{r + \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}}{\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}} + \sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}}\ dr$$

Integrando

$$
\begin{aligned}
w_1 &= \frac{18-\sqrt{30}}{36}\\
w_2 &= \frac{18+\sqrt{30}}{36}\\
w_3 &= \frac{18+\sqrt{30}}{36}\\
w_4 &= \frac{18-\sqrt{30}}{36}
\end{aligned}
$$

Tabla resumen

n	r	w
1	0	2
2	$-\sqrt{\frac{1}{3}}$ $\sqrt{\frac{1}{3}}$	1 1
3	$-\sqrt{\frac{3}{5}}$ 0 $\sqrt{\frac{3}{5}}$	$\frac{5}{9}$ $\frac{8}{9}$ $\frac{5}{9}$
4	$-\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}}$ $-\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}}$ $\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}}$ $\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18-\sqrt{30}}{36}$ $\frac{18+\sqrt{30}}{36}$ $\frac{18+\sqrt{30}}{36}$ $\frac{18-\sqrt{30}}{36}$