Esquema Tres niveles de tiempo

Hallar el perfil de flujo usando $\Delta x = 40$ m, $\Delta t = 10$ h y $D = 1 \times 10^{-3}$ m^2/s, para un tiempo final de 20 h

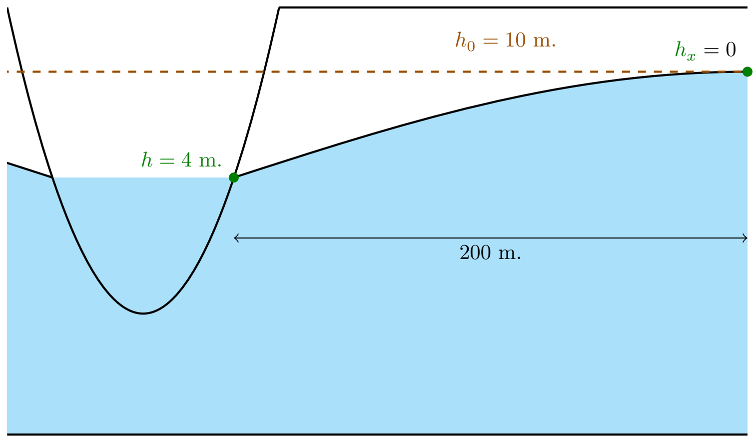

Figura 1: Representación gráfica del ejemplo

La ecuación y las condiciones de contorno serán

$$\frac{\partial h}{\partial t} - D\,\frac{\partial^2 h}{\partial x^2} = 0 \tag{1}$$
$$h(x,0) = 10 \tag{2}$$
$$h(0,t) = 4 \tag{3}$$
$$h_x(200,t) = 0 \tag{4}$$

Discretización espacial

$$N_{\text{elementos}} = \frac{L}{\Delta x} = \frac{200}{40} = 5$$
$$N_{\text{puntos}} = N_{\text{elementos}} + 1 = 5 + 1 = 6$$

Discretización temporal

$$N_{\text{elementos}} = \frac{t}{\Delta t} = \frac{20}{10} = 2$$
$$N_{\text{puntos}} = N_{\text{elementos}} + 1 = 2 + 1 = 3$$

Discretización numérica

$$\frac{\partial h}{\partial t} = \frac{3h_i^{n+1} - 4h_i^n + h_i^{n-1}}{2\Delta t}$$

a partir del esquema θ o esquema generalizado de Crank-Nicolson

$$\frac{\partial^2 h}{\partial x^2} = \theta\left(\frac{h_{i-1}^{n+1} - 2h_i^{n+1} + h_{i+1}^{n+1}}{\Delta x^2}\right) + (1-\theta)\left(\frac{h_{i-1}^{n} - 2h_i^{n} + h_{i+1}^{n}}{\Delta x^2}\right)$$

reemplazando $\theta = 1$ (puede usarse otro valor)

$$\frac{\partial^2 h}{\partial x^2} = \frac{h_{i-1}^{n+1} - 2h_i^{n+1} + h_{i+1}^{n+1}}{\Delta x^2}$$

Reemplazando en (1)

$$\frac{3h_i^{n+1} - 4h_i^n + h_i^{n-1}}{2\Delta t} - D\left(\frac{h_{i-1}^{n+1} - 2h_i^{n+1} + h_{i+1}^{n+1}}{2\Delta x^2}\right) = 0$$

Reordenando

$$-D\frac{\Delta t}{\Delta x^2}h_{i-1}^{n+1} + \left(3 + 2D\frac{\Delta t}{\Delta x^2}\right)h_i^{n+1} - D\frac{\Delta t}{\Delta x^2}h_{i+1}^{n+1} = 4h_i^n - h_i^{n-1}$$

Realizando un cambio de variable

$$a = -D\frac{\Delta t}{\Delta x^2}$$

$$b = 3 + 2D\frac{\Delta t}{\Delta x^2}$$

$$c = -D\frac{\Delta t}{\Delta x^2}$$

$$d_i = 4h_i^n - h_i^{n-1}$$

El esquema será

$$a\, h_{i-1}^{n+1} + b\, h_i^{n+1} + c\, h_{i+1}^{n+1} = d_i$$

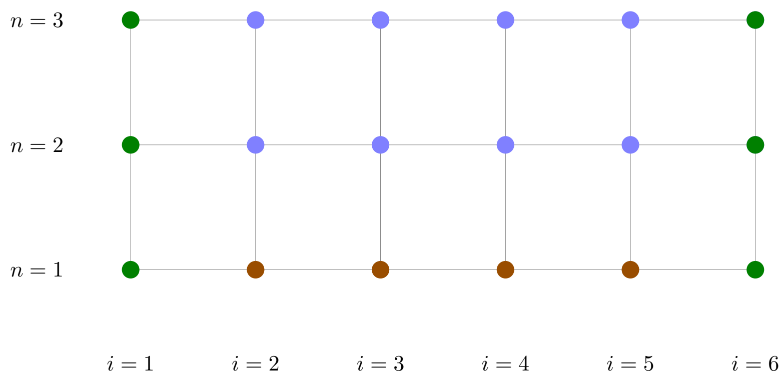

Figura 2: Mallado

3	h_1^3	h_2^3	h_3^3	h_4^3	h_5^3	h_6^3
2	h_1^2	h_2^2	h_3^2	h_4^2	h_5^2	h_6^2
1	h_1^1	h_2^1	h_3^1	h_4^1	h_5^1	h_6^1
	1	2	3	4	5	6

Figura 3: Matriz solución

El esquema es incondicionalmente estable para cualquier λ

$$D\frac{\Delta t}{\Delta x^2} = 0.001\left(\frac{36000}{40^2}\right) = 0.0225$$

Reemplazando las condiciones de contorno, para $i = 1$ y $n = 1, 2, 3$

$$
\begin{aligned}
h_1^1 &= 4 \\
h_1^2 &= 4 \\
h_1^3 &= 4
\end{aligned}
$$

Para $i = 2, 3, 4, 5$ y $n = 1$

$$
\begin{aligned}
h_2^1 &= 10 \\
h_3^1 &= 10 \\
h_4^1 &= 10 \\
h_5^1 &= 10
\end{aligned}
$$

Para $i = 6$ y $n = 1$, usando un esquema hacia atrás

$$h_x = \frac{\partial h}{\partial x} = \frac{h_6^1 - h_5^1}{\Delta x} = 0$$

reordenando

$$
\begin{aligned}
h_6^1 &= h_5^1 \\
&= 10
\end{aligned}
$$

3	4	h_2^3	h_3^3	h_4^3	h_5^3	h_6^3
2	4	h_2^2	h_3^2	h_4^2	h_5^2	h_6^2
1	4	10	10	10	10	10
	1	2	3	4	5	6

Figura 4: Matriz solución para $t = 0$ h

Las constantes a, b, c serán

$$
\begin{aligned}
a &= -0.0225 \\
b &= 3 + 2(0.0225) = 3.045 \\
c &= -0.0225
\end{aligned}
$$

Debido a que el esquema contiene el término h_i^{n-1}, no puede obtenerse soluciones para la fila $n = 2$, para esta fila se usará el esquema implícito de Euler (puede usarse cualquier otro esquema).

3	4	h_2^3	h_3^3	h_4^3	h_5^3	h_6^3
2	4	9.87075	9.99721	9.99994	9.99999	9.99999
1	4	10	10	10	10	10
	1	2	3	4	5	6

Figura 5: Matriz solución para $t = 10$ h

Para $i = 2$ y $n = 2$

$$-0.0225h_1^3 + 3.045h_2^3 - 0.0225h_3^3 = d_2$$
$$d_2 = 4h_2^2 - h_2^1 = 4(9.87075) - 10 = 29.483$$

Para $i = 3$ y $n = 2$

$$-0.0225h_2^3 + 3.045h_3^3 - 0.0225h_4^3 = d_3$$
$$d_3 = 4h_3^2 - h_3^1 = 4(9.99721) - 10 = 29.98884$$

Para $i = 4$ y $n = 2$

$$-0.0225h_3^3 + 3.045h_4^3 - 0.0225h_5^3 = d_4$$
$$d_4 = 4h_4^2 - h_4^1 = 4(9.99994) - 10 = 29.99976$$

Para $i = 5$ y $n = 2$

$$-0.0225h_4^3 + 3.045h_5^3 - 0.0225h_6^3 = d_5$$
$$d_5 = 4h_5^2 - h_5^1 = 4(9.99999) - 10 = 29.99996$$

Formando un sistema de ecuaciones

$$\begin{alignedat}{6}
-0.0225\,h_1^3 &+ 3.045\,h_2^3 &- 0.0225\,h_3^3 & & & &= 29.483\\
&- 0.0225\,h_2^3 &+ 3.045\,h_3^3 &- 0.0225\,h_4^3 & & &= 29.98884\\
& &- 0.0225\,h_3^3 &+ 3.045\,h_4^3 &- 0.0225\,h_5^3 & &= 29.99976\\
& & &- 0.0225\,h_4^3 &+ 3.045\,h_5^3 &- 0.0225\,h_6^3 &= 29.99996
\end{alignedat}$$

En forma matricial

$$\begin{bmatrix} -0.0225 & 3.045 & -0.0225 & 0 & 0 & 0 \\ 0 & -0.0225 & 3.045 & -0.0225 & 0 & 0 \\ 0 & 0 & -0.0225 & 3.045 & -0.0225 & 0 \\ 0 & 0 & 0 & -0.0225 & 3.045 & -0.0225 \end{bmatrix} \begin{bmatrix} h_1^3 \\ h_2^3 \\ h_3^3 \\ h_4^3 \\ h_5^3 \\ h_6^3 \end{bmatrix} = \begin{bmatrix} 29.483 \\ 29.98884 \\ 29.99976 \\ 29.99996 \end{bmatrix}$$

Agregando las dos ecuaciones faltantes

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -0.0225 & 3.045 & -0.0225 & 0 & 0 & 0 \\ 0 & -0.0225 & 3.045 & -0.0225 & 0 & 0 \\ 0 & 0 & -0.0225 & 3.045 & -0.0225 & 0 \\ 0 & 0 & 0 & -0.0225 & 3.045 & -0.0225 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} h_1^3 \\ h_2^3 \\ h_3^3 \\ h_4^3 \\ h_5^3 \\ h_6^3 \end{bmatrix} = \begin{bmatrix} 4 \\ 29.483 \\ 29.98884 \\ 29.99976 \\ 29.99996 \\ h_6^3 \end{bmatrix}$$

El sistema anterior puede transformarse en una tabla para aplicar el algoritmo de Thomas

	a	b	c	d	e	f
1		1	0	4		
2	-0.0225	3.045	-0.0225	29.483		
3	-0.0225	3.045	-0.0225	29.98884		
4	-0.0225	3.045	-0.0225	29.99976		
5	-0.0225	3.045	-0.0225	29.99996		
6	0	1		h_6^3		

Constantes e y f, hacia adelante

$$e_1 = \frac{d_1}{b_1} = \frac{4}{1} = 4$$
$$e_2 = \frac{d_2 - a_2 e_1}{b_2 + a_2 f_1} = \frac{29.483 - (-0.0225)(4)}{3.045 + (-0.0225)(0)} = 9.71198$$
$$e_3 = \frac{d_3 - a_3 e_2}{b_3 + a_3 f_2} = \frac{29.98884 - (-0.0225)(9.71198)}{3.045 + (-0.0225)(0.00738)} = 9.92085$$
$$e_4 = \frac{d_4 - a_4 e_3}{b_4 + a_4 f_3} = \frac{29.99976 - (-0.0225)(9.92085)}{3.045 + (-0.0225)(0.00738)} = 9.92598$$
$$e_5 = \frac{d_5 - a_5 e_4}{b_5 + a_5 f_4} = \frac{29.99996 - (-0.0225)(9.92598)}{3.045 + (-0.0225)(0.00738)} = 9.92608$$

$$f_1 = -\frac{c_1}{b_1} = -\frac{0}{1} = 0$$
$$f_2 = -\frac{c_2}{b_2 + a_2 f_1} = -\frac{-0.0225}{3.045 + (-0.0225)(0)} = 0.00738$$
$$f_3 = -\frac{c_3}{b_3 + a_3 f_2} = -\frac{-0.0225}{3.045 + (-0.0225)(0.00738)} = 0.00738$$
$$f_4 = -\frac{c_4}{b_4 + a_4 f_3} = -\frac{-0.0225}{3.045 + (-0.0225)(0.00738)} = 0.00738$$
$$f_5 = -\frac{c_5}{b_5 + a_5 f_4} = -\frac{-0.0225}{3.045 + (-0.0225)(0.00738)} = 0.00738$$

Incógnitas, hacia atrás

$$\begin{aligned} h_6^3 &= h_6^3 \\ h_5^3 &= e_5 + f_5 h_6^3 \\ h_4^3 &= e_4 + f_4 h_5^3 \\ h_3^3 &= e_3 + f_3 h_4^3 \\ h_2^3 &= e_2 + f_2 h_3^3 \\ h_1^3 &= e_1 + f_1 h_2^3 \end{aligned}$$

Debido a la condición de contorno del lado derecho, la primera ecuación cambia

$$h_6^3 = h_5^3$$
$$h_5^3 = e_5 + f_5 h_5^3 = 9.92608 + 0.00738 h_5^2 = 9.99987$$
$$h_4^3 = e_4 + f_4 h_5^3 = 9.92598 + 0.00738(9.99987) = 9.99977$$
$$h_3^3 = e_3 + f_3 h_4^3 = 9.92085 + 0.00738(9.99977) = 9.99464$$
$$h_2^3 = e_2 + f_2 h_3^3 = 9.71198 + 0.00738(9.99464) = 9.78574$$
$$h_1^3 = e_1 + f_1 h_2^3 = 4 + 0(9.78574) = 4$$

3	4	9.78574	9.99464	9.99977	9.99987	9.99987
2	4	9.87075	9.99721	9.99994	9.99999	9.99999
1	4	10	10	10	10	10
	1	2	3	4	5	6

Figura 6: Matriz solución para $t = 20$ h