Ecuación de momentum

Introducción

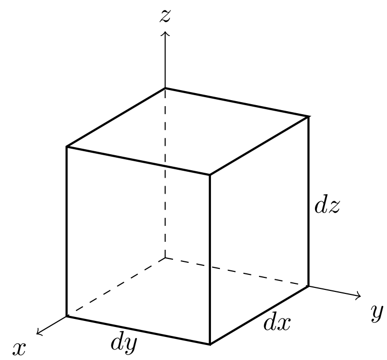

Figura 1: Elemento diferencial

Esta ecuación es equivalente al equilibrio de fuerzas

$$\vec{F} = m\,\vec{a} \tag{1}$$

Demostración

La ecuación (1) en la dirección x será

$$F_x = m\,a_x \tag{2}$$

La aceleración en la dirección x será

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

Reordenando y reemplazando términos conocidos

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \tag{3}$$

El lado derecho de la ecuación (2) será

$$\rho\,dx\,dy\,dz\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) \tag{4}$$

Equilibrio de presiones en la dirección x

$$-\,p(x+dx)\,dy\,dz + p(x)\,dy\,dz \tag{5}$$

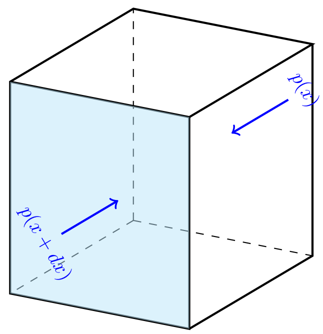

Figura 2: Presión en la dirección x

Equilibrio de fuerzas de compresión en la dirección xx

$$\sigma_{xx}(x + dx)\, dy\, dz - \sigma_{xx}(x)\, dy\, dz \tag{6}$$

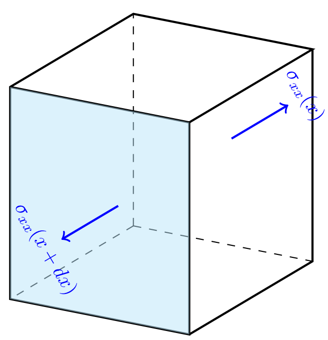

Figura 3: Esfuerzo de compresión en la dirección xx

Equilibrio de fuerzas de corte en la dirección yx

$$\tau_{yx}(y + dy)\, dx\, dz - \tau_{yx}(y)\, dx\, dz \tag{7}$$

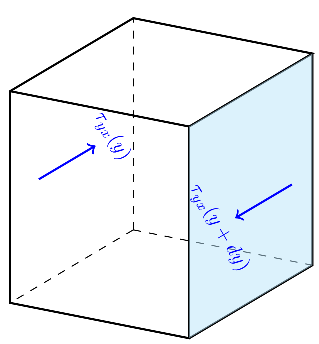

Figura 4: Esfuerzo de corte en la dirección yx

Equilibrio de fuerzas de corte en la dirección zx

$$\tau_{zx}(z + dz)\, dx\, dy - \tau_{zx}(z)\, dx\, dy \tag{8}$$

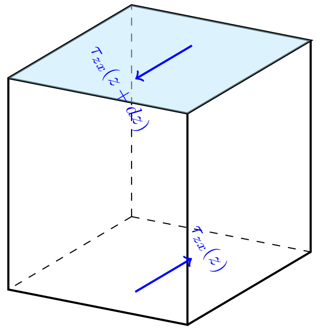

Figura 5: Esfuerzo de corte en la dirección zx

La fuerza de inercia en la dirección x

$$\rho\, dx\, dy\, dz\, g_x \tag{9}$$

Sumando las ecuaciones (5), (6), (7), (8) y (9), luego igualando a la ecuación (4)

$$-p(x+dx)\, dy\, dz + p(x)\, dy\, dz + \sigma_{xx}(x+dx)\, dy\, dz - \sigma_{xx}(x)\, dy\, dz + \tau_{yx}(y+dy)\, dx\, dz - \tau_{yx}(y)\, dx\, dz$$
$$+\, \tau_{zx}(z+dz)\, dx\, dy - \tau_{zx}(z)\, dx\, dy + \rho\, dx\, dy\, dz\, g_x = \rho\, dx\, dy\, dz\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

Dividiendo entre $dx\, dy\, dz$

$$-\frac{p(x+dx)}{dx} + \frac{p(x)}{dx} + \frac{\sigma_{xx}(x+dx)}{dx} - \frac{\sigma_{xx}(x)}{dx} + \frac{\tau_{yx}(y+dy)}{dy} - \frac{\tau_{yx}(y)}{dy} + \frac{\tau_{zx}(z+dz)}{dz} - \frac{\tau_{zx}(z)}{dz} + \rho\, g_x$$
$$= \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

Reordenando y agrupando

$$-\left[\frac{p(x+dx) - p(x)}{dx}\right] + \left[\frac{\sigma_{xx}(x+dx) - \sigma_{xx}(x)}{dx}\right] + \left[\frac{\tau_{yx}(y+dy) - \tau_{yx}(y)}{dy}\right] + \left[\frac{\tau_{zx}(z+dz) - \tau_{zx}(z)}{dz}\right] + \rho\, g_x$$
$$= \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

Usando la definición de derivada

$$-\frac{\partial p}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho\, g_x = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

Dividiendo entre la densidad

$$\frac{1}{\rho}\left(-\frac{\partial p}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + g_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

Siguiente el mismo procedimiento en las otras direcciones, se obtiene

$$\frac{1}{\rho}\left(-\frac{\partial p}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + g_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \tag{10}$$

$$\frac{1}{\rho}\left(-\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + g_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \tag{11}$$

$$\frac{1}{\rho}\left(-\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) + g_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \tag{12}$$

Referencias

[1] Bengt Andersson; et al. *Computational fluid dynamics for engineers*. Cambridge University Press, 2012.