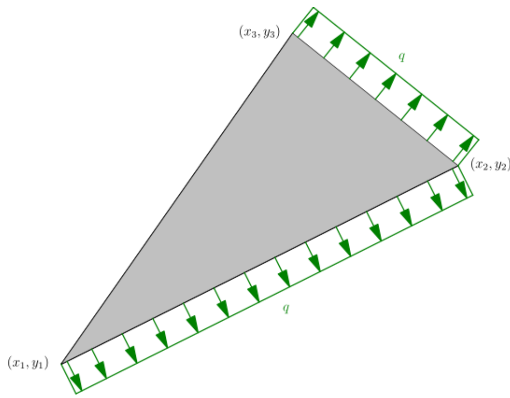
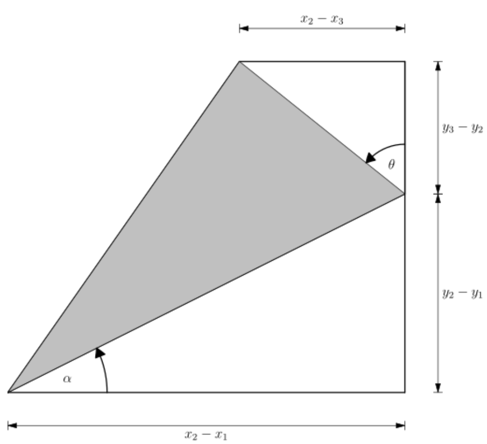
Introducción a elementos finitos
Primer Parcial II-2016

1. Calcular las fuerzas nodales de la placa de espesor t sometida a esfuerzo plano

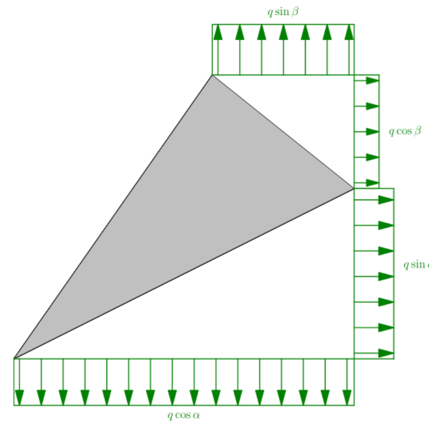


Solución



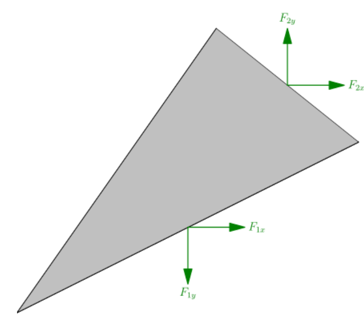
$$\sin\alpha = \frac{y_2 - y_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \quad \cos\alpha = \frac{x_2 - x_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

$$\sin\beta = \frac{x_2 - x_3}{\sqrt{(x_2 - x_3)^2 + (y_3 - y_2)^2}} \quad \cos\beta = \frac{y_3 - y_2}{\sqrt{(x_2 - x_3)^2 + (y_3 - y_2)^2}}$$

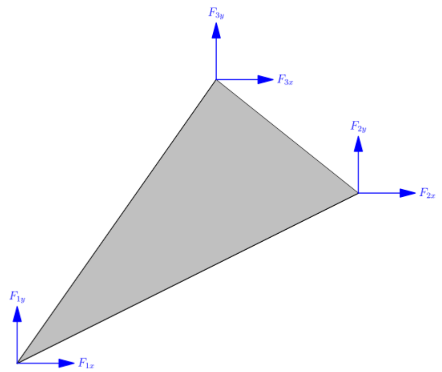


$$F_{1x} = q\sin\alpha\ t(y_2 - y_1) \quad F_{1y} = -q\cos\alpha\ t(x_2 - x_1)$$

$$F_{2x} = q\cos\beta\ t(y_3 - y_2) \quad F_{2y} = \quad q\sin\beta\ t(x_2 - x_3)$$



$$F_{1x} = \frac{F_{1x}}{2} \qquad F_{1y} = \frac{F_{1y}}{2}$$
$$F_{2x} = \frac{F_{1x} + F_{2x}}{2} \qquad F_{2y} = \frac{F_{1x} + F_{2x}}{2}$$
$$F_{3x} = \frac{F_{2x}}{2} \qquad F_{3y} = \frac{F_{2y}}{2}$$



2. Calcular la integral mediante la cuadratura de Newton-Cotes para $n = 3$, los pesos w_i y los puntos de muestreo r_i

$$I = \int_1^5 x^3 e^x \, dx$$

Solución

$$k = n - 1 = 3 - 1 = 2$$

Calculando r_i

$$\int_{-1}^{+1} P(r)\ r^0\ dr = 0$$
$$\int_{-1}^{+1} P(r)\ r^1\ dr = 0$$
$$\int_{-1}^{+1} P(r)\ r^2\ dr = 0$$

El polinomio es

$$P(r) = (r - r_1)(r - r_2)(r - r_3)$$

Reemplazando

$$\int_{-1}^{+1} (r - r_1)(r - r_2)(r - r_3)\ dr = 0$$
$$\int_{-1}^{+1} (r - r_1)(r - r_2)(r - r_3) r\ dr = 0$$
$$\int_{-1}^{+1} (r - r_1)(r - r_2)(r - r_3) r^2\ dr = 0$$

Integrando

$$\left[\frac{1}{4}r^4 - \frac{r_1 + r_2 + r_3}{3}r^3 + \frac{r_1r_2 + r_1r_3 + r_2r_3}{2}r^2 - r_1r_2r_3r\right]\Bigg|_{-1}^{+1}$$
$$= -\frac{2}{3}(3r_1r_2r_3 + r_1 + r_2 + r_3)$$

$$\left[\frac{1}{5}r^5 - \frac{r_1 + r_2 + r_3}{4}r^4 + \frac{r_1r_2 + r_1r_3 + r_2r_3}{3}r^3 - \frac{r_1r_2r_3}{2}r^2\right]\Bigg|_{-1}^{+1}$$
$$= \frac{2}{15}(5r_1r_2 + 5r_1r_3 + 5r_2r_3 + 3)$$

$$\left[\frac{1}{6}r^6 - \frac{r_1 + r_2 + r_3}{5}r^5 + \frac{r_1r_2 + r_1r_3 + r_2r_3}{4}r^4 - \frac{r_1r_2r_3}{3}r^3\right]\Bigg|_{-1}^{+1}$$
$$= -\frac{2}{15}(5r_1r_2r_3 + 3r_1 + 3r_2 + 3r_3)$$

Formando el sistema de ecuaciones

$$3r_1r_2r_3 + r_1 + r_2 + r_3 = 0$$
$$5r_1r_2 + 5r_1r_3 + 5r_2r_3 = -3$$
$$5r_1r_2r_3 + 3r_1 + 3r_2 + 3r_3 = 0$$

Resolviendo

$$
\begin{aligned}
r_1 &= -\sqrt{\frac{3}{5}} \\
r_2 &= 0 \\
r_3 &= \sqrt{\frac{3}{5}}
\end{aligned}
$$

Calculando w_i

$$
\begin{aligned}
w_1 &= \int_{-1}^{+1} \frac{r - r_2}{r_1 - r_2} \cdot \frac{r - r_3}{r_1 - r_3} \, dr \\
w_2 &= \int_{-1}^{+1} \frac{r - r_1}{r_2 - r_1} \cdot \frac{r - r_3}{r_2 - r_3} \, dr \\
w_3 &= \int_{-1}^{+1} \frac{r - r_2}{r_3 - r_2} \cdot \frac{r - r_1}{r_3 - r_1} \, dr
\end{aligned}
$$

Reemplazando e integrando

$$
\begin{aligned}
w_1 &= \int_{-1}^{+1} \frac{r - 0}{-\sqrt{\frac{3}{5}} - 0} \cdot \frac{r - \sqrt{\frac{3}{5}}}{-\sqrt{\frac{3}{5}} - \sqrt{\frac{3}{5}}} \, dr = \left(\frac{5}{18} r^3 - \frac{\sqrt{15}}{12} r^2 \right) \Bigg|_{-1}^{+1} = \frac{5}{9} \\
w_2 &= \int_{-1}^{+1} \frac{r + \sqrt{\frac{3}{5}}}{0 + \sqrt{\frac{3}{5}}} \cdot \frac{r - \sqrt{\frac{3}{5}}}{0 - \sqrt{\frac{3}{5}}} \, dr = \left(-\frac{5}{9} r^3 + r \right) \Bigg|_{-1}^{+1} = \frac{8}{9} \\
w_3 &= \int_{-1}^{+1} \frac{r - 0}{\sqrt{\frac{3}{5}} - 0} \cdot \frac{r + \sqrt{\frac{3}{5}}}{\sqrt{\frac{3}{5}} + \sqrt{\frac{3}{5}}} \, dr = \left(\frac{5}{18} r^3 + \frac{\sqrt{15}}{12} r^2 \right) \Bigg|_{-1}^{+1} = \frac{5}{9}
\end{aligned}
$$

Usando la fórmula

$$
I = w_1' f(r_1') + w_2' f(r_2') + w_3' f(r_3')
$$

Puntos de muestreo

$$
\begin{aligned}
r_1' &= \frac{b + a}{2} + \frac{b - a}{2} r_1 = \frac{5 + 1}{2} + \frac{5 - 1}{2} \left(-\sqrt{\frac{3}{5}} \right) = 1.45081 \\
r_2' &= \frac{b + a}{2} + \frac{b - a}{2} r_2 = \frac{5 + 1}{2} + \frac{5 - 1}{2} (0) = 3 \\
r_3' &= \frac{b + a}{2} + \frac{b - a}{2} r_3 = \frac{5 + 1}{2} + \frac{5 - 1}{2} \left(\sqrt{\frac{3}{5}} \right) = 4.54919
\end{aligned}
$$

Pesos

$$
\begin{aligned}
w_1' &= \frac{b - a}{2} w_1 = \frac{5 - 1}{2} \left(\frac{5}{9} \right) = 1.11111 \\
w_2' &= \frac{b - a}{2} w_2 = \frac{5 - 1}{2} \left(\frac{8}{9} \right) = 1.77778 \\
w_3' &= \frac{b - a}{2} w_2 = \frac{5 - 1}{2} \left(\frac{5}{9} \right) = 1.11111
\end{aligned}
$$

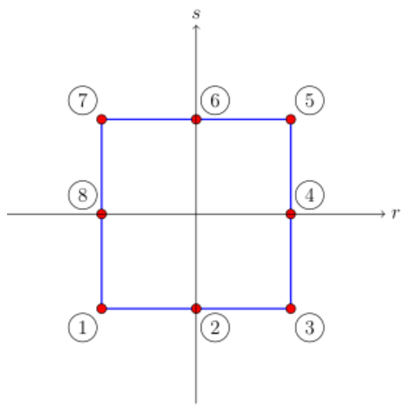
Reemplazando

$$
\begin{aligned}
I &= 1.11111 \left(1.45081^3 e^{1.45081} \right) + 1.77778 \left(3^3 e^3 \right) + 1.11111 \left(4.54919^3 e^{4.54919} \right) \\
&= 11819.38
\end{aligned}
$$

3. Calcular las funciones de forma $\boldsymbol{N}$

   a) Elemento bidimensional

   b) Elemento unidimensional formado por los nodos 3–4–5



Solución

a) Elemento bidimensional

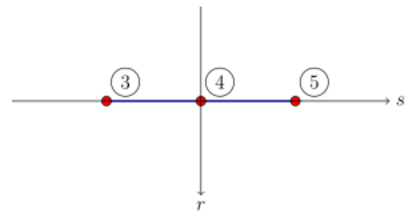
Coordenadas de los nodos

$$
\begin{array}{ll}
① = [r_1, s_1] = [-1, -1] & ⑤ = [r_5, s_5] = [1, 1] \\
② = [r_2, s_2] = [0, -1] & ⑥ = [r_6, s_6] = [0, 1] \\
③ = [r_3, s_3] = [1, -1] & ⑦ = [r_7, s_7] = [-1, -1] \\
④ = [r_4, s_4] = [1, 0] & ⑧ = [r_8, s_8] = [-1, 0]
\end{array}
$$

Reemplazando valores

$$
\begin{aligned}
N_1 &= \frac{r - r_2}{r_1 - r_2} \cdot \frac{r - r_3}{r_1 - r_3} \cdot \frac{s - s_8}{s_1 - s_8} \cdot \frac{s - s_7}{s_1 - s_7} \\
&= \frac{r - 0}{-1 - 0} \cdot \frac{r - 1}{-1 - 1} \cdot \frac{s - 0}{-1 - 0} \cdot \frac{s - 1}{-1 - 1} = \frac{1}{4} r(r - 1)s(s - 1) \\
N_2 &= \frac{r - r_3}{r_2 - r_3} \cdot \frac{r - r_1}{r_2 - r_1} \cdot \frac{s - s_6}{s_2 - s_6} \\
&= \frac{r - 1}{0 - 1} \cdot \frac{r - (-1)}{0 - (-1)} \cdot \frac{s - 1}{-1 - 1} = \frac{1}{2}(r - 1)(r + 1)(s - 1) \\
N_3 &= \frac{r - r_2}{r_3 - r_2} \cdot \frac{r - r_1}{r_3 - r_1} \cdot \frac{s - s_4}{s_3 - s_4} \cdot \frac{s - s_5}{s_3 - s_5} \\
&= \frac{r - 0}{1 - 0} \cdot \frac{r - (-1)}{1 - (-1)} \cdot \frac{s - 0}{-1 - 0} \cdot \frac{s - 1}{-1 - 1} = \frac{1}{4} r(r + 1)s(s - 1) \\
N_4 &= \frac{r - r_8}{r_4 - r_8} \cdot \frac{s - s_3}{s_4 - s_3} \cdot \frac{s - s_5}{s_4 - s_5} \\
&= \frac{r - (-1)}{1 - (-1)} \cdot \frac{s - (-1)}{0 - (-1)} \cdot \frac{s - 1}{0 - 1} = -\frac{1}{2}(r + 1)(s + 1)(s - 1) \\
N_5 &= \frac{r - r_4}{r_5 - r_4} \cdot \frac{r - r_3}{r_5 - r_3} \cdot \frac{s - s_6}{s_5 - s_6} \cdot \frac{s - s_7}{s_5 - s_7} \\
&= \frac{r - 0}{1 - 0} \cdot \frac{r - (-1)}{1 - (-1)} \cdot \frac{s - 0}{1 - 0} \cdot \frac{s - (-1)}{1 - (-1)} = \frac{1}{4} r(r + 1)s(s + 1) \\
N_6 &= \frac{r - r_5}{r_6 - r_5} \cdot \frac{r - r_7}{r_6 - r_7} \cdot \frac{s - s_2}{s_6 - s_2} \\
&= \frac{r - 1}{0 - 1} \cdot \frac{r - (-1)}{0 - (-1)} \cdot \frac{s - (-1)}{1 - (-1)} = -\frac{1}{2}(r - 1)(r + 1)(s + 1) \\
N_7 &= \frac{r - r_6}{r_7 - r_6} \cdot \frac{r - r_5}{r_7 - r_5} \cdot \frac{s - s_8}{s_7 - s_8} \cdot \frac{s - s_1}{s_7 - s_1} \\
&= \frac{r - 0}{-1 - 0} \cdot \frac{r - 1}{-1 - 1} \cdot \frac{s - 0}{1 - 0} \cdot \frac{s - (-1)}{1 - (-1)} = \frac{1}{4} r(r - 1)s(s + 1) \\
N_8 &= \frac{r - r_4}{r_8 - r_4} \cdot \frac{s - s_1}{s_8 - s_1} \cdot \frac{s - s_7}{s_8 - s_7} \\
&= \frac{r - 1}{-1 - 1} \cdot \frac{s - (-1)}{0 - (-1)} \cdot \frac{s - 1}{0 - 1} = \frac{1}{2}(r - 1)(s + 1)(s - 1)
\end{aligned}
$$

b) Elemento unidimensional



$$
\begin{array}{ll}
③ = r_3 = -1 & ⑤ = r_5 = 1 \\
④ = r_4 = 0 &
\end{array}
$$

Reemplazando valores

$$
\begin{aligned}
N_1 &= \frac{s - s_4}{s_3 - s_4} \cdot \frac{s - s_5}{s_3 - s_5} = \frac{s - 0}{-1 - 0} \cdot \frac{s - 1}{-1 - 1} = \frac{1}{2} s(s - 1) \\
N_2 &= \frac{s - s_3}{s_4 - s_3} \cdot \frac{s - s_5}{s_4 - s_5} = \frac{s - (-1)}{0 - (-1)} \cdot \frac{s - 1}{0 - 1} = -(s^2 - 1) \\
N_3 &= \frac{s - s_4}{s_5 - s_4} \cdot \frac{s - s_3}{s_5 - s_3} = \frac{s - 0}{1 - 0} \cdot \frac{s - (-1)}{1 - (-1)} = \frac{1}{2} s(s + 1)
\end{aligned}
$$