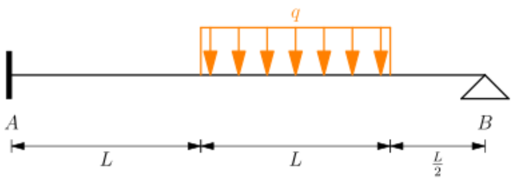
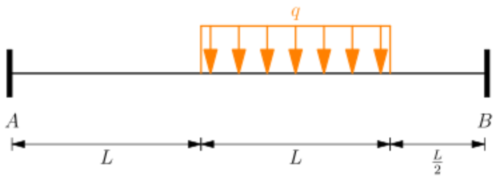
Introducción a elementos finitos
Tarea 2 I-2016

Resolver por el método de Castigliano



La estructura se dividirá en dos estructuras, las reacciones se obtendrán mediante superposición.
La primera estructura es



El momento de $0 \leqslant x \leqslant L$ es

$$M = -M_A + V_A x$$

El momento de $L \leqslant x \leqslant 2L$ es

$$\begin{aligned} M &= -M_A + V_A x - \frac{q}{2}(x-L)^2 \\ &= -\left(M_A + \frac{qL^2}{2}\right) + (V_A + qL)\,x - \frac{q}{2}x^2 \end{aligned}$$

El momento de $2L \leqslant x \leqslant \frac{5L}{2}$ es

$$\begin{aligned} M &= -M_A + V_A x - qL\left(x - \frac{3L}{2}\right) \\ &= -\left(M_A - \frac{3qL^2}{2}\right) + (V_A - qL)\,x \end{aligned}$$

El desplazamiento es cero en el punto A

$$\frac{\partial U_i}{\partial V_A} = 0$$
$$\frac{\partial U_i}{\partial M_A} = 0$$

Derivando U_i respecto de V_A

$$\begin{aligned} \frac{\partial U_i}{\partial V_A} &= \frac{1}{EI}\int_0^{\frac{5L}{2}} M\frac{\partial M}{\partial V_A}\;dx \\ &= \frac{1}{EI}\int_0^{L} (-M_A + V_A x)\,x\;dx \\ &+ \frac{1}{EI}\int_L^{2L}\left[-\left(M_A + \frac{qL^2}{2}\right) + (V_A+qL)\,x - \frac{q}{2}x^2\right] x\;dx \\ &+ \frac{1}{EI}\int_{2L}^{\frac{5L}{2}}\left[-\left(M_A - \frac{3qL^2}{2}\right) + (V_A - qL)\,x\right] x\;dx \end{aligned}$$

Derivando U_i respecto de M_A

$$\begin{aligned} \frac{\partial U_i}{\partial M_A} &= \frac{1}{EI}\int_0^{\frac{5L}{2}} M\frac{\partial M}{\partial M_A}\;dx \\ &= \frac{1}{EI}\int_0^{L} (-M_A + V_A x)\,(-1)\;dx \\ &+ \frac{1}{EI}\int_L^{2L}\left[-\left(M_A + \frac{qL^2}{2}\right) + (V_A+qL)\,x - \frac{q}{2}x^2\right](-1)\;dx \\ &+ \frac{1}{EI}\int_{2L}^{\frac{5L}{2}}\left[-\left(M_A - \frac{3qL^2}{2}\right) + (V_A - qL)\,x\right](-1)\;dx \end{aligned}$$

Multiplicando

$$\begin{aligned} &\frac{1}{EI}\int_0^{L} -M_A x + V_A x^2\;dx + \frac{1}{EI}\int_L^{2L} -\left(M_A + \frac{qL^2}{2}\right)x + (V_A+qL)\,x^2 \\ &-\frac{q}{2}x^3\;dx + \frac{1}{EI}\int_{2L}^{\frac{5L}{2}} -\left(M_A - \frac{3qL^2}{2}\right)x + (V_A - qL)\,x^2\;dx = 0 \\ &\frac{1}{EI}\int_0^{L} M_A - V_A x\;dx + \frac{1}{EI}\int_L^{2L}\left(M_A + \frac{qL^2}{2}\right) - (V_A+qL)\,x \\ &+\frac{q}{2}x^2\;dx + \frac{1}{EI}\int_{2L}^{\frac{5L}{2}}\left(M_A - \frac{3qL^2}{2}\right) - (V_A - qL)\,x\;dx = 0 \end{aligned}$$

Integrando

$$
\begin{aligned}
&\frac{1}{EI}\left[-\frac{M_A}{2}x^2+\frac{V_A}{3}x^3\right]\Big|_0^L \\
&+\frac{1}{EI}\left[-\frac{1}{2}\left(M_A+\frac{qL^2}{2}\right)x^2+\frac{1}{3}\left(V_A+qL\right)x^3-\frac{q}{8}x^4\right]\Big|_L^{2L} \\
&+\frac{1}{EI}\left[-\frac{1}{2}\left(M_A-\frac{3qL^2}{2}\right)x^2+\frac{1}{3}\left(V_A-qL\right)x^3\right]\Big|_{2L}^{\frac{5L}{2}}=0 \\
&\frac{1}{EI}\left(M_Ax-\frac{V_A}{2}x^2\right)\Big|_0^L \\
&+\frac{1}{EI}\left[\left(M_A+\frac{qL^2}{2}\right)x-\frac{1}{2}\left(V_A+qL\right)x^2+\frac{q}{6}x^3\right]\Big|_L^{2L} \\
&+\frac{1}{EI}\left[\left(M_A-\frac{3qL^2}{2}\right)x-\frac{1}{2}\left(V_A-qL\right)x^2\right]\Big|_{2L}^{\frac{5L}{2}}=0
\end{aligned}
$$

Simplificando

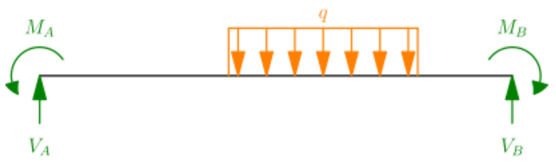
$$
\begin{aligned}
\frac{125L^3}{24}V_A-\frac{25L^2}{8}M_A&=\frac{55qL^4}{48} \\
-\frac{25L^2}{8}V_A+\frac{5L}{2}M_A&=-\frac{13qL^3}{24}
\end{aligned}
$$

Resolviendo

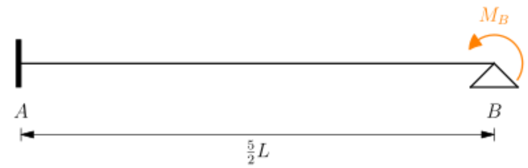
$$
\begin{aligned}
V_A&=\frac{9qL}{25} \\
M_A&=\frac{7qL^2}{30}
\end{aligned}
$$

Por equilibrio las reacciones son

$$
\begin{aligned}
V_A&=\frac{9qL}{25} & V_B&=\frac{16qL}{25} \\
M_A&=\frac{7qL^2}{30} & M_B&=\frac{qL^2}{3}
\end{aligned}
$$



La segunda estructura es



El momento de $0 \leqslant x \leqslant \frac{5L}{2}$ es

$$M=-\frac{qL^2}{3}+V_Bx$$

El desplazamiento vertical es cero en el punto B

$$\frac{\partial U_i}{\partial V_B}=0$$

Derivando U_i respecto de V_A

$$
\begin{aligned}
\frac{\partial U_i}{\partial V_B}&=\frac{1}{EI}\int_0^{\frac{5L}{2}}M\frac{\partial M}{\partial V_B}\ dx \\
&=\frac{1}{EI}\int_0^{\frac{5L}{2}}\left(-\frac{qL^2}{3}+V_Bx\right)x\ dx
\end{aligned}
$$

Multiplicando

$$\frac{1}{EI}\int_0^{\frac{5L}{2}}-\frac{qL^2}{3}x+V_Bx^2\ dx=0$$

Integrando

$$\frac{1}{EI}\left(-\frac{qL^2}{6}x^2+\frac{V_B}{3}x^3\right)\Big|_0^{\frac{5L}{2}}=0$$

Simplificando

$$\frac{1}{EI}\left(-\frac{25qL^4}{24}+\frac{125L^3}{24}V_B\right)=0$$

Resolviendo

$$V_B=\frac{qL}{5}$$

Por equilibrio las reacciones son

$$\begin{aligned} V_A &= \frac{qL}{5} & V_B &= \frac{qL}{5} \\ M_A &= \frac{qL^2}{6} & M_B &= \frac{qL^2}{3} \end{aligned}$$

Por superposición las reacciones son

$$\begin{aligned} V_A &= \frac{9qL}{25}+\frac{qL}{5}=\frac{14qL}{25} \\ M_A &= \frac{7qL^2}{30}+\frac{qL^2}{6}=\frac{2qL^2}{5} \\ V_B &= \frac{16qL}{25}-\frac{qL}{5}=\frac{11qL}{25} \\ M_B &= -\frac{qL^2}{3}+\frac{qL^2}{3}=0 \end{aligned}$$