Ecuación de Navier-Stokes en promedios de Reynolds

Introducción

Estas ecuaciones se obtienen a partir de

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+g_x=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} \tag{1}$$

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+g_y=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} \tag{2}$$

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial z}\right)+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+g_z=\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} \tag{3}$$

Demostración

En la dirección x

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+g_x=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}$$

Aproximando las variables mediante la descomposición de Reynolds

$$u=\overline{u}+u' \tag{4}$$

$$v=\overline{v}+v' \tag{5}$$

$$w=\overline{w}+w' \tag{6}$$

$$p=\overline{p}+p' \tag{7}$$

Reordenando la ecuación (1)

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+g_x=\frac{\partial u}{\partial t}+\frac{\partial(u\,u)}{\partial x}+\frac{\partial(v\,u)}{\partial y}+\frac{\partial(w\,u)}{\partial z}$$

Reemplazando las ecuaciones (4), (5), (6) y (7) en la ecuación (1)

$$\begin{aligned}&-\frac{1}{\rho}\left[\frac{\partial(\overline{p}+p')}{\partial x}\right]+\nu\left[\frac{\partial^2(\overline{u}+u')}{\partial x^2}+\frac{\partial^2(\overline{u}+u')}{\partial y^2}+\frac{\partial^2(\overline{u}+u')}{\partial z^2}\right]+g_x\\&=\frac{\partial(\overline{u}+u')}{\partial t}+\frac{\partial[(\overline{u}+u')(\overline{u}+u')]}{\partial x}+\frac{\partial[(\overline{v}+v')(\overline{u}+u')]}{\partial y}+\frac{\partial[(\overline{w}+w')(\overline{u}+u')]}{\partial z}\end{aligned}$$

Promediando ambos lados de la ecuación

$$\begin{aligned}&\overline{-\frac{1}{\rho}\left[\frac{\partial(\overline{p}+p')}{\partial x}\right]+\nu\left[\frac{\partial^2(\overline{u}+u')}{\partial x^2}+\frac{\partial^2(\overline{u}+u')}{\partial y^2}+\frac{\partial^2(\overline{u}+u')}{\partial z^2}\right]+g_x}\\&=\overline{\frac{\partial(\overline{u}+u')}{\partial t}+\frac{\partial[(\overline{u}+u')(\overline{u}+u')]}{\partial x}+\frac{\partial[(\overline{v}+v')(\overline{u}+u')]}{\partial y}+\frac{\partial[(\overline{w}+w')(\overline{u}+u')]}{\partial z}}\end{aligned}$$

Simplificando (usando álgebra de operadores de Reynolds)

$$-\frac{1}{\rho}\left(\frac{\partial\overline{p}}{\partial x}\right)+\nu\left(\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2\overline{u}}{\partial y^2}+\frac{\partial^2\overline{u}}{\partial z^2}\right)+g_x=\frac{\partial\overline{u}}{\partial t}+\frac{\partial(\overline{u}^2+u'^2)}{\partial x}+\frac{\partial(\overline{u}\,\overline{v}+u'v')}{\partial y}+\frac{\partial(\overline{u}\,\overline{w}+u'w')}{\partial z}$$

Expandiendo términos

$$-\frac{1}{\rho}\left(\frac{\partial\overline{p}}{\partial x}\right)+\nu\left(\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2\overline{u}}{\partial y^2}+\frac{\partial^2\overline{u}}{\partial z^2}\right)+g_x=\frac{\partial\overline{u}}{\partial t}+\frac{\partial\overline{u}^2}{\partial x}+\frac{\partial u'^2}{\partial x}+\frac{\partial(\overline{u}\,\overline{v})}{\partial y}+\frac{\partial(u'v')}{\partial y}+\frac{\partial(\overline{u}\,\overline{w})}{\partial z}+\frac{\partial(u'w')}{\partial z}$$

Reordenando

$$-\frac{1}{\rho}\left(\frac{\partial\overline{p}}{\partial x}\right)+\nu\left(\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2\overline{u}}{\partial y^2}+\frac{\partial^2\overline{u}}{\partial z^2}\right)+g_x=\frac{\partial\overline{u}}{\partial t}+\frac{\partial\overline{u}^2}{\partial x}+\frac{\partial(\overline{u}\,\overline{v})}{\partial y}+\frac{\partial(\overline{u}\,\overline{w})}{\partial z}+\frac{\partial u'^2}{\partial x}+\frac{\partial(u'v')}{\partial y}+\frac{\partial(u'w')}{\partial z}$$

Reordenando de nuevo

$$-\frac{1}{\rho}\left(\frac{\partial\overline{p}}{\partial x}\right)+\nu\left(\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2\overline{u}}{\partial y^2}+\frac{\partial^2\overline{u}}{\partial z^2}\right)+g_x=\frac{\partial\overline{u}}{\partial t}+\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{u}\frac{\partial\overline{v}}{\partial y}+\overline{u}\frac{\partial\overline{w}}{\partial z}+\frac{\partial u'^2}{\partial x}+\frac{\partial(u'v')}{\partial y}+\frac{\partial(u'w')}{\partial z}$$

Llevando al lado izquierdo los términos fluctuantes

$$-\frac{1}{\rho}\left(\frac{\partial\overline{p}}{\partial x}\right)+\nu\left(\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2\overline{u}}{\partial y^2}+\frac{\partial^2\overline{u}}{\partial z^2}\right)+\left(-\frac{\partial u'^2}{\partial x}-\frac{\partial(u'v')}{\partial y}-\frac{\partial(u'w')}{\partial z}\right)+g_x=\frac{\partial\overline{u}}{\partial t}+\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{u}\frac{\partial\overline{v}}{\partial y}+\overline{u}\frac{\partial\overline{w}}{\partial z}$$

Multiplicando y dividiendo por la densidad el segundo y tercer término del lado izquierdo

$$-\frac{1}{\rho}\left(\frac{\partial\overline{p}}{\partial x}\right)+\nu\left(\frac{\rho}{\rho}\right)\left(\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2\overline{u}}{\partial y^2}+\frac{\partial^2\overline{u}}{\partial z^2}\right)+\left(\frac{\rho}{\rho}\right)\left(-\frac{\partial u'^2}{\partial x}-\frac{\partial(u'v')}{\partial y}-\frac{\partial(u'w')}{\partial z}\right)+g_x$$

Reordenando

$$-\frac{1}{\rho}\left(\frac{\partial\overline{p}}{\partial x}\right)+\frac{1}{\rho}\left[\frac{\partial^2(\mu\,\overline{u})}{\partial x^2}+\frac{\partial^2(\mu\,\overline{u})}{\partial y^2}+\frac{\partial^2(\mu\,\overline{u})}{\partial z^2}\right]+\frac{1}{\rho}\left[-\frac{\partial(\rho\,u'^2)}{\partial x}-\frac{\partial(\rho\,u'v')}{\partial y}-\frac{\partial(\rho\,u'w')}{\partial z}\right]+g_x$$

Factorizando las derivadas parciales del segundo término

$$-\frac{1}{\rho}\left(\frac{\partial\overline{p}}{\partial x}\right)+\frac{1}{\rho}\left\{\frac{\partial}{\partial x}\left[\frac{\partial(\mu\,\overline{u})}{\partial x}\right]+\frac{\partial}{\partial y}\left[\frac{\partial(\mu\,\overline{u})}{\partial y}\right]+\frac{\partial}{\partial z}\left[\frac{\partial(\mu\,\overline{u})}{\partial z}\right]\right\}+\frac{1}{\rho}\left[-\frac{\partial(\rho\,u'^2)}{\partial x}-\frac{\partial(\rho\,u'v')}{\partial y}-\frac{\partial(\rho\,u'w')}{\partial z}\right]+g_x$$

Reordenando

$$-\frac{1}{\rho}\left(\frac{\partial\overline{p}}{\partial x}\right)+\frac{1}{\rho}\left\{\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial\overline{u}}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial\overline{u}}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial\overline{u}}{\partial z}\right)\right]\right\}+\frac{1}{\rho}\left[-\frac{\partial(\rho\,u'^2)}{\partial x}-\frac{\partial(\rho\,u'v')}{\partial y}-\frac{\partial(\rho\,u'w')}{\partial z}\right]+g_x$$

Sumando y restando términos que aparecen en los esfuerzos

$$-\frac{1}{\rho}\left(\frac{\partial\overline{p}}{\partial x}\right)+\frac{1}{\rho}\left\{\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial\overline{u}}{\partial x}+\frac{\partial\overline{u}}{\partial x}-\frac{\partial\overline{u}}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial\overline{u}}{\partial y}+\frac{\partial\overline{v}}{\partial x}-\frac{\partial\overline{v}}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial\overline{u}}{\partial z}+\frac{\partial\overline{w}}{\partial x}-\frac{\partial\overline{w}}{\partial x}\right)\right]\right\}$$
$$+\frac{1}{\rho}\left[-\frac{\partial(\rho\,u'^2)}{\partial x}-\frac{\partial(\rho\,u'v')}{\partial y}-\frac{\partial(\rho\,u'w')}{\partial z}\right]+g_x$$

Reagrupando

$$\frac{1}{\rho}\left\{-\frac{\partial\overline{p}}{\partial x}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial\overline{u}}{\partial x}+\frac{\partial\overline{u}}{\partial x}\right)-\mu\frac{\partial\overline{u}}{\partial x}\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial\overline{u}}{\partial y}+\frac{\partial\overline{v}}{\partial x}\right)-\mu\frac{\partial\overline{v}}{\partial x}\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial\overline{u}}{\partial z}+\frac{\partial\overline{w}}{\partial x}\right)-\mu\frac{\partial\overline{w}}{\partial x}\right]\right\}$$
$$+\frac{1}{\rho}\left[-\frac{\partial(\rho\,u'^2)}{\partial x}-\frac{\partial(\rho\,u'v')}{\partial y}-\frac{\partial(\rho\,u'w')}{\partial z}\right]+g_x$$

Reagrupando nuevamente

$$\frac{1}{\rho}\left\{-\frac{\partial\overline{p}}{\partial x}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial\overline{u}}{\partial x}+\frac{\partial\overline{u}}{\partial x}\right)-\rho\,u'^2-\mu\frac{\partial\overline{u}}{\partial x}\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial\overline{u}}{\partial y}+\frac{\partial\overline{v}}{\partial x}\right)-\rho\,u'v'-\mu\frac{\partial\overline{v}}{\partial x}\right]\right.$$
$$\left.+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial\overline{u}}{\partial z}+\frac{\partial\overline{w}}{\partial x}\right)-\rho\,u'w'-\mu\frac{\partial\overline{w}}{\partial x}\right]\right\}+g_x$$

Expandiendo términos y reagrupando

$$\frac{1}{\rho}\left\{-\frac{\partial\overline{p}}{\partial x}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial\overline{u}}{\partial x}+\frac{\partial\overline{u}}{\partial x}\right)-\rho\,u'^2\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial\overline{u}}{\partial y}+\frac{\partial\overline{v}}{\partial x}\right)-\rho\,u'v'\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial\overline{u}}{\partial z}+\frac{\partial\overline{w}}{\partial x}\right)-\rho\,u'w'\right]\right\}$$
$$-\nu\left(\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2\overline{v}}{\partial y\partial x}+\frac{\partial^2\overline{w}}{\partial z\partial x}\right)+g_x$$

Intercambiando el orden de las derivadas parciales y factorizando la derivada parcial respecto de x

$$\frac{1}{\rho}\left\{-\frac{\partial\overline{p}}{\partial x}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial\overline{u}}{\partial x}+\frac{\partial\overline{u}}{\partial x}\right)-\rho\,u'^2\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial\overline{u}}{\partial y}+\frac{\partial\overline{v}}{\partial x}\right)-\rho\,u'v'\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial\overline{u}}{\partial z}+\frac{\partial\overline{w}}{\partial x}\right)-\rho\,u'w'\right]\right\}$$
$$-\,\nu\,\frac{\partial}{\partial x}\left(\frac{\partial\overline{u}}{\partial x}+\frac{\partial\overline{v}}{\partial y}+\frac{\partial\overline{w}}{\partial z}\right)+g_x$$

Reemplazando la ecuación de continuidad

$$\frac{1}{\rho}\left\{-\frac{\partial\overline{p}}{\partial x}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial\overline{u}}{\partial x}+\frac{\partial\overline{u}}{\partial x}\right)-\rho\,u'^2\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial\overline{u}}{\partial y}+\frac{\partial\overline{v}}{\partial x}\right)-\rho\,u'v'\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial\overline{u}}{\partial z}+\frac{\partial\overline{w}}{\partial x}\right)-\rho\,u'w'\right]\right\}+g_x \tag{8}$$

Los esfuerzos serán

$$\sigma_{xx}=\overline{\sigma}_{xx}+\sigma'_{xx}=\mu\left(\frac{\partial\overline{u}}{\partial x}+\frac{\partial\overline{u}}{\partial x}\right)-\rho\,u'^2 \tag{9}$$
$$\tau_{yx}=\overline{\tau}_{yx}+\tau'_{yx}=\mu\left(\frac{\partial\overline{v}}{\partial x}+\frac{\partial\overline{u}}{\partial y}\right)-\rho\,u'v' \tag{10}$$
$$\tau_{zx}=\overline{\tau}_{zx}+\tau'_{zx}=\mu\left(\frac{\partial\overline{w}}{\partial x}+\frac{\partial\overline{u}}{\partial z}\right)-\rho\,u'w' \tag{11}$$

Reemplazando las ecuaciones (9), (10) y (11) en la ecuación (8)

$$\frac{1}{\rho}\left(-\frac{\partial\overline{p}}{\partial x}+\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)+g_x$$

Siguiente el mismo procedimiento en las otras direcciones, se obtiene

$$\frac{1}{\rho}\left(-\frac{\partial\overline{p}}{\partial x}+\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)+g_x=\frac{\partial\overline{u}}{\partial t}+\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y}+\overline{w}\frac{\partial\overline{u}}{\partial z} \tag{12}$$
$$\frac{1}{\rho}\left(-\frac{\partial\overline{p}}{\partial y}+\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}\right)+g_y=\frac{\partial\overline{v}}{\partial t}+\overline{u}\frac{\partial\overline{v}}{\partial x}+\overline{v}\frac{\partial\overline{v}}{\partial y}+\overline{w}\frac{\partial\overline{v}}{\partial z} \tag{13}$$
$$\frac{1}{\rho}\left(-\frac{\partial\overline{p}}{\partial z}+\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}\right)+g_z=\frac{\partial\overline{w}}{\partial t}+\overline{u}\frac{\partial\overline{w}}{\partial x}+\overline{v}\frac{\partial\overline{w}}{\partial y}+\overline{w}\frac{\partial\overline{w}}{\partial z} \tag{14}$$

Referencias

[1] Bengt Andersson; et al. *Computational fluid dynamics for engineers.* Cambridge University Press, 2012.