FMI, Info, Anul I
Logică matematică şi computaţională

Seminar 7

(S7.1) Să se arate că pentru orice formulă φ,

$$\vdash (\neg\varphi \to \varphi) \to \varphi.$$

Demonstraţie: Avem

(1)	$\{\neg\varphi \to \varphi, \neg\varphi\}$	$\vdash \neg\varphi$	Propoziţia 2.38.(ii)
(2)	$\{\neg\varphi \to \varphi, \neg\varphi\}$	$\vdash \neg\varphi \to \varphi$	Propoziţia 2.38.(ii)
(3)	$\{\neg\varphi \to \varphi, \neg\varphi\}$	$\vdash \varphi$	(MP): (1), (2)
(4)	$\{\neg\varphi \to \varphi, \neg\varphi\}$	$\vdash \neg\varphi \to (\varphi \to \neg(\varphi \to \varphi))$	(S6.2).(ii) şi Prop. 2.39.(ii)
(5)	$\{\neg\varphi \to \varphi, \neg\varphi\}$	$\vdash \varphi \to \neg(\varphi \to \varphi)$	(MP): (1), (4)
(6)	$\{\neg\varphi \to \varphi, \neg\varphi\}$	$\vdash \neg(\varphi \to \varphi)$	(MP): (3), (5)
(7)	$\{\neg\varphi \to \varphi\}$	$\vdash \varphi$	(6) şi (S6.1)
(8)		$\vdash (\neg\varphi \to \varphi) \to \varphi$	Teorema deducţiei.

□

(S7.2) Să se arate că pentru orice formule φ, ψ, χ avem:

(i) $\{\varphi \wedge \psi\} \vdash \varphi$;

(ii) $\{\varphi \wedge \psi\} \vdash \psi$;

(iii) $\{\varphi, \psi\} \vdash \varphi \wedge \psi$;

(iv) $\{\varphi, \psi\} \vdash \chi$ ddacă $\{\varphi \wedge \psi\} \vdash \chi$.

Demonstraţie: Reamintim că $\varphi \wedge \psi = \neg(\varphi \to \neg\psi)$. De asemenea, oriunde folosim o teoremă formală cunoscută, aplicăm implicit Propoziţia 2.39.(ii).
Demonstrăm (i):

(1)	$\{\neg(\varphi \to \neg\psi)\}$	$\vdash \neg(\varphi \to \neg\psi)$	Propoziţia 2.38.(ii)
(2)	$\{\neg(\varphi \to \neg\psi)\}$	$\vdash \neg\varphi \to (\varphi \to \neg\psi)$	(S6.2).(ii)
(3)	$\{\neg(\varphi \to \neg\psi)\}$	$\vdash (\neg\varphi \to (\varphi \to \neg\psi)) \to (\neg(\varphi \to \neg\psi) \to \neg\neg\varphi)$	(S6.3)
(4)	$\{\neg(\varphi \to \neg\psi)\}$	$\vdash \neg(\varphi \to \neg\psi) \to \neg\neg\varphi$	(MP): (2), (3)
(5)	$\{\neg(\varphi \to \neg\psi)\}$	$\vdash \neg\neg\varphi$	(MP): (1), (4)
(6)	$\{\neg(\varphi \to \neg\psi)\}$	$\vdash \neg\neg\varphi \to \varphi$	(S6.2).(iii)
(7)	$\{\neg(\varphi \to \neg\psi)\}$	$\vdash \varphi$	(MP): (5), (6).

Demonstrăm (ii):

(1)	$\{\neg(\varphi \to \neg\psi), \neg\psi\}$	$\vdash \neg\psi \to (\varphi \to \neg\psi)$	(A1)
(2)	$\{\neg(\varphi \to \neg\psi), \neg\psi\}$	$\vdash \neg\psi$	Propoziţia 2.38.(ii)
(3)	$\{\neg(\varphi \to \neg\psi), \neg\psi\}$	$\vdash \varphi \to \neg\psi$	(MP): (1), (2)
(4)	$\{\neg(\varphi \to \neg\psi), \neg\psi\}$	$\vdash \neg(\varphi \to \neg\psi)$	Propoziţia 2.38.(ii)
(5)	$\{\neg(\varphi \to \neg\psi), \neg\psi\}$	$\vdash \neg(\varphi \to \neg\psi) \to ((\varphi \to \neg\psi) \to \bot)$	(S6.2).(ii)
(6)	$\{\neg(\varphi \to \neg\psi), \neg\psi\}$	$\vdash (\varphi \to \neg\psi) \to \bot$	(MP): (4), (5)
(7)	$\{\neg(\varphi \to \neg\psi), \neg\psi\}$	$\vdash \bot$	(MP): (3), (6)
(8)	$\{\neg(\varphi \to \neg\psi)\}$	$\vdash \psi$	(7) şi (S6.1).

Demonstrăm (iii):

(1)	$\{\varphi, \psi, \neg\neg(\varphi \to \neg\psi)\}$	$\vdash \varphi$	Propoziţia 2.38.(ii)
(2)	$\{\varphi, \psi, \neg\neg(\varphi \to \neg\psi)\}$	$\vdash \psi$	Propoziţia 2.38.(ii)
(3)	$\{\varphi, \psi, \neg\neg(\varphi \to \neg\psi)\}$	$\vdash \neg\neg(\varphi \to \neg\psi)$	Propoziţia 2.38.(ii)
(4)	$\{\varphi, \psi, \neg\neg(\varphi \to \neg\psi)\}$	$\vdash \neg\neg(\varphi \to \neg\psi) \to (\varphi \to \neg\psi)$	(S6.2).(iii)
(5)	$\{\varphi, \psi, \neg\neg(\varphi \to \neg\psi)\}$	$\vdash \varphi \to \neg\psi$	(MP): (3), (4)
(6)	$\{\varphi, \psi, \neg\neg(\varphi \to \neg\psi)\}$	$\vdash \neg\psi$	(MP): (1), (5)
(7)	$\{\varphi, \psi, \neg\neg(\varphi \to \neg\psi)\}$	$\vdash \neg\psi \to (\psi \to \bot)$	(S6.2).(ii)
(8)	$\{\varphi, \psi, \neg\neg(\varphi \to \neg\psi)\}$	$\vdash \psi \to \bot$	(MP): (6), (7)
(9)	$\{\varphi, \psi, \neg\neg(\varphi \to \neg\psi)\}$	$\vdash \bot$	(MP): (2), (8)
(10)	$\{\varphi, \psi\}$	$\vdash \neg(\varphi \to \neg\psi)$	(9) şi (S6.1).

Demonstrăm (iv), implicaţia "$\Rightarrow$":

(1)	$\{\varphi, \psi\}$	$\vdash \chi$	Ipoteză
(2)	$\{\varphi\}$	$\vdash \psi \to \chi$	Teorema deducţiei
(3)		$\vdash \varphi \to (\psi \to \chi)$	Teorema deducţiei
(4)	$\{\varphi \wedge \psi\}$	$\vdash \varphi \to (\psi \to \chi)$	(3)
(5)	$\{\varphi \wedge \psi\}$	$\vdash \varphi$	(i)
(6)	$\{\varphi \wedge \psi\}$	$\vdash \psi \to \chi$	(MP): (4), (5)
(7)	$\{\varphi \wedge \psi\}$	$\vdash \psi$	(ii)
(8)	$\{\varphi \wedge \psi\}$	$\vdash \chi$	(MP): (6), (7).

Demonstrăm (iv), implicaţia "$\Leftarrow$":

$$
\begin{array}{llll}
(1) & \{\varphi \wedge \psi\} & \vdash \chi & \text{Ipoteză} \\
(2) & & \vdash (\varphi \wedge \psi) \rightarrow \chi & \text{Teorema deducţiei} \\
(3) & \{\varphi, \psi\} & \vdash (\varphi \wedge \psi) \rightarrow \chi & (2) \\
(4) & \{\varphi, \psi\} & \vdash \varphi \wedge \psi & (\text{iii}) \\
(5) & \{\varphi, \psi\} & \vdash \chi & \text{(MP): (3), (4).}
\end{array}
$$

□

(S7.3) Fie $n \in \mathbb{N}$ şi $\varphi_1, \ldots, \varphi_n$ formule. Să se arate că (Propoziţia 2.62 din curs):

(i) Pentru orice formulă ψ, $\{\varphi_1, \ldots, \varphi_n\} \vdash \psi$ dacă şi numai dacă $\vdash \varphi_1 \wedge \ldots \wedge \varphi_n \rightarrow \psi$ dacă şi numai dacă $\{\varphi_1 \wedge \ldots \wedge \varphi_n\} \vdash \psi$.

(ii) $\{\varphi_1, \ldots, \varphi_n\}$ este consistentă dacă şi numai dacă $\{\varphi_1 \wedge \ldots \wedge \varphi_n\}$ este consistentă.

Demonstraţie:

(i) Observăm mai întâi că ultima echivalenţă (între a doua şi a treia afirmaţie) este o instanţă a Teoremei deducţiei. E suficient, deci, să arătăm faptul că prima afirmaţie este echivalentă cu a treia. Demonstrăm prin inducţie după $n \geq 1$.

Pentru $n = 1$, enunţul este tautologic.

Fie $n \geq 1$. Presupunem adevărată concluzia pentru n şi o demonstrăm pentru $n + 1$. Avem:

$$
\begin{aligned}
\{\varphi_1, \ldots, \varphi_n, \varphi_{n+1}\} \vdash \psi &\Leftrightarrow \{\varphi_1, \ldots, \varphi_n\} \vdash \varphi_{n+1} \rightarrow \psi && \text{(din Teorema deducţiei)} \\
&\Leftrightarrow \{\varphi_1 \wedge \ldots \wedge \varphi_n\} \vdash \varphi_{n+1} \rightarrow \psi && \text{(din ipoteza de inducţie)} \\
&\Leftrightarrow \{\varphi_1 \wedge \ldots \wedge \varphi_n, \varphi_{n+1}\} \vdash \psi && \text{(din Teorema deducţiei)} \\
&\Leftrightarrow \{\varphi_1 \wedge \ldots \wedge \varphi_n \wedge \varphi_{n+1}\} \vdash \psi. && \text{(din (S7.2).(iv))}
\end{aligned}
$$

(ii) Avem că:

$$
\begin{aligned}
\{\varphi_1, \ldots, \varphi_n\} \text{ consistentă} &\Leftrightarrow \{\varphi_1, \ldots, \varphi_n\} \nvdash \perp && \text{(din Propoziţia 2.60)} \\
&\Leftrightarrow \{\varphi_1 \wedge \ldots \wedge \varphi_n\} \nvdash \perp && \text{(din punctul (i))} \\
&\Leftrightarrow \{\varphi_1 \wedge \ldots \wedge \varphi_n\} \text{ consistentă.} && \text{(din Propoziţia 2.60)}
\end{aligned}
$$

□