FMI, Info, Anul I
Logică matematică şi computaţională

Seminar 6

(S6.1) (Metoda reducerii la absurd)
Să se arate că pentru orice mulţime de formule Γ şi orice formule φ, ψ,

$$\Gamma \cup \{\neg\psi\} \vdash \neg(\varphi \to \varphi) \quad \Rightarrow \quad \Gamma \vdash \psi.$$

Demonstraţie: Avem

(1)	$\Gamma \cup \{\neg\psi\}$	$\vdash \neg(\varphi \to \varphi)$	Ipoteză
(2)	Γ	$\vdash \neg\psi \to \neg(\varphi \to \varphi)$	Teorema deducţiei
(3)	Γ	$\vdash (\neg\psi \to \neg(\varphi \to \varphi)) \to ((\varphi \to \varphi) \to \psi)$	(A3) şi Propoziţia 2.38.(i)
(4)	Γ	$\vdash (\varphi \to \varphi) \to \psi$	(MP): (2), (3)
(5)	Γ	$\vdash \varphi \to \varphi$	Propoziţiile 2.45 şi 2.39.(ii)
(6)	Γ	$\vdash \psi$	(MP): (4), (5).

$\square$

(S6.2) Să se arate că pentru orice formule φ, ψ,

(i) $\{\psi, \neg\psi\} \vdash \varphi$;

(ii) $\vdash \neg\psi \to (\psi \to \varphi)$;

(iii) $\vdash \neg\neg\varphi \to \varphi$;

(iv) $\vdash \varphi \to \neg\neg\varphi$.

Demonstraţie: Demonstrăm (i):

(1)		$\vdash \neg\psi \to (\neg\varphi \to \neg\psi)$	(A1)
(2)	$\{\neg\psi\}$	$\vdash \neg\varphi \to \neg\psi$	Teorema deducţiei
(3)	$\{\neg\psi\}$	$\vdash (\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$	(A3) şi Propoziţia 2.38.(i)
(4)	$\{\neg\psi\}$	$\vdash \psi \to \varphi$	(MP): (2), (3)
(5)	$\{\psi, \neg\psi\}$	$\vdash \varphi$	Teorema deducţiei.

Punctul (ii) se obţine din (i) aplicând de două ori Teorema deducţiei:

(1)	$\{\psi, \neg\psi\}$	$\vdash \varphi$	se aplică (i)
(2)	$\{\neg\psi\}$	$\vdash \psi \to \varphi$	Teorema deducţiei
(3)		$\vdash \neg\psi \to (\psi \to \varphi)$	Teorema deducţiei.

Demonstrăm în continuare (iii).

(1)	$\{\neg\varphi, \neg\neg\varphi\}$	$\vdash \neg(\varphi \to \varphi)$	se aplică (i)
(2)	$\{\neg\neg\varphi\}$	$\vdash \varphi$	(1) şi (S6.1)
(3)		$\vdash \neg\neg\varphi \to \varphi.$	Teorema deducţiei.

Demonstrăm (iv):

(1)	$\vdash \neg\neg\neg\varphi \to \neg\varphi$	se aplică (iii) cu $\varphi := \neg\varphi$
(2)	$\vdash (\neg\neg\neg\varphi \to \neg\varphi) \to (\varphi \to \neg\neg\varphi)$	(A3)
(3)	$\vdash \varphi \to \neg\neg\varphi$	(MP): (1), (2).

□

(S6.3) ("Reciproca" axiomei 3)
Să se arate că pentru orice formule φ, ψ,

$$\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi).$$

Demonstraţie:

(1)	$\{\varphi \to \psi, \neg\psi, \neg\neg\varphi\}$	$\vdash \varphi \to \psi$	Propoziţia 2.38.(ii)
(2)	$\{\varphi \to \psi, \neg\psi, \neg\neg\varphi\}$	$\vdash \neg\psi$	Propoziţia 2.38.(ii)
(3)	$\{\varphi \to \psi, \neg\psi, \neg\neg\varphi\}$	$\vdash \neg\neg\varphi$	Propoziţia 2.38.(ii)
(4)	$\{\varphi \to \psi, \neg\psi, \neg\neg\varphi\}$	$\vdash \neg\neg\varphi \to \varphi$	(S6.2).(iii) şi Propoziţia 2.39.(ii)
(5)	$\{\varphi \to \psi, \neg\psi, \neg\neg\varphi\}$	$\vdash \varphi$	(MP): (3), (4)
(6)	$\{\varphi \to \psi, \neg\psi, \neg\neg\varphi\}$	$\vdash \psi$	(MP): (1), (5)
(7)	$\{\varphi \to \psi, \neg\psi, \neg\neg\varphi\}$	$\vdash \neg\psi \to (\psi \to \neg(\varphi \to \varphi))$	(S6.2).(ii) şi Propoziţia 2.39.(ii)
(8)	$\{\varphi \to \psi, \neg\psi, \neg\neg\varphi\}$	$\vdash \psi \to \neg(\varphi \to \varphi)$	(MP): (2), (7)
(9)	$\{\varphi \to \psi, \neg\psi, \neg\neg\varphi\}$	$\vdash \neg(\varphi \to \varphi)$	(MP): (6), (8)
(10)	$\{\varphi \to \psi, \neg\psi\}$	$\vdash \neg\varphi$	(9) şi (S6.1)
(11)	$\{\varphi \to \psi\}$	$\vdash \neg\psi \to \neg\varphi$	Teorema deducţiei
(12)		$\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$	Teorema deducţiei.

□

(S6.4) Să se arate că pentru orice formule φ, ψ,

$$\{\psi, \neg\varphi\} \vdash \neg(\psi \to \varphi).$$

Demonstraţie: Avem

(1)	$\{\psi, \neg\varphi, \neg\neg(\psi \to \varphi)\}$	$\vdash \psi$	Propoziţia 2.38.(ii)
(2)	$\{\psi, \neg\varphi, \neg\neg(\psi \to \varphi)\}$	$\vdash \neg\varphi$	Propoziţia 2.38.(ii)
(3)	$\{\psi, \neg\varphi, \neg\neg(\psi \to \varphi)\}$	$\vdash \neg\neg(\psi \to \varphi)$	Propoziţia 2.38.(ii)
(4)	$\{\psi, \neg\varphi, \neg\neg(\psi \to \varphi)\}$	$\vdash \neg\neg(\psi \to \varphi) \to (\psi \to \varphi)$	(S6.2).(iii) şi Prop. 2.39.(ii)
(5)	$\{\psi, \neg\varphi, \neg\neg(\psi \to \varphi)\}$	$\vdash \psi \to \varphi$	(MP): (3), (4)
(6)	$\{\psi, \neg\varphi, \neg\neg(\psi \to \varphi)\}$	$\vdash \varphi$	(MP): (1), (5)
(7)	$\{\psi, \neg\varphi, \neg\neg(\psi \to \varphi)\}$	$\vdash \neg\varphi \to (\varphi \to \neg(\varphi \to \varphi))$	(S6.2).(ii) şi Prop. 2.39.(ii)
(8)	$\{\psi, \neg\varphi, \neg\neg(\psi \to \varphi)\}$	$\vdash \varphi \to \neg(\varphi \to \varphi)$	(MP): (2), (7)
(9)	$\{\psi, \neg\varphi, \neg\neg(\psi \to \varphi)\}$	$\vdash \neg(\varphi \to \varphi)$	(MP): (6), (8)
(10)	$\{\psi, \neg\varphi\}$	$\vdash \neg(\psi \to \varphi)$	(9) şi (S6.1).

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