-TEMA 3-
ALGEBRĂ LINIARĂ ȘI GEOMETRIE

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Apl Fie spațiul vectorial euclidian $E_3 = (\mathbb{R}^3/\mathbb{R}, \langle , \rangle)$ p.s.c

$B_0 = \{e_1, e_2, e_3\} \subset E_3$
baza canonică

Stabiliți dacă următoarele aplicații liniare sunt transformări ortogonale:

c) $T : E_3 \to E_3$

$$\begin{cases} T(e_1) = \frac{2}{3} e_1 + \frac{2}{3} e_2 - \frac{1}{3} e_3 \\ T(e_2) = \frac{2}{3} e_1 - \frac{1}{3} e_2 + \frac{2}{3} e_3 \\ T(e_3) = -\frac{1}{3} e_1 + \frac{2}{3} e_2 + \frac{2}{3} e_3 \end{cases}$$

Rezolvare c) $A=\begin{pmatrix} 2/3 & 2/3 & -1/3 \\ 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \end{pmatrix}$ -> m. asoc. endm. T în raport cu baza canonică B_0

$$\Rightarrow {}^tA\cdot A=\begin{pmatrix} 2/3 & 2/3 & -1/3 \\ 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \end{pmatrix}\cdot\begin{pmatrix} 2/3 & 2/3 & -1/3 \\ 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \end{pmatrix}=$$

$$=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}=I_3 \Rightarrow A \text{ m. ortogonală}$$

-> T este transformare ortogonală

Apl Fie forma pătratică $Q:\mathbb{R}^3\to\mathbb{R}$, $Q(x)=3x_1^2+4x_2^2+5x_3^2+4x_1x_2=4x_2x_3$, $(\forall)\, x=(x_1,x_2,x_3)$. Să se aducă Q la o formă canonică, utilizând:

a) Metoda Gauss;

b) Metoda Jacobi;

c) Metoda transformării ortogonale;

Rezolvare a) $Q(x) = 3x_1^2 + 4x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_2x_3$

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 5 \end{pmatrix}$$ $\to$ m. asoc. f.p. Q în raport cu baza canonică $B_0 = \{e_1, e_2, e_3\} \subset \mathbb{R}^3$

$$\Rightarrow 3\left(x_1^2 + \frac{4}{3}x_1x_2\right) + 4x_2^2 + 5x_3^2 - 4x_2x_3 =$$

$$= 3\left[\left(x_1 + \frac{2}{3}x_2\right)^2 - \frac{4}{9}x_2^2\right] + 4x_2^2 + 5x_3^2 - 4x_2x_3 =$$

$$= 3\left(x_1 + \frac{2}{3}x_2\right)^2 + \frac{8}{3}\left(x_2^2 - \frac{3}{2}x_2x_3\right) + 5x_3^2 =$$

$$= 3\left(x_1 + \frac{2}{3}x_2\right)^2 + \frac{8}{3}\left(x_2 - \frac{3}{4}x_3\right)^2 + \frac{7}{2}x_3^2$$

$\Rightarrow$ Sistemul
$$\begin{cases} x_1' = x_1 + \frac{2}{3}x_2 \\ x_2' = x_2 - \frac{3}{4}x_3 \\ x_3' = x_3 \end{cases}$$

$$\Rightarrow Q(x) = 3x_1' + \frac{8}{3}x_2' + \frac{4}{3}x_3'$$

Rezolvare b) $A=\begin{pmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 5 \end{pmatrix}$

$$\begin{cases} \Delta_1 = 3 \neq 0 \\ \Delta_2 = \begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix} = 12-4 = 8 \neq 0 \\ \Delta_3 = \begin{vmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 5 \end{vmatrix} = 60-12-20 = 28 \neq 0 \end{cases} \qquad \Delta_i \neq 0, (\forall) i = \overline{1,3}$$

$$\Rightarrow Q(x) = \frac{1}{\Delta_1}(x_1')^2 + \frac{\Delta_1}{\Delta_2}(x_2')^2 + \frac{\Delta_2}{\Delta_3}(x_3')^2 =$$

$$= \frac{1}{3}(x_1')^2 + \frac{8}{3}(x_2')^2 + \frac{2}{7}(x_3')^2 \quad (\forall) x = (x_1', x_2', x_3') \in \mathbb{R}^3$$

coord. în raport cu noua bază B'

Rezolvare c)

Determinăm valorile proprii coresp lui A

Polinomul caracteristic este $P(\lambda) = \det(A - \lambda I_3) =$

$$= \begin{vmatrix} 3-\lambda & 2 & 0 \\ 2 & 4-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{vmatrix} = (3-\lambda)(4-\lambda)(5-\lambda) - 8(4-\lambda) =$$

$$= [(3-\lambda)(5-\lambda) - 8](4-\lambda) =$$

$= (\lambda_1 - 1)(\lambda_2 - 7)(4 - \lambda_3)$

$\Rightarrow$ Ec. caract. $P(\lambda) = 0 \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 7 \\ \lambda_3 = 4 \end{cases}$ valorile proprii, Spec $= \{1, 7, 4\}$

$m_a(\lambda_1) = m_a(\lambda_2) = m_a(\lambda_3) = 1$

Subspații proprii

$$S_\lambda : \begin{cases} (3-\lambda)x + 2y = 0 \\ 2x + (4-\lambda)y - 2z = 0 \\ -2y + (5-\lambda)z = 0 \end{cases}$$

Atunci

$$S_{\lambda_1} : \begin{cases} 2x + 2y = 0 \\ 2x + 3y - 2z = 0 \\ -2y + 4z = 0 \end{cases}$$ $\{\lambda_1 = 1\}$ → sistem liniar omogen cu 3 ec. și 3 nec.

$$rg(A_f - \lambda_1 I_3) = 2$$

$$\Delta_p = \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} = 6 - 4 = 2 \neq 0 \Rightarrow \begin{cases} x, y \text{ nec. princip.} \\ z = \alpha, \alpha \in \mathbb{R} \text{ nec. sec.} \end{cases}$$

$$\begin{cases} 2x+2y=0 \\ 2x+3y=2\alpha \end{cases} \Rightarrow \begin{array}{l} y=2\alpha \\ z=\alpha, \alpha \in \mathbb{R} \end{array}$$

$$2x+4\alpha=0$$
$$2x=-4\alpha$$
$$x=-2\alpha$$

$$\begin{cases} x=-2\alpha \\ y=2\alpha \\ z=\alpha, \alpha \in \mathbb{R} \end{cases}$$

Deci $V_{\lambda_1}=\{(-2\alpha, 2\alpha, \alpha) / \alpha \in \mathbb{R}\}=\{\alpha \underbrace{(-2,2,1)}_{= v_1} / \alpha \in \mathbb{R}\}$

$S'_{\lambda_3}: \begin{cases} -x+2y=0 \\ 2x-2z=0 \\ -2y+z=0 \end{cases}$ → sistem liniar omogen cu 3 ec. și 3 nec.

$\{\lambda_3=4\}$

$$rg(A_f-\lambda I_3)=2$$

$\Delta_p=\begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix}=2-4=-2 \neq 0 \Rightarrow \begin{cases} x, y \text{ nec. princip} \\ z=\beta, \beta \in \mathbb{R} \text{ nec. sec.} \end{cases}$

$$-2y=-\beta \qquad -x+\beta=0$$
$$y=\frac{\beta}{2} \qquad x=\beta$$
$$z=\beta, \beta \in \mathbb{R}$$

$$\begin{cases} x=\beta \\ y=\beta/2 \\ z=\beta, \alpha \in \mathbb{R} \end{cases}$$

Deci $V_{\lambda_3} = \{(\beta, \beta/2, \beta) / \beta \in \mathbb{R}\} = \{\beta \underbrace{(1, 1/2, 1)}_{= v_3} / \beta \in \mathbb{R}\}$

$S_{\lambda_3} : \begin{cases} -4x + 2y = 0 \\ 2x - 3y - 2z = 0 \\ -2y - 2z = 0 \end{cases}$

$\{\lambda_3 = 7\}$

$rg(A_f - \lambda_3 I_3) = 2$

$D_p = \begin{vmatrix} -4 & 2 \\ 2 & -3 \end{vmatrix} = -12 - 4 = -16 \neq 0 \Rightarrow \begin{cases} x, y \text{ nec principale} \\ z = \mu, \mu \in \mathbb{R} \\ \text{nec. sec.} \end{cases}$

$\begin{cases} -4x + 2y = 0 \\ 2x - 3y = 2\mu \quad | \cdot 2 \end{cases} \Rightarrow \begin{cases} -4x + 2y = 0 \\ 4x - 6y = 4\mu \end{cases} \Rightarrow$

$\Rightarrow -4y = 4\mu$

$y = -\mu$

$z = \mu \in \mathbb{R}$

$-4x - 2\mu = 0$

$4x = -2\mu \Rightarrow x = -\mu/2$

$\begin{cases} x = -\mu/2 \\ y = -\mu \\ z = \mu \in \mathbb{R} \end{cases}$

Deci $V_{\lambda_2} = \{(-\mu/2, -\mu, \mu) / \mu \in \mathbb{R}\} =$

$$= \{\mu \underbrace{(-1/2, -1, 1)}_{= v_2} / \mu \in \mathbb{R}\}$$

$$B = \{v_1 = (-2, 2, 1),\ v_2 = (-1/2, -1, 1),\ v_3 = (1, 1/2, 1)\}$$

$$\Rightarrow B' = \left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\} = \left\{ \frac{(-2, 2, 1)}{3}, \frac{2}{3}(-1/2, -1, 1), \frac{2}{3}(1, 1/2, 1) \right\}$$

bază ortonormată

$$\|v_1\| = \sqrt{(-2)^2 + 2^2 + 1^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

$$\|v_3\| = \sqrt{1 + \frac{1}{4} + 1} = \sqrt{\frac{9}{4}} = \frac{3}{2}$$

$$\|v_2\| = \sqrt{\frac{1}{4} + 1 + 1} = \frac{3}{2}$$

Avem $Q(x) = \lambda_1 (x_1')^2 + \lambda_2 (x_2')^2 + \lambda_3 (x_3')^2$

$$= (x_1')^2 + 7(x_2')^2 + 4(x_3')^2$$

Apl Fie Γ de ecuație: $x_1^2+2x_1x_2-5x_2^2+4x_1-8x_2-17=0$. Să se aducă la o formă canonică conica Γ prin izometrii.

Rezolvare: $A=\begin{pmatrix}1 & 1\\ 1 & -5\end{pmatrix}$; $\delta=\det A=\begin{vmatrix}1 & 1\\ 1 & -5\end{vmatrix}=$

$=-5-1=-6$

$$A'=\begin{pmatrix}1 & 1 & 2\\ 1 & -5 & -4\\ 2 & -4 & -17\end{pmatrix};\ \Delta=\det A'=90$$

$$\left.\begin{matrix}\delta<0\\ \Delta\neq 0\end{matrix}\right| \Rightarrow \Gamma \text{ este HIPERBOLĂ}$$

Centrul conicei Γ este $P_0(x_1^0, x_2^0)$, unde coord (x_1^0, x_2^0) se determină ca sol. unică a sist. de ec. liniare

$$\begin{cases}\dfrac{\partial f}{\partial x_1}=0\\ \dfrac{\partial f}{\partial x_2}=0\end{cases} \Leftrightarrow \begin{cases}2x_1+2x_2+4=0\\ 2x_1-10x_2-8=0\end{cases}$$

$\Rightarrow -12x_2 - 12 = 0$

$12x_2 = -12$

$x_2 = -1$

$2x_1 + 10 - 8 = 0$

$2x_1 + 2 = 0$

$2x_1 = -2$

$x_1 = -1$

$\Rightarrow \begin{cases} x_1 = -1 \\ x_2 = -1 \end{cases}$

Deci: $P_0(-1,-1)$ centrul conicei

Efectuăm translația t

$t \begin{cases} x_1' = x_1 + 1 \\ x_2' = x_2 + 1 \end{cases} \Rightarrow \begin{cases} x_1 = x_1' - 1 \\ x_2 = x_2' - 1 \end{cases}$

$t(\Gamma): (x_1')^2 + 2x_1'x_2' - 5(x_2')^2 + \frac{\Delta}{\delta} \Rightarrow$

$\Rightarrow (x_1')^2 + 2x_1'x_2' - 5(x_2')^2 - 15 = 0$

Determinăm valorile proprii ale lui A

Ec. caracteristică $\det(A - \lambda I_2) = 0$, în $\mathbb{R}$

$\Leftrightarrow \begin{vmatrix} 1-\lambda & 1 \\ 1 & -5-\lambda \end{vmatrix} = 0 \Leftrightarrow (1-\lambda)(-5-\lambda) - 1 = 0$

$\Leftrightarrow \lambda^2+4\lambda-6=0$ $\Rightarrow \begin{cases} \lambda_1=-2+\sqrt{10} \\ \lambda_2=-2-\sqrt{10} \end{cases}$ valorile proprii

$\Delta=16+24=40$

Determinăm subsp. proprii corespunzătoare

$V_{\lambda_1}=\{v\in\mathbb{R}^2 / Av=\lambda_1 v\}$

$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$\Rightarrow \begin{cases} (3-\sqrt{10})v_1+v_2=0 \\ (3-\sqrt{10})v_1-(9-10)v_2=0 \end{cases}$ $\Rightarrow (3-\sqrt{10})v_1+v_2=0$

$v_1=\alpha\in\mathbb{R} \Rightarrow v_2=-\alpha(3-\sqrt{10})$

$\Rightarrow V_{\lambda_1}=<(1,3-\sqrt{10})>$

$V_{\lambda_2}=\{v\in\mathbb{R}^2 / Av=\lambda_2 v\}$

$\begin{cases} (3+\sqrt{10})v_1+v_2=0 \\ v_1+(3+\sqrt{10})v_2=0 \end{cases}$ $v_1=\alpha\in\mathbb{R} \Rightarrow v_2=-\alpha(3+\sqrt{10})$

$\Rightarrow V_{\lambda_2}=<(1,-3-\sqrt{10})>$

$\begin{cases} e_1=(1,-3+\sqrt{10}) \\ e_2=(1,-3-\sqrt{10}) \end{cases} \Rightarrow \begin{cases} e_1=\dfrac{e_1}{\|e_1\|}=\dfrac{1}{\sqrt{20-6\sqrt{10}}}(1,-3+\sqrt{10}) \\ e_2=\dfrac{e_2}{\|e_2\|}=\dfrac{1}{\sqrt{20+6\sqrt{10}}}(1,-3-\sqrt{10}) \end{cases}$

Rotația r

$$r \quad x_1'' = \frac{1}{\sqrt{20+6\sqrt{10}}}\left(1,-3-\sqrt{10}\right)$$

$$x_2'' = \frac{1}{\sqrt{20-6\sqrt{10}}}\left(1,-3+\sqrt{10}\right)$$

$$\Rightarrow (rot)(\Gamma): \left(-2+\sqrt{10}\right)\left(x_1''\right)^2 - \left(2+\sqrt{10}\right)\left(x_2''\right)^2 - 15 = 0 \,\Big|:15$$

$$\Rightarrow \frac{\left(x_1''\right)^2}{\frac{15}{-2+\sqrt{10}}} - \frac{\left(x_2''\right)^2}{\frac{15}{2+\sqrt{10}}} - 1 = 0 \Rightarrow \Gamma \text{ este HIPERBOLĂ}$$