ST 1210: Introduction to Probability and Statistics

LECTURE 3

Expected value

- The expected value of a random variable X is given by

$$E[X] = \sum xP(X = x)$$

- **Example**: The probabilities that a customer will buy 1, 2, 3, 4 or 5 items In a grocery store are $\frac{3}{10}, \frac{1}{10}, \frac{1}{10}, \frac{2}{10}, \frac{3}{10}$ respectively. What is the average number of items that a customer will buy?
- $E[X] = \sum xP(X = x)$=1($\frac{3}{10}$)+2($\frac{1}{10}$)+3($\frac{1}{10}$)+4($\frac{2}{10}$)+5($\frac{3}{10}$)=3.1

Properties of Expectation

- If $g(X) = aX + b$, then $\mathrm{E}[g(X)] = aE[X] + b$
- If $g(X) = aX + b$ then $\mathrm{Var}[g(X)] = a^2Var[X]$

Exercise:

- Show that $Var[X] = E[X^2] - (E[X])^2$

Bernoulli's probability distribution

- A random variable X which takes two values 0 and 1 with probability p and q respectively i.e. $P(X=1)=p$ and $P(X=0)=q$

- This is called a Bernoulli variate and is said to have a Bernoulli's distribution

Binomial probability distribution

Properties

1. Each trial results into two mutually disjoint outcomes which are success and failure
2. The number of trials is finite
3. The trials are independent of each other
4. The probability of success p is constant for each trial

Binomial probability distribution…

- Assume that there is a coin which is tossed *n*-times
- Tossing of this coin results into two independent outcomes H or T
- These outcomes can be regarded as success and failure being denoted by S or F respectively

Binomial probability distribution…

The same coin is tossed successively and independently *n* times.

We arbitrarily use *S* to denote the outcome *H* (heads) and *F* to denote the outcome *T* (tails). Then this experiment satisfies Conditions 1–4.

Tossing a thumbtack *n* times, with *S* = point up and *F* = point down, also results in a binomial experiment.

Binomial random variable and distribution

In most binomial experiments, it is the total number of S's, rather than knowledge of exactly which trials yielded S's, that is of interest.

Definition

The **binomial random variable *X*** associated with a binomial experiment consisting of n trials is defined as

$$X = \text{the number of } S\text{'s among the } n \text{ trials}$$

Binomial random variable and distribution…

Suppose, for example, that $n = 3$.

Then there are eight possible outcomes for the experiment:

$$SSS \quad SSF \quad SFS \quad SFF \quad FSS \quad FSF \quad FFS \quad FFF$$

From the definition of X, $X(SSF) = 2$, $X(SFF) = 1$, and so on. Possible values for X in an n-trial experiment are $x = 0, 1, 2, \ldots, n$.

We will often write $X \sim \text{Bin}(n, p)$ to indicate that X is a binomial rv based on n trials with success probability p.

Binomial random variable and distribution…

Notation

Because the pmf of a binomial random variable *X* depends on the two parameters *n* and *p*, we denote the pmf by

$b(x; n, p)$.

$$b(x;n,p) = \binom{n}{x} p^{x}(1-p)^{n-x}, \quad x = 0,1,\cdots,n.$$

Binomial random variable and distribution…

- In other words
- A random variable X is said to follow Binomial distribution if it assumes only non-negative values and its probability mass function is given by

$P(X = x) = C(n, x)p^x q^{n-x}, x = 0,2,3, \dots, n.$ and $p + q = 1$

- n and p are parameters and n is called the degree of the distribution
- A variable which follow the Binomial distribution is called Binomial variate.

Binomial random variable and distribution…

Example

Each of six randomly selected cola drinkers is given a glass containing cola S and one containing cola F. The glasses are identical in appearance except for a code on the bottom to identify the cola.

Suppose there is actually no tendency among cola drinkers to prefer one cola to the other.

Then $p = P$(a selected individual prefers S) = .5, so with
X = the number among the six who prefer S,
$X \sim \text{Bin}(6,.5)$.

Thus

$$P(X = 3) = b(3; 6, .5) = \binom{6}{3}(.5)^3(.5)^3 = 20(.5)^6 = .313$$

Binomial random variable and distribution… Example…

The probability that at least three prefer *S* is

$$P(3 \le X) = \sum_{x=3}^{6} b(x;\, 6,\, .5)$$

$$= \sum_{x=3}^{6} \binom{6}{x} (.5)^x (.5)^{6-x}$$

$$= .656$$

and the probability that at most one prefers *S* is

$$P(X \le 1) = \sum_{x=0}^{1} b(x;\, 6,\, .5)$$

$$= .109$$

Binomial random variable and distribution…

Example

- Prove that

$$P(X = x) = C(n, x)p^x q^{n-x}$$

is a probability mass function (pmf).

Proof

- Checking if $P(X = x) \geq 0$, It is clear from the expression that $C(n, x)p^x q^{n-x} \geq 0$

Binomial random variable and distribution…

- We also have to show that

$$\sum_{x=0}^{n} P(x) = \sum_{x=0}^{n} C(n,x) p^x q^{n-x}$$

- It follows that

$$\sum_{x=0}^{n} P(x) = \sum_{x=0}^{n} \frac{n!}{(n-x)!\,x!} p^x q^{n-x}$$

- Thus

$$\sum_{x=0}^{n} P(x) = \frac{n!}{(n-0)!\,0!} p^0 q^{n-0} + \frac{n!}{(n-1)!\,1!} p^1 q^{n-1} + \cdots + \frac{n!}{(n-n)!\,n!} p^n q^{n-n}$$

Binomial random variable and distribution…

- Implies

$$\sum_{x=0}^{n} P(x) = q^n + C(n,1)pq^{n-1} + \cdots + p^n$$

- It follows that

$$q^n + C(n,1)pq^{n-1} + \cdots + p^n = (q+p)^n$$

- This implies that

$$\sum_{x=0}^{n} P(x) = (q+p)^n$$

- But $q + p = 1$, Implies that

$$\sum_{x=0}^{n} P(x) = (1)^n = 1$$

Mean and variance for Binomial distribution

For $n = 1$, the binomial distribution becomes the Bernoulli distribution.

The mean value of a Bernoulli variable is $\mu = p$, so the expected number of S's on any single trial is p.

Since a binomial experiment consists of n trials, intuition suggests that for $X \sim \text{Bin}(n, p)$, $E(X) = np$, the product of the number of trials and the probability of success on a single trial.

$V(X)$ is not so intuitive.

Mean and variance for Binomial distribution…

Proposition

If $X \sim \text{Bin}(n, p)$, then $E(X) = np$, $V(X) = np(1 - p) = npq$, and $\sigma_X = \sqrt{npq}$ (where $q = 1 - p$).

- Now, from

$$E[X] = \sum_{x=0}^{n} xP(x)$$

Mean and variance for Binomial distribution…

$$E[X] = \sum_{x=0}^{n} xC(n,x)p^x q^{n-x}$$

- It follows that

$$E[X] = 0 + (1)C(n,x)p^x q^{n-x}$$

- Thus,

$$E[X] = 0 + (1)C(n,1)p^1 q^{n-1} + (2)C(n,2)p^2 q^{n-2} + \cdots$$
$$\ldots + (n)C(n,n)p^n q^0$$

- Therefore,

$$E[X] = np\left[q^{n-1} + \frac{C[(n-1),1]pq^{n-2}}{(1)(2)} + \cdots + p^{n-1}\right] = np(q+p)^{n-1} = np$$

Mean and variance for Binomial distribution…

- The variance

$$Var(X) = E(X^2) - (E[X])^2$$

- Let $E[X] = \mu$
- It follows that

$$Var(X) = E(X^2) - \mu^2$$

- Thus,

$$Var(X) = \sum_{x=0}^{n} x^2\, P(X = x) - \mu^2$$

Mean and variance for Binomial distribution…

- Thus,

$$Var(X) = \sum_{x=0}^{n} [x + x(x-1)]\, P(X = x) - \mu^{2}$$

- It follows that,

$$Var(X) = \sum_{x=0}^{n} xP(X = x) + \sum_{x=0}^{n} x(x-1)\, P(X = x) - \mu^{2}$$

- Therefore,

$$Var(X) = \mu + \sum_{n=2}^{n} x(x-1)C(n,x)\, p^{x} q^{n-x} - \mu^{2}$$

Mean and variance for Binomial distribution…

- It follows that,

$$Var(X) = \mu + [(2)(1)C(n,2)p^2q^{n-2} + (3)(2)C(n,3)p^3q^{n-3} + \cdots + n(n-1)C(n,n)p^nq^0] - \mu^2$$

- Thus,

$$Var(X) = \mu + n(n-1)p^2(p+q)^{n-2} - \mu^2$$

- Therefore,

$$Var(X) = \mu + n^2p^2 - np^2 - (p+q)^{n-2} - \mu^2$$

- But $n^2p^2 = \mu^2$, it follows that

$$Var(X) = \mu + \mu^2 - np^2 - (p+q)^{n-2} - \mu^2 = np(1-p) = npq$$

Mean and variance for Binomial distribution… Example

If 75% of all purchases at a certain store are made with a credit card and X is the number among ten randomly selected purchases made with a credit card, then
$X \sim \text{Bin}(10, .75)$.

Thus $E(X) = np = (10)(.75) = 7.5$,

$$V(X) = npq = 10(.75)(.25)$$

$$= 1.875,$$

and $\sigma = \sqrt{1.875}$

$$= 1.37.$$

Mean and variance for Binomial distribution…

Again, even though X can take on only integer values, $E(X)$ need not be an integer.

If we perform a large number of independent binomial experiments, each with $n = 10$ trials and $p = .75$, then the average number of S's per experiment will be close to 7.5.

The probability that X is within 1 standard deviation of its mean value is

$$
\begin{aligned}
P(7.5 - 1.37 \leq X \leq 7.5 + 1.37) &= P(6.13 \leq X \leq 8.87) \\
&= P(X = 7 \text{ or } 8) \\
&= .532.
\end{aligned}
$$

Poisson probability distribution

- A random variable X is said to follow Poisson distribution if it assumes only non-negative values
- Its probability distribution is given by
- $P(x,\boldsymbol{\lambda}) = P(X = x) = \begin{cases} \frac{e^{-\lambda}\boldsymbol{\lambda}^x}{x!}, & x = 0,1,2,\ldots \\ 0 & otherwise \end{cases}$
- Here $\boldsymbol{\lambda}$>0 is called the parameter of the distribution
- **Note**: You have to be able to prove that $P(X = x)$ is the probability mass function

Poisson distribution…

- Poisson distribution is a discrete probability distribution
- Consider

$$P(X = x) = \frac{e^{-\lambda}\boldsymbol{\lambda}^{x}}{x!}$$

- We need to show that

$$\sum_{x=0}^{\infty} P(X = x) = 1$$

- It imply that

$$\sum_{x=0}^{\infty} \frac{e^{-\lambda}\boldsymbol{\lambda}^{x}}{x!} = 1$$

Poisson distribution…

- Therefore,

$$\sum_{x=0}^{\infty} \frac{e^{-\lambda}\boldsymbol{\lambda}^{x}}{x!} = e^{-\lambda}\left[1 + \frac{\boldsymbol{\lambda}}{1!} + \frac{\boldsymbol{\lambda}^{2}}{2!} + \frac{\boldsymbol{\lambda}^{3}}{3!} + \cdots\right]$$

- But

$$1 + \frac{\boldsymbol{\lambda}}{1!} + \frac{\boldsymbol{\lambda}^{2}}{2!} + \frac{\boldsymbol{\lambda}^{3}}{3!} + \cdots = e^{\lambda}$$

- It follows that

$$\sum_{x=0}^{\infty} \frac{e^{-\lambda}\boldsymbol{\lambda}^{x}}{x!} = e^{-\lambda}[\,e^{\lambda}] = e^{-\lambda+\lambda} = e^{0} = 1$$

Mean of Poisson distribution

- The mean or expectation of a Poisson distribution is given by

$$E[X] = \sum_{x=0}^{\infty} xP(X)$$

- It follows that

$$E[X] = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\boldsymbol{\lambda}^{x}}{x!}$$

- Therefore

$$E[X] = e^{-\lambda} \sum_{x=0}^{\infty} \frac{x\boldsymbol{\lambda}^{x}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{x\boldsymbol{\lambda}^{x}}{x(x-1)!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\boldsymbol{\lambda}^{x}}{(x-1)!}$$

Mean of Poisson distribution…

- It follows that

$$e^{-\lambda}\sum_{x=1}^{\infty}\frac{\boldsymbol{\lambda}^{x}}{(x-1)!}=e^{-\lambda}\left[\frac{\boldsymbol{\lambda}^{1}}{0!}+\frac{\boldsymbol{\lambda}^{2}}{1!}+\frac{\boldsymbol{\lambda}^{3}}{2!}+\cdots\right]$$

- Thus

$$e^{-\lambda}\left[\frac{\boldsymbol{\lambda}^{1}}{0!}+\frac{\boldsymbol{\lambda}^{2}}{1!}+\frac{\boldsymbol{\lambda}^{3}}{2!}+\cdots\right]=\boldsymbol{\lambda}e^{-\lambda}\left[\frac{1}{0!}+\frac{\boldsymbol{\lambda}^{1}}{1!}+\frac{\boldsymbol{\lambda}^{2}}{2!}+\cdots\right]$$

- But

$$\frac{1}{0!}+\frac{\boldsymbol{\lambda}^{1}}{1!}+\frac{\boldsymbol{\lambda}^{2}}{2!}+\cdots=1+\frac{\boldsymbol{\lambda}}{1!}+\frac{\boldsymbol{\lambda}^{2}}{2!}+\cdots=e^{\boldsymbol{\lambda}}$$

Mean of Poisson distribution…

- It follows that

$$\boldsymbol{\lambda} e^{-\lambda}\left[\frac{1}{0!}+\frac{\boldsymbol{\lambda}^1}{1!}+\frac{\boldsymbol{\lambda}^2}{2!}+\cdots\right]=\boldsymbol{\lambda} e^{-\lambda}\left[e^{\lambda}\right]=\boldsymbol{\lambda} e^{-\lambda+\lambda}=\boldsymbol{\lambda} e^{0}=\boldsymbol{\lambda}$$

- Therefore, $E[X]=\boldsymbol{\lambda}$

Variance of Poisson distribution

- The variance of a probability distribution is given by

$$Var(X) = E[X^2] - (E[X])^2$$

- This implies that

$$Var(X) = \sum_{x=1}^{\infty} x^2 \frac{e^{-\lambda}\boldsymbol{\lambda}^x}{x!} - \boldsymbol{\lambda}^2$$

- It follows that

$$Var(X) = \sum_{x=1}^{\infty} \frac{[(x-1)+1](x)e^{-\lambda}\boldsymbol{\lambda}^x}{x(x-1)!} - \boldsymbol{\lambda}^2$$

Variance of Poisson distribution…

- Thus

$$Var(X) = e^{-\lambda}\left[\sum_{x=1}^{\infty}\frac{(x-1)\boldsymbol{\lambda}^{x}}{(x-1)!}+\sum_{x=1}^{\infty}\frac{\boldsymbol{\lambda}^{x}}{(x-1)!}\right]-\boldsymbol{\lambda}^{2}$$

- Then

$$Var(X) = e^{-\lambda}\left[\sum_{x=1}^{\infty}\frac{(x-1)\boldsymbol{\lambda}^{x}}{(x-1)(x-2)!}+\sum_{x=1}^{\infty}\frac{\boldsymbol{\lambda}^{x}}{(x-1)!}\right]-\boldsymbol{\lambda}^{2}$$

- It follows that

$$Var(X) = e^{-\lambda}\left[\sum_{x=1}^{\infty}\frac{\boldsymbol{\lambda}^{x}}{(x-2)!}+\sum_{x=1}^{\infty}\frac{\boldsymbol{\lambda}^{x}}{(x-1)!}\right]-\boldsymbol{\lambda}^{2}$$

Variance of Poisson distribution…

$$Var(X) = e^{-\lambda}\left[\left(\frac{\lambda^2}{0!} + \frac{\lambda^3}{1!} + \frac{\lambda^4}{2!} + \cdots\right) + \left(\frac{\lambda^1}{0!} + \frac{\lambda^2}{1!} + \frac{\lambda^3}{2!} + \cdots\right)\right] - \lambda^2$$

- It follows that

$$Var(X) = e^{-\lambda}\left[\lambda^2\left(1 + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \cdots\right) + \lambda\left(1 + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \cdots\right)\right] - \lambda^2$$

- Thus,

$$Var(X) = e^{-\lambda}\left[\lambda^2\left(e^{\lambda}\right) + \lambda\left(e^{\lambda}\right)\right] - \lambda^2 = \lambda^2 e^0 + \lambda e^0 - \lambda^2 = \lambda$$

Geometric probability distribution

- Consider a fair die which is thrown successively cy taking into account the number of throws it takes to throw a 3
- The probability of success in each trial is $\frac{1}{6}$ and that of failure is $\frac{5}{6}$
- Assume that the experiment repeated until there is success
- Thus a random variable X in this kind of experiment is said to follow a geometric distribution if it assumes only non-negative values and its probability distribution is given by

$$P(X = x) = P(x) = q^{x-1}p$$

- Where $x = 1,2,3, \ldots$ and $q = 1 - p$ where p is the probability of success of an outcome.

Geometric probability distribution…

- q is the probability of failure of an outcome
- Any variable which follow geometric distribution is known as geometric variate
- **Note**: You need to be able to prove that geometric distribution is a probability mass function
- It is clear that $P(X = x) = q^{x-1}p \geq 0$ for all $x = 1,2,3, \ldots$
- Therefore we need show that

$$\sum_{x=1}^{\infty} P(X = x) = 1$$

Geometric probability distribution…

- It follows that

$$\sum_{x=1}^{\infty} P(X = x) = \sum_{x=1}^{\infty} q^{x-1}p = p\sum_{x=1}^{\infty} q^{x-1}$$

- Then,

$$p\sum_{x=1}^{\infty} q^{x-1} = p[q^0 + q^1 + q^2 + \cdots] = p[(1-q)^{-1}]$$

- It follows that

$$p[(1-q)^{-1}] = \frac{p}{1-q} = \frac{p}{p} = 1$$

Mean of Geometric distribution

- The mean or expected value of a geometric distribution is given by

$$E[X] = \sum_{x=1}^{\infty} xP(X = x)$$

- It follows that

$$E[X] = \sum_{x=1}^{\infty} xq^{x-1}p$$

- It follows that

$$\sum_{x=1}^{\infty} xq^{x-1}p \ = p[(1)q^0 + (2)q^1 + (3)q^2 + \cdots]$$

Mean of Geometric distribution…

- But

$$(1)q^0 + (2)q^1 + (3)q^2 + \cdots = (1-q)^{-2} = \frac{1}{(1-q)^2}$$

- Then

$$\sum_{x=1}^{\infty} xq^{x-1}p \ = p[(1)q^0 + (2)q^1 + (3)q^2 + \cdots] = \frac{p}{(1-q)^2}$$

- It implies,

$$E[X] = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

Variance of Geometric probability distribution

- The variance for the probability distribution is given by

$$Var(X) = E[X^2] - (E[X])^2$$

- It therefore follows that

$$Var(X) = \sum_{x=1}^{\infty} x^2 q^{x-1} p - \left(\frac{1}{p}\right)^2$$

- Expanding and simplifying this expression you get

$$Var(X) = \frac{q}{p^2}$$

Variance of Geometric probability distribution…

- It follows that

$$Var(X) = \sum_{x=1}^{\infty} x^2 q^{x-1} p - \left(\frac{1}{p}\right)^2 = p \sum_{x=1}^{\infty} x^2 q^{x-1} - \left(\frac{1}{p}\right)^2$$

- Thus

$$p \sum_{x=1}^{\infty} x^2 q^{x-1} - \left(\frac{1}{p}\right)^2 = p \sum_{x=0}^{\infty} [x(x-1) + x] q^{x-1}] - \left(\frac{1}{p}\right)^2$$

- It follows that

$$p[(2)(1)q^1 + (3)(2)q^2 + (4)(3)q^3 + \cdots] + \frac{1}{p} - \left(\frac{1}{p}\right)^2$$

Variance of Geometric probability distribution…

- It follows that

$$Var[X] = 2pq[1 + 3q + 6q^2 + 10q^3 + \cdots] + \frac{1}{p} - \left(\frac{1}{p}\right)^2$$

- Thus

$$Var[X] = 2pq(1 - q)^{-3} + \frac{1}{p} - \left(\frac{1}{p}\right)^2 = \frac{q}{p^2}$$

- Therefore

$$Var[X] = \frac{q}{p^2}$$

Moment generating function

- Moments are expected values of random variables
- The moment generating function is given by
$$M_x(t) = E[e^{tx}]$$
- It follows that
$$M_x(t) = \sum_{all\ x} e^{tx} P(X = x)$$
- $E[X^k]$ is the k^{th} derivative of $M_x(t)$ at $t = 0$.
- If $x = c$ then, $M_x(t) = E[e^{ct}] = e^{ct}$

Moment generating function…

Example

- Let X be the random variable with probability mass function (pmf)

$$P(X = x) = \frac{3}{4}\left(\frac{1}{4}\right)^{x} \; for\; x = 0,1,2,\ldots$$

find (a) $M_x(t)$.

(b)$E[X]$

Solution

(a)The moment generating function is given by

$$M_x(t) = E[e^{tx}] = \sum_{x=0}^{\infty} e^{tx} P(X = x)$$

Moment generating function…

- It follows that

$$\sum_{x=0}^{\infty} e^{tx} P(X = x) = \sum_{x=0}^{\infty} e^{tx} \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{x} = \frac{3}{4}\sum_{x=0}^{\infty} e^{tx} \left(\frac{1}{4}\right)^{x}$$

- This is the same as

$$\frac{3}{4}\sum_{x=0}^{\infty} e^{tx} \left(\frac{1}{4}\right)^{x} = \frac{3}{4}\sum_{x=0}^{\infty} \left(\frac{e^{t}}{4}\right)^{x}$$

- Expanding this expression we obtain a geometric series of the form

$$\frac{3}{4}\sum_{x=0}^{\infty} \left(\frac{e^{t}}{4}\right)^{x} = \frac{3}{4}\left[1 + \left(\frac{e^{t}}{4}\right)^{2} + \left(\frac{e^{t}}{4}\right)^{3} + \cdots\right]$$

Moment generating function…

- If $\frac{e^t}{4} < 1$, then

$$\left[1 + \left(\frac{e^t}{4}\right)^2 + \left(\frac{e^t}{4}\right)^3 + \cdots\right]$$

is the geometric series giving sum to infinity given by

$$1 + \left(\frac{e^t}{4}\right)^2 + \left(\frac{e^t}{4}\right)^3 + \cdots = \frac{1}{1 - \frac{e^t}{4}}$$

- Therefore

$$M_x(t) = \frac{3}{4}\left[\frac{1}{1 - \frac{e^t}{4}}\right] = \frac{\frac{3}{4}}{1 - \frac{e^t}{4}} = \frac{3}{4}\left(1 - \frac{e^t}{4}\right)^{-1}$$

Moment generating function…

(b)Since $E[X]$ if the $E[X^k]$ when $k = 1$, it implies that

$$E[X] = \frac{d}{dt}M_x(t)|_{t=0}$$

- Now

$$\frac{d}{dt}M_x(t) = \frac{\left(-\frac{3}{4}\right)}{\left(1 - \frac{e^t}{4}\right)^2}\left(-\frac{e^t}{4}\right) = \frac{\left(\frac{3}{4}\right)e^t}{4\left(1 - \frac{e^t}{4}\right)^2}$$

- Thus,

$$E[X] = \frac{d}{dt}M_x(t)|_{t=0} = \frac{\left(\frac{3}{4}\right)e^0}{4\left(1 - \frac{e^0}{4}\right)^2} = \frac{1}{3}$$

Moment generating function…

Important note

- From $M_x(t) = E[e^{tx}]$, by expanding e^{tx}

$$E[e^{tx}] = E[(tx)^0 + \frac{(tx)^1}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots]$$

- It implies that

$$E[e^{tx}] = 1 + tE[X] + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \cdots$$

- Therefore, $M_x(t)$ can also be written as

$$M_x(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E[X^k]$$

Moment generating function…

Example

The value of a piece of factory equipment after three years of use is $100\left(\frac{1}{2}\right)^{X}$ where X is a random variable having moment generating function

$$M_x(t) = \frac{1}{1-2t}$$

for $t < \frac{1}{2}$. Calculate the expected value of this piece of equipment after three years of use.

Moment generating function…

Solution

- Let $Y = 100\left(\frac{1}{2}\right)^X$
- Therefore,

$$E[Y] = \mathrm{E}\left[100\left(\frac{1}{2}\right)^X\right]$$

- It implies that

$$\mathrm{E}\left[100\left(\frac{1}{2}\right)^X\right] = 100E\left[\left(\frac{1}{2}\right)^X\right] = 100E\left[e^{\ln\left(\frac{1}{2}\right)^X}\right]$$

Moment generating function…

- It follows that

$$100E\left[e^{\left(\ln\left[\frac{1}{2}\right]\right)^{X}}\right] = 100E\left[\, e^{X\ln\left[\frac{1}{2}\right]}\right]$$

- We therefore see that

$$E[Y] = 100E\left[\, e^{X\ln\left[\frac{1}{2}\right]}\right] = 100M_x\left(\ln\left[\frac{1}{2}\right]\right)$$

- But we are given that

$$M_x(t) = \frac{1}{1-2t}$$

- It follows that

$$E[Y] = 100\left(\frac{1}{1-2\left(\ln\left[\frac{1}{2}\right]\right)}\right) = 41.906$$

Probability generating function

- This is a technique which is helpful in setting up some discrete models and finding their parameters
- If the probability of obtaining $0,1,2,,\ldots,n$ successes as the result of some experiment with the probability $p_{0,}p_{1,}p_{2,}\ldots,\ p_n$ respectively then we can form the expression
$$G(t) = p_0 + p_1 t + p_2 t^2 + \cdots + p_n t^n$$
- This expression is called the probability generating function for this particular experiment
- The dummy variable t has no real life meaning but the power of t acts as a label for the probability

Probability generating function…

- Example:
- The probability generating function for the sum of two dice $(2,3,4,\ldots,12)$ is

$$G(t)=\frac{1}{36}t^2+\frac{2}{36}t^3+\frac{3}{36}t^4+\cdots+\frac{1}{36}t^{11}+\frac{1}{36}t^{12}$$

- We find that when $t=1, G(1)=1$
- Consider the discrete probability distribution below

x	0	1	2	3	…	n
P(X=x)	p_0	p_1	p_2	p_3	…	p_n

Probability generating function…

- It follows that

$$G(t) = p_0 + p_1 t + p_2 t^2 + \cdots + p_n t^n$$

- We then find that

$$G'(1) = E[X], G''(1) = E[X^2] - E[X]$$

- We also fond that

$$Var[X] = E[X^2] - E^2[X] = G''(1) + G'(1) - [G'(1)]^2$$

Joint probability distribution

- The joint probability mass function (pmf) of the random variables X and Y is given by
- $p_{X,Y}(x, y) = P(X = x, Y = y)$ and satisfies the following properties

  (a)$0 \leq p_{X,Y}(x, y) \leq 1$

  (b)$\sum_{x,y \in S} p_{X,Y}(x, y) = 1$ where S is the support of the random

vector (X, Y)

Joint probability distribution…

Example

- Consider two random variables X and Y such that

$$X = \begin{cases} 0 & un - infected\ patient \\ 1 & infected\ patient \end{cases}$$

and

$$Y = \begin{cases} 0 & has\ negative\ diagnostic\ test \\ 1 & has\ positive\ diagnostic\ test \end{cases}$$

Identify the probabilities from the table below

Joint probability distribution…

		Negative diagnostic test	Positive diagnostic test
		0	1
Infected patient	0	0.8750	0.0233
Un-infected patient	1	0.0055	0.0963

Solution

(i) $P(X = 0, Y = 0) = 0.8750$

(ii) $P(X = 0, Y = 1) = 0.0233$

(iii) $P(X = 1, Y = 0) = 0.0055$

(iv) $P(X = 1, Y = 1) = 0.0963$

Characteristic function

- The characteristic function of a random variable X denoted by $Q_x(t)$ is given by

$$Q_x(t) = E[e^{itx}]$$

End of lecture 3