ST 1210: Introduction to Probability and Statistics

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Lecture-1a: Basic Probability

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Basic probability

1. Concept of probability
2. Axioms of probability
3. Probability theorems and proof
4. Conditional probability

Concept of probability

Probability

Is the term that relates to the chance that a particular event will occur when some experiment is performed

Experiment

Is any process that produces an *observation* or *outcome*

Sample space

Is the set of all possible outcomes of an experiment and it is denoted by S

Random experiment

Is an experiment with more than one possible outcome

Sample spaces and Events

1. If the outcome of the experiment is the gender of the child, then $S = \{g, b\}$
2. If the experiment consists of flipping two coins and noting whether they land heads (H) or tails (T), then $S = \{(H, H), (H, T), (T, H), (T, T)\}$
3. If an experiment consists of rolling a single die then $S = \{1, 2, 3, 4, 5, 6\}$

Event

Is the set of possible outcome(s) of an experiment. If E is an **Event** then the outcome of an experiment is contained in E

Examples

- For an experiment with $S = \{g, b\}$ then, the events may be $E_1 = \{g\}$ and $E_2 = \{b\}$

Concept of probability cont . . .

- For an experiment with $S = \{(H,H),(H,T),(T,H),(T,T)\}$ then, an event may be $E = \{(H,H),(H,T)\}$ which imply that E is an event that *the first coin lands on heads*

The Union of Events

The union of two events E_1 and E_2 denoted by $E_1 \cup E_2$ consists of all outcomes that are in E_1 or in E_2 or in both E_1 and E_2.

Example

- Let $E_1 = \{g\}$ and $E_2 = \{b\}$ then $E_1 \cup E_2 = \{g, b\}$ which is the whole sample space.

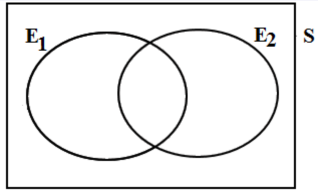
Null event

Is the event without any outcome and it is designated by $\emptyset$

Concept of probability cont ...

Venn diagram

Let E_1 and E_2 be two events and S be the sample space, then the Venn diagram representing these events and the sample space is illustrated as follows



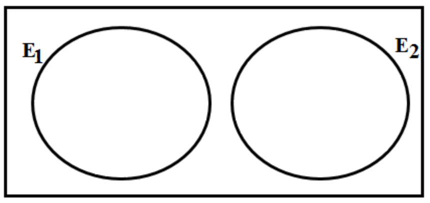
The Intersection of Events

The intersection of two events E_1 and E_2 denoted by $E_1 \cap E_2$ consists of all outcomes that are in both E_1 and E_2.

Concept of probability cont ...

Mutually exclusive or disjoint events

If E_1 and E_2 are any two events such that $E_1 \cap E_2 = \emptyset$ then E_1 and E_2 are called *mutually exclusive* or *disjoint* events.



Complement of an Event

For any event E, E^c is called a complement of an event E and it contains of all outcomes in the sample space which are not in E.

Concept of probability cont ...

Example

- If $E = \{g\}$ then $E^c = \{b\}$.

Note

The complement of a sample space is the null set, i.e. $S^c = \emptyset$.

Axioms of probability

Axioms of probability

Consider an experiment with a sample space S. For each event E there is a number $P(E)$ called the probability of an event with the following properties:

A1: The probability of E is a number between 0 and 1, i.e. $0 \leq P(E) \leq 1$

A2: The probability of the sample space is 1, i.e. $P(S) = 1$

A3: The probability of union of independent events is equal to the sum of probabilities of these events, i.e. if $E_1, E_2, \ldots, E_n$ are disjoint then $P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$

Note: $P(E)$ is the probability that an outcome of the experiment is contained in E.

Axioms of probability cont . . .

- Axiom-1 indicates that the probability that the outcome of the experiment is in E is some value between 0 and 1.
- Axiom-2 indicates that with probability 1, the outcome of the experiment will be an element of the sample space.
- Axiom-3 indicates that if $E_1, E_2, \ldots, E_n$ can not simultaneously occur then the probability that the outcome of the experiment is contained in either E_1 or E_2 or ... or E_n is equal to the sum of probability that it is in $E_1, E_2, \ldots, E_n$.
- With E and E^c being disjoint events, implies $E \cup E^c = S$
- By Axiom-2, $P(E \cup E^c) = P(S) = 1$
- But by Axiom-3, $P(E \cup E^c) = P(E) + P(E^c)$
- It follows that $1 = P(E) + P(E^c)$ which implies that $P(E^c) = 1 - P(E)$

Axioms of probability cont ...

Note

$P(E^c) = 1 - P(E)$ indicates that the probability that an outcome is not contained in E equals 1 minus the probability that it is contained in E.

Example

If the probability of obtaining head on the toss of a coin is 0.4, then the probability of obtaining a tail is 0.6.

Note

The probability of the union of events which are not necessarily disjoint is given by $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$. This is referred to as *addition rule of probability*.

Example

A certain retail establishment accepts EXIM or VISA credit card. A total of 22% of its customers carry an EXIM card, 58% carry a VISA credit card and 14% carry both. What is the probability that a customer will have atleast one of these cards?

Solution

- Let E_1 be an event that the customer has EXIM credit card and E_2 be an event that the customer has a VISA credit card
- It implies that $P(E_1) = 0.22$, $P(E_2) = 0.58$ and $P(E_1 \cap E_2) = 0.14$
- By additive rule
  $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$
- It implies that
  $P(E_1 \cup E_2) = 0.22 + 0.58 - 0.14 = 0.66$
- Therefore, 66% has atleast one of the two cards.

Axioms of probability cont . . .

Experiments having equally likely outcomes

If an experiment has equally likely outcomes with the sample space $S = \{1, 2, ..., N\}$ then $P(\{1\}) = P(\{2\}) = \ldots = P(\{N\})$. Therefore, $P(A) = \frac{n(A)}{N}$

Example

In a survey of 420 people in retirement centres, 144 are found to be smokers and 276 are not. If a member is selected in such a way that each of the members is likely to be one selected, what is the probability that a person is a smoker?

- This experiment has 420 possible outcomes and 144 outcomes in an event
  $P(\{\text{smoker}\}) = \frac{144}{420} = \frac{12}{35}$

Axioms of probability cont ...

Example

Consider an experiment which consists of rolling two dice. Find the probability that the sum of numbers showing up is 6.

Solution

- Let E be an event that the numbers showing up sum to 6.
- The sample space has 36 outcomes which are equally likely to occur.
- Thus, $P(E) = \frac{n(E)}{n(S)} = \frac{5}{36}$

Exercise

One man and one woman are to be selected from a group that consists of 10 married couples. If all possible selections are equally likely, what is the probability that the woman and the man selected are married to each other.

Probability theorems and proof

Theorem

Is a general proposition not self-evident but proved by a chain of reasoning; a truth established by means of accepted truths or a rule in algebra or other branches of mathematics expressed by symbols or formulae.

Theorem

The probability of impossible event is 0. i.e. $P(\emptyset) = 0$

Proof

- Consider the sample space S and and and event $\emptyset$
- Then S and $\emptyset$ are mutually exclusive events
- But $S \cup \emptyset = S$, it therefore implies that $P(S \cup \emptyset) = P(S)$
- By Axiom-2, $P(S) = 1$, it means $P(S \cup \emptyset) = P(S) = 1$
- Thus, $P(S \cup \emptyset) = 1$

Probability theorems and proof cont ...

- By Axiom-3, $P(S \cup \emptyset) = P(S) + P(\emptyset)$, it implies that $P(S) + P(\emptyset) = 1$
- But by Axiom-2, $P(S) = 1$, which implies $1 + P(\emptyset) = 1$
- Therefore, $P(\emptyset) = 0$ $\blacksquare$

Theorem

If S is the sample space and E is any event of the experiment, then $P(E^c) = 1 - P(E)$

Proof

- Since E and E^c are mutually exclusive, it implies $E \cup E^c = S$
- It follows that $P(E \cup E^c) = P(S)$
- By Axiom-2,$P(S) = 1$, which implies $P(E \cup E^c) = 1$
- By Axiom-3, $P(E \cup E^c) = P(E) + P(E^c)$, implies $P(E) + P(E^c) = 1$
- Therefore, $P(E^c) = 1 - P(E)$ $\blacksquare$

Conditional probability

Definition

When we determine a probability of an event given some partial information about the outcome of the experiment, such probability is referred to as *conditional probability*.

Example

Consider an experiment which consists of rolling two dice, what is the probability of an event that the two dice sum to 10 given that the first die lands on 4?

Solution

- Let E_1 denote the event that the first die lands on 4 and E_2 be the event that the sum of dice is 10.
- The probability of E_2 given that E_1 has occurred is the conditional probability denoted by $P(E_2|E_1)$ is defined by

Conditional probability cont …

$$\begin{aligned} P(E_2|E_1) &= \tfrac{P(E_1 \cap E_2)}{P(E_1)} \\ &= \tfrac{P(\{(4,6)\})}{P(\{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\})} \\ &= \tfrac{\frac{1}{36}}{\frac{6}{36}} \\ &= \frac{1}{6} \end{aligned}$$

Multiplication rule

The multiplication rule is given by $P(E_1 \cap E_2) = P(E_1)P(E_2|E_1)$. This implies that the probability that both E_1 and E_2 occur is equal to the probability that E_1 occurs multiplied by the conditional probability of E_2 given that E_1 occurs.

- It is useful for computing the probability of an intersection.

Conditional probability cont ...

Example

Suppose that two people are randomly chosen from a group of 4 women and 6 men

(a) What is the probability that both are women?

(b) What is the probability that one is a woman and the other is a man?

Solution (a)

- Let E_1 and E_2 denote respectively the events that the first person selected is a woman and that the second person selected is a woman
- It implies that $P(E_1 \cap E_2) = P(E_1)P(E_2|E_1)$, but $P(E_1) = \frac{4}{10}$
- Given that the first person selected is a woman, it follows that $P(E_2|E_1) = \frac{3}{9}$ and also $P(E_1 \cap E_2) = \frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$

Conditional probability cont ...

Solution (b)

- Let E_1 denote an event that the first person chosen is a man and E_2 denote an event that the second person chosen is a woman
- It implies that $P(E_1) = \frac{6}{10}$ and $P(E_2|E_1) = \frac{4}{9}$
- From $P(E_1 \cap E_2) = P(E_1)P(E_2|E_1)$,
  $P(\text{man then woman}) = P(E_1 \cap E_2) = \frac{6}{10} \times \frac{4}{9} = \frac{4}{15}$
- Also $P(\text{woman then man}) = \frac{4}{10} \times \frac{6}{10} = \frac{4}{15}$
- Since the pair chosen consists of a man and a woman then
  $P(1 \text{ woman and } 1 \text{ man}) = \frac{4}{15} + \frac{4}{15} = \frac{8}{15}$

Note

An event E_2 is independent of E_1 when $P(E_2|E_1) = P(E_2)$

Conditional probability cont ...

- Since $P(E_1 \cap E_2) = P(E_1)P(E_2|E_1)$, we see that E_2 is independent of E_1 when
  $P(E_1 \cap E_2) = P(E_1)P(E_2)$

Note

If E_1 and E_2 are independent events then $P(E_2|E_1^c) = P(E_2)$

Example

Suppose we roll a pair of fair dice, so each of the 36 possible outcomes is equally likely, let E_1 denote the event that the first die lands on 3, let E_2 be an event that the sum of the dice is 8 and let E_3 be the event that the sum of dice is 7,

(a) are E_1 and E_2 independent?

(b) are E_1 and E_3 independent?

Conditional probability cont ...

Solution (a)

- Since $E_1 \cap E_2$ is the event that the first die lands on 3 and the second die lands on 5, it follows that
  $P(E_1 \cap E_2) = P(\{(3,5)\}) = \frac{1}{36}$
- On the other hand
  $P(E_1) = P(\{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\}) = \frac{6}{36}$
- Likewise
  $P(E_2) = P(\{(2,6),(3,5),(4,4),(5,3),(6,2)\}) = \frac{5}{36}$
- Since $\frac{1}{36} \neq \frac{6}{36} \times \frac{5}{36}$, it implies $P(E_1 \cap E_2) \neq P(E_1)P(E_2)$ and so E_1 and E_2 are not independent

Solution (b)

Checking the independence of Events E_1 and E_3

- $P(E_1 \cap E_3) = P(\{(3,4)\}) = \frac{1}{36}$

Conditional probability cont ...

- $P(E_1) = \frac{1}{6}$ and $P(E_3) = P(\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}) = \frac{6}{36}$
- Thus, $P(E_1)P(E_3) = \frac{1}{6} \times \frac{6}{36} = \frac{6}{216} = \frac{1}{36} = P(E_1 \cap E_3)$
- Therefore, $P(E_1 \cap E_3) = P(E_1)P(E_3)$ and so E_1 and E_3 are independent events.

Note

If $A_1, A_2, \ldots, A_n$ are independent events, then $P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i)$

Example

A couple is planning on having 3 children. Assuming that each child is equally likely to be of either sex, and that the sexes of children are independent, find the probability that

(a) all three children will be girls

Conditional probability cont ...

(b) *atleast one child will be a girl*

Solution(a)

- Let A_i be an event that their $i^{th}-$child is a girl
- $$\begin{aligned} P(\text{all girls}) &= P(A_1 \cap A_2 \cap A_3) \\ &= P(A_1)P(A_2)P(A_3) \\ &= \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{8} \end{aligned}$$
- Therefore, $P(\text{all girls}) = \frac{1}{8}$

Solution(b)

- Let A_i be an event that their $i^{th}-$child is a boy
- $$\begin{aligned} P(\text{all boys}) &= P(A_1 \cap A_2 \cap A_3) \\ &= P(A_1)P(A_2)P(A_3) \\ &= \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{8} \end{aligned}$$
- Thus, $P(\text{all boys}) = \frac{1}{8}$
- Now, $P(\text{atleast 1 girl}) = 1 - P(\text{all boys}) = 1 - \frac{1}{8} = \frac{7}{8}$

Example

If A and B are independent events such that $P(B) \neq 0$, prove that A and B are independent if and only if $P(A|B) = P(A)$.

Proof

- **Case I**: Given that A and B are independent events, required to prove that $P(A|B) = P(A)$.
- Since A and B are independent, it implies that $P(A \cap B) = P(A)P(B)$
- Since $P(B) \neq 0$, it follows that $\frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$
- But $\frac{P(A \cap B)}{P(B)} = P(A|B)$, hence $P(A|B) = P(A)$

Conditional probability cont ...

- **Case II:** Suppose $P(A|B) = P(A)$, then required to show that A and B are independent
- Now if $P(A|B) = P(A)$, it implies $\frac{P(A \cap B)}{P(B)} = P(A)$
- Multiplying throughout by $P(B)$, it follows that
- Therefore, $(A \cap B) = P(A)P(B)$, which is the condition required for independent events ■

Thank you for your Attention.