ST 1210 Introduction to Probability and statistics

Lecture 2: Discrete Outcomes

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Random variables

Definition 1

If $\mathcal{S}$ is a sample space with a probability measure and X is a real-valued function defined over the elements of $\mathcal{S}$, then X is called a *random variable* (or *stochastic variable*).
In this course we shall always denote random variables by capital letters such as X, Y etc., and their *values* by the corresponding lowercase letters such as x and y, respectively.

Random variables cont ...

Example 2

Suppose that a coin is tossed twice so that the sample space is $\mathcal{S} = \{HH, HT, TH, TT\}$. Let X represent the number of heads that can come up. With each sample point we can associate a number for X as shown in the table:

Sample Point	HH	HT	TH	TT
Probability	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
x	2	1	1	0

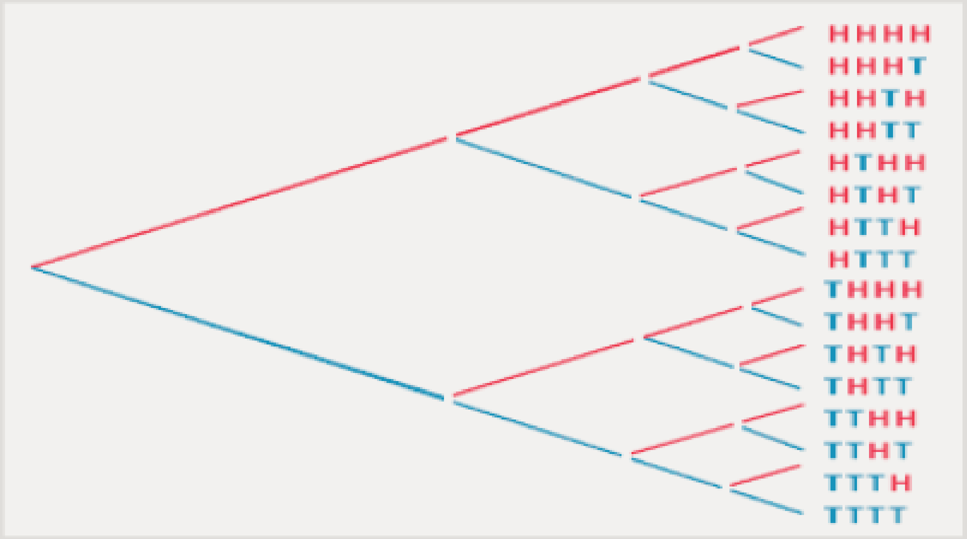
Thus, for example, in the case of HH (i.e., 2 heads), $X = 2$ while for TH (1 head), $X = 1$. It follows that X is a random variable. Also, we can write $P(X = 2) = \frac{1}{4}$, $P(X = 1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, and $P(X = 0) = \frac{1}{4}$.

Random variables cont ...

Example 3

A balanced coin is tossed four times. List the elements of the sample space that are presumed to be equally likely, as this is what we mean by a coin being balanced, and the corresponding values x of the random variable X, the total number of heads.

Solution. If H and T stand for heads and tails, the results are as shown in the following table:



Random variables cont ...

Elements of sample space	Probability	x		Elements of sample space	Probability	x
$HHHH$	$\frac{1}{16}$	4		$THHT$	$\frac{1}{16}$	2
$HHHT$	$\frac{1}{16}$	3		$THTH$	$\frac{1}{16}$	2
$HHTH$	$\frac{1}{16}$	3		$TTHH$	$\frac{1}{16}$	2
$HTHH$	$\frac{1}{16}$	3		$HTTT$	$\frac{1}{16}$	1
$THHH$	$\frac{1}{16}$	3		$THTT$	$\frac{1}{16}$	1
$HHTT$	$\frac{1}{16}$	2		$TTHT$	$\frac{1}{16}$	1
$HTHT$	$\frac{1}{16}$	2		$TTTH$	$\frac{1}{16}$	1
$HTTH$	$\frac{1}{16}$	2		$TTTT$	$\frac{1}{16}$	0

Thus, we can write $P(X=0)=\frac{1}{16}$, $P(X=1)=\frac{4}{16}$, $P(X=2)=\frac{6}{16}$, $P(X=3)=\frac{4}{16}$ and $P(X=4)=\frac{1}{16}$.

Random variables cont ...

Example 4

Two socks are selected at random and removed in succession from a drawer containing five brown socks and three green socks. List the elements of the sample space, the corresponding probabilities, and the corresponding values x of the random variable X is the number of brown socks selected.

Solution. If B and G stand for brown and green, then we have following probabilities

$$P(BB) = \frac{5}{8} \cdot \frac{4}{7} = \frac{20}{56}, P(BG) = \frac{5}{8} \cdot \frac{3}{7} = \frac{15}{56},$$

$$P(GB) = \frac{3}{8} \cdot \frac{5}{7} = \frac{15}{56}, \text{ and } P(GG) = \frac{3}{8} \cdot \frac{2}{7} = \frac{6}{56},$$

and the results are shown in the following table:

Elements of Sample Space	BB	BG	GB	GG
Probability	20/56	15/56	15/56	6/56
x	2	1	1	0

Random variables cont ...

Definition 5

If X is a discrete random variable, the function given by

$$f(x) = P(X = x)$$

for each x within the range of X is called the *probability distribution* (or *probability function*) of X.

Based on the postulates of probability, it immediately follows that

Theorem 6

A function can serve as the probability distribution of a discrete random variable X if and only if its values, $f(x)$, satisfy the conditions

1. $f(x) \geq 0$ *for each value within its domains;*
2. $\sum_x f(x) = 1$, *where the summation extends over all the values within its domain.*

Random variables cont ...

Example 7

Find a formula for the probability distribution of the total number of heads obtained in four tosses of a balanced coin.

Solution. We know that $P(X = 0) = \frac{1}{16}$, $P(X = 1) = \frac{4}{16}$, $P(X = 2) = \frac{6}{16}$, $P(X = 3) = \frac{4}{16}$ and $P(X = 4) = \frac{1}{16}$. Observing that the numerators of these five fractions, 1, 4, 6, 4, and 1, are the binomial coefficients $\binom{4}{0}$, $\binom{4}{1}$, $\binom{4}{2}$, $\binom{4}{3}$, and $\binom{4}{4}$, we find that the formula for the probability distribution can be written as

$$f(x) = \frac{\binom{4}{x}}{16} \text{ for } x = 0, 1, 2, 3, 4.$$

Random variables cont ...

Example 8

Check whether the function given by

$$f(x) = \frac{x+2}{25} \text{ for } x = 1, 2, 3, 4, 5$$

can serve as the probability distribution of a discrete random variable.

Solution. Substituting the different values of x, we get $f(1) = \frac{3}{25}$, $f(2) = \frac{4}{25}$, $f(3) = \frac{5}{25}$, $f(4) = \frac{6}{25}$, and $f(5) = \frac{7}{25}$. Since these values are all nonnegative, the first condition of Theorem 6 is satisfied, and since

$$f(1) + f(2) + f(3) + f(4) + f(5) = \frac{3}{25} + \frac{4}{25} + \frac{5}{25} + \frac{6}{25} + \frac{7}{25} = 1$$

the second conditions of Theorem 6 is satisfied. Thus, the given function can serve as the probability distribution of a random variable having the range $\{1, 2, 3, 4, 5\}$.

Random variables cont …

Example 9

Suppose that a pair of fair dice are to be tossed, and let the random variable X denote the sum of the points. Obtain the probability distribution for X.

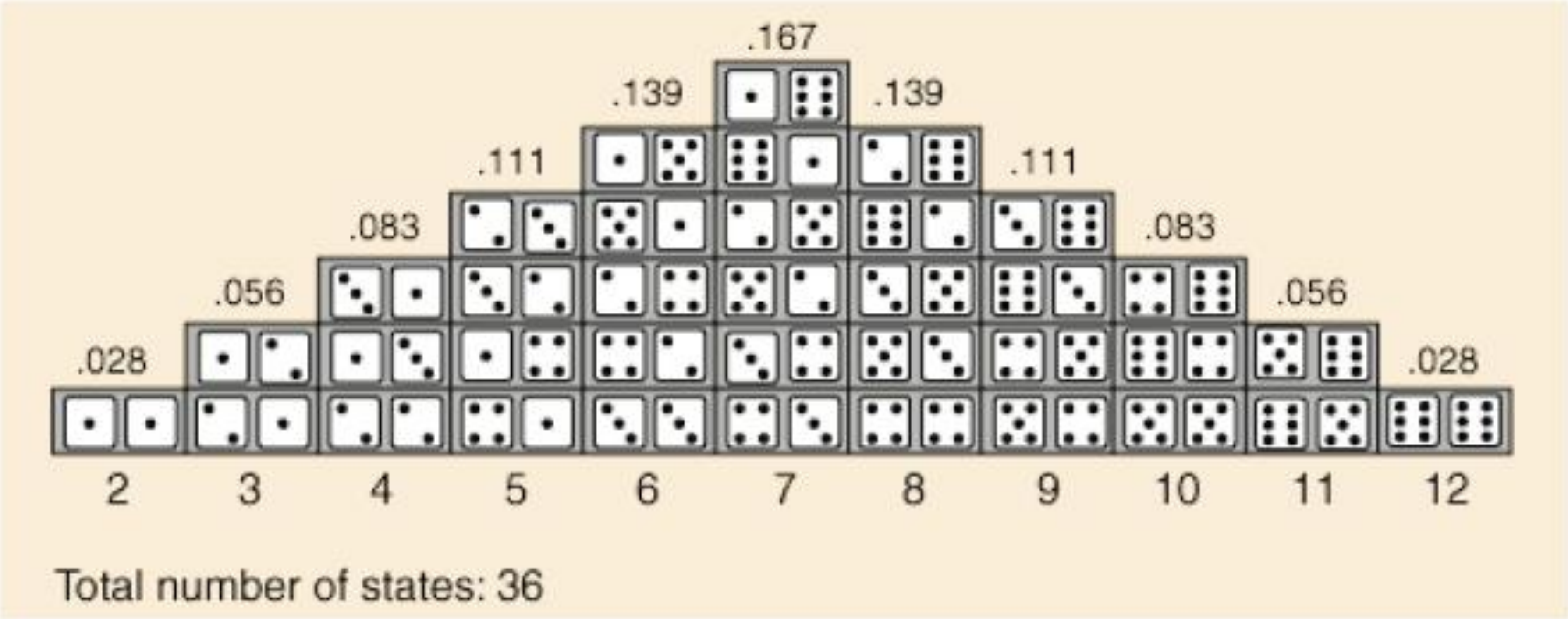
Solution. The random variable X is the sum of the coordinates for each point. Thus for $(3, 2)$ we have $X = 5$. Using the fact that all 36 sample points are equally probable, so that each sample point has probability $1/36$.

First Die

Second Die	1	2	3	4	5	6
1	1,1 2	1,2 3	1,3 4	1,4 5	1,5 6	1,6 7
2	2,1 3	2,2 4	2,3 5	2,4 6	2,5 7	2,6 8
3	3,1 4	3,2 5	3,3 6	3,4 7	3,5 8	3,6 9
4	4,1 5	4,2 6	4,3 7	4,4 8	4,5 9	4,6 10
5	5,1 6	5,2 7	5,3 8	5,4 9	5,5 10	5,6 11
6	6,1 7	6,2 8	6,3 9	6,4 10	6,5 11	6,6 12

The one is red is the sum

Random variables cont …



x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Random variables cont …

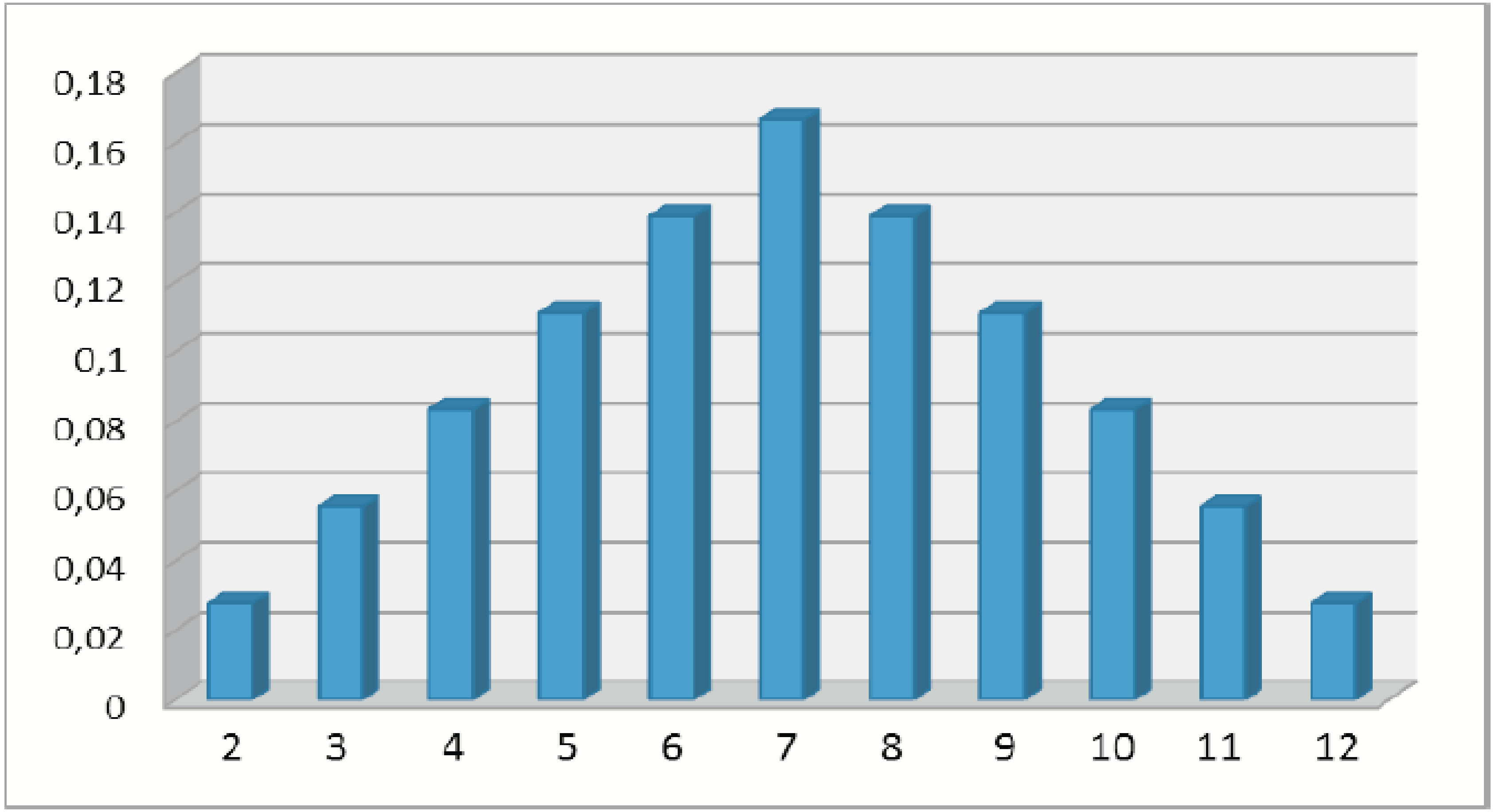


Figure 1 : Probability Bar Chart

Random variables cont …

Definition 10

If X is a discrete random variable, the function given by

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t) \text{ for } -\infty < x < \infty$$

where $f(t)$ is the value of the probability distribution of X at t, is called the *distribution function*, or the *cumulative distribution*, of X.

Random variables cont ...

Based on the postulates of probability and some of their immediate consequences, it follows that

Theorem 11

The values $F(x)$ of the distribution function of a discrete random variable X satisfy the conditions

1. $F(-\infty) = 0$ *and* $F(\infty) = 1$;
2. *if* $a < b$, *then* $F(a) \leq F(b)$ *for any real numbers* a *and* b.

If we are given the probability distribution of a discrete random variable, the corresponding distribution function is generally easy to find.

Random variables cont ...

Example 12

Find the distribution function of the total of heads obtained in four tosses of a balanced coin.

Solution. Given $f(0) = \frac{1}{16}$, $f(1) = \frac{4}{16}$, $f(2) = \frac{6}{16}$, $f(3) = \frac{4}{16}$, and $f(4) = \frac{1}{16}$ from Example 3, it follows that

$$F(0) = f(0) = \frac{1}{16},$$

$$F(1) = f(0) + f(1) = \frac{5}{16},$$

$$F(2) = f(0) + f(1) + f(2) = \frac{11}{16},$$

$$F(3) = f(0) + f(1) + f(2) + f(3) = \frac{15}{16},$$

$$F(4) = f(0) + f(1) + f(2) + f(3) + f(4) = 1.$$

Random variables cont …

Hence, the distribution function is given by

$$F(x) = \begin{cases} 0 & \text{for } x < 0, \\ \frac{1}{16} & \text{for } 0 \leq x < 1, \\ \frac{5}{16} & \text{for } 1 \leq x < 2, \\ \frac{11}{16} & \text{for } 2 \leq x < 3, \\ \frac{15}{16} & \text{for } 3 \leq x < 4, \\ 1 & \text{for } x \geq 4. \end{cases}$$

Observe that this distribution function is defined not only for the values taken on by the given random variable, but for all real numbers. For instance, we can write $F(1.7) = \frac{5}{16}$ and $F(100) = 1$, although the probabilities of getting at most 1.7 heads or at most 100 heads in four tosses of a balanced coin may not be of any real significance.

Random variables cont …

Example 13

Find the distribution function of the random variable X of Example 4 and plot its graph.

Solution. Based on the probabilities given in the following table

Elements of Sample Space	BB	BG	GB	GG
Probability	$20/56$	$15/56$	$15/56$	$6/56$
x	2	1	1	0

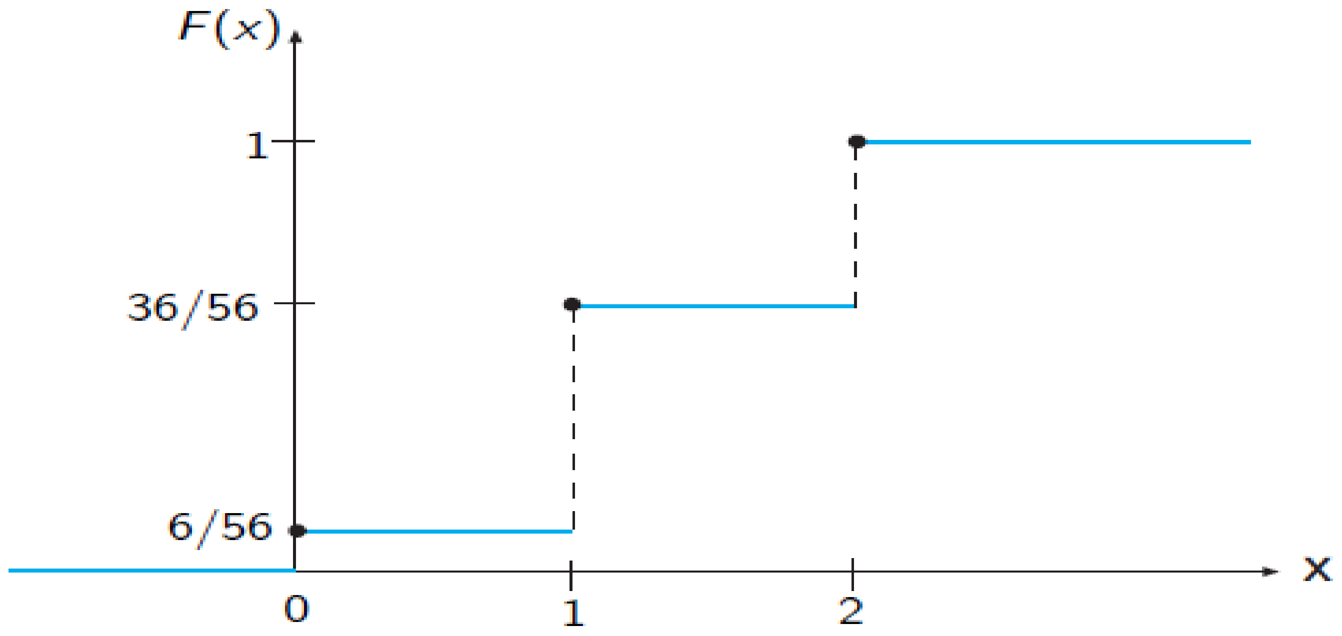
we can write $f(0) = \frac{6}{56}$, $f(1) = \frac{15}{56} + \frac{15}{56} = \frac{30}{56}$, and $f(2) = \frac{20}{56}$, so that

$$F(0) = f(0) = \frac{6}{56},$$
$$F(1) = f(0) + f(1) = \frac{36}{56},$$
$$F(2) = f(0) + f(1) + f(2) = 1.$$

Random variables cont ...

Hence, the distribution function of X is given by

$$F(x) = \begin{cases} 0 & \text{for } x < 0, \\ \frac{6}{56} & \text{for } 0 \leq x < 1, \\ \frac{36}{56} & \text{for } 1 \leq x < 2, \\ 1 & \text{for } x \geq 2. \end{cases}$$



Random variables cont ...

Example 14

Find the distribution function of the random variable that has the probability distribution

$$f(x) = \frac{x}{15} \text{ for } x = 1, 2, 3, 4, 5.$$

Solution. Since $f(1) = \frac{1}{15}$, $f(2) = \frac{2}{15}$, $f(3) = \frac{3}{15}$, $f(4) = \frac{4}{15}$, and $f(5) = \frac{5}{15}$, then

$$F(x) = \begin{cases} 0 & \text{for } x < 1, \\ \frac{1}{15} & \text{for } 1 \leq x < 2, \\ \frac{3}{15} & \text{for } 2 \leq x < 3, \\ \frac{6}{15} & \text{for } 3 \leq x < 4, \\ \frac{10}{15} & \text{for } 4 \leq x < 5, \\ 1 & \text{for } x \geq 5 \end{cases}$$

Random variables cont ...

Theorem 15

If the range of a random variable X consists of the values $x_1 < x_2 < x_3 < \cdots < x_n$, then $f(x_1) = F(x_1)$ and

$$f(x_i) = F(x_i) - F(x_{i-1}) \text{ for } i = 2, 3, \cdots, n.$$

Example 16

If X has the distribution function $F(1) = 0.25$, $F(2) = 0.61$, $F(3) = 0.83$, and $F(4) = 1$ for $x = 1, 2, 3, 4$, find the probability distribution of X.

Solution. We have

$$\begin{aligned}
f(1) &= F(1) = 0.25, \\
f(2) &= F(2) - F(1) = 0.61 - 0.25 = 0.36, \\
f(3) &= F(3) - F(2) = 0.83 - 0.61 = 0.22, \\
f(4) &= F(4) - F(3) = 1 - 0.83 = 0.17.
\end{aligned}$$

Random variables cont ...

Example 17

If X has the distribution function

$$F(x) = \begin{cases} 0 & \text{for } x < -1, \\ \frac{1}{4} & \text{for } \; -1 \leq x < 1, \\ \frac{1}{2} & \text{for } 1 \leq x < 3, \\ \frac{3}{4} & \text{for } 3 \leq x < 5, \\ 1 & \text{for } x \geq 5. \end{cases}$$

find

1. $P(X \leq 3)$, $P(X = 3)$, $P(X < 3)$;
2. $P(X \geq 1)$;
3. $P(-0.4 < X < 4)$;
4. $P(X = 5)$;
5. the probability distribution of X.

Random variables cont ...

$$F(x) = \begin{cases} 0 & \text{for } x < -1, \\ \frac{1}{4} & \text{for } -1 \leq x < 1, \\ \frac{1}{2} & \text{for } 1 \leq x < 3, \\ \frac{3}{4} & \text{for } 3 \leq x < 5, \\ 1 & \text{for } x \geq 5. \end{cases}$$

Solution.

1.

$$P(X \leq 3) = \frac{3}{4}$$

$$P(X = 3) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$$

$$P(X < 3) = \frac{1}{2}$$

Random variables cont ...

$$F(x) = \begin{cases} 0 & \text{for } x < -1, \\ \frac{1}{4} & \text{for } -1 \leq x < 1, \\ \frac{1}{2} & \text{for } 1 \leq x < 3, \\ \frac{3}{4} & \text{for } 3 \leq x < 5, \\ 1 & \text{for } x \geq 5. \end{cases}$$

2. $P(X \geq 1) = 1 - P(X < 1) = 1 - \frac{1}{4} = \frac{3}{4}$.
3. $P(-0.4 < X < 4) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$.
4. $P(X = 5) = 1 - \frac{3}{4} = \frac{1}{4}$.
5. $f(-1) = \frac{1}{4}$, $f(1) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$, $f(3) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$, $f(5) = 1 - \frac{3}{4} = \frac{1}{4}$, and 0 elsewhere.

Binomial distribution

Binomial distribution. We say that X has the binomial distribution with parameters n and p if X takes values in $\{0, 1, \ldots, n\}$ and

$$\mathbb{P}(X = k) = \binom{n}{k} p^k q^{n-k} \qquad \text{for } k = 0, 1, 2, \ldots, n.$$

Note that (2.14) gives rise to a mass function satisfying (2.6) since, by the binomial theorem,

$$\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = (p+q)^n = 1.$$

Poisson distribution

Poisson distribution. We say that X has the Poisson distribution with parameter λ (> 0) if X takes values in $\{0, 1, 2, \ldots\}$ and

$$\mathbb{P}(X = k) = \frac{1}{k!}\lambda^k e^{-\lambda} \qquad \text{for } k = 0, 1, 2, \ldots.$$

Again, this gives rise to a mass function since

$$\sum_{k=0}^{\infty} \frac{1}{k!}\lambda^k e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{1}{k!}\lambda^k = e^{-\lambda} e^{\lambda} = 1.$$

Geometric distribution

Geometric distribution. We say that X has the geometric distribution with parameter $p \in (0, 1)$ if X takes values in $\{1, 2, 3, \dots\}$ and

$$\mathbb{P}(X = k) = pq^{k-1} \qquad \text{for } k = 1, 2, 3, \dots.$$

As before, note that

$$\sum_{k=1}^{\infty} pq^{k-1} = \frac{p}{1-q} = 1.$$

Expectation

Consider a fair die. If it were thrown a large number of times, each of the possible outcomes $1, 2, \ldots, 6$ would appear on about one-sixth of the throws, and the average of the numbers observed would be approximately

$$\tfrac{1}{6} \cdot 1 + \tfrac{1}{6} \cdot 2 + \cdots + \tfrac{1}{6} \cdot 6 = \tfrac{7}{2},$$

which we call the *mean value*. This notion of mean value is easily extended to more general distributions as follows.

Definition *If X is a discrete random variable, the expectation of X is denoted by $\mathbb{E}(X)$ and defined by*

$$\mathbb{E}(X) = \sum_{x \in \operatorname{Im} X} x\mathbb{P}(X = x)$$

whenever this sum converges absolutely, in that $\sum_x |x\mathbb{P}(X = x)| < \infty$.

Expectation…

- This function can also be written as

$$\mathbb{E}(X) = \sum_{x} x\,\mathbb{P}(X = x)$$

- The expectation of X is often called the *expected value* or *mean* of X
- The physical analogy of '*expectation*' is the idea of '*centre of gravity*'

Expectation…

Theorem 2.30 *Let X be a discrete random variable and let $a, b \in \mathbb{R}$.*

(a) *If $\mathbb{P}(X \geq 0) = 1$ and $\mathbb{E}(X) = 0$, then $\mathbb{P}(X = 0) = 1$.*
(b) *We have that $\mathbb{E}(aX + b) = a\mathbb{E}(X) + b$.*

Example Suppose that X is a random variable with the Poisson distribution, parameter λ, and we wish to find the expected value of $Y = e^X$.

$$\begin{aligned}
\mathbb{E}(Y) &= \mathbb{E}(e^X) \\
&= \sum_{k=0}^{\infty} e^k \mathbb{P}(X = k) = \sum_{k=0}^{\infty} e^k \frac{1}{k!} \lambda^k e^{-\lambda} \\
&= e^{-\lambda} \sum_{k=0}^{\infty} \frac{1}{k!} (\lambda e)^k = e^{\lambda(e-1)}.
\end{aligned}$$

Expectation…

The expectation $\mathbb{E}(X)$ of a discrete random variable X is an indication of the 'centre' of the distribution of X. Another important quantity associated with X is the 'variance' of X, and this is a measure of the degree of dispersion of X about its expectation $\mathbb{E}(X)$.

Definition *The variance* $\operatorname{var}(X)$ *of a discrete random variable* X *is defined by*

$$\operatorname{var}(X) = \mathbb{E}\left([X - \mathbb{E}(X)]^2\right).$$

Expectation…

The equation above is not always the most convenient way to calculate the variance of a discrete random variable. We may expand the term $(x-\mu)^2$ in to obtain

$$\begin{aligned}\operatorname{var}(X) &= \sum_x (x^2 - 2\mu x + \mu^2)\mathbb{P}(X = x)\\ &= \sum_x x^2\mathbb{P}(X = x) - 2\mu \sum_x x\mathbb{P}(X = x) + \mu^2 \sum_x \mathbb{P}(X = x)\\ &= \mathbb{E}(X^2) - 2\mu^2 + \mu^2 \qquad \text{by (2.28) and (2.6)}\\ &= \mathbb{E}(X^2) - \mu^2,\end{aligned}$$

where $\mu = \mathbb{E}(X)$ as before. Thus we obtain the useful formula

$$\operatorname{var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2.$$

Expectation…

Example If X has the geometric distribution with parameter p $(= 1 - q)$, the mean of X is

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} kpq^{k-1}$$

$$= \frac{p}{(1-q)^2} = \frac{1}{p},$$

and the variance of X is

$$\operatorname{var}(X) = \sum_{k=1}^{\infty} k^2 pq^{k-1} - \frac{1}{p^2}$$

Expectation…

It then follows that

$$\sum_{k=1}^{\infty} k^2 q^{k-1} = q \sum_{k=1}^{\infty} k(k-1) q^{k-2} + \sum_{k=1}^{\infty} k q^{k-1}$$

$$= \frac{2q}{(1-q)^3} + \frac{1}{(1-q)^2}$$

Therefore,

$$\operatorname{var}(X) = p\left(\frac{2q}{p^3} + \frac{1}{p^2}\right) - \frac{1}{p^2}$$

$$= qp^{-2}.$$

Expectation…

Exercise

1. If X has the binomial distribution with parameters n and $p = 1 - q$, show that
   $\mathbb{E}(X) = np, \qquad \mathbb{E}(X^2) = npq + n^2p^2,$ and deduce the variance of X.

2. Show that $\mathrm{var}(aX + b) = a^2 \mathrm{var}(X)$ for $a, b \in \mathbb{R}$.

3. Find $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$ when X has the Poisson distribution with parameter λ, and hence show that the Poisson distribution has variance equal to its mean.