Objective: Determine how the components of stress are transformed under a rotation about the z-axis.

As discussed in Lectures 1-2, the stress state at a point can be described by six components of stress. Three normal stresses σ_{xx}, σ_{yy}, and σ_{zz} and three shear stresses σ_{xy}, σ_{yz}, and σ_{xz}. Recall from Lectures 1-2 that the stress components depend on the coordinate axes used. This derivation will determine how the components of stress are transformed under a rotation about the z-axis.

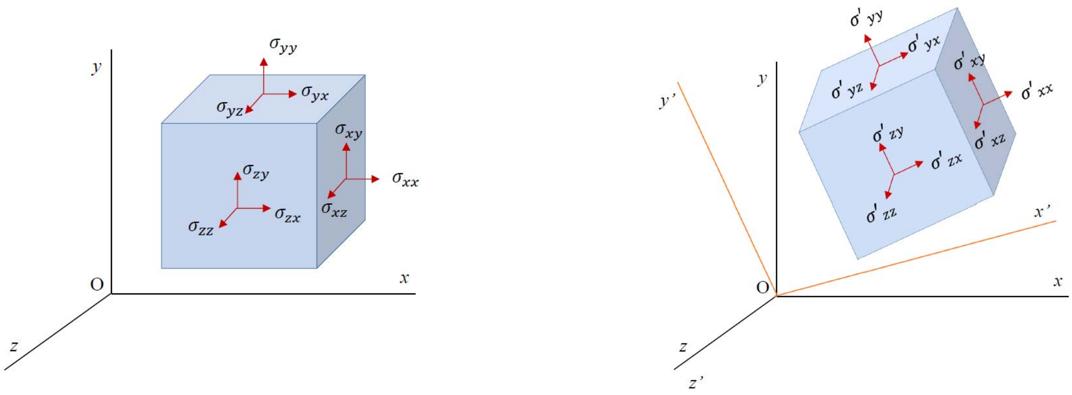

Figure 1: Shows the 6 components of stress acting at a points before (left) and after (right) a rotation about the z axis

For simplicity, we will only consider *plane stress* meaning that $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$ and that the only remaining components are σ_{xx}, σ_{yy}, and σ_{xy}. The simplified plane stress element can be redrawn as shown below in Fig. 2.

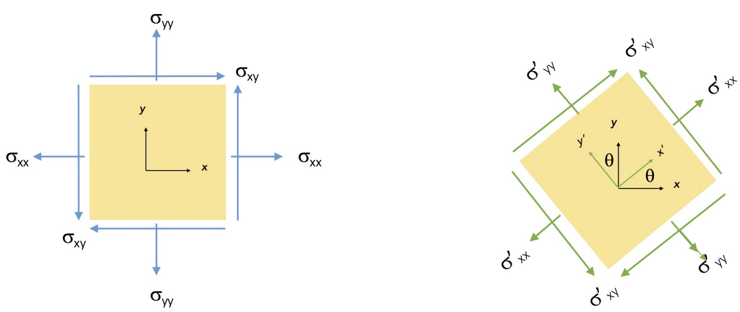

Figure 2: Shows a plane stress element at a points before (left) and after (right) a rotation about the z axis

In this set of notes, the stress components σ'_{xx}, σ'_{yy}, and σ'_{xy} will be determined in terms of the stress components σ_{xx}, σ_{yy}, and σ_{xy} and the rotation angle θ.

To determine the normal stress σ'_{xx} and the shearing stress σ'_{xy} the stress cube shown in Fig. 1 will be cut along a plane perpendicular to the x' axis.

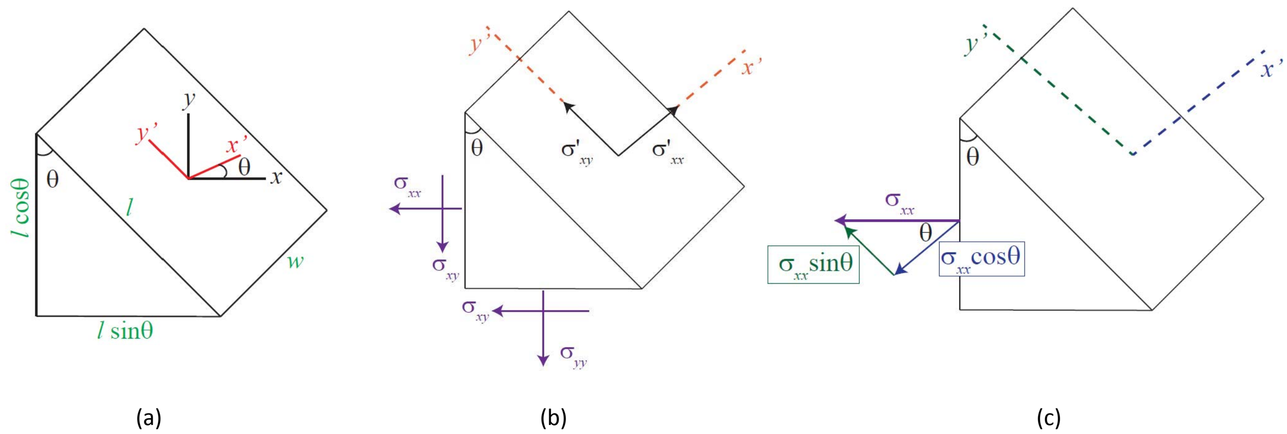

Figure 3: A prism with the inclined face cut perpendicular to the x' axis. (a) Shows the length of each side. (b) Shows the stresses acting on each face. (c) Shows the stress σ_{xx} broken into components in the x' and y' directions.

In Fig. 3 the x' axis is perpendicular to the inclined face. In Fig. 3 (a) of the figure above the length of each side is denoted. In Fig. 3 (b) the stresses acting on each face are shown. Since only the plane stress case is under consideration, there is no stress on the triangular faces as $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$. As shown in the Fig. 1, the normal stresses σ_{xx} and σ_{yy} and the shear stress σ_{xy} act on the two remaining square faces. The rotated stresses that we are attempting to determine act on the inclined face. By summing forces in the x' direction it is possible to obtain and equation for σ'_{xx} in terms of σ_{xx}, σ_{yy} and σ_{xy}. To sum forces along the x' direction the stresses on the side and bottom faces must be re-written in terms of the x'-y' coordinate axes. In Fig. 3 (c), this process is shown for the stress σ_{xx}. This processes has to be repeated for the remaining three stresses.

Summing the forces in the x' direction one obtains:

$$\begin{aligned}\sum F_{x'} &= 0\\ &= \sigma'_{xx}\,(lw) - \sigma_{xx}\cos\theta\,(lw\cos\theta) - \sigma_{xy}\cos\theta\,(lw\sin\theta) - \sigma_{xy}\sin\theta\,(lw\cos\theta) - \sigma_{yy}\sin\theta\,(lw\sin\theta)\end{aligned}$$

In each case the stress in the x' direction is multiplied with the area of the face (always in the parentheses). The entire equation can be divided by lw since it appears in every term. Then solving the equation for σ'_{xx}:

$$\boxed{\sigma'_{xx} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta} \tag{1}$$

The same process can be repeated in the y' direction to identify σ'_{xy} in terms of σ_{xx}, σ_{yy} and σ_{xy}.

$$\begin{aligned}\sum F_{y'} &= 0\\ &= \sigma'_{xy}\,(lw) + \sigma_{xx}\sin\theta\,(lw\cos\theta) + \sigma_{xy}\sin\theta\,(lw\sin\theta) - \sigma_{xy}\cos\theta\,(lw\cos\theta) - \sigma_{yy}\cos\theta\,(lw\sin\theta)\end{aligned}$$

$$\boxed{\sigma'_{xy} = (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta + \sigma_{xy}\left(\cos^2\theta - \sin^2\theta\right)} \tag{2}$$

For a short cut to determine σ'_{yy} notice that the y' axis is 90 degrees from the x' axis. The expression for σ'_{yy} is simply obtained by substituting $\theta + 90$ into the equation for σ'_{xx} (Eq. 1).

$$\boxed{\sigma'_{yy} = \sigma_{xx} \sin^2\theta + \sigma_{yy} \cos^2\theta - 2\sigma_{xy} \sin\theta\cos\theta} \tag{3}$$

These three equations can be collected into matrix form as presented in Lecture 7:

$$\begin{bmatrix} \sigma'_{xx} \\ \sigma'_{yy} \\ \sigma'_{xy} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$$

Using trigonometric relationships an alternative double angle formula can be derived:

$$\begin{bmatrix} \sigma'_{xx} \\ \sigma'_{yy} \\ \sigma'_{xy} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 + 2\cos 2\theta & 1 - 2\cos 2\theta & 2\sin 2\theta \\ 1 - 2\cos 2\theta & 1 + 2\cos 2\theta & -2\sin 2\theta \\ -\sin 2\theta & \sin 2\theta & 2\cos 2\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$$