Objective: Derive a relationship between E, ν, and G.

In Lecture 5, we introduced the Poisson effect. A slender bar that is subjected to an axial load, P, along the x-axis will elongate in the x direction and contract in the y and z direction, as shown in Fig. 1. For a cube element in the center of the bar that has an original side length of 1 this will result in a rectangle with a length of $1+\varepsilon_{xx}$ in the x direction and $1-\nu\varepsilon_{xx}$ in the y-direction and z-direction.

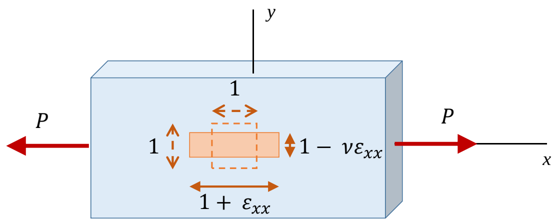

Figure 1: Strain in an axially loaded bar

To determine the relationship between E, ν and G, we will consider another case where the element is 45° from the x-axis as shown in Fig. 2. For this case a diamond deforms into a rhombus. Clearly the internal angles in the rhombus are no longer 90° indicating the presence of shear strains.

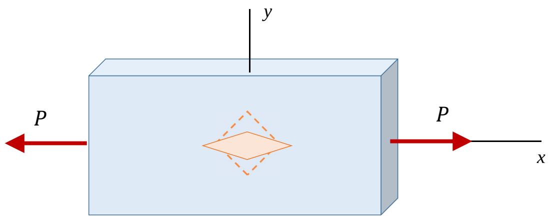

Figure 2: Strain in an axially loaded bar with an internal element oriented 45° from the x-axis

This should not come as a surprise based on what was discussed in Lectures 7-8 on stress transformation. Rotating a uniaxial stress state by 45° is completely equivalent.

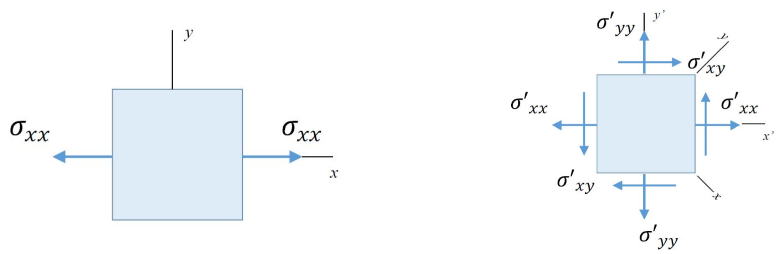

Figure 3: Shows a uniaxial plane stress element rotated through a 45° angle.

Calculating the stress state in rotated coordinate system one finds:

$$\begin{bmatrix} \sigma'_{xx} \\ \sigma'_{yy} \\ \sigma'_{xy} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \sigma_{xx} \\ \sigma_{xx} \\ -\sigma_{xx} \end{bmatrix} \tag{1}$$

If we focus in on the last equation $\sigma'_{xy} = \sigma_{xx}/2$ and use Hooke's Law to re-write the stresses in terms of strains one finds $2G\epsilon'_{xy} = \frac{-E}{2}\epsilon_{xx}$. Collecting all of the strain terms on one side:

$$\boxed{\frac{4G}{E} = -\frac{\epsilon_{xx}}{\epsilon'_{xy}}} \tag{2}$$

The missing component is some relationship between the normal strain applied in the un-rotated state and the shear strain in the rotated state. There are two approaches to determine this ratio: directly using the strain transformation equations or using a geometric approach. First, the approach using the strain transformation is presented.

Strain transformation For the unaxial stress state shown in Fig. 1 the strain state is known to be:

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & -\nu\epsilon_{xx} & 0 \\ 0 & 0 & -\nu\epsilon_{xx} \end{bmatrix} \tag{3}$$

With this information it is possible to perform a rotation of 45° about the z axis.

$$\begin{bmatrix} \epsilon'_{xx} \\ \epsilon'_{yy} \\ \epsilon'_{xy} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \epsilon'_{xx} \\ \epsilon'_{yy} \\ \epsilon'_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & -1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ -\nu\epsilon_{xx} \\ 0 \end{bmatrix}$$

$$\epsilon'_{xy} = -\frac{1}{2}\epsilon_{xx}(1+\nu)$$

$$\boxed{\frac{\epsilon'_{xy}}{\epsilon_{xx}} = -\frac{1}{2}(1+\nu)} \tag{4}$$

Now simply substitute Eq. 4 into Eq. 2 to find that:

$$\boxed{\frac{2G}{E} = \frac{1}{(1+\nu)}} \quad \text{or} \quad \boxed{\frac{E}{2G} = 1+\nu} \tag{5}$$

Geometric approach To perform the same calculation using geometry we will revisit Fig. 2, zooming in on the elements highlighted in the figure and labelling the side lengths:

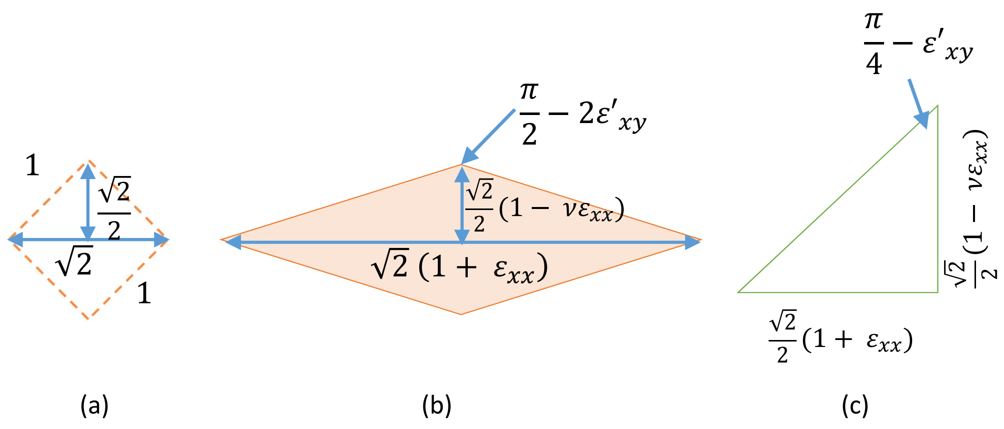

Figure 4: Zoom in on the element from Fig. 2 (a) Undeformed element (b) Deformed element (c) Right triangle with in deformed element

In Fig. 4 (a) the hypotenuse was calculated using the Pythagorean theorem and the height of the triangle was calculated by bisecting the right angle and creating two smaller right triangles. In Fig. 4 (b) the deformed rhombus is pictured. The horizontal centerline elongates by a factor of $1 + \varepsilon_{xx}$ while the vertical centerline shortens by a factor of $1 - \nu\varepsilon_{xx}$. Recall from Lecture 4 that shear strain is measured by a change in angles. In Fig. 4 (a) the upper angle is 90° while in the deformed state the angle is given by $\pi/2 + 2\epsilon'_{xy}$. To make the trigonometry easier, we will work on the right triangle formed by bisecting that upper angle, pictured in Fig. 4 (c).

Since the goal is to find a relationship between ϵ'_{xy} and ϵ_{xx} let's start by taking the tangent of the angle.

$$\tan\left(\pi/4 + \epsilon'_{xy}\right) = \frac{\frac{\sqrt{2}}{2}\left(1 + \epsilon_{xx}\right)}{\frac{\sqrt{2}}{2}\left(1 - \nu\epsilon_{xx}\right)}$$

To simplify this equation, use the sum angle formula for tangent:

$$\tan\left(\alpha + \beta\right) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\,.$$

Using this equation we find that:

$$\tan\left(\pi/4 + \epsilon'_{xy}\right) = \frac{1 + \tan\epsilon'_{xy}}{1 - \tan\epsilon'_{xy}}\,.$$

Using the assumption that $\epsilon'_{xy} \ll 1$ then $\tan\epsilon'_{xy} \approx \epsilon'_{xy}$ and

$$\tan\left(\pi/4 + \epsilon'_{xy}\right) = \frac{1 + \epsilon'_{xy}}{1 - \epsilon'_{xy}} = \frac{1 + \epsilon_{xx}}{1 - \nu\epsilon_{xx}}\,.$$

Cross multiplying and solving for ϵ_{xy}, one finds

$$\epsilon'_{xy} = \frac{\epsilon_{xx}\left(1 + \nu\right)}{2 + \left(1 - \nu\right)\epsilon_{xx}}\,.$$

Assuming that $\epsilon_{xx} \ll 1$ then $\epsilon'_{xy} = \frac{\epsilon_{xx}(1+\nu)}{2}$. Finally, plugging this relationship back into Eq. 2 one recovers Eq. 5.