Objective: Derive a relationship between E, ν, and G for a linear elastic isotropic material.

In Lecture 9, we introduced the Poisson effect. A slender bar that is subjected to an axial load, P, along the x-axis will elongate in the x direction and contract in the y and z direction, as shown in Fig. 1. For a cube element in the center of the bar that has an original side length of 1 this will result in a rectangle with a length of $1+\varepsilon_{xx}$ in the x direction and $1-\nu\varepsilon_{xx}$ in the y-direction and z-direction.

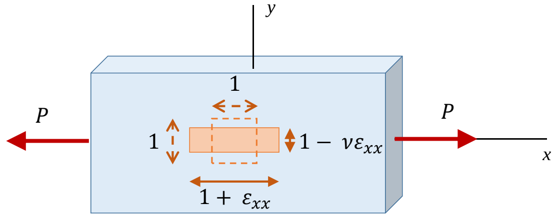

Figure 1: Strain in an axially loaded bar

To determine the relationship between E, ν and G, we will consider another case where the element is oriented 45° from the x-axis as shown in Fig. 2. For this case a diamond deforms into a rhombus. Clearly the internal angles in the rhombus are no longer 90° indicating the presence of shear strains.

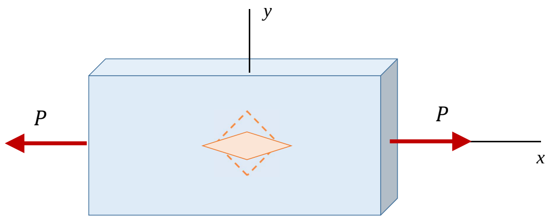

Figure 2: Strain in an axially loaded bar with an internal element oriented 45° from the x-axis

This should not come as a surprise based on what was discussed in during the lectures on stress transformation. We start with a uniaxial state of stress (Fig. 1) which can be represented as a plane stress element (Fig. 3(a)). Next, if we rotate the stress element by 45° we find that σ'_{xx}, σ'_{yy}, and σ'_{xy} are non-zero. Using Hooke's Law we would find that the shear strain, ε'_{xy}, is also non-zero.

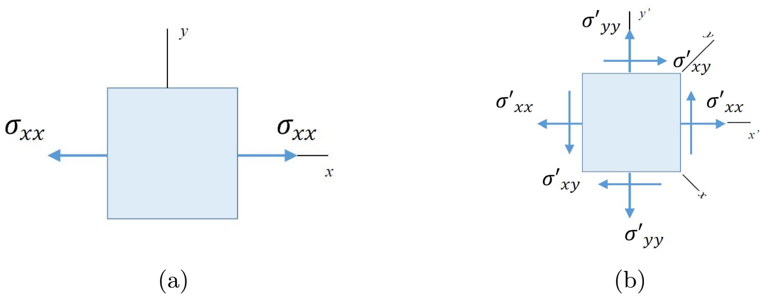

Figure 3: Shows a uniaxial plane stress element rotated through a 45° angle.

Calculating the stress state in rotated coordinate system one finds:

$$\begin{bmatrix} \sigma'_{xx} \\ \sigma'_{yy} \\ \sigma'_{xy} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \sigma_{xx} \\ \sigma_{xx} \\ -\sigma_{xx} \end{bmatrix} \tag{1}$$

If we focus in on the last equation $\sigma'_{xy} = -\sigma_{xx}/2$ and use Hooke's Law to re-write the stresses in terms of strains one finds $2G\epsilon'_{xy} = \frac{-E}{2}\epsilon_{xx}$. Collecting all of the strain terms on one side:

$$\boxed{\frac{4G}{E} = -\frac{\epsilon_{xx}}{\epsilon'_{xy}}} \tag{2}$$

The missing component is some relationship between the normal strain applied in the un-rotated state and the shear strain in the rotated state. There are two approaches to determine this ratio: directly using the strain transformation equations or using a geometric approach. First, the approach using the strain transformation is presented.

Strain transformation For the unaxial stress state shown in Fig. 1 the strain state is known to be

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & -\nu\epsilon_{xx} & 0 \\ 0 & 0 & -\nu\epsilon_{xx} \end{bmatrix}$$

from Hooke's Law.

With this information it is possible to perform a rotation of 45° about the z axis.

$$\begin{bmatrix} \epsilon'_{xx} \\ \epsilon'_{yy} \\ \epsilon'_{xy} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \epsilon'_{xx} \\ \epsilon'_{yy} \\ \epsilon'_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & -1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ -\nu\epsilon_{xx} \\ 0 \end{bmatrix}$$

$$\epsilon'_{xy} = -\frac{1}{2}\epsilon_{xx}(1+\nu)$$

$$\boxed{\frac{\epsilon'_{xy}}{\epsilon_{xx}} = -\frac{1}{2}(1+\nu)} \tag{3}$$

Now simply substitute Eq. 3 into Eq. 2 to find that:

$$\boxed{\frac{2G}{E} = \frac{1}{1+\nu}} \quad \text{or} \quad \boxed{G = \frac{E}{2(1+\nu)}} \tag{4}$$

Geometric approach To perform the same calculation using geometry we will revisit Fig. 2, zooming in on the elements highlighted in the figure and labelling the side lengths as shown in Fig. 4. In Fig. 4(a) the hypotenuse was calculated using the Pythagorean theorem and the height of the triangle was calculated by bisecting the right angle and creating two smaller right triangles. In Fig. 4(b) the deformed rhombus is pictured. The horizontal centerline elongates by a factor of $1 + \varepsilon_{xx}$ while the vertical centerline shortens by a factor of $1 - \nu\varepsilon_{xx}$. Recall from Lecture 4 that shear strain is measured by a change in angles. In Fig. 4(a) the upper angle is 90° while in the deformed state the angle is given by $\pi/2 - 2\epsilon'_{xy}$. To make the trigonometry easier, we will work on the right triangle formed by bisecting that upper angle, pictured in Fig. 4(c).

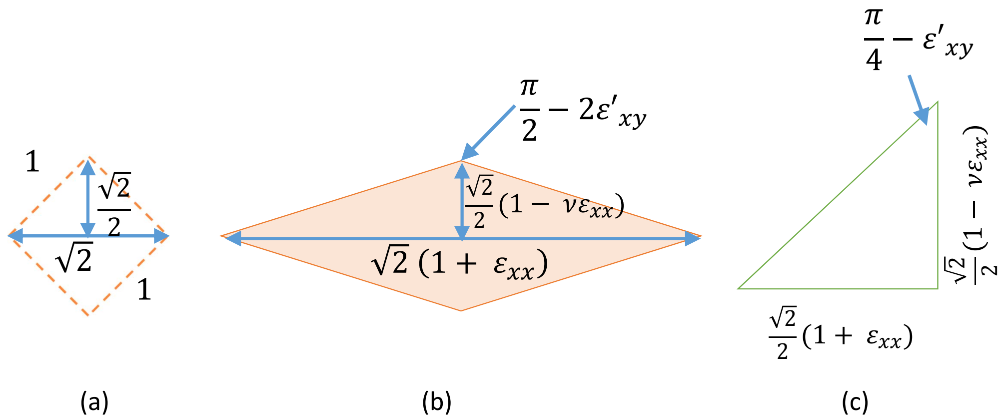

Figure 4: Zoom in on the element from Fig. 2 (a) Undeformed element (b) Deformed element (c) Right triangle within deformed element

Since the goal is to find a relationship between ϵ'_{xy} and ϵ_{xx} let's start by taking the tangent of the angle.

$$\tan\left(\pi/4 - \epsilon'_{xy}\right) = \frac{\frac{\sqrt{2}}{2}\left(1 + \epsilon_{xx}\right)}{\frac{\sqrt{2}}{2}\left(1 - \nu\epsilon_{xx}\right)} \tag{5}$$

To simplify this equation, use the difference angle formula for tangent:

$$\tan\left(\alpha - \beta\right) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}\,.$$

Using this equation we find that:

$$\tan\left(\pi/4 - \epsilon'_{xy}\right) = \frac{1 - \tan\epsilon'_{xy}}{1 + \tan\epsilon'_{xy}}\,.$$

Using the assumption that $\epsilon'_{xy} \ll 1$ then $\tan\epsilon'_{xy} \approx \epsilon'_{xy}$ and

$$\tan\left(\pi/4 + \epsilon'_{xy}\right) = \frac{1 - \epsilon'_{xy}}{1 + \epsilon'_{xy}}\,.$$

Now that we have simplified the left hand side of Eq. 5, let's re-write Eq. 5:

$$\frac{1 - \epsilon'_{xy}}{1 + \epsilon'_{xy}} = \frac{\cancel{\frac{\sqrt{2}}{2}}\left(1 + \epsilon_{xx}\right)}{\cancel{\frac{\sqrt{2}}{2}}\left(1 - \nu\epsilon_{xx}\right)}$$

Cross multiplying and solving for ϵ_{xy}, one finds

$$\epsilon'_{xy} = \frac{\epsilon_{xx}\left(1 + \nu\right)}{\left(\nu - 1\right)\epsilon_{xx} - 2}\,.$$

Assuming that $\epsilon_{xx} \ll 1$ then

$$\epsilon'_{xy} = \frac{-\epsilon_{xx}\left(1 + \nu\right)}{2}\,.$$

Solving for $\epsilon'_{xy}/\epsilon_{xx}$, Eq. 3 is recovered. Finally, plugging this relationship back into Eq. 2 one recovers Eq. 4.