1.

$$
\begin{aligned}
I &:= \int \frac{x+3}{x^2+2x+4} \mathrm{d}x \\
&= \frac{1}{2} \int \frac{2x+2+4}{x^2+2x+4} \mathrm{d}x \\
&= \frac{1}{2} \int \frac{1}{x^2+2x+4} \mathrm{d}(x^2+2x+4) + 2 \int \frac{1}{(x+1)^2+3} \mathrm{d}x \\
&= \frac{1}{2} \ln(x^2+2x+4) + \frac{2}{\sqrt{3}} \arctan \frac{x+1}{\sqrt{3}} + C
\end{aligned}
$$

2.

$$
\begin{aligned}
I &:= \int \frac{2x+3}{x^2+4x+6} \mathrm{d}x \\
&= \int \frac{2x+4-1}{x^2+4x+6} \mathrm{d}x \\
&= \int \frac{1}{x^2+4x+6} \mathrm{d}(x^2+4x+6) - \int \frac{1}{(x+2)^2+2} \mathrm{d}x \\
&= \ln(x^2+4x+6) - \frac{1}{\sqrt{2}} \arctan \frac{x+2}{\sqrt{2}} + C
\end{aligned}
$$

3.

法1：三角换元

令 $x = a\tan t$， $\mathrm{d}x = a\sec^2 t\mathrm{d}t$， $t = \arctan\frac{x}{a}$.

$$
\begin{aligned}
I &:= \int \frac{x^2}{(a^2+x^2)^2} \mathrm{d}x \\
&= \int \frac{a^2\tan^2 t}{(a^2+a^2\tan^2 t)^2} \cdot a\sec^2 t\mathrm{d}t \\
&= \frac{1}{a} \int \sin^2 t\mathrm{d}t \\
&= \frac{1}{a} \int \frac{1-\cos 2t}{2} \mathrm{d}t \\
&= \frac{1}{2a} t - \frac{1}{4a} \sin 2t + C \\
&= \frac{1}{2a} \arctan \frac{x}{a} - \frac{1}{4a} \cdot 2\sin t\cos t + C \\
&= \frac{1}{2a} \arctan \frac{x}{a} - \frac{1}{2} \cdot \frac{x}{a^2+x^2} + C
\end{aligned}
$$

法2：分部积分降次

$$
\begin{aligned}
I &:= \int \frac{x^2}{(a^2+x^2)^2} \mathrm{d}x \\
&= \frac{1}{2} \int \frac{x}{(a^2+x^2)^2} \mathrm{d}(a^2+x^2) \\
&= -\frac{1}{2} \int x\mathrm{d}\frac{1}{a^2+x^2} \\
&= -\frac{1}{2} \cdot \frac{x}{a^2+x^2} + \frac{1}{2} \int \frac{1}{a^2+x^2} \mathrm{d}x \\
&= -\frac{1}{2} \cdot \frac{x}{a^2+x^2} + \frac{1}{2a} \arctan \frac{x}{a} + C
\end{aligned}
$$

4.

$$
\begin{aligned}
I &:= \int \frac{1}{(a^2+x^2)^2}\mathrm{d}x \\
&= \frac{1}{a^2}\int \frac{a^2}{(a^2+x^2)^2}\mathrm{d}x \\
&= \frac{1}{a^2}\int \frac{a^2+x^2-x^2}{(a^2+x^2)^2}\mathrm{d}x \\
&= \frac{1}{a^2}\int \frac{1}{a^2+x^2}\mathrm{d}x - \frac{1}{a^2}\int \frac{x^2}{(a^2+x^2)^2}\mathrm{d}x \\
&= \frac{1}{2a^3}\arctan\frac{x}{a} + \frac{1}{2a^2}\cdot\frac{x}{a^2+x^2} + C
\end{aligned}
$$

5.

$$
\begin{aligned}
I &:= \int \frac{x+2}{(x^2+2x+10)^2}\mathrm{d}x \\
&= \frac{1}{2}\int \frac{(2x+2)+2}{(x^2+2x+10)^2}\mathrm{d}x \\
&= \frac{1}{2}\int \frac{1}{(x^2+2x+10)^2}\mathrm{d}(x^2+2+10) + \int \frac{1}{(x^2+2x+10)^2}\mathrm{d}x \\
&= -\frac{1}{2}\cdot\frac{1}{x^2+2x+10} + \int \frac{1}{[(x+1)^2+9]^2}\mathrm{d}(x+1) \\
&= -\frac{1}{2}\cdot\frac{1}{x^2+2x+10} + \frac{1}{54}\arctan\frac{x+1}{3} + \frac{1}{18}\cdot\frac{x+1}{9+(x+1)^2} + C
\end{aligned}
$$

6.

$$
\begin{aligned}
I &:= \int \frac{3x+6}{(x-1)^2(x^2+x+1)}\mathrm{d}x \\
&= \int \left[\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+x+1}\right]\mathrm{d}x
\end{aligned}
$$

要令

$$
\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+x+1} = \frac{3x+6}{(x-1)^2(x^2+x+1)}
$$

$$
A(x-1)(x^2+x+1) + B(x^2+x+1) + (Cx+D)(x-1)^2 = 3x+6
$$

也即令

$$
\begin{aligned}
3x+6 &= A(x^3-1) + B(x^2+x+1) + (Cx+D)(x^2-2x+1) \\
&= A(x^3-1) + B(x^2+x+1) + (Cx^3-2Cx^2+Cx+Dx^2-2Dx+D) \\
&= (A+C)x^3 + (B-2C+D)x^2 + (B+C-2D)x + (-A+B+D)
\end{aligned}
$$

得

$$
\begin{cases}
0 = A+C \\
0 = B-2C+D \\
3 = B+C-2D \\
6 = -A+B+D
\end{cases}
$$

解得

$$
\begin{cases}
A = -2 \\
B = 3 \\
C = 2 \\
D = 1
\end{cases}
$$

故

$$
\begin{aligned}
I &:= \int \frac{3x+6}{(x-1)^2(x^2+x+1)} \mathrm{d}x \\
&= \int \left[\frac{-2}{x-1} + \frac{3}{(x-1)^2} + \frac{2x+1}{x^2+x+1} \right] \mathrm{d}x \\
&= -2 \ln |x-1| - 3 \cdot \frac{1}{x-1} + \int \frac{1}{x^2+x+1} \mathrm{d}(x^2+x+1) \\
&= -2 \ln |x-1| - 3 \cdot \frac{1}{x-1} + \ln(x^2+x+1) + C
\end{aligned}
$$

7.

$$
\begin{aligned}
I &:= \int \frac{1}{1+x^3} \mathrm{d}x \\
&= \int \frac{1}{(1+x)(x^2-x+1)} \mathrm{d}x \\
&= \int \left(\frac{A}{1+x} + \frac{Bx+C}{x^2-x+1} \right) \mathrm{d}x
\end{aligned}
$$

要令

$$
\frac{A}{1+x} + \frac{Bx+C}{x^2-x+1} = \frac{1}{(1+x)(x^2-x+1)}
$$

即令

$$
\begin{gathered}
A(x^2-x+1) + (1+x)(Bx+C) = 1 \\
Ax^2 - Ax + A + Bx + C + Bx^2 + Cx = 1
\end{gathered}
$$

得

$$
\begin{cases}
0 = A + B \\
0 = -A + B + C \\
1 = A + C
\end{cases}
$$

解得

$$
\begin{cases}
A = \dfrac{1}{3} \\
B = -\dfrac{1}{3} \\
C = \dfrac{2}{3}
\end{cases}
$$

故

$$
\begin{aligned}
I &:= \int \frac{1}{1+x^3}\mathrm{d}x \\
&= \int \frac{1}{(1+x)(x^2-x+1)}\mathrm{d}x \\
&= \frac{1}{3}\int \left(\frac{1}{1+x}+\frac{-x+2}{x^2-x+1}\right)\mathrm{d}x \\
&= \frac{1}{3}\ln|1+x| - \int \frac{x-2}{x^2-x+1}\mathrm{d}x \\
&= \frac{1}{3}\ln|1+x| - \frac{1}{2}\int \frac{(2x-1)-3}{x^2-x+1}\mathrm{d}x \\
&= \frac{1}{3}\ln|1+x| - \frac{1}{2}\int \frac{1}{x^2-x+1}\mathrm{d}(x^2-x+1) + \frac{3}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\mathrm{d}\left(x-\frac{1}{2}\right) \\
&= \frac{1}{3}\ln|1+x| - \frac{1}{2}\ln(x^2-x+1) + \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + C
\end{aligned}
$$

8.

令

$$
\frac{x^2+ax+2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}
$$

得

$$
\begin{cases}
A+B=1 \\
C+B=a \\
A+C=2
\end{cases}
$$

要积分中不含反正切函数，就要 $C=0$

得

$$
\begin{cases}
A+B=1 \\
B=a \\
A=2
\end{cases}
$$

即 $a=-1$.

9.

法1：待定系数

令

$$
\begin{aligned}
f(x) &:= \frac{1}{1-x^4} \\
&= \frac{1}{(1+x^2)(1-x^2)} \\
&= \frac{1}{(1+x^2)(1+x)(1-x)} \\
&= \frac{Ax+B}{1+x^2} + \frac{C}{1+x} + \frac{D}{1-x}
\end{aligned}
$$

得

$$
(Ax+B)(1-x^2) + C(1+x^2)(1-x) + D(1+x^2)(1+x) = 1
$$

- 子法1：解方程组

$$\begin{cases} -A-C+D=0 \\ -B+C+D=0 \\ \quad A-C+D=0 \\ \quad B+C+D=1 \end{cases}$$

- 子法2：特殊值

  令 $x=1$ ，得 $4D=1$ ，即 $D=\frac{1}{4}$.

  令 $x=-1$ ，得 $4C=1$ ，即 $C=\frac{1}{4}$.

  令 $x=\mathrm{i}$ ，得 $2(A\mathrm{i}+B)=1$ ，即 $A=0,B=\frac{1}{2}$.

- 子法3：留数法

$$\begin{aligned} C &= \lim_{x\to-1} f(x)\cdot(x+1) \\ &= \lim_{x\to-1}\frac{1}{(1+x^2)(1-x)} \\ &= \frac{1}{4}. \\ D &= \lim_{x\to1} f(x)\cdot(1-x) \\ &= \lim_{x\to1}\frac{1}{(1+x^2)(1+x)} \\ &= \frac{1}{4}. \end{aligned}$$

  以此带到方程组，可以解出另外两个。

故

$$\begin{aligned} I &:= \int\frac{1}{1-x^4}\mathrm{d}x \\ &= \frac{1}{2}\int\frac{1}{1+x^2}\mathrm{d}x+\frac{1}{4}\int\frac{1}{1+x}\mathrm{d}x+\frac{1}{4}\int\frac{1}{1-x}\mathrm{d}x \\ &= \frac{1}{2}\arctan x+\frac{1}{4}\ln|1+x|-\frac{1}{4}\ln|1-x|+C \end{aligned}$$

法2：

$$\begin{aligned} I &:= \int\frac{1}{1-x^4}\mathrm{d}x \\ &= \int\frac{1}{(1+x^2)(1-x^2)}\mathrm{d}x \\ &= \frac{1}{2}\int\frac{(1+x^2)+(1-x^2)}{(1+x^2)(1-x^2)}\mathrm{d}x \\ &= \frac{1}{2}\int\frac{1}{1-x^2}\mathrm{d}x+\frac{1}{2}\int\frac{1}{1+x^2}\mathrm{d}x \\ &= \frac{1}{4}\ln\left|\frac{1+x}{1-x}\right|+\frac{1}{2}\arctan x+C \end{aligned}$$

10.

法1：倒代换

令 $x=\frac{1}{t}$ ， $\mathrm{d}x=-\frac{1}{t^2}\mathrm{d}t$.

$$
\begin{aligned}
I &:= \int \frac{1}{x^8(1+x^2)}\mathrm{d}x \\
&= -\int \frac{1}{t^{-8}\cdot(1+t^{-2})}\cdot\frac{1}{t^2}\mathrm{d}t \\
&= -\int \frac{t^8}{t^2+1}\mathrm{d}t \\
&= -\int \frac{t^8-1}{t^2+1}\mathrm{d}t - \int \frac{1}{t^2+1}\mathrm{d}t \\
&= -\int \frac{(t^4-1)(t^4+1)}{t^2+1}\mathrm{d}t - \arctan t \\
&= -\int \frac{(t^2-1)(t^2+1)(t^4+1)}{t^2+1}\mathrm{d}t - \arctan t \\
&= -\int (t^2-1)(t^4+1)\mathrm{d}t - \arctan t \\
&= -\int (t^6-t^4+t^2-1)\mathrm{d}t - \arctan t \\
&= -\frac{1}{7}t^7 + \frac{1}{5}t^5 - \frac{1}{3}t^3 + t - \arctan t + C \\
&= -\frac{1}{7}x^{-7} + \frac{1}{5}x^{-5} - \frac{1}{3}x^{-3} + x^{-1} - \arctan\frac{1}{x} + C
\end{aligned}
$$

法2：

$$
\begin{aligned}
I_n &:= \int \frac{1}{x^n(1+x^2)}\mathrm{d}x \\
&= \int \frac{1+x^2-x^2}{x^n(1+x^2)}\mathrm{d}x \\
&= \int \frac{1}{x^n}\mathrm{d}x - \int \frac{1}{x^{n-1}(1+x^2)}\mathrm{d}x \\
&= -\frac{1}{n-1}x^{1-n} - I_{n-2}, (n \geq 2)
\end{aligned}
$$

又

$$
I_0 := \int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C
$$

$$
\begin{aligned}
I_8 &:= \int \frac{1}{x^8(1+x^2)}\mathrm{d}x \\
&= -\frac{1}{7}x^{-7} - I_6 \\
&= -\frac{1}{7}x^{-7} + \frac{1}{5}x^{-5} + I_4 \\
&= -\frac{1}{7}x^{-7} + \frac{1}{5}x^{-5} - \frac{1}{3}x^{-3} - I_2\mathrm{d}x \\
&= -\frac{1}{7}x^{-7} + \frac{1}{5}x^{-5} - \frac{1}{3}x^{-3} + x^{-1} + I_0 \\
&= -\frac{1}{7}x^{-7} + \frac{1}{5}x^{-5} - \frac{1}{3}x^{-3} + x^{-1} + \arctan x + C
\end{aligned}
$$

其中有公式 $\arctan x + \arctan\frac{1}{x} = \frac{\pi}{2}$，故与倒代换做出的结果一致。

11.

$$
\begin{aligned}
I &:= \int \frac{1+x^4}{1+x^6}\mathrm{d}x \\
&= \int \frac{1+x^4}{(1+x^2)(x^4-x^2+1)}\mathrm{d}x \\
&= \int \frac{1+x^4-x^2+x^2}{(1+x^2)(x^4-x^2+1)}\mathrm{d}x \\
&= \arctan x + \int \frac{x^2}{(1+x^2)(x^4-x^2+1)}\mathrm{d}x \\
&= \arctan x + \int \frac{x^2}{\color{blue}{1+x^6}}\mathrm{d}x \\
&= \arctan x + \int \frac{x^2}{1+(x^3)^2}\mathrm{d}x \\
&= \arctan x + \frac{1}{3}\int \frac{1}{1+(x^3)^2}\mathrm{d}(x^3) \\
&= \arctan x + \frac{1}{3}\arctan x^3
\end{aligned}
$$

12.

$$
\begin{aligned}
I &:= \int \frac{1}{x(x^3+27)}\mathrm{d}x \\
&= \int \frac{x^2}{x^3(x^3+27)}\mathrm{d}x \\
&= \frac{1}{3}\int \frac{1}{x^3(x^3+27)}\mathrm{d}(x^3) \\
&= \frac{1}{3\times 27}\int \frac{27+x^3-x^3}{x^3(x^3+27)}\mathrm{d}(x^3) \\
&= \frac{1}{81}x^3 - \frac{1}{81}\ln|x^3+27| + C
\end{aligned}
$$

13.

尝试倒代换，不行

$$
\begin{aligned}
I &:= \int \frac{1+x^2}{1+x^4}\mathrm{d}x \\
&= -\int \frac{1+t^{-2}}{1+t^{-4}}\cdot t^{-2}\mathrm{d}t \\
&= -\int \frac{t^2+1}{t^4+1}\mathrm{d}t \\
&= -I \\
&= 0???
\end{aligned}
$$

法2：

$$
\begin{aligned}
I &:= \int \frac{1+x^2}{1+x^4}\mathrm{d}x \\
&= \int \frac{x^{-2}+1}{x^{-2}+x^2}\mathrm{d}x \\
&= \int \frac{1}{(-x^{-1}+x)^2+2}\mathrm{d}\left(-x^{-1}+x\right) \\
&= \frac{1}{\sqrt{2}}\arctan\frac{x-x^{-1}}{\sqrt{2}} + C
\end{aligned}
$$

14.

$$
\begin{aligned}
I &:= \int \frac{1-x^2}{1+x^4}\mathrm{d}x \\
&= \int \frac{x^{-2}-1}{x^{-2}+x^2}\mathrm{d}x \\
&= -\int \frac{1}{x^{-2}+x^2}\mathrm{d}(x^{-1}+x) \\
&= -\int \frac{1}{(x^{-1}+x)^2-2}\mathrm{d}(x^{-1}+x) \\
&= -\frac{1}{2\sqrt{2}}\ln\left|\frac{x^{-1}+x-\sqrt{2}}{x^{-1}+x+\sqrt{2}}\right| + C
\end{aligned}
$$

15.

$$
\begin{aligned}
I &:= \int \frac{1}{1+x^6}\mathrm{d}x \\
&= \int \frac{1}{(1+x^2)(x^4-x^2+1)}\mathrm{d}x \\
&= \int \frac{1+x^2-x^2}{(1+x^2)(x^4-x^2+1)}\mathrm{d}x \\
&= \int \frac{1}{x^4-x^2+1}\mathrm{d}x - \int \frac{x^2}{1+(x^3)^2}\mathrm{d}x \\
&= \frac{1}{2}\int \frac{(1-x^2)+(1+x^2)}{x^4-x^2+1}\mathrm{d}x - \frac{1}{3}\arctan x^3 \\
&= \frac{1}{2}I_1 + \frac{1}{2}I_2 - \frac{1}{3}\arctan x^3
\end{aligned}
$$

其中

$$
\begin{aligned}
I_1 &:= \int \frac{1-x^2}{x^4-x^2+1}\mathrm{d}x \\
&= \int \frac{x^{-2}-1}{x^2-1+x^{-2}}\mathrm{d}x \\
&= -\int \frac{1}{(x^{-1}+x)^2-3}\mathrm{d}(x^{-1}+x) \\
&= -\frac{1}{2\sqrt{3}}\ln\left|\frac{x^{-1}+x-\sqrt{3}}{x^{-1}+x+\sqrt{3}}\right| + C
\end{aligned}
$$

$$
\begin{aligned}
I_2 &:= \int \frac{1+x^2}{x^4-x^2+1}\mathrm{d}x \\
&= \int \frac{x^{-2}+1}{x^2-1+x^{-2}}\mathrm{d}x \\
&= \int \frac{1}{(-x^{-1}+x)^2+1}\mathrm{d}(-x^{-1}+x) \\
&= \arctan(-x^{-1}+x) + C
\end{aligned}
$$

故

$$
\begin{aligned}
I &:= \int \frac{1}{1+x^6}\mathrm{d}x \\
&= -\frac{1}{4\sqrt{3}}\ln\left|\frac{x^{-1}+x-\sqrt{3}}{x^{-1}+x+\sqrt{3}}\right| + \frac{1}{2}\arctan(-x^{-1}+x) - \frac{1}{3}\arctan x^3 + C
\end{aligned}
$$

16.

$$
\begin{aligned}
I &:= \int \frac{e^{3x}+e^{x}}{e^{4x}-e^{2x}+1}\mathrm{d}x \\
&= \int \frac{e^{2x}+1}{e^{4x}-e^{2x}+1}\mathrm{d}e^{x} \\
&= \int \frac{t^2+1}{t^4-t^2+1}\mathrm{d}t \\
&= \int \frac{1+t^{-2}}{t^2-1+t^{-2}}\mathrm{d}t \\
&= \int \frac{1}{(t-t^{-1})^2+1}\mathrm{d}(t-t^{-1}) \\
&= \arctan(t-t^{-1})+C \\
&= \arctan\left(e^{x}-e^{-x}\right)+C
\end{aligned}
$$

17.

令 $t=\sqrt{\tan x}$，$x=\arctan t^2$，$\mathrm{d}x=\frac{2t}{1+t^4}\mathrm{d}t.$

故

$$
\begin{aligned}
I &:= \int \frac{1}{\sqrt{\tan x}}\mathrm{d}x \\
&= \int \frac{1}{t}\cdot\frac{2t}{1+t^4}\mathrm{d}t \\
&= \int \frac{(1+t^2)+(1-t^2)}{1+t^4}\mathrm{d}t \\
&= \int \frac{1+t^2}{1+t^4}\mathrm{d}t+\int \frac{1-t^2}{1+t^4}\mathrm{d}t \\
&= \int \frac{t^{-2}+1}{t^{-2}+t^2}\mathrm{d}t+\int \frac{t^{-2}-1}{t^{-2}+t^2}\mathrm{d}t \\
&= \int \frac{1}{(-t^{-1}+t)^2+2}\mathrm{d}(-t^{-1}+t)+\int \frac{1}{(-t^{-1}-t)^2-2}\mathrm{d}(-t^{-1}-t) \\
&= \frac{1}{\sqrt{2}}\arctan\frac{-t^{-1}+t}{\sqrt{2}}+\frac{1}{2\sqrt{2}}\ln\left|\frac{-t^{-1}-t-\sqrt{2}}{-t^{-1}-t+\sqrt{2}}\right|+C \\
&= \frac{1}{\sqrt{2}}\arctan\frac{-1+\tan x}{\sqrt{2\tan x}}+\frac{1}{2\sqrt{2}}\ln\left|\frac{-1-\tan x-\sqrt{2\tan x}}{-1-\tan x+\sqrt{2\tan x}}\right|+C
\end{aligned}
$$

18.

如 17 令

$$
\begin{aligned}
I &:= \int \sqrt{\tan x}\mathrm{d}x \\
&= \int t\cdot\frac{2t}{1+t^4}\mathrm{d}t \\
&= \int \frac{t^2+1-1+t^2}{1+t^4}\mathrm{d}t \\
&= \int \frac{t^2+1}{1+t^4}\mathrm{d}t-\int \frac{1-t^2}{1+t^4}\mathrm{d}t \\
&= \frac{1}{\sqrt{2}}\arctan\frac{-1+\tan x}{\sqrt{2\tan x}}-\frac{1}{2\sqrt{2}}\ln\left|\frac{-1-\tan x-\sqrt{2\tan x}}{-1-\tan x+\sqrt{2\tan x}}\right|+C
\end{aligned}
$$