

凯哥不定积分笔记——2 三角函数

套路集合

- 套路1: 万能代换

令 $t = \tan \frac{x}{2}$, $x = 2 \arctan t$, $dx = \frac{2}{1+t^2} dt$.

$$\begin{aligned}\sin x &= 2 \sin \frac{x}{2} \cos \frac{x}{2} \\ &= \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}} \\ &= \frac{2 \tan \frac{x}{2}}{\tan^2 \frac{x}{2} + 1} \\ &= \frac{2t}{t^2 + 1}\end{aligned}$$

$$\begin{aligned}\cos x &= \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \\ &= \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}} \\ &= \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \\ &= \frac{1 - t^2}{1 + t^2}\end{aligned}$$

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- 套路2: 缩分母

- 共轭表达式:

对于分母为 $1 + \cos x$ 或 $1 + \sin x$ 的, 分式上下同乘 $1 - \cos x$ 或 $1 - \sin x$

- 倍角公式
- 辅助角公式

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- 套路3: 凑偶次方

如果被积函数 $R(\sin x, \cos x)$ 满足 $R(\sin x, -\cos x) = -R(\sin x, \cos x)$ 或 $R(-\sin x, \cos x) = -R(\sin x, \cos x)$. 这里把前者称为对 $\cos x$ 有“奇性”, 后者为对 $\sin x$ 有“奇性”。

做法是, 对 $\cos x$ 有奇性时凑 $d \sin x$; 对 $\sin x$ 有奇性时凑 $d \cos x$.

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- 套路4: 凑 $d \tan x$

如果被积函数满足 $R(-\sin x, -\cos x) = R(\sin x, \cos x)$, 考虑凑 $d \tan x$.

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- 套路5:

形如

$$\int \frac{A \sin x + B \cos x}{C \sin x + D \cos x} dx$$

的积分，可以通过假设

$$\text{分子} = p \cdot \text{分母} + q \cdot (\text{分母})'$$

然后使用待定系数解出 p, q ，从而化简积分。

- 套路6：统一角度

当被积函数出现不同角度，先统一角度。

- 套路7：积化和差公式

形如

$$\int \sin ax \sin bx dx, (a \neq b)$$

的积分，可以使用积化和差公式化简积分。

积化和差公式，立即推！

由和角公式有

$$\begin{cases} \sin(a+b) = \sin a \cos b + \cos a \sin b & (1) \\ \sin(a-b) = \sin a \cos b - \cos a \sin b & (2) \\ \cos(a+b) = \cos a \cos b - \sin a \sin b & (3) \\ \cos(a-b) = \cos a \cos b + \sin a \sin b & (4) \end{cases}$$

那么得

$$\begin{cases} (1) + (2) : \frac{1}{2} [\sin(a+b) + \sin(a-b)] = \sin a \cos b \\ (3) + (4) : \frac{1}{2} [\cos(a+b) + \cos(a-b)] = \cos a \cos b \\ (4) - (3) : \frac{1}{2} [\cos(a-b) - \cos(a+b)] = \sin a \sin b \end{cases}$$

题目列表：

1.

$$\int \frac{1}{3 + 5 \cos x} dx$$

2.

$$\int \frac{1}{1 + \sin x + \cos x} dx$$

3.

$$\int \frac{1}{1 + \cos x} dx$$

4.

$$\int \frac{\sin x}{1 + \sin x} dx$$

5.

$$\int \frac{1}{\sin x + \cos x} dx$$

6.

$$\int \frac{\cos x}{\sin x + \cos x} dx$$

7.

$$\int \frac{1}{\sin^2 x \cos x} dx$$

8.

$$\int \frac{\cos^3 x - 2 \cos x}{1 + \sin^2 x + \sin^4 x} dx$$

9.

$$\int \sec^3 x dx$$

10.

$$\int \sqrt{1 + x^2} dx$$

11.

$$\int \sec^5 x dx$$

12.

$$\int \sec^n x dx$$

13.

$$\int \tan^n x dx$$

14.

$$\int \csc^n x dx$$

15.

$$\int \cot^n x dx$$

16.

$$\int \frac{1}{\sin x \cos^2 x} dx$$

17.

$$\int \frac{5 + 4 \cos x}{(2 + \cos x)^2 \sin x} dx$$

18.

$$\int \frac{1}{1 + \cos^2 x} dx$$

19.

$$\int \frac{1}{(3 \sin x + 2 \cos x)^2} dx$$

20.

$$\int \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} dx$$

21.

$$\int \frac{1}{\sin^4 x \cos^2 x} dx$$

22.

$$\int \sin^4 x \cos^2 x dx$$

23.

$$\int \frac{1 + \sin x + \cos x}{1 + \sin^2 x} dx$$

24.

$$\int \frac{3 \sin x + 4 \cos x}{2 \sin x + \cos x} dx$$

25.

$$\int \frac{1}{\sin x \sin 2x} dx$$

26.

$$\int \frac{1}{2 \sin x + \sin 2x} dx$$

27.

$$\int \frac{\cos 2x - \sin 2x}{\sin x + \cos x} dx$$

28.

$$\int \sin 2x \sin 3x dx$$

29.

$$\int \frac{\sin x \cos x}{\sin x + \cos x} dx$$

30.

$$\int \frac{\sin x \cos x}{\sin x - \cos x} dx$$

31.

$$\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

32.

$$\int \frac{1}{\sin^6 x + \cos^6 x} dx$$

33.

$$\int \frac{1}{\sin^3 x + \cos^3 x} dx$$

解答之：

1.

$$\text{令 } t = \tan \frac{x}{2}, \quad dx = \frac{2}{1+t^2} dt, \quad \cos x = \frac{1-t^2}{1+t^2}.$$

$$\begin{aligned} I &:= \int \frac{1}{3+5\cos x} dx \\ &= \int \frac{1}{3+5 \cdot \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt \\ &= \int \frac{1}{4-t^2} dt \\ &= \frac{1}{4} \left[\int \frac{dt}{2-t} + \int \frac{dt}{2+t} \right] \\ &= \frac{1}{4} \ln \left| \frac{2+t}{2-t} \right| + C \\ &= \frac{1}{4} \ln \left| \frac{2+\tan \frac{x}{2}}{2-\tan \frac{x}{2}} \right| + C \end{aligned}$$

2.

$$\begin{aligned} I &:= \int \frac{1}{1+\sin x + \cos x} dx \\ &= \int \frac{1}{1+\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt \\ &= \int \frac{1}{1+t} dt \\ &= \ln |1+t| + C \\ &= \ln \left| 1 + \tan \frac{x}{2} \right| + C \end{aligned}$$

3.

因

$$\cos 2x = 2 \cos^2 x - 1$$

故

$$\begin{aligned}
 I &:= \int \frac{1}{1 + \cos x} dx \\
 &= \int \frac{1}{2 \cos^2 \frac{x}{2}} dx \\
 &= \tan \frac{x}{2} + C
 \end{aligned}$$

4.

$$\begin{aligned}
 I &:= \int \frac{\sin x + 1 - 1}{1 + \sin x} dx \\
 &= x - \int \frac{1}{1 + \sin x} dx \\
 &= x - \int \frac{1 - \sin x}{\cos^2 x} dx \\
 &= x - \tan x - \int \frac{d \cos x}{\cos^2 x} \\
 &= x - \tan x + \sec x + C
 \end{aligned}$$

5.

有公式

$$\int \csc x dx = \ln |\cot x - \csc x| + C \quad (1)$$

那么

$$\begin{aligned}
 I &:= \int \frac{1}{\sin x + \cos x} dx \\
 &= \int \frac{d(x + \frac{\pi}{4})}{\sqrt{2} \sin(x + \frac{\pi}{4})} \\
 &= \int \frac{d(x + \frac{\pi}{4})}{\sqrt{2} \sin(x + \frac{\pi}{4})} \\
 &= \frac{\sqrt{2}}{2} \ln \left| \cot \left(x + \frac{\pi}{4} \right) - \csc \left(x + \frac{\pi}{4} \right) \right|
 \end{aligned}$$

6.

法1: 共轭

又有公式

$$\int \sec x dx = \ln |\tan x + \sec x| + C \quad (2)$$

$$\begin{aligned}
 I &:= \int \frac{\cos x}{\sin x + \cos x} dx \\
 &= \int \frac{\cos x (\cos x - \sin x)}{\cos^2 x - \sin^2 x} dx \\
 &= \int \frac{\cos^2 x - \sin x \cos x}{\cos 2x} dx \\
 &= \frac{1}{2} \int \frac{\cos 2x + 1}{\cos 2x} dx - \frac{1}{2} \int \frac{\sin 2x}{\cos 2x} dx \\
 &= \frac{1}{2} x + \frac{1}{4} \ln |\sec 2x + \tan 2x| + \frac{1}{4} \ln |\cos 2x| + C
 \end{aligned}$$

法2: 组合积分法

$$I := \int \frac{\cos x}{\sin x + \cos x} dx$$

$$J := \int \frac{\sin x}{\sin x + \cos x} dx$$

那么,

$$I + J = x + C$$

$$I - J = \ln |\sin x + \cos x| + C$$

故

$$I := \int \frac{\cos x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}[(I + J) + (I - J)]$$

$$= \frac{1}{2}x + \ln |\sin x + \cos x| + C$$

7.

法1: 三角恒等式

$$I := \int \frac{1}{\sin^2 x \cos x} dx$$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos x} dx$$

$$= \int \sec x dx + \int \frac{\cos x}{\sin^2 x} dx$$

$$= \ln |\tan x + \sec x| + \int \frac{1}{\sin^2 x} d \sin x$$

$$= \ln |\tan x + \sec x| - \csc x + C$$

法2: 凑 $d \sin x$

$$I := \int \frac{1}{\sin^2 x \cos x} dx$$

$$= \int \frac{\cos x}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{1}{\sin^2 x (1 - \sin^2 x)} d \sin x$$

$$= \int \frac{1 - \sin^2 x + \sin^2 x}{\sin^2 x (1 - \sin^2 x)} d \sin x$$

$$= \int \frac{1}{\sin^2 x} d \sin x + \int \frac{1}{1 - \sin^2 x} d \sin x$$

$$= -\csc x - \frac{1}{2} \ln \left| \frac{1 - \sin x}{1 + \sin x} \right| + C$$

8.

$$\begin{aligned}
I &:= \int \frac{\cos^3 x - 2 \cos x}{1 + \sin^2 x + \sin^4 x} dx \\
&= \int \frac{\cos^2 x - 2}{1 + \sin^2 x + \sin^4 x} d \sin x \\
&= \int \frac{-(1 + 1 - \cos^2 x)}{1 + \sin^2 x + \sin^4 x} d \sin x \\
&= \int \frac{-(1 + \sin^2 x)}{1 + \sin^2 x + \sin^4 x} d \sin x \\
&= - \int \frac{1 + t^2}{1 + t^2 + t^4} dt \\
&= - \int \frac{t^{-2} + 1}{t^{-2} + 1 + t^2} dt \\
&= - \int \frac{1}{(t - t^{-1})^2 + 3} d(-t^{-1} + t) \\
&= - \frac{1}{\sqrt{3}} \arctan \frac{t - t^{-1}}{\sqrt{3}} + C \\
&= - \frac{1}{\sqrt{3}} \arctan \frac{\sin x - \csc x}{\sqrt{3}} + C
\end{aligned}$$

9.

法1: 凑 $d \sin x$

$$\begin{aligned}
I &:= \int \sec^3 x dx \\
&= \int \frac{\cos x}{\sec^2 x} dx \\
&= \int \frac{1}{(1 - \sin^2 x)^2} d \sin x \\
&= \int \frac{1}{[(1 - \sin x)(1 + \sin x)]^2} d \sin x \\
&= \int \left[\frac{1}{2} \left(\frac{1}{1 - \sin x} + \frac{1}{1 + \sin x} \right) \right]^2 d \sin x \\
&= -\frac{1}{4} \int \frac{d(1 - \sin x)}{(1 - \sin x)^2} + \frac{1}{4} \int \frac{d(1 + \sin x)}{(1 + \sin x)^2} + \frac{1}{2} \int \frac{d \sin x}{1 - \sin^2 x} \\
&= \frac{1}{4} \cdot \frac{1}{1 - \sin x} - \frac{1}{4} \cdot \frac{1}{1 + \sin x} + \frac{1}{4} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + C
\end{aligned}$$

法2: 直接分部积分+积分重现

$$\begin{aligned}
I &:= \int \sec^3 x dx \\
&= \int \sec x \cdot \sec^2 x dx \\
&= \int \sec x d \tan x \\
&= \sec x \tan x - \int \tan x d \sec x \\
&= \sec x \tan x - \int \tan^2 x \sec x dx \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx \\
&= \sec x \tan x + \int \sec x dx - \int \sec^3 x dx \\
&= \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x dx \\
&= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\tan x + \sec x| + C
\end{aligned}$$

10.

法1: 三角换元

$$\begin{aligned}
I &:= \int \sqrt{1+x^2} dx \\
&= \int \sec t \cdot \sec^2 t dt \\
&= \frac{1}{2} \sec t \tan t + \frac{1}{2} \ln |\tan t + \sec t| + C \\
&= \frac{1}{2} \cdot x \sqrt{1+x^2} + \frac{1}{2} \ln |x + \sqrt{1+x^2}| + C
\end{aligned}$$

法2: 直接分部积分 + 积分重现

有公式

$$\int \frac{1}{\sqrt{a^2+x^2}} dx = \ln(x + \sqrt{a^2+x^2}) + C \quad (3)$$

故

$$\begin{aligned}
I &:= \int \sqrt{1+x^2} dx \\
&= x \sqrt{1+x^2} - \int x d \sqrt{1+x^2} \\
&= x \sqrt{1+x^2} - \int \frac{x^2}{\sqrt{1+x^2}} dx \\
&= x \sqrt{1+x^2} - \int \frac{x^2+1-1}{\sqrt{1+x^2}} dx \\
&= x \sqrt{1+x^2} - \int \sqrt{1+x^2} dx + \int \frac{1}{\sqrt{1+x^2}} dx \\
&= \frac{1}{2} \cdot x \sqrt{1+x^2} + \frac{1}{2} \int \frac{1}{\sqrt{1+x^2}} dx \\
&= \frac{1}{2} \cdot x \sqrt{1+x^2} + \frac{1}{2} \ln(x + \sqrt{1+x^2}) + C
\end{aligned}$$

11.

先做12.

由12, 得

$$\begin{aligned}
 I_5 &:= \int \sec^5 x dx \\
 &= \frac{1}{4} \sec^3 \tan x + \frac{3}{4} I_3 \\
 &= \frac{1}{4} \sec^3 \tan x + \frac{3}{4} \left(\frac{1}{2} \sec x \tan x + \frac{1}{2} I_1 \right) \\
 &= \frac{1}{4} \sec^3 \tan x + \frac{3}{8} \sec x \tan x + \frac{3}{8} \ln |\tan x + \sec x| + C
 \end{aligned}$$

12.

$$\begin{aligned}
 I_n &:= \int \sec^n x dx \\
 &= \int \sec^{n-2} x d \tan x \\
 &= \sec^{n-2} x \tan x - \int \tan x d \sec^{n-2} x \\
 &= \sec^{n-2} x \tan x - (n-2) \int \tan^2 x \cdot \sec^{n-2} x dx \\
 &= \sec^{n-2} x \tan x - (n-2) \int (\sec^2 x - 1) \cdot \sec^{n-2} x dx \\
 &= \sec^{n-2} x \tan x - (n-2) \int \sec^n x dx + (n-2) \int \sec^{n-2} x dx \\
 &= \frac{1}{n-1} \cdot \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x dx, (n > 2)
 \end{aligned}$$

13.

$$\begin{aligned}
 I_n &:= \int \tan^n x dx \\
 &= \int \tan^{n-2} x (\sec^2 x - 1) dx \\
 &= \int \tan^{n-2} x d \tan x - I_{n-2} \\
 &= \frac{1}{n-1} \tan^{n-1} x - I_{n-2}, (n > 1)
 \end{aligned}$$

14.

$$\begin{aligned}
 I_n &:= \int \csc^n x dx \\
 &= \int \csc^{n-2} x \cdot \csc^2 x dx \\
 &= - \int \csc^{n-2} x d \cot x \\
 &= - \csc^{n-2} x \cot x + \int \cot x d \csc^{n-2} x \\
 &= - \csc^{n-2} x \cot x + \int \cot x \cdot (n-2) \csc^{n-3} x \cdot (-\cot x \csc x) dx \\
 &= - \csc^{n-2} x \cot x - (n-2) \int \cot^2 x \csc^{n-2} x dx \\
 &= - \csc^{n-2} x \cot x - (n-2) \int (\csc^2 x - 1) \csc^{n-2} x dx \\
 &= - \csc^{n-2} x \cot x - (n-2) \int \csc^n x dx + (n-2) I_{n-2} \\
 &= - \frac{1}{n-1} \cdot \csc^{n-2} x \cot x + \frac{n-2}{n-1} I_{n-2}, (n > 1)
 \end{aligned}$$

15.

$$\begin{aligned}
I_n &:= \int \cot^n x dx \\
&= \int \cot^{n-2} x \cdot \cot^2 x dx \\
&= \int \cot^{n-2} x \cdot (\csc^2 x - 1) dx \\
&= - \int \cot^{n-2} x d \cot x - I_{n-2} \\
&= -\frac{1}{n-1} \cot^{n-1} x - I_{n-2}, (n > 1)
\end{aligned}$$

16.

$$\begin{aligned}
I &:= \int \frac{1}{\sin x \cos^2 x} dx \\
&= \int \frac{\sin x}{\sin^2 x \cos^2 x} dx \\
&= - \int \frac{1}{\sin^2 x \cos^2 x} d \cos x \\
&= - \int \frac{1}{(1 - \cos^2 x) \cos^2 x} d \cos x \\
&= - \int \frac{1 - \cos^2 x + \cos^2 x}{(1 - \cos^2 x) \cos^2 x} d \cos x \\
&= - \int \frac{1}{\cos^2 x} d \cos x - \int \frac{1}{1 - \cos^2 x} d \cos x \\
&= \sec x + \frac{1}{2} \ln \left| \frac{\cos x - 1}{\cos x + 1} \right| + C
\end{aligned}$$

17.

$$\begin{aligned}
I &:= \int \frac{5 + 4 \cos x}{(2 + \cos x)^2 \sin x} dx \\
&= - \int \frac{5 + 4 \cos x}{(2 + \cos x)^2 \sin^2 x} d \cos x \\
&= - \int \frac{5 + 4 \cos x}{(2 + \cos x)^2 (1 - \cos^2 x)} d \cos x \\
&= - \int \frac{4 + 4 \cos x + \cos^2 x + 1 - \cos^2 x}{(2 + \cos x)^2 (1 - \cos^2 x)} d \cos x \\
&= - \int \frac{1}{1 - \cos^2 x} d \cos x - \int \frac{1}{(2 + \cos x)^2} d \cos x \\
&= \frac{1}{2} \ln \left| \frac{\cos x - 1}{\cos x + 1} \right| + \frac{1}{2 + \cos x} + C
\end{aligned}$$

18.

$$\begin{aligned}
I &:= \int \frac{1}{1 + \cos^2 x} dx \\
&= \int \frac{1}{\sec^2 x + 1} d \tan x \\
&= \int \frac{1}{\tan^2 x + 2} d \tan x \\
&= \frac{1}{\sqrt{2}} \arctan \frac{\tan x}{\sqrt{2}} + C
\end{aligned}$$

19.

$$\begin{aligned}
I &:= \int \frac{1}{(3 \sin x + 2 \cos x)^2} dx \\
&= \int \frac{1}{(3 \tan x + 2)^2} d \tan x \\
&= -\frac{1}{9 \tan x + 6} + C
\end{aligned}$$

20.

$$\begin{aligned}
I_1 &:= \int \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} dx, (a \neq 0, b \neq 0) \\
&= \int \frac{1}{a^2 \tan^2 x + b^2} d \tan x \\
&= \frac{1}{a^2} \int \frac{1}{\tan^2 x + \frac{b^2}{a^2}} d \tan x \\
&= \frac{1}{ab} \arctan \frac{a \tan x}{b} + C \\
I_2 &:= \int \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} dx, (a = 0, b \neq 0) \\
&= \frac{1}{b^2} \int \sec^2 x dx \\
&= \frac{1}{b^2} \tan x + C \\
I_2 &:= \int \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} dx, (a \neq 0, b = 0) \\
&= \frac{1}{a^2} \int \csc^2 x dx \\
&= \frac{1}{a^2} \ln |\cot x - \csc x| + C
\end{aligned}$$

21.

$$\begin{aligned}
I &:= \int \frac{1}{\sin^4 x \cos^2 x} dx \\
&= \int \frac{1}{\sin^4 x} d \tan x \\
&= \int \frac{(1 + \tan^2 x)^2}{\tan^4 x} d \tan x \\
&= \int \frac{d \tan x}{\tan^4 x} + 2 \int \frac{d \tan x}{\tan^2 x} + x \\
&= -\frac{1}{3} \cot^3 x - 2 \cot x + x + C
\end{aligned}$$

22.

法1: 凑 $d \tan x$

$$\begin{aligned}
I &:= \int \sin^4 x \cos^2 x dx \\
&= \int \sin^4 x \cos^4 x d \tan x \\
&= \int \tan^4 x (1 + \tan^2 x)^{-3} d \tan x \\
&= \text{getting nuts}
\end{aligned}$$

法2: 使用降次公式

$$\begin{aligned}
 I &:= \int \sin^4 x \cos^2 x dx \\
 &= \int \left(\frac{1}{2} \sin 2x \right)^2 \cdot \left(\frac{1 - \cos 2x}{2} \right) dx \\
 &= \frac{1}{8} \int \sin^2 2x dx - \frac{1}{8} \int \cos 2x \sin^2 2x dx \\
 &= \frac{1}{8} \int \frac{1 - \cos 4x}{2} dx - \frac{1}{16} \int \cos 2x \sin^2 2x d(2x) \\
 &= \frac{1}{16} x - \frac{1}{16} \int \cos 4x dx - \frac{1}{16} \int \sin^2 2x d \sin 2x \\
 &= \frac{1}{16} x - \frac{1}{64} \int \cos 4x d(4x) - \frac{1}{48} \sin^3 2x \\
 &= \frac{1}{16} x - \frac{1}{64} \sin 4x - \frac{1}{48} \sin^3 2x + C
 \end{aligned}$$

法3: 倍角法

令

$$y = \cos x + \mathrm{i} \sin x$$

有

$$\frac{1}{y} = \cos x - \mathrm{i} \sin x$$

由上二式得

$$\begin{cases} y + \frac{1}{y} = 2 \cos x \\ y - \frac{1}{y} = 2\mathrm{i} \sin x \end{cases}$$

由棣莫弗公式

$$(\cos x + \mathrm{i} \sin x)^n = \cos nx + \mathrm{i} \sin nx$$

有

$$\begin{cases} y^n + \frac{1}{y^n} = 2 \cos nx \\ y^n - \frac{1}{y^n} = 2\mathrm{i} \sin nx \end{cases}$$

$$\begin{aligned}
 2^4 \mathrm{i}^4 \sin^4 x \cdot 2^2 \cos^2 x &= \left(y - \frac{1}{y} \right)^4 \left(y + \frac{1}{y} \right)^2 \\
 &= \left(y - \frac{1}{y} \right)^2 \left(y^2 - \frac{1}{y^2} \right)^2 \\
 &= \left(y^2 - 2 + \frac{1}{y^2} \right) \left(y^4 - 2 + \frac{1}{y^4} \right) \\
 &= y^6 - 2y^2 + \frac{1}{y^2} - 2y^4 + 4 - 2\frac{1}{y^4} + y^2 - 2\frac{1}{y^2} + \frac{1}{y^6} \\
 &= \left(y^6 + \frac{1}{y^6} \right) - 2 \left(y^4 + \frac{1}{y^4} \right) - \left(y^2 + \frac{1}{y^2} \right) + 4 \\
 &= 2 \cos 6x - 4 \cos 4x - 2 \cos 2x + 4
 \end{aligned}$$

故

$$\begin{aligned} I &:= \int \sin^4 x \cos^2 x dx \\ &= \frac{1}{64} \int (2 \cos 6x - 4 \cos 4x - 2 \cos 2x + 4) dx \\ &= \frac{1}{192} \sin 6x - \frac{1}{64} \sin 4x - \frac{1}{64} \sin 2x + \frac{1}{16} x + C \end{aligned}$$

23.

$$\begin{aligned} I &:= \int \frac{1 + \sin x + \cos x}{1 + \sin^2 x} dx \\ &= \int \frac{1}{1 + \sin^2 x} dx + \int \frac{\sin x}{1 + \sin^2 x} dx + \int \frac{\cos x}{1 + \sin^2 x} dx \\ &= \int \frac{1}{\sec^2 x + \tan^2 x} d \tan x - \int \frac{1}{2 - \cos^2 x} d \cos x + \arctan \sin x \\ &= \frac{1}{2} \int \frac{1}{\frac{1}{2} + \tan^2 x} d \tan x + \frac{1}{2\sqrt{2}} \ln \left| \frac{\cos x - \sqrt{2}}{\cos x + \sqrt{2}} \right| + \arctan \sin x \\ &= \frac{\sqrt{2}}{2} \arctan(\sqrt{2} \tan x) + \frac{1}{2\sqrt{2}} \ln \left| \frac{\cos x - \sqrt{2}}{\cos x + \sqrt{2}} \right| + \arctan \sin x + C \end{aligned}$$

24.

$$\begin{aligned} I &:= \int \frac{3 \sin x + 4 \cos x}{2 \sin x + \cos x} dx \\ &= \int \frac{p \cdot (2 \sin x + \cos x) + q \cdot (2 \cos x - \sin x)}{2 \sin x + \cos x} dx \\ &= \int \frac{2 \cdot (2 \sin x + \cos x) + 1 \cdot (2 \cos x - \sin x)}{2 \sin x + \cos x} dx \\ &= 2x + \ln |2 \sin x + \cos x| + C \end{aligned}$$

25.

$$\begin{aligned} I &:= \int \frac{1}{\sin x \sin 2x} dx \\ &= \frac{1}{2} \int \frac{1}{\sin^2 x \cos x} dx \\ &= \frac{1}{2} \int \frac{1}{\sin^2 x (1 - \sin^2 x)} d \sin x \\ &= \frac{1}{2} \int \frac{1 - \sin^2 x + \sin^2 x}{\sin^2 x (1 - \sin^2 x)} d \sin x \\ &= \frac{1}{2} \int \frac{1}{\sin^2 x} d \sin x + \frac{1}{2} \int \frac{1}{1 - \sin^2 x} d \sin x \\ &= -\frac{1}{2} \csc x - \frac{1}{4} \ln \left| \frac{\sin x - 1}{\sin x + 1} \right| + C \end{aligned}$$

26.

$$\begin{aligned}
I &:= \int \frac{1}{2 \sin x + \sin 2x} dx \\
&= \frac{1}{2} \int \frac{1}{\sin x (1 + \cos x)} dx \\
&= \frac{1}{2} \int \frac{1}{2 \sin \frac{x}{2} \cos \frac{x}{2} \cdot 2 \cos^2 \frac{x}{2}} dx \\
&= \frac{1}{4} \int \frac{1}{\sin \frac{x}{2} \cos^3 \frac{x}{2}} d \frac{x}{2} \\
&= \frac{1}{4} \int \frac{1}{\sin t \cos^3 t} dt \\
&= \frac{1}{4} \int \frac{\sec^2 t}{\tan t} d \tan t \\
&= \frac{1}{4} \int \frac{1 + \tan^2 t}{\tan t} d \tan t \\
&= \frac{1}{4} \ln |\tan t| + \frac{1}{8} \tan^2 t + C \\
&= \frac{1}{4} \ln \left| \tan \frac{x}{2} \right| + \frac{1}{8} \tan^2 \frac{x}{2} + C
\end{aligned}$$

27.

$$\begin{aligned}
I &:= \int \frac{\cos 2x - \sin 2x}{\sin x + \cos x} dx \\
&= \int \frac{\cos^2 x - \sin^2 x - 2 \sin x \cos x}{\sin x + \cos x} dx \\
&= \int \frac{\cos^2 x - \sin^2 x - 2 \sin x \cos x}{\sin x + \cos x} dx \\
&= \int (\cos x - \sin x) dx - \int \frac{(\sin x + \cos x)^2 - 1}{\sin x + \cos x} dx \\
&= \sin x + \cos x - \int (\sin x + \cos x) dx + \int \frac{1}{\sin x + \cos x} dx \\
&= 2 \cos x + \int \frac{1}{\sqrt{2} \sin(x + \frac{\pi}{4})} d \left(x + \frac{\pi}{4} \right) \\
&= 2 \cos x + \frac{1}{\sqrt{2}} \ln \left| \cot \left(x + \frac{\pi}{4} \right) - \csc \left(x + \frac{\pi}{4} \right) \right| + C
\end{aligned}$$

28.

$$\begin{aligned}
I &:= \int \sin 2x \sin 3x dx \\
&= \frac{1}{2} \int [\cos x - \cos 5x] dx \\
&= \frac{1}{2} \sin x - \frac{1}{10} \sin 5x + C
\end{aligned}$$

29.

$$\begin{aligned}
I &:= \int \frac{\sin x \cos x}{\sin x + \cos x} dx \\
&= \frac{1}{2} \int \frac{(\sin x + \cos x)^2 - 1}{\sin x + \cos x} dx \\
&= \frac{1}{2} \int (\sin x + \cos x) dx - \frac{1}{2} \int \frac{1}{\sin x + \cos x} dx \\
&= \frac{1}{2} (-\cos x + \sin x) - \frac{\sqrt{2}}{4} \ln \left| \cot \left(x + \frac{\pi}{4} \right) - \csc \left(x + \frac{\pi}{4} \right) \right|
\end{aligned}$$

30.

$$\begin{aligned}
I &:= \int \frac{\sin x \cos x}{\sin x - \cos x} dx \\
&= \frac{1}{2} \int \frac{1 - (\sin x - \cos x)^2}{\sin x - \cos x} dx \\
&= \frac{1}{2} \int \frac{1}{\sin x - \cos x} dx - \frac{1}{2} \int (\sin x - \cos x) dx \\
&= \frac{\sqrt{2}}{4} \ln \left| \cot \left(x - \frac{\pi}{4} \right) - \csc \left(x - \frac{\pi}{4} \right) \right| + \frac{1}{2} (\cos x + \sin x) + C
\end{aligned}$$

31.

$$\begin{aligned}
I &:= \int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx \\
&= \int \frac{\sin x \cos x}{(\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x} dx \\
&= \int \frac{\frac{1}{2} \sin 2x}{1 - \frac{1}{2} \sin^2 2x} dx \\
&= \frac{1}{2} \int \frac{\sin 2x}{2 - \sin^2 2x} d(2x) \\
&= -\frac{1}{2} \int \frac{1}{1 + \cos^2 t} d \cos t \\
&= -\frac{1}{2} \arctan \cos t + C \\
&= -\frac{1}{2} \arctan \cos 2x + C
\end{aligned}$$

32.

有公式

$$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$$

那么

$$\begin{aligned}
I &:= \int \frac{1}{\sin^6 x + \cos^6 x} dx \\
&= \int \frac{1}{(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x)} dx \\
&= \int \frac{1}{\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x} dx \\
&= \int \frac{1}{(\sin^2 x + \cos^2 x)^2 - 3 \sin^2 x \cos^2 x} dx \\
&= \int \frac{1}{1 - 3 \sin^2 x \cos^2 x} dx \\
&= \int \frac{1}{1 - 3 \left(\frac{1}{2} \sin 2x\right)^2} dx \\
&= \int \frac{4}{4 - 3 \sin^2 2x} dx \\
&= 2 \int \frac{1}{4 - 3 \sin^2 t} dt \\
&= 2 \int \frac{1}{1 + 3 \cos^2 t} dt \\
&= 2 \int \frac{1}{\sec^2 t + 3} d \tan t \\
&= 2 \int \frac{1}{4 + \tan^2 t} d \tan t \\
&= \arctan \frac{\tan t}{2} + C \\
&= \arctan \frac{\tan 2x}{2} + C
\end{aligned}$$

33.

$$\begin{aligned}
I &:= \int \frac{1}{\sin^3 x + \cos^3 x} dx \\
&= \int \frac{1}{(\sin x + \cos x)(\sin^2 x - \sin x \cos x + \cos^2 x)} dx \\
&= \int \frac{1}{(\sin x + \cos x)(1 - \sin x \cos x)} dx \\
&= \frac{1}{3} \int \frac{(\sin x + \cos x)^2 + 2(1 - \sin x \cos x)}{(\sin x + \cos x)(1 - \sin x \cos x)} dx \\
&= \frac{1}{3} \int \frac{\sin x + \cos x}{1 - \sin x \cos x} dx + \frac{2}{3} \int \frac{1}{\sin x + \cos x} dx \\
&= I_1 + \frac{\sqrt{2}}{3} \ln \left| \cot \left(x + \frac{\pi}{4} \right) - \csc \left(x + \frac{\pi}{4} \right) \right|
\end{aligned}$$

其中

$$\begin{aligned}
I_1 &:= \frac{1}{3} \int \frac{\sin x + \cos x}{1 - \sin x \cos x} dx \\
&= \frac{1}{3} \int \frac{1}{(\sin x - \cos x)^2 + \frac{1 - (\sin x - \cos x)^2}{2}} d(\sin x - \cos x) \\
&= \frac{2}{3} \int \frac{1}{t^2 + 1} dt \\
&= \frac{2}{3} \arctan t + C \\
&= \frac{2}{3} \arctan(\sin x - \cos x) + C
\end{aligned}$$

代回得

$$I = \frac{2}{3} \arctan(\sin x - \cos x) + \frac{\sqrt{2}}{3} \ln \left| \cot \left(x + \frac{\pi}{4} \right) - \csc \left(x + \frac{\pi}{4} \right) \right| + C$$

公式证明：

$$\int \csc x dx = \ln |\cot x - \csc x| + C \quad (1)$$

证明：

这是一个对 $\sin x$ 有奇性的积分。

$$\begin{aligned} I &:= \int \csc x dx \\ &= \int \frac{\sin x}{\sin^2 x} dx \\ &= - \int \frac{1}{1 - \cos^2 x} d \cos x \\ &= \frac{1}{2} \ln \left| \frac{\cos x - 1}{\cos x + 1} \right| + C \\ &= \frac{1}{2} \ln \left| \frac{(\cos x - 1)^2}{\cos^2 x - 1} \right| + C \\ &= \ln \left| \frac{\cos x - 1}{\sin x} \right| + C \\ &= \ln |\cot x - \csc x| + C \end{aligned}$$

$$\int \sec x dx = \ln |\tan x + \sec x| + C \quad (2)$$

证明：

这是一个对 $\cos x$ 有奇性的积分。

$$\begin{aligned} I &:= \int \sec x dx \\ &= \int \frac{d \sin x}{\cos^2 x} \\ &= \int \frac{d \sin x}{1 - \sin^2 x} \\ &= \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + C \\ &= \frac{1}{2} \ln \left| \frac{(1 + \sin x)^2}{1 - \sin^2 x} \right| + C \\ &= \ln \left| \frac{1 + \sin x}{\cos x} \right| + C \\ &= \ln |\sec x + \tan x| + C \end{aligned}$$

$$\int \frac{1}{\sqrt{a^2 + x^2}} dx = \ln \left(x + \sqrt{a^2 + x^2} \right) + C \quad (3)$$

证明：

$$\begin{aligned}
I &:= \int \frac{1}{\sqrt{a^2 + x^2}} dx \\
&= \int \frac{a \sec^2 t}{\sqrt{a^2 + a^2 \tan^2 t}} dt \\
&= \int \sec t dt \\
&= \ln |\sec t + \tan t| + C \\
&= \ln \left| \frac{1}{a} \sqrt{a^2 + x^2} + \frac{x}{a} \right| + C \\
&= \ln \left| x + \sqrt{a^2 + x^2} \right| + C
\end{aligned}$$