凯哥不定积分笔记——3 分部积分与换元

目录

套路集合:

- 套路1：使用换元法打开局面
- 套路2：利用分部积分，对分母进行降阶
  - 使用

$$\frac{1}{x^2}\mathrm{d}x = -\mathrm{d}\frac{1}{x}$$

- 套路3：拆分积分，使得积分抵消
  - 有些积分题，需要把一个积分拆成两个积分的和，其中一个 I_1 不动，另一个使用分部积分，使得分部之后得到的新积分和 I_1 抵消。这些题一般含有 e^x ，应用如下式子。
  - $\left[e^x f(x)\right]' = e^x\left[f(x) + f'(x)\right]$
- 套路 4：形如

$$\int \frac{\mathrm{d}x}{(x+d)\sqrt{ax^2+bx+c}}$$

或

$$\int \frac{\mathrm{d}x}{(x+d)^2\sqrt{ax^2+bx+c}}$$

的积分，可以用倒代换 $x+d=\frac{1}{t}$.

- 分部积分的口诀：”反对幂三指“

谁在后面，就把谁凑到 d 里面去。

如

$$\int xe^x\mathrm{d}x$$

故这里要把 e^x 凑到 d 里去。

即

$$\int xe^x\mathrm{d}x = \int x\mathrm{d}e^x = xe^x - \int e^x\mathrm{d}x$$

题目列表:

1.

$$\int \sqrt{\frac{x}{x+1}}\mathrm{d}x$$

2.

$$\int \frac{1}{x} \cdot \sqrt{\frac{x+1}{x}}\mathrm{d}x$$

3.

$$\int \sqrt{\frac{1-x}{1+x}}\mathrm{d}x$$

4.

$$\int \left(\sqrt{\frac{1-x}{1+x}} + \sqrt{\frac{1+x}{1-x}}\right)\mathrm{d}x$$

5.

$$\int \frac{xe^x}{\sqrt{e^x-2}}\mathrm{d}x$$

6.

$$\int \frac{1}{\sqrt[3]{(x+1)^2(x-1)^4}}\mathrm{d}x$$

7.

$$\int \frac{1}{(1+\sqrt[3]{x}) \cdot \sqrt{x}}\mathrm{d}x$$

8.

$$\int \frac{1}{1+\exp\frac{x}{2}+\exp\frac{x}{3}+\exp\frac{x}{6}}\mathrm{d}x$$

9.

$$\int \frac{\sqrt{4-x^2}}{x^4}\mathrm{d}x$$

10.

$$\int \frac{1}{\sqrt{(x^2+1)^3}}\mathrm{d}x$$

11.

$$\int \frac{1}{\sqrt{x(4-x)}}\mathrm{d}x$$

12.

$$\int x\sqrt{2x-x^2}\mathrm{d}x$$

13.

$$\int \frac{1}{x^2\sqrt{x^2-1}}\mathrm{d}x$$

14.

$$\int \frac{1}{x\sqrt{2x^2+2x+1}}\mathrm{d}x$$

15.

$$\int \frac{\mathrm{d}x}{x^2\sqrt{2x^2+2x+1}}$$

16.

$$\int x\arctan x\mathrm{d}x$$

17.

$$\int x\ln(1+x^2)\arctan x\mathrm{d}x$$

18.

$$\int e^x\sin x\mathrm{d}x$$

19.

$$\int xe^x\sin x\mathrm{d}x$$

20.

$$\int \ln^2\left(x+\sqrt{1+x^2}\right)\mathrm{d}x$$

21.

$$\int e^{2x}\arctan\sqrt{e^x-1}\mathrm{d}x$$

22.

$$\int \frac{xe^{\arctan x}}{(1+x^2)^{\frac{3}{2}}}\mathrm{d}x$$

23.

$$\int \frac{e^{\arctan x}}{(1+x^2)^{\frac{3}{2}}}\mathrm{d}x$$

24.

$$\int \frac{\ln x}{(1+x^2)^{\frac{3}{2}}}\mathrm{d}x$$

25.

$$\int \frac{\arctan\sqrt{x-1}}{x\sqrt{x-1}}\mathrm{d}x$$

26.

$$\int \frac{\sqrt{x-1}\arctan\sqrt{x-1}}{x}\mathrm{d}x$$

27.

$$\int \ln\left(1+\sqrt{\frac{1+x}{x}}\right)\mathrm{d}x$$

28.

$$\int \arctan(1+\sqrt{x})\mathrm{d}x$$

29.

$$\int \frac{xe^x}{(1+x)^2}\mathrm{d}x$$

30.

$$\int \frac{x^2e^x}{(x+2)^2}\mathrm{d}x$$

31.

$$\int \frac{xe^x}{(1+e^x)^2}\mathrm{d}x$$

32.

$$\int \frac{x^2}{(x\sin x+\cos x)^2}\mathrm{d}x$$

33.

$$\int_0^{+\infty} \frac{e^{-x^2}}{\left(x^2+\frac{1}{2}\right)^2}\mathrm{d}x$$

34.

$$\int \frac{xe^x}{(1+x)^2}\mathrm{d}x$$

35.

$$\int e^x\left(\frac{1-x}{1+x^2}\right)^2\mathrm{d}x$$

36.

$$\int \frac{1+\sin x}{1+\cos x}\cdot e^x\mathrm{d}x$$

37.

$$\int \frac{e^{-\sin x}\cdot\sin 2x}{\sin^4\left(\frac{\pi}{4}-\frac{x}{2}\right)}\mathrm{d}x$$

38.

$$\int e^{-\frac{x}{2}} \cdot \frac{\cos x - \sin x}{\sqrt{\sin x}} \mathrm{d}x$$

39.

$$\int e^{\sin x} \cdot \frac{x \cos^3 x - \sin x}{\cos^2 x} \mathrm{d}x$$

40.

$$\int \left(\ln \ln x + \frac{1}{\ln x} \right) \mathrm{d}x$$

41.

已知 $f^{''}(x)$ 连续，$f^{'}(x) \neq 0$. 求

$$\int \left[\frac{f(x)}{f^{'}(x)} - \frac{f^2(x) f^{''}(x)}{[f^{'}(x)]^3} \right] \mathrm{d}x$$

42.

$$\int \frac{1 - \ln x}{(x - \ln x)^2} \mathrm{d}x$$

43.

$$\int \left(\frac{\arctan x}{\arctan x - x} \right)^2 \mathrm{d}x$$

一些要记的:

1.

$$\int \frac{\mathrm{d}x}{\sqrt{1+x^2}} = \ln\left(x + \sqrt{1+x^2}\right) + C$$

2.

$$f(x) := \ln\left(x + \sqrt{1+x^2}\right)$$

是奇函数。

3.

$$\ln\left(x + \sqrt{1+x^2}\right) \sim x, x \to 0$$

解答之:

1.

令 $t=\sqrt{\frac{x}{x+1}}$ ，$x=\frac{t^2}{1-t^2}$，$\mathrm{d}x=\frac{2t}{(1-t^2)^2}\mathrm{d}t.$

那么

$$\begin{aligned}I&:=\int\sqrt{\frac{x}{x+1}}\mathrm{d}x\\&=\int t\cdot\frac{2t}{(1-t^2)^2}\mathrm{d}t\end{aligned}$$

疯了。重来

令 $t=\sqrt{\frac{x}{x+1}}$ ，那么 $t^2=\frac{x}{x+1}=1-\frac{1}{x+1}$ ，得 $\frac{1}{x+1}=1-t^2$ ，即 $x+1=\frac{1}{1-t^2}$. 就此打住。

两边同时取微分得 $\mathrm{d}x=\mathrm{d}\frac{1}{1-t^2}$. 也是就此打住。如果把右边的微分展开，将会如上面那样复杂。直接来

$$\begin{aligned}I&:=\int\sqrt{\frac{x}{x+1}}\mathrm{d}x\\&=\int t\mathrm{d}\frac{1}{1-t^2}\\&=\frac{t}{1-t^2}-\int\frac{1}{1-t^2}\mathrm{d}t\\&=\frac{t}{1-t^2}+\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+C\\&=(x+1)\sqrt{\frac{x}{x+1}}+\frac{1}{2}\ln\left|\frac{\sqrt{\frac{x}{x+1}}-1}{\sqrt{\frac{x}{x+1}}+1}\right|+C\end{aligned}$$

2.

令 $t=\sqrt{\frac{x+1}{x}}$ ，那么 $t^2=1+\frac{1}{x}$ ，$x=\frac{1}{t^2-1}$ ，$\mathrm{d}x=\mathrm{d}\frac{1}{t^2-1}.$

$$\begin{aligned}I&:=\int\frac{1}{x}\cdot\sqrt{\frac{x+1}{x}}\mathrm{d}x\\&=\int(t^2-1)\cdot t\cdot\mathrm{d}\frac{1}{t^2-1}\\&=t-\int\frac{1}{t^2-1}\mathrm{d}(t^3-t)\\&=t-\int\frac{1}{t^2-1}\cdot(3t^2-1)\mathrm{d}t\\&=t-\int\frac{1}{t^2-1}\cdot(3t^2-3)\mathrm{d}t-2\int\frac{1}{t^2-1}\mathrm{d}t\\&=t-3t-\ln\left|\frac{t-1}{t+1}\right|+C\\&=-2t-\ln\left|\frac{t-1}{t+1}\right|+C\\&=-2\sqrt{\frac{x+1}{x}}-\ln\left|\frac{\sqrt{\frac{x+1}{x}}-1}{\sqrt{\frac{x+1}{x}}+1}\right|+C\end{aligned}$$

3.

令 $t=\sqrt{\frac{1-x}{1+x}}$ ，$t^2=\frac{1-x}{1+x}=\frac{1+x-2x}{1+x}=1-\frac{2x}{1+x}$ ，$\frac{1+x-1}{1+x}=1-\frac{1}{1+x}=\frac{1}{2}(1-t^2)$ ，$\frac{1}{1+x}=\frac{1}{2}(1+t^2)$，$1+x=\frac{2}{1+t^2}$. $\mathrm{d}x=\mathrm{d}\frac{2}{1+t^2}.$

$$
\begin{aligned}
I &:= \int \sqrt{\frac{1-x}{1+x}} \mathrm{d}x \\
&= \int t \mathrm{d} \frac{2}{1+t^2} \\
&= \frac{2t}{1+t^2} - 2\int \frac{1}{1+t^2} \mathrm{d}t \\
&= \frac{2t}{1+t^2} - 2\arctan t + C \\
&= (1+x)\sqrt{\frac{1-x}{1+x}} - 2\arctan \sqrt{\frac{1-x}{1+x}} + C
\end{aligned}
$$

4.

$$
\begin{aligned}
f(x) &:= \sqrt{\frac{1-x}{1+x}} + \sqrt{\frac{1+x}{1-x}} \\
&= \frac{1-x}{\sqrt{1-x^2}} + \frac{1+x}{\sqrt{1-x^2}} \\
&= \frac{2}{\sqrt{1-x^2}}
\end{aligned}
$$

故

$$
\begin{aligned}
I &:= \int \left(\sqrt{\frac{1-x}{1+x}} + \sqrt{\frac{1+x}{1-x}} \right) \mathrm{d}x \\
&= 2\int \frac{1}{\sqrt{1-x^2}} \mathrm{d}x \\
&= 2\arcsin x + C
\end{aligned}
$$

5.

令 $t = \sqrt{e^x - 2}$， $\ln(t^2+2) = x$，$\mathrm{d}x = \frac{2t}{t^2+2}\mathrm{d}t$.

$$
\begin{aligned}
I &:= \int \frac{xe^x}{\sqrt{e^x-2}} \mathrm{d}x \\
&= \int \frac{\ln(t^2+2)(t^2+2)}{t} \cdot \frac{2t}{t^2+2} \mathrm{d}t \\
&= 2\int \ln(t^2+2) \mathrm{d}t \\
&= 2t\ln(t^2+2) - 4\int \frac{t^2}{t^2+2} \mathrm{d}t \\
&= 2t\ln(t^2+2) - 4\int \frac{t^2+2-2}{t^2+2} \mathrm{d}t \\
&= 2t\ln(t^2+2) - 4t + 8\int \frac{1}{t^2+2} \mathrm{d}t \\
&= 2t\ln(t^2+2) - 4t + \frac{8}{\sqrt{2}} \arctan \frac{t}{\sqrt{2}} + C \\
&= 2x\sqrt{e^x-2} - 4\sqrt{e^x-2} + \frac{8}{\sqrt{2}} \arctan \frac{\sqrt{e^x-2}}{\sqrt{2}} + C
\end{aligned}
$$

6.

这里关键要把 $x+1, x-1$ 整出来。。

$$
\begin{aligned}
I &:= \int \frac{1}{\sqrt[3]{(x+1)^2(x-1)^4}} \mathrm{d}x \\
&= \int \frac{1}{(x-1)\sqrt[3]{(x+1)^2(x-1)}} \mathrm{d}x \\
&= \int \frac{1}{(x^2-1)\sqrt[3]{\frac{x-1}{x+1}}} \mathrm{d}x
\end{aligned}
$$

令 $t=\sqrt[3]{\frac{x-1}{x+1}}$ ，$1+x=\frac{2}{1-t^2}$，$x-1=\frac{2t^2}{1-t^2}$，$\mathrm{d}x=\mathrm{d}\frac{2}{1-t^3}$.

$$
\begin{aligned}
I &:= \int \frac{1}{\sqrt[3]{(x+1)^2(x-1)^4}} \mathrm{d}x \\
&= \int \frac{1}{(x^2-1)\sqrt[3]{\frac{x-1}{x+1}}} \mathrm{d}x \\
&= \int \frac{1}{\frac{4t^2}{(1-t^2)^2}\cdot t} \cdot \mathrm{d}\frac{2}{1-t^2} \\
&= \frac{1}{2}\int \frac{(1-t^2)^2}{t^3} \cdot \frac{2t}{(1-t^2)^2} \mathrm{d}t \\
&= \int \frac{1}{t^2} \mathrm{d}t \\
&= -t^{-1} + C \\
&= -\sqrt[3]{\frac{x+1}{x-1}} + C
\end{aligned}
$$

7.

令 $x=t^6$ ，$\mathrm{d}x=6t^5\mathrm{d}t$.

$$
\begin{aligned}
I &:= \int \frac{1}{(1+\sqrt[3]{x})\cdot\sqrt{x}} \mathrm{d}x \\
&= 6\int \frac{t^2}{1+t^2} \mathrm{d}t \\
&= 6t - 6\arctan t + C \\
&= 6\sqrt[6]{x} - 6\arctan\sqrt[6]{x} + C
\end{aligned}
$$

8.

令 $\exp\frac{x}{6}=t$，$x=6\ln t$ ，$\mathrm{d}x=\frac{6}{t}\mathrm{d}t$.

$$
\begin{aligned}
I &:= \int \frac{1}{1+\exp\frac{x}{2}+\exp\frac{x}{3}+\exp\frac{x}{6}}\mathrm{d}x \\
&= 6\int \frac{1}{1+t^3+t^2+t}\cdot\frac{1}{t}\mathrm{d}t \\
&= 6\int \frac{1-t}{1-t^4}\cdot\frac{1}{t}\mathrm{d}t \\
&= 6\int \frac{1-t}{(1+t^2)(1-t^2)}\cdot\frac{1}{t}\mathrm{d}t \\
&= 6\int \frac{1}{t(1+t^2)(1-t)}\mathrm{d}t \\
&= 6\int \left(\frac{A}{t}+\frac{Bt+C}{1+t^2}+\frac{D}{1-t}\right)\mathrm{d}t \\
&= 6\int \left(\frac{1}{t}+\frac{-\frac{1}{2}t+\frac{1}{2}}{1+t^2}+\frac{\frac{1}{2}}{1-t}\right)\mathrm{d}t \\
&= \int \left(\frac{6}{t}+\frac{-3t+3}{1+t^2}+\frac{3}{1-t}\right)\mathrm{d}t \\
&= 6\ln|t|-3\ln|t-1|-3\int\frac{t}{1+t^2}\mathrm{d}t+3\int\frac{1}{1+t^2}\mathrm{d}t \\
&= 6\ln|t|-3\ln|t-1|-\frac{3}{2}\ln|1+t^2|+3\arctan t+C \\
&= x-3\ln\left|\exp\frac{x}{6}-1\right|-\frac{3}{2}\ln\left|1+\exp\frac{x}{3}\right|+3\arctan\exp\frac{x}{6}+C
\end{aligned}
$$

9.

令 $x=2\sin t$， $\mathrm{d}x=2\cos t\mathrm{d}t.$

$$
\begin{aligned}
I &:= \int \frac{\sqrt{4-x^2}}{x^4}\mathrm{d}x \\
&= \int \frac{2\cos t}{16\sin^4 t}\cdot 2\cos t\mathrm{d}t \\
&= \frac{1}{4}\int \frac{\cos^2 t}{\sin^4 t}\mathrm{d}t \\
&= \frac{1}{4}\int \frac{1}{\tan^4 t}\mathrm{d}\tan t \\
&= -\frac{1}{12}\tan^{-3} t+C \\
&= -\frac{1}{12}\frac{\sqrt{(1-x^2/4)^3}}{(x^3/8)}+C \\
&= -\frac{1}{12}\frac{\sqrt{(4-x^2)^3}}{x^3}+C
\end{aligned}
$$

10.

令 $x=\tan t$， $\mathrm{d}x=\sec^2 t\mathrm{d}t.$

$$
\begin{aligned}
I &:= \int \frac{1}{\sqrt{(x^2+1)^3}} \mathrm{d}x \\
&= \int \frac{1}{\sec^3 t} \cdot \sec^2 t \mathrm{d}t \\
&= \int \cos t \mathrm{d}t \\
&= \sin t + C \\
&= \frac{\tan t}{\sqrt{\sec^2 t}} + C \\
&= \frac{x}{\sqrt{1+x^2}} + C
\end{aligned}
$$

11.

$$
\begin{aligned}
I &:= \int \frac{1}{\sqrt{x(4-x)}} \mathrm{d}x \\
&= \int \frac{1}{\sqrt{4x-x^2}} \mathrm{d}x \\
&= \int \frac{1}{\sqrt{4-(x-2)^2}} \mathrm{d}x
\end{aligned}
$$

令 $x-2=2\sin t$， $\mathrm{d}x = 2\cos t \mathrm{d}t.$

$$
\begin{aligned}
I &:= \int \frac{1}{\sqrt{x(4-x)}} \mathrm{d}x \\
&= \int \frac{1}{\sqrt{4-(x-2)^2}} \mathrm{d}x \\
&= \int \frac{2\cos t}{2\cos t} \mathrm{d}t \\
&= t + C \\
&= \arcsin \frac{x-2}{2} + C
\end{aligned}
$$

12.

$$
\begin{aligned}
I &:= \int x\sqrt{2x-x^2} \mathrm{d}x \\
&= \int x\sqrt{1-(x-1)^2} \mathrm{d}x
\end{aligned}
$$

令 $x-1=\sin t$， $\mathrm{d}x = \cos t \mathrm{d}t.$

$$
\begin{aligned}
I &:= \int x\sqrt{2x-x^2} \mathrm{d}x \\
&= \int x\sqrt{1-(x-1)^2} \mathrm{d}x \\
&= \int (\sin t + 1)\cos^2 t \mathrm{d}t \\
&= -\int \cos^2 t \mathrm{d}\cos t + \int \frac{\cos 2x + 1}{2} \mathrm{d}t \\
&= -\frac{1}{3}\cos^3 t + \frac{1}{4}\sin 2x + \frac{1}{2}t + C \\
&= -\frac{1}{3}\sqrt{(2x-x^2)^3} + \frac{1}{4}(x-1)\sqrt{2x-x^2} + \frac{1}{2}\arcsin(x-1) + C
\end{aligned}
$$

13.

法1：三角换元

令 $x=\sec t$ ，$\mathrm{d}x=\sec t\tan t\mathrm{d}t$.

$$\begin{aligned}
I:&=\int\frac{1}{x^2\sqrt{x^2-1}}\mathrm{d}x\\
&=\int\frac{\sec t\tan t}{\sec^2 t\tan t}\mathrm{d}t\\
&=\int\cos t\mathrm{d}t\\
&=\sin t+C\\
&=\sqrt{1-\sec^{-2}t}+C\\
&=\sqrt{1-x^{-2}}+C
\end{aligned}$$

法2：倒代换 + 三角换元

令 $x=\frac{1}{t}$ ，$\mathrm{d}x=-\frac{1}{t^2}\mathrm{d}t$.

$$\begin{aligned}
I:&=\int\frac{1}{x^2\sqrt{x^2-1}}\mathrm{d}x\\
&=-\int t^2\cdot\sqrt{t^{-2}-1}\cdot\frac{1}{t^2}\mathrm{d}t\\
&=-\int\frac{\sqrt{1-t^2}}{t}\mathrm{d}t
\end{aligned}$$

令 $t=\sin u$，$\mathrm{d}t=\cos u\mathrm{d}u$.

$$\begin{aligned}
I&=-\int\frac{\sqrt{1-t^2}}{t}\mathrm{d}t\\
&=-\int\frac{\cos^2 u}{\sin u}\mathrm{d}u\\
&=-\int\frac{1-\sin^2 u}{\sin u}\mathrm{d}u\\
&=-\int\csc u\mathrm{d}u+\int\sin u\mathrm{d}u\\
&=-\ln|\cot u-\csc u|-\cos u+C
\end{aligned}$$

回代疯了。

14.

这题用倒代换才好一点。

令 $x=t^{-1}$ ，$\mathrm{d}x=-t^{-2}\mathrm{d}t$.

$$\begin{aligned}
I:&=\int\frac{1}{x\sqrt{2x^2+2x+1}}\mathrm{d}x\\
&=-\int\frac{t}{\sqrt{2t^{-2}+2t^{-1}+1}}\cdot t^{-2}\mathrm{d}t\\
&=-\int\frac{1}{\sqrt{2+2t+t^2}}\mathrm{d}t\\
&=-\int\frac{1}{\sqrt{(t+1)^2+1}}\mathrm{d}t\\
&=-\ln\left[t+1+\sqrt{(t+1)^2+1}\right]+C\\
&=-\ln\left[x^{-1}+1+\sqrt{(x^{-1}+1)^2+1}\right]+C
\end{aligned}$$

15.

如 14 令

$$
\begin{aligned}
I &:= \int \frac{1}{x^2\sqrt{2x^2+2x+1}}\mathrm{d}x \\
&= -\int \frac{t^2}{\sqrt{2t^{-2}+2t^{-1}+1}} \cdot t^{-2}\mathrm{d}t \\
&= -\int \frac{t}{\sqrt{2+2t+t^2}}\mathrm{d}t \\
&= -\int \frac{t+1-1}{\sqrt{(t+1)^2+1}}\mathrm{d}t \\
&= -\frac{1}{2}\int \frac{1}{\sqrt{(t+1)^2+1}}\mathrm{d}[(t+1)^2] + \int \frac{1}{\sqrt{(t+1)^2+1}}\mathrm{d}t \\
&= -\frac{1}{2}\int \frac{\mathrm{d}u}{\sqrt{u+1}} + \ln\left(t+1+\sqrt{(t+1)^2+1}\right) + C \\
&= \cdots
\end{aligned}
$$

16.

$$
\begin{aligned}
I &:= \int x\arctan x\mathrm{d}x \\
&= \frac{1}{2}\int \arctan x\mathrm{d}(x^2+{\color{blue}1}) \\
&= \frac{1}{2}(x^2+1)\arctan x - \frac{1}{2}\int (x^2+{\color{blue}1}) \cdot \frac{1}{1+x^2}\mathrm{d}(x^2+1) \\
&= \frac{1}{2}(x^2+1)\arctan x - \frac{1}{2}x + C
\end{aligned}
$$

注意，适时地添加常数（蓝色部分），使积分大大简化。

17.

$$
\begin{aligned}
I &:= \int x\ln(1+x^2)\arctan x\mathrm{d}x \\
&= \frac{1}{2}\int \ln(1+x^2)\arctan x\mathrm{d}(x^2+1) \\
&= \frac{1}{2}(x^2+1)\ln(1+x^2)\arctan x - \frac{1}{2}\int (x^2+1)\mathrm{d}\ln(1+x^2)\arctan x \\
&= \frac{1}{2}(x^2+1)\ln(1+x^2)\arctan x - \frac{1}{2}\int (x^2+1)\cdot\left[\frac{2x\arctan x}{x^2+1} + \frac{\ln(1+x^2)}{x^2+1}\right]\mathrm{d}x \\
&= \frac{1}{2}(x^2+1)\ln(1+x^2)\arctan x - \frac{1}{2}\int \left[2x\arctan x + \ln(1+x^2)\right]\mathrm{d}x \\
&= \frac{1}{2}(x^2+1)\ln(1+x^2)\arctan x + I_1 + I_2
\end{aligned}
$$

其中

$$
\begin{aligned}
I_1 &:= \int x\arctan x\mathrm{d}x \\
&= \frac{1}{2}\int \arctan x\mathrm{d}(x^2+1) \\
&= \frac{1}{2}(x^2+1)\arctan x - \frac{1}{2}x + C_1
\end{aligned}
$$

$$
\begin{aligned}
I_2 &:= \frac{1}{2}\int \ln(1+x^2)\mathrm{d}x \\
&= x\ln(1+x^2) - \frac{1}{2}\int x\cdot\frac{2x}{1+x^2}\mathrm{d}x \\
&= \frac{1}{2}x\ln(1+x^2) - \int \frac{x^2+1-1}{1+x^2}\mathrm{d}x \\
&= \frac{1}{2}x\ln(1+x^2) - x + \arctan x + C_2
\end{aligned}
$$

故

$$I=\frac{1}{2}(x^2+1)\ln(1+x^2)\arctan x+\frac{1}{2}(x^2+1)\arctan x-\frac{3}{2}x+\frac{1}{2}x\ln(1+x^2)+\arctan x+C$$

18.

法1：分部积分直到积分重现

$$\begin{aligned}I:=&\int e^x\sin x\mathrm{d}x\\=&\int\sin x\mathrm{d}e^x\\=&e^x\sin x-\int e^x\mathrm{d}\sin x\\=&e^x\sin x-\int e^x\cos x\mathrm{d}x\\=&e^x\sin x-\int\cos x\mathrm{d}e^x\\=&e^x\sin x-e^x\cos x+\int e^x\mathrm{d}\cos x\\=&e^x\sin x-e^x\cos x-\int e^x\sin x\mathrm{d}x\\=&e^x\sin x-e^x\cos x-I\\=&\frac{1}{2}(e^x\sin x-e^x\cos x)+C\end{aligned}$$

法2：组合积分法

记

$$\begin{cases}I:=\int e^x\sin x\mathrm{d}x\\J:=\int e^x\cos x\mathrm{d}x\end{cases}$$

又

$$\begin{cases}(e^x\sin x)'=e^x\sin x+e^x\cos x\\(e^x\cos x)'=e^x\cos x-e^x\sin x\end{cases}$$

两边同时积分得

$$\begin{cases}e^x\sin x=\int e^x\sin x\mathrm{d}x+\int e^x\cos x\mathrm{d}x=I+J\\e^x\cos x=\int e^x\cos x\mathrm{d}x-\int e^x\sin x\mathrm{d}x=J-I\end{cases}$$

故

$$I=\frac{1}{2}[(I+J)-(J-I)]=\frac{1}{2}(e^x\sin x-e^x\cos x)+C$$

19.

法1：分部积分法

$$
\begin{aligned}
I &:= \int xe^x \sin x \mathrm{d}x \\
&= \int x \sin x \mathrm{d}e^x \\
&= xe^x \sin x - \int e^x \mathrm{d}(x \sin x) \\
&= xe^x \sin x - \int e^x (\sin x + x \cos x) \mathrm{d}x \\
&= xe^x \sin x - I_1 - \int xe^x \cos x \mathrm{d}x \\
&= xe^x \sin x - I_1 - \int x \cos x \mathrm{d}e^x \\
&= xe^x \sin x - I_1 - xe^x \cos x + \int e^x \mathrm{d}(x \cos x) \\
&= xe^x \sin x - I_1 - xe^x \cos x + \int e^x (\cos x - x \sin x) \mathrm{d}x \\
&= xe^x \sin x - I_1 - xe^x \cos x + J_1 - I \\
&= \frac{1}{2}(xe^x \sin x - I_1 - xe^x \cos x + J_1) \\
&= \frac{1}{2}(xe^x \sin x - xe^x \cos x + e^x \cos x) + C
\end{aligned}
$$

法2：组合积分法

之前用组合积分做过一次这个积分，但是很长，所以这里不再写了。

法3：利用上题结论

$$
\begin{aligned}
I &:= \int xe^x \sin x \mathrm{d}x \\
&= \int x \mathrm{d}\left[\frac{1}{2}(e^x \sin x - e^x \cos x)\right] \\
&= \frac{x}{2}(e^x \sin x - e^x \cos x) - \int \frac{1}{2}(e^x \sin x - e^x \cos x) \mathrm{d}x \\
&= \frac{x}{2}(e^x \sin x - e^x \cos x) - \frac{1}{2}\left[\frac{(I+J)-(J-I)}{2} - \frac{(I+J)+(J-I)}{2}\right] \\
&= \frac{x}{2}(e^x \sin x - e^x \cos x) + \frac{J-I}{2} \\
&= \frac{x}{2}(e^x \sin x - e^x \cos x) + \frac{1}{2}e^x \cos x + C
\end{aligned}
$$

20.

$$
\begin{aligned}
I &:= \int \ln^2\left(x + \sqrt{1+x^2}\right) \mathrm{d}x \\
&= x \ln^2\left(x + \sqrt{1+x^2}\right) - \int x \mathrm{d} \ln^2\left(x + \sqrt{1+x^2}\right) \\
&= x \ln^2\left(x + \sqrt{1+x^2}\right) - \int x \cdot \frac{1}{\sqrt{1+x^2}} \cdot 2 \ln\left(x + \sqrt{1+x^2}\right) \mathrm{d}x \\
&= x \ln^2\left(x + \sqrt{1+x^2}\right) - \int \frac{1}{\sqrt{1+x^2}} \cdot \ln\left(x + \sqrt{1+x^2}\right) \mathrm{d}(x^2+1) \\
&= x \ln^2\left(x + \sqrt{1+x^2}\right) - 2 \int \ln\left(x + \sqrt{1+x^2}\right) \mathrm{d}\sqrt{1+x^2} \\
&= x \ln^2\left(x + \sqrt{1+x^2}\right) - 2\sqrt{1+x^2} \ln\left(x + \sqrt{1+x^2}\right) + 2 \int \sqrt{1+x^2} \cdot \frac{1}{\sqrt{1+x^2}} \mathrm{d}x \\
&= x \ln^2\left(x + \sqrt{1+x^2}\right) - 2\sqrt{1+x^2} \ln\left(x + \sqrt{1+x^2}\right) + 2x + C
\end{aligned}
$$

21.

令 $t=\sqrt{e^x-1}$ ，$\ln(t^2+1)=x$，$\mathrm{d}x=\frac{2t}{t^2+1}\mathrm{d}t$.

$$
\begin{aligned}
I &:= \int e^{2x}\arctan\sqrt{e^x-1}\mathrm{d}x \\
&= \int 2t(t^2+1)\arctan t\mathrm{d}t \\
&= \int \arctan t\mathrm{d}\left(\frac{1}{2}t^4+t^2\right) \\
&= \left(\frac{1}{2}t^4+t^2\right)\arctan t-\int\left(\frac{1}{2}t^4+t^2\right)\cdot\frac{1}{1+t^2}\mathrm{d}t \\
&= \left(\frac{1}{2}t^4+t^2\right)\arctan t-\int\left(\frac{1}{2}t^4+\frac{1}{2}t^2+\frac{1}{2}t^2\right)\cdot\frac{1}{1+t^2}\mathrm{d}t \\
&= \left(\frac{1}{2}t^4+t^2\right)\arctan t-\frac{1}{2}\int t^2\mathrm{d}t-\frac{1}{2}\int\frac{t^2}{1+t^2}\mathrm{d}t \\
&= \left(\frac{1}{2}t^4+t^2\right)\arctan t-\frac{1}{6}t^3-\frac{1}{2}\int\frac{t^2+1-1}{1+t^2}\mathrm{d}t \\
&= \left(\frac{1}{2}t^4+t^2\right)\arctan t-\frac{1}{6}t^3-\frac{1}{2}t+\frac{1}{2}\arctan t+C \\
&= \left(\frac{1}{2}(e^x-1)^2+e^x-1\right)\arctan\sqrt{e^x-1}-\frac{1}{6}\sqrt{(e^x-1)^3}-\frac{1}{2}\sqrt{e^x-1}+\frac{1}{2}\arctan\sqrt{e^x-1}+C
\end{aligned}
$$

这里在换元之后凑微分，可把 $2t(t^2+1)$ 凑成 $\mathrm{d}\left[\frac{1}{2}(t^2+1)^2\right]$.

$$
\begin{aligned}
I &:= \int e^{2x}\arctan\sqrt{e^x-1}\mathrm{d}x \\
&= \int 2t(t^2+1)\arctan t\mathrm{d}t \\
&= \frac{1}{2}\int\arctan t\mathrm{d}\left[(t^2+1)^2\right] \\
&= \frac{1}{2}(t^2+1)^2\arctan t-\frac{1}{2}\int(t^2+1)\mathrm{d}t \\
&= \frac{1}{2}(t^2+1)^2\arctan t-\frac{1}{6}t^3-\frac{1}{2}t+C
\end{aligned}
$$

22.

令 $t=\arctan x$ ，$\mathrm{d}x=\sec^2 t\mathrm{d}t$.

$$
\begin{aligned}
I &:= \int\frac{xe^{\arctan x}}{(1+x^2)^{\frac{3}{2}}}\mathrm{d}x \\
&= \int\frac{\tan te^t}{\sec^3 t}\cdot\sec^2 t\mathrm{d}t \\
&= \int e^t\sin t\mathrm{d}t \\
&= \frac{e^t}{2}(\sin t-\cos t)+C \\
&= \frac{e^{\arctan x}}{2}\left(\frac{x-1}{\sqrt{1+x^2}}\right)+C
\end{aligned}
$$

23.

如 22. 令

$$
\begin{aligned}
I &:= \int\frac{e^{\arctan x}}{(1+x^2)^{\frac{x}{2}}}\mathrm{d}x \\
&= \int e^t\cos t\mathrm{d}t \\
&= \frac{e^t}{2}(\sin x+\cos t)+C \\
&= \frac{e^{\arctan x}}{2}\left(\frac{x+1}{\sqrt{1+x^2}}\right)+C
\end{aligned}
$$

24.

如 22 令

$$
\begin{aligned}
I :=& \int \frac{\ln x}{(1+x^2)^{\frac{3}{2}}}\mathrm{d}x \\
&= \int \cos t \ln \tan t \mathrm{d}t \\
&= \int \ln \tan t \mathrm{d} \sin t \\
&= \sin t \ln \tan t - \int \sin t \cdot \cot t \cdot \sec^2 t \mathrm{d}t \\
&= \sin t \ln \tan t - \int \frac{1}{\cos t}\mathrm{d}t \\
&= \sin t \ln \tan t - \ln|\sec t + \tan t| + C \\
&= \frac{t}{\sqrt{1+x^2}}\ln x - \ln|\sqrt{1+x^2}+x| + C
\end{aligned}
$$

25.

令 $t=\sqrt{x-1}$, $x=t^2+1$, $\mathrm{d}x=2t\mathrm{d}t$.

$$
\begin{aligned}
I :=& \int \frac{\arctan\sqrt{x-1}}{x\sqrt{x-1}}\mathrm{d}x \\
&= \int \frac{\arctan t}{(t^2+1)t}\cdot 2t\mathrm{d}t \\
&= 2\int \arctan t \mathrm{d} \arctan t \\
&= \arctan^2 t + C \\
&= \arctan^2 \sqrt{x-1} + C
\end{aligned}
$$

26.

如 25. 令

$$
\begin{aligned}
I :=& \int \frac{\sqrt{x-1}\arctan\sqrt{x-1}}{x}\mathrm{d}x \\
&= \int \frac{t\arctan t}{t^2+1}\cdot 2t\mathrm{d}t \\
&= 2\int \frac{(t^2+1-1)\arctan t}{t^2+1}\mathrm{d}t \\
&= 2\int \arctan t\mathrm{d}t - 2\int \arctan t \mathrm{d}\arctan t \\
&= 2t\arctan t - 2\int \frac{t}{t^2+1}\mathrm{d}t - \arctan^2 t \\
&= 2t\arctan t - \ln|t^2+1| - \arctan^2 t + C \\
&= 2\sqrt{x-1}\arctan\sqrt{x-1} - \ln|x| - \arctan^2\sqrt{x-1} + C
\end{aligned}
$$

27.

因为换元之后，$f(t)\mathrm{d}t$ 又要凑到 d 后，所以先不用解出来，直接分部积分。

令 $t=\sqrt{\frac{1+x}{x}}$ ， $x=\frac{1}{t^2-1}$.

$$
\begin{aligned}
I &:= \int \ln\left(1+\sqrt{\frac{1+x}{x}}\right)\mathrm{d}x \\
&= \int \ln(1+t)\mathrm{d}\frac{1}{t^2-1} \\
&= \frac{\ln(1+t)}{t^2-1} - \int \frac{1}{t^2-1}\cdot\frac{1}{1+t}\mathrm{d}t \\
&= \frac{\ln(1+t)}{t^2-1} - \int \frac{1}{t-1}\cdot\frac{1}{(1+t)^2}\mathrm{d}t
\end{aligned}
$$

因

$$
\begin{aligned}
\varphi(t) &:= \frac{1}{t-1}\cdot\frac{1}{(1+t)^2} \\
&= \frac{A}{t-1} + \frac{B}{1+t} + \frac{C}{(1+t)^2} \\
&= \frac{\frac{1}{4}}{t-1} + \frac{-\frac{1}{4}}{1+t} + \frac{-\frac{1}{2}}{(1+t)^2}
\end{aligned}
$$

故

$$
\begin{aligned}
I &:= \int \ln\left(1+\sqrt{\frac{1+x}{x}}\right)\mathrm{d}x \\
&= \frac{\ln(1+t)}{t^2-1} - \int \frac{1}{t-1}\cdot\frac{1}{(1+t)^2}\mathrm{d}t \\
&= \frac{\ln(1+t)}{t^2-1} - \frac{1}{4}\ln|t-1| + \frac{1}{4}\ln|t+1| - \frac{1}{2}\cdot\frac{1}{t+1} + C
\end{aligned}
$$

28.

令 $t=\sqrt{x}$ ， $t^2=x$ ， $\mathrm{d}x=2t\mathrm{d}t$.

$$
\begin{aligned}
I &:= \int \arctan(1+\sqrt{x})\mathrm{d}x \\
&= \int \arctan(1+t)\mathrm{d}t^2 \\
&= t^2\arctan(1+t) - \int \frac{t^2}{t^2+2t+2}\mathrm{d}t \\
&= t^2\arctan(1+t) - \int \frac{t^2+2t+2-(2t+2)}{t^2+2t+2}\mathrm{d}t \\
&= t^2\arctan(1+t) - t + \int \frac{1}{t^2+2t+2}\mathrm{d}(t^2+2t+2) \\
&= t^2\arctan(1+t) - t + \ln|t^2+2t+2| + C \\
&= x\arctan(1+\sqrt{x}) - \sqrt{x} + \ln|x+2\sqrt{x}+2| + C
\end{aligned}
$$

29.

使用了

$$
\frac{1}{x^2}\mathrm{d}x = -\mathrm{d}\frac{1}{x}
$$

对分母进行降阶

$$
\begin{aligned}
I &:= \int \frac{xe^x}{(1+x)^2}\mathrm{d}x \\
&= -\int xe^x \mathrm{d}\frac{1}{x+1} \\
&= -\frac{xe^x}{x+1} + \int \frac{1}{x+1}\mathrm{d}(xe^x) \\
&= -\frac{xe^x}{x+1} + \int \frac{e^x + xe^x}{x+1}\mathrm{d}x \\
&= -\frac{xe^x}{x+1} + e^x + C \\
&= \frac{e^x}{x+1} + C
\end{aligned}
$$

30.

$$
\begin{aligned}
I &:= \int \frac{x^2e^x}{(x+2)^2}\mathrm{d}x \\
&= -\int x^2e^x \mathrm{d}\frac{1}{x+2} \\
&= -\frac{x^2e^x}{x+2} + \int \frac{1}{x+2}\mathrm{d}(x^2e^x) \\
&= -\frac{x^2e^x}{x+2} + \int \frac{2xe^x + x^2e^x}{x+2}\mathrm{d}x \\
&= -\frac{x^2e^x}{x+2} + \int xe^x\mathrm{d}x \\
&= -\frac{x^2e^x}{x+2} + \int x\mathrm{d}e^x \\
&= -\frac{x^2e^x}{x+2} + xe^x - e^x + C
\end{aligned}
$$

31.

$$
\begin{aligned}
I &:= \int \frac{xe^x}{(1+e^x)^2}\mathrm{d}x \\
&= -\int x\mathrm{d}\frac{1}{1+e^x} \\
&= -\frac{x}{1+e^x} + \int \frac{1}{1+e^x}\mathrm{d}x \\
&= -\frac{x}{1+e^x} + \int \frac{1+e^x-e^x}{1+e^x}\mathrm{d}x \\
&= -\frac{x}{1+e^x} + x - \ln(1+e^x) + C
\end{aligned}
$$

32.

这题需要强制凑微分。

考虑要进行如下凑微分

$$
\frac{1}{f^2}\mathrm{d}x \overset{?}{=} -\mathrm{d}\frac{1}{f}
$$

需要乘上一阶导才能进行凑

$$
\frac{f'}{f^2}\mathrm{d}x = \frac{1}{f^2}\mathrm{d}f = -\mathrm{d}\frac{1}{f}
$$

所以要在原被积函数里乘上分母（的平方根）的导数，然后除它：

$$
\begin{aligned}
I &:= \int \frac{x^2}{(x\sin x+\cos x)^2}\mathrm{d}x \\
&= \int \frac{x}{\cos x}\cdot\frac{x\cos x}{(x\sin x+\cos x)^2}\mathrm{d}x \\
&= \int \frac{x}{\cos x}\cdot\frac{1}{(x\sin x+\cos x)^2}\mathrm{d}(x\sin x+\cos x) \\
&= -\int \frac{x}{\cos x}\mathrm{d}\frac{1}{x\sin x+\cos x} \\
&= -\frac{1}{x\sin x+\cos x}\cdot\frac{x}{\cos x}+\int\frac{1}{x\sin x+\cos x}\mathrm{d}\frac{x}{\cos x} \\
&= -\frac{1}{x\sin x+\cos x}\cdot\frac{x}{\cos x}+\int\frac{1}{x\sin x+\cos x}\cdot\frac{\cos x+x\sin x}{\cos^2 x}\mathrm{d}x \\
&= -\frac{1}{x\sin x+\cos x}\cdot\frac{x}{\cos x}+\int\sec^2 x\mathrm{d}x \\
&= -\frac{1}{x\sin x+\cos x}\cdot\frac{x}{\cos x}+\tan x+C
\end{aligned}
$$

33.

$$
\begin{aligned}
I &:= \int_0^{+\infty}\frac{e^{-x^2}}{\left(x^2+\frac{1}{2}\right)^2}\mathrm{d}x \\
&= \int_0^{+\infty}\frac{e^{-x^2}}{2x}\cdot\frac{2x}{\left(x^2+\frac{1}{2}\right)^2}\mathrm{d}x \\
&= -\int_0^{+\infty}\frac{e^{-x^2}}{2x}\mathrm{d}\frac{1}{x^2+\frac{1}{2}} \\
&\overset{???}{=} -\frac{e^{-x^2}}{2x}\cdot\frac{1}{x^2+\frac{1}{2}}\Bigg|_0^{+\infty}+\int_0^{+\infty}\frac{1}{x^2+\frac{1}{2}}\mathrm{d}\frac{e^{-x^2}}{2x}
\end{aligned}
$$

整出来一个极限不存在。。懵了，回看凯哥视频，发现要凑一些常数。

$$
\begin{aligned}
I &:= \int_0^{+\infty}\frac{e^{-x^2}}{\left(x^2+\frac{1}{2}\right)^2}\mathrm{d}x \\
&= \int_0^{+\infty}\frac{e^{-x^2}}{2x}\cdot\frac{2x}{\left(x^2+\frac{1}{2}\right)^2}\mathrm{d}x \\
&= -\int_0^{+\infty}\frac{e^{-x^2}}{2x}\mathrm{d}\frac{1}{x^2+\frac{1}{2}} \\
&= \int_0^{+\infty}\frac{e^{-x^2}}{2x}\mathrm{d}\left({\color{red}2}-\frac{1}{x^2+\frac{1}{2}}\right) \\
&= \int_0^{+\infty}\frac{e^{-x^2}}{2x}\mathrm{d}\left(\frac{2x^2+1-1}{x^2+\frac{1}{2}}\right) \\
&= \int_0^{+\infty}\frac{e^{-x^2}}{2x}\mathrm{d}\left(\frac{2x^2}{x^2+\frac{1}{2}}\right) \\
&= \frac{e^{-x^2}}{2x}\cdot\frac{2x^2}{x^2+\frac{1}{2}}\Bigg|_0^{+\infty}-\int_0^{+\infty}\frac{2x^2}{x^2+\frac{1}{2}}\mathrm{d}\frac{e^{-x^2}}{2x} \\
&= -\int_0^{+\infty}\frac{2x^2}{x^2+\frac{1}{2}}\cdot\frac{2xe^{-x^2}\cdot(-2x)-2e^{-x^2}}{4x^2}\mathrm{d}x \\
&= 2\int_0^{+\infty}e^{-x^2}\mathrm{d}x
\end{aligned}
$$

其中

令 $t=x^2$ ，$\sqrt{t}=x$ ，$\mathrm{d}x=\frac{1}{2\sqrt{t}}\mathrm{d}t.$

$$\begin{aligned}
I_1 &:= \int_0^{+\infty} e^{-x^2}\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^{+\infty} t^{-\frac{1}{2}}\,e^{-t}\,\mathrm{d}t \\
&= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{\pi}}{2}
\end{aligned}$$

故

$$I=\sqrt{\pi}$$

34.

这题在 29. 中用了凑微分来使分母降次。这里采用不同的办法，即套路 3

$$\begin{aligned}
I &:= \int \frac{xe^x}{(1+x)^2}\mathrm{d}x \\
&= \int \frac{(x+1)e^x - e^x}{(1+x)^2}\mathrm{d}x \\
&= \int \frac{e^x}{1+x}\mathrm{d}x - \int \frac{e^x}{(1+x)^2}\mathrm{d}x \\
&= \int \frac{e^x}{1+x}\mathrm{d}x + \int e^x\,\mathrm{d}\frac{1}{1+x} \\
&= \int \frac{e^x}{1+x}\mathrm{d}x + \frac{e^x}{1+x} - \int \frac{e^x}{1+x}\mathrm{d}x \\
&= \frac{e^x}{1+x} + C
\end{aligned}$$

35.

$$\begin{aligned}
I &:= \int e^x\left(\frac{1-x}{1+x^2}\right)^2\mathrm{d}x \\
&= \int e^x\left[\frac{1-2x+x^2}{(1+x^2)^2}\right]\mathrm{d}x \\
&= \int e^x\left[\frac{1}{1+x^2} - \frac{-2x}{(1+x^2)^2}\right]\mathrm{d}x \\
&= \frac{e^x}{1+x^2} + C
\end{aligned}$$

36.

这里要两个都尝试是不是对方的导数。。

$$\begin{aligned}
I &:= \int \frac{1+\sin x}{1+\cos x}\cdot e^x\,\mathrm{d}x \\
&= \int e^x\left[\frac{1}{1+\cos x} + \frac{\sin x}{1+\cos x}\right]\mathrm{d}x \\
&= \frac{e^x\sin x}{1+\cos x} + C
\end{aligned}$$

37.

$$
\begin{aligned}
I &:= \int \frac{e^{-\sin x}\cdot \sin 2x}{\sin^4\left(\frac{\pi}{4}-\frac{x}{2}\right)}\mathrm{d}x \\
&= \int \frac{e^{-\sin x}\cdot 2\sin x\cos x}{\left[\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]^2}\mathrm{d}x \\
&= \int \frac{e^{-\sin x}\cdot 2\sin x\cos x}{\left[\frac{1-\cos\left(\frac{\pi}{2}-x\right)}{2}\right]^2}\mathrm{d}x \\
&= 8\int \frac{e^{-\sin x}\cdot \sin x\cos x}{\left[1-\sin x\right]^2}\mathrm{d}x \\
&= 8\int \frac{te^{-t}}{(1-t)^2}\mathrm{d}t \\
&= 8\int \frac{ue^{u}}{(1+u)^2}\mathrm{d}u \\
&= 8\int e^u\cdot\left[\frac{1}{1+u}-\frac{1}{(1+u)^2}\right]\mathrm{d}u \\
&= \frac{8e^u}{1+u}+C \\
&= \frac{8e^{-t}}{1-t}+C \\
&= \frac{8e^{-\sin x}}{1-\sin x}+C
\end{aligned}
$$

38.

令 $t=-\frac{x}{2}$，$-2t=x$，$\mathrm{d}x=-2\mathrm{d}t$.

$$
\begin{aligned}
I &:= \int e^{-\frac{x}{2}}\cdot\frac{\cos x-\sin x}{\sqrt{\sin x}}\mathrm{d}x \\
&= -2\int e^{t}\cdot\frac{\cos 2t-\sin 2t}{\sqrt{\sin 2t}}\mathrm{d}t \\
&= -2\int e^{t}\cdot\frac{\cos 2t+\sin 2t}{\sqrt{-\sin 2t}}\mathrm{d}t
\end{aligned}
$$

尝试有点复杂，看凯哥的做法

$$
\begin{aligned}
I &:= \int e^{-\frac{x}{2}}\cdot\frac{\cos x-\sin x}{\sqrt{\sin x}}\mathrm{d}x \\
&= \int e^{-\frac{x}{2}}\frac{\cos x}{\sqrt{\sin x}}\mathrm{d}x-\int e^{-\frac{x}{2}}\cdot\sqrt{\sin x}\mathrm{d}x \\
&= I_1-I_2
\end{aligned}
$$

其中

$$
\begin{aligned}
I_1 &:= \int e^{-\frac{x}{2}}\frac{\cos x}{\sqrt{\sin x}}\mathrm{d}x \\
&= \int e^{-\frac{x}{2}}\frac{1}{\sqrt{\sin x}}\mathrm{d}\sin x \\
&= 2\int e^{-\frac{x}{2}}\,\mathrm{d}\sqrt{\sin x} \\
&= 2e^{-\frac{x}{2}}\sqrt{\sin x}-2\int\sqrt{\sin x}\,\mathrm{d}e^{-\frac{x}{2}} \\
&= 2e^{-\frac{x}{2}}\sqrt{\sin x}-2\int\sqrt{\sin x}\cdot e^{-\frac{x}{2}}\cdot\left(-\frac{1}{2}\right)\mathrm{d}x \\
&= 2e^{-\frac{x}{2}}\sqrt{\sin x}+I_2
\end{aligned}
$$

故

$$I=2e^{-\frac{x}{2}}\sqrt{\sin x}+C$$

39.

$$\begin{aligned}
I&:=\int e^{\sin x}\cdot\frac{x\cos^3 x-\sin x}{\cos^2 x}\mathrm{d}x\\
&=\int x\cos xe^{\sin x}\mathrm{d}x-\int e^{\sin x}\cdot\frac{\sin x}{\cos^2 x}\mathrm{d}x\\
&=\int x\mathrm{d}e^{\sin x}-\int e^{\sin x}\mathrm{d}\sec x\\
&=xe^{\sin x}-\int e^{\sin x}\mathrm{d}x-e^{\sin x}\sec x+\int\sec x\mathrm{d}e^{\sin x}\\
&=xe^{\sin x}-\int e^{\sin x}\mathrm{d}x-e^{\sin x}\sec x+\int\sec xe^{\sin x}\cdot\cos x\mathrm{d}x\\
&=xe^{\sin x}-e^{\sin x}\sec x+C
\end{aligned}$$

40.

$$\begin{aligned}
I&:=\int\left(\ln\ln x+\frac{1}{\ln x}\right)\mathrm{d}x\\
&=\int\ln\ln x\mathrm{d}x+\int\frac{1}{\ln x}\mathrm{d}x\\
&=x\ln\ln x-\int x\mathrm{d}\ln\ln x+\int\frac{1}{\ln x}\mathrm{d}x\\
&=x\ln\ln x-\int x\cdot\frac{1}{\ln x}\cdot\frac{1}{x}\mathrm{d}x+\int\frac{1}{\ln x}\mathrm{d}x\\
&=x\ln\ln x+C
\end{aligned}$$

41.

因

$$\begin{aligned}
I&:=\int\left[\frac{f(x)}{f'(x)}-\frac{f^2(x)f''(x)}{[f'(x)]^3}\right]\mathrm{d}x\\
&=\int\frac{f(x)}{f'(x)}\mathrm{d}x-\int\frac{f^2(x)f''(x)}{[f'(x)]^3}\mathrm{d}x\\
&=\int\frac{f(x)}{f'(x)}\mathrm{d}x-\int\frac{f^2(x)}{[f'(x)]^3}\mathrm{d}f'(x)\\
&=\int\frac{f(x)}{f'(x)}\mathrm{d}x+\frac{1}{2}\int f^2(x)\mathrm{d}\frac{1}{[f'(x)]^2}\\
&=\int\frac{f(x)}{f'(x)}\mathrm{d}x+\frac{1}{2}\frac{f^2(x)}{[f'(x)]^2}-\frac{1}{2}\int\frac{1}{[f'(x)]^2}\mathrm{d}f^2(x)\\
&=\int\frac{f(x)}{f'(x)}\mathrm{d}x+\frac{1}{2}\frac{f^2(x)}{[f'(x)]^2}-\frac{1}{2}\int\frac{2f(x)\cdot f'(x)}{[f'(x)]^2}\mathrm{d}x\\
&=\int\frac{f(x)}{f'(x)}\mathrm{d}x+\frac{1}{2}\frac{f^2(x)}{[f'(x)]^2}-\int\frac{f(x)}{f'(x)}\mathrm{d}x\\
&=\frac{1}{2}\frac{f^2(x)}{[f'(x)]^2}+C
\end{aligned}$$

42.

$$
\begin{aligned}
I &:= \int \frac{1-\ln x}{(x-\ln x)^2}\mathrm{d}x \\
&= \int \frac{(x-\ln x)+(1-x)}{(x-\ln x)^2}\mathrm{d}x \\
&= \int \frac{1}{x-\ln x}\mathrm{d}x + \int \frac{1-x}{(x-\ln x)^2}\mathrm{d}x
\end{aligned}
$$

关键是要凑得可以抵消，所以，如果按着第二个不动，要把第一个分部积分的话，就要把分母凑到微分号后，以使分母次数升高。

如果按着第一个不动，则要把分母次数降低。

按着第二个不动：

$$
\begin{aligned}
I &:= \int \frac{1-\ln x}{(x-\ln x)^2}\mathrm{d}x \\
&= \int \frac{(x-\ln x)+(1-x)}{(x-\ln x)^2}\mathrm{d}x \\
&= \int \frac{1}{x-\ln x}\mathrm{d}x + \int \frac{1-x}{(x-\ln x)^2}\mathrm{d}x \\
&= \frac{x}{x-\ln x} + \int \frac{x\cdot\left(1-\frac{1}{x}\right)}{(x-\ln x)^2}\mathrm{d}x + \int \frac{1-x}{(x-\ln x)^2}\mathrm{d}x \\
&= \frac{x}{x-\ln x} + \int \frac{x-1}{(x-\ln x)^2}\mathrm{d}x + \int \frac{1-x}{(x-\ln x)^2}\mathrm{d}x \\
&= \frac{x}{x-\ln x} + C
\end{aligned}
$$

按着第一个不动：

$$
\begin{aligned}
I &:= \int \frac{1-\ln x}{(x-\ln x)^2}\mathrm{d}x \\
&= \int \frac{(x-\ln x)+(1-x)}{(x-\ln x)^2}\mathrm{d}x \\
&= \int \frac{1}{x-\ln x}\mathrm{d}x + \int \frac{1-x}{(x-\ln x)^2}\mathrm{d}x \\
&= \int \frac{1}{x-\ln x}\mathrm{d}x - \int \frac{1-x}{1-\frac{1}{x}}\mathrm{d}\frac{1}{x-\ln x} \\
&= \int \frac{1}{x-\ln x}\mathrm{d}x - \frac{1-x}{1-\frac{1}{x}}\cdot\frac{1}{x-\ln x} + \int \frac{1}{x-\ln x}\mathrm{d}\frac{1-x}{1-\frac{1}{x}} \\
&= \int \frac{1}{x-\ln x}\mathrm{d}x - \frac{1-x}{1-\frac{1}{x}}\cdot\frac{1}{x-\ln x} + \int \frac{1}{x-\ln x}\mathrm{d}\frac{(1-x)x}{x-1} \\
&= \int \frac{1}{x-\ln x}\mathrm{d}x - \frac{1-x}{1-\frac{1}{x}}\cdot\frac{1}{x-\ln x} + \int \frac{1}{x-\ln x}\mathrm{d}(-x) \\
&= -\frac{1-x}{1-\frac{1}{x}}\cdot\frac{1}{x-\ln x} + C \\
&= \frac{x}{x-\ln x} + C
\end{aligned}
$$

43.

$$
\begin{aligned}
I &:= \int \left(\frac{\arctan x}{\arctan x - x}\right)^2 \mathrm{d}x \\
&= \int \left(\frac{\arctan x - x + x}{\arctan x - x}\right)^2 \mathrm{d}x \\
&= \int \left(1 + \frac{x}{\arctan x - x}\right)^2 \mathrm{d}x \\
&= \int \left(1 + \frac{2x}{\arctan x - x} + \frac{x^2}{(\arctan x - x)^2}\right) \mathrm{d}x \\
&= x + \int \frac{2x}{\arctan x - x} \mathrm{d}x + \int \frac{x^2}{(\arctan x - x)^2} \mathrm{d}x \\
&= x + \int \frac{1}{\arctan x - x} \mathrm{d}(x^2 + {\color{red}1}) + \int \frac{x^2}{(\arctan x - x)^2} \mathrm{d}x \\
&= x + \frac{x^2 + 1}{\arctan x - x} + \int \frac{(x^2 + 1)\left(\frac{1}{1+x^2} - 1\right)}{(\arctan x - x)^2} \mathrm{d}x + \int \frac{x^2}{(\arctan x - x)^2} \mathrm{d}x \\
&= x + \frac{x^2 + 1}{\arctan x - x} + \int \frac{-x^2}{(\arctan x - x)^2} \mathrm{d}x + \int \frac{x^2}{(\arctan x - x)^2} \mathrm{d}x \\
&= x + \frac{x^2 + 1}{\arctan x - x} + C
\end{aligned}
$$