凯哥不定积分笔记——2 三角函数

套路集合

- 套路1：万能代换

  令 $t=\tan\frac{x}{2}$ ，$x=2\arctan t$ ，$\mathrm{d}x=\frac{2}{1+t^2}\mathrm{d}t$.

$$\begin{aligned}\sin x&=2\sin\frac{x}{2}\cos\frac{x}{2}\\&=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}\\&=\frac{2\tan\frac{x}{2}}{\tan^2\frac{x}{2}+1}\\&=\frac{2t}{t^2+1}\end{aligned}$$

$$\begin{aligned}\cos x&=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\\&=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}\\&=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\\&=\frac{1-t^2}{1+t^2}\end{aligned}$$

- 套路2：缩分母
  - 共轭表达式：

    对于分母为 $1+\cos x$ 或 $1+\sin x$ 的，分式上下同乘 $1-\cos x$ 或 $1-\sin x$
  - 倍角公式
  - 辅助角公式

- 套路3：凑偶次方

  如果被积函数 $R(\sin x,\cos x)$ 满足 $R(\sin x,-\cos x)=-R(\sin x,\cos x)$ 或 $R(-\sin x,\cos x)=-R(\sin x,\cos x)$. 这里把 这里把前者称为对 $\cos x$ 有“奇性”，后者为对 $\sin x$ 有 “奇性”。

  做法是，对 $\cos x$ 有奇性时凑 $\mathrm{d}\sin x$ ；对 $\sin x$ 有奇性时凑 $\mathrm{d}\cos x$.

- 套路4：凑 $\mathrm{d}\tan x$

  如果被积函数满足 $R(-\sin x,-\cos x)=R(\sin x,\cos x)$，考虑凑 $\mathrm{d}\tan x$.

- 套路5：

  形如

$$\int\frac{A\sin x+B\cos x}{C\sin x+D\cos x}\mathrm{d}x$$

的积分，可以通过假设

$$\text{分子} = p \cdot \text{分母} + q \cdot (\text{分母})'$$

然后使用待定系数解出 p, q ，从而化简积分。

- 套路6：统一角度

  当被积函数出现不同角度，先统一角度。

- 套路7：积化和差公式

  形如

$$\int \sin ax \sin bx \mathrm{d}x, (a \neq b)$$

  的积分，可以使用积化和差公式化简积分。

  积化和差公式，立即推！

  由和角公式有

$$\begin{cases} \sin(a+b) = \sin a \cos b + \cos a \sin b & (1) \\ \sin(a-b) = \sin a \cos b - \cos a \sin b & (2) \\ \cos(a+b) = \cos a \cos b - \sin a \sin b & (3) \\ \cos(a-b) = \cos a \cos b + \sin a \sin b & (4) \end{cases}$$

  那么得

$$\begin{cases} (1)+(2): \frac{1}{2}[\sin(a+b)+\sin(a-b)] = \sin a \cos b \\ (3)+(4): \frac{1}{2}[\cos(a+b)+\cos(a-b)] = \cos a \cos b \\ (4)-(3): \frac{1}{2}[\cos(a-b)-\cos(a+b)] = \sin a \sin b \end{cases}$$

题目列表：

1.

$$\int \frac{1}{3+5\cos x} \mathrm{d}x$$

2.

$$\int \frac{1}{1+\sin x+\cos x} \mathrm{d}x$$

3.

$$\int \frac{1}{1+\cos x} \mathrm{d}x$$

4.

$$\int \frac{\sin x}{1+\sin x} \mathrm{d}x$$

5.

$$\int \frac{1}{\sin x+\cos x} \mathrm{d}x$$

6.

$$\int \frac{\cos x}{\sin x + \cos x} \mathrm{d}x$$

7.

$$\int \frac{1}{\sin^2 x \cos x} \mathrm{d}x$$

8.

$$\int \frac{\cos^3 x - 2\cos x}{1 + \sin^2 x + \sin^4 x} \mathrm{d}x$$

9.

$$\int \sec^3 x \mathrm{d}x$$

10.

$$\int \sqrt{1 + x^2} \mathrm{d}x$$

11.

$$\int \sec^5 x \mathrm{d}x$$

12.

$$\int \sec^n x \mathrm{d}x$$

13.

$$\int \tan^n x \mathrm{d}x$$

14.

$$\int \csc^n x \mathrm{d}x$$

15.

$$\int \cot^n x \mathrm{d}x$$

16.

$$\int \frac{1}{\sin x \cos^2 x} \mathrm{d}x$$

17.

$$\int \frac{5 + 4\cos x}{(2 + \cos x)^2 \sin x} \mathrm{d}x$$

18.

$$\int \frac{1}{1+\cos^2 x} \mathrm{d}x$$

19.

$$\int \frac{1}{(3\sin x + 2\cos x)^2} \mathrm{d}x$$

20.

$$\int \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} \mathrm{d}x$$

21.

$$\int \frac{1}{\sin^4 x \cos^2 x} \mathrm{d}x$$

22.

$$\int \sin^4 x \cos^2 x \mathrm{d}x$$

23.

$$\int \frac{1+\sin x + \cos x}{1+\sin^2 x} \mathrm{d}x$$

24.

$$\int \frac{3\sin x + 4\cos x}{2\sin x + \cos x} \mathrm{d}x$$

25.

$$\int \frac{1}{\sin x \sin 2x} \mathrm{d}x$$

26.

$$\int \frac{1}{2\sin x + \sin 2x} \mathrm{d}x$$

27.

$$\int \frac{\cos 2x - \sin 2x}{\sin x + \cos x} \mathrm{d}x$$

28.

$$\int \sin 2x \sin 3x \mathrm{d}x$$

29.

$$\int \frac{\sin x \cos x}{\sin x + \cos x} \mathrm{d}x$$

30.

$$\int \frac{\sin x \cos x}{\sin x - \cos x} \mathrm{d}x$$

31.

$$\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} \mathrm{d}x$$

32.

$$\int \frac{1}{\sin^6 x + \cos^6 x} \mathrm{d}x$$

33.

$$\int \frac{1}{\sin^3 x + \cos^3 x} \mathrm{d}x$$

解答之:

1.

令 $t = \tan \frac{x}{2}$ ，$\mathrm{d}x = \frac{2}{1+t^2}\mathrm{d}t$，$\cos x = \frac{1-t^2}{1+t^2}$.

$$\begin{aligned} I &:= \int \frac{1}{3+5\cos x}\mathrm{d}x \\ &= \int \frac{1}{3+5\cdot\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\mathrm{d}t \\ &= \int \frac{1}{4-t^2}\mathrm{d}t \\ &= \frac{1}{4}\left[\int \frac{\mathrm{d}t}{2-t} + \int \frac{\mathrm{d}t}{2+t}\right] \\ &= \frac{1}{4}\ln\left|\frac{2+t}{2-t}\right| + C \\ &= \frac{1}{4}\ln\left|\frac{2+\tan\frac{x}{2}}{2-\tan\frac{x}{2}}\right| + C \end{aligned}$$

2.

$$\begin{aligned} I &:= \int \frac{1}{1+\sin x+\cos x}\mathrm{d}x \\ &= \int \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\mathrm{d}t \\ &= \int \frac{1}{1+t}\mathrm{d}t \\ &= \ln|1+t| + C \\ &= \ln\left|1+\tan\frac{x}{2}\right| + C \end{aligned}$$

3.

因

$$\cos 2x = 2\cos^2 x - 1$$

故

$$\begin{aligned} I &:= \int \frac{1}{1+\cos x}\mathrm{d}x \\ &= \int \frac{1}{2\cos^2\frac{x}{2}}\mathrm{d}x \\ &= \tan\frac{x}{2} + C \end{aligned}$$

4.

$$\begin{aligned}
I &:= \int \frac{\sin x + 1 - 1}{1 + \sin x} \mathrm{d}x \\
&= x - \int \frac{1}{1 + \sin x} \mathrm{d}x \\
&= x - \int \frac{1 - \sin x}{\cos^2 x} \mathrm{d}x \\
&= x - \tan x - \int \frac{\mathrm{d} \cos x}{\cos^2 x} \\
&= x - \tan x + \sec x + C
\end{aligned}$$

5.

有公式

$$\int \csc x \mathrm{d}x = \ln|\cot x - \csc x| + C \tag{1}$$

那么

$$\begin{aligned}
I &:= \int \frac{1}{\sin x + \cos x} \mathrm{d}x \\
&= \int \frac{\mathrm{d}\left(x + \frac{\pi}{4}\right)}{\sqrt{2} \sin\left(x + \frac{\pi}{4}\right)} \\
&= \int \frac{\mathrm{d}\left(x + \frac{\pi}{4}\right)}{\sqrt{2} \sin\left(x + \frac{\pi}{4}\right)} \\
&= \frac{\sqrt{2}}{2} \ln\left|\cot\left(x + \frac{\pi}{4}\right) - \csc\left(x + \frac{\pi}{4}\right)\right|
\end{aligned}$$

6.

法1：共轭

又有公式

$$\int \sec x \mathrm{d}x = \ln|\tan x + \sec x| + C \tag{2}$$

$$\begin{aligned}
I &:= \int \frac{\cos x}{\sin x + \cos x} \mathrm{d}x \\
&= \int \frac{\cos x(\cos x - \sin x)}{\cos^2 x - \sin^2 x} \mathrm{d}x \\
&= \int \frac{\cos^2 x - \sin x \cos x}{\cos 2x} \mathrm{d}x \\
&= \frac{1}{2} \int \frac{\cos 2x + 1}{\cos 2x} \mathrm{d}x - \frac{1}{2} \int \frac{\sin 2x}{\cos 2x} \mathrm{d}x \\
&= \frac{1}{2} x + \frac{1}{4} \ln|\sec 2x + \tan 2x| + \frac{1}{4} \ln|\cos 2x| + C
\end{aligned}$$

法2：组合积分法

$$\begin{aligned}
I &:= \int \frac{\cos x}{\sin x + \cos x} \mathrm{d}x \\
J &:= \int \frac{\sin x}{\sin x + \cos x} \mathrm{d}x
\end{aligned}$$

那么，

$$
\begin{aligned}
&I+J=x+C \\
&I-J=\ln|\sin x+\cos x|+C
\end{aligned}
$$

故

$$
\begin{aligned}
I&:=\int\frac{\cos x}{\sin x+\cos x}\mathrm{d}x \\
&=\frac{1}{2}[(I+J)+(I-J)] \\
&=\frac{1}{2}x+\ln|\sin x+\cos x|+C
\end{aligned}
$$

7.

法1：三角恒等式

$$
\begin{aligned}
I&:=\int\frac{1}{\sin^2x\cos x}\mathrm{d}x \\
&=\int\frac{\sin^2x+\cos^2x}{\sin^2x\cos x}\mathrm{d}x \\
&=\int\sec x\mathrm{d}x+\int\frac{\cos x}{\sin^2x}\mathrm{d}x \\
&=\ln|\tan x+\sec x|+\int\frac{1}{\sin^2x}\mathrm{d}\sin x \\
&=\ln|\tan x+\sec x|-\csc x+C
\end{aligned}
$$

法2：凑 $\mathrm{d}\sin x$

$$
\begin{aligned}
I&:=\int\frac{1}{\sin^2x\cos x}\mathrm{d}x \\
&=\int\frac{\cos x}{\sin^2x\cos^2x}\mathrm{d}x \\
&=\int\frac{1}{\sin^2x(1-\sin^2x)}\mathrm{d}\sin x \\
&=\int\frac{1-\sin^2x+\sin^2x}{\sin^2x(1-\sin^2x)}\mathrm{d}\sin x \\
&=\int\frac{1}{\sin^2x}\mathrm{d}\sin x+\int\frac{1}{1-\sin^2x}\mathrm{d}\sin x \\
&=-\csc x-\frac{1}{2}\ln\left|\frac{1-\sin x}{1+\sin x}\right|+C
\end{aligned}
$$

8.

$$
\begin{aligned}
I &:= \int \frac{\cos^3 x - 2\cos x}{1+\sin^2 x + \sin^4 x} \mathrm{d}x \\
&= \int \frac{\cos^2 x - 2}{1+\sin^2 x + \sin^4 x} \mathrm{d}\sin x \\
&= \int \frac{-(1+1-\cos^2 x)}{1+\sin^2 x + \sin^4 x} \mathrm{d}\sin x \\
&= \int \frac{-(1+\sin^2 x)}{1+\sin^2 x + \sin^4 x} \mathrm{d}\sin x \\
&= -\int \frac{1+t^2}{1+t^2+t^4} \mathrm{d}t \\
&= -\int \frac{t^{-2}+1}{t^{-2}+1+t^2} \mathrm{d}t \\
&= -\int \frac{1}{(t-t^{-1})^2+3} \mathrm{d}\left(-t^{-1}+t\right) \\
&= -\frac{1}{\sqrt{3}} \arctan \frac{t-t^{-1}}{\sqrt{3}} + C \\
&= -\frac{1}{\sqrt{3}} \arctan \frac{\sin x - \csc x}{\sqrt{3}} + C
\end{aligned}
$$

9.

法1：凑 $\mathrm{d}\sin x$

$$
\begin{aligned}
I &:= \int \sec^3 x \mathrm{d}x \\
&= \int \frac{\cos x}{\sec^2 x} \mathrm{d}x \\
&= \int \frac{1}{(1-\sin^2 x)^2} \mathrm{d}\sin x \\
&= \int \frac{1}{[(1-\sin x)(1+\sin x)]^2} \mathrm{d}\sin x \\
&= \int \left[\frac{1}{2}\left(\frac{1}{1-\sin x} + \frac{1}{1+\sin x}\right)\right]^2 \mathrm{d}\sin x \\
&= -\frac{1}{4}\int \frac{\mathrm{d}(1-\sin x)}{(1-\sin x)^2} + \frac{1}{4}\int \frac{\mathrm{d}(1+\sin x)}{(1+\sin x)^2} + \frac{1}{2}\int \frac{\mathrm{d}\sin x}{1-\sin^2 x} \\
&= \frac{1}{4}\cdot\frac{1}{1-\sin x} - \frac{1}{4}\cdot\frac{1}{1+\sin x} + \frac{1}{4}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C
\end{aligned}
$$

法2：直接分部积分+积分重现

$$
\begin{aligned}
I &:= \int \sec^3 x \mathrm{d}x \\
&= \int \sec x \cdot \sec^2 x \mathrm{d}x \\
&= \int \sec x \mathrm{d} \tan x \\
&= \sec x \tan x - \int \tan x \mathrm{d} \sec x \\
&= \sec x \tan x - \int \tan^2 x \sec x \mathrm{d}x \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x \mathrm{d}x \\
&= \sec x \tan x + \int \sec x \mathrm{d}x - \int \sec^3 x \mathrm{d}x \\
&= \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x \mathrm{d}x \\
&= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\tan x + \sec x| + C
\end{aligned}
$$

10.

法1：三角换元

$$
\begin{aligned}
I &:= \int \sqrt{1+x^2} \mathrm{d}x \\
&= \int \sec t \cdot \sec^2 t \mathrm{d}t \\
&= \frac{1}{2} \sec t \tan t + \frac{1}{2} \ln|\tan t + \sec t| + C \\
&= \frac{1}{2} \cdot x\sqrt{1+x^2} + \frac{1}{2} \ln\left|x + \sqrt{1+x^2}\right| + C
\end{aligned}
$$

法2：直接分部积分 + 积分重现

有公式

$$
\int \frac{1}{\sqrt{a^2+x^2}} \mathrm{d}x = \frac{1}{a} \ln\left(x + \sqrt{a^2+x^2}\right) + C \tag{3}
$$

故

$$
\begin{aligned}
I &:= \int \sqrt{1+x^2} \mathrm{d}x \\
&= x\sqrt{1+x^2} - \int x \mathrm{d}\sqrt{1+x^2} \\
&= x\sqrt{1+x^2} - \int \frac{x^2}{\sqrt{1+x^2}} \mathrm{d}x \\
&= x\sqrt{1+x^2} - \int \frac{x^2+1-1}{\sqrt{1+x^2}} \mathrm{d}x \\
&= x\sqrt{1+x^2} - \int \sqrt{1+x^2} \mathrm{d}x + \int \frac{1}{\sqrt{1+x^2}} \mathrm{d}x \\
&= \frac{1}{2} \cdot x\sqrt{1+x^2} - \frac{1}{2} \int \frac{1}{\sqrt{1+x^2}} \mathrm{d}x
\end{aligned}
$$

公式证明：

$$\int \csc x \mathrm{d}x = \ln|\cot x - \csc x| + C \tag{1}$$

证明：

这是一个对 $\sin x$ 有奇性的积分。

$$\begin{aligned}
I &:= \int \csc x \mathrm{d}x \\
&= \int \frac{\sin x}{\sin^2 x} \mathrm{d}x \\
&= -\int \frac{1}{1-\cos^2 x} \mathrm{d}\cos x \\
&= \frac{1}{2} \ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + C \\
&= \frac{1}{2} \ln\left|\frac{(\cos x - 1)^2}{\cos^2 x - 1}\right| + C \\
&= \ln\left|\frac{\cos x - 1}{\sin x}\right| + C \\
&= \ln|\cot x - \csc x| + C
\end{aligned}$$

$$\int \sec x \mathrm{d}x = \ln|\tan x + \sec x| + C \tag{2}$$

证明：

这是一个对 $\cos x$ 有奇性的积分。

$$\begin{aligned}
I &:= \int \sec x \mathrm{d}x \\
&= \int \frac{\mathrm{d}\sin x}{\cos^2 x} \\
&= \int \frac{\mathrm{d}\sin x}{1-\sin^2 x} \\
&= \frac{1}{2} \ln\left|\frac{1+\sin x}{1-\sin x}\right| + C \\
&= \frac{1}{2} \ln\left|\frac{(1+\sin x)^2}{1-\sin^2 x}\right| + C \\
&= \ln\left|\frac{1+\sin x}{\cos x}\right| + C \\
&= \ln|\sec x + \tan x| + C
\end{aligned}$$

$$\int \frac{1}{\sqrt{a^2+x^2}} \mathrm{d}x = \ln\left(x + \sqrt{a^2+x^2}\right) + C \tag{3}$$

证明：

$$
\begin{aligned}
I &:= \int \frac{1}{\sqrt{a^2 + x^2}} \mathrm{d}x \\
&= \int \frac{a \sec^2 t}{\sqrt{a^2 + a^2 \tan^2 t}} \mathrm{d}t \\
&= \int \sec t \mathrm{d}t \\
&= \ln\lvert \sec t + \tan t \rvert + C \\
&= \ln\left\lvert \frac{1}{a}\sqrt{a^2 + x^2} + \frac{x}{a} \right\rvert + C \\
&= \color{blue}{\ln\left\lvert x + \sqrt{a^2 + x^2} \right\rvert + C}
\end{aligned}
$$