凯哥不定积分笔记——2 三角函数

套路集合

- 套路1：万能代换

  令 $t=\tan\frac{x}{2}$ ， $x=2\arctan t$ ， $\mathrm{d}x=\frac{2}{1+t^2}\mathrm{d}t$.

$$
\begin{aligned}
\sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2} \\
&= \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} \\
&= \frac{2\tan\frac{x}{2}}{\tan^2\frac{x}{2}+1} \\
&= \frac{2t}{t^2+1}
\end{aligned}
$$

$$
\begin{aligned}
\cos x &= \cos^2\frac{x}{2}-\sin^2\frac{x}{2} \\
&= \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} \\
&= \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} \\
&= \frac{1-t^2}{1+t^2}
\end{aligned}
$$

- 套路2：缩分母
  - 共轭表达式：

    对于分母为 $1+\cos x$ 或 $1+\sin x$ 的，分式上下同乘 $1-\cos x$ 或 $1-\sin x$
  - 倍角公式
  - 辅助角公式

- 套路3：凑偶次方

  如果被积函数 $R(\sin x,\cos x)$ 满足 $R(\sin x,-\cos x)=-R(\sin x,\cos x)$ 或 $R(-\sin x,\cos x)=-R(\sin x,\cos x)$. 这里把 这里把前者称为对 $\cos x$ 有“奇性”，后者为对 $\sin x$ 有 “奇性”。

  做法是，对 $\cos x$ 有奇性时凑 $\mathrm{d}\sin x$ ； 对 $\sin x$ 有奇性时凑 $\mathrm{d}\cos x$.

- 套路4：凑 $\mathrm{d}\tan x$

  如果被积函数满足 $R(-\sin x,-\cos x)=R(\sin x,\cos x)$，考虑凑 $\mathrm{d}\tan x$.

- 套路5：

  形如

$$\int \frac{A\sin x+B\cos x}{C\sin x+D\cos x}\mathrm{d}x$$

的积分，可以通过假设

$$\text{分子}=p\cdot\text{分母}+q\cdot(\text{分母})'$$

然后使用待定系数解出 p,q ，从而化简积分。

- 套路6：统一角度

  当被积函数出现不同角度，先统一角度。

- 套路7：积化和差公式

  形如

$$\int \sin ax\sin bx\mathrm{d}x,(a\neq b)$$

  的积分，可以使用积化和差公式化简积分。

  积化和差公式，立即推！

  由和角公式有

$$\begin{cases}\sin(a+b)=\sin a\cos b+\cos a\sin b & (1)\\ \sin(a-b)=\sin a\cos b-\cos a\sin b & (2)\\ \cos(a+b)=\cos a\cos b-\sin a\sin b & (3)\\ \cos(a-b)=\cos a\cos b+\sin a\sin b & (4)\end{cases}$$

  那么得

$$\begin{cases}(1)+(2):\dfrac{1}{2}[\sin(a+b)+\sin(a-b)]=\sin a\cos b\\ (3)+(4):\dfrac{1}{2}[\cos(a+b)+\cos(a-b)]=\cos a\cos b\\ (4)-(3):\dfrac{1}{2}[\cos(a-b)-\cos(a+b)]=\sin a\sin b\end{cases}$$

题目列表：

1.

$$\int \frac{1}{3+5\cos x}\mathrm{d}x$$

2.

$$\int \frac{1}{1+\sin x+\cos x}\mathrm{d}x$$

3.

$$\int \frac{1}{1+\cos x}\mathrm{d}x$$

4.

$$\int \frac{\sin x}{1+\sin x}\mathrm{d}x$$

5.

$$\int \frac{1}{\sin x+\cos x}\mathrm{d}x$$

6.

$$\int \frac{\cos x}{\sin x+\cos x}\mathrm{d}x$$

7.

$$\int \frac{1}{\sin^2 x\cos x}\mathrm{d}x$$

8.

$$\int \frac{\cos^3 x-2\cos x}{1+\sin^2 x+\sin^4 x}\mathrm{d}x$$

9.

$$\int \sec^3 x\mathrm{d}x$$

10.

$$\int \sqrt{1+x^2}\mathrm{d}x$$

11.

$$\int \sec^5 x\mathrm{d}x$$

12.

$$\int \sec^n x\mathrm{d}x$$

13.

$$\int \tan^n x\mathrm{d}x$$

14.

$$\int \csc^n x\mathrm{d}x$$

15.

$$\int \cot^n x\mathrm{d}x$$

16.

$$\int \frac{1}{\sin x\cos^2 x}\mathrm{d}x$$

17.

$$\int \frac{5+4\cos x}{(2+\cos x)^2\sin x}\mathrm{d}x$$

18.

$$\int \frac{1}{1+\cos^2 x} \mathrm{d}x$$

19.

$$\int \frac{1}{(3\sin x + 2\cos x)^2} \mathrm{d}x$$

20.

$$\int \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} \mathrm{d}x$$

21.

$$\int \frac{1}{\sin^4 x \cos^2 x} \mathrm{d}x$$

22.

$$\int \sin^4 x \cos^2 x \mathrm{d}x$$

23.

$$\int \frac{1+\sin x + \cos x}{1+\sin^2 x} \mathrm{d}x$$

24.

$$\int \frac{3\sin x + 4\cos x}{2\sin x + \cos x} \mathrm{d}x$$

25.

$$\int \frac{1}{\sin x \sin 2x} \mathrm{d}x$$

26.

$$\int \frac{1}{2\sin x + \sin 2x} \mathrm{d}x$$

27.

$$\int \frac{\cos 2x - \sin 2x}{\sin x + \cos x} \mathrm{d}x$$

28.

$$\int \sin 2x \sin 3x \mathrm{d}x$$

29.

$$\int \frac{\sin x \cos x}{\sin x + \cos x} \mathrm{d}x$$

30.

$$\int \frac{\sin x \cos x}{\sin x - \cos x} \mathrm{d}x$$

31.

$$\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\mathrm{d}x$$

32.

$$\int \frac{1}{\sin^6 x + \cos^6 x}\mathrm{d}x$$

33.

$$\int \frac{1}{\sin^3 x + \cos^3 x}\mathrm{d}x$$

解答之：

1.

令 $t = \tan\frac{x}{2}$ ，$\mathrm{d}x = \frac{2}{1+t^2}\mathrm{d}t$，$\cos x = \frac{1-t^2}{1+t^2}$.

$$\begin{aligned} I &:= \int \frac{1}{3+5\cos x}\mathrm{d}x \\ &= \int \frac{1}{3+5\cdot\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\mathrm{d}t \\ &= \int \frac{1}{4-t^2}\mathrm{d}t \\ &= \frac{1}{4}\left[\int\frac{\mathrm{d}t}{2-t}+\int\frac{\mathrm{d}t}{2+t}\right] \\ &= \frac{1}{4}\ln\left|\frac{2+t}{2-t}\right|+C \\ &= \frac{1}{4}\ln\left|\frac{2+\tan\frac{x}{2}}{2-\tan\frac{x}{2}}\right|+C \end{aligned}$$

2.

$$\begin{aligned} I &:= \int \frac{1}{1+\sin x+\cos x}\mathrm{d}x \\ &= \int \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\mathrm{d}t \\ &= \int \frac{1}{1+t}\mathrm{d}t \\ &= \ln|1+t|+C \\ &= \ln\left|1+\tan\frac{x}{2}\right|+C \end{aligned}$$

3.

因

$$\cos 2x = 2\cos^2 x - 1$$

故

$$
\begin{aligned}
I &:= \int \frac{1}{1+\cos x}\mathrm{d}x \\
&= \int \frac{1}{2\cos^2\frac{x}{2}}\mathrm{d}x \\
&= \tan\frac{x}{2} + C
\end{aligned}
$$

4.

$$
\begin{aligned}
I &:= \int \frac{\sin x + 1 - 1}{1+\sin x}\mathrm{d}x \\
&= x - \int \frac{1}{1+\sin x}\mathrm{d}x \\
&= x - \int \frac{1-\sin x}{\cos^2 x}\mathrm{d}x \\
&= x - \tan x - \int \frac{\mathrm{d}\cos x}{\cos^2 x} \\
&= x - \tan x + \sec x + C
\end{aligned}
$$

5.

有公式

$$
\int \csc x \mathrm{d}x = \ln|\cot x - \csc x| + C \tag{1}
$$

那么

$$
\begin{aligned}
I &:= \int \frac{1}{\sin x + \cos x}\mathrm{d}x \\
&= \int \frac{\mathrm{d}\left(x+\frac{\pi}{4}\right)}{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)} \\
&= \int \frac{\mathrm{d}\left(x+\frac{\pi}{4}\right)}{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)} \\
&= \frac{\sqrt{2}}{2}\ln\left|\cot\left(x+\frac{\pi}{4}\right) - \csc\left(x+\frac{\pi}{4}\right)\right|
\end{aligned}
$$

6.

法1：共轭

又有公式

$$
\int \sec x \mathrm{d}x = \ln|\tan x + \sec x| + C \tag{2}
$$

$$
\begin{aligned}
I &:= \int \frac{\cos x}{\sin x + \cos x}\mathrm{d}x \\
&= \int \frac{\cos x(\cos x - \sin x)}{\cos^2 x - \sin^2 x}\mathrm{d}x \\
&= \int \frac{\cos^2 x - \sin x\cos x}{\cos 2x}\mathrm{d}x \\
&= \frac{1}{2}\int \frac{\cos 2x + 1}{\cos 2x}\mathrm{d}x - \frac{1}{2}\int \frac{\sin 2x}{\cos 2x}\mathrm{d}x \\
&= \frac{1}{2}x + \frac{1}{4}\ln|\sec 2x + \tan 2x| + \frac{1}{4}\ln|\cos 2x| + C
\end{aligned}
$$

法2：组合积分法

$$
\begin{aligned}
I &:= \int \frac{\cos x}{\sin x + \cos x} \mathrm{d}x \\
J &:= \int \frac{\sin x}{\sin x + \cos x} \mathrm{d}x
\end{aligned}
$$

那么,

$$
\begin{aligned}
&I + J = x + C \\
&I - J = \ln|\sin x + \cos x| + C
\end{aligned}
$$

故

$$
\begin{aligned}
I &:= \int \frac{\cos x}{\sin x + \cos x} \mathrm{d}x \\
&= \frac{1}{2}[(I+J) + (I-J)] \\
&= \frac{1}{2}x + \ln|\sin x + \cos x| + C
\end{aligned}
$$

7.

法1：三角恒等式

$$
\begin{aligned}
I &:= \int \frac{1}{\sin^2 x \cos x} \mathrm{d}x \\
&= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos x} \mathrm{d}x \\
&= \int \sec x \mathrm{d}x + \int \frac{\cos x}{\sin^2 x} \mathrm{d}x \\
&= \ln|\tan x + \sec x| + \int \frac{1}{\sin^2 x} \mathrm{d}\sin x \\
&= \ln|\tan x + \sec x| - \csc x + C
\end{aligned}
$$

法2：凑 $\mathrm{d}\sin x$

$$
\begin{aligned}
I &:= \int \frac{1}{\sin^2 x \cos x} \mathrm{d}x \\
&= \int \frac{\cos x}{\sin^2 x \cos^2 x} \mathrm{d}x \\
&= \int \frac{1}{\sin^2 x (1 - \sin^2 x)} \mathrm{d}\sin x \\
&= \int \frac{1 - \sin^2 x + \sin^2 x}{\sin^2 x (1 - \sin^2 x)} \mathrm{d}\sin x \\
&= \int \frac{1}{\sin^2 x} \mathrm{d}\sin x + \int \frac{1}{1 - \sin^2 x} \mathrm{d}\sin x \\
&= -\csc x - \frac{1}{2}\ln\left|\frac{1 - \sin x}{1 + \sin x}\right| + C
\end{aligned}
$$

8.

$$
\begin{aligned}
I :&= \int \frac{\cos^3 x - 2\cos x}{1+\sin^2 x+\sin^4 x} \mathrm{d}x \\
&= \int \frac{\cos^2 x - 2}{1+\sin^2 x+\sin^4 x} \mathrm{d}\sin x \\
&= \int \frac{\color{blue}{-(1+1-\cos^2 x)}}{1+\sin^2 x+\sin^4 x} \mathrm{d}\sin x \\
&= \int \frac{\color{blue}{-(1+\sin^2 x)}}{1+\sin^2 x+\sin^4 x} \mathrm{d}\sin x \\
&= -\int \frac{1+t^2}{1+t^2+t^4} \mathrm{d}t \\
&= -\int \frac{t^{-2}+1}{t^{-2}+1+t^2} \mathrm{d}t \\
&= -\int \frac{1}{(t-t^{-1})^2+3} \mathrm{d}\left(-t^{-1}+t\right) \\
&= -\frac{1}{\sqrt{3}} \arctan \frac{t-t^{-1}}{\sqrt{3}} + C \\
&= -\frac{1}{\sqrt{3}} \arctan \frac{\sin x - \csc x}{\sqrt{3}} + C
\end{aligned}
$$

9.

法1：凑 $\mathrm{d}\sin x$

$$
\begin{aligned}
I :&= \int \sec^3 x \mathrm{d}x \\
&= \int \frac{\cos x}{\sec^2 x} \mathrm{d}x \\
&= \int \frac{1}{(1-\sin^2 x)^2} \mathrm{d}\sin x \\
&= \int \frac{1}{[(1-\sin x)(1+\sin x)]^2} \mathrm{d}\sin x \\
&= \int \left[\frac{1}{2}\left(\frac{1}{1-\sin x}+\frac{1}{1+\sin x}\right)\right]^2 \mathrm{d}\sin x \\
&= -\frac{1}{4}\int \frac{\mathrm{d}(1-\sin x)}{(1-\sin x)^2} + \frac{1}{4}\int \frac{\mathrm{d}(1+\sin x)}{(1+\sin x)^2} + \frac{1}{2}\int \frac{\mathrm{d}\sin x}{1-\sin^2 x} \\
&= \frac{1}{4}\cdot\frac{1}{1-\sin x} - \frac{1}{4}\cdot\frac{1}{1+\sin x} + \frac{1}{4}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C
\end{aligned}
$$

法2：直接分部积分+积分重现

$$
\begin{aligned}
I &:= \int \sec^3 x \mathrm{d}x \\
&= \int \sec x \cdot \sec^2 x \mathrm{d}x \\
&= \int \sec x \mathrm{d}\tan x \\
&= \sec x \tan x - \int \tan x \mathrm{d}\sec x \\
&= \sec x \tan x - \int \tan^2 x \sec x \mathrm{d}x \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x \mathrm{d}x \\
&= \sec x \tan x + \int \sec x \mathrm{d}x - \int \sec^3 x \mathrm{d}x \\
&= \frac{1}{2}\sec x \tan x + \frac{1}{2} \int \sec x \mathrm{d}x \\
&= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\tan x + \sec x| + C
\end{aligned}
$$

10.

法1：三角换元

$$
\begin{aligned}
I &:= \int \sqrt{1+x^2}\mathrm{d}x \\
&= \int \sec t \cdot \sec^2 t \mathrm{d}t \\
&= \frac{1}{2}\sec t \tan t + \frac{1}{2}\ln|\tan t + \sec t| + C \\
&= \frac{1}{2} \cdot x\sqrt{1+x^2} + \frac{1}{2}\ln\left|x + \sqrt{1+x^2}\right| + C
\end{aligned}
$$

法2：直接分部积分 + 积分重现

有公式

$$
\int \frac{1}{\sqrt{a^2+x^2}}\mathrm{d}x = \ln\left(x + \sqrt{a^2+x^2}\right) + C \tag{3}
$$

故

$$
\begin{aligned}
I &:= \int \sqrt{1+x^2}\mathrm{d}x \\
&= x\sqrt{1+x^2} - \int x\mathrm{d}\sqrt{1+x^2} \\
&= x\sqrt{1+x^2} - \int \frac{x^2}{\sqrt{1+x^2}}\mathrm{d}x \\
&= x\sqrt{1+x^2} - \int \frac{x^2+1-1}{\sqrt{1+x^2}}\mathrm{d}x \\
&= x\sqrt{1+x^2} - \int \sqrt{1+x^2}\mathrm{d}x + \int \frac{1}{\sqrt{1+x^2}}\mathrm{d}x \\
&= \frac{1}{2} \cdot x\sqrt{1+x^2} + \frac{1}{2}\int \frac{1}{\sqrt{1+x^2}}\mathrm{d}x \\
&= \frac{1}{2} \cdot x\sqrt{1+x^2} + \frac{1}{2}\ln\left(x + \sqrt{1+x^2}\right) + C
\end{aligned}
$$

11.

先做12.

由12，得

$$
\begin{aligned}
I_5 &:= \int \sec^5 x \mathrm{d}x \\
&= \frac{1}{4}\sec^3 \tan x + \frac{3}{4} I_3 \\
&= \frac{1}{4}\sec^3 \tan x + \frac{3}{4}\left(\frac{1}{2}\sec x \tan x + \frac{1}{2} I_1\right) \\
&= \frac{1}{4}\sec^3 \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\ln|\tan x + \sec x| + C
\end{aligned}
$$

12.

$$
\begin{aligned}
I_n &:= \int \sec^n x \mathrm{d}x \\
&= \int \sec^{n-2} x \mathrm{d} \tan x \\
&= \sec^{n-2} x \tan x - \int \tan x \mathrm{d} \sec^{n-2} x \\
&= \sec^{n-2} x \tan x - (n-2) \int \tan^2 x \cdot \sec^{n-2} x \mathrm{d}x \\
&= \sec^{n-2} x \tan x - (n-2) \int \left(\sec^2 x - 1\right) \cdot \sec^{n-2} x \mathrm{d}x \\
&= \sec^{n-2} x \tan x - (n-2) \int \sec^n x \mathrm{d}x + (n-2) \int \sec^{n-2} x \mathrm{d}x \\
&= \frac{1}{n-1} \cdot \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \mathrm{d}x, (n > 2)
\end{aligned}
$$

13.

$$
\begin{aligned}
I_n &:= \int \tan^n x \mathrm{d}x \\
&= \int \tan^{n-2} x (\sec^2 x - 1) \mathrm{d}x \\
&= \int \tan^{n-2} x \mathrm{d} \tan x - I_{n-2} \\
&= \frac{1}{n-1} \tan^{n-1} x - I_{n-2}, (n > 1)
\end{aligned}
$$

14.

$$
\begin{aligned}
I_n &:= \int \csc^n x \mathrm{d}x \\
&= \int \csc^{n-2} x \cdot \csc^2 x \mathrm{d}x \\
&= -\int \csc^{n-2} x \mathrm{d} \cot x \\
&= -\csc^{n-2} x \cot x + \int \cot x \mathrm{d} \csc^{n-2} x \\
&= -\csc^{n-2} x \cot x + \int \cot x \cdot (n-2) \csc^{n-3} x \cdot (-\cot x \csc x) \mathrm{d}x \\
&= -\csc^{n-2} x \cot x - (n-2) \int \cot^2 x \csc^{n-2} x \mathrm{d}x \\
&= -\csc^{n-2} x \cot x - (n-2) \int (\csc^2 - 1) \csc^{n-2} x \mathrm{d}x \\
&= -\csc^{n-2} x \cot x - (n-2) \int \csc^n x \mathrm{d}x + (n-2) I_{n-2} \\
&= -\frac{1}{n-1} \cdot \csc^{n-2} x \cot x + \frac{n-2}{n-1} I_{n-2}, (n > 1)
\end{aligned}
$$

15.

$$\begin{aligned}
I_n &:= \int \cot^n x \mathrm{d}x \\
&= \int \cot^{n-2} x \cdot \cot^2 x \mathrm{d}x \\
&= \int \cot^{n-2} x \cdot (\csc^2 x - 1) \mathrm{d}x \\
&= -\int \cot^{n-2} x \mathrm{d} \cot x - I_{n-2} \\
&= -\frac{1}{n-1} \cot^{n-1} x - I_{n-2}, (n > 1)
\end{aligned}$$

16.

$$\begin{aligned}
I &:= \int \frac{1}{\sin x \cos^2 x} \mathrm{d}x \\
&= \int \frac{\sin x}{\sin^2 x \cos^2 x} \mathrm{d}x \\
&= -\int \frac{1}{\sin^2 x \cos^2 x} \mathrm{d} \cos x \\
&= -\int \frac{1}{(1-\cos^2 x)\cos^2 x} \mathrm{d} \cos x \\
&= -\int \frac{1-\cos^2 x + \cos^2 x}{(1-\cos^2 x)\cos^2 x} \mathrm{d} \cos x \\
&= -\int \frac{1}{\cos^2 x} \mathrm{d} \cos x - \int \frac{1}{1-\cos^2 x} \mathrm{d} \cos x \\
&= \sec x + \frac{1}{2} \ln \left| \frac{\cos x - 1}{\cos x + 1} \right| + C
\end{aligned}$$

17.

$$\begin{aligned}
I &:= \int \frac{5 + 4\cos x}{(2+\cos x)^2 \sin x} \mathrm{d}x \\
&= -\int \frac{5 + 4\cos x}{(2+\cos x)^2 \sin^2 x} \mathrm{d} \cos x \\
&= -\int \frac{5 + 4\cos x}{(2+\cos x)^2 (1-\cos^2 x)} \mathrm{d} \cos x \\
&= -\int \frac{4 + 4\cos x + \cos^2 x + 1 - \cos^2 x}{(2+\cos x)^2 (1-\cos^2 x)} \mathrm{d} \cos x \\
&= -\int \frac{1}{1-\cos^2 x} \mathrm{d} \cos x - \int \frac{1}{(2+\cos x)^2} \mathrm{d} \cos x \\
&= \frac{1}{2} \ln \left| \frac{\cos x - 1}{\cos x + 1} \right| + \frac{1}{2+\cos x} + C
\end{aligned}$$

18.

$$\begin{aligned}
I &:= \int \frac{1}{1+\cos^2 x} \mathrm{d}x \\
&= \int \frac{1}{\sec^2 x + 1} \mathrm{d} \tan x \\
&= \int \frac{1}{\tan^2 x + 2} \mathrm{d} \tan x \\
&= \frac{1}{\sqrt{2}} \arctan \frac{\tan x}{\sqrt{2}} + C
\end{aligned}$$

19.

$$\begin{aligned}I &:= \int \frac{1}{(3\sin x + 2\cos x)^2}\mathrm{d}x \\ &= \int \frac{1}{(3\tan x + 2)^2}\mathrm{d}\tan x \\ &= -\frac{1}{9\tan x + 6} + C\end{aligned}$$

20.

$$\begin{aligned}I_1 &:= \int \frac{1}{a^2\sin^2 x + b^2\cos^2 x}\mathrm{d}x, (a \neq 0, b \neq 0) \\ &= \int \frac{1}{a^2\tan^2 x + b^2}\mathrm{d}\tan x \\ &= \frac{1}{a^2}\int \frac{1}{\tan^2 x + \frac{b^2}{a^2}}\mathrm{d}\tan x \\ &= \frac{1}{ab}\arctan\frac{a\tan x}{b} + C\end{aligned}$$

$$\begin{aligned}I_2 &:= \int \frac{1}{a^2\sin^2 x + b^2\cos^2 x}\mathrm{d}x, (a = 0, b \neq 0) \\ &= \frac{1}{b^2}\int \sec^2 x\mathrm{d}x \\ &= \frac{1}{b^2}\tan x + C\end{aligned}$$

$$\begin{aligned}I_3 &:= \int \frac{1}{a^2\sin^2 x + b^2\cos^2 x}\mathrm{d}x, (a \neq 0, b = 0) \\ &= \frac{1}{a^2}\int \csc^2 x\mathrm{d}x \\ &= \frac{1}{a^2}\ln|\cot x - \csc x| + C\end{aligned}$$

21.

$$\begin{aligned}I &:= \int \frac{1}{\sin^4 x\cos^2 x}\mathrm{d}x \\ &= \int \frac{1}{\sin^4 x}\mathrm{d}\tan x \\ &= \int \frac{\left(1 + \tan^2 x\right)^2}{\tan^4 x}\mathrm{d}\tan x \\ &= \int \frac{\mathrm{d}\tan x}{\tan^4 x} + 2\int \frac{\mathrm{d}\tan x}{\tan^2 x} + x \\ &= -\frac{1}{3}\cot^3 x - 2\cot x + x + C\end{aligned}$$

22.

法1：凑 $\mathrm{d}\tan x$

$$\begin{aligned}I &:= \int \sin^4 x\cos^2 x\mathrm{d}x \\ &= \int \sin^4 x\cos^4 x\mathrm{d}\tan x \\ &= \int \tan^4 x(1 + \tan^2 x)^{-3}\mathrm{d}\tan x \\ &= \text{getting nuts}\end{aligned}$$

法2：使用降次公式

$$
\begin{aligned}
I &:= \int \sin^4 x \cos^2 x \mathrm{d}x \\
&= \int \left(\frac{1}{2}\sin 2x\right)^2 \cdot \left(\frac{1-\cos 2x}{2}\right) \mathrm{d}x \\
&= \frac{1}{8}\int \sin^2 2x \mathrm{d}x - \frac{1}{8}\int \cos 2x \sin^2 2x \mathrm{d}x \\
&= \frac{1}{8}\int \frac{1-\cos 4x}{2}\mathrm{d}x - \frac{1}{16}\int \cos 2x \sin^2 2x \mathrm{d}(2x) \\
&= \frac{1}{16}x - \frac{1}{16}\int \cos 4x \mathrm{d}x - \frac{1}{16}\int \sin^2 2x \mathrm{d}\sin 2x \\
&= \frac{1}{16}x - \frac{1}{64}\int \cos 4x \mathrm{d}(4x) - \frac{1}{48}\sin^3 2x \\
&= \frac{1}{16}x - \frac{1}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C
\end{aligned}
$$

法3：倍角法

令

$$y = \cos x + \mathrm{i}\sin x$$

有

$$\frac{1}{y} = \cos x - \mathrm{i}\sin x$$

由上二式得

$$
\begin{cases}
y + \dfrac{1}{y} = 2\cos x \\
y - \dfrac{1}{y} = 2\mathrm{i}\sin x
\end{cases}
$$

由棣莫弗公式

$$(\cos x + \mathrm{i}\sin x)^n = \cos nx + \mathrm{i}\sin nx$$

有

$$
\begin{cases}
y^n + \dfrac{1}{y^n} = 2\cos nx \\
y^n - \dfrac{1}{y^n} = 2\mathrm{i}\sin nx
\end{cases}
$$

$$
\begin{aligned}
2^4\mathrm{i}^4 \sin^4 x \cdot 2^2 \cos^2 x &= \left(y - \frac{1}{y}\right)^4 \left(y + \frac{1}{y}\right)^2 \\
&= \left(y - \frac{1}{y}\right)^2 \left(y^2 - \frac{1}{y^2}\right)^2 \\
&= \left(y^2 - 2 + \frac{1}{y^2}\right)\left(y^4 - 2 + \frac{1}{y^4}\right) \\
&= y^6 - 2y^2 + \frac{1}{y^2} - 2y^4 + 4 - 2\frac{1}{y^4} + y^2 - 2\frac{1}{y^2} + \frac{1}{y^6} \\
&= \left(y^6 + \frac{1}{y^6}\right) - 2\left(y^4 + \frac{1}{y^4}\right) - \left(y^2 + \frac{1}{y^2}\right) + 4 \\
&= 2\cos 6x - 4\cos 4x - 2\cos 2x + 4
\end{aligned}
$$

故

$$
\begin{aligned}
I &:= \int \sin^4 x \cos^2 x \mathrm{d}x \\
&= \frac{1}{64}\int \left(2\cos 6x - 4\cos 4x - 2\cos 2x + 4\right)\mathrm{d}x \\
&= \frac{1}{192}\sin 6x - \frac{1}{64}\sin 4x - \frac{1}{64}\sin 2x + \frac{1}{16}x + C
\end{aligned}
$$

23.

$$
\begin{aligned}
I &:= \int \frac{1+\sin x+\cos x}{1+\sin^2 x}\mathrm{d}x \\
&= \int \frac{1}{1+\sin^2 x}\mathrm{d}x + \int \frac{\sin x}{1+\sin^2 x}\mathrm{d}x + \int \frac{\cos x}{1+\sin^2 x}\mathrm{d}x \\
&= \int \frac{1}{\sec^2 x+\tan^2 x}\mathrm{d}\tan x - \int \frac{1}{2-\cos^2 x}\mathrm{d}\cos x + \arctan\sin x \\
&= \frac{1}{2}\int \frac{1}{\frac{1}{2}+\tan^2 x}\mathrm{d}\tan x + \frac{1}{2\sqrt{2}}\ln\left|\frac{\cos x-\sqrt{2}}{\cos x+\sqrt{2}}\right| + \arctan\sin x \\
&= \frac{\sqrt{2}}{2}\arctan\left(\sqrt{2}\tan x\right) + \frac{1}{2\sqrt{2}}\ln\left|\frac{\cos x-\sqrt{2}}{\cos x+\sqrt{2}}\right| + \arctan\sin x + C
\end{aligned}
$$

24.

$$
\begin{aligned}
I &:= \int \frac{3\sin x+4\cos x}{2\sin x+\cos x}\mathrm{d}x \\
&= \int \frac{p\cdot(2\sin x+\cos x)+q\cdot(2\cos x-\sin x)}{2\sin x+\cos x}\mathrm{d}x \\
&= \int \frac{2\cdot(2\sin x+\cos x)+1\cdot(2\cos x-\sin x)}{2\sin x+\cos x}\mathrm{d}x \\
&= 2x + \ln|2\sin x+\cos x| + C
\end{aligned}
$$

25.

$$
\begin{aligned}
I &:= \int \frac{1}{\sin x\sin 2x}\mathrm{d}x \\
&= \frac{1}{2}\int \frac{1}{\sin^2 x\cos x}\mathrm{d}x \\
&= \frac{1}{2}\int \frac{1}{\sin^2 x(1-\sin^2 x)}\mathrm{d}\sin x \\
&= \frac{1}{2}\int \frac{1-\sin^2 x+\sin^2 x}{\sin^2 x(1-\sin^2 x)}\mathrm{d}\sin x \\
&= \frac{1}{2}\int \frac{1}{\sin^2 x}\mathrm{d}\sin x + \frac{1}{2}\int \frac{1}{1-\sin^2 x}\mathrm{d}\sin x \\
&= -\frac{1}{2}\csc x - \frac{1}{4}\ln\left|\frac{\sin x-1}{\sin x+1}\right| + C
\end{aligned}
$$

26.

$$
\begin{aligned}
I &:= \int \frac{1}{2\sin x+\sin 2x}\mathrm{d}x \\
&= \frac{1}{2}\int \frac{1}{\sin x(1+\cos x)}\mathrm{d}x \\
&= \frac{1}{2}\int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}\cdot 2\cos^2\frac{x}{2}}\mathrm{d}x \\
&= \frac{1}{4}\int \frac{1}{\sin\frac{x}{2}\cos^3\frac{x}{2}}\mathrm{d}\frac{x}{2} \\
&= \frac{1}{4}\int \frac{1}{\sin t\cos^3 t}\mathrm{d}t \\
&= \frac{1}{4}\int \frac{\sec^2 t}{\tan t}\mathrm{d}\tan t \\
&= \frac{1}{4}\int \frac{1+\tan^2 t}{\tan t}\mathrm{d}\tan t \\
&= \frac{1}{4}\ln|\tan t| + \frac{1}{8}\tan^2 t + C \\
&= \frac{1}{4}\ln\left|\tan\frac{x}{2}\right| + \frac{1}{8}\tan^2\frac{x}{2} + C
\end{aligned}
$$

27.

$$
\begin{aligned}
I &:= \int \frac{\cos 2x-\sin 2x}{\sin x+\cos x}\mathrm{d}x \\
&= \int \frac{\cos^2 x-\sin^2 x-2\sin x\cos x}{\sin x+\cos x}\mathrm{d}x \\
&= \int \frac{\cos^2 x-\sin^2 x-2\sin x\cos x}{\sin x+\cos x}\mathrm{d}x \\
&= \int(\cos x-\sin x)\mathrm{d}x - \int \frac{(\sin x+\cos x)^2-1}{\sin x+\cos x}\mathrm{d}x \\
&= \sin x+\cos x - \int(\sin x+\cos x)\mathrm{d}x + \int \frac{1}{\sin x+\cos x}\mathrm{d}x \\
&= 2\cos x + \int \frac{1}{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}\mathrm{d}\left(x+\frac{\pi}{4}\right) \\
&= 2\cos x + \frac{1}{\sqrt{2}}\ln\left|\cot\left(x+\frac{\pi}{4}\right)-\csc\left(x+\frac{\pi}{4}\right)\right| + C
\end{aligned}
$$

28.

$$
\begin{aligned}
I &:= \int \sin 2x\sin 3x\mathrm{d}x \\
&= \frac{1}{2}\int[\cos x-\cos 5x]\,\mathrm{d}x \\
&= \frac{1}{2}\sin x - \frac{1}{10}\sin 5x + C
\end{aligned}
$$

29.

$$
\begin{aligned}
I &:= \int \frac{\sin x\cos x}{\sin x+\cos x}\mathrm{d}x \\
&= \frac{1}{2}\int \frac{(\sin x+\cos x)^2-1}{\sin x+\cos x}\mathrm{d}x \\
&= \frac{1}{2}\int(\sin x+\cos x)\mathrm{d}x - \frac{1}{2}\int \frac{1}{\sin x+\cos x}\mathrm{d}x \\
&= \frac{1}{2}(-\cos x+\sin x) - \frac{\sqrt{2}}{4}\ln\left|\cot\left(x+\frac{\pi}{4}\right)-\csc\left(x+\frac{\pi}{4}\right)\right|
\end{aligned}
$$

30.

$$
\begin{aligned}
I &:= \int \frac{\sin x \cos x}{\sin x - \cos x} \mathrm{d}x \\
&= \frac{1}{2} \int \frac{1-(\sin x - \cos x)^2}{\sin x - \cos x} \mathrm{d}x \\
&= \frac{1}{2} \int \frac{1}{\sin x - \cos x} \mathrm{d}x - \frac{1}{2} \int (\sin x - \cos x) \mathrm{d}x \\
&= \frac{\sqrt{2}}{4} \ln \left| \cot\left(x - \frac{\pi}{4}\right) - \csc\left(x - \frac{\pi}{4}\right) \right| + \frac{1}{2}(\cos x + \sin x) + C
\end{aligned}
$$

31.

$$
\begin{aligned}
I &:= \int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} \mathrm{d}x \\
&= \int \frac{\sin x \cos x}{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x} \mathrm{d}x \\
&= \int \frac{\frac{1}{2}\sin 2x}{1 - \frac{1}{2}\sin^2 2x} \mathrm{d}x \\
&= \frac{1}{2} \int \frac{\sin 2x}{2 - \sin^2 2x} \mathrm{d}(2x) \\
&= -\frac{1}{2} \int \frac{1}{1 + \cos^2 t} \mathrm{d} \cos t \\
&= -\frac{1}{2} \arctan \cos t + C \\
&= -\frac{1}{2} \arctan \cos 2x + C
\end{aligned}
$$

32.

有公式

$$
a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)
$$

那么

$$
\begin{aligned}
I &:= \int \frac{1}{\sin^6 x + \cos^6 x} \mathrm{d}x \\
&= \int \frac{1}{(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x)} \mathrm{d}x \\
&= \int \frac{1}{\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x} \mathrm{d}x \\
&= \int \frac{1}{(\sin^2 x + \cos^2 x)^2 - 3\sin^2 x \cos^2 x} \mathrm{d}x \\
&= \int \frac{1}{1 - 3\sin^2 x \cos^2 x} \mathrm{d}x \\
&= \int \frac{1}{1 - 3\left(\frac{1}{2}\sin 2x\right)^2} \mathrm{d}x \\
&= \int \frac{4}{4 - 3\sin^2 2x} \mathrm{d}x \\
&= 2\int \frac{1}{4 - 3\sin^2 t} \mathrm{d}t \\
&= 2\int \frac{1}{1 + 3\cos^2 t} \mathrm{d}t \\
&= 2\int \frac{1}{\sec^2 t + 3} \mathrm{d}\tan t \\
&= 2\int \frac{1}{4 + \tan^2 t} \mathrm{d}\tan t \\
&= \arctan \frac{\tan t}{2} + C \\
&= \arctan \frac{\tan 2x}{2} + C
\end{aligned}
$$

33.

$$
\begin{aligned}
I &:= \int \frac{1}{\sin^3 x + \cos^3 x} \mathrm{d}x \\
&= \int \frac{1}{(\sin x + \cos x)(\sin^2 x - \sin x \cos x + \cos^2 x)} \mathrm{d}x \\
&= \int \frac{1}{(\sin x + \cos x)(1 - \sin x \cos x)} \mathrm{d}x \\
&= \frac{1}{3}\int \frac{(\sin x + \cos x)^2 + 2(1 - \sin x \cos x)}{(\sin x + \cos x)(1 - \sin x \cos x)} \mathrm{d}x \\
&= \frac{1}{3}\int \frac{\sin x + \cos x}{1 - \sin x \cos x} \mathrm{d}x + \frac{2}{3}\int \frac{1}{\sin x + \cos x} \mathrm{d}x \\
&= I_1 + \frac{\sqrt{2}}{3} \ln\left|\cot\left(x + \frac{\pi}{4}\right) - \csc\left(x + \frac{\pi}{4}\right)\right|
\end{aligned}
$$

其中

$$
\begin{aligned}
I_1 &:= \frac{1}{3}\int \frac{\sin x + \cos x}{1 - \sin x \cos x} \mathrm{d}x \\
&= \frac{1}{3}\int \frac{1}{(\sin x - \cos x)^2 + \frac{1 - (\sin x - \cos x)^2}{2}} \mathrm{d}(\sin x - \cos x) \\
&= \frac{2}{3}\int \frac{1}{t^2 + 1} \mathrm{d}t \\
&= \frac{2}{3}\arctan t + C \\
&= \frac{2}{3}\arctan(\sin x - \cos x) + C
\end{aligned}
$$

代回得

$$I=\frac{2}{3}\arctan(\sin x-\cos x)+\frac{\sqrt{2}}{3}\ln\left|\cot\left(x+\frac{\pi}{4}\right)-\csc\left(x+\frac{\pi}{4}\right)\right|+C$$

公式证明：

$$\int\csc x\mathrm{d}x=\ln|\cot x-\csc x|+C\tag{1}$$

证明：

这是一个对 $\sin x$ 有奇性的积分。

$$\begin{aligned}I&:=\int\csc x\mathrm{d}x\\&=\int\frac{\sin x}{\sin^2 x}\mathrm{d}x\\&=-\int\frac{1}{1-\cos^2 x}\mathrm{d}\cos x\\&=\frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right|+C\\&=\frac{1}{2}\ln\left|\frac{(\cos x-1)^2}{\cos^2 x-1}\right|+C\\&=\ln\left|\frac{\cos x-1}{\sin x}\right|+C\\&=\ln|\cot x-\csc x|+C\end{aligned}$$

$$\int\sec x\mathrm{d}x=\ln|\tan x+\sec x|+C\tag{2}$$

证明：

这是一个对 $\cos x$ 有奇性的积分。

$$\begin{aligned}I&:=\int\sec x\mathrm{d}x\\&=\int\frac{\mathrm{d}\sin x}{\cos^2 x}\\&=\int\frac{\mathrm{d}\sin x}{1-\sin^2 x}\\&=\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|+C\\&=\frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{1-\sin^2 x}\right|+C\\&=\ln\left|\frac{1+\sin x}{\cos x}\right|+C\\&=\ln|\sec x+\tan x|+C\end{aligned}$$

$$\int\frac{1}{\sqrt{a^2+x^2}}\mathrm{d}x=\ln\left(x+\sqrt{a^2+x^2}\right)+C\tag{3}$$

证明：

$$
\begin{aligned}
I &:= \int \frac{1}{\sqrt{a^2+x^2}} \mathrm{d}x \\
&= \int \frac{a \sec^2 t}{\sqrt{a^2+a^2 \tan^2 t}} \mathrm{d}t \\
&= \int \sec t \mathrm{d}t \\
&= \ln|\sec t + \tan t| + C \\
&= \ln\left|\frac{1}{a}\sqrt{a^2+x^2} + \frac{x}{a}\right| + C \\
&= \color{blue}{\ln\left|x + \sqrt{a^2+x^2}\right| + C}
\end{aligned}
$$