1.

令 $t=\sqrt{\frac{x}{x+1}}$，$x=\frac{t^2}{1-t^2}$，$\mathrm{d}x=\frac{2t}{(1-t^2)^2}\mathrm{d}t.$

那么

$$\begin{aligned} I &:= \int \sqrt{\frac{x}{x+1}}\mathrm{d}x \\ &= \int t\cdot\frac{2t}{(1-t^2)^2}\mathrm{d}t \end{aligned}$$

疯了。重来

令 $t=\sqrt{\frac{x}{x+1}}$，那么 $t^2=\frac{x}{x+1}=1-\frac{1}{x+1}$，得 $\frac{1}{x+1}=1-t^2$，即 $x+1=\frac{1}{1-t^2}$. 就此打住。

两边同时取微分得 $\mathrm{d}x=\mathrm{d}\frac{1}{1-t^2}$. 也是就此打住。如果把右边的微分展开，将会如上面那样复杂。直接来

$$\begin{aligned} I &:= \int \sqrt{\frac{x}{x+1}}\mathrm{d}x \\ &= \int t\mathrm{d}\frac{1}{1-t^2} \\ &= \frac{t}{1-t^2}-\int\frac{1}{1-t^2}\mathrm{d}t \\ &= \frac{t}{1-t^2}+\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+C \\ &= (x+1)\sqrt{\frac{x}{x+1}}+\frac{1}{2}\ln\left|\frac{\sqrt{\frac{x}{x+1}}-1}{\sqrt{\frac{x}{x+1}}+1}\right|+C \end{aligned}$$

2.

令 $t=\sqrt{\frac{x+1}{x}}$，那么 $t^2=1+\frac{1}{x}$，$x=\frac{1}{t^2-1}$，$\mathrm{d}x=\mathrm{d}\frac{1}{t^2-1}.$

$$\begin{aligned} I &:= \int\frac{1}{x}\cdot\sqrt{\frac{x+1}{x}}\mathrm{d}x \\ &= \int(t^2-1)\cdot t\cdot\mathrm{d}\frac{1}{t^2-1} \\ &= t-\int\frac{1}{t^2-1}\mathrm{d}(t^3-t) \\ &= t-\int\frac{1}{t^2-1}\cdot(3t^2-1)\mathrm{d}t \\ &= t-\int\frac{1}{t^2-1}\cdot(3t^2-3)\mathrm{d}t-2\int\frac{1}{t^2-1}\mathrm{d}t \\ &= t-3t-\ln\left|\frac{t-1}{t+1}\right|+C \\ &= -2t-\ln\left|\frac{t-1}{t+1}\right|+C \\ &= -2\sqrt{\frac{x+1}{x}}-\ln\left|\frac{\sqrt{\frac{x+1}{x}}-1}{\sqrt{\frac{x+1}{x}}+1}\right|+C \end{aligned}$$

3.

令 $t=\sqrt{\frac{1-x}{1+x}}$， $t^2=\frac{1-x}{1+x}=\frac{1+x-2x}{1+x}=1-\frac{2x}{1+x}$， $\frac{1+x-1}{1+x}=1-\frac{1}{1+x}=\frac{1}{2}(1-t^2)$，
$\frac{1}{1+x}=\frac{1}{2}(1+t^2)$， $1+x=\frac{2}{1+t^2}$. $\mathrm{d}x=\mathrm{d}\frac{2}{1+t^2}$.

$$
\begin{aligned}
I &:= \int \sqrt{\frac{1-x}{1+x}}\mathrm{d}x \\
&= \int t\mathrm{d}\frac{2}{1+t^2} \\
&= \frac{2t}{1+t^2} - 2\int \frac{1}{1+t^2}\mathrm{d}t \\
&= \frac{2t}{1+t^2} - 2\arctan t + C \\
&= (1+x)\sqrt{\frac{1-x}{1+x}} - 2\arctan\sqrt{\frac{1-x}{1+x}} + C
\end{aligned}
$$

4.

$$
\begin{aligned}
f(x) &:= \sqrt{\frac{1-x}{1+x}} + \sqrt{\frac{1+x}{1-x}} \\
&= \frac{1-x}{\sqrt{1-x^2}} + \frac{1+x}{\sqrt{1-x^2}} \\
&= \frac{2}{\sqrt{1-x^2}}
\end{aligned}
$$

故

$$
\begin{aligned}
I &:= \int \left(\sqrt{\frac{1-x}{1+x}} + \sqrt{\frac{1+x}{1-x}}\right)\mathrm{d}x \\
&= 2\int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x \\
&= 2\arcsin x + C
\end{aligned}
$$

5.

令 $t=\sqrt{e^x-2}$， $\ln(t^2+2)=x$， $\mathrm{d}x=\frac{2t}{t^2+2}\mathrm{d}t$.

$$
\begin{aligned}
I &:= \int \frac{xe^x}{\sqrt{e^x-2}}\mathrm{d}x \\
&= \int \frac{\ln(t^2+2)(t^2+2)}{t}\cdot\frac{2t}{t^2+2}\mathrm{d}t \\
&= 2\int \ln(t^2+2)\mathrm{d}t \\
&= 2t\ln(t^2+2) - 4\int \frac{t^2}{t^2+2}\mathrm{d}t \\
&= 2t\ln(t^2+2) - 4\int \frac{t^2+2-2}{t^2+2}\mathrm{d}t \\
&= 2t\ln(t^2+2) - 4t + 8\int \frac{1}{t^2+2}\mathrm{d}t \\
&= 2t\ln(t^2+2) - 4t + \frac{8}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}} + C \\
&= 2x\sqrt{e^x-2} - 4\sqrt{e^x-2} + \frac{8}{\sqrt{2}}\arctan\frac{\sqrt{e^x-2}}{\sqrt{2}} + C
\end{aligned}
$$

6.

这里关键要把 $x+1, x-1$ 整出来。。

$$
\begin{aligned}
I &:= \int \frac{1}{\sqrt[3]{(x+1)^2(x-1)^4}}\mathrm{d}x \\
&= \int \frac{1}{(x-1)\sqrt[3]{(x+1)^2(x-1)}}\mathrm{d}x \\
&= \int \frac{1}{(x^2-1)\sqrt[3]{\frac{x-1}{x+1}}}\mathrm{d}x
\end{aligned}
$$

令 $t=\sqrt[3]{\frac{x-1}{x+1}}$ ，$1+x=\frac{2}{1-t^2}$，$x-1=\frac{2t^2}{1-t^2}$，$\mathrm{d}x=\mathrm{d}\frac{2}{1-t^3}$.

$$
\begin{aligned}
I &:= \int \frac{1}{\sqrt[3]{(x+1)^2(x-1)^4}}\mathrm{d}x \\
&= \int \frac{1}{(x^2-1)\sqrt[3]{\frac{x-1}{x+1}}}\mathrm{d}x \\
&= \int \frac{1}{\frac{4t^2}{(1-t^2)^2}\cdot t}\cdot \mathrm{d}\frac{2}{1-t^2} \\
&= \frac{1}{2}\int \frac{(1-t^2)^2}{t^3}\cdot\frac{2t}{(1-t^2)^2}\mathrm{d}t \\
&= \int \frac{1}{t^2}\mathrm{d}t \\
&= -t^{-1}+C \\
&= -\sqrt[3]{\frac{x+1}{x-1}}+C
\end{aligned}
$$

7.

令 $x=t^6$ ，$\mathrm{d}x=6t^5\mathrm{d}t$.

$$
\begin{aligned}
I &:= \int \frac{1}{(1+\sqrt[3]{x})\cdot\sqrt{x}}\mathrm{d}x \\
&= 6\int \frac{t^2}{1+t^2}\mathrm{d}t \\
&= 6t-6\arctan t+C \\
&= 6\sqrt[6]{x}-6\arctan\sqrt[6]{x}+C
\end{aligned}
$$

8.

令 $\exp\frac{x}{6}=t$，$x=6\ln t$ ，$\mathrm{d}x=\frac{6}{t}\mathrm{d}t$.

$$
\begin{aligned}
I &:= \int \frac{1}{1+\exp\frac{x}{2}+\exp\frac{x}{3}+\exp\frac{x}{6}}\mathrm{d}x \\
&= 6\int \frac{1}{1+t^3+t^2+t}\cdot\frac{1}{t}\mathrm{d}t \\
&= 6\int \frac{1-t}{1-t^4}\cdot\frac{1}{t}\mathrm{d}t \\
&= 6\int \frac{1-t}{(1+t^2)(1-t^2)}\cdot\frac{1}{t}\mathrm{d}t \\
&= 6\int \frac{1}{t(1+t^2)(1-t)}\mathrm{d}t \\
&= 6\int \left(\frac{A}{t}+\frac{Bt+C}{1+t^2}+\frac{D}{1-t}\right)\mathrm{d}t \\
&= 6\int \left(\frac{1}{t}+\frac{-\frac{1}{2}t+\frac{1}{2}}{1+t^2}+\frac{\frac{1}{2}}{1-t}\right)\mathrm{d}t \\
&= \int \left(\frac{6}{t}+\frac{-3t+3}{1+t^2}+\frac{3}{1-t}\right)\mathrm{d}t \\
&= 6\ln|t|-3\ln|t-1|-3\int\frac{t}{1+t^2}\mathrm{d}t+3\int\frac{1}{1+t^2}\mathrm{d}t \\
&= 6\ln|t|-3\ln|t-1|-\frac{3}{2}\ln\left|1+t^2\right|+3\arctan t+C \\
&= x-3\ln\left|\exp\frac{x}{6}-1\right|-\frac{3}{2}\ln\left|1+\exp\frac{x}{3}\right|+3\arctan\exp\frac{x}{6}+C
\end{aligned}
$$

9.

令 $x=2\sin t$， $\mathrm{d}x=2\cos t\mathrm{d}t$.

$$
\begin{aligned}
I &:= \int\frac{\sqrt{4-x^2}}{x^4}\mathrm{d}x \\
&= \int\frac{2\cos t}{16\sin^4 t}\cdot 2\cos t\mathrm{d}t \\
&= \frac{1}{4}\int\frac{\cos^2 t}{\sin^4 t}\mathrm{d}t \\
&= \frac{1}{4}\int\frac{1}{\tan^4 t}\mathrm{d}\tan t \\
&= -\frac{1}{12}\tan^{-3} t+C \\
&= -\frac{1}{12}\frac{\sqrt{(1-x^2/4)^3}}{(x^3/8)}+C \\
&= -\frac{1}{12}\frac{\sqrt{(4-x^2)^3}}{x^3}+C
\end{aligned}
$$