1.

令 $t=\sqrt{\frac{x}{x+1}}$ ，$x=\frac{t^2}{1-t^2}$，$\mathrm{d}x=\frac{2t}{\left(1-t^2\right)^2}\mathrm{d}t.$

那么

$$\begin{aligned}I&:=\int\sqrt{\frac{x}{x+1}}\mathrm{d}x\\&=\int t\cdot\frac{2t}{(1-t^2)^2}\mathrm{d}t\end{aligned}$$

疯了。重来

令 $t=\sqrt{\frac{x}{x+1}}$ ，那么 $t^2=\frac{x}{x+1}=1-\frac{1}{x+1}$ ，得 $\frac{1}{x+1}=1-t^2$ ，即 $x+1=\frac{1}{1-t^2}$. 就此打住。

两边同时取微分得 $\mathrm{d}x=\mathrm{d}\frac{1}{1-t^2}$. 也是就此打住。如果把右边的微分展开，将会如上面那样复杂。直接来

$$\begin{aligned}I&:=\int\sqrt{\frac{x}{x+1}}\mathrm{d}x\\&=\int t\mathrm{d}\frac{1}{1-t^2}\\&=\frac{t}{1-t^2}-\int\frac{1}{1-t^2}\mathrm{d}t\\&=\frac{t}{1-t^2}+\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+C\\&=(x+1)\sqrt{\frac{x}{x+1}}+\frac{1}{2}\ln\left|\frac{\sqrt{\frac{x}{x+1}}-1}{\sqrt{\frac{x}{x+1}}+1}\right|+C\end{aligned}$$

2.

令 $t=\sqrt{\frac{x+1}{x}}$ ，那么 $t^2=1+\frac{1}{x}$ ，$x=\frac{1}{t^2-1}$ ，$\mathrm{d}x=\mathrm{d}\frac{1}{t^2-1}.$

$$\begin{aligned}I&:=\int\frac{1}{x}\cdot\sqrt{\frac{x+1}{x}}\mathrm{d}x\\&=\int(t^2-1)\cdot t\cdot\mathrm{d}\frac{1}{t^2-1}\\&=t-\int\frac{1}{t^2-1}\mathrm{d}(t^3-t)\\&=t-\int\frac{1}{t^2-1}\cdot(3t^2-1)\mathrm{d}t\\&=t-\int\frac{1}{t^2-1}\cdot(3t^2-3)\mathrm{d}t-2\int\frac{1}{t^2-1}\mathrm{d}t\\&=t-3t-\ln\left|\frac{t-1}{t+1}\right|+C\\&=-2t-\ln\left|\frac{t-1}{t+1}\right|+C\\&=-2\sqrt{\frac{x+1}{x}}-\ln\left|\frac{\sqrt{\frac{x+1}{x}}-1}{\sqrt{\frac{x+1}{x}}+1}\right|+C\end{aligned}$$

3.

令 $t=\sqrt{\frac{1-x}{1+x}}$ ，$t^2=\frac{1-x}{1+x}=\frac{1+x-2x}{1+x}=1-\frac{2x}{1+x}$ ，$\frac{1+x-1}{1+x}=1-\frac{1}{1+x}=\frac{1}{2}(1-t^2)$ ，$\frac{1}{1+x}=\frac{1}{2}(1+t^2)$，$1+x=\frac{2}{1+t^2}$. $\mathrm{d}x=\mathrm{d}\frac{2}{1+t^2}$.

$$
\begin{aligned}
I&:=\int\sqrt{\frac{1-x}{1+x}}\mathrm{d}x\\
&=\int t\mathrm{d}\frac{2}{1+t^2}\\
&=\frac{2t}{1+t^2}-2\int\frac{1}{1+t^2}\mathrm{d}t\\
&=\frac{2t}{1+t^2}-2\arctan t+C\\
&=(1+x)\sqrt{\frac{1-x}{1+x}}-2\arctan\sqrt{\frac{1-x}{1+x}}+C
\end{aligned}
$$

4.

$$
\begin{aligned}
f(x)&:=\sqrt{\frac{1-x}{1+x}}+\sqrt{\frac{1+x}{1-x}}\\
&=\frac{1-x}{\sqrt{1-x^2}}+\frac{1+x}{\sqrt{1-x^2}}\\
&=\frac{2}{\sqrt{1-x^2}}
\end{aligned}
$$

故

$$
\begin{aligned}
I&:=\int\left(\sqrt{\frac{1-x}{1+x}}+\sqrt{\frac{1+x}{1-x}}\right)\mathrm{d}x\\
&=2\int\frac{1}{\sqrt{1-x^2}}\mathrm{d}x\\
&=2\arcsin x+C
\end{aligned}
$$

5.

令 $t=\sqrt{e^x-2}$ ，$\ln(t^2+2)=x$，$\mathrm{d}x=\frac{2t}{t^2+2}\mathrm{d}t$.

$$
\begin{aligned}
I&:=\int\frac{xe^x}{\sqrt{e^x-2}}\mathrm{d}x\\
&=\int\frac{\ln(t^2+2)(t^2+2)}{t}\cdot\frac{2t}{t^2+2}\mathrm{d}t\\
&=2\int\ln(t^2+2)\mathrm{d}t\\
&=2t\ln(t^2+2)-4\int\frac{t^2}{t^2+2}\mathrm{d}t\\
&=2t\ln(t^2+2)-4\int\frac{t^2+2-2}{t^2+2}\mathrm{d}t\\
&=2t\ln(t^2+2)-4t+8\int\frac{1}{t^2+2}\mathrm{d}t\\
&=2t\ln(t^2+2)-4t+\frac{8}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}+C\\
&=2x\sqrt{e^x-2}-4\sqrt{e^x-2}+\frac{8}{\sqrt{2}}\arctan\frac{\sqrt{e^x-2}}{\sqrt{2}}+C
\end{aligned}
$$

6.

这里关键要把 $x+1,x-1$ 整出来。。

$$
\begin{aligned}
I &:= \int \frac{1}{\sqrt[3]{(x+1)^2(x-1)^4}} \mathrm{d}x \\
&= \int \frac{1}{(x-1)\sqrt[3]{(x+1)^2(x-1)}} \mathrm{d}x \\
&= \int \frac{1}{(x^2-1)\sqrt[3]{\frac{x-1}{x+1}}} \mathrm{d}x
\end{aligned}
$$

令 $t = \sqrt[3]{\frac{x-1}{x+1}}$ ，$1+x = \frac{2}{1-t^2}$，$x-1 = \frac{2t^2}{1-t^2}$，$\mathrm{d}x = \mathrm{d}\frac{2}{1-t^3}$.

$$
\begin{aligned}
I &:= \int \frac{1}{\sqrt[3]{(x+1)^2(x-1)^4}} \mathrm{d}x \\
&= \int \frac{1}{(x^2-1)\sqrt[3]{\frac{x-1}{x+1}}} \mathrm{d}x \\
&= \int \frac{1}{\frac{4t^2}{(1-t^2)^2} \cdot t} \cdot \mathrm{d}\frac{2}{1-t^2} \\
&= \frac{1}{2} \int \frac{(1-t^2)^2}{t^3} \cdot \frac{2t}{(1-t^2)^2} \mathrm{d}t \\
&= \int \frac{1}{t^2} \mathrm{d}t \\
&= -t^{-1} + C \\
&= -\sqrt[3]{\frac{x+1}{x-1}} + C
\end{aligned}
$$

7.

令 $x = t^6$ ，$\mathrm{d}x = 6t^5\mathrm{d}t$.

$$
\begin{aligned}
I &:= \int \frac{1}{(1+\sqrt[3]{x}) \cdot \sqrt{x}} \mathrm{d}x \\
&= 6 \int \frac{t^2}{1+t^2} \mathrm{d}t \\
&= 6t - 6 \arctan t + C \\
&= 6\sqrt[6]{x} - 6 \arctan \sqrt[6]{x} + C
\end{aligned}
$$

8.

令 $\exp \frac{x}{6} = t$，$x = 6 \ln t$ ，$\mathrm{d}x = \frac{6}{t}\mathrm{d}t$.

$$
\begin{aligned}
I &:= \int \frac{1}{1+\exp\frac{x}{2}+\exp\frac{x}{3}+\exp\frac{x}{6}}\mathrm{d}x \\
&= 6\int \frac{1}{1+t^3+t^2+t}\cdot\frac{1}{t}\mathrm{d}t \\
&= 6\int \frac{1-t}{1-t^4}\cdot\frac{1}{t}\mathrm{d}t \\
&= 6\int \frac{1-t}{(1+t^2)(1-t^2)}\cdot\frac{1}{t}\mathrm{d}t \\
&= 6\int \frac{1}{t(1+t^2)(1-t)}\mathrm{d}t \\
&= 6\int \left(\frac{A}{t}+\frac{Bt+C}{1+t^2}+\frac{D}{1-t}\right)\mathrm{d}t \\
&= 6\int \left(\frac{1}{t}+\frac{-\frac{1}{2}t+\frac{1}{2}}{1+t^2}+\frac{\frac{1}{2}}{1-t}\right)\mathrm{d}t \\
&= \int \left(\frac{6}{t}+\frac{-3t+3}{1+t^2}+\frac{3}{1-t}\right)\mathrm{d}t \\
&= 6\ln|t|-3\ln|t-1|-3\int\frac{t}{1+t^2}\mathrm{d}t+3\int\frac{1}{1+t^2}\mathrm{d}t \\
&= 6\ln|t|-3\ln|t-1|-\frac{3}{2}\ln|1+t^2|+3\arctan t+C \\
&= x-3\ln\left|\exp\frac{x}{6}-1\right|-\frac{3}{2}\ln\left|1+\exp\frac{x}{3}\right|+3\arctan\exp\frac{x}{6}+C
\end{aligned}
$$

9.

令 $x=2\sin t$ ， $\mathrm{d}x=2\cos t\mathrm{d}t$.

$$
\begin{aligned}
I &:= \int \frac{\sqrt{4-x^2}}{x^4}\mathrm{d}x \\
&= \int \frac{2\cos t}{16\sin^4 t}\cdot 2\cos t\mathrm{d}t \\
&= \frac{1}{4}\int\frac{\cos^2 t}{\sin^4 t}\mathrm{d}t \\
&= \frac{1}{4}\int\frac{1}{\tan^4 t}\mathrm{d}\tan t \\
&= -\frac{1}{12}\tan^{-3} t+C \\
&= -\frac{1}{12}\frac{\sqrt{(1-x^2/4)^3}}{(x^3/8)}+C \\
&= -\frac{1}{12}\frac{\sqrt{(4-x^2)^3}}{x^3}+C
\end{aligned}
$$

10.

令 $x=\tan t$ ， $\mathrm{d}x=\sec^2 t\mathrm{d}t$.

$$
\begin{aligned}
I &:= \int \frac{1}{\sqrt{(x^2+1)^3}} \mathrm{d}x \\
&= \int \frac{1}{\sec^3 t} \cdot \sec^2 t \mathrm{d}t \\
&= \int \cos t \mathrm{d}t \\
&= \sin t + C \\
&= \frac{\tan t}{\sqrt{\sec^2 t}} + C \\
&= \frac{x}{\sqrt{1+x^2}} + C
\end{aligned}
$$

11.

$$
\begin{aligned}
I &:= \int \frac{1}{\sqrt{x(4-x)}} \mathrm{d}x \\
&= \int \frac{1}{\sqrt{4x-x^2}} \mathrm{d}x \\
&= \int \frac{1}{\sqrt{4-(x-2)^2}} \mathrm{d}x
\end{aligned}
$$

令 $x-2 = 2\sin t$， $\mathrm{d}x = 2\cos t \mathrm{d}t$.

$$
\begin{aligned}
I &:= \int \frac{1}{\sqrt{x(4-x)}} \mathrm{d}x \\
&= \int \frac{1}{\sqrt{4-(x-2)^2}} \mathrm{d}x \\
&= \int \frac{2\cos t}{2\cos t} \mathrm{d}t \\
&= t + C \\
&= \arcsin \frac{x-2}{2} + C
\end{aligned}
$$

12.

$$
\begin{aligned}
I &:= \int x\sqrt{2x-x^2} \mathrm{d}x \\
&= \int x\sqrt{1-(x-1)^2} \mathrm{d}x
\end{aligned}
$$

令 $x-1 = \sin t$， $\mathrm{d}x = \cos t \mathrm{d}t$.

$$
\begin{aligned}
I &:= \int x\sqrt{2x-x^2} \mathrm{d}x \\
&= \int x\sqrt{1-(x-1)^2} \mathrm{d}x \\
&= \int (\sin t + 1)\cos^2 t \mathrm{d}t \\
&= -\int \cos^2 t \mathrm{d} \cos t + \int \frac{\cos 2x + 1}{2} \mathrm{d}t \\
&= -\frac{1}{3}\cos^3 t + \frac{1}{4}\sin 2x + \frac{1}{2}t + C \\
&= -\frac{1}{3}\sqrt{(2x-x^2)^3} + \frac{1}{4}(x-1)\sqrt{2x-x^2} + \frac{1}{2}\arcsin(x-1) + C
\end{aligned}
$$

13.

法1：三角换元

令 $x=\sec t$ ， $\mathrm{d}x=\sec t\tan t\mathrm{d}t.$

$$\begin{aligned}
I&:=\int\frac{1}{x^2\sqrt{x^2-1}}\mathrm{d}x\\
&=\int\frac{\sec t\tan t}{\sec^2 t\tan t}\mathrm{d}t\\
&=\int\cos t\mathrm{d}t\\
&=\sin t+C\\
&=\sqrt{1-\sec^{-2}t}+C\\
&=\sqrt{1-x^{-2}}+C
\end{aligned}$$

法2：倒代换 + 三角换元

令 $x=\frac{1}{t}$ ， $\mathrm{d}x=-\frac{1}{t^2}\mathrm{d}t.$

$$\begin{aligned}
I&:=\int\frac{1}{x^2\sqrt{x^2-1}}\mathrm{d}x\\
&=-\int t^2\cdot\sqrt{t^{-2}-1}\cdot\frac{1}{t^2}\mathrm{d}t\\
&=-\int\frac{\sqrt{1-t^2}}{t}\mathrm{d}t
\end{aligned}$$

令 $t=\sin u$ ， $\mathrm{d}t=\cos u\mathrm{d}u$.

$$\begin{aligned}
I&=-\int\frac{\sqrt{1-t^2}}{t}\mathrm{d}t\\
&=-\int\frac{\cos^2 u}{\sin u}\mathrm{d}u\\
&=-\int\frac{1-\sin^2 u}{\sin u}\mathrm{d}u\\
&=-\int\csc u\mathrm{d}u+\int\sin u\mathrm{d}u\\
&=-\ln|\cot u-\csc u|-\cos u+C
\end{aligned}$$

回代疯了。

14.

这题用倒代换才好一点。

令 $x=t^{-1}$ ， $\mathrm{d}x=-t^{-2}\mathrm{d}t.$

$$\begin{aligned}
I&:=\int\frac{1}{x\sqrt{2x^2+2x+1}}\mathrm{d}x\\
&=-\int\frac{t}{\sqrt{2t^{-2}+2t^{-1}+1}}\cdot t^{-2}\mathrm{d}t\\
&=-\int\frac{1}{\sqrt{2+2t+t^2}}\mathrm{d}t\\
&=-\int\frac{1}{\sqrt{(t+1)^2+1}}\mathrm{d}t\\
&=-\ln\left[t+1+\sqrt{(t+1)^2+1}\right]+C\\
&=-\ln\left[x^{-1}+1+\sqrt{(x^{-1}+1)^2+1}\right]+C
\end{aligned}$$

15.

如 14 令

$$
\begin{aligned}
I &:= \int \frac{1}{x^2\sqrt{2x^2+2x+1}}\mathrm{d}x \\
&= -\int \frac{t^2}{\sqrt{2t^{-2}+2t^{-1}+1}} \cdot t^{-2}\mathrm{d}t \\
&= -\int \frac{t}{\sqrt{2+2t+t^2}}\mathrm{d}t \\
&= -\int \frac{t+1-1}{\sqrt{(t+1)^2+1}}\mathrm{d}t \\
&= -\frac{1}{2}\int \frac{1}{\sqrt{(t+1)^2+1}}\mathrm{d}[(t+1)^2] + \int \frac{1}{\sqrt{(t+1)^2+1}}\mathrm{d}t \\
&= -\frac{1}{2}\int \frac{\mathrm{d}u}{\sqrt{u+1}} + \ln\left(t+1+\sqrt{(t+1)^2+1}\right) + C \\
&= \cdots
\end{aligned}
$$

16.

$$
\begin{aligned}
I &:= \int x\arctan x\mathrm{d}x \\
&= \frac{1}{2}\int \arctan x\mathrm{d}(x^2+{\color{blue}1}) \\
&= \frac{1}{2}(x^2+1)\arctan x - \frac{1}{2}\int (x^2+{\color{blue}1}) \cdot \frac{1}{1+x^2}\mathrm{d}(x^2+1) \\
&= \frac{1}{2}(x^2+1)\arctan x - \frac{1}{2}x + C
\end{aligned}
$$

注意，适时地添加常数（蓝色部分），使积分大大简化。

17.

$$
\begin{aligned}
I &:= \int x\ln(1+x^2)\arctan x\mathrm{d}x \\
&= \frac{1}{2}\int \ln(1+x^2)\arctan x\mathrm{d}(x^2+1) \\
&= \frac{1}{2}(x^2+1)\ln(1+x^2)\arctan x - \frac{1}{2}\int (x^2+1)\mathrm{d}\ln(1+x^2)\arctan x \\
&= \frac{1}{2}(x^2+1)\ln(1+x^2)\arctan x - \frac{1}{2}\int (x^2+1) \cdot \left[\frac{2x\arctan x}{x^2+1} + \frac{\ln(1+x^2)}{x^2+1}\right]\mathrm{d}x \\
&= \frac{1}{2}(x^2+1)\ln(1+x^2)\arctan x - \frac{1}{2}\int \left[2x\arctan x + \ln(1+x^2)\right]\mathrm{d}x \\
&= \frac{1}{2}(x^2+1)\ln(1+x^2)\arctan x + I_1 + I_2
\end{aligned}
$$

其中

$$
\begin{aligned}
I_1 &:= \int x\arctan x\mathrm{d}x \\
&= \frac{1}{2}\int \arctan x\mathrm{d}(x^2+1) \\
&= \frac{1}{2}(x^2+1)\arctan x - \frac{1}{2}x + C_1
\end{aligned}
$$

$$
\begin{aligned}
I_2 &:= \frac{1}{2}\int \ln(1+x^2)\mathrm{d}x \\
&= x\ln(1+x^2) - \frac{1}{2}\int x \cdot \frac{2x}{1+x^2}\mathrm{d}x \\
&= \frac{1}{2}x\ln(1+x^2) - \int \frac{x^2+1-1}{1+x^2}\mathrm{d}x \\
&= \frac{1}{2}x\ln(1+x^2) - x + \arctan x + C_2
\end{aligned}
$$

故

$$I=\frac{1}{2}(x^2+1)\ln(1+x^2)\arctan x+\frac{1}{2}(x^2+1)\arctan x-\frac{1}{2}x+\frac{1}{2}x\ln(1+x^2)-x+\arctan x+C$$