凯哥不定积分笔记——4

- 套路1：

  对复杂部分求导，期待奇迹发生

  - 一个模板

$$\int \frac{1+xf'(x)}{x(1+xe^{f(x)})}\mathrm{d}x$$

题目列表：

1.

$$\int \frac{\ln x}{\sqrt{1+[x(\ln x-1)]^2}}\mathrm{d}x$$

2.

$$\int \frac{x+1}{x\cdot(1+xe^x)}\mathrm{d}x$$

3.

$$\int \frac{1+x\cos x}{x\cdot(1+xe^{\sin x})}\mathrm{d}x$$

4.

$$\int \frac{1-\ln x}{(x-\ln x)^2}\mathrm{d}x$$

5.

$$\int \frac{e^x(x-1)}{(x-e^x)^2}\mathrm{d}x$$

6.

$$\int \frac{x+\sin x\cos x}{(\cos x-x\sin x)^2}\mathrm{d}x$$

解答之：

1.
注意到

$$\begin{aligned} f'(x) &:= [x(\ln x-1)]' \\ &= (\ln x-1)+x\cdot\frac{1}{x} \\ &= \ln x \end{aligned}$$

故

$$
\begin{aligned}
I &:= \int \frac{\ln x}{\sqrt{1+[x(\ln x-1)]^2}}\mathrm{d}x \\
&= \int \frac{1}{\sqrt{1+[x(\ln x-1)]^2}}\mathrm{d}[x(\ln x-1)] \\
&= \int \frac{1}{\sqrt{1+t^2}}\mathrm{d}t \\
&= \ln(t+\sqrt{1+t^2})+C \\
&= \ln\left\{[x(\ln x-1)]+\sqrt{1+[x(\ln x-1)]^2}\right\}+C
\end{aligned}
$$

2.

$$
\begin{aligned}
I &:= \int \frac{x+1}{x\cdot(1+xe^x)}\mathrm{d}x \\
&= \int \frac{e^x(x+1)}{xe^x\cdot(1+xe^x)}\mathrm{d}x \\
&= \int \frac{1}{xe^x\cdot(1+xe^x)}\mathrm{d}(xe^x) \\
&= \int \frac{1}{t(t+1)}\mathrm{d}t \\
&= \int \frac{t+1-t}{t(t+1)}\mathrm{d}t \\
&= \int \frac{1}{t}\mathrm{d}t - \int \frac{1}{t+1}\mathrm{d}t \\
&= \ln\left|\frac{t}{t+1}\right|+C \\
&= \ln\left|\frac{xe^x}{xe^x+1}\right|+C
\end{aligned}
$$

3.

$$
\begin{aligned}
I &:= \int \frac{1+x\cos x}{x\cdot(1+xe^{\sin x})}\mathrm{d}x \\
&= \int \frac{e^{\sin x}(1+x\cos x)}{xe^{\sin x}\cdot(1+xe^{\sin x})}\mathrm{d}x \\
&= \int \frac{1}{xe^{\sin x}\cdot(1+xe^{\sin x})}\mathrm{d}(xe^{\sin x}) \\
&= \ln\left|\frac{xe^{\sin x}}{xe^{\sin x}+1}\right|+C
\end{aligned}
$$

4.

此时发现分母复杂函数左边还多了个函数 x，考虑给它整没。

即

$$
\begin{aligned}
I &:= \int \frac{1-\ln x}{(x-\ln x)^2}\mathrm{d}x \\
&= \int \frac{1-\ln x}{x^2\left(1-\frac{\ln x}{x}\right)^2}\mathrm{d}x
\end{aligned}
$$

此时把复杂项求导看看

$$
\begin{aligned}
\left(\frac{\ln x}{x}\right)' &= \frac{\frac{1}{x}\cdot x-\ln x}{x^2} \\
&= \frac{1-\ln x}{x^2}
\end{aligned}
$$

bravo，故

$$
\begin{aligned}
I &:= \int \frac{1-\ln x}{(x-\ln x)^2}\mathrm{d}x \\
&= \int \frac{1}{\left(1-\frac{\ln x}{x}\right)^2}\mathrm{d}\frac{\ln x}{x} \\
&= -\frac{1}{\frac{\ln x}{x}-1}+C \\
&= \frac{x}{x-\ln x}+C
\end{aligned}
$$

5.

$$
\begin{aligned}
I &:= \int \frac{e^x(x-1)}{(x-e^x)^2}\mathrm{d}x \\
&= \int \frac{e^x(x-1)}{x^2\left(1-\frac{1}{x}\cdot e^x\right)^2}\mathrm{d}x \\
&= \int \frac{1}{\left(1-\frac{1}{x}\cdot e^x\right)^2}\mathrm{d}\frac{e^x}{x} \\
&= -\frac{1}{\frac{1}{x}\cdot e^x-1}+C \\
&= \frac{x}{x-e^x}+C
\end{aligned}
$$

6.

$$
\begin{aligned}
I &:= \int \frac{x+\sin x\cos x}{(\cos x - x\sin x)^2}\mathrm{d}x \\
&= \int \frac{x+\sin x\cos x}{\cos^2 x(1-x\tan x)^2}\mathrm{d}x
\end{aligned}
$$

又

$$
\begin{aligned}
(x\tan x)' &= \tan x + x\sec^2 x \\
&= \frac{\sin x\cos x + x}{\cos^2 x}
\end{aligned}
$$

故

$$
\begin{aligned}
I &:= \int \frac{x+\sin x\cos x}{(\cos x - x\sin x)^2}\mathrm{d}x \\
&= \int \frac{x+\sin x\cos x}{\cos^2 x(1-x\tan x)^2}\mathrm{d}x \\
&= \int \frac{1}{(x\tan x-1)^2}\mathrm{d}(x\tan x) \\
&= \frac{1}{1-x\tan x}+C
\end{aligned}
$$