Math 112 Lecture Notes

Companion notes to the required "Math 112 Course Notes"

Jake Callahan

Section 2.3
Quadratic Functions and Applications

Jake Callahan

Basics of quadratic functions

What is a quadratic function?

You're likely very familiar with quadratic functions, even if you don't know them by that name. They're also called **parabolas**. Quadratic functions are extremely common in most areas math is used. Let's write down a precise definition:

Definition: Quadratic Function

Any function of x such that the largest power of x is 2 is called a ***quadratic function*** or a ***quadratic equation***. The graph of a quadratic function is a parabola. See Figure 2.1 for examples of what these parabolas can look like. For even more examples, see the Wikipedia page on parabolas in the physical world.

Question: Why do we care about quadratic functions?

Answer: Lots of things in the real world are related to each other in a quadratic way or look like parabolas. See Figure 2.2 for examples of things in everyday life that can be modeled by quadratic functions.

For example:

- Many bridges are shaped like parabolas
- Economic laws (like the Laffer curve) are described by quadratic functions
- Satellite dishes are shaped like parabolas
- The motion of objects often follows a parabolic shape

Remember, functions model relationships between things! If you can use a quadratic function to model an object's height after throwing it in the air, you're describing the relationship between the height of the object (y or $f(x)$) and the time after it's been thrown (x).

Important characteristics of a quadratic function

Every quadratic function has three points that we care a lot about:

- The vertex, which describes the lowest (or highest) point of the parabola and gives the line on which the parabola is symmetric
- Two *roots*, *zeros*, or *x-intercepts* (different names for the same thing). These two points are the places where the parabola touches the x-axis (graphical interpretation) and where the equation is equal to 0: $f(x) = 0$ (algebraic interpretation).

Different forms of a quadratic equation

A quadratic function can be written in several different forms.

Definition: Standard form

The ***standard form*** of a quadratic function is

$$f(x) = ax^2 + bx + c,$$

where a, b, and c are constant numbers and a is not equal to 0.

Definition: Vertex form

The ***vertex form*** of a quadratic equation is

$$f(x) = a(x - h)^2 + k,$$

where a, h, and k are constant numbers and a is not equal to 0. You can read the vertex of the equation's graph from this form. The vertex is found at the point

$$(h, k).$$

Definition: Factored form

The ***factored form*** of a quadratic equation is

$$f(x) = a(x - r_1)(x - r_2),$$

where a is a constant number not equal to 0 and r_1 and r_2 are the *roots* of the quadratic equation.

Important skills for working with quadratic equations

Factoring (finding roots of a quadratic equation by hand)

Factoring allows us to take an equation in *standard from* and write it in *factored form*. When an equation is in factored form, we can easily read the roots of the equation off from the factors.

Formula: Factoring a polynomial

For a polynomial in standard form ($f(x) = ax^2 + bx + c$):

1. First see if there is a common factor between a, b, and c. If there is, factoring it out might simplify the problem.
2. Next, multipy a times c
3. List all the factor pairs of $a \cdot c$
4. Identify the two factors f_1 and f_2 of $a \cdot c$ that sum to b
5. Rewrite $f(x)$ using these two factors: $f(x) = ax^2 + f_1 x + f_2 x + c$
6. Factor out common terms to get factored form
7. To find the roots of the function, set each factor to 0 and solve for x.

See this page for a more in depth explanation of factoring.

Example 1 (Page 143, problem 2)
Factor the following:

$$3x^2 - x + 9x - 3.$$

Solution:

1. To do this exactly according to the formula above, first combine $-x$ term and the $9x$ term (we'll discuss how to take a shortcut at the end):

$$3x^2 + {\color{red}-x + 9x} - 3 = 3x^2 + 8x - 3$$

2. Now we have a function in standard form. Here, $a = 3$, $b = 8$, and $c = -3$. These don't share any common factors, so we can't factor anything out to start.
3. Multiply a times c. Here, that means multiplying 3 times -3 to get -9.
4. List factor pairs of -9:

$$\begin{array}{rcr} -1 & \text{and} & 9 \\ 1 & \text{and} & -9 \\ 3 & \text{and} & -3 \\ -3 & \text{and} & 3. \end{array}$$

5. Identify which factor pair sums to b. In this case, $b = 8$, and we see that -1 and 9 sum to 8.
6. Split up our x term using this factor pair:

$$3x^2 + {\color{red}8x} - 3 = 3x^2 + {\color{red}9x - 1x} - 3$$

7. Now we look at the two terms on the left and the two terms on the right. Notice that the two terms on the right share a common factor in $3x$. Factor that out. Also notice that the two terms on the right share a common factor in -1. Factor that out as well:

$$3x^2 + 9x - x - 3 = 3x\,(x + 3) + (-1)(x + 3)$$

8. Now we have two things being multiplied by $(x + 3)$, so we can factor $(x + 3)$ from both to get

$$3x\,(x + 3) + (-1)(x + 3) = (x + 3)(3x - 1).$$

9. Our function written in standard form is

$$f(x) = (x + 3)(3x - 1).$$

To find the roots of this function, set each factor to 0 and solve for x!
For a shortcut, observe that the function we got to in step 6 was exactly the function we started with! In this case, if you recognize that the original function is in the form we want for factoring, you can start on step 7 and go from there. If you don't recognize that, you can just apply the process from the start and you'll get the same answer.

Practice in groups: Page 143, questions 1, 2, 4-7.

Finding the vertex of a parabola

There are two ways to find the vertex of a parabola: One if the equation is in standard form and one if the equation is in vertex form.

Formula: Finding the vertex: Standard form

In standard form $f(x) = ax^2 + bx + c$, the x-coordinate for the vertex is

$$\frac{-b}{2a}.$$

The y-coordinate can be found by plugging $\frac{-b}{2a}$ into $f(x)$ and evaluating.

Think about this: Why is this the formula for the x-coordinate of the vertex? See this great video for an explanation.

Formula: Finding the vertex: Vertex form

If your equation is in vertex form $f(x) = a(x - h)^2 + k$, the coordinates for your vertex are

$$(h, k).$$

Using the Quadratic Formula to find zeros

Sometimes it's impossible to factor equations by hand. When that is the case, we can use the Quadratic Formula to find the zeros.

Formula: The Quadratic Formula

The zeros of a quadratic function are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The Quadratic Formula can be programmed into your calculator! This is allowed on the exam. Bring calculator transfer cables to class next time and we will transfer the program to your calculator. Instructions for programming the Quadratic Formula into your calculator yourself are on Page 8 of the class notes.

Example 2 (Page 143, problem 8)
Use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to solve the equation

$$-2x^2 - 8x + 3 = 0.$$

Find the exact solutions, and then approximate the solutions to two decimal places.

Solution:

1. First we need to identify a, b, and c. In this case:

$$a = -2, \quad b = -8, \quad c = 3.$$

2. Plug these into the quadratic formula and simplify:

$$\begin{aligned}
x &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(-2)(3)}}{2(-2)} \\
&= \frac{-(-8) \pm \sqrt{64 - 4(-2)(3)}}{2(-2)} && \text{(square } -8\text{)} \\
&= \frac{-(-8) \pm \sqrt{64 - (-24)}}{2(-2)} && \text{(multiply 4 by 2 by } -3 \text{ on top)} \\
&= \frac{-(-8) \pm \sqrt{64 - (-24)}}{-4} && \text{(multiply 2 by } -2 \text{ on bottom)} \\
&= \frac{8 \pm \sqrt{64+24}}{-4} && \text{(cancel negatives)} \\
&= \frac{8 \pm \sqrt{88}}{-4} && \text{(simplify square root)}
\end{aligned}$$

3. The equation has two roots:

$$r_1 = -\frac{8+\sqrt{88}}{4}$$
$$r_2 = -\frac{8-\sqrt{88}}{4}.$$

Rounding them to the nearest two decimal places gives

$$r_1 \approx -4.25$$
$$r_2 \approx 0.35.$$

Solving quadratic equations

Example 3 (Page 143, problem 6)
Solve

$$4p^2 - 11 = 0.$$

Do not round.

Solution:

$$\begin{aligned} 4p^2 - 11 &= 0 \\ 4p^2 - 11 + 11 &= 0 + +11 && \text{(add 11 to both sides)} \\ \frac{4p^2}{4} &= \frac{11}{4} && \text{(divide both sides by 4)} \\ \sqrt{p^2} &= \pm\sqrt{\frac{11}{4}} && \text{(take the square root of both sides)} \\ p &= \pm\sqrt{\frac{11}{4}}. \end{aligned}$$

Warning: Square roots

Notice that when we took the square root of $\frac{11}{4}$ in Example 3, we got both $\frac{11}{4}$ and $-\frac{11}{4}$ as answers. This is because squaring a number always makes it positive. This means, for example, that if $a^2 = 4$, then a could either be 2 or -2, since both 2^2 and $(-2)^2$ equal 4. When solving an equation that has a square, both the positive and negative square root will be answers!

Practice in groups: Page 143, problems 5 and 7.

Applications

Example 4: Page 150 A sports team has 1500 feet of fencing to enclose a rectangular field. What is the maximum area that can be enclosed? What are the dimensions (length and width) of the enclosed field?

Solution: Let's translate the problem into equations we know. We can find the perimeter of a shape by adding up the length of all sides. Since it's a rectangle, we have four sides and two sides are equal to the length of the rectangle (ℓ) and two sides are equal to the width (w). Then we know that the perimeter of the fence can be found using the equation

$$1500 = w + w + \ell + \ell = 2w + 2\ell.$$

The equation for area (A) is length(ℓ) times width (w), so we can find the area using the equation

$$A = \ell w.$$

We're trying to make A as big as we can possibly get it. We don't really know how to work with multiple variables, so let's try to make A a function of a single variable. Luckily, we have a second equation we can use. Let's solve for ℓ using our perimeter equation and substitute:

$$\begin{aligned} 1500 &= 2w + 2\ell \\ 1500 - 2w &= 2w + 2\ell - 2w && \text{(subtract } 2w \text{ from both sides)} \\ \frac{1500 - 2w}{2} &= \frac{2\ell}{2} && \text{(divide both sides by 2)} \\ 750 - w &= \ell && \text{(simplify the left side)} \end{aligned}$$

So we have that in order for our perimeter to be equal to 1500 feet, ℓ has to be equal to $750 - w$ feet for whatever number w we choose. Now let's substitute this into our area equation to get area as a function of width:

$$\begin{aligned} A &= \ell w \\ A &= (750 - w)\, w \\ A &= 750w - w^2 && \text{(distribute } w\text{)} \end{aligned}$$

Now we have a quadratic equation! We've shown that the area inside the fence can be related to the width of the perimeter in a quadratic way. Remember that the maximum or minimum of a quadratic is found at the vertex, depending on which way the parabola is pointing. This quadratic has a negative in front of the x^2 term, so the parabola opens down. That means the maximum is found at the vertex. We can use the vertex formula for standard form to get that our width must be

$$w = \frac{-b}{2a} = \frac{-750}{(-1)(2)} = 375 \text{ feet.}$$

Plug this back into our function for ℓ and simplify:

$$\begin{aligned} \ell &= 750 - w \\ &= 750 - 375 \\ &= 375 \text{ feet.} \end{aligned}$$

We've found that the fence encloses maximum area when the fence is 375 feet wide and 375 feet long.